

ALGEBRA I

First edition**13 units****71 lessons****About 11 hours of video****FIRST EDITION**

COURSE GUIDE

Algebra I teaches the language of algebra: how to write, read and solve equations and inequalities; how functions describe relationships and how their graphs show them; how to solve systems of equations; how exponents, polynomials and factoring work; how to simplify and solve with radicals; how to graph and solve quadratics; and a first look at rational expressions and at lines of fit for real data. These are the tools you will use in Geometry, Algebra II, the sciences and every kind of practical work with numbers.

The approach is simple and steady. You will not only learn how to do each step but also understand why it works, because key results are derived carefully, at a level a 14-year-old can follow. Then you will master the skills through daily practice, with answers to check and mistakes to correct. Along the way, short history notes show how algebra was discovered, from Diophantus and al-Khwarizmi to Viète and Descartes, and each lesson gives one practice habit, such as a mental-math trick, a form to recite until it is automatic, or a working habit like checking every answer by substituting.

Most videos are by Eddie Woo, a veteran classroom teacher in Australia. Math with Mr. J and Math Antics fill in. Eddie Woo says “gradient” for slope, “simultaneous equations” for systems, “index laws” for exponent rules, “surds” for radicals, “factorise” for factor and “turning point” for vertex. Each lesson notes where a video’s words differ from the course.

Start with Lesson 1.1 →

WHO IT IS FOR

For students in grades 8–9 (ages 13–15) who have completed a full Pre-Algebra course (fractions, integers, order of operations, one- and two-step equations and basic graphing); it prepares them for Geometry and Algebra II.

HOW TO USE THIS COURSE

1. The course fills one school year: 32 weeks, 5 days a week, about 45 minutes a day (sometimes a little longer).
2. Daily rhythm: start with the lesson's practice habit, read the core ideas with pencil in hand, work through the two examples yourself, do the practice set, check your answers, and correct every mistake.
3. Two-day lessons: on day 1, study the teaching and examples and do the first half of the practice; on day 2, review the core ideas and do the second half.
4. At the end of each unit, review with the mastery checklist, correct any weak spots, and then take the unit test.
5. The midterm exam comes after Unit 6 (end of week 15) and the final exam comes in week 32.
6. Calculators: Units 1–6 are done by hand unless a problem says otherwise; a scientific calculator is used from Unit 7 on. A graphing calculator or online graphing tool may be used only to check graphs, never instead of the algebra.
7. Use graph paper for every graph, and keep all your work in one notebook so you can look back at it.
8. Students who need more time may stretch the course; understanding each unit matters more than finishing on a fixed date.

SCOPE AND SEQUENCE

UNIT 1 • WEEKS 1–2

Foundations: Real Numbers and Expressions

Classify and order real numbers, name the properties of arithmetic, evaluate and simplify expressions, and translate words into algebra.

- 1.1 The Real Number System
- 1.2 Properties of Real Numbers
- 1.3 Order of Operations and Evaluating Expressions
- 1.4 Simplifying Expressions
- 1.5 Translating Words into Algebra

UNIT 2 • WEEKS 2–5

Solving Linear Equations

Solve linear equations of every common kind, rearrange formulas, and use equations to solve word problems.

- 2.1 Solving One- and Two-Step Equations
- 2.2 Multi-Step Equations
- 2.3 Equations with Variables on Both Sides
- 2.4 Equations with Fractions and Decimals
- 2.5 Literal Equations and Formulas
- 2.6 Problem Solving with Linear Equations

UNIT 3 • WEEKS 5–7

Linear Inequalities and Absolute Value

Solve and graph inequalities, compound inequalities, and absolute value equations and inequalities.

- 3.1 Solving Inequalities
- 3.2 Multi-Step Inequalities
- 3.3 Compound Inequalities
- 3.4 Absolute Value Equations
- 3.5 Absolute Value Inequalities

UNIT 5 • WEEKS 9–12

Linear Functions

Find slope, write and graph lines in slope-intercept, point-slope and standard form, and build linear models.

- 5.1 Rate of Change and Slope
- 5.2 Direct Variation
- 5.3 Slope-Intercept Form
- 5.4 Point-Slope Form
- 5.5 Standard Form and Intercepts
- 5.6 Parallel and Perpendicular Lines
- 5.7 Linear Models

UNIT 4 • WEEKS 7–9

Relations and Functions

Understand relations and functions, use function notation, find domain and range, and see sequences as functions.

- 4.1 The Coordinate Plane and Relations
- 4.2 What Is a Function?
- 4.3 Function Notation
- 4.4 Domain and Range
- 4.5 Sequences as Functions

UNIT 6 • WEEKS 12–15

Systems of Equations and Inequalities

Solve systems of linear equations by graphing, substitution and elimination, apply them to real problems, and graph systems of inequalities.

- 6.1 Solving Systems by Graphing
- 6.2 Solving Systems by Substitution
- 6.3 Solving Systems by Elimination
- 6.4 Elimination with Multiplication
- 6.5 Applications of Systems
- 6.6 Linear Inequalities and Systems of Inequalities

UNIT 7 • WEEKS 16–18

Exponents and Exponential Functions

Use the exponent rules, including zero and negative exponents, work in scientific notation, and model growth and decay with exponential functions and geometric sequences.

- 7.1 Properties of Exponents
- 7.2 Zero and Negative Exponents
- 7.3 Scientific Notation
- 7.4 Exponential Functions
- 7.5 Exponential Growth and Decay
- 7.6 Geometric Sequences

UNIT 9 • WEEKS 20–23

Factoring

Factor polynomials completely and use factoring to solve quadratic equations.

- 9.1 Factoring Out the Greatest Common Factor
- 9.2 Factoring $x^2 + bx + c$
- 9.3 Factoring $ax^2 + bx + c$
- 9.4 Factoring Special Products
- 9.5 Factoring by Grouping and Factoring Completely
- 9.6 Solving Equations by Factoring

UNIT 8 • WEEKS 18–20

Polynomials

Add, subtract, multiply and divide polynomials, including the special products.

- 8.1 Polynomials and Their Terms
- 8.2 Multiplying by a Monomial
- 8.3 Multiplying Binomials and Polynomials
- 8.4 Special Products
- 8.5 Dividing Polynomials by Monomials

UNIT 10 • WEEKS 23–25

Radicals

Simplify and operate with square roots, solve radical equations, and use the Pythagorean theorem, distance and midpoint formulas.

- 10.1 Simplifying Square Roots
- 10.2 Quotients and Rationalizing Denominators
- 10.3 Operations with Radical Expressions
- 10.4 Solving Radical Equations
- 10.5 The Pythagorean Theorem, Distance and Midpoint

UNIT 11 • WEEKS 25–28

Quadratic Functions and Equations

Graph parabolas and solve quadratic equations by graphing, square roots, completing the square and the quadratic formula, and use quadratic models.

- 11.1 Graphing $y = ax^2$
- 11.2 Graphing $y = ax^2 + bx + c$
- 11.3 Solving Quadratic Equations by Graphing and Square Roots
- 11.4 Completing the Square
- 11.5 The Quadratic Formula
- 11.6 The Discriminant and Choosing a Method
- 11.7 Vertex Form and Quadratic Models

UNIT 13 • WEEKS 31–32

Data and Statistics: Scatter Plots and Lines of Fit

Read scatter plots and correlation, write and use lines of fit, and choose among linear, exponential and quadratic models.

- 13.1 Scatter Plots and Correlation
- 13.2 Lines of Best Fit
- 13.3 Choosing a Model: Linear, Exponential or Quadratic

UNIT 12 • WEEKS 29–31

Rational Expressions: An Introduction

Work with inverse variation and simplify, combine and solve equations with rational expressions.

- 12.1 Inverse Variation
- 12.2 Simplifying Rational Expressions
- 12.3 Multiplying and Dividing Rational Expressions
- 12.4 Adding and Subtracting Rational Expressions
- 12.5 Solving Rational Equations

WEEKLY PACING CALENDAR

WEEK	PLAN
1	- Day 1: Lesson 1.1 The Real Number System - Day 2: Lesson 1.2 Properties of Real Numbers - Day 3: Lesson 1.3 Order of Operations and Evaluating Expressions - Day 4: Lesson 1.4 Simplifying Expressions (day 1 of 2) - Day 5: Lesson 1.4 Simplifying Expressions (day 2 of 2)

WEEK	PLAN
	<i>Families on a 4-day week can combine the two days of a two-day lesson when the student is ready, or stretch the course to about 40 weeks.</i>
2	- Day 1: Lesson 1.5 Translating Words into Algebra (day 1 of 2) - Day 2: Lesson 1.5 Translating Words into Algebra (day 2 of 2) - Day 3: Unit 1 review (mastery checklist and missed problems) - Day 4: Unit 1 test - Day 5: Lesson 2.1 Solving One- and Two-Step Equations
3	- Day 1: Lesson 2.2 Multi-Step Equations (day 1 of 2) - Day 2: Lesson 2.2 Multi-Step Equations (day 2 of 2) - Day 3: Lesson 2.3 Equations with Variables on Both Sides (day 1 of 2) - Day 4: Lesson 2.3 Equations with Variables on Both Sides (day 2 of 2) - Day 5: Lesson 2.4 Equations with Fractions and Decimals (day 1 of 2)
4	- Day 1: Lesson 2.4 Equations with Fractions and Decimals (day 2 of 2) - Day 2: Lesson 2.5 Literal Equations and Formulas (day 1 of 2) - Day 3: Lesson 2.5 Literal Equations and Formulas (day 2 of 2) - Day 4: Lesson 2.6 Problem Solving with Linear Equations (day 1 of 2) - Day 5: Lesson 2.6 Problem Solving with Linear Equations (day 2 of 2)
5	- Day 1: Unit 2 review (mastery checklist and missed problems) - Day 2: Unit 2 test - Day 3: Lesson 3.1 Solving Inequalities - Day 4: Lesson 3.2 Multi-Step Inequalities (day 1 of 2) - Day 5: Lesson 3.2 Multi-Step Inequalities (day 2 of 2)
6	- Day 1: Lesson 3.3 Compound Inequalities (day 1 of 2) - Day 2: Lesson 3.3 Compound Inequalities (day 2 of 2) - Day 3: Lesson 3.4 Absolute Value Equations (day 1 of 2) - Day 4: Lesson 3.4 Absolute Value Equations (day 2 of 2) - Day 5: Lesson 3.5 Absolute Value Inequalities (day 1 of 2)
7	- Day 1: Lesson 3.5 Absolute Value Inequalities (day 2 of 2) - Day 2: Unit 3 review (mastery checklist and missed problems) - Day 3: Unit 3 test - Day 4: Lesson 4.1 The Coordinate Plane and Relations - Day 5: Lesson 4.2 What Is a Function? (day 1 of 2)

WEEK	PLAN
8	- Day 1: Lesson 4.2 What Is a Function? (day 2 of 2) - Day 2: Lesson 4.3 Function Notation (day 1 of 2) - Day 3: Lesson 4.3 Function Notation (day 2 of 2) - Day 4: Lesson 4.4 Domain and Range (day 1 of 2) - Day 5: Lesson 4.4 Domain and Range (day 2 of 2)
9	- Day 1: Lesson 4.5 Sequences as Functions (day 1 of 2) - Day 2: Lesson 4.5 Sequences as Functions (day 2 of 2) - Day 3: Unit 4 review (mastery checklist and missed problems) - Day 4: Unit 4 test - Day 5: Lesson 5.1 Rate of Change and Slope (day 1 of 2)
10	- Day 1: Lesson 5.1 Rate of Change and Slope (day 2 of 2) - Day 2: Lesson 5.2 Direct Variation - Day 3: Lesson 5.3 Slope-Intercept Form (day 1 of 2) - Day 4: Lesson 5.3 Slope-Intercept Form (day 2 of 2) - Day 5: Lesson 5.4 Point-Slope Form (day 1 of 2)
11	- Day 1: Lesson 5.4 Point-Slope Form (day 2 of 2) - Day 2: Lesson 5.5 Standard Form and Intercepts (day 1 of 2) - Day 3: Lesson 5.5 Standard Form and Intercepts (day 2 of 2) - Day 4: Lesson 5.6 Parallel and Perpendicular Lines (day 1 of 2) - Day 5: Lesson 5.6 Parallel and Perpendicular Lines (day 2 of 2)
12	- Day 1: Lesson 5.7 Linear Models (day 1 of 2) - Day 2: Lesson 5.7 Linear Models (day 2 of 2) - Day 3: Unit 5 review (mastery checklist and missed problems) - Day 4: Unit 5 test - Day 5: Lesson 6.1 Solving Systems by Graphing (day 1 of 2)
13	- Day 1: Lesson 6.1 Solving Systems by Graphing (day 2 of 2) - Day 2: Lesson 6.2 Solving Systems by Substitution (day 1 of 2) - Day 3: Lesson 6.2 Solving Systems by Substitution (day 2 of 2) - Day 4: Lesson 6.3 Solving Systems by Elimination (day 1 of 2) - Day 5: Lesson 6.3 Solving Systems by Elimination (day 2 of 2)
14	- Day 1: Lesson 6.4 Elimination with Multiplication (day 1 of 2) - Day 2: Lesson 6.4 Elimination with Multiplication (day 2 of 2)

WEEK	PLAN
	- Day 3: Lesson 6.5 Applications of Systems (day 1 of 2) - Day 4: Lesson 6.5 Applications of Systems (day 2 of 2) - Day 5: Lesson 6.6 Linear Inequalities and Systems of Inequalities (day 1 of 2)
15	- Day 1: Lesson 6.6 Linear Inequalities and Systems of Inequalities (day 2 of 2) - Day 2: Unit 6 review (mastery checklist and missed problems) - Day 3: Unit 6 test - Day 4: Midterm review (Units 1–6) - Day 5: Midterm exam (Units 1–6) <i>Midterm week. Use any spare time to correct missed problems from Units 1–6.</i>
16	- Day 1: Lesson 7.1 Properties of Exponents (day 1 of 2) - Day 2: Lesson 7.1 Properties of Exponents (day 2 of 2) - Day 3: Lesson 7.2 Zero and Negative Exponents (day 1 of 2) - Day 4: Lesson 7.2 Zero and Negative Exponents (day 2 of 2) - Day 5: Lesson 7.3 Scientific Notation
17	- Day 1: Lesson 7.4 Exponential Functions (day 1 of 2) - Day 2: Lesson 7.4 Exponential Functions (day 2 of 2) - Day 3: Lesson 7.5 Exponential Growth and Decay (day 1 of 2) - Day 4: Lesson 7.5 Exponential Growth and Decay (day 2 of 2) - Day 5: Lesson 7.6 Geometric Sequences
18	- Day 1: Unit 7 review (mastery checklist and missed problems) - Day 2: Unit 7 test - Day 3: Lesson 8.1 Polynomials and Their Terms (day 1 of 2) - Day 4: Lesson 8.1 Polynomials and Their Terms (day 2 of 2) - Day 5: Lesson 8.2 Multiplying by a Monomial
19	- Day 1: Lesson 8.3 Multiplying Binomials and Polynomials (day 1 of 2) - Day 2: Lesson 8.3 Multiplying Binomials and Polynomials (day 2 of 2) - Day 3: Lesson 8.4 Special Products (day 1 of 2) - Day 4: Lesson 8.4 Special Products (day 2 of 2) - Day 5: Lesson 8.5 Dividing Polynomials by Monomials (day 1 of 2)
20	- Day 1: Lesson 8.5 Dividing Polynomials by Monomials (day 2 of 2) - Day 2: Unit 8 review (mastery checklist and missed problems) - Day 3: Unit 8 test

WEEK	PLAN
	- Day 4: Lesson 9.1 Factoring Out the Greatest Common Factor (day 1 of 2) - Day 5: Lesson 9.1 Factoring Out the Greatest Common Factor (day 2 of 2)
21	- Day 1: Lesson 9.2 Factoring $x^2 + bx + c$ (day 1 of 2) - Day 2: Lesson 9.2 Factoring $x^2 + bx + c$ (day 2 of 2) - Day 3: Lesson 9.3 Factoring $ax^2 + bx + c$ (day 1 of 2) - Day 4: Lesson 9.3 Factoring $ax^2 + bx + c$ (day 2 of 2) - Day 5: Lesson 9.4 Factoring Special Products (day 1 of 2)
22	- Day 1: Lesson 9.4 Factoring Special Products (day 2 of 2) - Day 2: Lesson 9.5 Factoring by Grouping and Factoring Completely (day 1 of 2) - Day 3: Lesson 9.5 Factoring by Grouping and Factoring Completely (day 2 of 2) - Day 4: Lesson 9.6 Solving Equations by Factoring (day 1 of 2) - Day 5: Lesson 9.6 Solving Equations by Factoring (day 2 of 2)
23	- Day 1: Unit 9 review (mastery checklist and missed problems) - Day 2: Unit 9 test - Day 3: Lesson 10.1 Simplifying Square Roots (day 1 of 2) - Day 4: Lesson 10.1 Simplifying Square Roots (day 2 of 2) - Day 5: Lesson 10.2 Quotients and Rationalizing Denominators (day 1 of 2)
24	- Day 1: Lesson 10.2 Quotients and Rationalizing Denominators (day 2 of 2) - Day 2: Lesson 10.3 Operations with Radical Expressions (day 1 of 2) - Day 3: Lesson 10.3 Operations with Radical Expressions (day 2 of 2) - Day 4: Lesson 10.4 Solving Radical Equations (day 1 of 2) - Day 5: Lesson 10.4 Solving Radical Equations (day 2 of 2)
25	- Day 1: Lesson 10.5 The Pythagorean Theorem, Distance and Midpoint (day 1 of 2) - Day 2: Lesson 10.5 The Pythagorean Theorem, Distance and Midpoint (day 2 of 2) - Day 3: Unit 10 review (mastery checklist and missed problems) - Day 4: Unit 10 test - Day 5: Lesson 11.1 Graphing $y = ax^2$ (day 1 of 2)
26	- Day 1: Lesson 11.1 Graphing $y = ax^2$ (day 2 of 2) - Day 2: Lesson 11.2 Graphing $y = ax^2 + bx + c$ (day 1 of 2) - Day 3: Lesson 11.2 Graphing $y = ax^2 + bx + c$ (day 2 of 2) - Day 4: Lesson 11.3 Solving Quadratic Equations by Graphing and Square Roots (day 1 of 2)

WEEK	PLAN
	Day 5: Lesson 11.3 Solving Quadratic Equations by Graphing and Square Roots (day 2 of 2)
27	- Day 1: Lesson 11.4 Completing the Square (day 1 of 2) - Day 2: Lesson 11.4 Completing the Square (day 2 of 2) - Day 3: Lesson 11.5 The Quadratic Formula (day 1 of 2) - Day 4: Lesson 11.5 The Quadratic Formula (day 2 of 2) - Day 5: Lesson 11.6 The Discriminant and Choosing a Method (day 1 of 2)
28	- Day 1: Lesson 11.6 The Discriminant and Choosing a Method (day 2 of 2) - Day 2: Lesson 11.7 Vertex Form and Quadratic Models (day 1 of 2) - Day 3: Lesson 11.7 Vertex Form and Quadratic Models (day 2 of 2) - Day 4: Unit 11 review (mastery checklist and missed problems) - Day 5: Unit 11 test
29	- Day 1: Lesson 12.1 Inverse Variation - Day 2: Lesson 12.2 Simplifying Rational Expressions (day 1 of 2) - Day 3: Lesson 12.2 Simplifying Rational Expressions (day 2 of 2) - Day 4: Lesson 12.3 Multiplying and Dividing Rational Expressions (day 1 of 2) - Day 5: Lesson 12.3 Multiplying and Dividing Rational Expressions (day 2 of 2)
30	- Day 1: Lesson 12.4 Adding and Subtracting Rational Expressions (day 1 of 2) - Day 2: Lesson 12.4 Adding and Subtracting Rational Expressions (day 2 of 2) - Day 3: Lesson 12.5 Solving Rational Equations (day 1 of 2) - Day 4: Lesson 12.5 Solving Rational Equations (day 2 of 2) - Day 5: Unit 12 review (mastery checklist and missed problems)
31	- Day 1: Unit 12 test - Day 2: Lesson 13.1 Scatter Plots and Correlation - Day 3: Lesson 13.2 Lines of Best Fit (day 1 of 2) - Day 4: Lesson 13.2 Lines of Best Fit (day 2 of 2) - Day 5: Lesson 13.3 Choosing a Model: Linear, Exponential or Quadratic
32	- Day 1: Unit 13 review (mastery checklist and missed problems) - Day 2: Unit 13 test - Day 3: Final review (Units 1–6) - Day 4: Final review (Units 7–13) - Day 5: Final exam (Units 1–13)

MATERIALS

Pencil and eraser

Work in pencil so mistakes can be corrected neatly.

A notebook for all work

A composition book or a binder; keep every lesson's work in order so you can review it.

Graph paper

Plenty of quarter-inch grid paper; use it for every graph.

Ruler or straightedge

For drawing axes, lines and boundaries neatly.

Scientific calculator

Needed from Unit 7 on for messy arithmetic, growth and decay, and decimal approximations of radicals.

Index cards

For the daily habits and key formulas, to review and recite until they are automatic.

Optional: graphing calculator or a free online graphing tool

Only for checking graphs, never instead of the algebra.

Optional: colored pencils

Helpful for graphing systems of equations and shading inequalities.

Optional: a tape measure and a ball

For the falling-object activities in the quadratics lessons.

GRADING

Grades reward both steady daily work and real mastery shown on tests. Daily practice counts for completion and corrections; unit tests and the two exams show what you have truly learned.

PART	%
Daily practice sets	20
Unit tests	50
Midterm exam	15
Final exam	15

Scale: A 90–100, B 80–89, C 70–79, D 60–69, below 60 redo

- Grade daily practice for completion and for corrections: a practice set is complete when every problem is attempted and every mistake has been corrected.
- A unit test scored below 80 may be retaken once after the student reviews and corrects the original test; record the higher score, up to a maximum of 90 on the retake.
- Mastery matters more than speed. It is better to spend an extra day on a hard lesson than to move on without understanding it.
- Give partial credit for correct work: a clear, correct method with a small arithmetic slip earns most of the points, while a bare answer with no work earns little.

A NOTE TO PARENTS

You do not need to be an algebra expert to teach this course well. Every lesson explains its ideas step by step, and every practice answer and every test key is provided. Your job is to keep the daily rhythm going, to check the work, and to ask good questions. If you learn alongside your student, that is a gift to both of you.

Help without doing the work. When your student is stuck, ask questions instead of giving answers: "What is the question asking?" "What does the variable stand for?" "Can you show me your steps?" "Did you check by substituting?" Make sure practice happens every day and that mistakes are corrected, not just marked wrong; a corrected mistake is often the best lesson of the day. Consistency matters more than speed, and slowing down when a topic is hard is perfectly fine.

Algebra I matters. It is the foundation for Geometry, Algebra II and the sciences, and it is used in the trades, in business and in managing money wisely. Even more, it trains clear, honest and orderly reasoning: saying exactly what you mean, taking one justified step at a time, and checking that your conclusion is true. Those habits will serve your student for life.

UNIT 1

FOUNDATIONS: REAL NUMBERS AND EXPRESSIONS

Algebra is the language of quantities that are unknown or that change. Before we solve equations, we need to know exactly what numbers we are working with and what rules they obey. In this unit you will sort the real numbers into families (natural, whole, integer, rational and irrational), learn why repeating decimals are fractions and why $\sqrt{2}$ is not, and name the properties, such as the commutative and distributive properties, that justify every step of algebra.

You will then sharpen three skills you will use every day for the rest of the course: following the order of operations, evaluating expressions and formulas by substitution, and simplifying expressions by distributing and combining like terms. Finally, you will learn to translate English sentences into algebra, the step that turns real problems about gardens, savings and travel into equations.

Much of this unit reviews and deepens Pre-Algebra, so it moves quickly. Parents: the habits built here (one step per line, parentheses when substituting, checking with a number) prevent most of the errors students make later in solving equations (Unit 2), working with polynomials (Unit 8) and factoring (Unit 9). Work by hand; no calculator is needed in this unit.

PREREQUISITES

- Add, subtract, multiply and divide integers using the sign rules.
- Compute with fractions and decimals, and convert between fractions, decimals and percents.
- Evaluate whole-number exponents and square roots of perfect squares.
- Use the order of operations with whole numbers.
- Use the distributive property and combine simple like terms (Pre-Algebra).
- Find the perimeter and area of rectangles, triangles and trapezoids.

MASTERY CHECKLIST

- ☐ I can classify a number as natural, whole, integer, rational, irrational and real.
- ☐ I can write a terminating or repeating decimal as a fraction and explain why it is rational.
- ☐ I can place a square root between two consecutive whole numbers and order real numbers.
- ☐ I can state the commutative, associative, distributive, identity and inverse properties and name the property used in a step.
- ☐ I can show with an example why subtraction and division are not commutative.
- ☐ I can simplify numerical expressions using the order of operations, including exponents and fraction bars.
- ☐ I can evaluate expressions and formulas by substitution, using parentheses around negative values.
- ☐ I can identify terms, coefficients and like terms, and combine like terms.
- ☐ I can simplify expressions by distributing, including distributing a negative.
- ☐ I can define a variable and translate phrases and sentences into expressions and equations.

LESSONS IN THIS UNIT

1.1 The Real Number System	45 minutes
1.2 Properties of Real Numbers	45 minutes
1.3 Order of Operations and Evaluating Expressions	45 minutes
1.4 Simplifying Expressions	2 sessions of 45 minutes
1.5 Translating Words into Algebra	2 sessions of 45 minutes

UNIT 1 TEST

Time allowed: 50 minutes. No calculator. Show your work for every problem; answers without work may receive partial credit only. Give exact answers (fractions or square roots) unless a problem asks for a decimal or an estimate, and include units in answers to word problems.

1. 5 pts

Which of these numbers are irrational? $\sqrt{36}$, $\sqrt{15}$, $0.\bar{7}$, π , $-\frac{2}{3}$

2. 6 pts

Write $0.\overline{45}$ as a fraction in lowest terms.

3. 6 pts

Order from least to greatest: $\sqrt{8}$, 2.8 , $\frac{17}{6}$, $2.\bar{8}$.

4. 5 pts

Name the property: $(x + 3) + 9 = x + (3 + 9)$.

5. 6 pts

Use the distributive property to find 7×199 mentally. Show the property you used.

6. 6 pts

Evaluate $24 \div (-2)^3 + 5 \times 3 - 1$.

7. (7 pts)

Evaluate $3a^2 - ab$ for $a = -2$ and $b = 5$.

8. (7 pts)

Samuel drops a pebble from a footbridge 100 feet above a river. Its height in feet after t seconds is $h = -16t^2 + 100$. How high is the pebble after 2 seconds?

9. (7 pts)

Simplify $5(2x - 3) - 3(x - 4)$.

10. (7 pts)

Simplify $-(4y - 1) + 2(y^2 + 3y) - y^2$.

11. (6 pts)

Write an expression: eight less than the product of 3 and a number n .

12. (8 pts)

Ruth spends \$18 on ingredients and sells muffins at a community bake sale for \$2 each. Let m = the number of muffins she sells. Write an equation that says her profit is \$54.

13. (8 pts)

The length of a rectangular pasture is 10 yards more than 3 times its width w (in yards). Write and simplify an expression for the perimeter of the pasture.

14. (8 pts)

Explain why $-(x - 5)$ equals $-x + 5$ and not $-x - 5$. Check your explanation with $x = 2$.

15. (8 pts)

Simplify $2[3x - (x - 4)] - 4(x + 2)$.

ANSWER KEY

1. $\sqrt{15}$ and π . ($\sqrt{36} = 6$, $0.\overline{7} = \frac{7}{9}$ and $-\frac{2}{3}$ are rational.)
2. $100x - x = 45$, so $99x = 45$ and $x = \frac{45}{99} = \frac{5}{11}$.
3. $\sqrt{8} \approx 2.828$, $\frac{17}{6} = 2.8\overline{3}$, $2.\overline{8} = 2.888\dots$, so the order is 2.8 , $\sqrt{8}$, $\frac{17}{6}$, $2.\overline{8}$.
4. Associative property of addition.
5. $7(200 - 1) = 1400 - 7 = 1393$
6. $24 \div (-8) + 15 - 1 = -3 + 15 - 1 = 11$
7. $3(-2)^2 - (-2)(5) = 12 - (-10) = 22$
8. $h = -16(2)^2 + 100 = -64 + 100 = 36$ feet
9. $10x - 15 - 3x + 12 = 7x - 3$
10. $-4y + 1 + 2y^2 + 6y - y^2 = y^2 + 2y + 1$
11. $3n - 8$
12. Money taken in: $2m$; profit: $2m - 18$. Equation: $2m - 18 = 54$.
13. Length $3w + 10$; $P = 2(3w + 10) + 2w = 6w + 20 + 2w = 8w + 20$ yards
14. The minus sign in front means multiply by -1 , and the distributive property says -1 multiplies every term: $(-1)(x) + (-1)(-5) = -x + 5$. Check with $x = 2$: $-(2 - 5) = -(-3) = 3$, and $-2 + 5 = 3$, while $-2 - 5 = -7$.
15. Inside the brackets: $3x - x + 4 = 2x + 4$. Then $2(2x + 4) - 4x - 8 = 4x + 8 - 4x - 8 = 0$.

45 MINUTES

1.1 THE REAL NUMBER SYSTEM

OBJECTIVES

- Classify a number as natural, whole, integer, rational, irrational and real.
- Explain why every terminating or repeating decimal is a rational number, and write such a decimal as a fraction.
- Recognize common irrational numbers such as $\sqrt{2}$ and π , and estimate a square root between two whole numbers.
- Order real numbers on the number line.

WATCH

Rational Numbers Explained | Math with Mr. J.

Math with Mr. J • 9:54<https://www.youtube.com/watch?v=wswkQxG-Kk8>

WHEN TO WATCH

After the Core ideas, before Example 1: what makes a number rational, including terminating and repeating decimals.

Note: Neither video sorts numbers into every set (natural, whole, integer, rational, irrational, real). Use the nested-sets diagram in the Core ideas for that.

BACKUP VIDEO

Irrational Numbers Explained | Math with Mr. J.

Math with Mr. J • 8:19<https://www.youtube.com/watch?v=GupzujexBFA>

Before Example 2: irrational numbers such as $\sqrt{2}$ and π .

KEY TERMS

Natural numbers

The counting numbers 1, 2, 3, 4, and so on.

Whole numbers

The natural numbers together with zero: 0, 1, 2, 3, and so on.

Integers

The whole numbers and their opposites: ..., -3, -2, -1, 0, 1, 2, 3, ...

Rational number

A number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Its decimal either ends (terminates) or repeats forever.

Irrational number

A real number that cannot be written as a fraction of integers. Its decimal goes on forever without ending and without repeating, like $\sqrt{2} = 1.41421356...$ or $\pi = 3.14159265...$

Real numbers

All the rational and irrational numbers together. Every point on the number line is a real number.

CORE IDEAS

Numbers come in families that fit inside one another like nested boxes. The **natural numbers** 1, 2, 3, ... are the numbers we count with. Add 0 and you get the **whole numbers**. Add the negatives -1, -2, -3, ... and you get the **integers**. Every natural number is a whole number, and every whole number is an integer, but not the other way around: -3 is an integer but not a whole number.

A **rational number** is any number you can write as a fraction $\frac{a}{b}$ of two integers, with $b \neq 0$ (the word *rational* comes from *ratio*). Every integer is rational, because $5 = \frac{5}{1}$ and $-3 = \frac{-3}{1}$. Every terminating decimal is rational too, because it is a number of tenths, hundredths or thousandths: $0.375 = \frac{375}{1000} = \frac{3}{8}$.

Why is a repeating decimal rational? Let $x = 0.272727...$ Because two digits repeat, multiply by 100: $100x = 27.272727...$ Now subtract the first equation from the second. The endless tails are identical, so they cancel exactly: $100x - x = 27$, so $99x = 27$ and $x = \frac{27}{99} = \frac{3}{11}$. The same trick works for any repeating decimal (multiply by 10 for one repeating digit, by 100 for two, and so on). It also works the other way: when you divide a by b with long division, the only possible remainders are 0, 1, 2, ..., $b - 1$. So either a remainder of 0 appears (the decimal terminates) or some remainder comes back, and from then on the digits repeat. That is why every fraction gives a terminating or repeating decimal.

An **irrational number** has a decimal that never ends and never repeats, so it cannot be written as a fraction. The square root of any whole number that is not a perfect square is irrational, such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{10}$; the number π is irrational too. Here is a sketch of the classical argument that $\sqrt{2}$ is irrational. Suppose $\sqrt{2} = \frac{a}{b}$, a fraction in lowest terms. Squaring gives $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$, which means a^2 is even, so a is even. Write $a = 2k$. Then $4k^2 = 2b^2$, so $b^2 = 2k^2$ and b is even too. But then a and b are both even, so the fraction was not in lowest terms after all. The supposition leads to a contradiction, so no such fraction exists.

The rational and irrational numbers together make up the **real numbers**, and every point on the number line is exactly one real number. To order real numbers, write each as a decimal to a few places. To place a square root, find the perfect squares on either side: since $25 < 30 < 36$, $\sqrt{30}$ is between $\sqrt{25} = 5$ and $\sqrt{36} = 6$, a little closer to 5 (it is about 5.48).

KEY FORMULAS

natural $\subset$ whole $\subset$ integers $\subset$ rational $\subset$ real

rational number $= \frac{a}{b}$, a and b integers, $b \neq 0$

If $m^2 < n < (m + 1)^2$, then $m < \sqrt{n} < m + 1$

From history. Greek mathematicians discovered more than 2,000 years ago that the diagonal of a square cannot be measured exactly by any fraction of its side; in modern language, $\sqrt{2}$ is irrational. Euclid's Elements, written about 300 BC, gives a whole book (Book X) to such "incommensurable" lengths.

Practice habit. Recite the perfect squares from $1^2 = 1$ to $20^2 = 400$ until they are automatic. Knowing them lets you place any square root between two whole numbers in seconds.

WORKED EXAMPLES

Example 1. Classify each number in every set it belongs to (natural, whole, integer, rational, irrational, real): -4 , 0 , $\frac{3}{5}$, $\sqrt{16}$, $\sqrt{7}$, $0.\bar{8}$, π .

Show solution

1. -4 is a negative integer: integer, rational, real.
2. 0 is a whole number but not a natural (counting) number: whole, integer, rational, real.
3. $\frac{3}{5}$ is a fraction of integers but not an integer: rational, real.
4. $\sqrt{16} = 4$ because $4^2 = 16$: natural, whole, integer, rational, real.
5. $\sqrt{7}$: 7 is not a perfect square, so $\sqrt{7}$ is irrational and real.
6. $0.\bar{8} = \frac{8}{9}$ is a repeating decimal: rational, real.
7. π never ends and never repeats: irrational, real.

Answer: -4 : integer, rational, real. 0 : whole, integer, rational, real. $\frac{3}{5}$: rational, real. $\sqrt{16}$: natural, whole, integer, rational, real. $\sqrt{7}$: irrational, real. $0.\bar{8}$: rational, real. π : irrational, real.

Example 2. The Miller family is laying a square brick patio with an area of 45 square feet. Is the side length rational or irrational? Between which two consecutive whole numbers of feet does it lie?

Show solution

1. The side of a square with area 45 is $\sqrt{45}$ feet, because side $\times$ side = 45.
2. 45 is not a perfect square, so $\sqrt{45}$ is irrational.
3. Find the perfect squares around 45: $6^2 = 36$ and $7^2 = 49$.
4. Since $36 < 45 < 49$, we get $6 < \sqrt{45} < 7$.
5. 45 is closer to 49 than to 36, so $\sqrt{45}$ is closer to 7 (it is about 6.7).
6. Check: $6.7 \times 6.7 = 44.89$, very close to 45.

Answer: Irrational; the side is between 6 feet and 7 feet (about 6.7 feet).

PRACTICE

1. **Easy** List every set that -7 belongs to: natural, whole, integer, rational, irrational, real.

Show answer

Integer, rational, real.

2. **Easy** Is $\sqrt{49}$ a rational number?

Show answer

$$\text{Yes. } \sqrt{49} = 7 = \frac{7}{1}.$$

3. **Easy** Is $\sqrt{10}$ a rational number?

Show answer

No. 10 is not a perfect square, so $\sqrt{10}$ is irrational.

4. **Easy** Write 0.6 as a fraction in lowest terms.

Show answer

$$0.6 = \frac{6}{10} = \frac{3}{5}$$

5. **Medium** Write $0.\bar{5}$ as a fraction in lowest terms.

Show answer

$$10x - x = 5.\bar{5} - 0.\bar{5}, \text{ so } 9x = 5 \text{ and } x = \frac{5}{9}.$$

6. **Medium** Write $0.\overline{36}$ as a fraction in lowest terms.

Show answer

$$100x - x = 36, \text{ so } 99x = 36 \text{ and } x = \frac{36}{99} = \frac{4}{11}.$$

7. **Medium** Between which two consecutive whole numbers is $\sqrt{30}$?

Show answer

Between 5 and 6, because $25 < 30 < 36$.

8. **Medium** Henry is cutting a shelf board $2\frac{5}{8}$ feet long. Write the length as a decimal.

Show answer

$$\frac{5}{8} = 0.625, \text{ so the board is } 2.625 \text{ feet long (a terminating decimal, so rational).}$$

9. **Medium** At a church bake sale, Ruth cuts a pan of brownies into 3 equal pieces for a family. Her brother says each piece is exactly 0.33 of the pan. Is he right?

Show answer

No. Each piece is $\frac{1}{3} = 0.\overline{3} = 0.333\dots$ of the pan; 0.33 is only an approximation.

10. **Hard** Grace plans a square herb garden with an area of 20 square feet. Between which two consecutive whole numbers of feet is the length of one side?

Show answer

The side is $\sqrt{20}$ feet. Since $16 < 20 < 25$, it is between 4 and 5 feet (about 4.5 feet).

11. **Hard** Order from least to greatest: 2.3, $\sqrt{5}$, $\frac{7}{3}$, $2.\overline{2}$.

Show answer

As decimals: 2.3, $\sqrt{5} \approx 2.236$, $\frac{7}{3} = 2.333\dots$, $2.\overline{2} = 2.222\dots$ So the order is $2.\overline{2}$, $\sqrt{5}$, 2.3, $\frac{7}{3}$.

12. **Hard** Is every integer a whole number?

Show answer

No. For example, -3 is an integer but not a whole number. (Every whole number is an integer, but not the other way around.)

13. **Hard** Explain why every terminating decimal, such as 0.48, is a rational number.

Show answer

A terminating decimal is a whole number of tenths, hundredths, thousandths and so on, so it can be written as a fraction with denominator 10, 100, 1000, ...

For example, $0.48 = \frac{48}{100} = \frac{12}{25}$. A fraction of integers is rational by definition.

14. **Challenge** Write $0.1\overline{6} = 0.1666\dots$ as a fraction in lowest terms.

Show answer

Let $x = 0.1666\dots$. Then $100x = 16.666\dots$ and $10x = 1.666\dots$. Subtract: $90x = 15$, so $x = \frac{15}{90} = \frac{1}{6}$.

15. **Challenge** Name one rational number and one irrational number that are both between 1.4 and 1.5. Explain how you know the irrational one is in that interval.

Show answer

Rational: 1.45 (which is $\frac{29}{20}$). Irrational: $\sqrt{2}$, because $1.4^2 = 1.96 < 2$ and $1.5^2 = 2.25 > 2$, so $1.4 < \sqrt{2} < 1.5$.

COMMON MISTAKES

- Thinking every square root is irrational. $\sqrt{36} = 6$ is rational; only square roots of numbers that are not perfect squares (like $\sqrt{35}$) are irrational. Check whether the number under the root is a perfect square first.
- Calling a repeating decimal irrational because it goes on forever. A decimal that repeats, like $0.\overline{27}$, is rational; irrational decimals never end *and* never repeat.
- Forgetting that integers are rational. Any integer n can be written as $\frac{n}{1}$, so it belongs in the rational set as well.
- Treating a rounded value as exact, such as saying $\pi = 3.14$ or $\frac{1}{3} = 0.33$. Use $\approx$ for approximations and keep the exact form when the problem asks for it.

Make it a game • Nested Number Rings. Write ten numbers on slips: 7, 0, -4, $\frac{1}{3}$, 0.25, $0.777\dots$, -2.5, $\sqrt{2}$, π and $\sqrt{9}$. Make nested rings from string or paper circles: natural, whole, integers, rational, real. Beat the clock: place all ten in 60 seconds. Then pick a repeating decimal and turn it into a fraction aloud. Earn one point per slip placed correctly.

45 MINUTES

1.2 PROPERTIES OF REAL NUMBERS

OBJECTIVES

- State the commutative, associative, distributive, identity and inverse properties with variables.
- Name the property that justifies each step when rewriting an expression.
- Use the properties to do arithmetic mentally.
- Explain why subtraction and division are not commutative, and decide whether a set is closed under an operation.

WATCH

Review: Commutative, Associative & Distributive Laws

Eddie Woo · 12:15<https://www.youtube.com/watch?v=lob0bmworcM>

WHEN TO WATCH

Before Example 1: the commutative, associative and distributive properties.

Note: Eddie Woo teaches in Australia: he says 'laws' for properties and 'pronumeral' for variable. The inverse properties are taught from the Core ideas.

BACKUP VIDEO

What's the Identity Property? | Identity Property of Addition and Multiplication | Math with Mr. J

Math with Mr. J · 3:13<https://www.youtube.com/watch?v=kaKi4EpsFqc>

Before the practice set: the identity properties of addition and multiplication.

KEY TERMS

Commutative property

Changing the order of two numbers does not change a sum or product: $a + b = b + a$ and $ab = ba$.

Associative property

Changing the grouping of three numbers does not change a sum or product: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

Distributive property

Multiplying a sum by a number is the same as multiplying each part and adding: $a(b + c) = ab + ac$.

Identity

A number that leaves others unchanged: 0 is the additive identity ($a + 0 = a$) and 1 is the multiplicative identity ($a \cdot 1 = a$).

Inverse

The additive inverse (opposite) of a is $-a$, because $a + (-a) = 0$. The multiplicative inverse (reciprocal) of a nonzero a is $\frac{1}{a}$, because $a \cdot \frac{1}{a} = 1$.

Closure

A set is closed under an operation if doing that operation on any two members of the set always gives another member of the set.

CORE IDEAS

Algebra rests on a short list of rules that are true for **every** real number. Because they are always true, we write them with letters: $a + b = b + a$ says the order of a sum never matters, whatever numbers a and b are. These properties are the reasons behind every step you will take when you simplify expressions and solve equations, so it pays to know them by name.

The **commutative** properties are about *order* (think of commuting, moving back and forth): $a + b = b + a$ and $ab = ba$. The **associative** properties are about *grouping* (who is associated with whom inside parentheses): $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$. A quick test: if the numbers changed places, it is commutative; if only the parentheses moved, it is associative.

The **distributive** property links multiplication and addition: $a(b + c) = ab + ac$. Picture a rectangle a units tall and $b + c$ units wide, split into two smaller rectangles of widths b and c . The whole area $a(b + c)$ equals the two parts $ab + ac$. The same works with subtraction: $a(b - c) = ab - ac$. This is a mental-math tool: $7 \times 98 = 7(100 - 2) = 700 - 14 = 686$.

The **identity** properties say adding 0 or multiplying by 1 changes nothing. The **inverse** properties say every number has an opposite that adds to 0 and every nonzero number has a reciprocal that multiplies to 1: $5 + (-5) = 0$ and $5 \cdot \frac{1}{5} = 1$. Zero has no reciprocal, because 0 times any number is 0, never 1. These are exactly the tools you use to undo operations when solving equations.

Why are subtraction and division **not** commutative? Try one example: $5 - 3 = 2$ but $3 - 5 = -2$. In fact $b - a$ is always the opposite of $a - b$, so they agree only when $a = b$. Likewise $8 \div 2 = 4$ but $2 \div 8 = \frac{1}{4}$. One counterexample is enough to show a rule is not always true. The fix is to think of subtraction as adding the opposite, $a - b = a + (-b)$, and division as multiplying by the reciprocal; then the commutative and associative properties of addition and multiplication apply.

Closure, stated simply: when you add, subtract or multiply two integers you always get an integer, so the integers are closed under those operations. They are not closed under division, because $1 \div 2 = \frac{1}{2}$ is not an integer. The real numbers are closed under addition, subtraction, multiplication, and division by any nonzero number.

KEY FORMULAS

$$a + b = b + a \quad ab = ba$$

$$(a + b) + c = a + (b + c) \quad (ab)c = a(bc)$$

$$a(b + c) = ab + ac$$

$$a + 0 = a \quad a \cdot 1 = a$$

$$a + (-a) = 0 \quad a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$$

Practice habit. Hunt for friendly pairs before you compute: $4 \times 25 = 100$, $8 \times 125 = 1000$, $2 \times 5 = 10$. Use the commutative and associative properties to put them together, as in $4 \times 37 \times 25 = (4 \times 25) \times 37 = 3700$.

WORKED EXAMPLES

Example 1. Name the property that justifies each step: $3 + (x + 7) = 3 + (7 + x) = (3 + 7) + x = 10 + x$.

Show solution

1. $3 + (x + 7) = 3 + (7 + x)$: the order of x and 7 changed, so this is the commutative property of addition.
2. $3 + (7 + x) = (3 + 7) + x$: only the parentheses moved, so this is the associative property of addition.
3. $(3 + 7) + x = 10 + x$: this is simple arithmetic, $3 + 7 = 10$.

Answer: Commutative property of addition; associative property of addition; arithmetic (addition of numbers).

Example 2. Samuel buys 4 tickets to a community concert at \$9.95 each. Use the distributive property to find the total cost mentally.

Show solution

1. Write the price as a friendly difference: $9.95 = 10 - 0.05$.
2. Total cost: $4(10 - 0.05)$.
3. Distribute: $4 \times 10 - 4 \times 0.05 = 40 - 0.20$.
4. Subtract: $40 - 0.20 = 39.80$.
5. Check by estimating: 4 tickets at about \$10 is about \$40, and the answer is just under that.

Answer: \$39.80

PRACTICE

1. Easy Name the property: $7 + 5 = 5 + 7$.

Show answer

Commutative property of addition.

2. Easy Name the property: $(2 \cdot 9) \cdot 5 = 2 \cdot (9 \cdot 5)$.

Show answer

Associative property of multiplication.

3. **Easy** Find the additive inverse (opposite) of $-\frac{3}{4}$.

Show answer

$$\frac{3}{4}, \text{ because } -\frac{3}{4} + \frac{3}{4} = 0.$$

4. **Easy** Find the multiplicative inverse (reciprocal) of $-\frac{2}{5}$.

Show answer

$$-\frac{5}{2}, \text{ because } -\frac{2}{5} \cdot \left(-\frac{5}{2}\right) = 1.$$

5. **Medium** Use the distributive property to rewrite $-3(2x - 5)$ without parentheses.

Show answer

$$-3 \cdot 2x - (-3)(5) = -6x + 15$$

6. **Medium** Use the commutative and associative properties to find $25 \times 17 \times 4$ mentally.

Show answer

$$(25 \times 4) \times 17 = 100 \times 17 = 1700$$

7. **Medium** Use the distributive property to find 8×103 mentally.

Show answer

$$8(100 + 3) = 800 + 24 = 824$$

8. **Medium** Is the set of whole numbers closed under subtraction?

Show answer

No. For example, $3 - 5 = -2$, which is not a whole number.

9. **Medium** Henry buys 3 bags of chicken feed at \$12 each and 3 bags of grit at \$8 each. Use the distributive property to find the total cost.

Show answer

$$3(12) + 3(8) = 3(12 + 8) = 3(20) = 60, \text{ so } \$60.$$

10. **Hard** Ruth buys 6 loaves of bread for a church potluck at \$3.98 each. Use the distributive property to find the total cost mentally.

Show answer

$$6(4 - 0.02) = 24 - 0.12 = 23.88, \text{ so } \$23.88.$$

11. **Hard** For a choir concert, chairs are set up in three rooms: 17 in the first, 48 in the second and 33 in the third. Use the commutative and associative properties to add them mentally. How many chairs are there?

Show answer

$$17 + 48 + 33 = (17 + 33) + 48 = 50 + 48 = 98 \text{ chairs.}$$

12. **Hard** Name the property used in each step: $5x + 3 + 2x = 5x + 2x + 3 = (5 + 2)x + 3 = 7x + 3$.

Show answer

Commutative property of addition (3 and $2x$ trade places); distributive property ($5x + 2x = (5 + 2)x$); arithmetic ($5 + 2 = 7$).

13. **Hard** Use an example to show that division is not associative.

Show answer

$(24 \div 6) \div 2 = 4 \div 2 = 2$, but $24 \div (6 \div 2) = 24 \div 3 = 8$. Since $2 \neq 8$, changing the grouping changed the answer, so division is not associative.

14. **Challenge** Explain why 0 has no multiplicative inverse.

Show answer

A multiplicative inverse of 0 would be a number r with $0 \cdot r = 1$. But 0 times any number is 0, never 1, so no such number exists.

15. **Challenge** Is the set of irrational numbers closed under multiplication?

Show answer

No. $\sqrt{2}$ is irrational, but $\sqrt{2} \cdot \sqrt{2} = 2$, which is rational.

COMMON MISTAKES

- Mixing up commutative and associative. Ask: did the numbers change *order* (commutative) or did only the *parentheses* move (associative)?
- Distributing to only the first term, as in $4(x + 3) = 4x + 3$. The number outside multiplies *every* term inside: $4(x + 3) = 4x + 12$.

- Assuming subtraction and division can be reordered, as in $10 - 4 = 4 - 10$. Rewrite subtraction as adding the opposite before rearranging: $10 - 4 = 10 + (-4) = -4 + 10$.
- Changing the sign when finding a reciprocal. The reciprocal of $-\frac{2}{5}$ is $-\frac{5}{2}$ (same sign); the opposite is $\frac{2}{5}$ (opposite sign).

Make it a game • Name That Property. Quick-draw duel. The parent says a rewrite such as ' $4 + x$ becomes $x + 4$ ' or ' $3(x + 2)$ becomes $3x + 6$ ', and you name the property within three seconds. Five right in a row wins the round. Then swap: you invent the rewrites, and the parent must name each property. Challenge: agree on one rewrite that uses no property at all.

45 MINUTES

1.3 ORDER OF OPERATIONS AND EVALUATING EXPRESSIONS

OBJECTIVES

- Simplify numerical expressions with integers, fractions and exponents using the order of operations.
- Explain the difference between -3^2 and $(-3)^2$.
- Evaluate expressions and formulas by substitution, using parentheses around substituted values.

WATCH

Order of Operations with Integers | Positives and Negatives | PEMDAS | Math with Mr. J

Math with Mr. J • 9:02

<https://www.youtube.com/watch?v=AtPS0pWthEo>

WHEN TO WATCH

Before Example 1: the order of operations with positive and negative numbers.

BACKUP VIDEO

Exponents with Negative Bases | Math with Mr. J

Math with Mr. J • 9:02

https://www.youtube.com/watch?v=Yg_bC19aNjY

Before the practice set: why -3^2 and $(-3)^2$ are different.

KEY TERMS

Order of operations

The agreed order for doing operations: grouping symbols first, then exponents, then multiplication and division from left to right, then addition and subtraction from left to right.

Grouping symbols

Symbols that say "do this part first": parentheses (), brackets [], absolute value bars and the fraction bar.

Evaluate

To find the value of an expression by replacing each variable with a given number and doing the arithmetic.

Substitution

Replacing a variable with a number (or another expression).

CORE IDEAS

Does $8 + 4 \times 3$ equal 36 or 20? If everyone chose differently, the same expression would mean different things. So mathematicians agreed on an order: (1) work inside **grouping symbols**, innermost first; (2) evaluate **exponents**; (3) **multiply and divide** from left to right; (4) **add and subtract** from left to right. So $8 + 4 \times 3 = 8 + 12 = 20$. Notice that multiplication does not always come before division; they are equal partners and you go left to right. The same is true for addition and subtraction: $10 - 4 + 2 = 6 + 2 = 8$, not $10 - 6 = 4$.

A **fraction bar** is a grouping symbol. It means "divide the whole top by the whole bottom," so simplify the numerator and the denominator completely before dividing: $\frac{6 + 10}{2 \times 4} = \frac{16}{8} = 2$.

Watch negative signs with exponents. An exponent applies only to what is directly in front of it. In -3^2 the exponent belongs to the 3 alone, so $-3^2 = -(3 \cdot 3) = -9$. In $(-3)^2$ the parentheses make the base -3 , so $(-3)^2 = (-3)(-3) = 9$. Fractions work the same way: $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

To **evaluate** an expression, substitute the given value for each variable and then follow the order of operations. Always put a substituted number in parentheses, especially a negative one. For $x = -4$, the expression x^2 becomes $(-4)^2 = 16$; leaving out the parentheses gives $-4^2 = -16$, which is wrong. Parentheses also keep juxtaposed products clear: $3x$ with $x = -4$ is $3(-4) = -12$, not $3 - 4$.

A **formula** is an equation that tells how to compute one quantity from others, such as $d = rt$ (distance = rate $\times$ time) or $F = \frac{9}{5}C + 32$ (Fahrenheit from Celsius). Evaluating a formula is the same skill: substitute, then simplify. Include units in your answer.

KEY FORMULAS

Grouping $\rightarrow$ Exponents $\rightarrow$ Multiply/Divide (left to right) $\rightarrow$ Add/Subtract (left to right)

$$-a^2 = -(a^2), \quad (-a)^2 = a^2$$

Practice habit. Write one step per line and do only one operation (or one group of operations) per line. When you substitute, write the parentheses first, then put the number inside them.

WORKED EXAMPLES

Example 1. Evaluate $18 - 3(4 - 6)^2 \div 2$.

Show solution

1. Parentheses first: $4 - 6 = -2$, giving $18 - 3(-2)^2 \div 2$.
2. Exponent: $(-2)^2 = 4$, giving $18 - 3 \cdot 4 \div 2$.
3. Multiply and divide left to right: $3 \cdot 4 = 12$, then $12 \div 2 = 6$, giving $18 - 6$.
4. Subtract: $18 - 6 = 12$.

Answer: 12

Example 2. Henry tosses a baseball straight up at 48 feet per second from a height of 5 feet. Its height in feet after t seconds is $h = -16t^2 + 48t + 5$. How high is the ball after 2 seconds?

Show solution

1. Substitute $t = 2$ with parentheses: $h = -16(2)^2 + 48(2) + 5$.
2. Exponent: $(2)^2 = 4$, so $h = -16(4) + 48(2) + 5$.
3. Multiply: $-16(4) = -64$ and $48(2) = 96$, so $h = -64 + 96 + 5$.
4. Add left to right: $-64 + 96 = 32$, then $32 + 5 = 37$.
5. Check that it makes sense: the ball started at 5 feet and is still in the air, so a height of 37 feet is reasonable.

Answer: 37 feet

PRACTICE

1. Easy Evaluate $8 + 4 \times 3$.

Show answer

$$8 + 12 = 20$$

2. Easy Evaluate $(8 + 4) \times 3$.

Show answer

$$12 \times 3 = 36$$

3. Easy Evaluate $20 - 12 \div 4 + 1$.

Show answer

$$20 - 3 + 1 = 18$$

4. Easy Evaluate (a) -5^2 and (b) $(-5)^2$.

Show answer

$$(a) -25; (b) 25$$

5. Medium Evaluate $2x^2 - 3x + 1$ for $x = -2$.

Show answer

$$2(-2)^2 - 3(-2) + 1 = 8 + 6 + 1 = 15$$

6. Medium Evaluate $\frac{a+b}{a-b}$ for $a = 7$ and $b = -3$.

Show answer

$$\frac{7 + (-3)}{7 - (-3)} = \frac{4}{10} = \frac{2}{5}$$

7. Medium Evaluate $\frac{1}{2} + \frac{3}{4} \cdot \frac{2}{3}$.

Show answer

$$\text{Multiply first: } \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}; \text{ then } \frac{1}{2} + \frac{1}{2} = 1.$$

8. Medium Evaluate $3[10 - 2(5 - 1)]^2$.

Show answer

$$3[10 - 2(4)]^2 = 3[2]^2 = 3 \cdot 4 = 12$$

9. Medium On a winter morning the temperature on the Miller farm is -10 degrees Celsius. Use $F = \frac{9}{5}C + 32$ to find the temperature in degrees Fahrenheit.

Show answer

$$F = \frac{9}{5}(-10) + 32 = -18 + 32 = 14 \text{ degrees Fahrenheit.}$$

10. **Hard** Grace's family sells loaves of bread at a bake sale for \$3 each. Each loaf costs \$0.75 in ingredients, and the table rental is \$20. Their profit for n loaves is $P = 3n - (0.75n + 20)$. Find the profit for 48 loaves.

Show answer

$$P = 3(48) - (0.75(48) + 20) = 144 - (36 + 20) = 144 - 56 = 88, \text{ so } \$88.$$

11. **Hard** Evaluate $\frac{x^2 - y^2}{x + 2y}$ for $x = -4$ and $y = 3$.

Show answer

$$\frac{(-4)^2 - (3)^2}{-4 + 2(3)} = \frac{16 - 9}{2} = \frac{7}{2}$$

12. **Hard** A garden bed is shaped like a trapezoid with parallel sides of 10 feet and 14 feet and a height of 8 feet. Use $A = \frac{1}{2}h(b_1 + b_2)$ to find its area.

Show answer

$$A = \frac{1}{2}(8)(10 + 14) = 4 \cdot 24 = 96 \text{ square feet.}$$

13. **Hard** A classmate says $-3^2 = 9$. Explain the mistake.

Show answer

The exponent applies only to the 3, not to the negative sign, so $-3^2 = -(3^2) = -9$. Only $(-3)^2$, with parentheses, equals 9.

14. **Challenge** Insert one pair of parentheses to make the statement true:
 $2 + 3 \times 4 - 1 = 19$.

Show answer

$$(2 + 3) \times 4 - 1 = 5 \times 4 - 1 = 20 - 1 = 19$$

15. **Challenge** Evaluate $-x^2 - (-x)^3$ for $x = -2$.

Show answer

$$-(-2)^2 - (-(-2))^3 = -4 - (2)^3 = -4 - 8 = -12$$

COMMON MISTAKES

- Always multiplying before dividing (or adding before subtracting). These pairs are equal partners: go left to right. $12 \div 4 \times 3 = 3 \times 3 = 9$, not $12 \div 12 = 1$.
- Substituting a negative without parentheses, so that x^2 with $x = -4$ becomes $-4^2 = -16$. Write $(-4)^2 = 16$.

- Dividing before finishing the top and bottom of a fraction. Treat the fraction bar as parentheses around the numerator and around the denominator.
- Subtracting a whole group carelessly, as in $144 - (36 + 20)$. Finish the parentheses first (56), then subtract.

Make it a game • Minus Three Squared. Each player writes five expressions using negatives, fractions and exponents, and two of them must be -3^2 and $(-3)^2$. Swap papers and simplify against a two-minute timer. One point per correct answer. Bonus point if you can say aloud, in one sentence, why -3^2 and $(-3)^2$ give different answers. Beat your own total next day.

1.4 SIMPLIFYING EXPRESSIONS

OBJECTIVES

- Identify the terms, coefficients, constants and like terms of an expression.
- Explain why like terms can be combined, using the distributive property.
- Simplify expressions by distributing, including distributing a negative, and combining like terms.
- Check a simplification by substituting a number into the original and the simplified expression.

WATCH

Simplifying Algebraic Expressions | Distributive Property & Combining Like Terms | Math with Mr. J

Math with Mr. J • 10:22

<https://www.youtube.com/watch?v=1MmhAq-XVNO>

WHEN TO WATCH

Before Example 1: the distributive property and combining like terms.

Note: Math Antics says 'polynomial' for an expression with several terms. That word is defined properly in Lesson 8.1.

BACKUP VIDEO

Algebra Basics: The Distributive Property - Math Antics

mathantics • 11:54

<https://www.youtube.com/watch?v=v-6MShC82ow>

If distributing a negative is still shaky, before Example 2.

KEY TERMS

Term

A number, a variable, or a product of numbers and variables. Terms are the parts of an expression separated by + and – signs; in $4x^2 - x + 9$ the terms are $4x^2$, $-x$ and 9.

Coefficient

The number multiplying the variable part of a term. In $4x^2$ it is 4; in $-x$ it is -1 .

Constant

A term with no variable, such as 9.

Like terms

Terms with exactly the same variables raised to exactly the same powers, such as $3x^2$ and $-5x^2$. Constants are like terms with one another.

Simplify

To rewrite an expression in an equal but shorter form, with no parentheses left to remove and no like terms left to combine.

CORE IDEAS

An expression is built from **terms** joined by plus and minus signs. Read each sign as belonging to the term after it: $5x - 3y + 2$ is $5x + (-3y) + 2$, so its terms are $5x$, $-3y$ and 2. The **coefficient** of $-3y$ is -3 , and a variable alone, like x , has coefficient 1 (and $-x$ has coefficient -1).

Why can we combine **like terms**? It is the distributive property used backward: $3x + 5x = (3 + 5)x = 8x$. Think of x as a kind of object, say a bag of apples: 3 bags plus 5 bags is 8 bags. But $3x + 5y$ is 3 bags of apples and 5 boxes of oranges; they are different kinds, so it stays $3x + 5y$. Likewise x and x^2 are not like terms, since x^2 means $x \cdot x$, a different quantity. To combine like terms, add their coefficients and keep the variable part exactly the same.

To remove parentheses, **distribute**: multiply the number in front by every term inside. A minus sign in front of parentheses means multiply by -1 , which changes the sign of every term inside: $-(x - 4) = -1 \cdot x + (-1)(-4) = -x + 4$. Likewise $-4(x - 1) = -4x + 4$, because $(-4)(-1) = 4$.

Put it together. To simplify $3(2x - 5) - 4(x - 1)$: distribute the 3 to get $6x - 15$; distribute the -4 (keep the minus sign with the 4) to get $-4x + 4$. Now $6x - 15 - 4x + 4$. Group like terms: $(6x - 4x) + (-15 + 4) = 2x - 11$. Simplified expressions are usually written with the variable terms first, highest power first, and the constant last.

Check your work by substitution. Pick an easy value such as $x = 2$. The original: $3(2 \cdot 2 - 5) - 4(2 - 1) = 3(-1) - 4(1) = -7$. The simplified form: $2(2) - 11 = -7$.

They agree, so the simplification is very likely right. (Avoid checking with 0 or 1, which can hide mistakes.)

KEY FORMULAS

$$ax + bx = (a + b)x$$

$$-(a + b) = -a - b \quad -(a - b) = -a + b$$

$$c(a - b) = ca - cb$$

Practice habit. Check every simplification by substituting a small number such as $x = 2$ or $x = 3$ into both the original and your answer. If the two values differ, there is a mistake to find.

WORKED EXAMPLES

Example 1. Simplify $7a + 3b - 2a + 5 - b$.

Show solution

1. Identify like terms: $7a$ and $-2a$; $3b$ and $-b$; the constant 5.
2. Group them (commutative and associative properties):
 $(7a - 2a) + (3b - b) + 5$.
3. Combine coefficients: $7 - 2 = 5$ and $3 - 1 = 2$.
4. Result: $5a + 2b + 5$.
5. Check with $a = 2$, $b = 3$: original $14 + 9 - 4 + 5 - 3 = 21$; simplified
 $10 + 6 + 5 = 21$. ✓

Answer: $5a + 2b + 5$

Example 2. Henry is building a rectangular goat pen. Its length is $3x - 2$ feet and its width is $x + 4$ feet. Write and simplify an expression for the perimeter. Then find the perimeter when $x = 6$.

Show solution

1. Perimeter of a rectangle: $P = 2(\text{length}) + 2(\text{width})$.
2. Substitute: $P = 2(3x - 2) + 2(x + 4)$.
3. Distribute: $P = 6x - 4 + 2x + 8$.
4. Combine like terms: $P = 8x + 4$.
5. For $x = 6$: $P = 8(6) + 4 = 52$.
6. Check with the sides: length $3(6) - 2 = 16$, width $6 + 4 = 10$, perimeter $2(16) + 2(10) = 52$. ✓

Answer: $P = 8x + 4$ feet; when $x = 6$ the perimeter is 52 feet.

PRACTICE

1. Easy What is the coefficient of x in $4x^2 - x + 9$?

Show answer

-1 (the term $-x$ means $-1 \cdot x$)

2. Easy Simplify $5x + 9x$.

Show answer

$14x$

3. Easy Simplify $8y - 3 + 2y + 7$.

Show answer

$10y + 4$

4. Easy Simplify $4(3m - 2)$.

Show answer

$12m - 8$

5. Medium Simplify $-(6 - 2k)$.

Show answer

$-6 + 2k = 2k - 6$

6. Medium Simplify $2x^2 + 5x - x^2 + 3x - 4$.

Show answer

$$(2x^2 - x^2) + (5x + 3x) - 4 = x^2 + 8x - 4$$

7. Medium Simplify $5(a + 2) - 3a$.

Show answer

$$5a + 10 - 3a = 2a + 10$$

8. Medium Simplify $\frac{1}{2}(8n + 6) - n$.

Show answer

$$4n + 3 - n = 3n + 3$$

9. Medium Ruth is putting a border around a triangular flower bed whose sides measure x , $2x - 3$ and $x + 7$ feet. Write a simplified expression for the length of border she needs.

Show answer

$$x + (2x - 3) + (x + 7) = 4x + 4 \text{ feet}$$

10. **Hard** Simplify $6(x - 2) - 2(3x + 1)$.

Show answer

$$6x - 12 - 6x - 2 = -14 \text{ (the } x \text{ terms cancel)}$$

11. **Hard** Simplify $-2(3y - 4) - (5 - y) + 3y$.

Show answer

$$-6y + 8 - 5 + y + 3y = -2y + 3$$

12. **Hard** Grace buys n notebooks at \$3 each and $n + 2$ pens at \$1.50 each for the school year. Write a simplified expression for the total cost in dollars.

Show answer

$$3n + 1.5(n + 2) = 3n + 1.5n + 3 = 4.5n + 3 \text{ dollars}$$

13. **Hard** A classmate simplified $3x + 4y$ to $7xy$. Explain why this is wrong.

Show answer

$3x$ and $4y$ are not like terms because their variables are different, so they cannot be combined; $3x + 4y$ is already simplified. A check shows it: with $x = 2$ and $y = 1$, $3x + 4y = 10$ but $7xy = 14$. (With $x = 1$ and $y = 1$ both happen to equal 7, which is why checking with 1 can hide a mistake.)

14. **Challenge** Simplify $4[2x - 3(x - 2)] - 5x$.

Show answer

Inside the brackets: $2x - 3x + 6 = -x + 6$. Then $4(-x + 6) - 5x = -4x + 24 - 5x = -9x + 24$.

15. **Challenge** Samuel has \$40 and saves \$15 each week. His sister Ruth has \$25 and saves \$10 each week. Write a simplified expression for how many more dollars Samuel has than Ruth after w weeks.

Show answer

$$(40 + 15w) - (25 + 10w) = 40 + 15w - 25 - 10w = 5w + 15 \text{ dollars}$$

COMMON MISTAKES

- Distributing a negative to only the first term: $-(x - 4) = -x - 4$ is wrong. The -1 multiplies every term, so $-(x - 4) = -x + 4$.
- Losing the minus sign in front of a number being distributed. In $3(2x - 5) - 4(x - 1)$, distribute -4 , not 4: $-4x + 4$.
- Combining unlike terms, such as $2x + 3x^2 = 5x^3$ or $3x + 4 = 7x$. Only terms with the same variable part combine.
- Dropping the sign that belongs to a term when rearranging. Circle each term together with the sign in front of it before you move it.

Make it a game • Term Detective. The parent writes a messy expression such as $4x + 3 - 2x + 7y - 5 + y$. Use colored pencils to circle each term, one color per kind of like term, and label every coefficient and constant. Then simplify. Finish by explaining why $4x + 2x$ equals $6x$ using the distributive property. Do three expressions and try to beat your time.

1.5 TRANSLATING WORDS INTO ALGEBRA

OBJECTIVES

- Define a variable clearly, stating what it stands for and its units.
- Translate verbal phrases into algebraic expressions, paying attention to order and grouping.
- Translate sentences and word situations into equations.
- Explain why letters can stand for numbers and why "less than" reverses the order.

WATCH

Writing Algebraic Expressions | Writing Expressions with Variables | Math with Mr. J.

Math with Mr. J • 12:52

https://www.youtube.com/watch?v=o_Ubm7OI8t4

WHEN TO WATCH

Before Example 1: turning word phrases into expressions.

Note: Eddie Woo teaches in Australia: he says 'pronomeral' for variable.

BACKUP VIDEO

Translating Words into Algebra

Eddie Woo • 5:27

<https://www.youtube.com/watch?v=40T1hPkw928>

Before Example 2: a second set of phrases, including 'less than' and 'quotient'.

KEY TERMS

Variable

A letter that stands for a number, either one we do not know yet or one that can change.

Algebraic expression

A combination of numbers, variables and operations with no equal sign, such as $2n - 5$.

Equation

A statement that two expressions are equal, such as $2n - 5 = 11$. It is a complete sentence in mathematics.

Define a variable

To state in words exactly what a letter stands for, for example "Let w = the number of weeks Samuel saves."

CORE IDEAS

Why do letters stand for numbers? A letter holds the place of a number we do not know yet, or of a number that can change. It lets us write one short sentence that is true for many numbers at once. Instead of saying "the cost is 6 dollars times the number of jars, whatever that number turns out to be," we write $6j$. Any time you could ask "how many?" or "how much?" you can name that amount with a letter and work with it as if it were a number.

Always **define your variable** before you use it, with units: "Let j = the number of jars sold" or "Let t = the time in hours." A clear definition keeps you from mixing up quantities and lets someone else follow your work.

Learn the common word clues. Addition: *sum, plus, more than, increased by, total*. Subtraction: *difference, minus, less than, decreased by, fewer than*. Multiplication: *product, times, twice* (times 2), *of* (as in half of). Division: *quotient, divided by, per, ratio*. The word *is* (or *equals, gives, the result is*) becomes the equal sign and turns an expression into an equation.

Order matters for subtraction and division. "Five less than a number" means start with the number and take 5 away, so it is $n - 5$, not $5 - n$. Test it with a real number: five less than 12 is 7, and $12 - 5 = 7$, but $5 - 12 = -7$. The same goes for "subtracted from": "3 subtracted from x " is $x - 3$. But "the difference of a and b " keeps the order you read: $a - b$.

Grouping matters too. "Three times the sum of a number and 4" multiplies the whole sum, so it needs parentheses: $3(n + 4)$. "The sum of three times a number and 4" is $3n + 4$. Words like *the sum of, the difference of* and *the quantity* tell you that a group is being treated as one thing. Before writing, ask: what is the last operation being described? That operation goes on the outside.

For a word problem, read the whole situation, decide what is unknown, define a variable for it, write every other quantity in terms of that variable, and then write

the equation the story describes. Check your equation by trying a sensible number and seeing whether the arithmetic matches the story.

KEY FORMULAS

$$5 \text{ less than } n \rightarrow n - 5$$

$$3 \text{ times the sum of } n \text{ and } 4 \rightarrow 3(n + 4)$$

$$\text{is} \rightarrow =$$

From history. Algebra was not always written with letters. Diophantus of Alexandria, in his *Arithmetica* (about AD 250), used a special abbreviation for an unknown number, which let him write problems far more briefly than in full sentences. In 1591 the French mathematician François Viète used letters for both unknown and known quantities, so that one statement could describe a whole family of problems. In *La Géométrie* (1637), René Descartes began the custom still used today: x , y and z for unknowns and a , b and c for known quantities.

Practice habit. Before writing any equation, write a "Let" sentence that defines your variable with units, such as "Let m = the number of muffins sold." Then test your expression with an easy number to be sure the order is right.

WORKED EXAMPLES

Example 1. Let n be a number. Translate (a) "five less than twice a number" and (b) "twice the difference of a number and five."

Show solution

1. (a) "Twice a number" is $2n$.
2. "Five less than" that means take 5 away from $2n$: $2n - 5$.
3. Test with $n = 10$: twice 10 is 20, and five less is 15; $2(10) - 5 = 15$. ✓
4. (b) "The difference of a number and five" is a group: $n - 5$.
5. "Twice" that whole group: $2(n - 5)$.
6. Test with $n = 10$: the difference is 5, twice that is 10; $2(10 - 5) = 10$. ✓ The two phrases give different results, so the parentheses matter.

Answer: (a) $2n - 5$; (b) $2(n - 5)$

Example 2. The Miller family rents a booth at the county fair for \$35 and sells jars of honey for \$6 each. Define a variable, write an expression for their profit, and write an equation that says their profit is \$145.

Show solution

1. Define the variable: let j = the number of jars of honey sold.
2. Money taken in: \$6 per jar, so $6j$ dollars.
3. Profit = money taken in minus the booth rent: $6j - 35$ dollars.
4. "Profit is 145" becomes the equation $6j - 35 = 145$.
5. Check that the equation fits the story: if they sold 30 jars, they took in $6(30) = 180$ dollars, and $180 - 35 = 145$. So 30 jars gives exactly \$145 profit, which makes sense.

Answer: Let j = the number of jars sold. Profit: $6j - 35$ dollars.

Equation: $6j - 35 = 145$.

PRACTICE

1. Easy Write an expression: the sum of a number n and 8.

Show answer

$$n + 8$$

2. Easy Write an expression: 7 less than a number x .

Show answer

$$x - 7$$

3. Easy Write an expression: the product of 4 and a number y .

Show answer

$$4y$$

4. Easy Write an expression: a number k divided by 3.

Show answer

$$\frac{k}{3}$$

5. **Medium** Write an expression: three more than half a number n .

Show answer

$$\frac{n}{2} + 3$$

6. **Medium** Write an expression: three times the sum of a number n and 4.

Show answer

$$3(n + 4)$$

7. **Medium** Write an equation: six more than four times a number n is 30.

Show answer

$$4n + 6 = 30$$

8. **Medium** Write an equation: the quotient of a number n and 5, decreased by 2, is 9.

Show answer

$$\frac{n}{5} - 2 = 9$$

9. **Medium** Ruth is 3 years older than twice the age of her brother. Let b = her brother's age in years. Write an expression for Ruth's age.

Show answer

$$2b + 3 \text{ years}$$

10. **Hard** Tickets to a choir concert cost \$8 for adults and \$5 for students. Let a = the number of adult tickets and s = the number of student tickets. Write an equation that says the ticket sales totaled \$1,240.

Show answer

$$8a + 5s = 1240$$

11. **Hard** A train travels at 60 miles per hour for h hours and then at 45 miles per hour for 2 more hours. Write a simplified expression for the total distance in miles.

Show answer

$$60h + 45(2) = 60h + 90 \text{ miles}$$

12. **Hard** Write an equation: the square of a number n , decreased by 10, is equal to six times the number.

Show answer

$$n^2 - 10 = 6n$$

13. **Hard** Explain why "5 less than n " is written $n - 5$ and not $5 - n$. Use a number to support your answer.

Show answer

"5 less than n " means start with n and take away 5. For example, 5 less than 12 is 7, and $12 - 5 = 7$, while $5 - 12 = -7$. Since subtraction is not commutative, the order matters.

14. **Challenge** The sum of three consecutive integers is 72. Let n = the smallest integer. Write an equation for this situation.

Show answer

The integers are n , $n + 1$ and $n + 2$, so $n + (n + 1) + (n + 2) = 72$, or $3n + 3 = 72$.

15. **Challenge** Grace has \$50 and saves \$12 a week. Her brother Samuel has \$20 and saves \$18 a week. Let w = the number of weeks. Write an equation that says they have the same amount of money after w weeks.

Show answer

Grace: $50 + 12w$; Samuel: $20 + 18w$. Equation: $50 + 12w = 20 + 18w$.

COMMON MISTAKES

- Reversing "less than" or "subtracted from": "8 less than x " is $x - 8$, not $8 - x$. Test the phrase with a real number to check the order.
- Leaving out needed parentheses: "twice the sum of n and 3" is $2(n + 3)$, not $2n + 3$. If the words say "the sum" or "the difference" is being multiplied, put that group in parentheses.
- Using a variable without saying what it means, or letting it stand for two different things in one problem. Write a "Let ..." sentence with units first.
- Writing an expression when an equation is asked for (or the reverse). If the words include "is" or "equals" connecting two amounts, the answer needs an equal sign.

Make it a game · Say It in Symbols. The parent reads a phrase such as 'five less than twice a number'. First say what your variable stands for and its units, then write the expression within ten seconds. Watch the order with 'less than'. After six phrases, switch: you read an expression in words and the parent writes it. Disputes are settled by trying a number.

UNIT 1

FOUNDATIONS: REAL NUMBERS AND EXPRESSIONS

Algebra is the language of quantities that are unknown or that change. Before we solve equations, we need to know exactly what numbers we are working with and what rules they obey. In this unit you will sort the real numbers into families (natural, whole, integer, rational and irrational), learn why repeating decimals are fractions and why $\sqrt{2}$ is not, and name the properties, such as the commutative and distributive properties, that justify every step of algebra.

You will then sharpen three skills you will use every day for the rest of the course: following the order of operations, evaluating expressions and formulas by substitution, and simplifying expressions by distributing and combining like terms. Finally, you will learn to translate English sentences into algebra, the step that turns real problems about gardens, savings and travel into equations.

Much of this unit reviews and deepens Pre-Algebra, so it moves quickly. Parents: the habits built here (one step per line, parentheses when substituting, checking with a number) prevent most of the errors students make later in solving equations (Unit 2), working with polynomials (Unit 8) and factoring (Unit 9). Work by hand; no calculator is needed in this unit.

PREREQUISITES

- Add, subtract, multiply and divide integers using the sign rules.
- Compute with fractions and decimals, and convert between fractions, decimals and percents.
- Evaluate whole-number exponents and square roots of perfect squares.
- Use the order of operations with whole numbers.
- Use the distributive property and combine simple like terms (Pre-Algebra).
- Find the perimeter and area of rectangles, triangles and trapezoids.

MASTERY CHECKLIST

- ☐ I can classify a number as natural, whole, integer, rational, irrational and real.
- ☐ I can write a terminating or repeating decimal as a fraction and explain why it is rational.
- ☐ I can place a square root between two consecutive whole numbers and order real numbers.
- ☐ I can state the commutative, associative, distributive, identity and inverse properties and name the property used in a step.
- ☐ I can show with an example why subtraction and division are not commutative.
- ☐ I can simplify numerical expressions using the order of operations, including exponents and fraction bars.
- ☐ I can evaluate expressions and formulas by substitution, using parentheses around negative values.
- ☐ I can identify terms, coefficients and like terms, and combine like terms.
- ☐ I can simplify expressions by distributing, including distributing a negative.
- ☐ I can define a variable and translate phrases and sentences into expressions and equations.

LESSONS IN THIS UNIT

1.1 The Real Number System	45 minutes
1.2 Properties of Real Numbers	45 minutes
1.3 Order of Operations and Evaluating Expressions	45 minutes
1.4 Simplifying Expressions	2 sessions of 45 minutes
1.5 Translating Words into Algebra	2 sessions of 45 minutes

UNIT 1 TEST

Time allowed: 50 minutes. No calculator. Show your work for every problem; answers without work may receive partial credit only. Give exact answers (fractions or square roots) unless a problem asks for a decimal or an estimate, and include units in answers to word problems.

1. 5 pts

Which of these numbers are irrational? $\sqrt{36}$, $\sqrt{15}$, $0.\bar{7}$, π , $-\frac{2}{3}$

2. 6 pts

Write $0.\overline{45}$ as a fraction in lowest terms.

3. 6 pts

Order from least to greatest: $\sqrt{8}$, 2.8 , $\frac{17}{6}$, $2.\bar{8}$.

4. 5 pts

Name the property: $(x + 3) + 9 = x + (3 + 9)$.

5. 6 pts

Use the distributive property to find 7×199 mentally. Show the property you used.

6. 6 pts

Evaluate $24 \div (-2)^3 + 5 \times 3 - 1$.

7. (7 pts)

Evaluate $3a^2 - ab$ for $a = -2$ and $b = 5$.

8. (7 pts)

Samuel drops a pebble from a footbridge 100 feet above a river. Its height in feet after t seconds is $h = -16t^2 + 100$. How high is the pebble after 2 seconds?

9. (7 pts)

Simplify $5(2x - 3) - 3(x - 4)$.

10. (7 pts)

Simplify $-(4y - 1) + 2(y^2 + 3y) - y^2$.

11. (6 pts)

Write an expression: eight less than the product of 3 and a number n .

12. (8 pts)

Ruth spends \$18 on ingredients and sells muffins at a community bake sale for \$2 each. Let m = the number of muffins she sells. Write an equation that says her profit is \$54.

13. (8 pts)

The length of a rectangular pasture is 10 yards more than 3 times its width w (in yards). Write and simplify an expression for the perimeter of the pasture.

14. (8 pts)

Explain why $-(x - 5)$ equals $-x + 5$ and not $-x - 5$. Check your explanation with $x = 2$.

15. (8 pts)

Simplify $2[3x - (x - 4)] - 4(x + 2)$.

ANSWER KEY

1. $\sqrt{15}$ and π . ($\sqrt{36} = 6$, $0.\overline{7} = \frac{7}{9}$ and $-\frac{2}{3}$ are rational.)
2. $100x - x = 45$, so $99x = 45$ and $x = \frac{45}{99} = \frac{5}{11}$.
3. $\sqrt{8} \approx 2.828$, $\frac{17}{6} = 2.8\overline{3}$, $2.\overline{8} = 2.888\dots$, so the order is 2.8 , $\sqrt{8}$, $\frac{17}{6}$, $2.\overline{8}$.
4. Associative property of addition.
5. $7(200 - 1) = 1400 - 7 = 1393$
6. $24 \div (-8) + 15 - 1 = -3 + 15 - 1 = 11$
7. $3(-2)^2 - (-2)(5) = 12 - (-10) = 22$
8. $h = -16(2)^2 + 100 = -64 + 100 = 36$ feet
9. $10x - 15 - 3x + 12 = 7x - 3$
10. $-4y + 1 + 2y^2 + 6y - y^2 = y^2 + 2y + 1$
11. $3n - 8$
12. Money taken in: $2m$; profit: $2m - 18$. Equation: $2m - 18 = 54$.
13. Length $3w + 10$; $P = 2(3w + 10) + 2w = 6w + 20 + 2w = 8w + 20$ yards
14. The minus sign in front means multiply by -1 , and the distributive property says -1 multiplies every term: $(-1)(x) + (-1)(-5) = -x + 5$. Check with $x = 2$: $-(2 - 5) = -(-3) = 3$, and $-2 + 5 = 3$, while $-2 - 5 = -7$.
15. Inside the brackets: $3x - x + 4 = 2x + 4$. Then $2(2x + 4) - 4x - 8 = 4x + 8 - 4x - 8 = 0$.

UNIT 2

SOLVING LINEAR EQUATIONS

In this unit you learn to solve linear equations of every common kind: one- and two-step equations, equations that need distributing and combining like terms, equations with the variable on both sides, and equations with fractions and decimals. The key idea is the balance: an equation stays true when you do the same thing to both sides. You will also see why some equations are true for every number and some for none.

Next you solve formulas for one of their letters, such as rearranging $d = rt$ to find time, and you use a four-step plan to turn word problems about money, travel, mixtures and percents into equations. These skills build directly on the expressions of Unit 1.

Almost everything later in the course depends on this unit. Inequalities (Unit 3) are solved with the same moves, solving for y prepares you to graph lines (Unit 5), and systems of equations (Unit 6) are solved by reducing them to the one-variable equations you master here. Parents: a short daily check of one problem by substitution is the best way to build accuracy.

PREREQUISITES

- Add, subtract, multiply and divide integers and fractions using the sign rules
- Use the order of operations to evaluate expressions (Lesson 1.3)
- Combine like terms and use the distributive property, including with a negative (Lesson 1.4)
- Translate words into expressions and equations (Lesson 1.5)
- Find the least common multiple of two or three numbers
- Convert between percents and decimals

MASTERY CHECKLIST

- ☐ I can explain the properties of equality using the balance picture.
- ☐ I can solve one- and two-step equations and check by substituting.
- ☐ I can solve multi-step equations by distributing and combining like terms first.
- ☐ I can solve equations with variables on both sides.
- ☐ I can recognize an identity (all real numbers) and a contradiction (no solution) and explain why each happens.
- ☐ I can clear fractions with the LCD and decimals with a power of 10, and explain why this keeps the same solution.
- ☐ I can solve a formula or literal equation for a named variable, including

ax for

$by =$

c

y .

- ☐ I can use the four-step plan to solve consecutive integer, perimeter, money and percent problems.
- ☐ I can solve distance–rate–time and mixture problems.
- ☐ I can check an answer in the words of a word problem.

LESSONS IN THIS UNIT

2.1 Solving One- and Two-Step Equations	45 minutes
2.2 Multi-Step Equations	2 sessions of 45 minutes
2.3 Equations with Variables on Both Sides	2 sessions of 45 minutes
2.4 Equations with Fractions and Decimals	2 sessions of 45 minutes
2.5 Literal Equations and Formulas	2 sessions of 45 minutes
2.6 Problem Solving with Linear Equations	2 sessions of 45 minutes

UNIT 2 TEST

Time allowed: 60 minutes. No calculator. Show your work on every problem, one step per line, and check your solutions. Give exact answers (fractions or decimals as appropriate); include units on word problems.

1. 5 pts

Solve $4x - 9 = 27$.

2. 5 pts

Solve $-\frac{x}{6} + 3 = 8$.

3. 6 pts

Solve $5(2x - 3) - 4x = 21$.

4. 6 pts

Solve $8 - 3(y + 2) = -13$.

5. 6 pts

Solve $7n - 4 = 3n + 20$.

6. 6 pts

Solve $2(4x - 1) = 8x + 5$.

7. (6 pts)

Solve $3(x - 2) + x = 4(x - 1) - 2$.

8. (6 pts)

Solve $\frac{x}{4} + \frac{2}{3} = \frac{x}{6} + 1$.

9. (6 pts)

Solve $0.2(x + 5) = 0.15x + 1.6$.

10. (6 pts)

Solve $5x - 2y = 8$ for y .

11. (7 pts)

(a) Solve the triangle area formula $A = \frac{1}{2}bh$ for b . (b) A triangular boat sail has an area of 54 square feet and a height of 12 feet. Find the length of its base.

12. (7 pts)

A rectangular vegetable garden is 5 feet longer than twice its width. Its perimeter is 70 feet. Find the width and the length.

13. (7 pts)

A youth group is raising \$1,200 for a service trip. They already have \$330 and earn \$15 for each car they wash. How many cars must they wash to reach their goal?

14. (8 pts)

A baker has 6 pounds of a nut mix that is 20% almonds. How many pounds of almonds should she add to make a mix that is 50% almonds?

15. (7 pts)

A replacement part for a tractor is on sale for 25% off. The sale price is \$138. What was the original price?

16. (6 pts)

When Ruth solved an equation, the variable terms cancelled and she was left with $4 = 9$. Explain what this means about the equation's solutions and why.

ANSWER KEY

1. $4x = 36$, so $x = 9$
2. $-\frac{x}{6} = 5$, so $x = -30$
3. $10x - 15 - 4x = 21$, so $6x = 36$ and $x = 6$
4. $8 - 3y - 6 = -13$, so $2 - 3y = -13$, $-3y = -15$ and $y = 5$
5. $4n = 24$, so $n = 6$
6. $8x - 2 = 8x + 5$ leaves $-2 = 5$, which is false: no solution
7. Both sides simplify to $4x - 6$, so the solution is all real numbers
8. Multiply by 12: $3x + 8 = 2x + 12$, so $x = 4$
9. Multiply by 100: $20x + 100 = 15x + 160$, so $5x = 60$ and $x = 12$
10. $-2y = -5x + 8$, so $y = \frac{5}{2}x - 4$
11. (a) $b = \frac{2A}{h}$; (b) $b = \frac{2(54)}{12} = 9$ feet
12. $2w + 2(2w + 5) = 70$, so $6w + 10 = 70$ and $w = 10$: width 10 feet, length 25 feet
13. $330 + 15c = 1200$, so $15c = 870$ and $c = 58$ cars
14. $0.20(6) + x = 0.50(6 + x)$, so $1.2 + x = 3 + 0.5x$, $0.5x = 1.8$ and $x = 3.6$ pounds
15. $0.75p = 138$, so $75p = 13800$ and $p = 184$; the original price was \$184
16. The equation has no solution. The steps were all legal, so any solution of the original equation would also make $4 = 9$ true. Since $4 = 9$ is false for every value of the variable, no number can be a solution.

2.1 SOLVING ONE- AND TWO-STEP EQUATIONS

OBJECTIVES

- Explain how the properties of equality keep an equation balanced.
- Solve one-step equations using inverse operations.
- Solve two-step equations by undoing operations in reverse order.
- Check a solution by substituting it into the original equation.

WATCH

Algebra Basics: Solving 2-Step Equations - Math Antics

mathantics • 10:29

<https://www.youtube.com/watch?v=LDliYKYvvdA>

WHEN TO WATCH

Before Example 1: two-step equations.

BACKUP VIDEO

Algebra Basics: Solving Basic Equations Part 1 - Math Antics

mathantics • 11:08

<https://www.youtube.com/watch?v=l3XzepN03KQ>

Review before the Core ideas, if one-step equations need a refresher.

KEY TERMS

equation

A statement that two expressions are equal, such as $3x - 7 = 14$.

inverse operations

Operations that undo each other: addition and subtraction undo each other, and so do multiplication and division.

solution

A value of the variable that makes the equation true.

properties of equality

The rules that say you may add, subtract, multiply or divide both sides of an equation by the same number (never dividing by 0) and the two sides stay equal.

equivalent equations

Equations that have exactly the same solution, such as $3x = 21$ and $x = 7$.

CORE IDEAS

Picture an equation as a balance scale. The left side and the right side weigh exactly the same. If you add 5 pounds to one pan, the scale tips, unless you add 5 pounds to the other pan too. That is the whole idea behind solving equations: whatever you do to one side, do to the other, and the scale stays level.

These moves are called the **properties of equality**. If $a = b$, then $a + c = b + c$, $a - c = b - c$, and $ac = bc$; also $\frac{a}{c} = \frac{b}{c}$, as long as $c \neq 0$. Why is dividing by 0 left out? Division by 0 has no meaning, so it cannot give a true equation.

To solve, we want the variable alone on one side. We get it there with **inverse operations**. In $x + 9 = 23$, the 9 is added to x , so we subtract 9 from both sides: $x = 14$. In $-4y = 36$, y is multiplied by -4 , so we divide both sides by -4 : $y = -9$. Each step produces an equivalent equation, one with the same solution but simpler to read.

A two-step equation such as $3x - 7 = 14$ was built in a certain order: start with x , multiply by 3, then subtract 7. To undo it, reverse the order, just as you take off your shoes before your socks. First add 7 to both sides to get $3x = 21$, then divide both sides by 3 to get $x = 7$. Undo the operations in the reverse of the order they were done. In $3x - 7 = 14$ that means undoing the subtraction first and the multiplication second.

Always **check** by substituting your answer back into the original equation. For $x = 7$: $3(7) - 7 = 21 - 7 = 14$. Both sides equal 14, so 7 is the solution. Checking catches sign slips and arithmetic errors before they cost you.

KEY FORMULAS

If $a = b$, then $a + c = b + c$ and $a - c = b - c$.

If $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$ ($c \neq 0$).

From history. The word *algebra* comes from a book written in Baghdad about 820 by Muhammad ibn Musa al-Khwarizmi. Its title used two Arabic words: *al-jabr*, meaning "restoring" (moving a subtracted term to the other side, where it becomes added), and *al-muqabala*, meaning "balancing" (cancelling equal amounts from both sides). These are exactly the moves you make when you solve an equation today.

Practice habit. Check every solution by substituting it into the original equation, not into a later step. Write the check on its own line: left side, right side, and a check mark when they match.

WORKED EXAMPLES

Example 1. Solve $3x - 7 = 14$ and check.

Show solution

1. Add 7 to both sides: $3x - 7 + 7 = 14 + 7$, so $3x = 21$.
2. Divide both sides by 3: $\frac{3x}{3} = \frac{21}{3}$, so $x = 7$.
3. Check: $3(7) - 7 = 21 - 7 = 14$. The left side equals the right side.

Answer: $x = 7$

Example 2. Grace has \$18 saved. She plans to save \$6 each week from babysitting. After how many weeks will she have \$60 in all?

Show solution

1. Let w be the number of weeks.
2. Write the equation: $18 + 6w = 60$.
3. Subtract 18 from both sides: $6w = 42$.
4. Divide both sides by 6: $w = 7$.
5. Check: $18 + 6(7) = 18 + 42 = 60$. Correct.

Answer: 7 weeks

PRACTICE

1. Easy Solve $x + 9 = 23$.

Show answer

$$x = 14$$

2. Easy Solve $n - 12 = -5$.

Show answer

$$n = 7$$

3. Easy Solve $-4y = 36$.

Show answer

$$y = -9$$

4. Easy Solve $\frac{x}{5} = -3$.

Show answer

$$x = -15$$

5. Medium Solve $2x + 11 = 3$.

Show answer

$$2x = -8, \text{ so } x = -4$$

6. Medium Solve $15 - 3m = 27$.

Show answer

$$-3m = 12, \text{ so } m = -4$$

7. Medium Solve $\frac{k}{4} + 6 = 1$.

Show answer

$$\frac{k}{4} = -5, \text{ so } k = -20$$

8. Medium Solve $\frac{2}{3}x = 18$.

Show answer

$$\text{Multiply both sides by } \frac{3}{2}: x = 27$$

9. Medium A church choir orders new robes at \$35 each, plus a single \$20 shipping fee. The total bill is \$545. How many robes did the choir order?

Show answer

$$35r + 20 = 545, \text{ so } 35r = 525 \text{ and } r = 15 \text{ robes}$$

10. **Hard** Solve $-\frac{3}{4}p - 5 = 13$.

Show answer

$$-\frac{3}{4}p = 18, \text{ so } p = 18 \cdot \left(-\frac{4}{3}\right) = -24$$

11. **Hard** Samuel rents a canoe for a \$12 flat fee plus \$8 per hour. He pays \$44 in all. For how many hours did he rent the canoe?

Show answer

$$12 + 8h = 44, \text{ so } 8h = 32 \text{ and } h = 4 \text{ hours}$$

12. **Hard** Solve $\frac{x-4}{3} = -2$.

Show answer

$$\text{Multiply by 3: } x - 4 = -6, \text{ so } x = -2$$

13. **Hard** The Miller family's rectangular garden has a perimeter of 70 feet. Its length is 20 feet. Use $P = 2l + 2w$ to find the width.

Show answer

$$70 = 40 + 2w, \text{ so } 2w = 30 \text{ and } w = 15 \text{ feet}$$

14. **Challenge** Henry solved $5 - 2x = 11$ like this: $-2x = 6$, so $x = 3$. Find his mistake, explain it, and give the correct solution.

Show answer

His first step is right, but $6 \div (-2) = -3$, not 3; he dropped the negative sign when dividing by -2 . The solution is $x = -3$. Check: $5 - 2(-3) = 5 + 6 = 11$.

15. **Challenge** Solve $50 - 3x = 2$ the way al-Khwarizmi would: first "restore" by adding $3x$ to both sides, then finish solving.

Show answer

$$50 = 2 + 3x, \text{ then } 48 = 3x, \text{ so } x = 16. \text{ Check: } 50 - 48 = 2.$$

COMMON MISTAKES

- Doing an operation to only one side. Whatever you do to the left side, write the same thing on the right side on the same line.
- Undoing in the wrong order, such as dividing $3x - 7 = 14$ by 3 first and forgetting to divide the 7. Undo the operations in the reverse of the order they were done: in $3x - 7 = 14$, add 7 first, then divide by 3.

- Losing a negative sign when dividing by a negative number. Say the sign rule out loud: a positive divided by a negative is negative.
- Checking in a later step instead of the original equation. A mistake in an early step will not show up unless you check in the original.

Make it a game • Quiz the Parent. You are the teacher. Write three equations such as $x + 7 = 15$ and $3x - 4 = 11$ on a board. Solve each aloud, saying 'I do the same thing to both sides' at every step. The parent follows along but sneaks in one wrong step. Catch it and say which property of equality was broken. Check every answer by substitution.

2.2 MULTI-STEP EQUATIONS

OBJECTIVES

- Simplify each side of an equation by distributing and combining like terms before solving.
- Solve multi-step equations that include negative coefficients and parentheses.
- Write and solve multi-step equations for word problems.
- Check solutions by substituting into the original equation.

WATCH

Solving Multi-Step Equations Using the Distributive Property & Combining Like Terms

Math with Mr. J · 6:15

<https://www.youtube.com/watch?v=zKDIRT72bSg>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'brackets' for parentheses.

BACKUP VIDEO

Solving Linear Equations w/ Brackets

Eddie Woo · 7:04

<https://www.youtube.com/watch?v=EMFAKQ99v6M>

Before the practice set: equations with parentheses that must be distributed first.

KEY TERMS

multi-step equation

An equation that needs more than two steps to solve, often because one side must be simplified first.

like terms

Terms with exactly the same variable part, such as $4x$ and $-9x$, which can be combined into one term.

distributive property

The rule $a(b + c) = ab + ac$; it lets you remove parentheses by multiplying each term inside.

coefficient

The number multiplying a variable; in $-6x$ the coefficient is -6 .

CORE IDEAS

A multi-step equation is a two-step equation in disguise. Before you can undo anything, you must see what was done to the variable, and parentheses and scattered like terms hide that. So the plan is: first **simplify each side**, then **solve** as in Lesson 2.1, then **check**.

Simplifying a side does not change its value; it only rewrites it. For example, $4(x - 3) + 2x$ equals $4x - 12 + 2x$ by the distributive property, which equals $6x - 12$ by combining like terms. Because each rewritten side has the same value as before for every x , the new equation has exactly the same solutions as the old one.

Watch negatives carefully when you distribute. A minus sign in front of parentheses means multiply every term inside by -1 : $-(2x + 3) = -2x - 3$, and $-3(4 - m) = -12 + 3m$. A good habit is to draw arrows from the number outside to each term inside so you do not skip one.

Sometimes there is a shortcut. In $3(x + 4) = 27$ you can divide both sides by 3 first to get $x + 4 = 9$, then $x = 5$. Distributing first, $3x + 12 = 27$, gives the same answer. Both are legal because both use the properties of equality; choose whichever keeps the numbers simpler.

When the coefficient of the variable is negative, as in $-9a = -36$, divide by the negative number itself: $a = 4$. Dividing by 9 instead would leave $-a = -4$, which needs one more step.

KEY FORMULAS

$$a(b + c) = ab + ac \qquad -(b + c) = -b - c$$

Practice habit. Write one step per line and keep the equal signs lined up down the page. When you distribute, draw a small arrow from the outside number to each term inside the parentheses.

WORKED EXAMPLES

Example 1. Solve $4(x - 3) + 2x = 18$ and check.

Show solution

1. Distribute: $4x - 12 + 2x = 18$.
2. Combine like terms: $6x - 12 = 18$.
3. Add 12 to both sides: $6x = 30$.
4. Divide both sides by 6: $x = 5$.
5. Check: $4(5 - 3) + 2(5) = 4(2) + 10 = 8 + 10 = 18$. Correct.

Answer: $x = 5$

Example 2. Henry is building a rectangular pen for goats. Its length is 7 feet more than twice its width, and he has exactly 86 feet of fencing for the perimeter. Find the width and the length.

Show solution

1. Let w be the width in feet. Then the length is $2w + 7$.
2. Perimeter is $2(\text{width}) + 2(\text{length})$: $2w + 2(2w + 7) = 86$.
3. Distribute: $2w + 4w + 14 = 86$.
4. Combine like terms: $6w + 14 = 86$.
5. Subtract 14: $6w = 72$, so $w = 12$.
6. Length: $2(12) + 7 = 31$.
7. Check: $2(12) + 2(31) = 24 + 62 = 86$ feet. Correct.

Answer: Width 12 feet, length 31 feet

PRACTICE

1. Easy Solve $5x + 2x - 4 = 24$.

Show answer

$$7x - 4 = 24, \text{ so } 7x = 28 \text{ and } x = 4$$

2. Easy Solve $3(x + 4) = 27$.

Show answer

$$x + 4 = 9, \text{ so } x = 5$$

3. Easy Solve $-2(y - 5) = 14$.

Show answer

$$-2y + 10 = 14, \text{ so } -2y = 4 \text{ and } y = -2$$

4. Easy Solve $9 - x + 4 = 6$.

Show answer

$$13 - x = 6, \text{ so } -x = -7 \text{ and } x = 7$$

5. Medium Solve $6(2n - 1) - 4n = 34$.

Show answer

$$12n - 6 - 4n = 34, \text{ so } 8n = 40 \text{ and } n = 5$$

6. Medium Solve $-3(4 - m) + 2m = 18$.

Show answer

$$-12 + 3m + 2m = 18, \text{ so } 5m = 30 \text{ and } m = 6$$

7. Medium Solve $5 - (2x + 3) = -10$.

Show answer

$$5 - 2x - 3 = -10, \text{ so } 2 - 2x = -10, -2x = -12 \text{ and } x = 6$$

8. Medium A youth group buys 6 pizzas, all the same price, and a \$9 case of lemonade. A \$5 coupon comes off the total, and they pay \$76. What is the price of one pizza?

Show answer

$$6p + 9 - 5 = 76, \text{ so } 6p = 72 \text{ and } p = 12; \text{ each pizza costs } \$12$$

9. Medium Solve $2(3x + 1) + 3(x - 4) = 26$.

Show answer

$$6x + 2 + 3x - 12 = 26, \text{ so } 9x - 10 = 26, 9x = 36 \text{ and } x = 4$$

10. Hard Solve $-4(2a - 3) - (a + 6) = -30$.

Show answer

$$-8a + 12 - a - 6 = -30, \text{ so } -9a + 6 = -30, -9a = -36 \text{ and } a = 4$$

11. **Hard** A triangular flower bed has sides of x , $x + 3$ and $2x - 1$ feet. Its perimeter is 30 feet. How long is the longest side?

Show answer

$4x + 2 = 30$, so $x = 7$; the sides are 7, 10 and 13 feet, so the longest side is 13 feet

12. **Hard** Solve $7 - 2[3 - (x + 1)] = 11$.

Show answer

Inside: $3 - x - 1 = 2 - x$. Then $7 - 4 + 2x = 11$, so $3 + 2x = 11$ and $x = 4$

13. **Hard** Samuel worked at a farm stand for 3 days, working the same number of hours each day. He earned \$9 per hour plus a \$15 bonus each day, for \$261 in all. How many hours did he work each day?

Show answer

$3(9h + 15) = 261$, so $9h + 15 = 87$, $9h = 72$ and $h = 8$ hours

14. **Challenge** Ruth solved $3(x + 2) = 18$ by dividing both sides by 3 first. Her brother distributed first. Show both methods and explain why they must give the same answer.

Show answer

Dividing first: $x + 2 = 6$, so $x = 4$. Distributing first: $3x + 6 = 18$, $3x = 12$, so $x = 4$. Both methods only use the properties of equality or rewrite a side without changing its value, so every step keeps the same solution; the answers must agree.

15. **Challenge** Solve $-(3 - 2x) + 4(x - 2) - 2(x + 1) = 9$.

Show answer

$-3 + 2x + 4x - 8 - 2x - 2 = 9$, so $4x - 13 = 9$, $4x = 22$ and $x = \frac{11}{2}$

COMMON MISTAKES

- Distributing to only the first term, as in writing $-3(4 - m) = -12 - m$. Multiply the outside number by every term inside: $-3(4 - m) = -12 + 3m$.
- Forgetting that a minus sign in front of parentheses changes every sign inside: $5 - (2x + 3) = 5 - 2x - 3$, not $5 - 2x + 3$.
- Moving terms across the equal sign before each side is simplified, which makes it easy to lose a term. Simplify each side completely first, then solve.

- Combining unlike terms, such as $6x - 12$ into $-6x$. Only terms with the same variable part can be combined.

Make it a game · Draw and Solve Sprint. Write six multi-step equations on slips, with parentheses and negative coefficients, such as $-2(x - 3) + 5 = 9$. Draw one at a time and solve it on paper, showing the distributing and combining lines. Check each by substituting. Two points for a correct answer, one for a correct method with an arithmetic slip. Record your total and try to beat it.

2.3 EQUATIONS WITH VARIABLES ON BOTH SIDES

OBJECTIVES

- Solve equations with variable terms on both sides by collecting them on one side.
- Recognize identities (all real numbers are solutions) and contradictions (no solution).
- Explain why an equation can have one solution, no solution or infinitely many solutions.
- Write and solve equations with variables on both sides from word problems.

WATCH

An Intro to Solving Equations with Variables on Both Sides | Multi-Step Equations | Math with Mr. J

Math with Mr. J · 8:58

<https://www.youtube.com/watch?v=eZsyV0ISzV8>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: the backup also shows the same idea for inequalities, which comes in Unit 3.

BACKUP VIDEO

Equations/Inequalities with No/Infinite Solutions

Eddie Woo · 6:41

<https://www.youtube.com/watch?v=E74XdxKnnOs>

After Example 2: equations with no solution or infinitely many solutions (objective 3).

KEY TERMS

identity

An equation that is true for every value of the variable, such as
 $2(x + 3) = 2x + 6$. Its solution is all real numbers.

contradiction

An equation that is false for every value of the variable, such as
 $2x + 5 = 2x - 1$. It has no solution.

collect variable terms

Use addition or subtraction on both sides so that all the variable terms end up on one side of the equation.

CORE IDEAS

In an equation like $5x - 8 = 2x + 13$ the unknown appears on both sides. The properties of equality still work, and $2x$ is just an amount like any other, so we may subtract $2x$ from both sides. That gives $3x - 8 = 13$, a two-step equation you already know how to solve: $3x = 21$, so $x = 7$.

Which side should the variables go to? Either one works. A handy choice is to move the smaller variable term, so the coefficient that remains is positive. In $9 - y = 2y + 3$, add y to both sides to get $9 = 3y + 3$, then $6 = 3y$ and $y = 2$. It is fine for the answer to appear as $2 = y$.

Sometimes the variable terms cancel completely. Try $2(x + 3) = 2x + 6$. Distributing gives $2x + 6 = 2x + 6$, and subtracting $2x$ leaves $6 = 6$. That is always true, no matter what x was. The two sides were the same expression written two ways, so **every** real number is a solution. Such an equation is an **identity**.

Now try $2x + 5 = 2x - 1$. Subtracting $2x$ leaves $5 = -1$, which is never true. Why? The left side is always 6 more than the right side: whatever number $2x$ is, adding 5 can never give the same result as subtracting 1. So **no** value of x works. Such an equation is a **contradiction**, and we say it has no solution.

So a linear equation has exactly one solution when the variable terms on the two sides have different coefficients; it is an identity or a contradiction when the coefficients are equal, depending on whether the constants also match. When you get a statement with no variable left, read it: true means all real numbers, false means no solution.

KEY FORMULAS

Variables cancel and a true statement remains (like $6 = 6$): all real numbers.

Variables cancel and a false statement remains (like $5 = -1$): no solution.

From history. Al-Khwarizmi's "balancing" step, *al-muqabala*, is exactly what you do in this lesson: when the same kind of quantity appears on both sides, you remove it from both sides at once. In $5x - 8 = 2x + 13$, taking $2x$ from each side is a balancing step.

Practice habit. Before you start, simplify each side, then decide which side the variables will go to and write that move under both sides. Check your answer in the original equation, computing the left and right sides separately.

WORKED EXAMPLES

Example 1. Solve $5x - 8 = 2x + 13$ and check.

Show solution

1. Subtract $2x$ from both sides: $3x - 8 = 13$.
2. Add 8 to both sides: $3x = 21$.
3. Divide both sides by 3: $x = 7$.
4. Check: left side $5(7) - 8 = 35 - 8 = 27$; right side $2(7) + 13 = 14 + 13 = 27$.
Both are 27.

Answer: $x = 7$

Example 2. Ruth has \$40 saved and adds \$6 each week. Her sister Grace has \$70 saved and adds \$3 each week. After how many weeks will they have the same amount, and how much will each have?

Show solution

1. Let w be the number of weeks.
2. Ruth's amount is $40 + 6w$; Grace's amount is $70 + 3w$.
3. Set them equal: $40 + 6w = 70 + 3w$.
4. Subtract $3w$ from both sides: $40 + 3w = 70$.
5. Subtract 40: $3w = 30$, so $w = 10$.
6. Amount: $40 + 6(10) = 100$.
7. Check: Grace has $70 + 3(10) = 100$. Both have \$100.

Answer: After 10 weeks; each will have \$100

PRACTICE

1. Easy Solve $7x = 3x + 20$.

Show answer

$$4x = 20, \text{ so } x = 5$$

2. Easy Solve $6n - 4 = 2n + 12$.

Show answer

$$4n - 4 = 12, \text{ so } 4n = 16 \text{ and } n = 4$$

3. Easy Solve $9 - y = 2y + 3$.

Show answer

$$9 = 3y + 3, \text{ so } 6 = 3y \text{ and } y = 2$$

4. Easy Solve $3(x + 2) = 3x + 6$.

Show answer

$$3x + 6 = 3x + 6 \text{ leaves } 6 = 6, \text{ always true: all real numbers}$$

5. Medium Solve $4x + 1 = 4x - 7$.

Show answer

$$\text{Subtracting } 4x \text{ leaves } 1 = -7, \text{ which is false: no solution}$$

6. Medium Solve $5(a - 2) = 3a + 4$.

Show answer

$$5a - 10 = 3a + 4, \text{ so } 2a = 14 \text{ and } a = 7$$

7. Medium Solve $-2m + 11 = 5m - 17$.

Show answer

$$11 = 7m - 17, \text{ so } 28 = 7m \text{ and } m = 4$$

8. Medium Solve $2(3x - 1) = 4(x + 3)$.

Show answer

$$6x - 2 = 4x + 12, \text{ so } 2x = 14 \text{ and } x = 7$$

9. **Medium** Shop A rents canoes for \$15 plus \$6 per hour. Shop B charges \$25 plus \$4 per hour. For how many hours will the two shops charge the same amount?

Show answer

$$15 + 6h = 25 + 4h, \text{ so } 2h = 10 \text{ and } h = 5 \text{ hours (both cost \$45)}$$

10. **Hard** Solve $3(2y - 4) - y = 5(y - 2) - 2$.

Show answer

Left: $5y - 12$. Right: $5y - 12$. The sides are identical, so the solution is all real numbers

11. **Hard** Solve $4(2 - x) + 3x = 2(x + 7) - 9$.

Show answer

$$8 - x = 2x + 5, \text{ so } 3 = 3x \text{ and } x = 1$$

12. **Hard** On a farm, Tank A holds 500 gallons of water and is being drained at 15 gallons per minute. At the same time, Tank B holds 200 gallons and is being filled at 10 gallons per minute. After how many minutes will the two tanks hold the same amount?

Show answer

$$500 - 15t = 200 + 10t, \text{ so } 300 = 25t \text{ and } t = 12 \text{ minutes (both hold 320 gallons)}$$

13. **Hard** Grace is 14 years old and her father is 42. In how many years will her father be exactly twice as old as Grace?

Show answer

$$42 + y = 2(14 + y), \text{ so } 42 + y = 28 + 2y \text{ and } y = 14 \text{ years (Grace 28, father 56)}$$

14. **Challenge** Explain in words, without solving, why $2x + 3 = 2x + 5$ has no solution.

Show answer

Whatever number $2x$ is, adding 3 to it always gives a result 2 less than adding 5 to it. The two sides can never be equal, so no value of x makes the equation true.

15. **Challenge** For what value of a does the equation $ax + 7 = 5x - 2$ have no solution?

Show answer

$a = 5$: then subtracting $5x$ leaves $7 = -2$, which is false

COMMON MISTAKES

- Subtracting a variable term from one side only, or adding it to one side and subtracting it from the other. Write the same move under both sides.
- Writing $x = 0$ when the variables cancel. If the variables disappear, read the leftover statement: true means all real numbers, false means no solution.
- Confusing the two special cases. Remember: $6 = 6$ is always true (identity, all real numbers); $5 = -1$ is never true (contradiction, no solution).
- Moving a term and changing its sign incorrectly. Instead of "moving" terms, think "subtract $2x$ from both sides" so the sign takes care of itself.

Make it a game • Three Piles. Make three labeled piles: One Answer, All Real Numbers, No Solution. The parent writes eight equations with variables on both sides, such as $3x + 2 = 3x + 5$ and $2(x + 1) = 2x + 2$. Solve each one, then drop it on the right pile. Finally, write one equation for each pile yourself and stump the parent.

2.4 EQUATIONS WITH FRACTIONS AND DECIMALS

OBJECTIVES

- Clear fractions from an equation by multiplying both sides by the least common denominator.
- Clear decimals from an equation by multiplying both sides by a power of 10.
- Explain why multiplying both sides by a nonzero number keeps the same solution.
- Solve word problems involving fractions, decimals, money and percents.

WATCH

Equations with Fractions (1 of 3: Simple Example)

Eddie Woo · 6:16

<https://www.youtube.com/watch?v=HqNOvsJprls>

WHEN TO WATCH

Before Example 1: clearing fractions by multiplying by the common denominator.

Note: Neither video clears decimals by multiplying by 10 or 100. Teach that from the Core ideas; it is the same idea.

BACKUP VIDEO

Equations with Fractions (2 of 3: Trickier Example)

Eddie Woo · 3:13

<https://www.youtube.com/watch?v=PU76yTefGkw>

Before the practice set: a harder fraction equation.

KEY TERMS

least common denominator (LCD)

The smallest number that every denominator in the equation divides into evenly; for denominators 3, 4 and 2 it is 12.

clearing fractions

Multiplying every term on both sides of an equation by the LCD so that no fractions remain.

clearing decimals

Multiplying every term on both sides by 10, 100 or 1000 so that every decimal becomes a whole number.

power of 10

A number such as 10, 100 or 1000; multiplying by it moves the decimal point to the right 1, 2 or 3 places.

CORE IDEAS

You can solve $\frac{2x}{3} - \frac{1}{4} = \frac{x}{2} + 1$ by working with fractions the whole way, but it is easy to slip. A cleaner plan is to turn it into an equation with no fractions at all, then solve as before.

The tool is the multiplication property of equality: if $a = b$, then $ca = cb$. Why does this keep the same solution? Any number that makes $a = b$ true also makes $ca = cb$ true. And going back, if $ca = cb$ with $c \neq 0$, we can divide by c to get $a = b$ again. So the two equations have exactly the same solutions. (With $c = 0$ this fails: $0 \cdot a = 0 \cdot b$ is true for every x , so it would wipe out all information. Never multiply by 0.)

Which number should we multiply by? The **LCD**, because it is the smallest number that every denominator divides evenly, so every fraction becomes a whole number. In the equation above, the denominators are 3, 4 and 2, so the LCD is 12. Multiply **every term** by 12: $12 \cdot \frac{2x}{3} - 12 \cdot \frac{1}{4} = 12 \cdot \frac{x}{2} + 12 \cdot 1$, which is $8x - 3 = 6x + 12$. Then $2x = 15$ and $x = \frac{15}{2}$.

When a numerator has more than one term, keep it in parentheses as you multiply. In $\frac{x+2}{3} = \frac{x-1}{2}$, multiplying by 6 gives $2(x+2) = 3(x-1)$, not $2x+2 = 3x-1$. The fraction bar acts like a grouping symbol.

Decimals are fractions with denominators 10, 100, 1000 and so on, so the same idea works. Find the decimal with the most places after the point and multiply every term by the matching power of 10. In $1.25p + 3.50 = 11.00$, two decimal places means multiply by 100: $125p + 350 = 1100$. Money problems almost always use 100, since cents are hundredths of a dollar.

Percents are hundredths too: 20% means 0.20. If a pie costs \$9.60 after 20% off, the sale price is 80% of the original, so $0.80p = 9.60$. Multiply by 100 to get $80p = 960$, so $p = 12$. The original price was \$12.

KEY FORMULAS

If $a = b$ and $c \neq 0$, then $ca = cb$ has the same solutions.

Clear fractions: multiply every term by the LCD. Clear decimals: multiply every term by 10, 100, or 1000.

Practice habit. Before multiplying, list the denominators and write the LCD in the margin. Then draw an arrow to every term on both sides, including whole-number terms, so none is skipped.

WORKED EXAMPLES

Example 1. Solve $\frac{2x}{3} - \frac{1}{4} = \frac{x}{2} + 1$ and check.

Show solution

1. The denominators are 3, 4 and 2, so the LCD is 12.
2. Multiply every term by 12: $8x - 3 = 6x + 12$.
3. Subtract $6x$ from both sides: $2x - 3 = 12$.
4. Add 3: $2x = 15$.
5. Divide by 2: $x = \frac{15}{2} = 7.5$.
6. Check: left side $\frac{2(7.5)}{3} - \frac{1}{4} = 5 - 0.25 = 4.75$; right side $\frac{7.5}{2} + 1 = 3.75 + 1 = 4.75$. They match.

Answer: $x = \frac{15}{2}$

Example 2. Ruth buys apples at \$1.25 per pound and a \$3.50 basket to carry them. She spends \$11.00 in all. How many pounds of apples did she buy?

Show solution

1. Let p be the number of pounds.
 2. Write the equation: $1.25p + 3.50 = 11.00$.
 3. The decimals have two places, so multiply every term by 100:
 $125p + 350 = 1100$.
 4. Subtract 350: $125p = 750$.
 5. Divide by 125: $p = 6$.
 6. Check: $1.25(6) + 3.50 = 7.50 + 3.50 = 11.00$. Correct.
-

Answer: 6 pounds

PRACTICE

1. Easy Solve $\frac{x}{4} + 2 = 5$.

Show answer

Multiply by 4: $x + 8 = 20$, so $x = 12$

2. Easy Solve $\frac{1}{2}x + \frac{1}{3} = \frac{5}{6}$.

Show answer

Multiply by 6: $3x + 2 = 5$, so $3x = 3$ and $x = 1$

3. Easy Solve $0.3x + 1.2 = 2.7$.

Show answer

Multiply by 10: $3x + 12 = 27$, so $3x = 15$ and $x = 5$

4. Easy Solve $0.05n = 2.5$.

Show answer

Multiply by 100: $5n = 250$, so $n = 50$

5. Medium Solve $\frac{x}{3} - \frac{x}{5} = 4$.

Show answer

Multiply by 15: $5x - 3x = 60$, so $2x = 60$ and $x = 30$

6. Medium Solve $\frac{3}{4}y - 2 = \frac{1}{2}y + 1$.

Show answer

Multiply by 4: $3y - 8 = 2y + 4$, so $y = 12$

7. Medium Solve $1.5x - 0.25 = 0.75x + 2$.

Show answer

Multiply by 100: $150x - 25 = 75x + 200$, so $75x = 225$ and $x = 3$

8. Medium Solve $\frac{x+2}{3} = \frac{x-1}{2}$.

Show answer

Multiply by 6: $2(x+2) = 3(x-1)$, so $2x+4 = 3x-3$ and $x = 7$

9. Medium At a church bake sale, a pie is marked 20% off. The sale price is \$9.60. What was the original price?

Show answer

$0.80p = 9.60$, so $80p = 960$ and $p = 12$; the original price was \$12.00

10. Hard Solve $\frac{2x-1}{5} - \frac{x}{2} = -1$.

Show answer

Multiply by 10: $2(2x-1) - 5x = -10$, so $4x-2-5x = -10$, $-x = -8$ and $x = 8$

11. Hard Henry deposits money in a savings account that pays 4% simple interest per year. After one year, his balance (deposit plus interest) is \$520. How much did he deposit?

Show answer

$x + 0.04x = 520$, so $1.04x = 520$, $104x = 52000$ and $x = 500$; he deposited \$500

12. Hard Solve $0.06x + 0.04(1000 - x) = 52$.

Show answer

Multiply by 100: $6x + 4(1000 - x) = 5200$, so $6x + 4000 - 4x = 5200$, $2x = 1200$ and $x = 600$

13. **Hard** On a car trip, the Miller family drove one-third of the distance on the interstate, one-fourth of it on country roads, and the last 50 miles through towns. How long was the whole trip?

Show answer

$\frac{x}{3} + \frac{x}{4} + 50 = x$; multiply by 12: $4x + 3x + 600 = 12x$, so $600 = 5x$ and $x = 120$ miles

14. **Challenge** Explain why multiplying both sides of an equation by the LCD does not change its solution, and why you must never multiply both sides by 0.

Show answer

If a number makes the two sides equal, multiplying both equal sides by the same number keeps them equal, so it is still a solution. Going back, if the new sides are equal you can divide by the nonzero LCD to return to the original equation, so no new solutions appear. Multiplying by 0 makes both sides 0, which is true for every x , so it destroys the information in the equation and cannot be undone.

15. **Challenge** Solve $\frac{x}{2} - \frac{x-3}{4} = \frac{x+1}{6}$.

Show answer

Multiply by 12: $6x - 3(x-3) = 2(x+1)$, so $6x - 3x + 9 = 2x + 2$, $3x + 9 = 2x + 2$ and $x = -7$

COMMON MISTAKES

- Multiplying only the fractions by the LCD and forgetting whole-number terms. In $\frac{x}{4} + 2 = 5$, every term, including 2 and 5, must be multiplied by 4.
- Dropping the parentheses around a numerator with two terms. $-\frac{x-3}{4}$ times 12 is $-3(x-3) = -3x + 9$, not $-3x - 3$.
- Multiplying by too small a power of 10. Look for the decimal with the most places: $1.5x - 0.25$ needs 100, not 10.
- Setting up a discount as $0.20p = 9.60$. The \$9.60 is what is left after 20% off, which is 80% of the original, so write $0.80p = 9.60$.

Make it a game • Name the Multiplier. Before you solve anything, name the number that clears the fractions or decimals. The parent shows six equations such as $x/4 + 1/6 = 2/3$ and $0.05x + 1.2 = 0.3$. Say the LCD or power of 10 in five seconds for one point, then solve for a second point. Check your answers in the original equation.

2.5 LITERAL EQUATIONS AND FORMULAS

OBJECTIVES

- Solve a literal equation or formula for a named variable.
- Rewrite an equation such as $ax + by = c$ in the form " $y = \dots$ ".
- State when a solved formula requires a variable not to be zero.
- Use a rearranged formula to answer practical questions.

WATCH

Solving a Formula for a Given Variable (TTP Video 18)

Professor Leonard · 17:57

<https://www.youtube.com/watch?v=GjFsY9PjlfM>

WHEN TO WATCH

Before Example 1. It solves several formulas for a named variable, exactly as this lesson does.

Note: Alternate-tier primary (Professor Leonard), about 18 minutes. No approved channel has a short video on solving a formula for a given variable. Eddie Woo teaches in Australia: 'changing the subject' means solving for a variable.

BACKUP VIDEO

Equations & Formulas

Eddie Woo · 7:15

<https://www.youtube.com/watch?v=Pp0ou1nz18w>

Before the Core ideas: what a formula is and what the 'subject' of a formula means.

KEY TERMS

literal equation

An equation with two or more variables (letters), such as $d = rt$ or $2x + 3y = 6$.

formula

A literal equation that states a rule relating quantities, such as $A = lw$ for the area of a rectangle.

solve for a variable

Rewrite the equation so that one
chosen variable is alone on one side
and does not appear on the other side.

CORE IDEAS

A formula like $d = rt$ (distance equals rate times time) is built to find d . But suppose you know the distance and the rate and want the time. You could substitute the numbers and solve each time, or you could solve the formula for t once and use it again and again. Solving a formula for a different letter is called solving a **literal equation**.

Why does this work? The other letters stand for numbers, so they obey exactly the same rules as numbers. Compare $5 = 2t$ with $d = rt$. To solve $5 = 2t$ you divide both sides by 2. To solve $d = rt$ you divide both sides by r : $t = \frac{d}{r}$. The steps are identical; only the numbers have been replaced by letters.

Use the same plan as with numbers: find the variable you want, see what has been done to it, and undo those operations in reverse order. In $P = 2l + 2w$, the width w is multiplied by 2 and then $2l$ is added. So subtract $2l$ first, $P - 2l = 2w$, then divide by 2: $w = \frac{P - 2l}{2}$.

Dividing by a letter comes with a warning: we cannot divide by 0. When we write $t = \frac{d}{r}$ we are assuming $r \neq 0$, which makes sense, since a car with rate 0 never covers any distance. Get in the habit of noticing which letters must not be zero.

An important case for later units is solving $ax + by = c$ for y . Subtract ax from both sides: $by = c - ax$. Then divide **every** term by b : $y = \frac{c - ax}{b}$, with $b \neq 0$. For example, $2x - 5y = 10$ becomes $-5y = -2x + 10$, and dividing each term by -5 gives $y = \frac{2}{5}x - 2$. Writing equations as " $y = \dots$ " will make graphing lines much easier in Unit 5.

To convert between Celsius and Fahrenheit, start from $C = \frac{5}{9}(F - 32)$. Multiply both sides by $\frac{9}{5}$ to undo the $\frac{5}{9}$: $\frac{9}{5}C = F - 32$. Then add 32: $F = \frac{9}{5}C + 32$.

KEY FORMULAS

$$d = rt \quad A = lw \quad P = 2l + 2w \quad I = Prt$$

$$C = \frac{5}{9}(F - 32) \iff F = \frac{9}{5}C + 32$$

$$A = \frac{1}{2}h(b_1 + b_2)$$

$$ax + by = c \implies y = \frac{c - ax}{b} \quad (b \neq 0)$$

From history. Using letters for known quantities as well as unknown ones is the idea that makes formulas possible. François Viète, a French mathematician, began doing this in 1591, and René Descartes in 1637 started the custom of using letters near the start of the alphabet, such as a, b, c , for known quantities and x, y, z for unknowns. That is why we write $ax + by = c$.

Practice habit. Circle the variable you are solving for before you begin. Then solve a matching number equation in the margin (such as $5 = 2t$) and copy the same moves with the letters.

WORKED EXAMPLES

Example 1. Solve $P = 2l + 2w$ for w .

Show solution

1. The variable w is multiplied by 2, and then $2l$ is added.
2. Subtract $2l$ from both sides: $P - 2l = 2w$.
3. Divide both sides by 2: $\frac{P - 2l}{2} = w$.
4. Check with numbers: if $l = 12$ and $w = 8$, then $P = 40$, and $\frac{40 - 24}{2} = 8$.

Correct.

Answer: $w = \frac{P - 2l}{2}$

Example 2. Ruth's recipe from a European cookbook says to bake bread at 180 degrees Celsius. (a) Solve $C = \frac{5}{9}(F - 32)$ for F . (b) Use your formula to find the oven temperature in degrees Fahrenheit.

Show solution

1. (a) Multiply both sides by $\frac{9}{5}$: $\frac{9}{5}C = F - 32$.
2. Add 32 to both sides: $F = \frac{9}{5}C + 32$.
3. (b) Substitute $C = 180$: $F = \frac{9}{5}(180) + 32$.
4. Compute: $\frac{9}{5}(180) = 9 \times 36 = 324$, and $324 + 32 = 356$.
5. Check: $\frac{5}{9}(356 - 32) = \frac{5}{9}(324) = 5 \times 36 = 180$. Correct.

Answer: (a) $F = \frac{9}{5}C + 32$; (b) 356 degrees Fahrenheit

PRACTICE

1. Easy Solve $d = rt$ for t .

Show answer

$$t = \frac{d}{r}$$

2. Easy Solve $A = lw$ for l .

Show answer

$$l = \frac{A}{w}$$

3. Easy Solve $P = a + b + c$ for c .

Show answer

$$c = P - a - b$$

4. Easy Solve $3x + y = 7$ for y .

Show answer

$$y = -3x + 7$$

5. Medium Solve $I = Prt$ for r .

Show answer

Divide both sides by Pt : $r = \frac{I}{Pt}$

6. Medium Solve $2x - 5y = 10$ for y .

Show answer

$-5y = -2x + 10$, so $y = \frac{2}{5}x - 2$

7. Medium Solve $ax + by = c$ for y . What must be true about b ?

Show answer

$by = c - ax$, so $y = \frac{c - ax}{b}$, with $b \neq 0$

8. Medium (a) Solve $d = rt$ for r . (b) The Miller family drives 312 miles to visit their grandparents in 6 hours. What was their average speed?

Show answer

(a) $r = \frac{d}{t}$; (b) $r = \frac{312}{6} = 52$ miles per hour

9. Medium The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$. Solve for h .

Show answer

Multiply by 2: $2A = h(b_1 + b_2)$, so $h = \frac{2A}{b_1 + b_2}$

10. Hard Solve $A = \frac{1}{2}h(b_1 + b_2)$ for b_1 .

Show answer

$\frac{2A}{h} = b_1 + b_2$, so $b_1 = \frac{2A}{h} - b_2$

11. Hard Grace wants to earn \$90 in simple interest by depositing \$1,500 for 2 years. Use $I = Prt$ to find the annual interest rate she needs, as a percent.

Show answer

$r = \frac{I}{Pt} = \frac{90}{1500 \times 2} = \frac{90}{3000} = 0.03$, which is 3%

12. **Hard** The amount in a simple-interest account is $A = P + Prt$. Solve for r .

Show answer

$$A - P = Prt, \text{ so } r = \frac{A - P}{Pt}$$

13. **Hard** Henry has 64 feet of fencing for a rectangular chicken run. (a) Solve $P = 2l + 2w$ for l . (b) Find the length if the width is 12 feet.

Show answer

$$(a) l = \frac{P - 2w}{2}; (b) l = \frac{64 - 24}{2} = 20 \text{ feet}$$

14. **Challenge** Solve $ax + b = cx + d$ for x . What must be true for your answer to make sense?

Show answer

$$ax - cx = d - b, \text{ so } x(a - c) = d - b \text{ and } x = \frac{d - b}{a - c}, \text{ with } a \neq c$$

15. **Challenge** Samuel solved $2x + 3y = 6$ for y and wrote $y = 2 - 2x$. Find his mistake, explain it, and give the correct answer.

Show answer

From $3y = 6 - 2x$ he divided only the 6 by 3. Every term must be divided by 3:
 $y = 2 - \frac{2}{3}x$. Check with $x = 3$: the original gives $6 + 3y = 6$, so $y = 0$, and
 $2 - \frac{2}{3}(3) = 0$.

COMMON MISTAKES

- Dividing only one term by a coefficient. In $3y = 6 - 2x$, divide every term by 3:
 $y = 2 - \frac{2}{3}x$.
- Forgetting that Pt means P times t , so to solve $I = Prt$ for r you must divide by both: $r = \frac{I}{Pt}$, not $\frac{I}{P} - t$.
- Leaving the variable you want on both sides. The solved variable must appear on one side only.
- Ignoring division by zero. When you divide by a letter, note that it cannot be 0 (for example $r \neq 0$ in $t = \frac{d}{r}$).

Make it a game • Spin the Letter. Write familiar formulas on cards: $d = rt$, $A = lw$, $P = 2l + 2w$, $C = 2\pi r$ and $3x + 2y = 12$. Draw a card, then draw a letter slip, and solve the formula for that letter within 45 seconds. Check by substituting a test number for each letter. Score a point for each rearrangement, and play ten rounds.

2.6 PROBLEM SOLVING WITH LINEAR EQUATIONS

OBJECTIVES

- Use a four-step plan to solve word problems: understand, define a variable, write and solve, check in the words.
- Write expressions for consecutive integers, consecutive even integers and consecutive odd integers.
- Solve perimeter, money, distance–rate–time, mixture and percent problems with linear equations.
- Check an answer against the words of the problem, not only the equation.

WATCH

Equations: Interpreting & Solving Word Problems

Eddie Woo · 8:36

<https://www.youtube.com/watch?v=cjzlZuqP5cU>

WHEN TO WATCH

Before Example 1: reading a word problem and writing the equation.

Note: Neither video does consecutive-integer problems. Use Example 2 for those.

BACKUP VIDEO

Solving Equation Problems (3 of 3: Translating to an equation)

Eddie Woo · 6:01

https://www.youtube.com/watch?v=C65vdjQEI_c

Before the practice set: more sentences translated into equations.

KEY TERMS

consecutive integers

Integers that follow one after another, such as 28, 29, 30; written n , $n + 1$, $n + 2$.

consecutive even (or odd) integers

Even (or odd) integers that follow one after another, such as 36, 38, 40 or 23, 25; written n , $n + 2$, $n + 4$.

distance–rate–time

The relationship $d = rt$: distance equals rate (speed) times time.

mixture problem

A problem where two ingredients are combined, and the amount of something (value, or a pure substance) in the parts adds up to the amount in the whole.

CORE IDEAS

Word problems are where algebra earns its keep. Use the same four steps every time.

1. Understand: read the problem twice, and say in your own words what is known and what is asked. **2. Define a variable:** write "Let $n = \dots$ " with units, and write every other unknown in terms of it. **3. Write and solve** an equation. **4. Check in the words:** put your answer back into the story, not just the equation, and ask whether it makes sense.

Why check in the words and not just the equation? If you wrote the wrong equation, your answer will satisfy that wrong equation perfectly. Only the original words can catch a setup mistake. A fence cannot have a negative length, and a number of children cannot be 7.5.

Consecutive integers. If the first integer is n , the next is $n + 1$, then $n + 2$.

Consecutive even integers differ by 2, so they are $n, n + 2, n + 4$. Consecutive odd integers also differ by 2, so they are also $n, n + 2, n + 4$. What makes them odd is the starting value n , not the expressions.

Distance, rate and time. Make a small table with a row for each traveler and columns for rate, time and distance, using $d = rt$ in each row. Then ask how the distances are related. When two travelers go in opposite directions, their distances add to the total gap. When one catches up with the other from the same starting point, their distances are equal.

Mixtures and money. Each part contributes (amount) times (price or percent). The parts add up to the whole: value of part 1 + value of part 2 = value of mixture. For example, mixing p pounds of \$3 peanuts with 10 pounds of \$8 cashews to get a \$5 mix gives $3p + 8(10) = 5(p + 10)$. Percent problems work the same way with decimals: 10% of 40 gallons is $0.10(40)$ gallons.

Percent change. A 15% raise means the new amount is 115% of the old: $1.15w$. A 25% discount means you pay 75%: $0.75p$. Writing the whole change as one multiplier keeps the equation simple.

KEY FORMULAS

Consecutive integers: $n, n + 1, n + 2$ Consecutive even or odd: $n, n + 2, n + 4$

$$d = rt$$

value of part 1 + value of part 2 = value of the mixture

From history. Word problems are very old. Leonardo of Pisa, known as Fibonacci, filled his book *Liber Abaci* (1202) with practical problems about buying, selling, trading and travel, written to teach merchants how to calculate. Many of today's mixture and money problems look much like his.

Practice habit. Write the four steps down the side of your paper for every word problem: Understand, Let, Solve, Check in the words. Before solving, estimate the answer so you can tell whether your result is reasonable.

WORKED EXAMPLES

Example 1. The sum of three consecutive integers is 87. Find the integers.

Show solution

1. Understand: three integers in a row add to 87.
2. Let n be the first integer. Then the others are $n + 1$ and $n + 2$.
3. Write: $n + (n + 1) + (n + 2) = 87$.
4. Combine: $3n + 3 = 87$.
5. Subtract 3: $3n = 84$, so $n = 28$.
6. The integers are 28, 29 and 30.
7. Check in the words: they are consecutive, and $28 + 29 + 30 = 87$.

Answer: 28, 29 and 30

Example 2. Samuel leaves home driving at 45 miles per hour. One hour later, his sister Ruth leaves home and follows the same route at 60 miles per hour. How many hours after Ruth leaves will she catch up with Samuel?

Show solution

1. Let t be the number of hours Ruth drives. Samuel drives one hour longer, so $t + 1$ hours.
 2. Ruth's distance: $60t$. Samuel's distance: $45(t + 1)$.
 3. When she catches up, the distances are equal: $60t = 45(t + 1)$.
 4. Distribute: $60t = 45t + 45$.
 5. Subtract $45t$: $15t = 45$, so $t = 3$.
 6. Check in the words: Ruth drives $60 \times 3 = 180$ miles; Samuel drives $45 \times 4 = 180$ miles. They are at the same place.
-

Answer: 3 hours

PRACTICE

1. **Easy** The sum of two consecutive integers is 63. Find the integers.

Show answer

$$n + (n + 1) = 63, \text{ so } 2n = 62 \text{ and } n = 31: 31 \text{ and } 32$$

2. **Easy** When a number is tripled and then increased by 7, the result is 43. Find the number.

Show answer

$$3n + 7 = 43, \text{ so } 3n = 36 \text{ and } n = 12$$

3. **Easy** A rectangular picture frame is 3 times as long as it is wide. Its perimeter is 48 inches. Find the width and the length.

Show answer

$$2w + 2(3w) = 48, \text{ so } 8w = 48 \text{ and } w = 6: \text{ width } 6 \text{ inches, length } 18 \text{ inches}$$

4. **Easy** Grace has \$35 and saves \$15 each week. After how many weeks will she have \$200?

Show answer

$$35 + 15w = 200, \text{ so } 15w = 165 \text{ and } w = 11 \text{ weeks}$$

5. **Medium** The sum of three consecutive even integers is 114. Find the integers.

Show answer

$$n + (n + 2) + (n + 4) = 114, \text{ so } 3n + 6 = 114 \text{ and } n = 36: 36, 38 \text{ and } 40$$

6. **Medium** For two consecutive odd integers, the larger one plus twice the smaller one equals 71. Find the integers.

Show answer

$$(n + 2) + 2n = 71, \text{ so } 3n = 69 \text{ and } n = 23: 23 \text{ and } 25$$

7. **Medium** At a church bake sale, cookies sell for \$0.75 each and brownies for \$1.25 each. The Miller children sold 20 more cookies than brownies and took in \$83 in all. How many brownies and how many cookies did they sell?

Show answer

$$1.25b + 0.75(b + 20) = 83, \text{ so } 2b + 15 = 83 \text{ and } b = 34: 34 \text{ brownies and } 54 \text{ cookies}$$

8. **Medium** Each of the two equal sides of an isosceles triangle is 4 centimeters longer than the base. The perimeter is 38 centimeters. How long is the base?

Show answer

$$b + 2(b + 4) = 38, \text{ so } 3b + 8 = 38 \text{ and } b = 10 \text{ centimeters}$$

9. **Medium** Two trains leave the same station at noon and travel in opposite directions, one at 55 miles per hour and the other at 65 miles per hour. After how many hours will they be 300 miles apart?

Show answer

$$55t + 65t = 300, \text{ so } 120t = 300 \text{ and } t = 2.5 \text{ hours}$$

10. **Hard** After a 15% raise, Henry earns \$13.80 per hour at the hardware store. What did he earn per hour before the raise?

Show answer

$$1.15w = 13.80, \text{ so } 115w = 1380 \text{ and } w = 12; \text{ he earned } \$12.00 \text{ per hour}$$

11. **Hard** How many pounds of peanuts costing \$3 per pound must be mixed with 10 pounds of cashews costing \$8 per pound to make a mix worth \$5 per pound?

Show answer

$$3p + 8(10) = 5(p + 10), \text{ so } 3p + 80 = 5p + 50, 30 = 2p \text{ and } p = 15 \text{ pounds}$$

12. **Hard** A farmer has 40 gallons of a liquid fertilizer that is 10% nitrogen. How many gallons of a 25% nitrogen fertilizer should he add to get a mixture that is 15% nitrogen?

Show answer

$0.10(40) + 0.25x = 0.15(40 + x)$, so $4 + 0.25x = 6 + 0.15x$, $0.10x = 2$ and $x = 20$ gallons

13. **Hard** Samuel earned money mowing lawns this summer. He gave 10% of it to his church, put half of it into savings, and had \$96 left to spend. How much did he earn?

Show answer

$x - 0.10x - 0.50x = 96$, so $0.40x = 96$ and $x = 240$; he earned \$240

14. **Challenge** Henry drives to his grandparents' farm at 60 miles per hour and returns home by the same road at 40 miles per hour. The round trip takes 5 hours of driving. How far away is the farm?

Show answer

$\frac{d}{60} + \frac{d}{40} = 5$; multiply by 120: $2d + 3d = 600$, so $d = 120$ miles

15. **Challenge** Grace says three consecutive odd integers should be written n , $n + 1$, $n + 2$. Explain why that is wrong, and then explain why the sum of three consecutive odd integers can never be 100.

Show answer

Odd integers are 2 apart (like 23, 25, 27), so they are n , $n + 2$, $n + 4$; $n + 1$ would be even. If $3n + 6 = 100$, then $3n = 94$ and $n = \frac{94}{3}$, which is not an integer. Also, the sum of three odd numbers is always odd, and 100 is even.

COMMON MISTAKES

- Writing consecutive odd integers as n , $n + 1$, $n + 2$. Odd integers are 2 apart: n , $n + 2$, $n + 4$.
- Using the same time for both travelers when one left earlier. If Samuel left an hour before Ruth, his time is $t + 1$.
- Stopping after finding n when the problem asks for several quantities. Reread the question and answer every part, with units.

- Checking only in the equation. If the equation was set up wrong, the answer will still fit it; check that the answer makes the original story true.

Make it a game · Mystery Neighbors. The parent secretly picks three consecutive odd integers and tells you only their sum, such as 75. Define your variable, write x , $x + 2$ and $x + 4$, form the equation, solve, and check it in the words of the problem. Then swap roles. For a final round, write a distance-rate-time story that the parent must solve with the same four steps.

UNIT 2

SOLVING LINEAR EQUATIONS

In this unit you learn to solve linear equations of every common kind: one- and two-step equations, equations that need distributing and combining like terms, equations with the variable on both sides, and equations with fractions and decimals. The key idea is the balance: an equation stays true when you do the same thing to both sides. You will also see why some equations are true for every number and some for none.

Next you solve formulas for one of their letters, such as rearranging $d = rt$ to find time, and you use a four-step plan to turn word problems about money, travel, mixtures and percents into equations. These skills build directly on the expressions of Unit 1.

Almost everything later in the course depends on this unit. Inequalities (Unit 3) are solved with the same moves, solving for y prepares you to graph lines (Unit 5), and systems of equations (Unit 6) are solved by reducing them to the one-variable equations you master here. Parents: a short daily check of one problem by substitution is the best way to build accuracy.

PREREQUISITES

- Add, subtract, multiply and divide integers and fractions using the sign rules
- Use the order of operations to evaluate expressions (Lesson 1.3)
- Combine like terms and use the distributive property, including with a negative (Lesson 1.4)
- Translate words into expressions and equations (Lesson 1.5)
- Find the least common multiple of two or three numbers
- Convert between percents and decimals

MASTERY CHECKLIST

- ☐ I can explain the properties of equality using the balance picture.
- ☐ I can solve one- and two-step equations and check by substituting.
- ☐ I can solve multi-step equations by distributing and combining like terms first.
- ☐ I can solve equations with variables on both sides.
- ☐ I can recognize an identity (all real numbers) and a contradiction (no solution) and explain why each happens.
- ☐ I can clear fractions with the LCD and decimals with a power of 10, and explain why this keeps the same solution.
- ☐ I can solve a formula or literal equation for a named variable, including

ax for

$by =$

c

y .

- ☐ I can use the four-step plan to solve consecutive integer, perimeter, money and percent problems.
- ☐ I can solve distance–rate–time and mixture problems.
- ☐ I can check an answer in the words of a word problem.

LESSONS IN THIS UNIT

2.1 Solving One- and Two-Step Equations	45 minutes
2.2 Multi-Step Equations	2 sessions of 45 minutes
2.3 Equations with Variables on Both Sides	2 sessions of 45 minutes
2.4 Equations with Fractions and Decimals	2 sessions of 45 minutes
2.5 Literal Equations and Formulas	2 sessions of 45 minutes
2.6 Problem Solving with Linear Equations	2 sessions of 45 minutes

UNIT 2 TEST

Time allowed: 60 minutes. No calculator. Show your work on every problem, one step per line, and check your solutions. Give exact answers (fractions or decimals as appropriate); include units on word problems.

1. 5 pts

Solve $4x - 9 = 27$.

2. 5 pts

Solve $-\frac{x}{6} + 3 = 8$.

3. 6 pts

Solve $5(2x - 3) - 4x = 21$.

4. 6 pts

Solve $8 - 3(y + 2) = -13$.

5. 6 pts

Solve $7n - 4 = 3n + 20$.

6. 6 pts

Solve $2(4x - 1) = 8x + 5$.

7. (6 pts)

Solve $3(x - 2) + x = 4(x - 1) - 2$.

8. (6 pts)

Solve $\frac{x}{4} + \frac{2}{3} = \frac{x}{6} + 1$.

9. (6 pts)

Solve $0.2(x + 5) = 0.15x + 1.6$.

10. (6 pts)

Solve $5x - 2y = 8$ for y .

11. (7 pts)

(a) Solve the triangle area formula $A = \frac{1}{2}bh$ for b . (b) A triangular boat sail has an area of 54 square feet and a height of 12 feet. Find the length of its base.

12. (7 pts)

A rectangular vegetable garden is 5 feet longer than twice its width. Its perimeter is 70 feet. Find the width and the length.

13. (7 pts)

A youth group is raising \$1,200 for a service trip. They already have \$330 and earn \$15 for each car they wash. How many cars must they wash to reach their goal?

14. (8 pts)

A baker has 6 pounds of a nut mix that is 20% almonds. How many pounds of almonds should she add to make a mix that is 50% almonds?

15. (7 pts)

A replacement part for a tractor is on sale for 25% off. The sale price is \$138. What was the original price?

16. (6 pts)

When Ruth solved an equation, the variable terms cancelled and she was left with $4 = 9$. Explain what this means about the equation's solutions and why.

ANSWER KEY

1. $4x = 36$, so $x = 9$
2. $-\frac{x}{6} = 5$, so $x = -30$
3. $10x - 15 - 4x = 21$, so $6x = 36$ and $x = 6$
4. $8 - 3y - 6 = -13$, so $2 - 3y = -13$, $-3y = -15$ and $y = 5$
5. $4n = 24$, so $n = 6$
6. $8x - 2 = 8x + 5$ leaves $-2 = 5$, which is false: no solution
7. Both sides simplify to $4x - 6$, so the solution is all real numbers
8. Multiply by 12: $3x + 8 = 2x + 12$, so $x = 4$
9. Multiply by 100: $20x + 100 = 15x + 160$, so $5x = 60$ and $x = 12$
10. $-2y = -5x + 8$, so $y = \frac{5}{2}x - 4$
11. (a) $b = \frac{2A}{h}$; (b) $b = \frac{2(54)}{12} = 9$ feet
12. $2w + 2(2w + 5) = 70$, so $6w + 10 = 70$ and $w = 10$: width 10 feet, length 25 feet
13. $330 + 15c = 1200$, so $15c = 870$ and $c = 58$ cars
14. $0.20(6) + x = 0.50(6 + x)$, so $1.2 + x = 3 + 0.5x$, $0.5x = 1.8$ and $x = 3.6$ pounds
15. $0.75p = 138$, so $75p = 13800$ and $p = 184$; the original price was \$184
16. The equation has no solution. The steps were all legal, so any solution of the original equation would also make $4 = 9$ true. Since $4 = 9$ is false for every value of the variable, no number can be a solution.

UNIT 3

LINEAR INEQUALITIES AND ABSOLUTE VALUE

In Unit 2 you solved equations, which usually have one answer. Many real questions are not about one exact number but about a range: How many hours must I work to earn at least enough? How many guests fit our budget? How far can a board's length be from 36 inches and still be acceptable? This unit teaches you to write and solve inequalities, which answer "how much is enough" and "how much is too much" questions, and to show their solutions on a number line and in interval notation.

You will start with one-step and multi-step inequalities and learn why multiplying or dividing by a negative number reverses the inequality sign. Then you will join inequalities with "and" and "or," and finally study absolute value as distance, which leads to equations with two solutions and to inequalities that describe tolerances in carpentry, baking and manufacturing.

These skills are used all through the rest of the course: domain and range in Unit 4, graphing inequalities in two variables and systems of inequalities in Unit 6, and later work in Geometry and Algebra II. Parents: the key habits to watch for are checking answers with a test number and reversing the sign only when multiplying or dividing by a negative.

PREREQUISITES

- Solve one-step, two-step and multi-step linear equations, including equations with variables on both sides (Lessons 2.1–2.3).
- Clear fractions from an equation by multiplying by the least common denominator (Lesson 2.4).
- Distribute, including distributing a negative, and combine like terms (Lesson 1.4).
- Compare and order integers and fractions on a number line (Lesson 1.1).
- Translate words into algebraic expressions and equations (Lessons 1.5 and 2.6).

MASTERY CHECKLIST

- ☐ I can graph an inequality on a number line using open and closed circles.
- ☐ I can solve one- and two-step inequalities and explain why multiplying or dividing by a negative number reverses the sign.
- ☐ I can solve multi-step inequalities with distributing and variables on both sides.
- ☐ I can recognize when an inequality is true for all real numbers or has no solution, and explain why.
- ☐ I can translate "at least," "at most" and "no more than" into inequality symbols and give sensible answers to word problems.
- ☐ I can solve and graph "and" and "or" compound inequalities, including three-part inequalities.
- ☐ I can write solution sets in interval notation, such as
 $[-2, 5]$
 $(-\infty, 3) \cup$
 $[6, \infty)$
- ☐ I can explain absolute value as distance and solve absolute value equations by isolating the bars and using two cases.
- ☐ I can recognize absolute value equations and inequalities with no solution.
- ☐ I can solve absolute value inequalities, knowing that "less

than" gives an "and" inequality and "greater than" gives an "or" inequality.

- ☐ I can write and solve absolute value inequalities for tolerances.

LESSONS IN THIS UNIT

3.1 Solving Inequalities	45 minutes
3.2 Multi-Step Inequalities	2 sessions of 45 minutes
3.3 Compound Inequalities	2 sessions of 45 minutes
3.4 Absolute Value Equations	2 sessions of 45 minutes
3.5 Absolute Value Inequalities	2 sessions of 45 minutes

UNIT 3 TEST

You have 50 minutes. No calculator. Show all your work, one step per line. Give exact answers; write inequality answers in inequality form (interval notation is also accepted unless a problem asks for one form). Check your solutions with a test number when you can. For word problems, define your variable and answer in a sentence with units. The test is worth 100 points.

1. **5 pts**
Solve $4x - 7 > 13$ and describe the graph.
2. **6 pts**
Solve $9 - 3x \leq 24$.
3. **7 pts**
Solve $2(3x - 4) \geq 5x - 1$.
4. **6 pts**
Solve $3(x + 2) - x < 2(x + 4)$.

5. (7 pts)

Grace is saving for a \$240 telescope. She already has \$72 and saves \$14 each week. Write and solve an inequality to find the least number of weeks she must save to have at least enough.

6. (7 pts)

Solve $-5 \leq 3x + 4 < 19$. Write the answer in interval notation.

7. (7 pts)

Solve $2x + 3 < -5$ or $5 - x \leq 1$.

8. (6 pts)

Solve $|x - 6| = 10$.

9. (7 pts)

Solve $5|x + 2| - 3 = 22$.

10. (5 pts)

Solve $|4x - 1| + 9 = 2$.

11. (7 pts)

Solve $|x + 5| \leq 3$.

12. (8 pts)

Solve $|3x - 6| > 12$.

13. (8 pts)

A church choir is renting a coach bus for a concert trip. The bus costs \$180 plus \$6 per rider. The bus company requires a total bill of at least \$300, and the choir can spend at most \$420. Write and solve a compound inequality for the number of riders r .

14. (8 pts)

A bakery's loaves of bread should weigh 24 ounces, with a tolerance of 0.75 ounce. Write an absolute value inequality for the acceptable weights w and solve it.

15. 6 pts

Henry solved $-4x > 12$ and wrote $x > -3$. Explain his mistake and give the correct answer, using a test number to show which answer is right.

ANSWER KEY

1. $4x > 20$, so $x > 5$; open circle at 5, shaded to the right
2. $-3x \leq 15$, so $x \geq -5$
3. $6x - 8 \geq 5x - 1$, so $x \geq 7$
4. $2x + 6 < 2x + 8$ gives $6 < 8$, always true: all real numbers
5. $72 + 14w \geq 240$, so $14w \geq 168$ and $w \geq 12$; at least 12 weeks
6. $-9 \leq 3x < 15$, so $-3 \leq x < 5$, or $[-3, 5)$
7. $2x < -8$ gives $x < -4$; $-x \leq -4$ gives $x \geq 4$; so $x < -4$ or $x \geq 4$
8. $x - 6 = 10$ or $x - 6 = -10$, so $x = 16$ or $x = -4$
9. $5|x + 2| = 25$, so $|x + 2| = 5$; $x = -7$ or $x = 3$
10. $|4x - 1| = -7$; an absolute value cannot be negative, so there is no solution
11. $-3 \leq x + 5 \leq 3$, so $-8 \leq x \leq -2$
12. $3x - 6 < -12$ or $3x - 6 > 12$, so $x < -2$ or $x > 6$
13. $300 \leq 180 + 6r \leq 420$, so $120 \leq 6r \leq 240$ and $20 \leq r \leq 40$; from 20 to 40 riders
14. $|w - 24| \leq 0.75$, so $23.25 \leq w \leq 24.75$ ounces
15. Dividing by the negative number -4 reverses the order of numbers, so the sign must flip: $x < -3$. Test $x = -5$ (which satisfies $x < -3$): $-4(-5) = 20 > 12$, true. Test $x = 0$ (which satisfies Henry's answer): $-4(0) = 0 > 12$, false. So the correct answer is $x < -3$.

3.1 SOLVING INEQUALITIES

OBJECTIVES

- Graph the solution of an inequality on a number line using open and closed circles.
- Solve one- and two-step inequalities using inverse operations.
- Explain why multiplying or dividing both sides by a negative number reverses the inequality sign.
- Check a solution of an inequality with a test number.

WATCH

Algebra Basics: Inequalities In Algebra - Math Antics

[mathantics](#) • 14:11

<https://www.youtube.com/watch?v=RyesLifeUBw>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

When to Flip the Inequality Sign | Math with Mr. J

[Math with Mr. J](#) • 8:04

<https://www.youtube.com/watch?v=76RZWFrJkg4>

Before Example 2: why the inequality sign flips when you multiply or divide by a negative.

KEY TERMS

inequality

A statement that two quantities are not equal, using one of the symbols $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to) or $\geq$ (greater than or equal to).

solution set

The collection of all solutions of an inequality, such as every number greater than 4.

solution of an inequality

Any number that makes the inequality true when you substitute it for the variable. Most inequalities have infinitely many solutions.

open circle

A hollow dot on a number line graph showing that the endpoint is NOT a solution (used with $<$ and $>$).

closed circle

A filled-in dot on a number line graph showing that the endpoint IS a solution (used with $\leq$ and $\geq$).

CORE IDEAS

An equation like $x + 3 = 7$ has one solution, $x = 4$. An inequality like $x + 3 > 7$ has a whole crowd of solutions: 4.1, 5, 10, 1000 and every other number greater than 4. Because we cannot list them all, we write the answer as a simpler inequality, $x > 4$, and draw it on a number line.

To graph a solution, put a circle at the boundary number and shade the side that works. Use an **open circle** for $<$ or $>$ (the boundary itself is not included) and a **closed circle** for $\leq$ or $\geq$ (the boundary is included). Shade to the right for numbers greater than the boundary and to the left for numbers less than it. Tip: write the variable first, as in $x > 4$. Then $x > 4$ says " x is bigger than 4," and bigger numbers lie to the right, so you shade to the right.

You solve inequalities with the same inverse operations you use for equations. Adding or subtracting the same number on both sides keeps an inequality true: if Ruth is taller than Henry, they are still in the same order if both stand on 6-inch boxes. Multiplying or dividing both sides by the same **positive** number also keeps the order: if $3 < 5$, then $3 \times 2 = 6 < 10 = 5 \times 2$.

Multiplying or dividing by a **negative** number is different. Start with the true statement $2 < 5$ and multiply both sides by -1 . You get -2 and -5 . On the number line -2 is to the right of -5 , so $-2 > -5$. The order flipped! Here is why: multiplying by -1 reflects every number across 0 like a mirror, so whatever was on the right ends up on the left. Multiplying by any other negative number is the same as multiplying by a positive number and then by -1 , so it also flips the order. That is why the rule is: **when you multiply or divide both sides by a negative number, reverse the inequality sign.**

Here is the same idea in algebra. Saying $a < b$ means $b - a$ is positive. If c is negative, then $cb - ca = c(b - a)$ is a negative times a positive, which is negative. So cb is less than ca , that is, $ca > cb$. The sign really must turn around.

Always check your answer with a **test number**. Pick an easy number in the shaded region and substitute it into the ORIGINAL inequality; it should make a true statement. Then substitute the boundary number: it should make the two sides equal. If a number on the wrong side also works, you probably forgot to reverse the sign.

KEY FORMULAS

If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Practice habit. Check every inequality with two numbers: substitute the boundary number (the two sides should be equal) and one easy test number from your shaded region (it should give a true statement). This catches a forgotten sign flip right away.

WORKED EXAMPLES

Example 1. Solve $-3x + 5 \leq 17$ and graph the solution on a number line.

Show solution

1. Subtract 5 from both sides: $-3x \leq 12$.
2. Divide both sides by -3 . Dividing by a negative reverses the sign: $x \geq -4$.
3. Graph: a closed circle at -4 (because of $\geq$), shaded to the right.
4. Check the boundary: $-3(-4) + 5 = 12 + 5 = 17$, and $17 \leq 17$ is true (the sides are equal).
5. Check a test number, $x = 0$: $-3(0) + 5 = 5$, and $5 \leq 17$ is true.

Answer: $x \geq -4$; closed circle at -4 , shaded to the right

Example 2. Grace has \$45 saved and earns \$12 per hour babysitting. She wants to have at least \$165 to buy a sewing machine. Write and solve an inequality to find how many hours she must work.

Show solution

1. Let h be the number of hours Grace works.
2. Her total money is $45 + 12h$, and "at least \$165" means ≥ 165 : $45 + 12h \geq 165$.
3. Subtract 45 from both sides: $12h \geq 120$.
4. Divide both sides by 12 (positive, so the sign stays): $h \geq 10$.
5. Check the boundary: $45 + 12(10) = 45 + 120 = 165$. Check $h = 11$:
 $45 + 132 = 177 \geq 165$, true.
6. Grace must work at least 10 hours.

Answer: $h \geq 10$; she must work at least 10 hours

PRACTICE

1. Easy Solve $x + 7 > 12$.

Show answer

$$x > 5$$

2. Easy Solve $n - 4 \leq -9$.

Show answer

$$n \leq -5$$

3. Easy Solve $5y \geq 35$ and describe its graph.

Show answer

$$y \geq 7; \text{ closed circle at 7, shaded to the right}$$

4. Easy A number line graph has an open circle at 3 and is shaded to the left. Write the inequality it shows, using x .

Show answer

$$x < 3$$

5. Medium Solve $-6x < 42$.

Show answer

$$\text{Divide by } -6 \text{ and reverse the sign: } x > -7$$

6. Medium Solve $\frac{m}{-4} \geq 3$.

Show answer

Multiply by -4 and reverse the sign: $m \leq -12$

7. Medium Solve $2x - 9 > 7$.

Show answer

$$2x > 16, \text{ so } x > 8$$

8. Medium Solve $11 - 4p \leq 3$.

Show answer

$$-4p \leq -8, \text{ so } p \geq 2$$

9. Medium A freight elevator can safely carry at most 1,200 pounds. Henry weighs 160 pounds and is moving boxes of books that weigh 40 pounds each. What is the greatest number of boxes he can take in one trip with him?

Show answer

$$160 + 40b \leq 1200, \text{ so } 40b \leq 1040 \text{ and } b \leq 26; \text{ at most 26 boxes}$$

10. **Hard** Is $x = -3$ a solution of $4 - 2x > 10$? Answer yes or no and show why.

Show answer

$$\text{No. } 4 - 2(-3) = 4 + 6 = 10, \text{ and } 10 > 10 \text{ is false.}$$

11. **Hard** Solve $\frac{2}{3}x + 1 < -5$.

Show answer

$$\frac{2}{3}x < -6, \text{ so } x < -9$$

12. **Hard** A church youth group has \$320 toward a service trip and needs to raise more than \$800 in all. They sell homemade pies for \$12 each. What is the least number of pies they must sell?

Show answer

$$320 + 12p > 800, \text{ so } 12p > 480 \text{ and } p > 40; \text{ they must sell at least 41 pies}$$

13. **Hard** At 6 p.m. the temperature on the Miller farm is 14°F , and it falls 3°F each hour. Write and solve an inequality to find when the temperature will be below -4°F .

Show answer

$14 - 3h < -4$, so $-3h < -18$ and $h > 6$; more than 6 hours after 6 p.m. (after midnight)

14. **Challenge** Explain why the solution of $-2x < 6$ is $x > -3$ and not $x < -3$. Use a test number in your explanation.

Show answer

Dividing by -2 flips the order of numbers on the number line, so the sign must reverse. Test $x = 0$ (which satisfies $x > -3$): $-2(0) = 0 < 6$ is true. Test $x = -5$ (which satisfies $x < -3$): $-2(-5) = 10 < 6$ is false. So the solutions are the numbers greater than -3 .

15. **Challenge** Solve $7 - \frac{x}{5} \geq 9$ and check the boundary.

Show answer

$-\frac{x}{5} \geq 2$, multiply by -5 and reverse: $x \leq -10$. Check: $7 - \frac{-10}{5} = 7 + 2 = 9$.

COMMON MISTAKES

- Forgetting to reverse the sign when multiplying or dividing by a negative number. Before you divide, look at the sign of the number you are dividing by, and check with a test number afterward.
- Reversing the sign when it is not needed, such as when subtracting a number or when only the number on the other side is negative (in $3x < -12$ you divide by positive 3, so the answer is $x < -4$ with no flip). Only multiplying or dividing by a negative flips the sign.
- Using the wrong circle. Use an open circle for $<$ and $>$, and a closed circle for $\leq$ and $\geq$.
- Giving a non-whole answer to a counting word problem. If $p > 40$ pies, the least number of pies is 41, not 40 and not 40.5.

Make it a game • Human Number Line. Lay out a number line on the floor with paper slips from -5 to 5. The parent calls an inequality such as $x \geq -2$, or one to solve first like $2x - 3 < 7$. Stand on a few numbers that work, then name the one boundary. Hold a coin in your hand for a closed circle and keep your hand empty for an open circle.

3.2 MULTI-STEP INEQUALITIES

OBJECTIVES

- Solve inequalities that require distributing, combining like terms, or collecting variables on one side.
- Recognize and explain inequalities that are always true (all real numbers) or never true (no solution).
- Translate phrases such as "at least," "at most" and "no more than" into inequality symbols.
- Write and solve inequalities for word problems and give answers that make sense in the situation.

WATCH

How to Solve Inequalities with Variables on Both Sides | Math with Mr. J.

Math with Mr. J. • 7:54

https://www.youtube.com/watch?v=Dug_SB85ndQ

WHEN TO WATCH

Before Example 1: inequalities with variables on both sides.

BACKUP VIDEO

Solving and Graphing Two-Step Inequalities | Math with Mr. J.

Math with Mr. J. • 7:47

<https://www.youtube.com/watch?v=yPmyccAl11M>

Before the Core ideas, if two-step inequalities and their graphs need a review.

KEY TERMS

at least

That number or more; written with $\geq$.
"At least 10" includes 10.

at most / no more than

That number or less; written with $\leq$.
"No more than \$50" includes \$50.

all real numbers

The answer when an inequality is true no matter what number replaces the variable, such as $2x + 6 > 2x$.

no solution

The answer when an inequality is false for every number, such as $2x + 1 > 2x + 5$.

CORE IDEAS

Multi-step inequalities are solved with the same plan you used for multi-step equations in Unit 2: distribute, combine like terms on each side, collect the variable terms on one side and the numbers on the other, then divide. The one new rule from Lesson 3.1 still applies: if you multiply or divide both sides by a negative number, reverse the sign.

When variables appear on both sides, you choose which side to collect them on. A helpful trick is to move the variable terms to the side where the coefficient will be **positive**. For $6x - 1 > 9x + 14$, subtracting $6x$ from both sides gives $-1 > 3x + 14$, then $-15 > 3x$, then $-5 > x$. You never divided by a negative, so there was no sign to flip. Reading $-5 > x$ from right to left gives $x < -5$, the same answer you get by the other route.

Sometimes the variable disappears. Look at $2(x + 3) > 2x + 1$. Distributing gives $2x + 6 > 2x + 1$, and subtracting $2x$ leaves $6 > 1$. That statement is true and has no x in it at all, which means it is true **whatever** x is. Why? Both sides contain the same $2x$, and the left side always has 6 added while the right has only 1, so the left side is always 5 more. The solution is **all real numbers**.

If the leftover statement is false, such as $-5 \geq 3$, then no number can ever make the inequality true, and there is **no solution**. For example, $3x - 5 \geq 3(x + 1)$ becomes $3x - 5 \geq 3x + 3$; the left side is always 8 less than the right side, so it can never be greater than or equal to it.

Word problems often use phrases instead of symbols. "At least" and "no less than" mean $\geq$. "At most" and "no more than" mean $\leq$. "More than" means $>$ and "less than" or "fewer than" means $<$. After solving, ask whether the answer must be a whole number. If you find $g \leq 60.8$ guests, the most guests you can have is 60; if you find $p > 40$ pies, the fewest pies is 41.

To compare two plans, write an inequality that says one plan costs less than the other. The boundary number, where the two sides are equal, is the break-even point:

on one side of it the first plan is cheaper, and on the other side the second plan is cheaper.

KEY FORMULAS

at least $\rightarrow \geq$ at most, no more than $\rightarrow \leq$ more than $\rightarrow >$ less than $\rightarrow <$

Practice habit. Write one step per line and keep the inequality symbols lined up under each other. When variables are on both sides, collect them on the side where the coefficient stays positive, so you avoid dividing by a negative.

WORKED EXAMPLES

Example 1. Solve $4(x - 3) - 2 > 6x + 8$.

Show solution

1. Distribute: $4x - 12 - 2 > 6x + 8$.
2. Combine like terms: $4x - 14 > 6x + 8$.
3. Subtract $4x$ from both sides (this keeps the x -coefficient positive):
 $-14 > 2x + 8$.
4. Subtract 8 from both sides: $-22 > 2x$.
5. Divide both sides by 2: $-11 > x$, which is the same as $x < -11$.
6. Check the boundary $x = -11$: left side $4(-14) - 2 = -58$; right side $6(-11) + 8 = -58$. Equal, as expected.
7. Check a test number, $x = -12$: left side $4(-15) - 2 = -62$; right side $6(-12) + 8 = -64$; $-62 > -64$ is true.

Answer: $x < -11$

Example 2. Henry is buying chicken feed. Store A sells it for \$15 a bag with free delivery. Store B sells it for \$12 a bag but charges \$45 for delivery. For how many bags is Store B the cheaper choice?

Show solution

1. Let b be the number of bags.
2. Cost at Store A: $15b$. Cost at Store B: $12b + 45$.
3. Store B is cheaper when $12b + 45 < 15b$.
4. Subtract $12b$ from both sides: $45 < 3b$.
5. Divide by 3: $15 < b$, so $b > 15$.
6. Check the boundary $b = 15$: Store A costs $15(15) = 225$ dollars and Store B costs $12(15) + 45 = 225$ dollars. Equal: this is the break-even point.
7. Check $b = 16$: Store A costs \$240 and Store B costs $192 + 45 = 237$ dollars. Store B is cheaper.
8. Store B is cheaper when Henry buys more than 15 bags, that is, 16 bags or more.

Answer: $b > 15$; Store B is cheaper for 16 or more bags

PRACTICE

1. Easy Solve $3x + 4 \leq 2x + 9$.

Show answer

$$x \leq 5$$

2. Easy Solve $2(x + 5) > 16$.

Show answer

$$2x + 10 > 16, \text{ so } 2x > 6 \text{ and } x > 3$$

3. Easy Solve $5n - 3 \geq 2n + 12$.

Show answer

$$3n \geq 15, \text{ so } n \geq 5$$

4. Easy Solve $-(y - 4) < 9$.

Show answer

$$-y + 4 < 9, \text{ so } -y < 5 \text{ and } y > -5$$

5. Medium Solve $7 - 2(a + 3) \geq 11$.

Show answer

$$1 - 2a \geq 11, \text{ so } -2a \geq 10 \text{ and } a \leq -5$$

6. Medium Solve $6x - 1 > 9x + 14$.

Show answer

$$-15 > 3x, \text{ so } x < -5$$

7. Medium Solve $4(x - 2) \leq 4x + 3$.

Show answer

$$4x - 8 \leq 4x + 3 \text{ gives } -8 \leq 3, \text{ which is always true: all real numbers}$$

8. Medium Solve $5(2 - x) > -5x + 12$.

Show answer

$$10 - 5x > -5x + 12 \text{ gives } 10 > 12, \text{ which is never true: no solution}$$

9. Medium Ruth is planning a church potluck. The hall costs \$60, and plates, cups and napkins cost \$4 per guest. Her budget is no more than \$300. What is the greatest number of guests she can plan for?

Show answer

$$60 + 4g \leq 300, \text{ so } 4g \leq 240 \text{ and } g \leq 60; \text{ at most 60 guests}$$

10. **Hard** Solve $3(2x - 1) - (x + 4) \geq 2(x + 1)$.

Show answer

$$5x - 7 \geq 2x + 2, \text{ so } 3x \geq 9 \text{ and } x \geq 3$$

11. **Hard** Solve $\frac{x}{2} - 3 < \frac{x}{3} + 1$.

Show answer

$$\text{Multiply by 6: } 3x - 18 < 2x + 6, \text{ so } x < 24$$

12. **Hard** Grace scored 84, 91 and 78 on her first three history tests. What is the lowest score she can earn on the fourth test so that her average for all four is at least 85?

Show answer

$$\frac{84 + 91 + 78 + s}{4} \geq 85, \text{ so } 253 + s \geq 340 \text{ and } s \geq 87; \text{ she needs at least 87}$$

13. **Hard** To haul lumber, Samuel can rent a trailer from one shop for \$25 plus \$0.50 per mile, or from another shop for \$40 plus \$0.30 per mile. For how many miles is the second shop cheaper?

Show answer

$40 + 0.3m < 25 + 0.5m$, so $15 < 0.2m$ and $m > 75$; for more than 75 miles

14. **Challenge** Explain why $2(x + 3) > 2x$ is true for all real numbers, but $2(x + 3) < 2x$ has no solution.

Show answer

Both become comparisons of $2x + 6$ and $2x$. For any number x , $2x + 6$ is exactly 6 more than $2x$, so it is always greater. That makes $2x + 6 > 2x$ always true and $2x + 6 < 2x$ never true. Algebraically, subtracting $2x$ leaves $6 > 0$ (true) or $6 < 0$ (false).

15. **Challenge** For which values of k is the inequality $3(x - k) \leq 3x - 12$ true for every real number x ?

Show answer

$3x - 3k \leq 3x - 12$ gives $-3k \leq -12$, so $k \geq 4$

COMMON MISTAKES

- Distributing a negative to only the first term. In $7 - 2(a + 3)$ the -2 multiplies both a and 3 , giving $7 - 2a - 6$.
- Writing " $x =$ all real numbers" or " $x = 0$ " when the variable disappears. If the leftover statement is true, the answer is all real numbers; if it is false, there is no solution.
- Mixing up "at least" and "at most." "At least 85" means 85 or more ($\geq$); "at most \$300" means \$300 or less ($\leq$). Say the phrase aloud with a sample number to test it.
- Leaving a fraction or the wrong whole number in a counting answer. Decide whether to round down (at most) or up (at least) by checking the boundary in the words of the problem.

Make it a game • Always, Never, Sometimes. Play two truths and a tall tale. Write three statements about inequalities you solved, such as ' $-2x > 6$ gives $x < -3$ ' and ' $3x + 1 > 3x$ is true for every number'. Two are true and one is a tall tale. The parent finds the tall tale and explains the fix. Include a sentence using 'at least' and one using 'no more than'.

3.3 COMPOUND INEQUALITIES

OBJECTIVES

- Explain the difference between an "and" compound inequality (intersection) and an "or" compound inequality (union).
- Solve and graph compound inequalities, including three-part inequalities such as $-7 < 2x - 3 \leq 5$.
- Write solution sets in inequality form and in interval notation, such as $[-2, 5)$.
- Write and solve compound inequalities for real situations.

WATCH

How to Solve and Graph Compound Inequalities | Math with Mr. J

Math with Mr. J • 10:38

<https://www.youtube.com/watch?v=aj2TRbLw9tM>

WHEN TO WATCH

Before Example 1: 'and' and 'or' inequalities and their graphs.

BACKUP VIDEO

An Intro to Writing Inequalities in Interval Notation | Math with Mr. J

Math with Mr. J • 5:19

<https://www.youtube.com/watch?v=sJ7Sk-F-Jt4>

After Example 2: interval notation, if your student will use it later.

KEY TERMS

compound inequality

Two inequalities joined by the word "and" or the word "or."

intersection ("and")

The numbers that make BOTH inequalities true. Its graph is the overlap of the two graphs, often a piece between two numbers.

union ("or")

The numbers that make AT LEAST ONE of the inequalities true. Its graph includes everything shaded in either graph.

interval notation

A short way to write a set of numbers using its endpoints: a bracket [or] means the endpoint is included, a parenthesis (or) means it is not.

Example: $[-2, 5)$ means $-2 \leq x < 5$.

infinity (∞)

A symbol meaning "without end." It is not a number, so it always gets a parenthesis: $[4, \infty)$ means $x \geq 4$.

CORE IDEAS

Sometimes one inequality is not enough. A loaf of bread might need to bake at a temperature of at least 350°F **and** at most 425°F . A number x that must satisfy $x \geq -2$ **and** $x < 5$ has to pass both tests at once. On the number line, the numbers that pass both tests are the overlap of the two graphs: the piece from -2 to 5 , with a closed circle at -2 and an open circle at 5 . We write this "between" statement in one line as $-2 \leq x < 5$.

An **or** inequality is satisfied when at least one part is true. For $x < -1$ or $x \geq 4$, the number -3 works (first part), the number 6 works (second part), but the number 0 fails both. The graph is two rays pointing away from each other: an open circle at -1 shaded left, and a closed circle at 4 shaded right.

To solve a three-part inequality like $-7 < 2x - 3 \leq 5$, do the same operation to all three parts, because it is really two inequalities ($-7 < 2x - 3$ and $2x - 3 \leq 5$) being solved together. Add 3 to every part: $-4 < 2x \leq 8$. Divide every part by 2 : $-2 < x \leq 4$. If you must divide by a negative, BOTH signs reverse; then rewrite the answer so the smaller number is on the left. For example, $-10 < -2x \leq 4$ becomes $5 > x \geq -2$, which reads better as $-2 \leq x < 5$.

To solve an "or" inequality, solve each part separately and keep the word "or." For $5x + 2 < -8$ or $4x - 1 > 11$: the first part gives $5x < -10$, so $x < -2$; the second gives $4x > 12$, so $x > 3$. The answer is $x < -2$ or $x > 3$.

Interval notation lists the left endpoint, then the right endpoint. Use a bracket when the endpoint is included ($\leq$ or $\geq$) and a parenthesis when it is not ($<$ or $>$). So $-2 <$

$x \leq 4$ is $(-\infty, 4]$, $x \geq 4$ is $[4, \infty)$, and $x < 3$ is $(-\infty, 3)$. Infinity always gets a parenthesis because no number reaches it. For an "or" answer, join the two intervals with the union symbol $\cup$: $x < -2$ or $x > 3$ is $(-\infty, -2) \cup (3, \infty)$.

Watch for special cases. " $x < -3$ and $x > 3$ " has **no solution**: no number is both below -3 and above 3 , so the graphs never overlap. " $x > -3$ or $x > 2$ " is just $x > -3$, because every number greater than 2 is already greater than -3 . And " $x < 5$ or $x > 1$ " is true for **all real numbers**, because every number is either less than 5 or greater than 1 (or both).

KEY FORMULAS

$$a < x < b \text{ means } x > a \text{ and } x < b$$

$$-2 \leq x < 5 \longleftrightarrow [-2, 5) \quad x \geq 4 \longleftrightarrow [4, \infty) \quad x < 3 \longleftrightarrow (-\infty, 3)$$

$$x < -2 \text{ or } x > 3 \longleftrightarrow (-\infty, -2) \cup (3, \infty)$$

Practice habit. Draw a quick number line sketch for every compound inequality, then test one number from each region it creates (left of both endpoints, between them, right of both). Only the regions that pass belong in your answer.

WORKED EXAMPLES

Example 1. Solve $-7 < 2x - 3 \leq 5$. Graph the solution and write it in interval notation.

Show solution

1. Add 3 to all three parts: $-4 < 2x \leq 8$.
2. Divide all three parts by 2: $-2 < x \leq 4$.
3. Graph: an open circle at -2 , a closed circle at 4 , and the segment between them shaded.
4. Interval notation: $(-2, 4]$.
5. Check $x = 4$: $2(4) - 3 = 5$, and $-7 < 5 \leq 5$ is true, so 4 is included.
6. Check $x = -2$: $2(-2) - 3 = -7$, and $-7 < -7$ is false, so -2 is not included.
7. Check $x = 0$: $2(0) - 3 = -3$, and $-7 < -3 \leq 5$ is true.

Answer: $-2 < x \leq 4$, or $(-2, 4]$

Example 2. Ruth's bakery sells cookies for \$8 a dozen, plus a \$5 fee for a gift box. A customer wants to spend more than \$45 but no more than \$85 on one boxed order. Write and solve a compound inequality, then list the whole numbers of dozens that fit.

Show solution

1. Let d be the number of dozens. The cost is $8d + 5$.
2. "More than \$45 but no more than \$85" means $45 < 8d + 5 \leq 85$.
3. Subtract 5 from all three parts: $40 < 8d \leq 80$.
4. Divide all three parts by 8: $5 < d \leq 10$.
5. Check $d = 10$: $8(10) + 5 = 85$, allowed (no more than \$85).
6. Check $d = 5$: $8(5) + 5 = 45$, not allowed (it must be more than \$45).
7. The whole numbers that fit are 6, 7, 8, 9 and 10 dozen.

Answer: $5 < d \leq 10$, or $(5, 10]$; the customer can buy 6, 7, 8, 9 or 10 dozen

PRACTICE

1. Easy Write $-3 < x \leq 6$ in interval notation.

Show answer

$$(-3, 6]$$

2. Easy Write the interval $[2, \infty)$ as an inequality in x .

Show answer

$$x \geq 2$$

3. Easy Solve $4 < x + 3 < 10$.

Show answer

Subtract 3 from each part: $1 < x < 7$, or $(1, 7)$

4. Easy Solve $x - 2 < -3$ or $x + 1 \geq 5$, and describe the graph.

Show answer

$x < -1$ or $x \geq 4$; open circle at -1 shaded left, closed circle at 4 shaded right

5. Medium Solve $-6 \leq 3x \leq 12$ and write the answer in interval notation.

Show answer

$$-2 \leq x \leq 4, \text{ or } [-2, 4]$$

6. Medium Solve $1 \leq 2x + 5 < 13$ and write the answer in interval notation.

Show answer

$$-4 \leq 2x < 8, \text{ so } -2 \leq x < 4, \text{ or } [-2, 4)$$

7. Medium Solve $5x + 2 < -8$ or $4x - 1 > 11$.

Show answer

$$x < -2 \text{ or } x > 3, \text{ or } (-\infty, -2) \cup (3, \infty)$$

8. Medium Solve $-4 < -2x + 6 \leq 10$.

Show answer

$$-10 < -2x \leq 4; \text{ dividing by } -2 \text{ reverses both signs: } 5 > x \geq -2, \text{ so } -2 \leq x < 5$$

9. Medium Grace's piano teacher charges \$25 per lesson. Grace's family wants to spend at least \$100 and no more than \$175 on lessons each month. Write and solve a compound inequality for the number of lessons n per month.

Show answer

$$100 \leq 25n \leq 175, \text{ so } 4 \leq n \leq 7; \text{ from 4 to 7 lessons}$$

10. **Hard** Solve $x + 4 < 1$ and $2x > 6$.

Show answer

$$x < -3 \text{ and } x > 3; \text{ no number is both, so there is no solution}$$

11. **Hard** Solve $3x - 1 \geq 5$ or $x + 7 > 4$. Simplify your answer to a single inequality.

Show answer

$$x \geq 2 \text{ or } x > -3; \text{ every number } \geq 2 \text{ is already } > -3, \text{ so the answer is } x \geq 2$$

12. **Hard** To earn a B, Henry's average on four quizzes must be at least 80 and less than 90. His first three scores are 76, 85 and 88. Each quiz is worth at most 100 points. What scores s on the fourth quiz will give him a B?

Show answer

$$320 \leq 249 + s < 360, \text{ so } 71 \leq s < 111; \text{ since } s \leq 100, \text{ any score from 71 to 100}$$

13. **Hard** A seed packet says the soil should be from 50°F to 86°F , inclusive. Fahrenheit and Celsius are related by $F = \frac{9}{5}T + 32$, where T is the Celsius temperature. Solve $50 \leq \frac{9}{5}T + 32 \leq 86$ to find the range in degrees Celsius.

Show answer

$$18 \leq \frac{9}{5}T \leq 54; \text{ multiply by } \frac{5}{9}: 10 \leq T \leq 30, \text{ from } 10^{\circ}\text{C to } 30^{\circ}\text{C}$$

14. **Challenge** Explain why " $x > 2$ and $x < -1$ " has no solution, but " $x > 2$ or $x < -1$ " has infinitely many solutions.

Show answer

"And" requires a number to be greater than 2 and less than -1 at the same time; every number greater than 2 is to the right of -1 , so the two graphs never overlap. "Or" only requires one part to be true, so every number to the right of 2 and every number to the left of -1 works.

15. **Challenge** Solve $-3 \leq \frac{4-x}{2} < 5$ and write the answer in interval notation.

Show answer

Multiply by 2: $-6 \leq 4 - x < 10$; subtract 4: $-10 \leq -x < 6$; multiply by -1 and reverse: $10 \geq x > -6$, so $-6 < x \leq 10$, or $(-6, 10]$

COMMON MISTAKES

- Doing an operation to only two of the three parts of $a < \text{expression} < b$. Whatever you do to the middle, do to both outside parts too.
- Reversing only one sign when dividing a three-part inequality by a negative. Both signs reverse; then rewrite it with the smaller number on the left.
- Using a bracket with infinity, as in $[4, \infty]$. Infinity is not a number you can reach, so it always takes a parenthesis: $[4, \infty)$.
- Mixing up "and" with "or." "And" keeps only the overlap (usually a piece between two numbers); "or" keeps everything in either graph (usually two rays). Test a number from each region to be sure.

Make it a game • And or Or Drawing. Take turns drawing a compound inequality on a number line without telling your partner the words. The guesser writes the inequality, states whether it is 'and' or 'or', and gives the interval notation. Include at least one three-part inequality, such as $-7 < 2x - 3 \leq 5$, and solve it before drawing. Score one point for each correct reading.

3.4 ABSOLUTE VALUE EQUATIONS

OBJECTIVES

- Interpret absolute value as distance from zero, and $|x - a|$ as the distance between x and a .
- Explain why an equation $|x| = c$ with $c > 0$ has two solutions, and why it has none when $c < 0$.
- Solve absolute value equations by isolating the absolute value and splitting into two cases.
- Check every solution by substituting it into the original equation.

WATCH

[Absolute Value Equations | An Introduction | Math with Mr. J](#)

[Math with Mr. J](#) • 10:54

<https://www.youtube.com/watch?v=cOPVyzw8hg8>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

[Absolute Value Equations | One Solution or No Solution | Math with Mr. J](#)

[Math with Mr. J](#) • 8:02

<https://www.youtube.com/watch?v=6ElcTLnFWGc>

Before Example 2: when an absolute value equation has one solution or no solution.

KEY TERMS

absolute value

The distance of a number from 0 on the number line, written $|x|$. Because it is a distance, it is never negative: $|5| = 5$ and $|-5| = 5$.

absolute value equation

An equation with the variable inside absolute value bars, such as $|2x - 5| = 9$.

isolate the absolute value

Use inverse operations to get the absolute value expression alone on one side, as in $|x - 1| = 5$, before splitting into cases.

case

One of the two possibilities you solve separately: the expression inside the bars equals c , or it equals $-c$.

CORE IDEAS

The **absolute value** of a number is its distance from 0. Both 5 and -5 are 5 units from 0, so $|5| = 5$ and $|-5| = 5$. A distance can be 0 but never negative, so $|x| \geq 0$ for every number x .

Why does $|x| = 5$ have **two** solutions? The equation asks, "Which numbers are 5 units from 0?" Walk 5 units right from 0 and you reach 5; walk 5 units left and you reach -5 . There are exactly two such points, one on each side of 0, so $x = 5$ or $x = -5$. In general, if $c > 0$, then $|x| = c$ means $x = c$ or $x = -c$. If $c = 0$, only $x = 0$ works. If $c < 0$, nothing works, because no distance is negative: $|x| = -4$ has **no solution**.

The distance between two numbers on the number line is the absolute value of their difference. For example, 3 and 10 are $|10 - 3| = 7$ units apart. So $|x - 4| = 7$ asks, "Which numbers are 7 units from 4?" Going right gives $4 + 7 = 11$ and going left gives $4 - 7 = -3$. Thinking of distance lets you check your answers by picture.

For an equation like $|ax + b| = c$ with $c > 0$, the expression inside the bars must equal either c or $-c$, because those are the only two numbers whose absolute value is c . So you write **two cases**, $ax + b = c$ and $ax + b = -c$, and solve each one. For $|2x - 5| = 9$: $2x - 5 = 9$ gives $x = 7$, and $2x - 5 = -9$ gives $x = -2$.

If anything is outside the bars, **isolate the absolute value first**. In $3|2x + 1| - 5 = 16$, add 5 and then divide by 3 to get $|2x + 1| = 7$ before you split into cases. Do not split into cases while the 3 and the -5 are still outside the bars, and do not move the -5 inside them: the bars act like a grouping symbol whose result is never negative, so $3|2x + 1| - 5$ is not the same as $3(2x + 1) - 5$. Only after isolating can you see whether the right side is negative, which tells you right away that there is no solution.

Check every answer in the ORIGINAL equation. This is especially important when a variable appears outside the bars too, as in $|x + 1| = 2x - 4$. There the right side

might come out negative for one of your answers, which cannot equal an absolute value, so that answer must be thrown out.

KEY FORMULAS

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

If $c > 0$: $|ax + b| = c$ means $ax + b = c$ or $ax + b = -c$

$|x - a|$ = the distance between x and a

Practice habit. Say the four steps aloud until they are automatic: isolate the bars, check the sign of the other side, split into two cases, check both answers. Writing the two cases side by side keeps your work neat.

WORKED EXAMPLES

Example 1. The thermostat in the church fellowship hall is set to 68°F. The furnace switches on or off whenever the temperature differs from 68°F by exactly 3 degrees. Write and solve an absolute value equation for those two temperatures.

Show solution

1. Let t be the temperature in °F. "Differs from 68 by exactly 3" means the distance between t and 68 is 3: $|t - 68| = 3$.
2. Case 1: $t - 68 = 3$, so $t = 71$.
3. Case 2: $t - 68 = -3$, so $t = 65$.
4. Check: $|71 - 68| = |3| = 3$ and $|65 - 68| = |-3| = 3$. Both work.
5. The furnace switches at 65°F and at 71°F.

Answer: $|t - 68| = 3$; $t = 65$ or $t = 71$ (65°F and 71°F)

Example 2. Solve $3|2x + 1| - 5 = 16$.

Show solution

1. Add 5 to both sides: $3|2x + 1| = 21$.
 2. Divide both sides by 3: $|2x + 1| = 7$. The right side is positive, so there are two cases.
 3. Case 1: $2x + 1 = 7$, so $2x = 6$ and $x = 3$.
 4. Case 2: $2x + 1 = -7$, so $2x = -8$ and $x = -4$.
 5. Check $x = 3$: $3|2(3) + 1| - 5 = 3|7| - 5 = 21 - 5 = 16$. True.
 6. Check $x = -4$: $3|2(-4) + 1| - 5 = 3|-7| - 5 = 21 - 5 = 16$. True.
-

Answer: $x = -4$ or $x = 3$

PRACTICE

1. Easy Solve $|x| = 9$.

Show answer

$$x = -9 \text{ or } x = 9$$

2. Easy Solve $|x| = -4$.

Show answer

No solution; an absolute value is never negative

3. Easy Solve $|x + 2| = 6$.

Show answer

$$x + 2 = 6 \text{ or } x + 2 = -6, \text{ so } x = -8 \text{ or } x = 4$$

4. Easy Evaluate $|3 - 10| - |-4|$.

Show answer

$$7 - 4 = 3$$

5. Medium Solve $|2x - 5| = 9$.

Show answer

$$2x - 5 = 9 \text{ gives } x = 7; 2x - 5 = -9 \text{ gives } x = -2$$

6. Medium Solve $|x| + 8 = 15$.

Show answer

$$|x| = 7, \text{ so } x = -7 \text{ or } x = 7$$

7. Medium Solve $4|x - 1| = 20$.

Show answer

$$|x - 1| = 5, \text{ so } x = -4 \text{ or } x = 6$$

8. Medium Solve $|3x + 6| = 0$.

Show answer

$$3x + 6 = 0, \text{ so } x = -2 \text{ (only one solution)}$$

9. Medium On a family road trip, the Millers plan to stop at a rest area at mile marker 132. Their car is exactly 18 miles from the rest area. Write and solve an absolute value equation for the mile markers where the car could be.

Show answer

$$|m - 132| = 18, \text{ so } m = 114 \text{ or } m = 150$$

10. **Hard** Solve $2|x + 3| + 7 = 5$.

Show answer

$$2|x + 3| = -2, \text{ so } |x + 3| = -1: \text{ no solution}$$

11. **Hard** Solve $-3|4 - x| + 10 = -2$.

Show answer

$$-3|4 - x| = -12, \text{ so } |4 - x| = 4; 4 - x = 4 \text{ gives } x = 0 \text{ and } 4 - x = -4 \text{ gives } x = 8$$

12. **Hard** A machine at a mill fills bags of flour that should weigh 16 ounces. A bag is right at the edge of the allowed limit when its weight differs from 16 ounces by exactly 0.4 ounce. Write and solve an absolute value equation for those weights.

Show answer

$$|w - 16| = 0.4, \text{ so } w = 15.6 \text{ or } w = 16.4 \text{ ounces}$$

13. **Hard** On a two-day fishing trip, Henry caught 10 fish the first day. His average number of fish per day for the two days differed from his goal of 20 by exactly 3. Solve $\left| \frac{x + 10}{2} - 20 \right| = 3$ to find how many fish x he could have caught on the second day.

Show answer

$$\frac{x + 10}{2} = 23 \text{ gives } x = 36; \frac{x + 10}{2} = 17 \text{ gives } x = 24; \text{ so 24 or 36 fish}$$

14. **Challenge** Explain, using distance on a number line, why $|x - 3| = 5$ has two solutions but $|x - 3| = -5$ has none.

Show answer

$|x - 3|$ is the distance from x to 3. There are two numbers 5 units from 3: one to the right ($3 + 5 = 8$) and one to the left ($3 - 5 = -2$). A distance can never be negative, so no number is -5 units from 3.

15. **Challenge** Solve $|x + 1| = 2x - 4$. Check each possible answer.

Show answer

Case $x + 1 = 2x - 4$ gives $x = 5$; check: $|6| = 6$ and $2(5) - 4 = 6$, true. Case $x + 1 = -(2x - 4)$ gives $3x = 3$, $x = 1$; check: $|2| = 2$ but $2(1) - 4 = -2$, false. Only $x = 5$.

COMMON MISTAKES

- Solving only one case. $|x + 2| = 6$ has two answers; always write both cases, $x + 2 = 6$ and $x + 2 = -6$.
- Splitting into cases before isolating the bars. In $3|2x + 1| - 5 = 16$, first get $|2x + 1| = 7$; writing $3(2x + 1) - 5 = -16$ for the second case gives a wrong answer.
- Making the expression inside the bars negative instead of the other side. The second case of $|2x - 5| = 9$ is $2x - 5 = -9$, not $-2x - 5 = 9$.
- Solving $|x + 3| = -1$ as if it had answers. Once the bars are isolated, if the other side is negative, stop: there is no solution.

Make it a game • Stand c Steps Away. Use a floor number line. The parent stands on a number a and calls out a distance c . You walk to every spot exactly c steps away and say the equation $|x - a| = c$. Next, call a distance of -2 and explain why no one can stand there. Finish by solving $|2x - 1| + 3 = 8$ and checking both answers.

3.5 ABSOLUTE VALUE INEQUALITIES

OBJECTIVES

- Explain why $|x - a| < c$ means "within c of a " (an "and" inequality) and $|x - a| > c$ means "more than c away from a " (an "or" inequality).
- Solve and graph absolute value inequalities, isolating the absolute value first.
- Recognize absolute value inequalities with no solution or with all real numbers as solutions.
- Write and solve absolute value inequalities for tolerances in carpentry, baking and manufacturing.

WATCH

Solving a Basic Absolute Value Inequality

Eddie Woo · 8:51

https://www.youtube.com/watch?v=_KUgda3t4kY

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he sometimes says 'modulus' for absolute value.

BACKUP VIDEO

Absolute Value Equations & Inequalities (2 of 4: Visualising the inequality)

Eddie Woo · 5:52

<https://www.youtube.com/watch?v=juljg2hdWIM>

After Example 1: picturing $|x - a| < c$ as a distance on the number line.

KEY TERMS

within c of a

At a distance of at most c (or less than c) from the number a ; written $|x - a| \leq c$ (or $|x - a| < c$).

tolerance

The amount a measurement is allowed to differ from its target value. A board cut to 36 inches with a tolerance of $\frac{1}{8}$ inch must satisfy $|L - 36| \leq \frac{1}{8}$.

target value

The ideal or intended measurement,
the center of the allowed interval.

CORE IDEAS

Remember from Lesson 3.4 that $|x - a|$ is the distance between x and a on the number line. So $|x - 3| < 5$ asks, "Which numbers are less than 5 units from 3?" Picture standing at 3 and stretching 5 units each way: you reach $3 - 5 = -2$ on the left and $3 + 5 = 8$ on the right. Every number strictly between them is closer than 5, so the solution is $-2 < x < 8$. Notice this is an **and** inequality: x must be greater than -2 AND less than 8.

Algebraically, $|u| < c$ (with $c > 0$) means the number u is less than c units from 0, which means $-c < u < c$. So to solve $|x - 3| < 5$, write $-5 < x - 3 < 5$ and add 3 to all three parts: $-2 < x < 8$. The same works for any expression inside the bars: $|ax + b| < c$ becomes $-c < ax + b < c$.

Now flip the question: $|x - 3| > 5$ asks for numbers MORE than 5 units from 3. Those numbers are either far to the left (less than -2) or far to the right (greater than 8); a number cannot be in both places at once. This is an **or** inequality: $x < -2$ or $x > 8$. Algebraically, $|u| > c$ means $u < -c$ or $u > c$. A memory aid: "less th**AND**, great**OR**."

As with equations, **isolate the absolute value first**. For $|2x + 1| - 4 \geq 5$, add 4 to get $|2x + 1| \geq 9$, then split into $2x + 1 \leq -9$ or $2x + 1 \geq 9$.

Watch for a negative number on the other side once the bars are isolated. Since a distance is never negative, $|x + 1| < -3$ asks for a distance less than a negative number, which is impossible: **no solution**. But $|x - 5| > -4$ asks for a distance greater than a negative number, and every distance is at least 0, so it is true for **all real numbers**.

Absolute value inequalities are the natural language of **tolerances**. A carpenter who needs a shelf 36 inches long, give or take $\frac{1}{8}$ inch, needs $|L - 36| \leq \frac{1}{8}$, so $35\frac{7}{8} \leq L \leq 36\frac{1}{8}$. A part that should be thrown out is one that is MORE than the tolerance away from the target, which is an "or" inequality. In general, $|x - \text{target}| \leq \text{tolerance}$.

KEY FORMULAS

If $c > 0$: $|u| < c$ means $-c < u < c$ (and)

If $c > 0$: $|u| > c$ means $u < -c$ or $u > c$ (or)

$$|x - \text{target}| \leq \text{tolerance}$$

Practice habit. Before solving, read $|x - a| < c$ aloud as "the distance from x to a is less than c " and sketch the target a with the two endpoints $a - c$ and $a + c$. The sketch tells you at once whether the answer is the piece between (and) or the two outside rays (or).

WORKED EXAMPLES

Example 1. Samuel is cutting boards for bookshelves. Each board should be 36 inches long, with a tolerance of $\frac{1}{8}$ inch. Write an absolute value inequality for the acceptable lengths L and solve it.

Show solution

1. The distance between L and 36 must be at most $\frac{1}{8}$: $|L - 36| \leq \frac{1}{8}$.
2. "Less than or equal" gives an "and" inequality: $-\frac{1}{8} \leq L - 36 \leq \frac{1}{8}$.
3. Add 36 to all three parts: $35\frac{7}{8} \leq L \leq 36\frac{1}{8}$.
4. Check an endpoint: $|36\frac{1}{8} - 36| = \frac{1}{8}$, which is allowed. Check a board of 36.5 inches: $|36.5 - 36| = 0.5 > \frac{1}{8}$, so it is rejected, and indeed 36.5 is outside the interval.
5. Acceptable boards are from $35\frac{7}{8}$ inches to $36\frac{1}{8}$ inches long (35.875 to 36.125 inches).

Answer: $|L - 36| \leq \frac{1}{8}$, so $35\frac{7}{8} \leq L \leq 36\frac{1}{8}$ inches

Example 2. Solve $|2x + 1| - 4 \geq 5$ and graph the solution.

Show solution

1. Isolate the absolute value: add 4 to both sides to get $|2x + 1| \geq 9$.
 2. "Greater than or equal" gives an "or" inequality: $2x + 1 \leq -9$ or $2x + 1 \geq 9$.
 3. First part: $2x \leq -10$, so $x \leq -5$.
 4. Second part: $2x \geq 8$, so $x \geq 4$.
 5. Graph: a closed circle at -5 shaded left, and a closed circle at 4 shaded right.
 6. Check $x = -6$: $|2(-6) + 1| - 4 = |-11| - 4 = 7 \geq 5$, true. Check $x = 0$:
 $|1| - 4 = -3 \geq 5$, false, as expected since 0 lies between -5 and 4 .
 7. Check the boundary $x = 4$: $|9| - 4 = 5 \geq 5$, true.
-

Answer: $x \leq -5$ or $x \geq 4$, or $(-\infty, -5] \cup [4, \infty)$

PRACTICE

1. Easy Solve $|x| < 4$.

Show answer

$$-4 < x < 4$$

2. Easy Solve $|x| \geq 6$.

Show answer

$$x \leq -6 \text{ or } x \geq 6$$

3. Easy Solve $|x - 2| \leq 3$ and describe the graph.

Show answer

$-3 \leq x - 2 \leq 3$, so $-1 \leq x \leq 5$; closed circles at -1 and 5 with the segment between shaded

4. Easy Write an absolute value inequality for "all numbers whose distance from 7 is at most 5," then solve it.

Show answer

$$|x - 7| \leq 5, \text{ so } 2 \leq x \leq 12$$

5. Medium Solve $|x + 4| > 2$.

Show answer

$$x + 4 < -2 \text{ or } x + 4 > 2, \text{ so } x < -6 \text{ or } x > -2$$

6. Medium Solve $|3x| - 1 < 8$.

Show answer

$$|3x| < 9, \text{ so } -9 < 3x < 9 \text{ and } -3 < x < 3$$

7. Medium Solve $|2x - 1| \geq 7$.

Show answer

$$2x - 1 \leq -7 \text{ or } 2x - 1 \geq 7, \text{ so } x \leq -3 \text{ or } x \geq 4$$

8. Medium Solve $|5 - x| < 2$.

Show answer

$$-2 < 5 - x < 2, \text{ so } -7 < -x < -3; \text{ multiply by } -1 \text{ and reverse: } 3 < x < 7$$

9. Medium Ruth's bread recipe says the dough should rise at 78°F, give or take 4 degrees. Write an absolute value inequality for the acceptable temperatures t and solve it.

Show answer

$$|t - 78| \leq 4, \text{ so } 74 \leq t \leq 82 \text{ (from } 74^\circ\text{F to } 82^\circ\text{F)}$$

10. **Hard** Solve $|x - 5| + 6 > 2$.

Show answer

$$|x - 5| > -4; \text{ an absolute value is always at least } 0, \text{ which is greater than } -4, \\ \text{so all real numbers}$$

11. **Hard** Solve $2|x - 3| + 1 \leq 9$.

Show answer

$$2|x - 3| \leq 8, \text{ so } |x - 3| \leq 4 \text{ and } -1 \leq x \leq 7$$

12. **Hard** A machine shop makes bolts that should be 0.50 inch in diameter, with a tolerance of 0.02 inch. Write an absolute value inequality for the acceptable diameters d and solve it.

Show answer

$$|d - 0.50| \leq 0.02, \text{ so } 0.48 \leq d \leq 0.52 \text{ inch}$$

13. **Hard** A sensor in Henry's grain bin sounds an alarm when the moisture content m of the corn is more than 1.5 percentage points away from 14%. Write an absolute value inequality for the moisture levels that sound the alarm and solve it.

Show answer

$$|m - 14| > 1.5, \text{ so } m < 12.5 \text{ or } m > 15.5 \text{ (below 12.5\% or above 15.5\%)}$$

14. **Challenge** Write an absolute value inequality whose solution is $-3 < x < 9$. (Hint: find the number halfway between -3 and 9 and the distance from it to each endpoint.)

Show answer

$$\text{The center is } \frac{-3 + 9}{2} = 3 \text{ and each endpoint is 6 away, so } |x - 3| < 6$$

15. **Challenge** Explain why $|x - a| < c$ (with $c > 0$) is solved with an "and" inequality, but $|x - a| > c$ is solved with an "or" inequality.

Show answer

$|x - a|$ is the distance from x to a . Numbers less than c away from a form one piece between $a - c$ and $a + c$, so x must be greater than $a - c$ AND less than $a + c$. Numbers more than c away lie in two separate pieces, either left of $a - c$ OR right of $a + c$; no number can be in both pieces, so the word is "or."

COMMON MISTAKES

- Using "and" for a "greater than" inequality. $|x| > 6$ is NOT $-6 > x > 6$ (no number is both less than -6 and greater than 6). Write $x < -6$ or $x > 6$.
- Splitting before isolating. In $2|x - 3| + 1 \leq 9$, first get $|x - 3| \leq 4$, then split.
- Forgetting to reverse the signs when dividing by a negative inside a three-part inequality, as in $-7 < -x < -3$. Reverse both signs: $7 > x > 3$, then write $3 < x < 7$.
- Answering "no solution" for $|x - 5| > -4$. Being greater than a negative number is always true for an absolute value; only "less than a negative number" has no solution.

Make it a game • Within Tolerance. Pretend you run a factory that cuts paper strips exactly 15 cm long, with a tolerance of 0.5 cm. Cut ten strips by eye. The parent measures each one and calls it accepted or rejected. Write an absolute value inequality for the accepted strips and another for the rejected ones. Then change the tolerance and see how many strips still pass.

UNIT 3

LINEAR INEQUALITIES AND ABSOLUTE VALUE

In Unit 2 you solved equations, which usually have one answer. Many real questions are not about one exact number but about a range: How many hours must I work to earn at least enough? How many guests fit our budget? How far can a board's length be from 36 inches and still be acceptable? This unit teaches you to write and solve inequalities, which answer "how much is enough" and "how much is too much" questions, and to show their solutions on a number line and in interval notation.

You will start with one-step and multi-step inequalities and learn why multiplying or dividing by a negative number reverses the inequality sign. Then you will join inequalities with "and" and "or," and finally study absolute value as distance, which leads to equations with two solutions and to inequalities that describe tolerances in carpentry, baking and manufacturing.

These skills are used all through the rest of the course: domain and range in Unit 4, graphing inequalities in two variables and systems of inequalities in Unit 6, and later work in Geometry and Algebra II. Parents: the key habits to watch for are checking answers with a test number and reversing the sign only when multiplying or dividing by a negative.

PREREQUISITES

- Solve one-step, two-step and multi-step linear equations, including equations with variables on both sides (Lessons 2.1–2.3).
- Clear fractions from an equation by multiplying by the least common denominator (Lesson 2.4).
- Distribute, including distributing a negative, and combine like terms (Lesson 1.4).
- Compare and order integers and fractions on a number line (Lesson 1.1).
- Translate words into algebraic expressions and equations (Lessons 1.5 and 2.6).

MASTERY CHECKLIST

- ☐ I can graph an inequality on a number line using open and closed circles.
- ☐ I can solve one- and two-step inequalities and explain why multiplying or dividing by a negative number reverses the sign.
- ☐ I can solve multi-step inequalities with distributing and variables on both sides.
- ☐ I can recognize when an inequality is true for all real numbers or has no solution, and explain why.
- ☐ I can translate "at least," "at most" and "no more than" into inequality symbols and give sensible answers to word problems.
- ☐ I can solve and graph "and" and "or" compound inequalities, including three-part inequalities.
- ☐ I can write solution sets in interval notation, such as
 $[-2, 5]$
 $(-\infty, 3) \cup$
 $[6, \infty)$
- ☐ I can explain absolute value as distance and solve absolute value equations by isolating the bars and using two cases.
- ☐ I can recognize absolute value equations and inequalities with no solution.
- ☐ I can solve absolute value inequalities, knowing that "less

than" gives an "and" inequality and "greater than" gives an "or" inequality.

- ☐ I can write and solve absolute value inequalities for tolerances.

LESSONS IN THIS UNIT

3.1 Solving Inequalities	45 minutes
3.2 Multi-Step Inequalities	2 sessions of 45 minutes
3.3 Compound Inequalities	2 sessions of 45 minutes
3.4 Absolute Value Equations	2 sessions of 45 minutes
3.5 Absolute Value Inequalities	2 sessions of 45 minutes

UNIT 3 TEST

You have 50 minutes. No calculator. Show all your work, one step per line. Give exact answers; write inequality answers in inequality form (interval notation is also accepted unless a problem asks for one form). Check your solutions with a test number when you can. For word problems, define your variable and answer in a sentence with units. The test is worth 100 points.

1. 5 pts

Solve $4x - 7 > 13$ and describe the graph.

2. 6 pts

Solve $9 - 3x \leq 24$.

3. 7 pts

Solve $2(3x - 4) \geq 5x - 1$.

4. 6 pts

Solve $3(x + 2) - x < 2(x + 4)$.

5. (7 pts)

Grace is saving for a \$240 telescope. She already has \$72 and saves \$14 each week. Write and solve an inequality to find the least number of weeks she must save to have at least enough.

6. (7 pts)

Solve $-5 \leq 3x + 4 < 19$. Write the answer in interval notation.

7. (7 pts)

Solve $2x + 3 < -5$ or $5 - x \leq 1$.

8. (6 pts)

Solve $|x - 6| = 10$.

9. (7 pts)

Solve $5|x + 2| - 3 = 22$.

10. (5 pts)

Solve $|4x - 1| + 9 = 2$.

11. (7 pts)

Solve $|x + 5| \leq 3$.

12. (8 pts)

Solve $|3x - 6| > 12$.

13. (8 pts)

A church choir is renting a coach bus for a concert trip. The bus costs \$180 plus \$6 per rider. The bus company requires a total bill of at least \$300, and the choir can spend at most \$420. Write and solve a compound inequality for the number of riders r .

14. (8 pts)

A bakery's loaves of bread should weigh 24 ounces, with a tolerance of 0.75 ounce. Write an absolute value inequality for the acceptable weights w and solve it.

15. 6 pts

Henry solved $-4x > 12$ and wrote $x > -3$. Explain his mistake and give the correct answer, using a test number to show which answer is right.

ANSWER KEY

1. $4x > 20$, so $x > 5$; open circle at 5, shaded to the right
2. $-3x \leq 15$, so $x \geq -5$
3. $6x - 8 \geq 5x - 1$, so $x \geq 7$
4. $2x + 6 < 2x + 8$ gives $6 < 8$, always true: all real numbers
5. $72 + 14w \geq 240$, so $14w \geq 168$ and $w \geq 12$; at least 12 weeks
6. $-9 \leq 3x < 15$, so $-3 \leq x < 5$, or $[-3, 5)$
7. $2x < -8$ gives $x < -4$; $-x \leq -4$ gives $x \geq 4$; so $x < -4$ or $x \geq 4$
8. $x - 6 = 10$ or $x - 6 = -10$, so $x = 16$ or $x = -4$
9. $5|x + 2| = 25$, so $|x + 2| = 5$; $x = -7$ or $x = 3$
10. $|4x - 1| = -7$; an absolute value cannot be negative, so there is no solution
11. $-3 \leq x + 5 \leq 3$, so $-8 \leq x \leq -2$
12. $3x - 6 < -12$ or $3x - 6 > 12$, so $x < -2$ or $x > 6$
13. $300 \leq 180 + 6r \leq 420$, so $120 \leq 6r \leq 240$ and $20 \leq r \leq 40$; from 20 to 40 riders
14. $|w - 24| \leq 0.75$, so $23.25 \leq w \leq 24.75$ ounces
15. Dividing by the negative number -4 reverses the order of numbers, so the sign must flip: $x < -3$. Test $x = -5$ (which satisfies $x < -3$): $-4(-5) = 20 > 12$, true. Test $x = 0$ (which satisfies Henry's answer): $-4(0) = 0 > 12$, false. So the correct answer is $x < -3$.

UNIT 4

RELATIONS AND FUNCTIONS

In this unit you move from solving single equations to studying how one quantity depends on another. You begin with the coordinate plane and relations (sets of ordered pairs), then learn the central idea of the course: a function, a rule that gives exactly one output for each input. You will test tables, mappings and graphs to see whether they are functions, and learn why the vertical line test works.

Next you learn function notation, $f(x)$, which names the output of a function and lets you evaluate and solve with it. You find the domain and range of functions from lists, graphs, equations and real situations, and you see that a sequence is a function whose inputs are the counting numbers, with arithmetic sequences as the main example.

These ideas are the language of the rest of algebra. Unit 5 studies linear functions, Unit 7 exponential functions and geometric sequences, and Unit 11 quadratic functions, all using the notation, domain and range and graph-reading skills from this unit. Parents: the main habit to watch for is careful substitution with parentheses and checking answers in the original rule.

PREREQUISITES

- Plot points on the coordinate plane (Pre-Algebra).
- Evaluate expressions by substitution, using parentheses for negatives (Lesson 1.3).
- Solve multi-step linear equations (Lessons 2.2 and 2.3).
- Solve linear inequalities (Lessons 3.1 and 3.2).
- Read and write compound inequalities and interval notation (Lesson 3.3).

MASTERY CHECKLIST

- ☐ I can plot ordered pairs and name the quadrant or axis of a point.
- ☐ I can show a relation as a list of pairs, a table, a mapping diagram and a graph.
- ☐ I can name the independent and dependent variables in a real situation.
- ☐ I can decide whether a relation is a function and explain my reasoning.
- ☐ I can use the vertical line test and explain why it works.
- ☐ I can evaluate $f(x)$ for numbers, negatives and simple expressions, and I know $f(x)$ does not mean f times x .
- ☐ I can solve $f(x)$ for the input k .
- ☐ I can find the domain and range from pairs, graphs, equations and real situations.
- ☐ I can write domains and ranges with inequalities or interval notation.
- ☐ I can describe where a graph is increasing, decreasing or constant.
- ☐ I can find the common difference, explicit formula and recursive rule

of an arithmetic sequence and use them to find any term.

LESSONS IN THIS UNIT

4.1	The Coordinate Plane and Relations	45 minutes
4.2	What Is a Function?	2 sessions of 45 minutes
4.3	Function Notation	2 sessions of 45 minutes
4.4	Domain and Range	2 sessions of 45 minutes
4.5	Sequences as Functions	2 sessions of 45 minutes

UNIT 4 TEST

You have 50 minutes. No calculator. Show your work for every problem. Give exact answers, and include units in word-problem answers.

1. **5 pts**
In which quadrant is the point $(-6, -2)$?
2. **5 pts**
Start at $(-1, 3)$. Move 4 units right and 7 units down. What point do you reach?
3. **6 pts**
Is the relation $\{(1, 2), (3, 4), (1, 5)\}$ a function?
4. **6 pts**
Is the graph of the horizontal line $y = 4$ a function?
5. **6 pts**
Let $f(x) = 4x - 9$. Find $f(-3)$.
6. **7 pts**
Let $g(x) = x^2 - 5x + 1$. Find $g(-2)$.

7. (7 pts)

Let $f(x) = -3x + 8$. Solve $f(x) = 23$.

8. (7 pts)

Find the domain and range of $\{(4, 0), (-1, 2), (6, 2), (0, -3)\}$.

9. (7 pts)

Find the domain of $f(x) = \frac{7}{2x - 6}$.

10. (7 pts)

Find the domain of $f(x) = \sqrt{x + 9}$.

11. (8 pts)

A youth group rents a lodge for a retreat. The cost is \$200 plus \$35 per person, so $C(p) = 35p + 200$. If the total cost is \$900, how many people went on the retreat?

12. (8 pts)

Find the 50th term of the arithmetic sequence 11, 8, 5, 2, ...

13. (8 pts)

Henry's tomato plant is 4 inches tall at the end of week 1 and grows 3 inches each week. (a) Write an explicit formula for its height a_n in inches at the end of week n . (b) In which week will it be 31 inches tall?

14. (8 pts)

A rain barrel holds 60 gallons and drains at 4 gallons per minute, so the water left after t minutes is $V(t) = 60 - 4t$. Find the reasonable domain and range.

15. (5 pts)

A circle has center $(0, 0)$ and radius 2. Use the vertical line test to explain why it is not the graph of a function.

ANSWER KEY

1. Quadrant III, since both coordinates are negative.
2. $(-1 + 4, 3 - 7) = (3, -4)$
3. No. The input 1 has two outputs, 2 and 5.
4. Yes. Every vertical line crosses it exactly once.
5. $f(-3) = 4(-3) - 9 = -21$
6. $g(-2) = (-2)^2 - 5(-2) + 1 = 4 + 10 + 1 = 15$
7. $-3x + 8 = 23$, $-3x = 15$, so $x = -5$.
8. Domain $\{-1, 0, 4, 6\}$; range $\{-3, 0, 2\}$
9. $2x - 6 = 0$ when $x = 3$, so all real numbers except 3.
10. $x + 9 \geq 0$, so $x \geq -9$, or $[-9, \infty)$.
11. $35p + 200 = 900$, $35p = 700$, so $p = 20$ people.
12. $d = -3$, so $a_{50} = 11 + 49(-3) = 11 - 147 = -136$.
13. (a) $a_n = 4 + (n - 1)(3) = 3n + 1$ (b) $3n + 1 = 31$, $n = 10$: week 10.
14. Empty when $60 - 4t = 0$, $t = 15$. Domain $0 \leq t \leq 15$ minutes; range $0 \leq V \leq 60$ gallons.
15. The vertical line $x = 0$ crosses the circle at $(0, 2)$ and $(0, -2)$. Those points have the same input, 0, but different outputs, so the input 0 has two outputs and the graph is not a function.

4.1 THE COORDINATE PLANE AND RELATIONS

OBJECTIVES

- Plot ordered pairs and name the quadrant or axis where a point lies.
- Show a relation as a set of ordered pairs, a table, a mapping diagram and a graph.
- Identify the independent and dependent variables in a real situation.

WATCH

Algebra Basics: Graphing On The Coordinate Plane - Math Antics

[mathantics](#) · 10:14

<https://www.youtube.com/watch?v=9Uc62CuQjc4>

WHEN TO WATCH

Before Example 1: plotting ordered pairs on the coordinate plane.

Note: Eddie Woo teaches in Australia: he says 'number plane' for the coordinate plane.

BACKUP VIDEO

Functions & Relations (1 of 2: Introductory Concepts)

[Eddie Woo](#) · 10:11

https://www.youtube.com/watch?v=k5tnlUqF_hM

Before Example 2: what a relation is, with ordered pairs and mapping diagrams.

KEY TERMS

ordered pair

A pair of numbers (x, y) written in a set order: first the horizontal position x , then the vertical position y .

quadrant

One of the four regions the axes cut the plane into, numbered I, II, III, IV counterclockwise starting at the upper right.

coordinate plane

A flat grid made by a horizontal number line (the x-axis) and a vertical number line (the y-axis) that cross at the origin $(0, 0)$.

relation

Any set of ordered pairs. It can be shown as a list, a table, a mapping diagram or a graph.

independent variable

The input quantity that you choose or that changes on its own, usually x .

dependent variable

The output quantity whose value depends on the input, usually y .

CORE IDEAS

You already know the coordinate plane from Pre-Algebra. To plot (x, y) , start at the origin, move x units left or right along the x -axis, then y units up or down. A good memory aid is "walk before you climb": the horizontal move comes first because x is written first.

Why order matters. The pair $(3, 5)$ means 3 right and 5 up, while $(5, 3)$ means 5 right and 3 up. These are different places on the grid, so we call them *ordered* pairs. Swapping the numbers moves the point unless the two numbers are equal.

Why the quadrants have sign patterns. Points to the right of the y -axis have positive x ; points to the left have negative x . Points above the x -axis have positive y ; points below have negative y . Putting these together gives Quadrant I $(+, +)$, II $(-, +)$, III $(-, -)$ and IV $(+, -)$. A point with a 0 coordinate, such as $(0, -4)$ or $(6, 0)$, lies on an axis and is in no quadrant.

A **relation** is simply a collection of ordered pairs. The same relation can be written four ways. List: $\{(1, 4), (2, 7), (3, 10)\}$. Table: x : 1, 2, 3 and y : 4, 7, 10. Mapping diagram: an oval of inputs 1, 2, 3 with arrows to an oval of outputs 4, 7, 10. Graph: the three points plotted on the plane. Each form shows exactly the same information, so you can move from one to another.

In real situations one quantity usually depends on the other. If Henry earns \$9 for each hour he mows, the pay depends on the hours, so hours is the **independent variable** (the input, x) and pay is the **dependent variable** (the output, y). Ask yourself, "Which one causes or decides the other?" That one is independent and goes on the horizontal axis.

KEY FORMULAS

I : $(+, +)$ II : $(-, +)$ III : $(-, -)$ IV : $(+, -)$

From history. In 1637 the French thinker René Descartes published *La Géométrie*, which joined algebra and geometry by naming points with numbers measured from fixed lines. Pierre de Fermat worked out similar ideas at about the same time. In Descartes' honor, the coordinate plane is often called the Cartesian plane.

Practice habit. Walk before you climb: when plotting, always make the horizontal move for x first, then the vertical move for y . Say the moves aloud ("left 3, up 4") until it is automatic.

WORKED EXAMPLES

Example 1. Plot $A(-3, 4)$ and $B(5, -2)$. Name the quadrant of each point.

Show solution

1. For A : start at the origin, walk 3 units left (x is -3), then climb 4 units up (y is 4).
2. The signs are $(-, +)$, so A is in Quadrant II.
3. For B : walk 5 units right, then 2 units down.
4. The signs are $(+, -)$, so B is in Quadrant IV.

Answer: A is in Quadrant II and B is in Quadrant IV.

Example 2. Samuel has \$20 in savings and adds \$15 at the end of each week. After how many weeks will he have \$140? Begin by making a table for weeks 0 through 4, writing the relation as ordered pairs and naming the independent and dependent variables.

Show solution

1. Each week adds \$15: week 0: \$20, week 1: \$35, week 2: \$50, week 3: \$65, week 4: \$80.
2. Ordered pairs (week, savings): $(0, 20), (1, 35), (2, 50), (3, 65), (4, 80)$.
3. The number of weeks decides the savings, so weeks is the independent variable and savings is the dependent variable.
4. Let w be the number of weeks. Savings are $20 + 15w$, so solve $20 + 15w = 140$.
5. $15w = 120$, so $w = 8$.
6. Check: $20 + 15(8) = 20 + 120 = 140$. ✓

Answer: Table and pairs as shown; weeks is independent, savings is dependent; he will have \$140 after 8 weeks.

PRACTICE

1. Easy In which quadrant is the point $(-3, 5)$?

Show answer

Quadrant II, since x is negative and y is positive.

2. Easy A point is 3 units to the left of the origin and 6 units below it. Write its coordinates.

Show answer

$(-3, -6)$

3. Easy Start at $(2, -1)$. Move 5 units up and 4 units left. What point do you reach?

Show answer

$(2 - 4, -1 + 5) = (-2, 4)$

4. **Easy** What is the name of the point where the x-axis and y-axis cross, and what are its coordinates?

Show answer

The origin, $(0, 0)$.

5. **Medium** Ruth is baking bread. Each loaf uses 3 cups of flour. Name the independent variable and the dependent variable.

Show answer

Independent: the number of loaves. Dependent: the cups of flour used (it depends on how many loaves she bakes).

6. **Medium** A relation is given by the table x : 1, 2, 3, 4 and y : 3, 5, 7, 9. Write an equation for y in terms of x .

Show answer

Each y is twice x plus 1: $y = 2x + 1$.

7. **Medium** A mapping diagram shows the arrows $-2 \rightarrow 5$, $0 \rightarrow 1$, $3 \rightarrow 5$ and $3 \rightarrow 8$. Which input is paired with two different outputs?

Show answer

The input 3 (it goes to both 5 and 8).

8. **Medium** Henry earns \$9 per hour mowing lawns. The relation pairs hours worked with pay: $(1, 9)$, $(2, 18)$, $(3, 27)$, $(4, 36)$. How much does he earn for 5 hours?

Show answer

$9 \times 5 = 45$, so \$45.

9. **Medium** How many units apart are the points $(-3, 2)$ and $(5, 2)$?

Show answer

They are on the same horizontal line, so the distance is $5 - (-3) = 8$ units.

10. **Hard** The points $(-2, 3)$, $(4, 3)$, $(4, -1)$ and $(-2, -1)$ are the corners of a rectangle. Find its area.

Show answer

Width $4 - (-2) = 6$, height $3 - (-1) = 4$, area $6 \times 4 = 24$ square units.

11. **Hard** Three corners of a rectangle are $(1, 2)$, $(1, -3)$ and $(6, -3)$. Find the fourth corner.

Show answer

$(6, 2)$: it shares its x with $(6, -3)$ and its y with $(1, 2)$.

12. **Hard** Explain why $(3, 5)$ and $(5, 3)$ are different points.

Show answer

The first number tells the horizontal move and the second the vertical move. $(3, 5)$ is 3 right and 5 up; $(5, 3)$ is 5 right and 3 up, so they land in different places.

13. **Hard** At the church bake sale, muffins sell for \$2 each. The relation (muffins sold, dollars collected) has the table x : 10, 20, 30 and y : 20, 40, 60. How many muffins were sold if \$74 was collected?

Show answer

$2m = 74$, so $m = 37$ muffins.

14. **Challenge** The point (a, b) is in Quadrant II. In which quadrant is the point (b, a) ?

Show answer

In Quadrant II, $a < 0$ and $b > 0$. So (b, a) has a positive first coordinate and a negative second coordinate: Quadrant IV.

COMMON MISTAKES

- Using the first number for the up-or-down move. The first number (x) always tells the left-or-right move and the second number (y) the up-or-down move; walking first, then climbing, keeps them straight.
- Saying a point like $(0, 5)$ is in Quadrant I. A point with a zero coordinate lies on an axis and is in no quadrant.
- Numbering the quadrants clockwise. They go counterclockwise: I upper right, II upper left, III lower left, IV lower right.
- Mixing up the variables in a word problem. Ask which quantity decides the other; that one is independent and goes on the x -axis.

Make it a game • Plot Battleship. Each player draws a four-quadrant grid from -6 to 6 and hides a three-point ship in a straight line. Take turns calling ordered pairs. Before the other player answers hit or miss, they must name the quadrant or axis of your shot. Wrong quadrant means the shot is lost. First to sink a ship wins.

4.2 WHAT IS A FUNCTION?

OBJECTIVES

- State the definition of a function: each input has exactly one output.
- Decide whether a list of pairs, a table or a mapping diagram is a function.
- Use the vertical line test on a graph and explain why it works.

WATCH

Algebra Basics: What Are Functions? - Math Antics

[mathantics](#) · 11:34

<https://www.youtube.com/watch?v=52tpYl2tTqk>

WHEN TO WATCH

Before Example 1: the definition of a function as an input-output rule.

BACKUP VIDEO

Functions & Relations (2 of 2: Vertical Line Test)

[Eddie Woo](#) · 6:57

<https://www.youtube.com/watch?v=dmOVijQv5OE>

Before Example 2: the vertical line test.

KEY TERMS

input

A value of the independent variable, usually the x -value of an ordered pair.

output

A value of the dependent variable, usually the y -value of an ordered pair.

function

A relation in which each input is paired with exactly one output.

vertical line test

A graph is a function exactly when no vertical line crosses the graph at more than one point.

CORE IDEAS

A **function** is a relation with a promise: give it an input, and it gives back exactly one output. Think of a vending machine. If pressing B4 sometimes gave you crackers and sometimes gave you juice, the machine would be broken. A function never "breaks" that way: the same input always produces the same single output.

Testing a list or a table. Look for any input (x -value) that appears more than once. If a repeated input always has the same output, the relation can still be a function (the pair is just listed twice). If a repeated input has two different outputs, it is **not** a function. For example, $\{(2, 3), (2, -3), (5, 1)\}$ is not a function because the input 2 goes to both 3 and -3 .

Repeated outputs are fine. In $\{(-1, 4), (1, 4), (3, 4)\}$ three different inputs share the output 4. That is allowed: many people can share a birthday, but no person has two birthdays. In a mapping diagram, a function is one where each input has exactly one arrow leaving it; several arrows may land on the same output.

The vertical line test, and why it works. Every point on the vertical line $x = a$ has the same first coordinate, a . So if a vertical line crosses a graph at two points, say (a, b) and (a, c) with $b \neq c$, then the input a has two outputs, and the graph is not a function. If every vertical line crosses the graph at most once, then no input has two outputs, and the graph is a function. The test is just the definition of a function, seen in a picture.

Examples: a straight slanted line passes the test (each vertical line hits it once). A U-shaped curve through $(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)$ also passes. A circle fails, because a vertical line through its middle hits it twice, once on top and once on the bottom. A vertical line such as $x = 3$ fails badly: the input 3 has infinitely many outputs.

Practice habit. Before deciding, scan the inputs first: underline or circle any x -value that appears twice, then compare its outputs. Only repeated inputs can break a function.

WORKED EXAMPLES

Example 1. Decide whether each relation is a function. (a)

$\{(-2, 1), (0, 3), (1, 1), (4, 5)\}$ (b) the table x : 2, 3, 3, 6 and y : 1, 4, -4 , 0 (c) the graph of $x = y^2$, which passes through $(0, 0)$, $(1, 1)$, $(1, -1)$, $(4, 2)$ and $(4, -2)$.

Show solution

1. (a) The inputs are $-2, 0, 1, 4$. No input repeats, so each has one output. The output 1 repeats, which is allowed. Function.
2. (b) The input 3 appears twice, with outputs 4 and -4 . One input, two outputs: not a function.
3. (c) Draw the vertical line $x = 1$. It crosses the graph at $(1, 1)$ and $(1, -1)$.
4. Two points on one vertical line means the input 1 has two outputs: not a function.

Answer: (a) yes (b) no (c) no

Example 2. Ruth's family keeps chickens. She records the eggs collected each week: week 1: 30 eggs, week 2: 34 eggs, week 3: 30 eggs, week 4: 36 eggs. (a) Is the number of eggs a function of the week? (b) Is the week a function of the number of eggs?

Show solution

1. (a) Inputs are weeks 1, 2, 3, 4. Each week appears once with one egg count. Function.
2. (b) Now the inputs are egg counts: 30, 34, 30, 36. The input 30 appears twice.
3. 30 eggs is paired with week 1 and with week 3, two different outputs.
4. So the week is not a function of the number of eggs.

Answer: (a) yes (b) no

PRACTICE

1. Easy Is $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$ a function?

Show answer

Yes. No input repeats.

2. Easy Is $\{(2, 3), (2, -3), (5, 1)\}$ a function?

Show answer

No. The input 2 has two outputs, 3 and -3 .

3. Easy Is the relation given by the table $x: -1, 0, 1, 2$ and $y: 3, 3, 3, 3$ a function?

Show answer

Yes. Each input has exactly one output; sharing the output 3 is allowed.

4. Easy Each child in the Miller family is paired with the month of his or her birthday: Anna $\rightarrow$ March, Ben $\rightarrow$ July, Clara $\rightarrow$ March. Is this relation a function?

Show answer

Yes. Each child has exactly one birthday month; two children sharing March is allowed.

5. Medium Is the relation given by the table $x: 4, 1, 0, 1$ and $y: 2, 1, 0, -1$ a function?

Show answer

No. The input 1 is paired with both 1 and -1 .

6. Medium Is the graph of the vertical line $x = 3$ a function?

Show answer

No. The vertical line $x = 3$ itself crosses the graph at infinitely many points, so the input 3 has infinitely many outputs.

7. Medium The graph is a circle with center $(0, 0)$ and radius 5. Is it a function?

Show answer

No. The vertical line $x = 0$ crosses it at $(0, 5)$ and $(0, -5)$.

8. Medium The graph is a U-shaped curve through $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$ that keeps rising on both sides. Is it a function?

Show answer

Yes. Every vertical line crosses the curve exactly once.

9. **Medium** At a farm stand, apples cost \$1.50 per pound. The relation pairs each weight of apples (in pounds) with its price. Is the price a function of the weight?

Show answer

Yes. Each weight has exactly one price (weight times \$1.50).

10. **Hard** Find every value of k that makes $\{(3, 1), (k, 4), (5, 2)\}$ **not** a function.

Show answer

If $k = 3$, the input 3 goes to 1 and 4; if $k = 5$, the input 5 goes to 4 and 2. So $k = 3$ or $k = 5$.

11. **Hard** Explain why the vertical line test tells whether a graph is a function.

Show answer

All points on a vertical line have the same x -value. If the line hits the graph twice, that one input has two different outputs, so the graph is not a function. If no vertical line hits it twice, every input has only one output.

12. **Hard** Grace records the temperature: 8 a.m. 50°F, 10 a.m. 58°F, noon 64°F, 2 p.m. 64°F, 4 p.m. 58°F. Is the relation (temperature $\rightarrow$ time) a function?

Show answer

No. The input 58°F is paired with both 10 a.m. and 4 p.m. (Also 64°F goes to noon and 2 p.m.)

13. **Challenge** Is y a function of x in the equation $y^2 = x$? (Hint: try $x = 4$.)

Show answer

No. When $x = 4$, both $y = 2$ and $y = -2$ work, since $2^2 = 4$ and $(-2)^2 = 4$.

14. **Challenge** Henry drives from home to town. With time as the input and his distance from home as the output, explain why the graph of any real trip must pass the vertical line test.

Show answer

At any single moment, Henry is at only one distance from home. So each time (input) has exactly one distance (output), and no vertical line can hit the graph twice.

COMMON MISTAKES

- Thinking a repeated output breaks a function. Only a repeated **input** with different outputs breaks it; $\{(1, 4), (2, 4)\}$ is a function.
- Using a horizontal line instead of a vertical line for the test. Inputs are x -values, and points with the same x lie on a vertical line.
- Calling a relation "not a function" because the same pair is listed twice, such as $(2, 5)$ and $(2, 5)$. The input 2 still has only one output.
- Switching inputs and outputs in a word problem. Read which quantity is the input before testing.

Make it a game • The Cheating Machine. The parent becomes a function machine behind a book, secretly following a rule such as 'double and add one'. Feed in inputs and write the input-output pairs. At some point the machine will cheat and give two different outputs for one input. Shout when you catch it, and explain why the pairs are not a function.

4.3 FUNCTION NOTATION

OBJECTIVES

- Read and write function notation such as $f(x)$ and explain what it means.
- Evaluate a function for numbers, negative numbers and expressions such as $f(a + 1)$.
- Solve equations of the form $f(x) = k$.
- Write and use a function rule for a word situation.

WATCH

Function Notation

Eddie Woo · 9:38

<https://www.youtube.com/watch?v=2xEPJ97dJ-U>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

5.3 Functions - Evaluating Expressions with Function Notation

Eddie Woo · 11:21

<https://www.youtube.com/watch?v=XU4BAXFhNIU>

Before the practice set: more evaluating with $f(x)$ notation, including substituting expressions.

KEY TERMS

function notation

A way of naming a function and its output: $f(x)$, read "f of x," means the output of the function f when the input is x .

function rule

An equation, such as $f(x) = 3x - 5$, that tells how to get the output from the input.

evaluate a function

Find the output for a given input by substituting the input into the rule.

CORE IDEAS

In Lesson 4.2 you learned that a function gives exactly one output for each input.

Function notation names that output. If the function is called f , then $f(x)$, read "f of x," is the output when the input is x . So $f(x) = 3x - 5$ says: to find the output, multiply the input by 3 and subtract 5. It means the same as $y = 3x - 5$, but it also shows the input.

$f(x)$ is **not** f times x . The parentheses here do not mean multiplication; they hold the input. f is the name of the rule, not a number. Writing $f(3)$ means "the output when the input is 3." With $f(x) = 3x - 5$, we get $f(3) = 3(3) - 5 = 4$; it is an output, not " f times 3." The statement $f(3) = 4$ also tells you that the point $(3, 4)$ is on the graph of f .

Why we use parentheses when substituting. To evaluate, replace every x with the input, wrapped in parentheses. With $g(x) = x^2 + 1$, $g(-3) = (-3)^2 + 1 = 9 + 1 = 10$. Without parentheses you might write -3^2 , which means $-(3^2) = -9$, a different number. The input can even be an expression: if $f(x) = 2x - 3$, then $f(a + 1) = 2(a + 1) - 3 = 2a + 2 - 3 = 2a - 1$.

Two different questions. "Find $f(4)$ " gives you the input and asks for the output: substitute. "Solve $f(x) = 19$ " gives you the output and asks for the input: set the rule equal to 19 and solve the equation, just as in Unit 2. For $f(x) = 2x + 7$, $2x + 7 = 19$ gives $2x = 12$ and $x = 6$, so $f(6) = 19$.

Functions can have other names and letters that remind you of the quantities: $C(n)$ for the cost of n items, $h(t)$ for height at time t , $S(w)$ for savings after w weeks. For example, if a church van rents for \$30 plus \$12 per hour, the cost of renting it for n hours is $C(n) = 12n + 30$, and $C(5) = 12(5) + 30 = 90$ means 5 hours cost \$90.

KEY FORMULAS

$$f(\text{input}) = \text{output}$$

$$f(a) = b \quad \text{means the point } (a, b) \text{ is on the graph of } f$$

Practice habit. Write one step per line and always put the input in parentheses before simplifying: $f(-2) = 3(-2) - 5$. This one habit prevents most sign errors.

WORKED EXAMPLES

Example 1. Let $f(x) = x^2 - 3x + 2$. Find $f(-2)$ and $f(4)$.

Show solution

1. Replace each x with (-2) : $f(-2) = (-2)^2 - 3(-2) + 2$.
2. Simplify: $4 + 6 + 2 = 12$.
3. Replace each x with (4) : $f(4) = (4)^2 - 3(4) + 2$.
4. Simplify: $16 - 12 + 2 = 6$.

Answer: $f(-2) = 12$ and $f(4) = 6$

Example 2. For a church bake sale, the youth group spends \$45 on ingredients and sells muffins for \$3 each. Their profit from selling n muffins is $P(n) = 3n - 45$. (a) Find $P(40)$, the profit in dollars from selling 40 muffins. (b) How many muffins must they sell to make a profit of \$120?

Show solution

1. (a) $P(40) = 3(40) - 45 = 120 - 45 = 75$.
2. Meaning: selling 40 muffins gives a profit of \$75.
3. (b) Now the output is given. Solve $P(n) = 120$: $3n - 45 = 120$.
4. Add 45 to both sides: $3n = 165$.
5. Divide by 3: $n = 55$.
6. Check: $P(55) = 3(55) - 45 = 165 - 45 = 120$. ✓

Answer: (a) $P(40) = 75$: 40 muffins give a \$75 profit. (b) 55 muffins

PRACTICE

1. Easy Let $f(x) = 3x - 5$. Find $f(4)$.

Show answer

$$f(4) = 3(4) - 5 = 7$$

2. Easy Let $f(x) = 3x - 5$. Find $f(-2)$.

Show answer

$$f(-2) = 3(-2) - 5 = -11$$

3. Easy Let $g(x) = x^2 + 1$. Find $g(-3)$.

Show answer

$$g(-3) = (-3)^2 + 1 = 10$$

4. Easy Let $h(t) = 20 - 4t$. Find $h(5)$.

Show answer

$$h(5) = 20 - 4(5) = 0$$

5. Medium Let $f(x) = 2x + 7$. Solve $f(x) = 19$.

Show answer

$$2x + 7 = 19, 2x = 12, \text{ so } x = 6.$$

6. Medium Let $f(x) = 2x - 3$. Find and simplify $f(a + 1)$.

Show answer

$$2(a + 1) - 3 = 2a + 2 - 3 = 2a - 1$$

7. Medium Let $g(x) = x^2 - 4x$. Find $g(-1)$.

Show answer

$$g(-1) = (-1)^2 - 4(-1) = 1 + 4 = 5$$

8. Medium A church van rents for \$30 plus \$12 per hour, so the cost for n hours is $C(n) = 12n + 30$. Find $C(5)$, the cost in dollars of renting the van for 5 hours.

Show answer

$$C(5) = 12(5) + 30 = 90, \text{ so } \$90.$$

9. Medium Let $f(x) = -2x + 9$. Solve $f(x) = -3$.

Show answer

$$-2x + 9 = -3, -2x = -12, \text{ so } x = 6.$$

10. Hard Let $f(x) = 5x - 1$ and $g(x) = x^2$. Find $f(2) + g(-3)$.

Show answer

$$f(2) = 9 \text{ and } g(-3) = 9, \text{ so the sum is } 18.$$

11. **Hard** Samuel's savings after w weeks are $S(w) = 40 + 15w$ dollars. After how many weeks will he have \$205?

Show answer

$$40 + 15w = 205, 15w = 165, \text{ so } w = 11 \text{ weeks.}$$

12. **Hard** Let $f(x) = 4x - 3$. Find and simplify $f(2a) - f(a)$.

Show answer

$$f(2a) = 8a - 3 \text{ and } f(a) = 4a - 3, \text{ so } (8a - 3) - (4a - 3) = 4a.$$

13. **Hard** A classmate says that if $f(x) = x + 2$, then $f(3)$ means $f \cdot 3$. Explain what $f(3)$ really means and find its value.

Show answer

The parentheses hold the input; f is the name of the rule, not a number. $f(3)$ is the output when the input is 3: $f(3) = 3 + 2 = 5$.

14. **Challenge** A stone is dropped from a bridge 64 feet above the water. Its height in feet after t seconds is $h(t) = -16t^2 + 64$. Find $h(1.5)$, its height after 1.5 seconds.

Show answer

$$h(1.5) = -16(2.25) + 64 = -36 + 64 = 28 \text{ feet.}$$

15. **Challenge** A function has the rule $f(x) = 3x + b$, and $f(2) = 11$. Find $f(-1)$.

Show answer

$$3(2) + b = 11 \text{ gives } b = 5, \text{ so } f(x) = 3x + 5 \text{ and } f(-1) = -3 + 5 = 2.$$

COMMON MISTAKES

- Reading $f(x)$ as f times x . The parentheses hold the input; $f(3)$ is an output, not a product.
- Dropping parentheses when substituting a negative: $(-3)^2 = 9$, but $-3^2 = -9$. Always write the input in parentheses first.
- Confusing "find $f(6)$ " with "solve $f(x) = 6$." The first gives the input and asks for the output; the second gives the output and asks for the input.
- Forgetting to distribute when the input is an expression: $f(a + 1)$ with $f(x) = 2x - 3$ is $2(a + 1) - 3$, not $2a + 1 - 3$.

Make it a game · f of Anything. Write $f(x) = 2x^2 - 3x + 1$ at the top of a page. Put input slips in a cup: 2, -1, 0, a, a + 1 and -x. Draw one slip at a time and write f of that input as fast as you can, with the parent checking against the rule. Beat your own record for six slips.

4.4 DOMAIN AND RANGE

OBJECTIVES

- Find the domain and range of a relation given as pairs, a mapping or a graph.
- Find the domain of a function from its equation by avoiding division by zero and square roots of negative numbers.
- Choose a reasonable domain and range for a real situation and write them with inequalities or interval notation.
- Describe where the graph of a real situation is increasing, decreasing or constant.

WATCH

Domain and Range (1 of 2: Introduction)

Eddie Woo · 9:47

https://www.youtube.com/watch?v=EwZcwr_OPxA

WHEN TO WATCH

Before Example 1.

Note: Neither video covers where a graph is increasing or decreasing. Teach that from the Core ideas.

BACKUP VIDEO

Domain and Range (2 of 2: Examples)

Eddie Woo · 9:04

https://www.youtube.com/watch?v=mOVbl_kDvKY

Before the practice set: more domain and range examples.

KEY TERMS

domain

The set of all inputs (x -values) of a relation or function.

range

The set of all outputs (y -values) of a relation or function.

reasonable domain

The inputs that make sense in a real situation, such as times that are not negative or a number of people that is a whole number.

interval notation

A short way to write a set of numbers between two endpoints: a square bracket includes the endpoint and a parenthesis leaves it out, as in $[-2, 5)$. The symbol ∞ (infinity) means the numbers go on forever and always gets a parenthesis.

increasing, decreasing, constant

Reading a graph from left to right, it is increasing where it rises, decreasing where it falls and constant where it stays level.

CORE IDEAS

The **domain** is every input a function can take; the **range** is every output it actually gives. For a list of pairs, just collect them. For $\{(-2, 5), (0, 3), (3, 5), (4, -1)\}$, the domain is $\{-2, 0, 3, 4\}$ and the range is $\{-1, 3, 5\}$. List each value once, even if it repeats.

From a graph, imagine squashing the graph flat onto each axis. The shadow it makes on the x-axis is the domain; the shadow on the y-axis is the range. A segment from $(-3, 1)$ to $(2, 6)$ with both endpoints included has domain $-3 \leq x \leq 2$ and range $1 \leq y \leq 6$. An arrow on a graph means it keeps going forever in that direction.

From an equation, start by assuming every real number is allowed, then throw out inputs that cause trouble. In Algebra I there are two kinds of trouble. (1) Division by zero: $\frac{5}{x-4}$ has no value when $x = 4$, so the domain is all real numbers except 4. (2) A square root of a negative number: $\sqrt{x-3}$ needs $x-3 \geq 0$, so the domain is $x \geq 3$.

Why division by zero is not allowed. Dividing asks a multiplication question: $\frac{12}{3} = 4$ because $4 \times 3 = 12$. Then $\frac{5}{0}$ would have to be a number that gives 5 when multiplied by 0. But any number times 0 is 0, never 5, so no such number exists. **Why no square root of a negative.** A square root of -9 would be a number whose square is -9 . But a

positive number squared is positive, a negative number squared is positive, and $0^2 = 0$, so no real number works.

Real situations often limit the domain more than the equation does. If a 15-gallon water tank drains at 3 gallons per minute, the amount left is $V(t) = 15 - 3t$. Time cannot be negative, and once the tank is empty the rule stops applying: $15 - 3t = 0$ when $t = 5$. So the reasonable domain is $0 \leq t \leq 5$ and the range is $0 \leq V \leq 15$. If the input counts people or items, only whole numbers make sense.

Graphs of real situations tell stories. Reading left to right, a graph that rises is **increasing** (the output grows as the input grows), one that falls is **decreasing**, and a flat piece is **constant**. The draining tank's graph is decreasing for its whole domain.

Write domains and ranges with inequalities ($-2 \leq x < 5$) or interval notation ($[-2, 5)$), which you met in Lesson 3.3. For "all numbers 3 or less" write $x \leq 3$ or $(-\infty, 3]$. "All real numbers" is $(-\infty, \infty)$.

KEY FORMULAS

$\frac{a}{b}$ is undefined when $b = 0$

$\sqrt{a}$ is a real number only when $a \geq 0$

Practice habit. Draw a quick sketch. For a graph, shade the shadow on each axis before you write the domain and range; for a situation, mark where the input starts and where the story ends.

WORKED EXAMPLES

Example 1. Find the domain of each function. (a) $f(x) = \frac{3}{x-1}$ (b)

$$g(x) = \sqrt{x+4}$$

Show solution

1. (a) Trouble comes when the denominator is 0: $x - 1 = 0$ when $x = 1$.
2. Every other number works, so the domain is all real numbers except 1: $x \neq 1$.
3. (b) The expression under the square root must not be negative: $x + 4 \geq 0$.
4. Subtract 4 from both sides: $x \geq -4$, or $[-4, \infty)$.
5. Check a boundary value: $g(-4) = \sqrt{0} = 0$, which is fine; $g(-5) = \sqrt{-1}$ is not a real number.

Answer: (a) all real numbers except 1, $x \neq 1$ (b) $x \geq -4$, or $[-4, \infty)$

Example 2. The Millers' car starts a trip with a full 12-gallon tank and uses 1 gallon every 30 miles. The gas left after d miles is $G(d) = 12 - \frac{d}{30}$ gallons. Find the reasonable domain and range.

Show solution

1. Distance cannot be negative, so $d \geq 0$.
2. The tank is empty when $12 - \frac{d}{30} = 0$, so $\frac{d}{30} = 12$ and $d = 360$.
3. Reasonable domain: $0 \leq d \leq 360$.
4. Outputs: $G(0) = 12$ and $G(360) = 0$, so the range is $0 \leq G \leq 12$.
5. Notice also that as d grows, the gas left goes down, so the graph is decreasing over its whole domain.

Answer: Domain $0 \leq d \leq 360$ miles; range $0 \leq G \leq 12$ gallons.

PRACTICE

1. Easy Find the domain and range of $\{(-2, 5), (0, 3), (3, 5), (4, -1)\}$.

Show answer

Domain $\{-2, 0, 3, 4\}$; range $\{-1, 3, 5\}$

2. **Easy** A mapping diagram shows $1 \rightarrow 10$, $2 \rightarrow 20$ and $3 \rightarrow 10$. Find the domain and range.

Show answer

Domain $\{1, 2, 3\}$; range $\{10, 20\}$

3. **Easy** Write " x is at least -2 and less than 5 " in interval notation.

Show answer

$[-2, 5)$

4. **Easy** Write the interval $(-\infty, 3]$ as an inequality in x .

Show answer

$x \leq 3$

5. **Medium** A graph is the line segment from $(-3, 1)$ to $(2, 6)$, including both endpoints. Find its domain and range.

Show answer

Domain $-3 \leq x \leq 2$; range $1 \leq y \leq 6$

6. **Medium** Find the domain of $f(x) = \frac{5}{x-4}$.

Show answer

All real numbers except 4, since $x - 4 = 0$ when $x = 4$.

7. **Medium** Find the domain of $f(x) = \sqrt{x-3}$.

Show answer

$x - 3 \geq 0$, so $x \geq 3$, or $[3, \infty)$.

8. **Medium** The graph of $y = x^2 - 4$ is a U-shaped curve opening upward with its lowest point at $(0, -4)$, rising forever on both sides. Find its domain and range.

Show answer

Domain: all real numbers; range: $y \geq -4$, or $[-4, \infty)$.

9. **Medium** A 15-gallon water tank on a farm truck drains at 3 gallons per minute, so the water left after t minutes is $V(t) = 15 - 3t$. Find the reasonable domain and range.

Show answer

Empty when $15 - 3t = 0$, $t = 5$. Domain $0 \leq t \leq 5$ minutes; range $0 \leq V \leq 15$ gallons.

10. **Hard** Grace walks to church. Her distance from home rises steadily from 0 to 0.5 mile during the first 10 minutes, stays at 0.5 mile from minute 10 to minute 15 while she talks with a neighbor, then rises steadily to 1 mile at minute 25. Describe where the graph is increasing, decreasing or constant.

Show answer

Increasing from 0 to 10 minutes, constant from 10 to 15 minutes, increasing from 15 to 25 minutes. It is never decreasing.

11. **Hard** Find the domain of $f(x) = \frac{6}{x+2}$.

Show answer

$x + 2 = 0$ when $x = -2$, so all real numbers except -2 .

12. **Hard** Find the domain of $f(x) = \sqrt{2x+8}$.

Show answer

$2x + 8 \geq 0$, $2x \geq -8$, so $x \geq -4$.

13. **Hard** Explain why 0 is not in the domain of $f(x) = \frac{1}{x}$.

Show answer

$f(0)$ would be $\frac{1}{0}$, a number that gives 1 when multiplied by 0. Every number times 0 is 0, so no such number exists and $f(0)$ is undefined.

14. **Challenge** A ball is thrown upward from a roof 48 feet high. Its height in feet after t seconds is $h(t) = -16t^2 + 32t + 48$. It reaches its highest point at $t = 1$ and hits the ground at $t = 3$. Find the reasonable domain and range.

Show answer

$h(1) = -16 + 32 + 48 = 64$ and $h(3) = -144 + 96 + 48 = 0$. Domain $0 \leq t \leq 3$ seconds; range $0 \leq h \leq 64$ feet.

15. **Challenge** Find the domain of $f(x) = \frac{\sqrt{x-1}}{x-5}$.

Show answer

Need $x - 1 \geq 0$, so $x \geq 1$, and $x - 5 \neq 0$, so $x \neq 5$. Domain: $x \geq 1$ and $x \neq 5$.

COMMON MISTAKES

- Swapping domain and range. Domain is the inputs (x , read across); range is the outputs (y , read up and down).
- Setting the square root's inside greater than 0 instead of greater than or equal to 0. $\sqrt{0} = 0$ is allowed, so use $\geq$.
- Excluding the wrong number for a fraction. Set the **denominator** equal to 0 and solve: for $\frac{6}{x+2}$ the excluded value is -2 , not 2.
- Ignoring the situation. A time cannot be negative and a tank cannot hold less than 0 gallons, so a reasonable domain often stops where the story stops.

Make it a game • Forbidden Inputs. Challenge your parent with function riddles. 'Write me a function that cannot accept 5.' 'Write one whose domain is $x \geq -3$.' The parent writes a rule and you test it by plugging in the forbidden numbers. Then swap and build riddles of your own. Remember the two big rules: no dividing by zero and no square roots of negative numbers.

4.5 SEQUENCES AS FUNCTIONS

OBJECTIVES

- Recognize an arithmetic sequence and find its common difference.
- Write the explicit formula $a_n = a_1 + (n - 1)d$ and use it to find any term.
- Write and use a recursive rule for a sequence.
- Explain why a sequence is a function whose domain is the counting numbers.

WATCH

Introduction to Arithmetic Progressions (1 of 2: Pattern & Formula)

Eddie Woo · 8:04

<https://www.youtube.com/watch?v=ELVleTjIW-A>

WHEN TO WATCH

Before Example 1: the arithmetic sequence pattern and the n th-term formula.

Note: Eddie Woo teaches in Australia: he says 'arithmetic progression (AP)' for arithmetic sequence.

BACKUP VIDEO

Math Antics - Number Patterns

mathantics · 13:53

<https://www.youtube.com/watch?v=V7C7bXm4VI>

Before the Core ideas, as a gentler warm-up on number patterns.

KEY TERMS

sequence

An ordered list of numbers. Each number is a term; a_n (read "a sub n") is the term in position n .

common difference

The fixed number d added each time in an arithmetic sequence; $d = a_n - a_{n-1}$ for every term after the first.

arithmetic sequence

A sequence in which you add the same number each time to get the next term.

explicit formula

A rule that finds any term directly from its position n , such as $a_n = 4n + 2$.

recursive rule

A rule that gives the first term and tells how to get each term from the one before it, such as $a_1 = 6$, $a_n = a_{n-1} + 4$.

CORE IDEAS

A **sequence** is a list in order: 6, 10, 14, 18, ... We call the terms $a_1 = 6$, $a_2 = 10$, $a_3 = 14$, and so on. The small number n is the position, and a_n is the term in that position.

A **sequence is a function**. The input is the position n and the output is the term a_n . Each position has exactly one term, so it passes the test from Lesson 4.2. The domain is the counting numbers 1, 2, 3, ... (there is no position 2.5 or position -1). The notation a_n works just like $f(n)$. Its graph is a set of separate dots at $n = 1, 2, 3, \dots$, not a connected line.

In an **arithmetic sequence** the same number d , the **common difference**, is added each time. In 6, 10, 14, 18 we add 4, so $d = 4$. In 20, 17, 14, 11 we add -3 , so $d = -3$. To find d , subtract any term from the term after it, and check that the difference is the same all the way along.

Why the explicit formula has $n - 1$. Start at a_1 . To reach a_2 you add d once; to reach a_3 you add d twice; to reach a_4 , three times. The number of jumps is always one less than the position, because the first term needs no jumps. So $a_n = a_1 + (n - 1)d$. For 6, 10, 14, 18: $a_n = 6 + (n - 1)(4) = 6 + 4n - 4 = 4n + 2$. Check: $a_1 = 4(1) + 2 = 6$. ✓

A **recursive rule** describes the step-by-step pattern: give a starting term and how to get the next one. For 6, 10, 14, 18 it is $a_1 = 6$ and $a_n = a_{n-1} + 4$, where a_{n-1} means the term just before a_n . Recursive rules are natural for building a list, but to find the 100th term you would need all 99 before it. The explicit formula jumps straight there: $a_{100} = 4(100) + 2 = 402$.

Not every sequence is arithmetic. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... has the recursive rule $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$: each term is the sum of the two before it. The differences 0, 1, 1, 2, 3, 5 are not constant, so it is not arithmetic.

KEY FORMULAS

$$a_n = a_1 + (n - 1)d$$

$$a_1 = \text{first term}, \quad a_n = a_{n-1} + d$$

$$d = a_n - a_{n-1}$$

From history. Leonardo of Pisa, known as Fibonacci, wrote *Liber Abaci* in 1202, a book that helped bring Hindu-Arabic numerals into use in Europe. One of its puzzles asks how many pairs of rabbits there will be month by month if each mature pair produces a new pair every month. The counts give 1, 1, 2, 3, 5, 8, ..., where each number is the sum of the two before it.

Practice habit. Recite the explicit formula $a_n = a_1 + (n - 1)d$ until it is automatic, and always check your formula by plugging in $n = 1$ and $n = 2$ to see that it gives the first two terms.

WORKED EXAMPLES

Example 1. For the arithmetic sequence 6, 10, 14, 18, ..., (a) find the common difference, (b) write an explicit formula, and (c) find a_{25} .

Show solution

1. (a) $10 - 6 = 4$, $14 - 10 = 4$, $18 - 14 = 4$, so $d = 4$.
2. (b) $a_n = a_1 + (n - 1)d = 6 + (n - 1)(4)$.
3. Distribute and combine: $6 + 4n - 4 = 4n + 2$.
4. Check: $a_1 = 4(1) + 2 = 6$ ✓ and $a_2 = 4(2) + 2 = 10$ ✓.
5. (c) $a_{25} = 4(25) + 2 = 100 + 2 = 102$.

Answer: (a) $d = 4$ (b) $a_n = 4n + 2$ (c) $a_{25} = 102$

Example 2. Ruth is training for a race. She runs 2 miles each day in week 1 and adds half a mile to her daily run each week. (a) Write an explicit formula for her daily distance in miles in week n . (b) In which week will she run 6 miles a day?

Show solution

1. This is arithmetic with $a_1 = 2$ and $d = 0.5$. (Its recursive rule is $a_1 = 2$,
 $a_n = a_{n-1} + 0.5$.)
2. (a) $a_n = 2 + (n - 1)(0.5)$.
3. Distribute and combine: $2 + 0.5n - 0.5 = 0.5n + 1.5$.
4. (b) Solve $0.5n + 1.5 = 6$: $0.5n = 4.5$, so $n = 9$.
5. Check: $a_9 = 2 + 8(0.5) = 2 + 4 = 6$. ✓

Answer: (a) $a_n = 0.5n + 1.5$ (b) week 9

PRACTICE

1. **Easy** Find the common difference of the arithmetic sequence 4, 11, 18, 25, ...

Show answer

$$d = 11 - 4 = 7$$

2. **Easy** Write the next three terms of 20, 17, 14, 11, ...

Show answer

Subtract 3 each time: 8, 5, 2.

3. **Easy** Is 2, 4, 8, 16, ... an arithmetic sequence?

Show answer

No. The differences 2, 4, 8 are not the same.

4. **Easy** A sequence has the explicit formula $a_n = 3n + 2$. Find a_{10} .

Show answer

$$a_{10} = 3(10) + 2 = 32$$

5. **Medium** Write an explicit formula for 5, 9, 13, 17, ... and simplify it.

Show answer

$$a_n = 5 + (n - 1)(4) = 4n + 1$$

6. **Medium** Find the 30th term of the arithmetic sequence 7, 4, 1, -2 , ...

Show answer

$$d = -3, \text{ so } a_{30} = 7 + 29(-3) = 7 - 87 = -80.$$

7. **Medium** Write the first four terms of the sequence $a_1 = 6$, $a_n = a_{n-1} - 4$.

Show answer

$$6, 2, -2, -6$$

8. **Medium** The first row of a church auditorium has 14 seats, and each row after that has 2 more seats than the row in front of it. How many seats are in row 20?

Show answer

$$a_{20} = 14 + 19(2) = 52 \text{ seats.}$$

9. **Medium** Which term of the sequence 3, 8, 13, 18, ... is 98?

Show answer

$$a_n = 3 + (n - 1)(5) = 5n - 2. \text{ } 5n - 2 = 98 \text{ gives } n = 20: \text{ the 20th term.}$$

10. **Hard** Henry is building a ladder. The bottom rung is 12 inches above the ground, and the rungs are 10 inches apart. How high above the ground is the 15th rung?

Show answer

$$a_{15} = 12 + 14(10) = 152 \text{ inches.}$$

11. **Hard** Samuel has \$25 in his savings at the end of week 1 and deposits \$12 at the end of every week after that. In which week does his savings first reach at least \$200?

Show answer

$$a_n = 25 + 12(n - 1) = 12n + 13. \text{ } a_{15} = 193 \text{ and } a_{16} = 205, \text{ so week 16.}$$

12. **Hard** In an arithmetic sequence, $a_4 = 19$ and $a_9 = 44$. Write an explicit formula for a_n .

Show answer

$$\text{From term 4 to term 9 is 5 jumps: } 5d = 25, d = 5. \text{ Then } a_1 = 19 - 3(5) = 4, \text{ so } a_n = 4 + (n - 1)(5) = 5n - 1.$$

13. **Hard** Explain why the explicit formula is $a_n = a_1 + (n - 1)d$ and not $a_n = a_1 + nd$.

Show answer

You start at a_1 with no jumps. Each later term adds one more d , so a_2 has 1 jump, a_3 has 2 jumps, and a_n has $n - 1$ jumps. Using nd would add one jump too many.

14. **Challenge** The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ..., where each term is the sum of the two before it. Find the 12th term.

Show answer

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144: the 12th term is 144.

15. **Challenge** Is 200 a term of the arithmetic sequence 5, 12, 19, 26, ...?

Show answer

$a_n = 5 + (n - 1)(7) = 7n - 2$. $7n - 2 = 200$ gives $n = \frac{202}{7}$, not a counting number, so no.

COMMON MISTAKES

- Using nd instead of $(n - 1)d$. The first term has no jumps, so the n th term has only $n - 1$ jumps.
- Getting the sign of d wrong for a decreasing sequence. Always compute d as a later term minus the term before it: for 7, 4, 1, $d = 4 - 7 = -3$.
- Confusing the position n with the term a_n . In "which term is 98?" the 98 is a_n ; you solve for n .
- Connecting the dots of a sequence's graph. The domain is only counting numbers, so the graph is separate points.

Make it a game • Button Staircase. Build a staircase from buttons or beans: 3 in the first row, 7 in the second, 11 in the third, and so on. Find the common difference. Before building further, predict the number in row 20 with $a_n = a_1 + (n - 1)d$. Then keep building as far as you can to check. Closest estimate before the formula earns a point.

UNIT 4

RELATIONS AND FUNCTIONS

In this unit you move from solving single equations to studying how one quantity depends on another. You begin with the coordinate plane and relations (sets of ordered pairs), then learn the central idea of the course: a function, a rule that gives exactly one output for each input. You will test tables, mappings and graphs to see whether they are functions, and learn why the vertical line test works.

Next you learn function notation, $f(x)$, which names the output of a function and lets you evaluate and solve with it. You find the domain and range of functions from lists, graphs, equations and real situations, and you see that a sequence is a function whose inputs are the counting numbers, with arithmetic sequences as the main example.

These ideas are the language of the rest of algebra. Unit 5 studies linear functions, Unit 7 exponential functions and geometric sequences, and Unit 11 quadratic functions, all using the notation, domain and range and graph-reading skills from this unit. Parents: the main habit to watch for is careful substitution with parentheses and checking answers in the original rule.

PREREQUISITES

- Plot points on the coordinate plane (Pre-Algebra).
- Evaluate expressions by substitution, using parentheses for negatives (Lesson 1.3).
- Solve multi-step linear equations (Lessons 2.2 and 2.3).
- Solve linear inequalities (Lessons 3.1 and 3.2).
- Read and write compound inequalities and interval notation (Lesson 3.3).

MASTERY CHECKLIST

- ☐ I can plot ordered pairs and name the quadrant or axis of a point.
- ☐ I can show a relation as a list of pairs, a table, a mapping diagram and a graph.
- ☐ I can name the independent and dependent variables in a real situation.
- ☐ I can decide whether a relation is a function and explain my reasoning.
- ☐ I can use the vertical line test and explain why it works.
- ☐ I can evaluate $f(x)$ for numbers, negatives and simple expressions, and I know $f(x)$ does not mean f times x .
- ☐ I can solve $f(x)$ for the input k .
- ☐ I can find the domain and range from pairs, graphs, equations and real situations.
- ☐ I can write domains and ranges with inequalities or interval notation.
- ☐ I can describe where a graph is increasing, decreasing or constant.
- ☐ I can find the common difference, explicit formula and recursive rule

of an arithmetic sequence and use them to find any term.

LESSONS IN THIS UNIT

4.1 The Coordinate Plane and Relations	45 minutes
4.2 What Is a Function?	2 sessions of 45 minutes
4.3 Function Notation	2 sessions of 45 minutes
4.4 Domain and Range	2 sessions of 45 minutes
4.5 Sequences as Functions	2 sessions of 45 minutes

UNIT 4 TEST

You have 50 minutes. No calculator. Show your work for every problem. Give exact answers, and include units in word-problem answers.

1. **5 pts**
In which quadrant is the point $(-6, -2)$?
2. **5 pts**
Start at $(-1, 3)$. Move 4 units right and 7 units down. What point do you reach?
3. **6 pts**
Is the relation $\{(1, 2), (3, 4), (1, 5)\}$ a function?
4. **6 pts**
Is the graph of the horizontal line $y = 4$ a function?
5. **6 pts**
Let $f(x) = 4x - 9$. Find $f(-3)$.
6. **7 pts**
Let $g(x) = x^2 - 5x + 1$. Find $g(-2)$.

7. (7 pts)

Let $f(x) = -3x + 8$. Solve $f(x) = 23$.

8. (7 pts)

Find the domain and range of $\{(4, 0), (-1, 2), (6, 2), (0, -3)\}$.

9. (7 pts)

Find the domain of $f(x) = \frac{7}{2x - 6}$.

10. (7 pts)

Find the domain of $f(x) = \sqrt{x + 9}$.

11. (8 pts)

A youth group rents a lodge for a retreat. The cost is \$200 plus \$35 per person, so $C(p) = 35p + 200$. If the total cost is \$900, how many people went on the retreat?

12. (8 pts)

Find the 50th term of the arithmetic sequence 11, 8, 5, 2, ...

13. (8 pts)

Henry's tomato plant is 4 inches tall at the end of week 1 and grows 3 inches each week. (a) Write an explicit formula for its height a_n in inches at the end of week n . (b) In which week will it be 31 inches tall?

14. (8 pts)

A rain barrel holds 60 gallons and drains at 4 gallons per minute, so the water left after t minutes is $V(t) = 60 - 4t$. Find the reasonable domain and range.

15. (5 pts)

A circle has center $(0, 0)$ and radius 2. Use the vertical line test to explain why it is not the graph of a function.

ANSWER KEY

1. Quadrant III, since both coordinates are negative.
2. $(-1 + 4, 3 - 7) = (3, -4)$
3. No. The input 1 has two outputs, 2 and 5.
4. Yes. Every vertical line crosses it exactly once.
5. $f(-3) = 4(-3) - 9 = -21$
6. $g(-2) = (-2)^2 - 5(-2) + 1 = 4 + 10 + 1 = 15$
7. $-3x + 8 = 23$, $-3x = 15$, so $x = -5$.
8. Domain $\{-1, 0, 4, 6\}$; range $\{-3, 0, 2\}$
9. $2x - 6 = 0$ when $x = 3$, so all real numbers except 3.
10. $x + 9 \geq 0$, so $x \geq -9$, or $[-9, \infty)$.
11. $35p + 200 = 900$, $35p = 700$, so $p = 20$ people.
12. $d = -3$, so $a_{50} = 11 + 49(-3) = 11 - 147 = -136$.
13. (a) $a_n = 4 + (n - 1)(3) = 3n + 1$ (b) $3n + 1 = 31$, $n = 10$: week 10.
14. Empty when $60 - 4t = 0$, $t = 15$. Domain $0 \leq t \leq 15$ minutes; range $0 \leq V \leq 60$ gallons.
15. The vertical line $x = 0$ crosses the circle at $(0, 2)$ and $(0, -2)$. Those points have the same input, 0, but different outputs, so the input 0 has two outputs and the graph is not a function.

LINEAR FUNCTIONS

This unit is about straight lines: the simplest functions and among the most useful. You will learn to measure how steep a line is (its slope), see why every pair of points on a line gives the same slope, and write equations of lines in three forms: slope-intercept $y = mx + b$, point-slope $y - y_1 = m(x - x_1)$ and standard $Ax + By = C$. You will also learn what slope says about parallel and perpendicular lines, and why.

Lines describe any situation with a constant rate of change: wages by the hour, a fee plus a charge per mile, water filling a barrel, a building fund that grows by the same amount each month. In the last lesson you will build linear models from words, tables and data, interpret the slope and intercept with units, and make predictions while thinking about when a line is a sensible model.

This unit builds on the coordinate plane, functions and arithmetic sequences from Unit 4 and the equation-solving skills of Units 2 and 3. Everything here is used right away in Unit 6, where two lines are solved together as a system, and again in Unit 13, where lines are fitted to data. Geometry will also use slopes to prove that lines are parallel or perpendicular.

PREREQUISITES

- Plot and read ordered pairs in all four quadrants (Lesson 4.1).
- Add, subtract, multiply and divide integers and fractions, including subtracting negatives.
- Solve multi-step linear equations and solve a formula for one variable (Lessons 2.2–2.5).
- Use function notation and evaluate a function (Lesson 4.3).
- Write the explicit formula of an arithmetic sequence (Lesson 4.5).

MASTERY CHECKLIST

- ☐ I can find a rate of change from a table, a graph or a word situation, with units.
- ☐ I can find the slope through two points and tell whether it is positive, negative, zero or undefined.
- ☐ I can explain, using similar slope triangles, why a line has the same slope between any two of its points.
- ☐ I can recognize a direct variation $y = kx$ and use it to solve proportion problems.
- ☐ I can graph $y = mx + b$ from the slope and y -intercept and explain why b is the y -intercept.
- ☐ I can derive point-slope form from the slope formula and write the equation of a line through a point with a given slope or through two points.
- ☐ I can find intercepts from standard form, graph with them and write horizontal and vertical lines.
- ☐ I can convert among slope-intercept, point-slope and standard

form and choose the most useful form.

- ☐ I can write equations of lines parallel or perpendicular to a given line and explain why perpendicular slopes multiply to -1 .
- ☐ I can write a linear model from words, a table or two data points, interpret its slope and intercept, and judge when a linear model makes sense.

LESSONS IN THIS UNIT

5.1	Rate of Change and Slope	2 sessions of 45 minutes
5.2	Direct Variation	45 minutes
5.3	Slope-Intercept Form	2 sessions of 45 minutes
5.4	Point-Slope Form	2 sessions of 45 minutes
5.5	Standard Form and Intercepts	2 sessions of 45 minutes
5.6	Parallel and Perpendicular Lines	2 sessions of 45 minutes
5.7	Linear Models	2 sessions of 45 minutes

UNIT 5 TEST

Time: 50 minutes. No calculator. Show your work on every problem. Give exact answers, using fractions in lowest terms rather than rounded decimals, unless a problem says otherwise. Write equations of lines in the form each problem asks for, and include units in answers to word problems.

1. **5 pts**

Find the slope of the line through $(-1, 6)$ and $(3, -2)$.

2. **5 pts**

A table shows x : 0, 3, 6, 9 and y : 4, 6, 8, 10. Find the rate of change.

3. **6 pts**

y varies directly with x , and $y = 14$ when $x = 4$. Find y when $x = 10$.

4. **7 pts**

Samuel's pay for yard work varies directly with the hours he works. He earned \$52 for 4 hours. How much will he earn for 11 hours?

5. **6 pts**

Give the slope and y -intercept of $y = -\frac{1}{4}x + 6$, and describe its graph.

6. **7 pts**

Write $4x - 3y = 9$ in slope-intercept form.

7. **7 pts**

Write an equation in slope-intercept form for the line through $(-3, 2)$ with slope $\frac{2}{3}$.

8. **8 pts**

Write an equation in slope-intercept form for the line through $(-2, 9)$ and $(4, -3)$.

9. **7 pts**

Find the x - and y -intercepts of $5x - 2y = 20$.

10. **6 pts**

Write the equations of the horizontal line and the vertical line that pass through $(6, -1)$.

11. **8 pts**

Write an equation in slope-intercept form for the line through $(8, 3)$ perpendicular to $y = 4x - 1$.

12. 7 pts

Write an equation in slope-intercept form for the line through $(2, -1)$ parallel to $3x + 2y = 4$.

13. 8 pts

The cost of renting a hall for a church potluck is linear in the number of guests. For 40 guests it costs \$310, and for 70 guests it costs \$430. (a) Write a model for the cost y for x guests. (b) Find the cost for 100 guests.

14. 7 pts

Ruth spends \$36 on fabric for a quilt: cotton at \$6 per yard and flannel at \$9 per yard. (a) Write an equation in standard form using c yards of cotton and f yards of flannel. (b) Find the c -intercept and the f -intercept.

15. 6 pts

Explain why the y -intercept of $y = mx + b$ is b , and why a vertical line has an undefined slope.

ANSWER KEY

1. $m = \frac{-2 - 6}{3 - (-1)} = \frac{-8}{4} = -2$
2. Each increase of 3 in x raises y by 2, so the rate is $\frac{2}{3}$.
3. $k = \frac{14}{4} = \frac{7}{2}$, so $y = \frac{7}{2}x$ and $y = \frac{7}{2}(10) = 35$.
4. $k = \frac{52}{4} = 13$ dollars per hour, so $13 \cdot 11 = 143$. He earns \$143.
5. Slope $-\frac{1}{4}$, y -intercept 6; the line through $(0, 6)$ and $(4, 5)$, falling gently from left to right.
6. $-3y = -4x + 9$, so $y = \frac{4}{3}x - 3$
7. $y - 2 = \frac{2}{3}(x + 3)$, so $y - 2 = \frac{2}{3}x + 2$ and $y = \frac{2}{3}x + 4$
8. $m = \frac{-3 - 9}{4 - (-2)} = \frac{-12}{6} = -2$; $y - 9 = -2(x + 2)$, so $y = -2x + 5$
9. $y = 0$: $5x = 20$, $x = 4$. $x = 0$: $-2y = 20$, $y = -10$. x -intercept 4, y -intercept -10 .
10. Horizontal: $y = -1$; vertical: $x = 6$
11. Slope $-\frac{1}{4}$: $y - 3 = -\frac{1}{4}(x - 8)$, so $y - 3 = -\frac{1}{4}x + 2$ and $y = -\frac{1}{4}x + 5$
12. $2y = -3x + 4$ gives slope $-\frac{3}{2}$. $y + 1 = -\frac{3}{2}(x - 2)$, so $y + 1 = -\frac{3}{2}x + 3$ and $y = -\frac{3}{2}x + 2$
13. (a) $m = \frac{430 - 310}{70 - 40} = 4$; $y - 310 = 4(x - 40)$, so $y = 4x + 150$ (\$4 per guest plus a \$150 base charge). (b) $4(100) + 150 = 550$, so \$550.
14. (a) $6c + 9f = 36$ (or, dividing by 3, $2c + 3f = 12$); (b) c -intercept 6 (6 yards of cotton, no flannel) and f -intercept 4 (4 yards of flannel, no cotton).
15. The graph crosses the y -axis where $x = 0$, and then $y = m \cdot 0 + b = b$. On a vertical line every point has the same x -coordinate, so the run between any two points is 0, and rise divided by 0 is undefined.

5.1 RATE OF CHANGE AND SLOPE

OBJECTIVES

- Find a rate of change from a table, a graph or a word situation, with units.
- Calculate the slope of a line through two points with the slope formula.
- Explain why every pair of points on a line gives the same slope, using similar slope triangles.
- Classify a slope as positive, negative, zero or undefined and describe what each looks like.

WATCH

Algebra Basics: Slope And Distance - Math Antics

[mathantics](#) · 12:00

<https://www.youtube.com/watch?v=rpMu98yRk40>

WHEN TO WATCH

Before Example 1: slope as rise over run.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Introduction to Gradient (1 of 2: Definition & Meaning)

[Eddie Woo](#) · 11:36

<https://www.youtube.com/watch?v=jY25uxFoLG0>

Before Example 2: slope as a rate of change.

KEY TERMS

rate of change

How much the output changes for each one-unit change in the input; the change in output divided by the change in input.

slope

The steepness and direction of a line: the rise (vertical change) divided by the run (horizontal change) between any two points on it. Usually written m .

rise

The vertical change between two points, $y_2 - y_1$. Up is positive; down is negative.

run

The horizontal change between two points, $x_2 - x_1$. Right is positive; left is negative.

slope triangle

A right triangle drawn under (or over) a line, with a horizontal leg (the run), a vertical leg (the rise) and its long side lying on the line.

undefined slope

The slope of a vertical line. The run is 0, and we cannot divide by 0, so the slope is not a number.

CORE IDEAS

A **rate of change** tells how fast one quantity changes compared with another. If a train travels 120 miles in 2 hours and 300 miles in 5 hours, then in those 3 extra hours it went 180 extra miles, so its rate is $\frac{180 \text{ miles}}{3 \text{ hours}} = 60 \text{ miles per hour}$. In general, $\text{rate of change} = \frac{\text{change in output}}{\text{change in input}}$. From a table, pick two rows, subtract the outputs, subtract the inputs (in the same order) and divide. When this rate is the same for every pair of rows, the points lie on a straight line, just like the arithmetic sequences of Lesson 4.5, which add the same amount each step.

On a graph, the rate of change of a line is called its **slope**, written m . Walk from one point to another: the vertical change is the **rise** and the horizontal change is the **run**, and $\text{slope} = \text{rise} \div \text{run}$. For points (x_1, y_1) and (x_2, y_2) , the rise is $y_2 - y_1$ and the run is $x_2 - x_1$, which gives the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. The small numbers below the letters are subscripts: x_1 just means "the x of the first point."

Does it matter which point you call the first? No, as long as you keep the same order on top and bottom. Switching the points gives $\frac{y_1 - y_2}{x_1 - x_2}$. Each of these differences is the opposite of the original one, so the top and bottom both change sign, and a negative divided by a negative gives the same quotient: $\frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}$.

Here is the big **why**: a line has only one slope, no matter which two points you pick. Draw two slope triangles on the same line, a small one between two close points and a big one between two far-apart points. Each has a right angle where the horizontal and vertical legs meet. Each also has an angle where the line crosses its horizontal

leg, and these two angles are equal, because the horizontal legs are parallel and the line is straight (it meets every horizontal line at the same angle). Two triangles with two equal angles have all three angles equal, so they are **similar**: one is a scaled copy of the other. In a scaled copy every length is multiplied by the same number, so if the big triangle is k times as large, its rise and run are both k times as large and $\frac{k \cdot \text{rise}}{k \cdot \text{run}} = \frac{\text{rise}}{\text{run}}$. For example, on the line through $(0, 1)$, $(1, 4)$ and $(3, 10)$, the triangle from $(0, 1)$ to $(1, 4)$ has rise 3 and run 1, and the triangle from $(1, 4)$ to $(3, 10)$ has rise 6 and run 2. Both give slope 3.

There are four kinds of slope. Reading left to right, a line with **positive slope** goes up, a line with **negative slope** goes down, and a horizontal line has **zero slope** because its rise is 0 and $\frac{0}{\text{run}} = 0$. A vertical line has run 0, and division by 0 has no answer, so its slope is **undefined**. Be careful: zero slope and undefined slope are not the same thing. A larger absolute value of the slope means a steeper line: slope -5 is steeper than slope 2.

In a real situation, slope carries units: output units per input unit, such as miles per hour, dollars per week or gallons per minute. A negative rate means the quantity is decreasing, like water draining from a tank.

KEY FORMULAS

$$\text{rate of change} = \frac{\text{change in output}}{\text{change in input}}$$

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

Euclid connection. Euclid's Elements, Book VI, Proposition 4, proves that triangles with equal angles have proportional sides. That is exactly the fact that makes every slope triangle on a line give the same slope.

Practice habit. Before you substitute, label the points: write (x_1, y_1) over the first point and (x_2, y_2) over the second. Then put negative coordinates in parentheses, so $5 - (-3)$ never turns into $5 - 3$.

WORKED EXAMPLES

Example 1. Find the slope of the line through $(-1, 4)$ and $(3, -4)$.

Show solution

1. Label the points: $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (3, -4)$.
2. Substitute into the slope formula: $m = \frac{-4 - 4}{3 - (-1)}$.
3. Simplify the top and bottom: $m = \frac{-8}{4}$.
4. Divide: $m = -2$.
5. Check by using the points in the other order: $\frac{4 - (-4)}{-1 - 3} = \frac{8}{-4} = -2$. ✓
6. The slope is negative, so the line goes down from left to right: 2 units down for every 1 unit right.

Answer: $m = -2$

Example 2. Ruth saves money from her babysitting. After 2 weeks she has \$46, after 5 weeks she has \$73 and after 9 weeks she has \$109. Is she saving at a constant rate? If so, what is the rate?

Show solution

1. Input: weeks. Output: dollars saved.
2. From week 2 to week 5: change in dollars $= 73 - 46 = 27$; change in weeks $= 5 - 2 = 3$.
3. Rate: $\frac{27}{3} = 9$ dollars per week.
4. From week 5 to week 9: change in dollars $= 109 - 73 = 36$; change in weeks $= 9 - 5 = 4$.
5. Rate: $\frac{36}{4} = 9$ dollars per week.
6. The rate is the same for both pairs, so the rate is constant and the three points lie on a line with slope 9.
7. Check from week 2 to week 9: $\frac{109 - 46}{9 - 2} = \frac{63}{7} = 9$. ✓

Answer: Yes; she saves at a constant rate of \$9 per week.

PRACTICE

1. **Easy** Find the slope of the line through (1, 2) and (3, 8).

Show answer

$$m = \frac{8 - 2}{3 - 1} = \frac{6}{2} = 3$$

2. **Easy** Find the slope of the line through (−2, 5) and (2, −3).

Show answer

$$m = \frac{-3 - 5}{2 - (-2)} = \frac{-8}{4} = -2$$

3. **Easy** Find the slope of the line through (4, 7) and (−1, 7). Describe the line.

Show answer

$$m = \frac{7 - 7}{-1 - 4} = \frac{0}{-5} = 0; \text{ the line is horizontal.}$$

4. **Easy** Find the slope of the line through (3, −2) and (3, 6). Describe the line.

Show answer

The run is $3 - 3 = 0$, so the slope $\frac{8}{0}$ is undefined; the line is vertical.

5. **Medium** Find the slope of the line through (−3, −4) and (5, 2).

Show answer

$$m = \frac{2 - (-4)}{5 - (-3)} = \frac{6}{8} = \frac{3}{4}$$

6. **Medium** A table shows x : 0, 2, 4, 6 and y : 10, 7, 4, 1. Find the rate of change of y with respect to x .

Show answer

Each time x goes up 2, y goes down 3, so the rate is $\frac{-3}{2} = -\frac{3}{2}$.

7. **Medium** The Miller family drives to visit their grandparents. After 2 hours they have gone 110 miles, and after 5 hours they have gone 275 miles. Assuming a steady speed, what is their rate of travel?

Show answer

$$\frac{275 - 110}{5 - 2} = \frac{165}{3} = 55 \text{ miles per hour}$$

8. **Medium** A water tank on a farm holds 240 gallons. Fifteen minutes after a valve is opened, it holds 180 gallons. Find the rate of change of the water in the tank, assuming it drains steadily.

Show answer

$$\frac{180 - 240}{15 - 0} = \frac{-60}{15} = -4 \text{ gallons per minute (the tank loses 4 gallons each minute).}$$

9. **Medium** On a hiking trail, Henry is at an elevation of 1,200 feet at mile 1 and 2,100 feet at mile 4. Find the average rate of climb in feet per mile.

Show answer

$$\frac{2100 - 1200}{4 - 1} = \frac{900}{3} = 300 \text{ feet per mile}$$

10. **Hard** Find the value of k so that the line through $(2, k)$ and $(5, 10)$ has slope 3.

Show answer

$$\frac{10 - k}{5 - 2} = 3, \text{ so } 10 - k = 9 \text{ and } k = 1.$$

11. **Hard** Do the points $(1, 2)$, $(3, 8)$ and $(6, 17)$ all lie on one line? Answer yes or no and explain using slopes.

Show answer

Yes. From $(1, 2)$ to $(3, 8)$ the slope is $\frac{6}{2} = 3$; from $(3, 8)$ to $(6, 17)$ it is $\frac{9}{3} = 3$. The slopes match and the segments share the point $(3, 8)$.

12. **Hard** A table shows x : 1, 2, 3, 4 and y : 2, 5, 10, 17. Is the rate of change constant? Answer yes or no.

Show answer

No. The rates between rows are 3, 5 and 7, which are not equal, so the points do not lie on a line.

13. **Challenge** A line passes through $(-1, 4)$ and has slope $-\frac{3}{2}$. The point $(5, y)$ is also on the line. Find y .

Show answer

$$\frac{y - 4}{5 - (-1)} = -\frac{3}{2}, \text{ so } \frac{y - 4}{6} = -\frac{3}{2}, y - 4 = -9 \text{ and } y = -5.$$

14. **Challenge** Explain why you get the same slope no matter which two points of a line you use.

Show answer

Draw a slope triangle between each pair of points. Each triangle has a right angle, and the angle where the line meets the horizontal leg is the same in both, because the line is straight and the horizontal legs are parallel. So the triangles are similar, and one is a scaled copy of the other. Scaling multiplies the rise and the run by the same number, so rise divided by run does not change.

COMMON MISTAKES

- Putting the change in x on top. Slope is rise over run: the y -values go in the numerator. Say " y over x " as you write it.
- Subtracting in different orders, such as $\frac{y_2 - y_1}{x_1 - x_2}$, which flips the sign. Label the points first and keep the same point first on top and bottom.
- Losing a sign when subtracting a negative, as in $5 - (-3)$. Always put a substituted negative in parentheses; $5 - (-3) = 8$.
- Mixing up zero slope and undefined slope. Horizontal lines have zero rise, so slope 0; vertical lines have zero run, so the slope is undefined (we cannot divide by 0).

Make it a game • Slope Scavenger Hunt. Find three slopes in your home or yard, such as a stair, a ramp, or a ladder against a wall. Measure the rise and the run with a tape measure, keeping the same units. Calculate each slope and give it a rate-of-change sentence with units. Then name which of your three is steepest and predict its rise over a run of 10 feet.

5.2 DIRECT VARIATION

OBJECTIVES

- Decide whether an equation, a table or a graph shows direct variation.
- Find the constant of variation k and write the equation $y = kx$.
- Explain why the graph of a direct variation is a line through the origin with slope k .
- Solve recipe, wage and unit-price problems with direct variation.

WATCH

Direct Linear Variation (1 of 2: Introduction)

Eddie Woo · 7:45

<https://www.youtube.com/watch?v=3l4LD23Qpls>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope, and some examples use metric units.

BACKUP VIDEO

Direct Linear Variation (2 of 2: Example questions)

Eddie Woo · 5:41

<https://www.youtube.com/watch?v=6E6W6evLcfc>

Before the practice set: worked direct variation questions.

KEY TERMS

direct variation

A relationship that can be written $y = kx$ for some nonzero number k . We say " y varies directly with x " or " y is directly proportional to x ."

constant of variation

The fixed number k in $y = kx$. It equals $\frac{y}{x}$ for every pair with $x \neq 0$.

origin

The point $(0, 0)$, where the x -axis and y -axis cross.

unit rate

A rate for one unit of the input, such as dollars per pound or miles per hour. In a direct variation the unit rate is k .

CORE IDEAS

Many everyday quantities grow together in the simplest possible way: twice as much of one means twice as much of the other. If apples cost \$2 a pound, 3 pounds cost \$6 and 6 pounds cost \$12. A relationship like this is called a **direct variation**, and it can always be written $y = kx$, where k is a fixed nonzero number called the **constant of variation**. Here, $\text{cost} = 2 \cdot \text{pounds}$, so $k = 2$.

To find k , divide both sides of $y = kx$ by x (when $x \neq 0$): $k = \frac{y}{x}$. So a table shows direct variation exactly when the ratio $\frac{y}{x}$ is the same for every row. For x : 2, 5, 8 and y : 6, 15, 24, every ratio is 3, so $y = 3x$. For x : 1, 2, 3 and y : 5, 7, 9 the ratios are 5, 3.5 and 3, so it is not direct variation, even though y goes up steadily.

Why does doubling x double y ? If $y = kx$ and you replace x by $2x$, the new output is $k(2x) = 2(kx) = 2y$. The same argument works for tripling, halving or multiplying by any number. That is why direct variation fits recipes (double the batch, double the flour), wages (twice the hours, twice the pay) and unit prices.

The graph of $y = kx$ is a straight line through the origin with slope k . **Why the origin?** When $x = 0$, $y = k \cdot 0 = 0$: zero pounds of apples cost nothing. **Why slope k ?** Use the points $(0, 0)$ and $(1, k)$, which are both on the graph: $m = \frac{k - 0}{1 - 0} = k$. So the constant of variation is the slope and also the unit rate. A line like $y = 2x + 3$ is straight, but it misses the origin, so it is not a direct variation: doubling x from 1 to 2 changes y from 5 to 7, not to 10.

Because $\frac{y}{x}$ is always the same number, any two pairs give a proportion: $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. In practice it is usually clearest to find k first from the pair you know, write $y = kx$, and then substitute the new value.

KEY FORMULAS

$$y = kx, \quad k \neq 0$$

$$k = \frac{y}{x} \quad (x \neq 0)$$

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Practice habit. Find the unit rate first. Before answering "how much for 15?", ask "how much for 1?" Then multiply. For example, 4 pounds for \$6.20 means \$1.55 per pound.

WORKED EXAMPLES

Example 1. y varies directly with x , and $y = 12$ when $x = 4$. Write the direct variation equation, then find y when $x = 10$.

Show solution

1. Start with $y = kx$.
2. Substitute the known pair: $12 = k \cdot 4$.
3. Divide both sides by 4: $k = 3$.
4. Write the equation: $y = 3x$.
5. Substitute $x = 10$: $y = 3 \cdot 10 = 30$.
6. Check the ratios: $\frac{12}{4} = 3$ and $\frac{30}{10} = 3$. ✓

Answer: $y = 3x$; $y = 30$

Example 2. Grace's pay varies directly with the number of hours she works at a bakery. She earned \$58.50 for 6.5 hours. How much will she earn for 15 hours?

Show solution

1. Let h be hours and p be pay in dollars, so $p = kh$.
2. Find k : $k = \frac{58.50}{6.5} = 9$. She earns \$9 per hour.
3. Write the equation: $p = 9h$.
4. Substitute $h = 15$: $p = 9 \cdot 15 = 135$.
5. Check with a proportion: $\frac{58.50}{6.5} = 9$ and $\frac{135}{15} = 9$. ✓

Answer: \$135

PRACTICE

1. Easy Find the constant of variation for $y = -4x$.

Show answer

$$k = -4$$

2. Easy Is $y = 2x + 3$ a direct variation? Answer yes or no.

Show answer

No. When $x = 0$, $y = 3$, so the graph does not pass through the origin.

3. Easy y varies directly with x , and $y = 20$ when $x = 5$. Find k .

Show answer

$$k = \frac{20}{5} = 4$$

4. Easy A table shows x : 2, 4, 6 and y : 7, 14, 21. Find the constant of variation.

Show answer

Every ratio $\frac{y}{x}$ is $\frac{7}{2}$, so $k = \frac{7}{2} = 3.5$.

5. Medium A table shows x : 1, 2, 3 and y : 4, 7, 10. Does it show direct variation? Answer yes or no.

Show answer

No. $\frac{4}{1} = 4$ but $\frac{7}{2} = 3.5$; the ratios are not equal.

6. Medium y varies directly with x , and $y = -18$ when $x = 6$. Find y when $x = -5$.

Show answer

$k = \frac{-18}{6} = -3$, so $y = -3x$ and $y = -3(-5) = 15$.

7. Medium y varies directly with x , and $y = 9$ when $x = 12$. Write the direct variation equation.

Show answer

$$k = \frac{9}{12} = \frac{3}{4}, \text{ so } y = \frac{3}{4}x.$$

8. **Medium** A pancake recipe uses 3 cups of flour to make 12 pancakes. The flour varies directly with the number of pancakes. How many cups of flour are needed for 20 pancakes?

Show answer

$$k = \frac{3}{12} = \frac{1}{4} \text{ cup per pancake, so } \frac{1}{4} \cdot 20 = 5 \text{ cups.}$$

9. **Medium** At a farm stand, 4 pounds of apples cost \$6.20. The cost varies directly with the weight. What do 7 pounds cost?

Show answer

$$k = \frac{6.20}{4} = 1.55 \text{ dollars per pound, so } 1.55 \cdot 7 = 10.85. \text{ The cost is \$10.85.}$$

10. **Hard** A graph of a direct variation is a line through the origin and the point $(4, -6)$. Write its equation.

Show answer

$$k = \frac{-6}{4} = -\frac{3}{2}, \text{ so } y = -\frac{3}{2}x.$$

11. **Hard** A train moving at a constant speed travels 210 miles in 3.5 hours. Distance varies directly with time. How many hours will it take to travel 330 miles?

Show answer

$$k = \frac{210}{3.5} = 60 \text{ miles per hour, so } 330 = 60t \text{ and } t = 5.5 \text{ hours.}$$

12. **Hard** The amount of paint needed varies directly with the area painted. Two gallons cover 700 square feet. How many gallons are needed to paint a barn wall with an area of 1,225 square feet?

Show answer

$$k = \frac{2}{700} = \frac{1}{350} \text{ gallon per square foot, so } \frac{1225}{350} = 3.5 \text{ gallons.}$$

13. **Challenge** y varies directly with x . If x is multiplied by 3, by what number is y multiplied?

Show answer

$$\text{By 3: } k(3x) = 3(kx) = 3y.$$

14. **Challenge** Explain why the graph of every direct variation passes through the origin, and why $y = x + 3$ is not a direct variation even though its graph is a straight line.

Show answer

In $y = kx$, putting $x = 0$ gives $y = k \cdot 0 = 0$, so $(0, 0)$ is always on the graph.

For $y = x + 3$, $x = 0$ gives $y = 3$, so the line misses the origin; also $\frac{y}{x}$ is not constant (for example $\frac{4}{1} = 4$ but $\frac{5}{2} = 2.5$), and doubling x does not double y .

COMMON MISTAKES

- Calling every straight line a direct variation. It must pass through the origin; check that $\frac{y}{x}$ is the same for every pair.
- Finding k upside down as $\frac{x}{y}$. From $y = kx$, divide by x : $k = \frac{y}{x}$, output over input.
- Checking only that y increases by a constant amount. Equal steps make a line, but direct variation also needs $y = 0$ when $x = 0$.

Make it a game • Match the Constant. Make twelve cards: four equations, four tables and four quick graph sketches. Some show direct variation, such as $y = 3x$, and some are impostors, such as $y = 3x + 1$. Turn all cards face down. Flip two at a time and keep a pair only if both show direct variation with the same constant k . State k aloud.

5.3 SLOPE-INTERCEPT FORM

OBJECTIVES

- Identify the slope and y -intercept of a line written as $y = mx + b$, and explain why b is the y -intercept.
- Graph a line from its slope and y -intercept.
- Write the equation of a line from its graph, or from its slope and y -intercept, and rewrite an equation in slope-intercept form.
- Interpret m and b in a real situation, such as a starting fee plus a rate.

WATCH

Basic Linear Functions - Math Antics

[mathantics](#) · 13:24

<https://www.youtube.com/watch?v=MXV65i9g1Xg>

WHEN TO WATCH

Before Example 1: $y = mx + b$, slope and y -intercept.

BACKUP VIDEO

How to Graph Lines in Slope-Intercept Form ($y=mx+b$) | Math with Mr. J

[Math with Mr. J](#) · 8:28

<https://www.youtube.com/watch?v=vRY4ofFiWeo>

Before Example 2: graphing a line from slope-intercept form.

KEY TERMS

y -intercept

The y -coordinate of the point where a graph crosses the y -axis. At that point $x = 0$.

linear equation

An equation whose graph is a straight line, such as $y = 2x - 3$.

slope-intercept form

The form $y = mx + b$ of a linear equation, where m is the slope and b is the y -intercept.

fixed fee

A one-time amount charged no matter how much you use; in a cost equation $y = mx + b$ it is b .

CORE IDEAS

Every non-vertical line can be written in **slope-intercept form**, $y = mx + b$. The two numbers tell you everything: m is the slope and b is the y -intercept. For $y = 4x - 7$, the slope is 4 and the y -intercept is -7 (notice that -7 comes with its sign, because $4x - 7 = 4x + (-7)$).

Why is b the y -intercept? A graph crosses the y -axis at the point where $x = 0$, because every point on the y -axis has x -coordinate 0. Put $x = 0$ into $y = mx + b$: $y = m \cdot 0 + b = b$. So the line passes through $(0, b)$.

Why is m the slope? Suppose a line has slope m and crosses the y -axis at $(0, b)$. Take any other point (x, y) on the line. By the slope formula from Lesson 5.1, $\frac{y - b}{x - 0} = m$. Multiply both sides by x : $y - b = mx$. Add b : $y = mx + b$. So every point on the line satisfies this equation. You can also see the slope in the equation itself: when x increases by 1, the output changes from $mx + b$ to $m(x + 1) + b = mx + b + m$, so y goes up by exactly m for each step of 1 to the right.

Graphing from $y = mx + b$: (1) plot the y -intercept $(0, b)$; (2) write the slope as a fraction $\frac{\text{rise}}{\text{run}}$ and move from $(0, b)$ by that rise and run to plot a second point; (3) repeat once more as a check; (4) draw the line through the points. For $m = -\frac{2}{3}$, go down 2 and right 3 (or up 2 and left 3). For a whole-number slope such as $m = 5$, think $\frac{5}{1}$.

Writing an equation from a graph: read b where the line crosses the y -axis, pick two points where the line crosses grid corners, find the slope, and write $y = mx + b$. If an equation is not in slope-intercept form, such as $3x + 2y = 8$, solve it for y as in Lesson 2.5: $2y = -3x + 8$, so $y = -\frac{3}{2}x + 4$.

In real situations, $y = mx + b$ often means a **starting amount plus a rate**. A bike shop that charges a \$8 fee plus \$6 per hour has cost $C = 6h + 8$: the slope 6 is the rate in dollars per hour, and the intercept 8 is the cost at $h = 0$, the fee you pay before you ride at all. When $b = 0$, the equation $y = mx$ is a direct variation from Lesson 5.2.

KEY FORMULAS

$$y = mx + b \quad (m = \text{slope}, b = y\text{-intercept})$$

Practice habit. Graph from the intercept, then check with a third point. After plotting $(0, b)$ and one slope step, take a second slope step and make sure all three points line up; then substitute one point into the equation as a final check.

WORKED EXAMPLES

Example 1. Identify the slope and y -intercept of $y = -\frac{2}{3}x + 4$, then graph the line.

Show solution

1. Compare with $y = mx + b$: $m = -\frac{2}{3}$ and $b = 4$.
2. Plot the y -intercept $(0, 4)$.
3. Read the slope as rise -2 over run 3 : from $(0, 4)$ go down 2 and right 3 to $(3, 2)$.
4. Take another step: down 2 and right 3 to $(6, 0)$.
5. Check $(6, 0)$ in the equation: $-\frac{2}{3} \cdot 6 + 4 = -4 + 4 = 0$. ✓
6. Draw the line through $(0, 4)$, $(3, 2)$ and $(6, 0)$; it falls from left to right.

Answer: Slope $-\frac{2}{3}$, y -intercept 4; the graph is a line through $(0, 4)$, $(3, 2)$ and $(6, 0)$, falling from left to right.

Example 2. A bike-rental shop charges an \$8 fee plus \$6 per hour. Write an equation for the total cost C of renting a bike for h hours, and find the cost of a 3.5-hour ride.

Show solution

1. The rate is \$6 per hour, so $m = 6$.
2. The fee is paid even at 0 hours, so $b = 8$.
3. Write the equation: $C = 6h + 8$.
4. Substitute $h = 3.5$: $C = 6(3.5) + 8$.
5. Multiply: $6(3.5) = 21$, so $C = 21 + 8 = 29$.
6. Check in words: 3.5 hours at \$6 is \$21, plus the \$8 fee is \$29. ✓

Answer: $C = 6h + 8$; a 3.5-hour ride costs \$29.

PRACTICE

1. **Easy** Give the slope and y -intercept of $y = 4x - 7$.

Show answer

Slope 4, y -intercept -7

2. **Easy** Give the slope and y -intercept of $y = -x + 2$.

Show answer

Slope -1 , y -intercept 2

3. **Easy** Write the equation of the line with slope 3 and y -intercept -5 .

Show answer

$$y = 3x - 5$$

4. **Easy** Give the slope and y -intercept of $y = 6$.

Show answer

$y = 0x + 6$: slope 0, y -intercept 6 (a horizontal line).

5. **Medium** Graph $y = \frac{3}{4}x - 2$.

Show answer

A line through $(0, -2)$ and $(4, 1)$ (up 3, right 4), also through $(8, 4)$; slope $\frac{3}{4}$, rising from left to right.

6. **Medium** Write $2x + y = 9$ in slope-intercept form.

Show answer

Subtract $2x$: $y = -2x + 9$

7. **Medium** Write $3x + 2y = 8$ in slope-intercept form.

Show answer

$$2y = -3x + 8, \text{ so } y = -\frac{3}{2}x + 4$$

8. **Medium** A line on a graph crosses the y -axis at $(0, 5)$ and also passes through $(2, -1)$. Write its equation in slope-intercept form.

Show answer

$$m = \frac{-1 - 5}{2 - 0} = -3 \text{ and } b = 5, \text{ so } y = -3x + 5.$$

9. **Medium** A plumber charges a \$75 visit fee plus \$60 per hour. (a) Write an equation for the total cost C for h hours of work. (b) Find the cost of a 2.5-hour job.

Show answer

(a) $C = 60h + 75$; (b) $C = 60(2.5) + 75 = 150 + 75 = 225$, so \$225.

10. **Hard** A candle is 12 inches tall when lit and burns down 0.75 inch per hour. (a) Write an equation for its height h after t hours. (b) After how many hours will it be 3 inches tall?

Show answer

(a) $h = -0.75t + 12$; (b) $3 = -0.75t + 12$, so $-0.75t = -9$ and $t = 12$ hours.

11. **Hard** Henry's savings account balance after w weeks is $S = 15w + 120$ dollars. What do the numbers 15 and 120 mean in this situation?

Show answer

15 is the slope: Henry adds \$15 to the account each week. 120 is the S -intercept: he had \$120 in the account at the start ($w = 0$).

12. **Hard** Write the equation in slope-intercept form of the line through $(-2, 1)$ and $(2, 7)$.

Show answer

$$m = \frac{7 - 1}{2 - (-2)} = \frac{6}{4} = \frac{3}{2}. \text{ Then } 7 = \frac{3}{2}(2) + b, \text{ so } b = 4: y = \frac{3}{2}x + 4.$$

13. **Challenge** The line $y = mx + 3$ passes through $(4, -5)$. Find m .

Show answer

$$-5 = 4m + 3, \text{ so } 4m = -8 \text{ and } m = -2.$$

14. **Challenge** Explain why b is the y -intercept of $y = mx + b$, and why y increases by m every time x increases by 1.

Show answer

The graph meets the y -axis where $x = 0$, and then $y = m \cdot 0 + b = b$, so it crosses at $(0, b)$. If x becomes $x + 1$, the output becomes $m(x + 1) + b = mx + b + m$, which is the old y plus m .

COMMON MISTAKES

- Dropping the sign of b : in $y = 4x - 7$ the y -intercept is -7 , not 7 . Rewrite as $y = 4x + (-7)$ if it helps.
- Reading the slope from an equation that is not solved for y . In $3x + 2y = 8$ the slope is not 3 ; solve for y first to get $y = -\frac{3}{2}x + 4$.
- Moving the run vertically and the rise horizontally when graphing. Rise is up or down (the numerator); run is left or right (the denominator).
- Plotting the intercept on the x -axis. The point $(0, b)$ is on the y -axis.

Make it a game • Name That Line. Secretly write an equation $y = mx + b$ and draw its line on graph paper without any labels. Your partner finds two points, reads the slope and y -intercept, and guesses the equation. Score two points for a correct guess. Add a story twist: b is a fixed fee, like a taxi's starting charge, and m is the rate per mile.

5.4 POINT-SLOPE FORM

OBJECTIVES

- Derive point-slope form $y - y_1 = m(x - x_1)$ from the slope formula.
- Write the equation of a line through a given point with a given slope.
- Write the equation of a line through two given points.
- Convert an equation from point-slope form to slope-intercept form and check it.

WATCH

Point-Gradient Form (2 of 2: $y - y_o = m(x - x_o)$)

Eddie Woo · 9:52

<https://www.youtube.com/watch?v=yx7-HQJNQNY>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he calls it 'point-gradient form' and writes the point as (x_o, y_o) instead of (x_1, y_1) .

BACKUP VIDEO

Slope-Intercept Form from Two Points | Write the Equation of a Line Given Two Points

Math with Mr. J · 11:40

<https://www.youtube.com/watch?v=mFNwaQUyBlw>

Before Example 2: writing the equation of a line through two points.

KEY TERMS

point-slope form

The form $y - y_1 = m(x - x_1)$ of a linear equation, where m is the slope and (x_1, y_1) is a known point on the line.

given point

A specific point (x_1, y_1) that the line is known to pass through; its coordinates are numbers, while x and y stay as variables.

equivalent equations

Equations with exactly the same solutions, so they have the same graph, such as $y - 4 = 3(x - 2)$ and $y = 3x - 2$.

CORE IDEAS

Slope-intercept form is easy to use when you know where the line crosses the y -axis. But often you know some other point, like $(2, -3)$, and the slope. **Point-slope form** is built for exactly that situation, and it comes straight out of the slope formula.

Where it comes from: suppose a line has slope m and passes through a known point (x_1, y_1) . Let (x, y) stand for any other point on the line. The slope between the two points must be m , so $\frac{y - y_1}{x - x_1} = m$. Multiply both sides by $(x - x_1)$ to clear the fraction: $y - y_1 = m(x - x_1)$. That is point-slope form. The fraction version could not be used at the point (x_1, y_1) itself (it would give $\frac{0}{0}$), but after multiplying, the point works too: $y_1 - y_1 = 0$ and $m(x_1 - x_1) = 0$. So the equation describes every point of the line, including the given one.

Using it with a point and a slope: substitute the numbers for m , x_1 and y_1 and leave x and y as letters. Watch the signs: the form has minus signs built in, so a point with negative coordinates such as $(-4, 2)$ gives $y - 2 = m(x - (-4))$, which simplifies to $y - 2 = m(x + 4)$. Reading backwards, $y + 5 = \frac{1}{2}(x - 6)$ passes through $(6, -5)$, not $(-6, 5)$.

Using it with two points: first find the slope with the slope formula, then use either point in point-slope form. The two choices look different at first, but both simplify to the same slope-intercept equation, because they describe the same line.

Converting to slope-intercept form: distribute m , then add y_1 to both sides. For $y - 4 = 3(x - 2)$: $y - 4 = 3x - 6$, so $y = 3x - 2$. Then check by substituting the original point (or both points): $3(2) - 2 = 4$. ✓

In word problems, point-slope form is handy whenever you know a rate and one situation that is not the starting point, such as "on day 4 the plant was 9 cm tall, and it grows 1.5 cm per day." Two situations, like the costs of two different orders, give two points and therefore the slope.

KEY FORMULAS

$$y - y_1 = m(x - x_1)$$

$$\frac{y - y_1}{x - x_1} = m \implies y - y_1 = m(x - x_1)$$

Practice habit. Recite the three forms of a line until they are automatic: slope-intercept $y = mx + b$, point-slope $y - y_1 = m(x - x_1)$, and standard $Ax + By = C$ (coming in Lesson 5.5). After writing any line through given points, substitute every given point to check.

WORKED EXAMPLES

Example 1. Write an equation in point-slope form for the line through $(2, -3)$ with slope 4. Then write it in slope-intercept form.

Show solution

1. Identify: $m = 4$, $x_1 = 2$, $y_1 = -3$.
2. Substitute into $y - y_1 = m(x - x_1)$: $y - (-3) = 4(x - 2)$.
3. Simplify the left side: $y + 3 = 4(x - 2)$. This is point-slope form.
4. Distribute: $y + 3 = 4x - 8$.
5. Subtract 3 from both sides: $y = 4x - 11$.
6. Check the point $(2, -3)$: $4(2) - 11 = 8 - 11 = -3$. ✓

Answer: $y + 3 = 4(x - 2)$, or $y = 4x - 11$

Example 2. Ruth's bakery finds that the cost of making cakes is linear. Making 10 cakes costs \$85, and making 25 cakes costs \$160. Write an equation in slope-intercept form for the cost y of making x cakes.

Show solution

1. Write the two situations as points: $(10, 85)$ and $(25, 160)$.
2. Find the slope: $m = \frac{160 - 85}{25 - 10} = \frac{75}{15} = 5$. Each cake costs \$5 to make.
3. Use point-slope form with $(10, 85)$: $y - 85 = 5(x - 10)$.
4. Distribute: $y - 85 = 5x - 50$.
5. Add 85 to both sides: $y = 5x + 35$.
6. Check the other point: $5(25) + 35 = 125 + 35 = 160$. ✓
7. Meaning: \$5 per cake plus \$35 of fixed costs (such as running the oven) even before the first cake.

Answer: $y = 5x + 35$

PRACTICE

1. **Easy** Write an equation in point-slope form for the line through $(3, 1)$ with slope 2.

Show answer

$$y - 1 = 2(x - 3)$$

2. **Easy** Write an equation in point-slope form for the line through $(-4, 2)$ with slope -3 .

Show answer

$$y - 2 = -3(x + 4)$$

3. **Easy** Name a point on the line $y + 5 = \frac{1}{2}(x - 6)$ and give its slope.

Show answer

$$\text{Point } (6, -5), \text{ slope } \frac{1}{2}$$

4. **Easy** Write $y - 4 = 3(x - 2)$ in slope-intercept form.

Show answer

$$y - 4 = 3x - 6, \text{ so } y = 3x - 2$$

5. **Medium** Write an equation in slope-intercept form for the line through $(-2, -1)$ with slope $\frac{5}{2}$.

Show answer

$$y + 1 = \frac{5}{2}(x + 2), \text{ so } y + 1 = \frac{5}{2}x + 5 \text{ and } y = \frac{5}{2}x + 4$$

6. **Medium** Write an equation in slope-intercept form for the line through $(1, 3)$ and $(4, 9)$.

Show answer

$$m = \frac{9 - 3}{4 - 1} = 2; y - 3 = 2(x - 1), \text{ so } y = 2x + 1$$

7. **Medium** Write an equation in slope-intercept form for the line through $(-3, 4)$ and $(3, 0)$.

Show answer

$$m = \frac{0 - 4}{3 - (-3)} = -\frac{2}{3}; y - 0 = -\frac{2}{3}(x - 3), \text{ so } y = -\frac{2}{3}x + 2$$

8. **Medium** Write the equation of the line through $(5, -2)$ with slope 0.

Show answer

$$y + 2 = 0(x - 5), \text{ so } y = -2 \text{ (a horizontal line)}$$

9. **Medium** Samuel transplanted a tomato seedling. On day 4 it was 9 cm tall, and it grows 1.5 cm per day. Write an equation in slope-intercept form for its height h on day d .

Show answer

$$h - 9 = 1.5(d - 4), \text{ so } h = 1.5d + 3$$

10. **Hard** A moving-truck rental costs \$62 for 40 miles and \$83 for 100 miles, and the cost is linear in the miles driven. (a) Write an equation in slope-intercept form for the cost y of driving x miles. (b) Find the cost of driving 150 miles.

Show answer

$$\begin{aligned} \text{(a) } m &= \frac{83 - 62}{100 - 40} = \frac{21}{60} = 0.35; y - 62 = 0.35(x - 40), \text{ so } y = 0.35x + 48. \\ \text{(b) } &0.35(150) + 48 = 52.5 + 48 = 100.5, \text{ so } \$100.50. \end{aligned}$$

11. **Hard** Water heating in a pot warms at a steady rate. After 3 minutes it is 74°F , and after 8 minutes it is 104°F . (a) Write an equation for the temperature T after t minutes. (b) At this rate, after how many minutes does the water reach 212°F , its boiling point?

Show answer

(a) $m = \frac{104 - 74}{8 - 3} = 6$; $T - 74 = 6(t - 3)$, so $T = 6t + 56$. (b) $212 = 6t + 56$, so $6t = 156$ and $t = 26$ minutes.

12. **Hard** Write an equation in slope-intercept form for the line through $(-6, 1)$ and $(4, -4)$.

Show answer

$$m = \frac{-4 - 1}{4 - (-6)} = -\frac{1}{2}; y - 1 = -\frac{1}{2}(x + 6), \text{ so } y = -\frac{1}{2}x - 2$$

13. **Challenge** The point $(a, 11)$ lies on the line through $(-1, -1)$ and $(2, 5)$. Find a .

Show answer

$$m = \frac{5 - (-1)}{2 - (-1)} = 2, \text{ so the line is } y + 1 = 2(x + 1), \text{ or } y = 2x + 1. \text{ Then } 11 = 2a + 1, \text{ so } a = 5.$$

14. **Challenge** Explain where point-slope form comes from, and why the given point (x_1, y_1) satisfies $y - y_1 = m(x - x_1)$ even though it cannot be used in $\frac{y - y_1}{x - x_1} = m$.

Show answer

For any point (x, y) on the line other than (x_1, y_1) , the slope formula gives $\frac{y - y_1}{x - x_1} = m$; multiplying both sides by $x - x_1$ gives $y - y_1 = m(x - x_1)$. At (x_1, y_1) the fraction would be $\frac{0}{0}$, which is undefined, but in the multiplied form both sides are 0, so the point satisfies it.

COMMON MISTAKES

- Sign errors with negative coordinates: through $(-4, 2)$ the equation is $y - 2 = m(x + 4)$, not $y - 2 = m(x - 4)$. Write $x - (-4)$ first, then simplify.
- Reading the point backwards from the equation: $y + 5 = \frac{1}{2}(x - 6)$ goes through $(6, -5)$. Rewrite it as $y - (-5) = \frac{1}{2}(x - 6)$ to see this.
- Distributing m to only the first term: $3(x - 2) = 3x - 6$, not $3x - 2$.

- Not checking. Substitute each given point into the final equation; if one fails, look for a sign or arithmetic slip.

Make it a game · Pencil Through the Point. Put a button on a graph paper point such as (2, 3). The parent names a slope such as $-1/2$. Lay a pencil through the button at that slope, then write the equation in point-slope form. Check it by finding another point on the pencil and substituting. Then convert it to slope-intercept form. Do five points, one new slope each.

5.5 STANDARD FORM AND INTERCEPTS

OBJECTIVES

- Find the x - and y -intercepts of a line and use them to graph it.
- Write and recognize equations of horizontal lines $y = k$ and vertical lines $x = h$.
- Convert among standard form, slope-intercept form and point-slope form.
- Choose the most useful form of a line for a given problem and explain the choice.

WATCH

Sketching Linear Graphs by x/y-Intercepts Method (1 of 2: Explanation)

Eddie Woo · 8:00

<https://www.youtube.com/watch?v=sMe14ltgG4o>

WHEN TO WATCH

Before Example 1: graphing a line from its two intercepts.

Note: Eddie Woo teaches in Australia: his 'general form' is $ax + by + c = 0$, which is standard form with everything moved to one side.

BACKUP VIDEO

Equations of Straight Lines (2 of 2: General Form)

Eddie Woo · 7:28

<https://www.youtube.com/watch?v=Y26rz6iPxPw>

After Example 1: rewriting equations of lines into one form.

KEY TERMS

standard form

The form $Ax + By = C$ of a linear equation, where A , B and C are integers, A and B are not both 0, and usually $A \geq 0$.

x -intercept

The x -coordinate of the point where a graph crosses the x -axis. At that point $y = 0$.

y-intercept

The y -coordinate of the point where a graph crosses the y -axis. At that point $x = 0$.

horizontal line

A flat line $y = k$; every point on it has the same y -coordinate k , and its slope is 0.

vertical line

An up-and-down line $x = h$; every point on it has the same x -coordinate h , and its slope is undefined.

CORE IDEAS

A third common way to write a line is **standard form**, $Ax + By = C$, such as $3x - 4y = 12$. By custom, A , B and C are integers and A is not negative; for a bare equation we also divide out any common factor, so $6x - 8y = 24$ is usually written $3x - 4y = 12$. In a word problem it is fine to keep the coefficients that come from the prices, such as $4x + 10y = 60$, because each number then has a meaning. Standard form keeps x and y side by side, which fits situations where two amounts share a total: 4 dollars per flower times x flowers plus 10 dollars per shrub times y shrubs equals 60 dollars is $4x + 10y = 60$.

Intercepts. Every point on the x -axis has y -coordinate 0, so to find where a line crosses the x -axis, set $y = 0$ and solve for x . Every point on the y -axis has x -coordinate 0, so to find the y -intercept, set $x = 0$ and solve for y . In standard form this is quick, because one term simply disappears. For $3x - 4y = 12$: $y = 0$ gives $3x = 12$, so $x = 4$; $x = 0$ gives $-4y = 12$, so $y = -3$. Plot $(4, 0)$ and $(0, -3)$ and draw the line through them. (Two points determine a line, but a third point is a good check.)

Horizontal and vertical lines. The line through $(2, -4)$ and $(7, -4)$ has every y -coordinate equal to -4 , so its equation is simply $y = -4$; its slope is $\frac{0}{5} = 0$, and indeed $y = 0x - 4$. The line through $(-3, 1)$ and $(-3, 7)$ has every x -coordinate equal to -3 , so its equation is $x = -3$. Its run is 0, so its slope is undefined, and it cannot be written as $y = mx + b$: the single input $x = -3$ has infinitely many outputs, so it is not a function (it fails the vertical line test from Lesson 4.2). Standard form handles both kinds: $y = -4$ is $0x + 1y = -4$ and $x = -3$ is $1x + 0y = -3$.

Converting. To go from standard form to slope-intercept form, solve for y : $5x + 2y = 8$ becomes $2y = -5x + 8$, then $y = -\frac{5}{2}x + 4$. This also shows that the slope of $Ax + By = C$ is $-\frac{A}{B}$ when $B \neq 0$. To go the other way, collect x and y on the left and clear fractions by multiplying by the LCD: $y = \frac{2}{3}x - 4$ becomes $3y = 2x - 12$, then $-2x + 3y = -12$, and multiplying by -1 gives $2x - 3y = 12$. A point-slope equation converts the same way after distributing.

Which form when? Use slope-intercept form $y = mx + b$ when you want to graph from the slope or read off a rate and a starting value. Use point-slope form $y - y_1 = m(x - x_1)$ when you know a point and the slope (or two points) and need to write the equation. Use standard form $Ax + By = C$ when you want both intercepts quickly, when the situation is "two kinds of things that add to a total", or when the line is vertical. All three describe the same line, so you can always convert.

KEY FORMULAS

$$Ax + By = C$$

x -intercept: set $y = 0$; y -intercept: set $x = 0$

horizontal: $y = k$ ($m = 0$); vertical: $x = h$ (m undefined)

Practice habit. Cover-up method for intercepts: in $Ax + By = C$, cover the y -term with a finger and solve what is left for the x -intercept; then cover the x -term for the y -intercept. For $3x - 4y = 12$: $3x = 12$ gives 4, and $-4y = 12$ gives -3 .

WORKED EXAMPLES

Example 1. Find the intercepts of $3x - 4y = 12$ and use them to graph the line.

Show solution

1. x -intercept: set $y = 0$. Then $3x - 4(0) = 12$, so $3x = 12$ and $x = 4$. Point $(4, 0)$.
2. y -intercept: set $x = 0$. Then $3(0) - 4y = 12$, so $-4y = 12$ and $y = -3$. Point $(0, -3)$.
3. Plot $(4, 0)$ and $(0, -3)$ and draw the line through them.
4. Find a third point as a check: the slope from $(0, -3)$ to $(4, 0)$ is $\frac{3}{4}$, so $(8, 3)$ should be on the line.
5. Check $(8, 3)$: $3(8) - 4(3) = 24 - 12 = 12$. ✓

Answer: x -intercept 4, y -intercept -3 ; the graph is the line through $(4, 0)$ and $(0, -3)$, slope $\frac{3}{4}$.

Example 2. Grace has \$60 to spend on plants for the church garden. Flowering plants cost \$4 each and shrubs cost \$10 each. Write an equation in standard form if she spends all \$60 on x flowering plants and y shrubs. Then find both intercepts and say what they mean.

Show solution

1. Cost of flowers: $4x$. Cost of shrubs: $10y$. Together they make \$60.
2. Equation: $4x + 10y = 60$.
3. x -intercept: set $y = 0$. Then $4x = 60$, so $x = 15$: she could buy 15 flowering plants and no shrubs.
4. y -intercept: set $x = 0$. Then $10y = 60$, so $y = 6$: she could buy 6 shrubs and no flowering plants.
5. Check a point between them, $(5, 4)$: $4(5) + 10(4) = 20 + 40 = 60$. ✓ So 5 flowering plants and 4 shrubs also use exactly \$60.
6. Only whole-number points on the line make sense here, since she cannot buy part of a plant.

Answer: $4x + 10y = 60$; x -intercept 15 (15 flowering plants, no shrubs), y -intercept 6 (6 shrubs, no flowering plants).

PRACTICE

1. Easy Find the x - and y -intercepts of $2x + 5y = 10$.

Show answer

$y = 0$: $2x = 10$, $x = 5$. $x = 0$: $5y = 10$, $y = 2$. x -intercept 5, y -intercept 2.

2. Easy Find the x - and y -intercepts of $x - 3y = 9$.

Show answer

x -intercept 9; $-3y = 9$ gives y -intercept -3 .

3. Easy Write the equation of the horizontal line through $(2, -4)$.

Show answer

$$y = -4$$

4. Easy Write the equation of the vertical line through $(-3, 7)$.

Show answer

$$x = -3$$

5. Medium Write $y = \frac{2}{3}x - 4$ in standard form with integer coefficients.

Show answer

Multiply by 3: $3y = 2x - 12$, so $-2x + 3y = -12$, or $2x - 3y = 12$.

6. Medium Write $5x + 2y = 8$ in slope-intercept form.

Show answer

$$2y = -5x + 8, \text{ so } y = -\frac{5}{2}x + 4$$

7. Medium Find the intercepts of $-4x + 6y = 24$, then describe its graph.

Show answer

x -intercept -6 , y -intercept 4; the line through $(-6, 0)$ and $(0, 4)$, rising from left to right with slope $\frac{2}{3}$.

8. Medium Find the slope of the line $6x + 3y = 15$.

Show answer

$3y = -6x + 15$, so $y = -2x + 5$ and the slope is -2 .

9. **Medium** Write the equation of the line through $(2, 5)$ and $(2, -1)$.

Show answer

Both points have $x = 2$, so the line is vertical: $x = 2$.

10. **Hard** At a choir concert, adult tickets cost \$8 and student tickets cost \$5. Ticket sales totaled \$400. (a) Write an equation in standard form using a adult tickets and s student tickets. (b) If 30 adult tickets were sold, how many student tickets were sold?

Show answer

(a) $8a + 5s = 400$; (b) $8(30) + 5s = 400$, so $240 + 5s = 400$, $5s = 160$ and $s = 32$ student tickets.

11. **Hard** Write an equation in standard form for the line through $(-2, 3)$ and $(4, -1)$.

Show answer

$m = \frac{-1 - 3}{4 - (-2)} = -\frac{2}{3}$; $y - 3 = -\frac{2}{3}(x + 2)$. Multiply by 3: $3y - 9 = -2x - 4$, so $2x + 3y = 5$.

12. **Hard** Farmer Miller spends \$180 on feed. Corn costs \$12 per bag and oats cost \$15 per bag, so $12c + 15t = 180$, where c is bags of corn and t is bags of oats. Find the c -intercept and the t -intercept.

Show answer

c -intercept: $12c = 180$, so $c = 15$ (15 bags of corn, no oats). t -intercept: $15t = 180$, so $t = 12$ (12 bags of oats, no corn).

13. **Hard** Ruth spends \$24 on trail mix ingredients for a family hike: raisins at \$3 per pound and almonds at \$8 per pound. If she buys 2.4 pounds of almonds, how many pounds of raisins does she buy?

Show answer

$3r + 8a = 24$ with $a = 2.4$: $3r + 19.2 = 24$, so $3r = 4.8$ and $r = 1.6$ pounds.

14. **Challenge** Write an equation in standard form for the line with x -intercept 6 and y -intercept -4 .

Show answer

Points $(6, 0)$ and $(0, -4)$: $m = \frac{-4 - 0}{0 - 6} = \frac{2}{3}$, so $y = \frac{2}{3}x - 4$, which is
 $2x - 3y = 12$.

15. **Challenge** Explain why you find the x -intercept by setting $y = 0$, and why a vertical line cannot be written in slope-intercept form.

Show answer

The x -intercept is on the x -axis, and every point on the x -axis has y -coordinate 0. A vertical line has run 0, so its slope is undefined and there is no number m to use; also, one x -value has many y -values, so y cannot equal a single expression $mx + b$.

COMMON MISTAKES

- Setting the wrong variable to 0. For the x -intercept set $y = 0$; for the y -intercept set $x = 0$. Ask yourself, "Which coordinate is 0 on this axis?"
- Mixing up $x = 3$ and $y = 3$. $x = 3$ is vertical (all points have x equal to 3); $y = 3$ is horizontal.
- Reading the slope of $Ax + By = C$ as A . Solve for y first; the slope is $-\frac{A}{B}$.
- Leaving fractions in standard form. Multiply every term by the LCD so that A , B and C are integers.

Make it a game • Intercept Sprint. Use the cover-up method. The parent writes an equation such as $3x + 4y = 12$. In 30 seconds, find both intercepts and sketch the line. Between rounds the parent calls ' $y = -2$ ' or ' $x = 5$ ', and you show the line with your arm, flat for horizontal and straight up for vertical. Five lines in a row earns a victory lap.

5.6 PARALLEL AND PERPENDICULAR LINES

OBJECTIVES

- Decide from their slopes whether two lines are parallel, perpendicular or neither.
- Explain why parallel lines have equal slopes and why perpendicular slopes multiply to -1 , using a slope triangle turned 90° .
- Write the equation of a line through a given point that is parallel or perpendicular to a given line.

WATCH

Perpendicular Lines (2 of 2: Understanding the relationship)

Eddie Woo · 6:35

<https://www.youtube.com/watch?v=JEKkpGTtEA>

WHEN TO WATCH

Before the perpendicular-lines example: why perpendicular slopes are negative reciprocals.

Note: Parallel lines (equal slopes) are taught from the Core ideas. Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Perpendicular Lines (1 of 2: Using congruent triangles)

Eddie Woo · 6:54

<https://www.youtube.com/watch?v=Ut4bl3m95xU>

Optional, after Example 2: a congruent-triangles proof of the same fact.

KEY TERMS

parallel lines

Lines in the same plane that never meet. Non-vertical parallel lines have equal slopes and different y -intercepts.

perpendicular lines

Lines that meet at a right angle (90°).

opposite reciprocal

The number you get by flipping a fraction and changing its sign. The opposite reciprocal of $\frac{2}{5}$ is $-\frac{5}{2}$; of 3 it is $-\frac{1}{3}$.

right angle

An angle of 90° , a quarter turn, like the corner of a square.

CORE IDEAS

Parallel lines have equal slopes. Slope measures steepness and direction, so two lines that run side by side forever must have the same slope. The algebra shows why such lines never meet: take $y = 2x + 1$ and $y = 2x - 6$. At every x , the first output minus the second is $(2x + 1) - (2x - 6) = 7$. The gap is always 7, never 0, so the lines never cross. If the slopes were different, the gap would change by the difference of the slopes for every step to the right, so by moving right or left far enough it would shrink to 0, and the lines would meet. Two equations with the same slope and the same intercept are the same line, not two parallel lines. All vertical lines ($x = h$) are parallel to one another too, even though their slopes are undefined.

Perpendicular lines have slopes that multiply to -1 . Here is why. Draw a slope triangle for a line: start at a point, go right a units (the run) and up b units (the rise), so the slope is $\frac{b}{a}$. Now turn the whole picture a quarter turn (90°) counterclockwise around the starting point. The run arrow, which pointed right a units, now points up a units. The rise arrow, which pointed up b units, now points left b units. The turned line is perpendicular to the original, because a quarter turn is a right angle. Its new run is $-b$ (left) and its new rise is a (up), so its slope is $\frac{a}{-b} = -\frac{a}{b}$. Multiply the two slopes: $\frac{b}{a} \cdot \left(-\frac{a}{b}\right) = -1$.

So to get a perpendicular slope, **flip the fraction and change the sign** (take the opposite reciprocal). Slope 3 becomes $-\frac{1}{3}$; slope $-\frac{2}{5}$ becomes $\frac{5}{2}$. One special case: horizontal lines (slope 0) and vertical lines (undefined slope) are perpendicular, but their slopes cannot be multiplied, so the rule "product = -1 " applies only when neither line is vertical or horizontal.

Writing equations. To write a line through a given point parallel or perpendicular to a given line: (1) find the slope of the given line, solving for y first if it is in standard form; (2) keep the slope for a parallel line, or take the opposite reciprocal for a

perpendicular line; (3) use point-slope form with the given point; (4) convert to the form asked for and check the point.

Testing figures. Slopes let you check corners and sides on a coordinate grid. If two sides of a triangle have slopes whose product is -1 , the triangle has a right angle where those sides meet. A builder or surveyor laying out a garden or a foundation on a grid can use this to confirm a square corner.

KEY FORMULAS

parallel: $m_1 = m_2$

perpendicular: $m_1 \cdot m_2 = -1$, so $m_2 = -\frac{1}{m_1}$

Euclid connection. Euclid's fifth postulate says that if a line crossing two other lines makes the two interior angles on one side add to less than two right angles, those two lines, extended far enough, meet on that side. A form equivalent to it, often taught in school, is that through a point not on a line there is exactly one line parallel to the given line, which is why each problem in this lesson that asks for a parallel line through a given point has exactly one answer.

Practice habit. Say the rule out loud as you work: "parallel: same slope; perpendicular: flip and switch the sign." Then check: multiply the two slopes, and the product should be -1 for perpendicular lines.

WORKED EXAMPLES

Example 1. Write an equation in slope-intercept form for the line through $(1, 5)$ that is parallel to $y = 3x - 2$.

Show solution

1. The given line has slope 3, so a parallel line also has slope 3.
2. Use point-slope form with $(1, 5)$: $y - 5 = 3(x - 1)$.
3. Distribute: $y - 5 = 3x - 3$.
4. Add 5: $y = 3x + 2$.
5. Check the point: $3(1) + 2 = 5$. ✓ The y -intercept is 2, not -2 , so the lines are different and parallel.

Answer: $y = 3x + 2$

Example 2. On a town map with a grid measured in blocks, Mill Road runs along the line $2x - 5y = 10$. The town wants a footpath from the church at $(4, -1)$ that meets Mill Road at a right angle. Write the equation of the footpath's line in slope-intercept form.

Show solution

1. Find the slope of Mill Road by solving for y : $-5y = -2x + 10$.
2. Divide by -5 : $y = \frac{2}{5}x - 2$, so the road's slope is $\frac{2}{5}$.
3. Perpendicular slope: flip and change the sign to get $-\frac{5}{2}$. Check: $\frac{2}{5} \cdot \left(-\frac{5}{2}\right) = -1$. ✓
4. Point-slope form with $(4, -1)$: $y + 1 = -\frac{5}{2}(x - 4)$.
5. Distribute: $y + 1 = -\frac{5}{2}x + 10$.
6. Subtract 1: $y = -\frac{5}{2}x + 9$.
7. Check the church: $-\frac{5}{2}(4) + 9 = -10 + 9 = -1$. ✓

Answer: $y = -\frac{5}{2}x + 9$

PRACTICE

1. Easy What is the slope of a line parallel to $y = -4x + 1$?

Show answer

$$-4$$

2. Easy What is the slope of a line perpendicular to $y = 3x - 7$?

Show answer

$$-\frac{1}{3}$$

3. Easy What is the slope of a line perpendicular to $y = -\frac{2}{5}x + 3$?

Show answer

$$\frac{5}{2}$$

4. **Easy** Are the lines $y = 2x + 1$ and $y = 2x - 6$ parallel, perpendicular or neither?

Show answer

Parallel: both slopes are 2 and the y -intercepts differ.

5. **Medium** Are the lines $y = 4x - 3$ and $y = -\frac{1}{4}x + 2$ parallel, perpendicular or neither?

Show answer

Perpendicular: $4 \cdot \left(-\frac{1}{4}\right) = -1$.

6. **Medium** Are the lines $3x + y = 5$ and $6x + 2y = 7$ parallel, perpendicular or neither?

Show answer

Parallel: $y = -3x + 5$ and $y = -3x + \frac{7}{2}$ have the same slope -3 and different intercepts.

7. **Medium** Write an equation in slope-intercept form for the line through $(3, 4)$ parallel to $y = -2x + 7$.

Show answer

$$y - 4 = -2(x - 3), \text{ so } y = -2x + 10$$

8. **Medium** Write an equation in slope-intercept form for the line through $(2, -5)$ perpendicular to $y = \frac{1}{3}x + 4$.

Show answer

$$\text{Slope } -3: y + 5 = -3(x - 2), \text{ so } y = -3x + 1$$

9. **Medium** Write the equation of the line through $(-2, 6)$ perpendicular to the vertical line $x = 4$.

Show answer

A line perpendicular to a vertical line is horizontal: $y = 6$

10. **Hard** Write an equation in slope-intercept form for the line through $(-1, 3)$ parallel to $4x - 2y = 9$.

Show answer

$$-2y = -4x + 9 \text{ gives } y = 2x - \frac{9}{2}, \text{ slope } 2. \text{ Then } y - 3 = 2(x + 1), \text{ so } y = 2x + 5.$$

11. **Hard** On a field map measured in yards, a fence runs along $y = \frac{3}{4}x + 2$. Farmer Miller wants a second fence parallel to it that passes through the well at $(8, 1)$. Write its equation in slope-intercept form.

Show answer

$$y - 1 = \frac{3}{4}(x - 8), \text{ so } y - 1 = \frac{3}{4}x - 6 \text{ and } y = \frac{3}{4}x - 5$$

12. **Hard** On a carpenter's drawing, a roof rafter runs along the line through $(0, 0)$ and $(12, 9)$. A support brace must meet the rafter at a right angle at the point $(12, 9)$. Write the brace's equation in slope-intercept form.

Show answer

$$\text{Rafter slope } \frac{9}{12} = \frac{3}{4}, \text{ so brace slope } -\frac{4}{3}. y - 9 = -\frac{4}{3}(x - 12), \text{ so } y - 9 = -\frac{4}{3}x + 16 \text{ and } y = -\frac{4}{3}x + 25.$$

13. **Hard** Henry marks the corners of a triangular garden plot at $A(1, 1)$, $B(4, 3)$ and $C(2, 6)$ on a grid in meters. Is the corner at B a right angle? Answer yes or no.

Show answer

$$\text{Yes. Slope of } AB = \frac{3 - 1}{4 - 1} = \frac{2}{3}; \text{ slope of } BC = \frac{6 - 3}{2 - 4} = -\frac{3}{2}; \text{ the product is } -1.$$

14. **Challenge** Explain why the slopes of two perpendicular lines (neither one vertical) multiply to -1 .

Show answer

Take a slope triangle with run a and rise b , so the slope is $\frac{b}{a}$. Turning it 90° counterclockwise makes the run point up a and the rise point left b , so the perpendicular line has run $-b$ and rise a , slope $-\frac{a}{b}$. Then $\frac{b}{a} \cdot \left(-\frac{a}{b}\right) = -1$.

15. **Challenge** Find a so that the line through $(1, 2)$ and $(3, a)$ is perpendicular to $y = -\frac{2}{3}x + 1$.

Show answer

The needed slope is $\frac{3}{2}$, so $\frac{a - 2}{3 - 1} = \frac{3}{2}$, giving $a - 2 = 3$ and $a = 5$.

COMMON MISTAKES

- Flipping without changing the sign (or changing the sign without flipping). The perpendicular slope to $\frac{2}{5}$ is $-\frac{5}{2}$, not $\frac{5}{2}$ or $-\frac{2}{5}$. Check that the product is -1 .
- Reading the slope from standard form without solving for y . In $4x - 2y = 9$ the slope is 2, not 4.
- Using the given line's y -intercept for the new line. The new line goes through the given point; use point-slope form with that point.
- Calling two equations for the same line "parallel." Parallel lines have the same slope but different intercepts.

Make it a game · Slope Partner Reflex. Write slopes on ten cards, such as $2/3$, -4 and $1/5$. The parent flashes a card and you say the parallel slope and the perpendicular slope in three seconds. A wrong answer sends you back to the start of the pile. Once finished, turn an index card by 90 degrees over a slope triangle to see why perpendicular slopes multiply to -1 .

5.7 LINEAR MODELS

OBJECTIVES

- Decide whether a linear model fits a situation or a table by checking for a constant rate of change.
- Write a linear function from a verbal description, a table or two data points.
- Interpret the slope and intercept of a linear model with units, and use the model to make predictions.
- Write an arithmetic sequence as a linear function of n .

WATCH

Linear Modelling (1 of 3: Finding the constancy of 'm' in linear functions)

Eddie Woo · 5:32

<https://www.youtube.com/watch?v=2ujC6TR2laE>

WHEN TO WATCH

Before Example 1: checking that a situation has a constant rate of change.

Note: Eddie Woo teaches in Australia: his prices are in Australian dollars.

BACKUP VIDEO

Linear Modelling (3 of 3: Using the Gradient to find a relation between two variables)

Eddie Woo · 8:46

<https://www.youtube.com/watch?v=vDzDShA39DY>

Before Example 2: using the rate of change to write a linear model.

KEY TERMS

linear model

A linear equation or function, such as $y = mx + b$, used to describe a real situation and make predictions.

constant rate of change

A rate that stays the same: the output changes by the same amount for every equal change in the input. This is what makes a situation linear.

initial value

The value of the output when the input is 0; in $y = mx + b$ it is b .

prediction

An output value found by substituting an input into a model, often for an input not in the data.

CORE IDEAS

A **linear model** is a line used to describe something real: the cost of a rental, the water in a barrel, the money in a building fund. A linear model makes sense exactly when the situation has a **constant rate of change**. If a barrel gains 4 gallons every hour, the amount is linear; if a population doubles every year, it is not, because the amount added keeps growing (that kind of growth comes in Unit 7).

From words: look for a starting amount and a rate. "A \$25 sign-up fee plus \$30 per month" gives $C = 30n + 25$: the rate is the slope and the starting amount is the intercept.

From a table: check that equal steps in the input give equal steps in the output. For x : 0, 2, 4, 6 and y : 5, 13, 21, 29, each step of 2 in x adds 8 to y , so the rate is $\frac{8}{2} = 4$, and the value at $x = 0$ is 5, giving $y = 4x + 5$. If the inputs are not evenly spaced, compute $\frac{\text{change in } y}{\text{change in } x}$ for each pair of neighboring rows; the model is linear only if every one of these rates is the same.

From two data points: find the slope with the slope formula, then use point-slope form (Lesson 5.4) and simplify. Always say what the slope and intercept mean, with units. In a model of a building fund, $y = 500x + 2600$ (dollars after x months), the slope means the fund grows \$500 per month and the intercept means it started with \$2,600.

Predicting and its limits. To predict, substitute an input and compute; to find when something happens, set the output equal to a value and solve. But a model is only trustworthy where the constant rate really holds. A child who grows 2.5 inches a year will not keep that up for 30 years, and a draining tank cannot hold a negative number of gallons. Choose a sensible domain, and be cautious about predictions far outside your data.

Arithmetic sequences are linear functions. In Lesson 4.5 you wrote

$a_n = a_1 + (n - 1)d$. Distribute and rearrange: $a_n = dn + (a_1 - d)$. That is the form

$y = mx + b$ with slope $m = d$, the common difference, and intercept $b = a_1 - d$, the term that would come just before the first. The only difference is the domain: a sequence uses only the counting numbers $n = 1, 2, 3, \dots$, so its graph is a row of separate dots lying on a line.

KEY FORMULAS

$$y = mx + b \quad (m = \text{rate of change, } b = \text{initial value})$$

$$a_n = a_1 + (n - 1)d = dn + (a_1 - d)$$

Practice habit. Estimate first, then compute. Before solving, guess the answer roughly ("about \$500 a month, so \$7,500 more takes about 15 months") and compare your exact answer with the guess; a big disagreement means look again.

WORKED EXAMPLES

Example 1. Rain fills a barrel at a steady rate. A table shows hours x : 0, 2, 4, 6 and gallons y : 5, 13, 21, 29. Write a linear model and predict the amount after 8 hours.

Show solution

1. Check the rate: each 2-hour step adds $13 - 5 = 8$, $21 - 13 = 8$ and $29 - 21 = 8$ gallons. The rate is constant, so a linear model fits.
2. Slope: $m = \frac{8}{2} = 4$ gallons per hour.
3. Initial value: at $x = 0$, $y = 5$, so $b = 5$ (the barrel already had 5 gallons).
4. Model: $y = 4x + 5$.
5. Check a table row: $4(6) + 5 = 29$. ✓
6. Predict: $y = 4(8) + 5 = 37$ gallons.

Answer: $y = 4x + 5$; about 37 gallons after 8 hours.

Example 2. A church building fund grows at a steady rate. Four months after the fund began it held \$4,600, and nine months after it began it held \$7,100. Write a linear model for the amount y after x months, explain the slope and intercept, and find when the fund will reach its goal of \$10,100.

Show solution

1. Points: $(4, 4600)$ and $(9, 7100)$.
2. Slope: $m = \frac{7100 - 4600}{9 - 4} = \frac{2500}{5} = 500$.
3. Point-slope form: $y - 4600 = 500(x - 4)$.
4. Distribute and add 4600: $y = 500x - 2000 + 4600 = 500x + 2600$.
5. Check the other point: $500(9) + 2600 = 4500 + 2600 = 7100$. ✓
6. Meaning: the fund grows \$500 per month (slope) and started with \$2,600 (intercept).
7. Goal: $10100 = 500x + 2600$, so $500x = 7500$ and $x = 15$.
8. Check: $500(15) + 2600 = 10100$. ✓

Answer: $y = 500x + 2600$; \$500 is added each month and the fund began with \$2,600; it reaches \$10,100 after 15 months.

PRACTICE

1. **Easy** A fitness club charges a \$25 sign-up fee plus \$30 per month. Write a model for the total cost C after n months.

Show answer

$$C = 30n + 25$$

2. **Easy** A table shows x : 0, 1, 2, 3 and y : 12, 9, 6, 3. Write a linear equation for the table.

Show answer

Rate -3 , starting value 12: $y = -3x + 12$

3. **Easy** A table shows x : 0, 1, 2, 3 and y : 2, 4, 8, 16. Is a linear model appropriate? Answer yes or no.

Show answer

No. The changes are 2, 4 and 8, not constant (each y doubles instead).

4. **Easy** The cost of renting a truck is $y = 0.15x + 40$ dollars for x miles. What do 0.15 and 40 mean?

Show answer

0.15 is the rate, \$0.15 per mile driven; 40 is the fixed charge, \$40 even if the truck is driven 0 miles.

5. **Medium** Write the arithmetic sequence 7, 11, 15, 19, ... as a linear function of n .

Show answer

$$a_n = 7 + (n - 1)4 = 4n + 3$$

6. **Medium** A table shows x : 1, 3, 5, 7 and y : 10, 16, 22, 28. Write a linear equation for the table.

Show answer

$$\text{Slope } \frac{6}{2} = 3; y - 10 = 3(x - 1), \text{ so } y = 3x + 7$$

7. **Medium** A stock tank on a ranch drains at a steady rate. After 2 minutes it holds 410 gallons, and after 7 minutes it holds 285 gallons. (a) Write a linear model for the gallons y after x minutes. (b) When will the tank be empty?

Show answer

$$\begin{aligned} \text{(a) } m &= \frac{285 - 410}{7 - 2} = -25; y - 410 = -25(x - 2), \text{ so } y = -25x + 460. \text{ (b)} \\ 0 &= -25x + 460, \text{ so } x = 18.4 \text{ minutes.} \end{aligned}$$

8. **Medium** Grace sells bread at a farmers market. She pays a \$20 booth fee and makes \$4 on each loaf she sells. (a) Write a model for her profit P after selling L loaves. (b) How many loaves must she sell to make a profit of \$100?

Show answer

$$\text{(a) } P = 4L - 20; \text{ (b) } 100 = 4L - 20, \text{ so } 4L = 120 \text{ and } L = 30 \text{ loaves.}$$

9. **Medium** A linear function f has $f(2) = 7$ and $f(6) = 19$. Write $f(x)$ and find $f(10)$.

Show answer

$$m = \frac{19 - 7}{6 - 2} = 3, \text{ so } f(x) = 3x + 1 \text{ and } f(10) = 31.$$

10. **Hard** On a mountain drive, the Miller family notes that it is 68°F at an elevation of 1,000 feet and 54°F at 5,000 feet. Assume temperature is linear in elevation. (a) Write a model for the temperature T at elevation x feet. (b) Predict the temperature at 8,000 feet.

Show answer

$$(a) m = \frac{54 - 68}{5000 - 1000} = -0.0035 \text{ (a drop of } 3.5^{\circ} \text{ per 1,000 feet);}$$
$$T = -0.0035x + 71.5. (b) -0.0035(8000) + 71.5 = -28 + 71.5 = 43.5^{\circ}\text{F.}$$

11. **Hard** Samuel is 54 inches tall at age 10 and has been growing about 2.5 inches per year. The model $h = 2.5(a - 10) + 54$ predicts his height at age 40 to be 129 inches (10 feet 9 inches). Explain what went wrong.

Show answer

The model assumes a constant rate of growth forever, but people stop growing in their late teens or early twenties. A linear model fits only over the years when the rate really is about constant, so predicting far outside the data gives nonsense.

12. **Hard** A bakery's daily cost is linear in the number of loaves baked: 50 loaves cost \$110 and 120 loaves cost \$187. (a) Write a model for the cost y of baking x loaves. (b) Find the cost of baking 200 loaves.

Show answer

$$(a) m = \frac{187 - 110}{120 - 50} = \frac{77}{70} = 1.1; y - 110 = 1.1(x - 50), \text{ so } y = 1.1x + 55 \text{ (\$1.10 per loaf plus \$55 of fixed daily costs).}$$
$$(b) 1.1(200) + 55 = 275, \text{ so \$275.}$$

13. **Challenge** An arithmetic sequence has $a_4 = 23$ and $a_{10} = 53$. (a) Write a_n as a linear function of n . (b) Find a_{25} .

Show answer

$$(a) d = \frac{53 - 23}{10 - 4} = 5 \text{ and } a_1 = 23 - 3(5) = 8, \text{ so } a_n = 8 + 5(n - 1) = 5n + 3. (b)$$
$$a_{25} = 5(25) + 3 = 128.$$

14. **Challenge** Ruth measures a bean plant: week 1, 3.2 cm; week 3, 6.0 cm; week 4, 7.4 cm; week 6, 10.2 cm. (a) Show that a linear model fits and write it. (b) Predict the height in week 10.

Show answer

(a) Rates: $\frac{2.8}{2} = 1.4$, $\frac{1.4}{1} = 1.4$, $\frac{2.8}{2} = 1.4$ cm per week, all equal;
 $y - 3.2 = 1.4(x - 1)$, so $y = 1.4x + 1.8$. (b) $1.4(10) + 1.8 = 15.8$ cm.

COMMON MISTAKES

- Using a linear model when the rate is not constant. Check the differences (or the rates, if the inputs are unevenly spaced) before writing $y = mx + b$.
- Treating the first value in a table as the intercept when the table does not start at $x = 0$. In x : 1, 3, 5, 7 and y : 10, 16, 22, 28, the intercept is 7, not 10; use point-slope form.
- Forgetting units when interpreting. Say "\$1.10 per loaf" and "\$55 of fixed cost," not just "1.1 and 55."
- Trusting predictions far beyond the data. Ask whether the constant rate could really continue that long.

Make it a game • Beans in the Jar. Start a jar with 12 beans and add 3 more each minute, using a timer. After four rounds, write the linear model from the table, naming the initial value and constant rate of change. Before counting further, predict the beans at minute 25. Then keep adding to test it. Make up a second jar with a different start and rate.

LINEAR FUNCTIONS

This unit is about straight lines: the simplest functions and among the most useful. You will learn to measure how steep a line is (its slope), see why every pair of points on a line gives the same slope, and write equations of lines in three forms: slope-intercept $y = mx + b$, point-slope $y - y_1 = m(x - x_1)$ and standard $Ax + By = C$. You will also learn what slope says about parallel and perpendicular lines, and why.

Lines describe any situation with a constant rate of change: wages by the hour, a fee plus a charge per mile, water filling a barrel, a building fund that grows by the same amount each month. In the last lesson you will build linear models from words, tables and data, interpret the slope and intercept with units, and make predictions while thinking about when a line is a sensible model.

This unit builds on the coordinate plane, functions and arithmetic sequences from Unit 4 and the equation-solving skills of Units 2 and 3. Everything here is used right away in Unit 6, where two lines are solved together as a system, and again in Unit 13, where lines are fitted to data. Geometry will also use slopes to prove that lines are parallel or perpendicular.

PREREQUISITES

- Plot and read ordered pairs in all four quadrants (Lesson 4.1).
- Add, subtract, multiply and divide integers and fractions, including subtracting negatives.
- Solve multi-step linear equations and solve a formula for one variable (Lessons 2.2–2.5).
- Use function notation and evaluate a function (Lesson 4.3).
- Write the explicit formula of an arithmetic sequence (Lesson 4.5).

MASTERY CHECKLIST

- ☐ I can find a rate of change from a table, a graph or a word situation, with units.
- ☐ I can find the slope through two points and tell whether it is positive, negative, zero or undefined.
- ☐ I can explain, using similar slope triangles, why a line has the same slope between any two of its points.
- ☐ I can recognize a direct variation $y = kx$ and use it to solve proportion problems.
- ☐ I can graph $y = mx + b$ from the slope and y -intercept and explain why b is the y -intercept.
- ☐ I can derive point-slope form from the slope formula and write the equation of a line through a point with a given slope or through two points.
- ☐ I can find intercepts from standard form, graph with them and write horizontal and vertical lines.
- ☐ I can convert among slope-intercept, point-slope and standard

form and choose the most useful form.

- ☐ I can write equations of lines parallel or perpendicular to a given line and explain why perpendicular slopes multiply to -1 .
- ☐ I can write a linear model from words, a table or two data points, interpret its slope and intercept, and judge when a linear model makes sense.

LESSONS IN THIS UNIT

5.1	Rate of Change and Slope	2 sessions of 45 minutes
5.2	Direct Variation	45 minutes
5.3	Slope-Intercept Form	2 sessions of 45 minutes
5.4	Point-Slope Form	2 sessions of 45 minutes
5.5	Standard Form and Intercepts	2 sessions of 45 minutes
5.6	Parallel and Perpendicular Lines	2 sessions of 45 minutes
5.7	Linear Models	2 sessions of 45 minutes

UNIT 5 TEST

Time: 50 minutes. No calculator. Show your work on every problem. Give exact answers, using fractions in lowest terms rather than rounded decimals, unless a problem says otherwise. Write equations of lines in the form each problem asks for, and include units in answers to word problems.

1. **5 pts**

Find the slope of the line through $(-1, 6)$ and $(3, -2)$.

2. **5 pts**

A table shows x : 0, 3, 6, 9 and y : 4, 6, 8, 10. Find the rate of change.

3. **6 pts**

y varies directly with x , and $y = 14$ when $x = 4$. Find y when $x = 10$.

4. **7 pts**

Samuel's pay for yard work varies directly with the hours he works. He earned \$52 for 4 hours. How much will he earn for 11 hours?

5. **6 pts**

Give the slope and y -intercept of $y = -\frac{1}{4}x + 6$, and describe its graph.

6. **7 pts**

Write $4x - 3y = 9$ in slope-intercept form.

7. **7 pts**

Write an equation in slope-intercept form for the line through $(-3, 2)$ with slope $\frac{2}{3}$.

8. **8 pts**

Write an equation in slope-intercept form for the line through $(-2, 9)$ and $(4, -3)$.

9. **7 pts**

Find the x - and y -intercepts of $5x - 2y = 20$.

10. **6 pts**

Write the equations of the horizontal line and the vertical line that pass through $(6, -1)$.

11. **8 pts**

Write an equation in slope-intercept form for the line through $(8, 3)$ perpendicular to $y = 4x - 1$.

12. 7 pts

Write an equation in slope-intercept form for the line through $(2, -1)$ parallel to $3x + 2y = 4$.

13. 8 pts

The cost of renting a hall for a church potluck is linear in the number of guests. For 40 guests it costs \$310, and for 70 guests it costs \$430. (a) Write a model for the cost y for x guests. (b) Find the cost for 100 guests.

14. 7 pts

Ruth spends \$36 on fabric for a quilt: cotton at \$6 per yard and flannel at \$9 per yard. (a) Write an equation in standard form using c yards of cotton and f yards of flannel. (b) Find the c -intercept and the f -intercept.

15. 6 pts

Explain why the y -intercept of $y = mx + b$ is b , and why a vertical line has an undefined slope.

ANSWER KEY

$$1. m = \frac{-2 - 6}{3 - (-1)} = \frac{-8}{4} = -2$$

2. Each increase of 3 in x raises y by 2, so the rate is $\frac{2}{3}$.

$$3. k = \frac{14}{4} = \frac{7}{2}, \text{ so } y = \frac{7}{2}x \text{ and } y = \frac{7}{2}(10) = 35.$$

$$4. k = \frac{52}{4} = 13 \text{ dollars per hour, so } 13 \cdot 11 = 143. \text{ He earns \$143.}$$

5. Slope $-\frac{1}{4}$, y -intercept 6; the line through $(0, 6)$ and $(4, 5)$, falling gently from left to right.

$$6. -3y = -4x + 9, \text{ so } y = \frac{4}{3}x - 3$$

$$7. y - 2 = \frac{2}{3}(x + 3), \text{ so } y - 2 = \frac{2}{3}x + 2 \text{ and } y = \frac{2}{3}x + 4$$

$$8. m = \frac{-3 - 9}{4 - (-2)} = \frac{-12}{6} = -2; y - 9 = -2(x + 2), \text{ so } y = -2x + 5$$

9. $y = 0$: $5x = 20$, $x = 4$. $x = 0$: $-2y = 20$, $y = -10$. x -intercept 4, y -intercept -10 .

10. Horizontal: $y = -1$; vertical: $x = 6$

$$11. \text{Slope } -\frac{1}{4}: y - 3 = -\frac{1}{4}(x - 8), \text{ so } y - 3 = -\frac{1}{4}x + 2 \text{ and } y = -\frac{1}{4}x + 5$$

$$12. 2y = -3x + 4 \text{ gives slope } -\frac{3}{2}. y + 1 = -\frac{3}{2}(x - 2), \text{ so } y + 1 = -\frac{3}{2}x + 3 \text{ and } y = -\frac{3}{2}x + 2$$

13. (a) $m = \frac{430 - 310}{70 - 40} = 4$; $y - 310 = 4(x - 40)$, so $y = 4x + 150$ (\$4 per guest plus a \$150 base charge). (b) $4(100) + 150 = 550$, so \$550.

14. (a) $6c + 9f = 36$ (or, dividing by 3, $2c + 3f = 12$); (b) c -intercept 6 (6 yards of cotton, no flannel) and f -intercept 4 (4 yards of flannel, no cotton).

15. The graph crosses the y -axis where $x = 0$, and then $y = m \cdot 0 + b = b$. On a vertical line every point has the same x -coordinate, so the run between any two points is 0, and rise divided by 0 is undefined.

SYSTEMS OF EQUATIONS AND INEQUALITIES

In Unit 5 you studied one line at a time. In this unit you put two lines together and ask which ordered pairs make both equations true. You will solve these systems three ways: by graphing (where do the lines cross?), by substitution and by elimination, and you will learn why a system of two lines always has exactly one solution, no solution or infinitely many. Along the way you will see why each method is allowed: equal quantities may replace each other, and adding equals to equals keeps an equation balanced.

Systems are some of the most useful tools in algebra, because many real questions have two unknowns: how many adult and children's tickets were sold, how fast a boat goes and how fast the river flows, when a small business breaks even, or which of two plans costs less. The unit ends with linear inequalities in two variables and systems of inequalities, which describe all the choices that fit a budget or a garden plot.

This unit builds on solving equations (Unit 2), inequalities (Unit 3) and graphing lines (Unit 5). It prepares you for Geometry, where lines meet and cross, and for Algebra II, where you will solve larger systems and systems involving curves. All work in this unit is done by hand.

PREREQUISITES

- Solve multi-step linear equations, including equations with variables on both sides, identities and contradictions (Lessons 2.2–2.3).
- Solve a literal equation such as $ax + by = c$ for y (Lesson 2.5).
- Solve and graph linear inequalities, reversing the sign when multiplying or dividing by a negative (Lessons 3.1–3.2).
- Graph a line from slope-intercept form and from intercepts (Lessons 5.3 and 5.5).
- Recognize parallel lines by their equal slopes (Lesson 5.6).
- Write a linear model from a word problem with a starting amount and a rate (Lesson 5.7).

MASTERY CHECKLIST

- ☐ I can check whether an ordered pair is a solution of a system of equations or inequalities.
- ☐ I can solve a system of linear equations by graphing and check the crossing point in both equations.
- ☐ I can tell from slopes and y-intercepts whether a system has one solution, no solution or infinitely many, and explain why.
- ☐ I can solve a system by substitution, using parentheses when I substitute an expression.
- ☐ I can solve a system by elimination, adding or subtracting the equations, and explain why this is allowed.
- ☐ I can multiply one or both equations by a nonzero number before eliminating, and explain why the solutions do not change.
- ☐ I can recognize no solution or infinitely many solutions when both variables disappear.
- ☐ I can choose the best method (graphing, substitution or elimination) for a given system.
- ☐ I can write and solve systems for ticket, coin, mixture, break-even, current and wind, and plan-comparison problems.
- ☐ I can graph a linear inequality in two variables with a dashed or

solid boundary and the correct shading.

- I can graph a system of linear inequalities, find its solution region and use it for budget and planting constraints.

LESSONS IN THIS UNIT

6.1 Solving Systems by Graphing	2 sessions of 45 minutes
6.2 Solving Systems by Substitution	2 sessions of 45 minutes
6.3 Solving Systems by Elimination	2 sessions of 45 minutes
6.4 Elimination with Multiplication	2 sessions of 45 minutes
6.5 Applications of Systems	2 sessions of 45 minutes
6.6 Linear Inequalities and Systems of Inequalities	2 sessions of 45 minutes

UNIT 6 TEST

Time allowed: 60 minutes. No calculator. Show your work for every problem: write the system you use, one step per line, and check your solutions. All answers in this test are exact; give money to the nearest cent and include units in word-problem answers. For graphs, use graph paper and a ruler, and label the boundary lines.

1. **5 pts**

Is $(-1, 4)$ a solution of the system $y = -2x + 2$ and $x + y = 3$?

2. **6 pts**

Solve by graphing: $y = x - 3$ and $y = -2x + 3$.

3. **5 pts**

Without solving, tell how many solutions the system $y = \frac{2}{3}x + 1$ and $2x - 3y = -3$ has.

4. (6 pts)

Solve by substitution: $y = x - 4$ and $3x + 2y = 7$.

5. (6 pts)

Solve by substitution: $x - 3y = 2$ and $2x - 6y = 5$.

6. (6 pts)

Solve by elimination: $5x + 2y = 9$ and $3x - 2y = 15$.

7. (7 pts)

Solve by elimination: $4x + 3y = 1$ and $6x + 5y = 1$.

8. (7 pts)

Choose the best method and solve: $x = 2y - 1$ and $3x + 4y = 17$.

9. (8 pts)

A pancake breakfast at church served 150 people. Adults paid \$6 and children paid \$4, and the breakfast took in \$780. How many adults and how many children were served?

10. (7 pts)

A plane flies 1,800 miles with the wind in 4 hours and makes the return trip against the wind in 5 hours. Find the speed of the plane in still air and the speed of the wind.

11. (7 pts)

Grace mixes raisins costing \$3 per pound with almonds costing \$8 per pound to make 10 pounds of trail mix worth \$5 per pound. How many pounds of each does she use?

12. (7 pts)

Samuel sells jars of honey. He spent \$150 on equipment, each jar costs him \$3 to fill, and he sells each jar for \$8. How many jars must he sell to break even?

13. (6 pts)

Is $(-2, 3)$ a solution of the system $y > -x$ and $3x + y \leq 0$?

14. (7 pts)

Graph the system $y \leq \frac{1}{2}x + 2$ and $y > -x - 1$. Give the point where the boundary lines meet.

15. (5 pts)

Ruth wants to spend no more than \$30 on seed packets for the church garden. Carrot seeds cost \$2 a packet and bean seeds cost \$3 a packet. Write an inequality using x = carrot packets and y = bean packets.

16. (5 pts)

Henry solves a system by elimination and ends with $0 = 0$. Grace graphs the same system. What will Grace see, and why do the two results agree?

ANSWER KEY

1. Yes: $-2(-1) + 2 = 4$ and $-1 + 4 = 3$ are both true.
2. The lines cross at $(2, -1)$; check: $2 - 3 = -1$ and $-2(2) + 3 = -1$.
3. Infinitely many: $2x - 3y = -3$ becomes $y = \frac{2}{3}x + 1$, the same line.
4. $3x + 2(x - 4) = 7$, so $5x = 15$, $x = 3$, $y = -1$: $(3, -1)$.
5. $x = 3y + 2$; $2(3y + 2) - 6y = 5$ gives $4 = 5$, false: no solution.
6. Add: $8x = 24$, $x = 3$; $15 + 2y = 9$, $y = -3$: $(3, -3)$.
7. Multiply by 3 and by 2: $12x + 9y = 3$, $12x + 10y = 2$; subtract: $y = -1$; $4x - 3 = 1$, $x = 1$: $(1, -1)$.
8. Substitution, since x is isolated: $3(2y - 1) + 4y = 17$, $10y = 20$, $y = 2$, $x = 3$: $(3, 2)$.
9. $a + c = 150$, $6a + 4c = 780$; $6a + 4(150 - a) = 780$ gives $2a = 180$, $a = 90$, $c = 60$. 90 adults and 60 children.
10. $p + w = 450$, $p - w = 360$; add: $2p = 810$, $p = 405$, $w = 45$. Plane 405 mph, wind 45 mph.
11. $r + a = 10$, $3r + 8a = 50$; subtract $3r + 3a = 30$: $5a = 20$, $a = 4$, $r = 6$. 6 pounds of raisins and 4 pounds of almonds.
12. $8x = 150 + 3x$, so $5x = 150$, $x = 30$ jars.
13. Yes: $3 > 2$ is true, and $3(-2) + 3 = -3 \leq 0$ is true.
14. Solid line $y = \frac{1}{2}x + 2$ through $(0, 2)$ and $(2, 3)$, shaded below; dashed line $y = -x - 1$ through $(0, -1)$ and $(-1, 0)$, shaded above. The lines meet at $(-2, 1)$, and the solution region is the wedge to the right of that point between the lines; $(0, 0)$ is in it.
15. $2x + 3y \leq 30$
16. Grace will see only one line, because both equations describe the same line. Getting $0 = 0$ means the equations are true for exactly the same pairs, so every point on that line is a solution: infinitely many solutions, which matches the graph.

6.1 SOLVING SYSTEMS BY GRAPHING

OBJECTIVES

- Check whether an ordered pair is a solution of a system of two linear equations.
- Solve a system of linear equations by graphing both lines and finding the point where they cross.
- Explain why a system of two linear equations has exactly one solution, no solution or infinitely many solutions, and tell which case from the slopes and y-intercepts.
- Describe the limits of graphing when the solution does not land on grid points.

WATCH

Intro to Simultaneous Equations (2 of 2: Solving visually)

Eddie Woo · 8:57

https://www.youtube.com/watch?v=CC_1Roehip4

WHEN TO WATCH

Before Example 1: solving a system by graphing both lines.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations. Neither video covers parallel or identical lines (no solution, infinitely many). Teach that from Example 2.

BACKUP VIDEO

Intro to Simultaneous Equations (1 of 2: Understanding multiple solutions)

Eddie Woo · 11:04

<https://www.youtube.com/watch?v=URAPBVkQodM>

Before the Core ideas: what it means for an ordered pair to solve two equations at once.

KEY TERMS

system of linear equations

Two or more linear equations that use the same variables and are considered together, such as $y = x + 1$ and $y = -2x + 7$.

solution of a system

An ordered pair that makes every equation in the system true at the same time. On a graph it is a point that lies on both lines.

point of intersection

The point where two graphs cross or meet.

consistent and inconsistent

A system is consistent if it has at least one solution and inconsistent if it has no solution.

dependent system

A system whose two equations describe the same line, so every point on the line is a solution (infinitely many solutions).

CORE IDEAS

In Unit 5 you graphed one linear equation at a time. Each graph was the picture of every ordered pair that makes that equation true: points on the line work, and points off the line do not. Now we ask a new question about **two** equations at once: is there an ordered pair (x, y) that makes **both** true? Two equations considered together are called a **system**.

Here is the key reason graphing works. A point that makes the first equation true must lie on the first line. A point that makes the second equation true must lie on the second line. So a point that makes both true must lie on both lines at once, and the only places that can happen are where the lines meet. **The solution of a system is the point of intersection.**

To solve by graphing: (1) write each equation in slope-intercept form $y = mx + b$ if it is not already; (2) graph both lines carefully on the same axes, using the y-intercept and the slope; (3) read the point where they cross; (4) **check** the point in both original equations. The check matters because a hand-drawn graph can be off by a little.

Two different lines in a plane can meet in only one way or not at all. If their slopes are different, one line is steeper than the other, so they must cross, and they cross exactly once: **one solution**. If their slopes are equal but the y-intercepts differ, the lines are parallel and never meet: **no solution**. If the slopes and the y-intercepts are both equal, the two equations are really the same line, so every point on it is a solution: **infinitely many solutions**.

Why can two lines never cross in exactly two points? Through any two points there is only one straight line. If two lines shared two points, they would both be the line through those two points, so they would be the same line and share every point. That is why a linear system has 0, 1 or infinitely many solutions, never 2 or 3.

Graphing gives a clear picture, but it has limits. If the lines cross at a point such as $(2.5, -1.5)$ or $\left(\frac{4}{3}, \frac{7}{3}\right)$, you can only estimate it from a hand-drawn graph. In the next lessons you will learn substitution and elimination, which find exact answers every time. Graphing is still the best way to *see* what a system means.

KEY FORMULAS

different slopes $\Rightarrow$ exactly one solution

same slope, different y -intercepts $\Rightarrow$ no solution (parallel lines)

same slope, same y -intercept $\Rightarrow$ infinitely many solutions (same line)

From history. Solving a system by graphing uses the coordinate plane that René Descartes described in *La Géométrie* (1637); Pierre de Fermat had similar ideas at the same time. Joining algebra and geometry meant that a question about equations (which numbers make both true?) could be pictured as a question about lines (where do they cross?).

Practice habit. Check by substituting: after you read a solution from a graph, put it into BOTH original equations. A point that works in only one equation is not a solution of the system.

WORKED EXAMPLES

Example 1. Solve the system by graphing: $y = x + 1$ and $y = -2x + 7$.

Show solution

1. First line: $y = x + 1$ has y-intercept 1 and slope 1. Plot $(0, 1)$, then go up 1 and right 1 to $(1, 2)$, $(2, 3)$, $(3, 4)$.
2. Second line: $y = -2x + 7$ has y-intercept 7 and slope -2 . Plot $(0, 7)$, then go down 2 and right 1 to $(1, 5)$, $(2, 3)$, $(3, 1)$.
3. Both lists contain $(2, 3)$, so the lines cross at $(2, 3)$.
4. Check in the first equation: $3 = 2 + 1$ is true.
5. Check in the second equation: $3 = -2(2) + 7 = -4 + 7$ is true.

Answer: $(2, 3)$

Example 2. Grace has \$20 saved and adds \$5 each week. Her brother Henry has \$50 saved and adds \$2 each week. Let x be the number of weeks and y the amount saved in dollars. Write a system, solve it by graphing, and tell when they will have the same amount.

Show solution

1. Grace: $y = 5x + 20$. Henry: $y = 2x + 50$.
2. Make a table for Grace: x : 0, 5, 10, 15 and y : 20, 45, 70, 95.
3. Make a table for Henry: x : 0, 5, 10, 15 and y : 50, 60, 70, 80.
4. Both tables contain $(10, 70)$. On the graph, Grace's steeper line starts lower and crosses Henry's line at $(10, 70)$.
5. Check Grace: $5(10) + 20 = 70$. Check Henry: $2(10) + 50 = 70$. Both are true.
6. Answer in words: after 10 weeks they will each have \$70.

Answer: $(10, 70)$: after 10 weeks each will have \$70.

PRACTICE

1. Easy Is $(3, 1)$ a solution of the system $x + y = 4$ and $x - y = 2$?

Show answer

Yes: $3 + 1 = 4$ and $3 - 1 = 2$ are both true.

2. Easy Is $(2, 5)$ a solution of the system $y = 2x + 1$ and $y = -x + 6$?

Show answer

No: $2(2) + 1 = 5$ is true, but $-2 + 6 = 4$, not 5.

3. Easy Solve by graphing: $y = x$ and $y = -x + 4$.

Show answer

$(2, 2)$; check: $2 = 2$ and $2 = -2 + 4$.

4. Easy Without graphing, how many solutions does the system $y = 3x + 2$ and $y = 3x - 5$ have?

Show answer

No solution: same slope 3, different y-intercepts, so the lines are parallel.

5. Medium Solve by graphing: $y = 2x - 3$ and $y = -x + 3$.

Show answer

$(2, 1)$; check: $2(2) - 3 = 1$ and $-2 + 3 = 1$.

6. Medium Solve by graphing: $y = \frac{1}{2}x - 1$ and $y = -x + 5$.

Show answer

$(4, 1)$; check: $\frac{1}{2}(4) - 1 = 1$ and $-4 + 5 = 1$.

7. Medium Solve by graphing: $x + y = 5$ and $x - y = 1$. (First write each in slope-intercept form.)

Show answer

$y = -x + 5$ and $y = x - 1$ cross at $(3, 2)$.

8. Medium How many solutions does the system $y = -2x + 4$ and $4x + 2y = 8$ have?

Show answer

Infinitely many: $4x + 2y = 8$ becomes $y = -2x + 4$, the same line.

9. **Medium** Samuel wants to rent a canoe. River Outfitters charges \$10 plus \$5 per hour. Lakeside Rentals charges \$20 plus \$3 per hour. Write a system with x = hours and y = cost in dollars, and solve it by graphing. For how many hours do the two cost the same, and what is that cost?

Show answer

$y = 5x + 10$ and $y = 3x + 20$ cross at $(5, 35)$: 5 hours, \$35.

10. **Hard** Ruth plants a tree 3 feet tall that grows 2 feet per year. Her neighbor's tree is 9 feet tall and grows 1 foot per year. Write a system with x = years and y = height in feet and solve it by graphing. After how many years are the trees the same height, and how tall are they?

Show answer

$y = 2x + 3$ and $y = x + 9$ cross at $(6, 15)$: after 6 years, both are 15 feet tall.

11. **Hard** Henry graphs $y = -x + 1$ and $y = x - 4$. The lines seem to cross between grid lines, near $(2.5, -1.5)$. Check by substituting: is $(2.5, -1.5)$ exactly the solution?

Show answer

Yes: $-2.5 + 1 = -1.5$ and $2.5 - 4 = -1.5$. (This is why graphing alone is not always enough: the point is between grid lines.)

12. **Hard** The Miller family is choosing between two hay suppliers. Farm A charges \$4 per bale plus a \$30 delivery fee. Farm B charges \$5 per bale plus a \$10 delivery fee. Write a system with x = bales and y = total cost and solve it by graphing. For how many bales is the cost the same, and what is it?

Show answer

$y = 4x + 30$ and $y = 5x + 10$ cross at $(20, 110)$: 20 bales, \$110.

13. **Hard** Explain why a system of two linear equations can never have exactly two solutions.

Show answer

Only one straight line passes through two given points. If two lines shared two points, they would both be that one line, so they would be the same line and share every point (infinitely many solutions). So two lines meet in 0, 1 or infinitely many points.

14. **Challenge** For what value of k does the system $y = kx + 3$ and $y = 4x - 1$ have no solution?

Show answer

$k = 4$: equal slopes with different y-intercepts (3 and -1) give parallel lines.

15. **Challenge** The system $y = 2x + b$ and $y = -x + 7$ has a solution with $x = 3$. Find b .

Show answer

From the second equation $y = -3 + 7 = 4$; then $4 = 2(3) + b$, so $b = -2$.

COMMON MISTAKES

- Checking a point in only one equation. A solution of a system must make both equations true, so always substitute into both.
- Graphing carelessly, then trusting the picture. Use a ruler, plot at least three points per line, and check the crossing point algebraically.
- Assuming lines that look parallel never meet. Compare the slopes: if the slopes are different (even slightly, like 2 and 2.1), the lines cross somewhere.
- Calling a system with the same line 'no solution.' If both equations simplify to the same line, every point on it works, so there are infinitely many solutions.

Make it a game • Where the Lines Meet. The parent writes a system such as $y = x + 1$ and $y = -2x + 7$. First test five ordered pairs and say 'solution' or 'not a solution' for each, checking both equations. Then graph the two lines with different colored pencils and find the crossing point. Finally, write your own system with a secret solution and let the parent find it.

6.2 SOLVING SYSTEMS BY SUBSTITUTION

OBJECTIVES

- Solve a system of linear equations by solving one equation for a variable and substituting into the other.
- Explain why substitution is allowed and why it turns a system into a single equation in one variable.
- Recognize algebraically when a system has no solution or infinitely many solutions.
- Write and solve systems by substitution for word problems.

WATCH

Solving Simultaneous Equations by Substitution

Eddie Woo · 5:47

<https://www.youtube.com/watch?v=9Si7GW69UDE>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations.

BACKUP VIDEO

Simultaneous Equations (2 of 2: By substitution)

Eddie Woo · 11:02

<https://www.youtube.com/watch?v=gQ3T4D6Qdyo>

Before the practice set: a longer worked example of substitution.

KEY TERMS

substitution method

A way to solve a system: solve one equation for one variable, replace that variable in the other equation with the expression, and solve the resulting one-variable equation.

substitution property of equality

If two quantities are equal, either one may replace the other in any equation or expression.

back-substitute

After finding one variable, put its value into an equation to find the other variable.

identity

An equation that is true for every value, such as $4 = 4$. In a system it means the two equations describe the same line.

contradiction

An equation that is never true, such as $10 = 3$. In a system it means the lines are parallel and there is no solution.

CORE IDEAS

Graphing shows a solution but can only estimate it when the crossing point is not on the grid. **Substitution** finds the exact answer with algebra you already know from Unit 2.

Look at the system $y = 2x - 1$ and $3x + y = 14$. If (x, y) is a solution, then at that point y really is equal to $2x - 1$. Equal quantities may replace each other (the substitution property of equality), so in the second equation we may write $2x - 1$ in place of y : $3x + (2x - 1) = 14$. Now there is only one variable, and you know how to solve that: $5x - 1 = 14$, so $5x = 15$ and $x = 3$.

We still need y . Put $x = 3$ back into the easier equation: $y = 2(3) - 1 = 5$. The solution is the ordered pair $(3, 5)$. Check it in both original equations: $5 = 2(3) - 1$ and $3(3) + 5 = 14$. Both are true.

The steps: (1) solve one equation for one variable, choosing a variable whose coefficient is 1 or -1 if you can, so you avoid fractions; (2) substitute that expression, **in parentheses**, into the *other* equation; (3) solve for the remaining variable; (4) back-substitute to find the second variable; (5) write the ordered pair and check it in both equations.

Sometimes both variables disappear. If you are left with a false statement such as $10 = 3$, no ordered pair can ever make both equations true, so there is **no solution**; the lines are parallel. If you are left with a true statement such as $4 = 4$, the second equation says nothing new: every pair that satisfies one equation satisfies the other, so there are **infinitely many solutions**; the equations describe the same line. This is the same thing that happened with contradictions and identities in Lesson 2.3.

Substitution is the natural choice when one equation is already solved for a variable (like $y = 2x - 1$) or when a variable has a coefficient of 1 or -1 . In the next two lessons you will learn elimination, which is often faster when no variable is easy to isolate.

KEY FORMULAS

If $y = (\text{expression in } x)$, replace y by $(\text{expression in } x)$ in the other equation.

false statement (e.g. $10 = 3$) $\Rightarrow$ no solution; true statement (e.g. $4 = 4$) $\Rightarrow$ infinitely many

From history. Diophantus of Alexandria, in his *Arithmetica* (about AD 250), had an abbreviation for only one unknown number. When a problem had two unknowns, he wrote the second one in terms of the first, which is the heart of the substitution method.

Practice habit. Use parentheses every time you substitute an expression, and write one step per line. Writing $3x + (2x - 1)$ or $4x - 3(1 - 2x)$ keeps the signs right.

WORKED EXAMPLES

Example 1. Solve by substitution: $y = 2x - 1$ and $3x + y = 14$.

Show solution

1. The first equation is already solved for y .
2. Substitute into the second equation: $3x + (2x - 1) = 14$.
3. Combine like terms: $5x - 1 = 14$.
4. Add 1: $5x = 15$, so $x = 3$.
5. Back-substitute: $y = 2(3) - 1 = 5$.
6. Check: $5 = 2(3) - 1$ is true, and $3(3) + 5 = 14$ is true.

Answer: $(3, 5)$

Example 2. The perimeter of a rectangular vegetable garden is 56 feet. Its length is 4 feet more than twice its width. Find the width and the length.

Show solution

1. Let w = width and l = length, in feet.
2. Perimeter: $2l + 2w = 56$. Length: $l = 2w + 4$.
3. Substitute: $2(2w + 4) + 2w = 56$.
4. Distribute and combine: $4w + 8 + 2w = 56$, so $6w + 8 = 56$.
5. Subtract 8: $6w = 48$, so $w = 8$.
6. Back-substitute: $l = 2(8) + 4 = 20$.
7. Check in the words: 20 is 4 more than twice 8, and $2(20) + 2(8) = 40 + 16 = 56$.

Answer: Width 8 feet, length 20 feet.

PRACTICE

1. Easy Solve by substitution: $y = x + 2$ and $x + y = 10$.

Show answer

$$x + (x + 2) = 10, \text{ so } x = 4, y = 6: (4, 6).$$

2. Easy Solve by substitution: $x = 3y$ and $x + y = 12$.

Show answer

$$3y + y = 12, \text{ so } y = 3, x = 9: (9, 3).$$

3. Easy Solve by substitution: $y = 4x$ and $2x + y = 18$.

Show answer

$$2x + 4x = 18, \text{ so } x = 3, y = 12: (3, 12).$$

4. Easy Solve by substitution: $y = -x + 7$ and $y = 2x + 1$.

Show answer

$$-x + 7 = 2x + 1, \text{ so } 3x = 6, x = 2, y = 5: (2, 5).$$

5. Medium Solve by substitution: $x + 2y = 7$ and $3x - y = 7$.

Show answer

$$x = 7 - 2y; 3(7 - 2y) - y = 7, \text{ so } 21 - 7y = 7, y = 2, x = 3: (3, 2).$$

6. Medium Solve by substitution: $2x + y = 1$ and $4x - 3y = 17$.

Show answer

$$y = 1 - 2x; 4x - 3(1 - 2x) = 17, \text{ so } 10x - 3 = 17, x = 2, y = -3: (2, -3).$$

7. Medium Solve by substitution: $y = 3x - 2$ and $6x - 2y = 4$.

Show answer

$$6x - 2(3x - 2) = 4 \text{ gives } 4 = 4, \text{ always true: infinitely many solutions (the same line).}$$

8. Medium Solve by substitution: $y = -2x + 5$ and $4x + 2y = 3$.

Show answer

$$4x + 2(-2x + 5) = 3 \text{ gives } 10 = 3, \text{ never true: no solution (parallel lines).}$$

9. Medium Two numbers add up to 30. One number is 6 more than the other. Write a system and solve it by substitution to find the two numbers.

Show answer

$$x + y = 30, x = y + 6: 2y + 6 = 30, \text{ so } y = 12, x = 18. \text{ The numbers are 12 and 18.}$$

10. Hard Solve by substitution: $y = \frac{1}{2}x + 1$ and $3x - 4y = 2$.

Show answer

$$3x - 4\left(\frac{1}{2}x + 1\right) = 2, \text{ so } x - 4 = 2, x = 6, y = 4: (6, 4).$$

11. Hard A farm has 60 animals, all chickens and goats. Together they have 170 legs. How many chickens and how many goats are there?

Show answer

$$c + g = 60, 2c + 4g = 170; c = 60 - g \text{ gives } 120 + 2g = 170, \text{ so } g = 25, c = 35: \\ 35 \text{ chickens and 25 goats.}$$

12. Hard Samuel cuts a 12-foot board into two pieces. The long piece is 4 feet less than three times the short piece. How long is each piece?

Show answer

$$s + l = 12, l = 3s - 4: 4s - 4 = 12, \text{ so } s = 4, l = 8. \text{ Short piece 4 feet, long piece 8 feet.}$$

13. **Hard** Grace solves a system by substitution, and the variables disappear, leaving $0 = 5$. What does this tell her about the system, and why?

Show answer

The system has no solution. The statement $0 = 5$ is false no matter what x and y are, so no ordered pair makes both equations true. On a graph the two lines are parallel.

14. **Challenge** Solve by substitution: $3x + 2y = 4$ and $2x - 5y = 9$.

Show answer

$x = \frac{4 - 2y}{3}$; multiply through by 3 after substituting: $8 - 4y - 15y = 27$, so $y = -1$, $x = 2$: $(2, -1)$.

15. **Challenge** For what value of a does the system $y = 2x + 3$ and $4x - 2y = a$ have infinitely many solutions?

Show answer

$4x - 2(2x + 3) = a$ gives $-6 = a$, so $a = -6$ (then the equations describe the same line).

COMMON MISTAKES

- Substituting back into the same equation you solved. That only gives an identity like $y = y$. Substitute into the *other* equation.
- Forgetting parentheses: writing $4x - 3 \cdot 1 - 2x$ instead of $4x - 3(1 - 2x)$. Put the whole expression in parentheses, then distribute.
- Stopping after finding one variable. A solution of a system is an ordered pair; back-substitute to find the other value.
- Writing ' $x = 4 = 4$ ' or 'no answer' when the variables cancel. Read the result: a true statement means infinitely many solutions; a false one means no solution.

Make it a game • Trade Fair Teaching. You are the teacher. Solve $y = 2x + 1$ and $3x + y = 16$ on a board for the parent. Explain why you may trade y for $2x + 1$, as if swapping a coin for one of equal value. The parent asks 'why is that allowed?' and sneaks in one wrong step. Catch the error, then check the solution in both equations.

6.3 SOLVING SYSTEMS BY ELIMINATION

OBJECTIVES

- Solve a system of linear equations by adding or subtracting the equations to eliminate one variable.
- Explain why adding two true equations gives another true equation.
- Decide whether to add or subtract by looking at the coefficients of one variable.
- Recognize no solution and infinitely many solutions when both variables are eliminated.

WATCH

Solving Simultaneous Equations by Elimination

Eddie Woo · 5:26

https://www.youtube.com/watch?v=_fTFFBJmLM8

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations.

BACKUP VIDEO

Simultaneous Equations (1 of 2: By elimination)

Eddie Woo · 6:15

<https://www.youtube.com/watch?v=GM6SUFcsk5Q>

Before the practice set: another elimination example.

KEY TERMS

elimination method

A way to solve a system by adding or subtracting the equations so that one variable drops out, leaving one equation in one variable.

opposite coefficients

Coefficients that add to zero, like $3y$ and $-3y$. Adding the equations eliminates that variable.

equal coefficients

Coefficients that are the same, like $2x$ and $2x$. Subtracting one equation from the other eliminates that variable.

addition property of equality

If $a = b$ and $c = d$, then $a + c = b + d$: adding equal amounts to both sides keeps an equation true.

CORE IDEAS

Look at the system $x + y = 10$ and $x - y = 4$. Neither equation is solved for a variable, but notice the $+y$ and $-y$. If we add the left sides and add the right sides, the y terms cancel: $(x + y) + (x - y) = 10 + 4$, so $2x = 14$ and $x = 7$. Then $7 + y = 10$ gives $y = 3$.

Why are we allowed to add equations? In Unit 2 you learned that adding the same number to both sides of an equation keeps it true. Suppose (x, y) is a solution of the system. Then the second equation says $x - y$ and 4 are the *same number* with two different names. So when we add $x - y$ to the left side of the first equation and 4 to the right side, we are adding equal amounts to both sides. The balance stays level, and the new equation $2x = 14$ is true for every solution of the system.

The same reasoning works for subtracting: taking away equal amounts from both sides keeps the balance level. So you may subtract one equation from another. When you subtract, subtract **every** term on the left and the whole right side, like distributing a -1 .

How to choose: look for a variable whose coefficients are **opposites** (like $5y$ and $-5y$) and **add** the equations; or a variable whose coefficients are **equal** (like $3x$ and $3x$) and **subtract**. Line up the x terms, the y terms and the constants in columns first, so the equations look like $ax + by = c$.

After one variable is found, back-substitute into either original equation to find the other, write the ordered pair, and check it in **both** original equations.

If both variables vanish, read what is left, just as with substitution. A false statement such as $0 = -4$ means **no solution** (parallel lines). A true statement such as $0 = 0$ means **infinitely many solutions** (the same line).

KEY FORMULAS

If $a = b$ and $c = d$, then $a + c = b + d$ and $a - c = b - d$.

opposite coefficients $\Rightarrow$ add; equal coefficients $\Rightarrow$ subtract

From history. Al-Khwarizmi (Baghdad, about 820) named two basic moves of algebra: al-jabr, "restoring," and al-muqabala, "balancing," which cancels equal amounts from both sides of an equation. Elimination is that same balancing idea used on two equations at once.

Euclid connection. Euclid's Elements begins with common notions such as "if equals are added to equals, the wholes are equal" and "if equals are subtracted from equals, the remainders are equal." These are exactly the facts that make the elimination method work.

Practice habit. Keep the equal signs lined up: write the two equations one above the other with x terms, y terms and constants in neat columns before you add or subtract.

WORKED EXAMPLES

Example 1. Solve by elimination: $x + y = 10$ and $x - y = 4$.

Show solution

1. The y terms are opposites ($+y$ and $-y$), so add the equations.
2. Add: $(x + y) + (x - y) = 10 + 4$, so $2x = 14$.
3. Divide by 2: $x = 7$.
4. Back-substitute into $x + y = 10$: $7 + y = 10$, so $y = 3$.
5. Check: $7 + 3 = 10$ is true, and $7 - 3 = 4$ is true.

Answer: $(7, 3)$

Example 2. At a church bake sale, Ruth pays \$18 for 3 loaves of bread and 2 jars of jam. Henry pays \$27 for 3 loaves of bread and 5 jars of jam. Find the price of one loaf and of one jar.

Show solution

1. Let b = price of a loaf and j = price of a jar, in dollars.
2. Ruth: $3b + 2j = 18$. Henry: $3b + 5j = 27$.
3. The b terms are equal ($3b$ and $3b$), so subtract Ruth's equation from Henry's:
 $(3b + 5j) - (3b + 2j) = 27 - 18$.
4. Simplify: $3j = 9$, so $j = 3$.
5. Back-substitute: $3b + 2(3) = 18$, so $3b = 12$ and $b = 4$.
6. Check: $3(4) + 2(3) = 12 + 6 = 18$ and $3(4) + 5(3) = 12 + 15 = 27$.

Answer: A loaf costs \$4 and a jar of jam costs \$3.

PRACTICE

1. Easy Solve by elimination: $x + y = 8$ and $x - y = 2$.

Show answer

Add: $2x = 10$, so $x = 5$, $y = 3$: $(5, 3)$.

2. Easy Solve by elimination: $2x + y = 11$ and $-2x + 3y = 1$.

Show answer

Add: $4y = 12$, so $y = 3$; $2x + 3 = 11$, $x = 4$: $(4, 3)$.

3. Easy Solve by elimination: $3x + 2y = 16$ and $x + 2y = 8$.

Show answer

Subtract: $2x = 8$, so $x = 4$; $4 + 2y = 8$, $y = 2$: $(4, 2)$.

4. Easy Decide whether to add or subtract to eliminate y , then solve: $5x - 4y = 7$ and $2x + 4y = 14$.

Show answer

Add (the y coefficients are opposites): $7x = 21$, $x = 3$; $2(3) + 4y = 14$, $y = 2$: $(3, 2)$.

5. Medium Solve by elimination: $4x + 3y = 5$ and $4x - y = 9$.

Show answer

Subtract: $4y = -4$, so $y = -1$; $4x + 1 = 9$, $x = 2$: $(2, -1)$.

6. Medium Solve by elimination: $3x + 5y = 7$ and $-3x + 2y = 7$.

Show answer

Add: $7y = 14$, so $y = 2$; $3x + 10 = 7$, $x = -1$: $(-1, 2)$.

7. Medium Solve by elimination: $2x + 3y = 6$ and $2x + 3y = 10$.

Show answer

Subtract: $0 = -4$, which is false: no solution (parallel lines).

8. Medium Solve by elimination: $x - 2y = 3$ and $-x + 2y = -3$.

Show answer

Add: $0 = 0$, always true: infinitely many solutions (the same line).

9. Medium The sum of two numbers is 45 and their difference is 11. Use elimination to find the numbers.

Show answer

$x + y = 45$, $x - y = 11$; add: $2x = 56$, $x = 28$, $y = 17$. The numbers are 28 and 17.

10. **Hard** For a choir concert, the Miller family buys 2 adult tickets and 3 student tickets for \$36. Their neighbors buy 2 adult tickets and 1 student ticket for \$28. Find the price of an adult ticket and of a student ticket.

Show answer

$2a + 3s = 36$, $2a + s = 28$; subtract: $2s = 8$, $s = 4$; $2a = 24$, $a = 12$. Adult \$12, student \$4.

11. **Hard** On a service project, Henry and Samuel worked a total of 26 hours. Henry worked 4 more hours than Samuel. Use elimination to find each one's hours.

Show answer

$h + s = 26$, $h - s = 4$; add: $2h = 30$, $h = 15$, $s = 11$. Henry 15 hours, Samuel 11 hours.

12. **Hard** Solve by elimination: $3x = 2y + 4$ and $5x + 2y = 28$.

Show answer

Rewrite the first as $3x - 2y = 4$; add: $8x = 32$, $x = 4$; $12 - 2y = 4$, $y = 4$: $(4, 4)$.

13. **Hard** Explain why it is legal to add the equations $x + y = 10$ and $x - y = 4$ to get $2x = 14$.

Show answer

For a solution of the system, $x - y$ and 4 are the same number. Adding $x - y$ to the left side of $x + y = 10$ and 4 to the right side adds equal amounts to both sides, so the result $2x = 14$ is still true.

14. **Challenge** Solve by elimination: $0.3x + 0.2y = 1.3$ and $0.3x - 0.5y = -2.2$.

Show answer

Subtract: $0.7y = 3.5$, so $y = 5$; $0.3x + 1 = 1.3$, $x = 1$: $(1, 5)$.

15. **Challenge** The system $ax + by = 7$ and $ax - by = 1$ has the solution $(2, 3)$. Find a and b .

Show answer

$2a + 3b = 7$ and $2a - 3b = 1$; add: $4a = 8$, $a = 2$; then $3b = 3$, $b = 1$.

COMMON MISTAKES

- Subtracting only the first term. When you subtract an equation, change the sign of every term, including the constant: $(3b + 5j) - (3b + 2j) = 3j$, and $27 - 18 = 9$.
- Adding when the coefficients are equal (or subtracting when they are opposites), which doubles the variable instead of eliminating it. Check that the chosen terms will become 0 before you combine.
- Lining up the wrong terms. Rewrite both equations in the form $ax + by = c$ so x is under x , y under y , and constants on the right.
- Finding one variable and stopping. Back-substitute for the other, and check the pair in both original equations.

Make it a game • Secret Pair Riddles. The parent secretly picks two numbers and gives you the riddle as two equations, such as $x + y = 12$ and $x - y = 4$. Add the equations to eliminate one variable, solve, and back-substitute. Check both numbers in both equations. Then it is your turn to write a riddle from a secret pair. Make the third riddle trickier with a coefficient of 2.

6.4 ELIMINATION WITH MULTIPLICATION

OBJECTIVES

- Multiply one or both equations by a nonzero number to create equal or opposite coefficients, then solve by elimination.
- Explain why multiplying an equation by a nonzero number does not change its solutions.
- Choose the best method (graphing, substitution or elimination) for a given system and justify the choice.
- Solve word problems with systems that need multiplication before elimination.

WATCH

Solving Simultaneous Equations by Elimination (2 of 2: Worked example)

Eddie Woo · 7:25

<https://www.youtube.com/watch?v=kGkc2HYQngY>

WHEN TO WATCH

Before Example 1: multiplying an equation before eliminating.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations.

BACKUP VIDEO

Solving Simultaneous Equations by Elimination (1 of 2: Reviewing other methods)

Eddie Woo · 7:45

<https://www.youtube.com/watch?v=82isSVjefVk>

After Example 2: comparing graphing, substitution and elimination.

KEY TERMS

equivalent equations

Equations that have exactly the same solutions, such as $x + 2y = 5$ and $2x + 4y = 10$. Their graphs are the same line.

multiplication property of equality

Multiplying both sides of an equation by the same nonzero number keeps it true and does not change its solutions.

least common multiple (LCM)

The smallest positive number that two numbers both divide into evenly; used to pick multipliers, e.g. the LCM of 2 and 3 is 6.

CORE IDEAS

Elimination is quick when a variable already has equal or opposite coefficients. But in $x + 2y = 5$ and $3x - 4y = 5$ nothing cancels yet. The fix: multiply the first equation by 2 to get $2x + 4y = 10$. Now the y terms, $4y$ and $-4y$, are opposites, and adding gives $5x = 15$.

Why is multiplying allowed? If $x + 2y$ and 5 are equal, then twice the one equals twice the other, so $2x + 4y = 10$ is true for exactly the same pairs (x, y) . Nothing is lost, because you could divide by 2 to get back the original equation. On a graph, $x + 2y = 5$ and $2x + 4y = 10$ are the same line. That is why the multiplier must not be 0: multiplying by 0 gives $0 = 0$ and throws the equation's information away.

Sometimes you must multiply **both** equations. In $2x + 3y = 12$ and $3x + 4y = 17$, make the x coefficients both 6 (the LCM of 2 and 3): multiply the first by 3 and the second by 2, getting $6x + 9y = 36$ and $6x + 8y = 34$. Subtract to get $y = 2$, then back-substitute to get $x = 3$. Using a negative multiplier, such as -3 , makes opposites so you can always add instead of subtract.

Multiply **every** term, on both sides. A common slip is multiplying the left side and forgetting the constant on the right.

Choosing a method. Graphing is best for a picture or a quick estimate, and when the answer is likely to be whole numbers. Substitution is best when one equation is already solved for a variable (like $y = 3x - 1$) or a variable has coefficient 1 or -1 . Elimination is best when both equations are in standard form $Ax + By = C$, especially when no variable has coefficient 1. All three methods give the same answer, so pick the one with the least work and the fewest fractions.

KEY FORMULAS

$$a = b \text{ and } k \neq 0 \Rightarrow ka = kb \text{ (same solutions)}$$

Make the coefficients of one variable equal or opposite by using their LCM, then add or subtract.

Practice habit. Before you multiply, write the multiplier beside each equation (for example " $\times 3$ " and " $\times (-2)$ "), then multiply every term, including the number on the right side.

WORKED EXAMPLES

Example 1. Solve by elimination: $x + 2y = 5$ and $3x - 4y = 5$.

Show solution

1. Multiply the first equation by 2: $2x + 4y = 10$.
2. Now the y terms are opposites. Add to $3x - 4y = 5$: $5x = 15$.
3. Divide by 5: $x = 3$.
4. Back-substitute into $x + 2y = 5$: $3 + 2y = 5$, so $2y = 2$ and $y = 1$.
5. Check: $3 + 2(1) = 5$ is true, and $3(3) - 4(1) = 9 - 4 = 5$ is true.

Answer: $(3, 1)$

Example 2. At a farm stand, Grace pays \$8.50 for 3 pounds of apples and 2 pounds of pears. Her mother pays \$13.00 for 2 pounds of apples and 5 pounds of pears. Find the price per pound of apples and of pears.

Show solution

1. Let a = price per pound of apples and p = price per pound of pears, in dollars.
2. Grace: $3a + 2p = 8.50$. Her mother: $2a + 5p = 13$.
3. Make the a coefficients both 6: multiply the first by 2 and the second by 3.
4. This gives $6a + 4p = 17$ and $6a + 15p = 39$.
5. Subtract the first from the second: $11p = 22$, so $p = 2$.
6. Back-substitute: $3a + 2(2) = 8.50$, so $3a = 4.50$ and $a = 1.50$.
7. Check: $3(1.50) + 2(2) = 4.50 + 4 = 8.50$ and $2(1.50) + 5(2) = 3 + 10 = 13$.

Answer: Apples cost \$1.50 per pound and pears cost \$2.00 per pound.

PRACTICE

1. Easy Solve by elimination: $x + y = 5$ and $3x + 2y = 12$.

Show answer

Multiply the first by 2: $2x + 2y = 10$; subtract from the second: $x = 2$, then $y = 3$: $(2, 3)$.

2. Easy Solve by elimination: $2x + y = 7$ and $3x - 2y = 7$.

Show answer

Multiply the first by 2: $4x + 2y = 14$; add: $7x = 21$, $x = 3$, $y = 1$: $(3, 1)$.

3. Easy Solve by elimination: $x - 3y = -1$ and $2x + 5y = 9$.

Show answer

Multiply the first by -2 : $-2x + 6y = 2$; add: $11y = 11$, $y = 1$, $x = 2$: $(2, 1)$.

4. Easy Multiply the first equation by 5 and solve by elimination: $4x + y = 6$ and $3x - 5y = 16$.

Show answer

$20x + 5y = 30$; add: $23x = 46$, $x = 2$; $8 + y = 6$, $y = -2$: $(2, -2)$.

5. Medium Solve by elimination: $2x + 3y = 12$ and $3x + 4y = 17$.

Show answer

Multiply by 3 and by 2: $6x + 9y = 36$ and $6x + 8y = 34$; subtract: $y = 2$; $2x + 6 = 12$, $x = 3$: $(3, 2)$.

6. Medium Solve by elimination: $3x - 2y = 5$ and $-6x + 4y = -10$.

Show answer

Multiply the first by 2: $6x - 4y = 10$; add: $0 = 0$, always true: infinitely many solutions.

7. Medium Solve by elimination: $4x + 6y = 9$ and $2x + 3y = 5$.

Show answer

Multiply the second by -2 : $-4x - 6y = -10$; add: $0 = -1$, false: no solution.

8. **Medium** Choose the best method and solve: $y = 3x - 1$ and $2x + 5y = 12$.

Show answer

Substitution, since y is already isolated: $2x + 5(3x - 1) = 12$, $17x = 17$, $x = 1$, $y = 2$: (1, 2).

9. **Medium** At a church potluck there are only casseroles and salads, 32 dishes in all. There are 3 times as many casseroles as salads. Choose a method and find how many of each there are.

Show answer

Substitution: $c = 3s$ and $c + s = 32$ give $4s = 32$, $s = 8$, $c = 24$. There are 24 casseroles and 8 salads.

10. **Hard** A carpenter needs 60 hours to build 4 chairs and 3 tables, and 54 hours to build 5 chairs and 2 tables. How many hours does one chair take, and one table?

Show answer

$4c + 3t = 60$, $5c + 2t = 54$; multiply by 2 and 3: $8c + 6t = 120$, $15c + 6t = 162$; subtract: $7c = 42$, $c = 6$, $t = 12$. A chair takes 6 hours and a table 12 hours.

11. **Hard** Grace buys 2 bags of flour and 3 bags of sugar for \$19.00. Ruth buys 5 bags of flour and 2 bags of sugar for \$25.50. Find the price of one bag of flour and one bag of sugar.

Show answer

$2f + 3s = 19$, $5f + 2s = 25.5$; multiply by 2 and 3: $4f + 6s = 38$, $15f + 6s = 76.5$; subtract: $11f = 38.5$, $f = 3.50$, $s = 4$. Flour \$3.50, sugar \$4.00.

12. **Hard** Solve: $\frac{1}{2}x + \frac{1}{3}y = 4$ and $x - y = 3$.

Show answer

Multiply the first by 6: $3x + 2y = 24$; multiply the second by 2: $2x - 2y = 6$; add: $5x = 30$, $x = 6$, $y = 3$: (6, 3).

13. **Hard** Henry multiplies the equation $x + 2y = 5$ by 3 to get $3x + 6y = 15$. Explain why the new equation has exactly the same solutions, and why he must never multiply by 0.

Show answer

If $x + 2y$ equals 5, then 3 times it equals 3 times 5, so every solution of the old equation solves the new one; dividing by 3 goes back, so every solution of the new one solves the old one. Both are the same line. Multiplying by 0 gives $0 = 0$, which is true for every pair, so the equation's information would be lost.

14. **Challenge** Solve by elimination: $7x + 4y = 2$ and $5x + 3y = 1$.

Show answer

Multiply by 3 and by 4: $21x + 12y = 6$, $20x + 12y = 4$; subtract: $x = 2$;
 $10 + 3y = 1$, $y = -3$: $(2, -3)$.

15. **Challenge** The line $ax + by = 11$ passes through the points $(1, 3)$ and $(-2, 5)$. Find a and b .

Show answer

$a + 3b = 11$ and $-2a + 5b = 11$; multiply the first by 2 and add: $11b = 33$,
 $b = 3$, $a = 2$. The line is $2x + 3y = 11$.

COMMON MISTAKES

- Multiplying only the left side. Multiply every term, including the constant on the right: $2(x + 2y) = 2(5)$ gives $2x + 4y = 10$, not $2x + 4y = 5$.
- Sign errors with a negative multiplier. Multiplying $x - 3y = -1$ by -2 gives $-2x + 6y = 2$; write each product carefully.
- Choosing multipliers that make the coefficients equal and then adding (or opposite and then subtracting). Decide first: equal means subtract, opposite means add.
- Using a harder method than needed. If a variable is already isolated, substitution is usually faster; if both equations are in standard form, elimination usually is.

Make it a game • Find the Slip. The parent writes a worked elimination solution with one hidden mistake, such as multiplying only one side of an equation by 3. Read it line by line, find the slip, and explain why you must multiply every term in the equation by the same nonzero number. Then reverse roles and plant a sneaky error of your own.

6.5 APPLICATIONS OF SYSTEMS

OBJECTIVES

- Define two variables and write a system of two equations from a word problem.
- Solve ticket, coin, mixture, break-even, current and wind, and plan-comparison problems with systems.
- Explain why the speed with a current is $b + c$ and against it is $b - c$.
- Interpret the solution of a system in the words of the problem and check it against the facts given.

WATCH

Forming Simultaneous Equations (1 of 2: Fast & Slow Walkers)

Eddie Woo · 8:52

<https://www.youtube.com/watch?v=eETzBdTefQ4>

WHEN TO WATCH

Before Example 2 (the boat problem): a distance-rate-time system.

Note: Neither video does ticket, coin or mixture problems. Use Example 1 for those. Eddie Woo teaches in Australia: his distances are in kilometres.

BACKUP VIDEO

Problem Solving w/ Linear Functions

Eddie Woo · 9:04

<https://www.youtube.com/watch?v=9KwQj83wTjM>

After Example 2: a second travel problem set up as a system.

KEY TERMS

break-even point

The number of items at which a business's total income (revenue) equals its total cost, so it has neither gained nor lost money.

revenue

The money a business takes in from sales: price per item times the number sold.

mixture problem

A problem in which two ingredients with different prices or strengths are combined; one equation counts the amounts and the other counts the value or the pure ingredient.

current and wind

A river's current or the wind adds to a boat's or plane's own speed when moving with it and subtracts when moving against it.

CORE IDEAS

Many real problems have **two** unknown quantities and give **two** facts about them. One equation with one variable could not hold both facts easily, but a system can: each fact becomes one equation. Use the four-step plan from Lesson 2.6: understand the problem, define two variables (with units), write two equations, solve, and check the answer **in the words** of the problem.

Ticket and coin problems usually give a *count* fact and a *value* fact. If 200 tickets were sold at \$8 and \$5, the count equation is $a + c = 200$ and the value equation is $8a + 5c = 1360$ (number of each type times its price, added up). Coins work the same way: dimes are worth 0.10 dollars each and quarters 0.25 dollars each. Multiplying the value equation by 100 to work in cents clears the decimals.

Mixture problems follow the same pattern. If you mix oats and corn, one equation adds the pounds; the other adds the cost (or, with percents, the amount of the pure ingredient). The key fact is $\text{amount of pure ingredient} = \text{percent} \times \text{total amount}$, so 20 quarts of punch that is 25% juice contains $0.25 \times 20 = 5$ quarts of juice.

Break-even problems compare a cost line and a revenue line. A small business has a fixed starting cost plus a cost per item, $C = F + vx$, and earns $R = px$. Where the lines cross, $C = R$: below that number of items the business loses money, and above it the business makes a profit. **Comparing two plans** works the same way: the solution tells where the two plans cost the same, and on each side one plan is cheaper.

Current and wind problems. Suppose a boat moves at b miles per hour in still water and the river flows at c miles per hour. Going downstream, the water carries the boat along, so each hour the boat covers its own b miles plus the c miles the water moves it: speed $b + c$. Going upstream, the water pushes back c miles each hour: speed $b - c$. Find each actual speed from $d = rt$, then solve the system $b + c =$

downstream speed and $b - c =$ upstream speed by elimination. Planes with and against the wind work the same way.

After solving, ask whether the answer makes sense: numbers of tickets and coins must be whole numbers, amounts cannot be negative, and the current must be slower than the boat.

KEY FORMULAS

$$\text{total value} = (\text{number}) \times (\text{value of each})$$

$$\text{amount of pure ingredient} = (\text{percent as a decimal}) \times (\text{total amount})$$

$$\text{break-even: } C = R, \quad C = F + vx, \quad R = px$$

$$\text{with the current: } b + c, \quad \text{against the current: } b - c, \quad d = rt$$

From history. Word problems with two unknowns are very old. Leonardo of Pisa (Fibonacci) filled his *Liber Abaci* (1202) with practical problems about prices, trade and money for merchants, many of them solved by reasoning that we would now write as a system of equations.

Practice habit. Check in the words, not just the equations: after solving, reread the problem and make sure your two numbers fit every fact given (the count, the total value, the times and distances).

WORKED EXAMPLES

Example 1. The church choir sold 200 tickets to its spring concert. Adult tickets cost \$8 and children's tickets cost \$5. Ticket sales totaled \$1,360. How many of each kind were sold?

Show solution

1. Let a = number of adult tickets and c = number of children's tickets.
2. Count: $a + c = 200$. Value: $8a + 5c = 1360$.
3. Multiply the count equation by 5: $5a + 5c = 1000$.
4. Subtract from the value equation: $3a = 360$, so $a = 120$.
5. Back-substitute: $120 + c = 200$, so $c = 80$.
6. Check in the words: $120 + 80 = 200$ tickets, and $8(120) + 5(80) = 960 + 400 = 1360$ dollars.

Answer: 120 adult tickets and 80 children's tickets.

Example 2. The Miller family's boat travels 36 miles downstream in 2 hours. The return trip upstream takes 3 hours. Find the speed of the boat in still water and the speed of the current.

Show solution

1. Let b = speed of the boat in still water and c = speed of the current, in miles per hour.
2. Downstream speed: $36 \div 2 = 18$ mph, so $b + c = 18$.
3. Upstream speed: $36 \div 3 = 12$ mph, so $b - c = 12$.
4. Add the equations: $2b = 30$, so $b = 15$.
5. Back-substitute: $15 + c = 18$, so $c = 3$.
6. Check: downstream $(15 + 3)(2) = 36$ miles; upstream $(15 - 3)(3) = 36$ miles.

Answer: The boat goes 15 miles per hour in still water; the current is 3 miles per hour.

PRACTICE

1. **Easy** Ruth has 20 coins, all dimes and quarters, worth \$3.50 in all. How many dimes and how many quarters does she have?

Show answer

$d + q = 20$ and $10d + 25q = 350$ (cents): $15q = 150$, so $q = 10$, $d = 10$. 10 dimes and 10 quarters.

2. **Easy** At a bake sale, cookies sell for \$1 and brownies for \$2. The youth group sold 75 items and took in \$110. How many cookies and how many brownies were sold?

Show answer

$c + b = 75$ and $c + 2b = 110$; subtract: $b = 35$, $c = 40$. 40 cookies and 35 brownies.

3. **Easy** Grace starts a bread business. She spends \$60 on pans, and each loaf costs \$2 in ingredients. She sells each loaf for \$5. How many loaves must she sell to break even?

Show answer

$C = 60 + 2x$, $R = 5x$; $5x = 60 + 2x$, so $3x = 60$, $x = 20$ loaves.

4. **Easy** Lawn service A charges \$25 per mowing. Service B charges a \$40 sign-up fee plus \$20 per mowing. For how many mowings do the two cost the same, and what is that cost?

Show answer

$25x = 40 + 20x$, so $x = 8$; cost $25(8) = 200$. 8 mowings, \$200.

5. **Medium** A farmer mixes oats costing \$0.40 per pound with corn costing \$0.25 per pound to make 60 pounds of feed costing \$0.30 per pound. How many pounds of each should he use?

Show answer

$o + c = 60$ and $0.40o + 0.25c = 18$; with $c = 60 - o$: $40o + 1500 - 25o = 1800$, $o = 20$, $c = 40$. 20 pounds of oats and 40 pounds of corn.

6. **Medium** A plane flies 1,200 miles with the wind in 3 hours and returns against the wind in 4 hours. Find the plane's speed in still air and the speed of the wind.

Show answer

$p + w = 400$ and $p - w = 300$; add: $2p = 700$, $p = 350$, $w = 50$. Plane 350 mph, wind 50 mph.

7. **Medium** How many quarts of a drink that is 10% fruit juice and how many quarts of a drink that is 30% fruit juice should Ruth mix to make 20 quarts of punch that is 25% fruit juice?

Show answer

$x + y = 20$ and $0.10x + 0.30y = 5$; multiply the second by 10: $x + 3y = 50$; subtract: $2y = 30$, $y = 15$, $x = 5$. 5 quarts of the 10% drink and 15 quarts of the 30% drink.

8. **Medium** The Miller family gives \$1,200 to two missions. They give \$300 more to the first mission than to the second. How much does each mission receive?

Show answer

$f + s = 1200$, $f - s = 300$; add: $2f = 1500$, $f = 750$, $s = 450$. First \$750, second \$450.

9. **Hard** Henry builds birdhouses. He spent \$90 on tools, and the wood for each birdhouse costs \$6. He sells each birdhouse for \$15. Find the break-even point: the number of birdhouses and the revenue at that point.

Show answer

$C = 90 + 6x$, $R = 15x$; $9x = 90$, $x = 10$; $R = 15(10) = 150$. 10 birdhouses, \$150.

10. **Hard** A motorboat travels 48 miles upstream in 4 hours and 48 miles downstream in 3 hours. Find the speed of the boat in still water and the speed of the current.

Show answer

Upstream $48 \div 4 = 12$, downstream $48 \div 3 = 16$: $b - c = 12$, $b + c = 16$; $b = 14$, $c = 2$. Boat 14 mph, current 2 mph.

11. **Hard** The county fair sold 450 tickets for a show. Adult tickets were \$12 and student tickets \$7, and sales totaled \$4,400. How many of each were sold?

Show answer

$a + s = 450$, $12a + 7s = 4400$; $12a + 7(450 - a) = 4400$ gives $5a = 1250$,
 $a = 250$, $s = 200$. 250 adult and 200 student tickets.

12. **Hard** Grace graphs her bread business's cost line $C = 60 + 2x$ and revenue line $R = 5x$. Explain what the point where the lines cross means, and what happens when she sells more loaves than that.

Show answer

At the crossing point, cost equals revenue, so she has made back exactly what she spent: no gain and no loss (20 loaves, \$100). For more loaves the revenue line is above the cost line, because each loaf brings in \$5 but costs only \$2, so she makes a profit of \$3 on each extra loaf.

13. **Challenge** Samuel saves \$2,000 in two accounts. One pays 3% simple interest per year and the other pays 5%. After one year he earns \$84 in interest. How much is in each account?

Show answer

$x + y = 2000$, $0.03x + 0.05y = 84$; times 100: $3x + 5y = 8400$; subtract
 $3x + 3y = 6000$: $2y = 2400$, $y = 1200$, $x = 800$. \$800 at 3% and \$1,200 at 5%.

14. **Challenge** A rectangular garden bed has a perimeter of 38 feet. If the length were doubled and the width tripled, the perimeter would be 92 feet. Find the length and width of the bed.

Show answer

$2l + 2w = 38$ gives $l + w = 19$; $2(2l) + 2(3w) = 92$ gives $2l + 3w = 46$; subtract
 $2l + 2w = 38$: $w = 8$, $l = 11$. Length 11 feet, width 8 feet.

COMMON MISTAKES

- Writing both equations about the count (or both about the value). Most problems need one equation for *how many* and one for *how much they are worth*.
- Mixing dollars and cents in the same equation, such as $d + q = 20$ and $10d + 25q = 3.50$. Use cents everywhere (350) or dollars everywhere (0.10, 0.25, 3.50).

- Using the total distance as a speed in current and wind problems. First find each actual speed with $r = d \div t$, then set $b + c$ and $b - c$ equal to those speeds.
- Stopping at the numbers. State the answer in words with units and check it against every fact in the problem.

Make it a game • Ticket Booth Proof. Pretend you run a ticket booth. Adult tickets cost 8 pretend dollars and child tickets cost 5. The parent buys 40 tickets for 245 pretend dollars. Define two variables, write the system, and solve it. Then prove it by counting out paper tickets and pretend bills. Make up a coin or mixture problem and swap.

6.6 LINEAR INEQUALITIES AND SYSTEMS OF INEQUALITIES

OBJECTIVES

- Graph a linear inequality in two variables with a dashed or solid boundary line and correct shading, using a test point.
- Explain why the points above the line $y = mx + b$ are exactly the points with $y > mx + b$.
- Graph a system of linear inequalities and identify its solution region, checking a point in every inequality.
- Write and interpret systems of inequalities for budget and planting constraints.

WATCH

Inequality Regions (1 of 3: Comparison to 1-dimensional inequalities)

Eddie Woo · 7:19

<https://www.youtube.com/watch?v=A9bZfj8fHz0>

WHEN TO WATCH

Before Example 1: graphing a linear inequality as a region.

Note: Eddie Woo teaches in Australia: he says 'region' for the solution area of an inequality.

BACKUP VIDEO

Multiple Inequality Regions (1 of 2: Intersection of regions)

Eddie Woo · 11:31

https://www.youtube.com/watch?v=_gmg_uYsoNQ

Before Example 2: the overlap of two inequality regions.

KEY TERMS

linear inequality in two variables

An inequality such as $y < 2x - 1$ or $3x + 2y \geq 6$. Its solutions are all ordered pairs that make it true.

boundary line

The line you get by replacing the inequality symbol with an equal sign. It splits the plane into two sides.

half-plane

All the points on one side of a line in the coordinate plane.

test point

A point not on the boundary line that you substitute into the inequality to decide which side to shade. $(0, 0)$ is easiest when the line does not pass through it.

system of linear inequalities

Two or more linear inequalities considered together. Its solutions are the points in the region where all the shaded half-planes overlap.

constraint

A limit in a real situation, such as a budget or the space in a garden, written as an inequality.

CORE IDEAS

An equation like $y = 2x - 1$ has a line of solutions. An inequality like $y < 2x - 1$ has a whole region of solutions. Start with the **boundary line** $y = 2x - 1$. Every point in the plane is either on this line, above it or below it.

Why does $y > mx + b$ mean above the line? Pick any x , say $x = 3$. The point of the line $y = 2x - 1$ with that x is $(3, 5)$. Points straight above it, like $(3, 6)$ or $(3, 9)$, have the same x but a bigger y , so they satisfy $y > 2x - 1$. Points straight below, like $(3, 4)$ or $(3, 0)$, have a smaller y , so they satisfy $y < 2x - 1$. This works for every x , so $y > mx + b$ is the whole half-plane above the line and $y < mx + b$ is the whole half-plane below it.

The boundary line is **dashed** for $<$ or $>$, because points on the line make the two sides equal, not less or greater, so they are not solutions. It is **solid** for $\leq$ or $\geq$, because points on the line are included.

A **test point** confirms which side to shade, and it is the safest method when the inequality is in standard form like $2x + 3y > 6$ (where solving for y might involve dividing by a negative and reversing the sign). Substitute a point not on the line, usually $(0, 0)$. If it makes the inequality true, shade the side containing it; if false, shade the other side. A vertical boundary such as $x \leq -2$ is a solid line $x = -2$ with the left side shaded.

For a **system** of inequalities, graph each one on the same axes. The solutions of the system are the points that satisfy *every* inequality, so they lie where all the shaded

regions **overlap**. Pick one point in the overlap and check it in each inequality. If the boundary lines are parallel and the regions never overlap, the system has **no solution**.

In real problems, the variables are often counts of things, so they cannot be negative. Add the constraints $x \geq 0$ and $y \geq 0$, which keep the region in the first quadrant, and remember that some answers must be whole numbers (you cannot buy half a bag of feed).

KEY FORMULAS

$y > mx + b$: dashed line, shade above	$y \geq mx + b$: solid line, shade above
$y < mx + b$: dashed line, shade below	$y \leq mx + b$: solid line, shade below
solution of a system of inequalities = the region where all shadings overlap	

Practice habit. Test a point twice: use $(0, 0)$ (or another point off the line) to decide where to shade, and then check one point from your final shaded region in every inequality.

WORKED EXAMPLES

Example 1. Graph $y < 2x - 1$.

Show solution

1. Boundary line: $y = 2x - 1$, with y-intercept -1 and slope 2. It passes through $(0, -1)$, $(1, 1)$ and $(2, 3)$.
2. The symbol is $<$, so the points on the line are not solutions: draw the line dashed.
3. Test $(0, 0)$: $0 < 2(0) - 1$ means $0 < -1$, which is false.
4. So shade the side that does not contain $(0, 0)$: the side below and to the right of the line.
5. Check a shaded point, $(3, 0)$: $0 < 2(3) - 1 = 5$ is true.

Answer: A dashed line through $(0, -1)$ and $(1, 1)$ (slope 2), with the region below the line shaded.

Example 2. Ruth is buying tomato plants at \$3 each and pepper plants at \$2 each. She can spend at most \$24, and her garden has room for at most 10 plants. Let x = tomato plants and y = pepper plants. Write a system of inequalities, describe its graph, and decide whether she can buy 4 tomato plants and 5 pepper plants.

Show solution

1. Budget: $3x + 2y \leq 24$. Space: $x + y \leq 10$. Counts cannot be negative: $x \geq 0$, $y \geq 0$.
2. Graph $3x + 2y = 24$ as a solid line through $(8, 0)$ and $(0, 12)$. Test $(0, 0)$: $0 \leq 24$ is true, so shade the side toward the origin.
3. Graph $x + y = 10$ as a solid line through $(10, 0)$ and $(0, 10)$. Test $(0, 0)$: $0 \leq 10$ is true, so shade toward the origin.
4. The two lines meet where $x + y = 10$ and $3x + 2y = 24$. Subtracting $2(x + y) = 20$ from the second gives $x = 4$, so $y = 6$: the point $(4, 6)$.
5. The overlap in the first quadrant is a region with corners $(0, 0)$, $(8, 0)$, $(4, 6)$ and $(0, 10)$.
6. Check $(4, 5)$: $3(4) + 2(5) = 22 \leq 24$ is true, and $4 + 5 = 9 \leq 10$ is true. Both counts are nonnegative.
7. So yes, 4 tomato plants and 5 pepper plants fit both her budget (\$22) and her space (9 plants).

Answer: Yes. The system is $3x + 2y \leq 24$, $x + y \leq 10$, $x \geq 0$, $y \geq 0$; the region has corners $(0, 0)$, $(8, 0)$, $(4, 6)$, $(0, 10)$, and $(4, 5)$ lies in it.

PRACTICE

1. Easy Is $(2, 1)$ a solution of $y > x - 3$?

Show answer

Yes: $1 > 2 - 3 = -1$ is true.

2. Easy Is $(1, 4)$ a solution of $2x + y \leq 5$?

Show answer

No: $2(1) + 4 = 6$, and $6 \leq 5$ is false.

3. Easy Graph $y \geq x + 1$.

Show answer

A solid line through $(0, 1)$ and $(1, 2)$ (slope 1), with the region above the line shaded.

4. Easy For $y < -3x + 2$, is the boundary line dashed or solid, and do you shade above or below it?

Show answer

Dashed (the symbol is $<$), and shade below the line.

5. Medium Graph $2x + 3y > 6$.

Show answer

A dashed line through $(3, 0)$ and $(0, 2)$; testing $(0, 0)$ gives $0 > 6$, false, so shade the side away from the origin (above the line).

6. Medium Graph $x \leq -2$ in the coordinate plane.

Show answer

A solid vertical line $x = -2$, with the region to the left of the line shaded.

7. Medium A graph shows a dashed line through $(0, 2)$ and $(1, 5)$, with the region below the line shaded. Write the inequality.

Show answer

Slope $\frac{5 - 2}{1 - 0} = 3$, y-intercept 2, dashed and below: $y < 3x + 2$.

8. Medium Is $(3, 1)$ a solution of the system $y < x$ and $x + y \geq 5$?

Show answer

No: $1 < 3$ is true, but $3 + 1 = 4$ and $4 \geq 5$ is false.

9. Medium Graph the system $y \geq x - 2$ and $y < -x + 4$, and name one point in the solution region.

Show answer

Solid line $y = x - 2$ shaded above; dashed line $y = -x + 4$ shaded below. The lines meet at $(3, 1)$, and the solution region is the wedge to the left of that point between the lines. One point: $(0, 0)$, since $0 \geq -2$ and $0 < 4$.

10. **Hard** Henry has at most \$24 to spend on feed. Chicken feed costs \$4 a bag and rabbit feed costs \$6 a bag. (a) Write an inequality using x = bags of chicken feed and y = bags of rabbit feed. (b) Can he buy 3 bags of chicken feed and 2 bags of rabbit feed?

Show answer

(a) $4x + 6y \leq 24$. (b) Yes: $4(3) + 6(2) = 24$, and $24 \leq 24$ is true (he spends exactly \$24).

11. **Hard** For a bake sale, the youth group sells pies for \$10 each and cookies for \$6 a dozen. They want to raise at least \$120, and they have time to bake at most 20 items (pies plus dozens of cookies). Write a system of inequalities and use it to decide whether 6 pies and 10 dozen cookies meet both goals.

Show answer

System: $10p + 6c \geq 120$, $p + c \leq 20$, $p \geq 0$, $c \geq 0$. Check (6, 10): $60 + 60 = 120 \geq 120$ and $6 + 10 = 16 \leq 20$. Yes, it meets both.

12. **Hard** Samuel plans x rows of beans and y rows of corn. He wants at least 2 rows of corn and at most 12 rows in all, and $x \geq 0$. The system is $y \geq 2$, $x + y \leq 12$, $x \geq 0$. Find the corners of the solution region and the greatest number of bean rows he can plant.

Show answer

Corners (0, 2), (10, 2) and (0, 12). The greatest x in the region is at (10, 2): 10 rows of beans.

13. **Hard** Explain why the graph of $y > 2x + 1$ is the region above the line $y = 2x + 1$.

Show answer

For any x , the point on the line has $y = 2x + 1$. Points directly above it have the same x and a larger y , so $y > 2x + 1$ there; points below have a smaller y . Since this is true for every x , the solutions fill the whole region above the line.

14. **Challenge** Write a system of three inequalities whose solution region is the triangle with corners (0, 0), (4, 0) and (0, 4), including its edges.

Show answer

$x \geq 0$, $y \geq 0$, and $x + y \leq 4$ (that is, $y \leq -x + 4$).

15. **Challenge** Does the system $y > 2x + 3$ and $y < 2x - 1$ have any solutions?

Explain briefly.

Show answer

No solution. The boundary lines are parallel (both slope 2), and $y = 2x + 3$ is always 4 units above $y = 2x - 1$. A point cannot be above the higher line and below the lower line at the same time, so the shaded regions never overlap.

COMMON MISTAKES

- Using a solid line for $<$ or $>$. Dashed means the boundary is not included; use solid only for $\leq$ and $\geq$.
- Shading by the symbol alone when the inequality is not solved for y . In $-2y > 4x$, for example, dividing by -2 reverses the sign. Use a test point to be sure.
- Choosing a test point on the boundary line, which tells you nothing. If the line passes through $(0, 0)$, test a point such as $(1, 0)$ or $(0, 1)$.
- Shading the union instead of the overlap for a system. The solution is only where every inequality is true, so check a point from your region in each one.

Make it a game • Inequality Treasure Map. The parent hides a button on a graph paper point and gives clues such as $y > x - 2$, $y \leq -x + 6$ and $x \geq 0$. Shade the region for each clue, using dashed or solid boundaries. Pick a test point to decide the shading. Then guess the treasure's location, and each guess must satisfy every clue before it can be spoken.

SYSTEMS OF EQUATIONS AND INEQUALITIES

In Unit 5 you studied one line at a time. In this unit you put two lines together and ask which ordered pairs make both equations true. You will solve these systems three ways: by graphing (where do the lines cross?), by substitution and by elimination, and you will learn why a system of two lines always has exactly one solution, no solution or infinitely many. Along the way you will see why each method is allowed: equal quantities may replace each other, and adding equals to equals keeps an equation balanced.

Systems are some of the most useful tools in algebra, because many real questions have two unknowns: how many adult and children's tickets were sold, how fast a boat goes and how fast the river flows, when a small business breaks even, or which of two plans costs less. The unit ends with linear inequalities in two variables and systems of inequalities, which describe all the choices that fit a budget or a garden plot.

This unit builds on solving equations (Unit 2), inequalities (Unit 3) and graphing lines (Unit 5). It prepares you for Geometry, where lines meet and cross, and for Algebra II, where you will solve larger systems and systems involving curves. All work in this unit is done by hand.

PREREQUISITES

- Solve multi-step linear equations, including equations with variables on both sides, identities and contradictions (Lessons 2.2–2.3).
- Solve a literal equation such as $ax + by = c$ for y (Lesson 2.5).
- Solve and graph linear inequalities, reversing the sign when multiplying or dividing by a negative (Lessons 3.1–3.2).
- Graph a line from slope-intercept form and from intercepts (Lessons 5.3 and 5.5).
- Recognize parallel lines by their equal slopes (Lesson 5.6).
- Write a linear model from a word problem with a starting amount and a rate (Lesson 5.7).

MASTERY CHECKLIST

- ☐ I can check whether an ordered pair is a solution of a system of equations or inequalities.
- ☐ I can solve a system of linear equations by graphing and check the crossing point in both equations.
- ☐ I can tell from slopes and y-intercepts whether a system has one solution, no solution or infinitely many, and explain why.
- ☐ I can solve a system by substitution, using parentheses when I substitute an expression.
- ☐ I can solve a system by elimination, adding or subtracting the equations, and explain why this is allowed.
- ☐ I can multiply one or both equations by a nonzero number before eliminating, and explain why the solutions do not change.
- ☐ I can recognize no solution or infinitely many solutions when both variables disappear.
- ☐ I can choose the best method (graphing, substitution or elimination) for a given system.
- ☐ I can write and solve systems for ticket, coin, mixture, break-even, current and wind, and plan-comparison problems.
- ☐ I can graph a linear inequality in two variables with a dashed or

solid boundary and the correct shading.

- I can graph a system of linear inequalities, find its solution region and use it for budget and planting constraints.

LESSONS IN THIS UNIT

6.1 Solving Systems by Graphing	2 sessions of 45 minutes
6.2 Solving Systems by Substitution	2 sessions of 45 minutes
6.3 Solving Systems by Elimination	2 sessions of 45 minutes
6.4 Elimination with Multiplication	2 sessions of 45 minutes
6.5 Applications of Systems	2 sessions of 45 minutes
6.6 Linear Inequalities and Systems of Inequalities	2 sessions of 45 minutes

UNIT 6 TEST

Time allowed: 60 minutes. No calculator. Show your work for every problem: write the system you use, one step per line, and check your solutions. All answers in this test are exact; give money to the nearest cent and include units in word-problem answers. For graphs, use graph paper and a ruler, and label the boundary lines.

1. **5 pts**

Is $(-1, 4)$ a solution of the system $y = -2x + 2$ and $x + y = 3$?

2. **6 pts**

Solve by graphing: $y = x - 3$ and $y = -2x + 3$.

3. **5 pts**

Without solving, tell how many solutions the system $y = \frac{2}{3}x + 1$ and $2x - 3y = -3$ has.

4. (6 pts)

Solve by substitution: $y = x - 4$ and $3x + 2y = 7$.

5. (6 pts)

Solve by substitution: $x - 3y = 2$ and $2x - 6y = 5$.

6. (6 pts)

Solve by elimination: $5x + 2y = 9$ and $3x - 2y = 15$.

7. (7 pts)

Solve by elimination: $4x + 3y = 1$ and $6x + 5y = 1$.

8. (7 pts)

Choose the best method and solve: $x = 2y - 1$ and $3x + 4y = 17$.

9. (8 pts)

A pancake breakfast at church served 150 people. Adults paid \$6 and children paid \$4, and the breakfast took in \$780. How many adults and how many children were served?

10. (7 pts)

A plane flies 1,800 miles with the wind in 4 hours and makes the return trip against the wind in 5 hours. Find the speed of the plane in still air and the speed of the wind.

11. (7 pts)

Grace mixes raisins costing \$3 per pound with almonds costing \$8 per pound to make 10 pounds of trail mix worth \$5 per pound. How many pounds of each does she use?

12. (7 pts)

Samuel sells jars of honey. He spent \$150 on equipment, each jar costs him \$3 to fill, and he sells each jar for \$8. How many jars must he sell to break even?

13. (6 pts)

Is $(-2, 3)$ a solution of the system $y > -x$ and $3x + y \leq 0$?

14. (7 pts)

Graph the system $y \leq \frac{1}{2}x + 2$ and $y > -x - 1$. Give the point where the boundary lines meet.

15. (5 pts)

Ruth wants to spend no more than \$30 on seed packets for the church garden. Carrot seeds cost \$2 a packet and bean seeds cost \$3 a packet. Write an inequality using x = carrot packets and y = bean packets.

16. (5 pts)

Henry solves a system by elimination and ends with $0 = 0$. Grace graphs the same system. What will Grace see, and why do the two results agree?

ANSWER KEY

1. Yes: $-2(-1) + 2 = 4$ and $-1 + 4 = 3$ are both true.
2. The lines cross at $(2, -1)$; check: $2 - 3 = -1$ and $-2(2) + 3 = -1$.
3. Infinitely many: $2x - 3y = -3$ becomes $y = \frac{2}{3}x + 1$, the same line.
4. $3x + 2(x - 4) = 7$, so $5x = 15$, $x = 3$, $y = -1$: $(3, -1)$.
5. $x = 3y + 2$; $2(3y + 2) - 6y = 5$ gives $4 = 5$, false: no solution.
6. Add: $8x = 24$, $x = 3$; $15 + 2y = 9$, $y = -3$: $(3, -3)$.
7. Multiply by 3 and by 2: $12x + 9y = 3$, $12x + 10y = 2$; subtract: $y = -1$; $4x - 3 = 1$, $x = 1$: $(1, -1)$.
8. Substitution, since x is isolated: $3(2y - 1) + 4y = 17$, $10y = 20$, $y = 2$, $x = 3$: $(3, 2)$.
9. $a + c = 150$, $6a + 4c = 780$; $6a + 4(150 - a) = 780$ gives $2a = 180$, $a = 90$, $c = 60$. 90 adults and 60 children.
10. $p + w = 450$, $p - w = 360$; add: $2p = 810$, $p = 405$, $w = 45$. Plane 405 mph, wind 45 mph.
11. $r + a = 10$, $3r + 8a = 50$; subtract $3r + 3a = 30$: $5a = 20$, $a = 4$, $r = 6$. 6 pounds of raisins and 4 pounds of almonds.
12. $8x = 150 + 3x$, so $5x = 150$, $x = 30$ jars.
13. Yes: $3 > 2$ is true, and $3(-2) + 3 = -3 \leq 0$ is true.
14. Solid line $y = \frac{1}{2}x + 2$ through $(0, 2)$ and $(2, 3)$, shaded below; dashed line $y = -x - 1$ through $(0, -1)$ and $(-1, 0)$, shaded above. The lines meet at $(-2, 1)$, and the solution region is the wedge to the right of that point between the lines; $(0, 0)$ is in it.
15. $2x + 3y \leq 30$
16. Grace will see only one line, because both equations describe the same line. Getting $0 = 0$ means the equations are true for exactly the same pairs, so every point on that line is a solution: infinitely many solutions, which matches the graph.

EXPONENTS AND EXPONENTIAL FUNCTIONS

In this unit you take a fresh look at exponents. You will learn why the rules for multiplying, dividing and raising powers work, what zero and negative exponents mean and why, and how scientific notation lets us write the mass of the earth or the size of a bacterium without a long string of zeros.

Then exponents move from the base to the variable. In an exponential function such as $y = a \cdot b^x$, each step multiplies by the same factor instead of adding the same amount. These functions describe growing towns, savings accounts earning compound interest, tractors losing value, radioactive substances with a half-life, and doubling patterns. Geometric sequences are the same idea counted in whole steps, the multiplying partner of the arithmetic sequences from Unit 4.

The exponent rules are the foundation for polynomials (Unit 8) and factoring (Unit 9), and exponential models return in Unit 13 when you choose between linear, quadratic and exponential models. From this unit on, a scientific calculator is allowed for messy arithmetic and expected for growth and decay problems.

PREREQUISITES

- Evaluate whole-number powers such as 2^5 and $(-3)^2$ (Pre-Algebra).
- Multiply and divide fractions and decimals and convert percents to decimals.
- Use the order of operations and evaluate expressions by substitution (Lesson 1.3).
- Use function notation and make tables and graphs of functions (Lessons 4.3 and 4.4).
- Write and use arithmetic sequences (Lesson 4.5) and linear functions (Unit 5).

MASTERY CHECKLIST

- ☐ I can explain and use the product, quotient and power rules for exponents.
- ☐ I can simplify products and quotients of monomials.
- ☐ I can explain why $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ and write expressions with only positive exponents.
- ☐ I can write numbers in scientific notation and in standard form.
- ☐ I can multiply and divide numbers in scientific notation and use them in science problems.
- ☐ I can make a table and describe the graph of $y = ab^x$ and tell growth from decay.
- ☐ I can tell whether a table is linear or exponential by checking differences and ratios.
- ☐ I can write and use growth and decay models $y = a(1 \pm r)^t$.
- ☐ I can use the compound interest formula with a calculator and round money to the nearest cent.
- ☐ I can solve half-life and depreciation problems.

- I can find the common ratio and any term of a geometric sequence using

$$a_n = a_1 \cdot r^{n-1}.$$

LESSONS IN THIS UNIT

7.1 Properties of Exponents	2 sessions of 45 minutes
7.2 Zero and Negative Exponents	2 sessions of 45 minutes
7.3 Scientific Notation	45 minutes
7.4 Exponential Functions	2 sessions of 45 minutes
7.5 Exponential Growth and Decay	2 sessions of 45 minutes
7.6 Geometric Sequences	45 minutes

UNIT 7 TEST

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem. Give exact answers unless a problem asks you to round; round money to the nearest cent. Write answers with exponents using positive exponents only, and assume no variable equals zero.

1. **5 pts**

Simplify $(4x^3y)(-2x^5y^6)$.

2. **6 pts**

Simplify $\frac{(3a^4b^2)^3}{9a^5b}$.

3. **5 pts**

Evaluate $5^{-2} + 5^0$.

4. **7 pts**

Simplify $\frac{15x^{-4}y^3}{5x^2y^{-2}}$ so that all exponents are positive.

5. (6 pts)

(a) Write 0.000058 in scientific notation. (b) Write 7.4×10^6 in standard form.

6. (7 pts)

Divide $\frac{4.8 \times 10^9}{1.2 \times 10^{-3}}$. Write the answer in scientific notation.

7. (8 pts)

The moon is about 3.84×10^5 kilometers from the earth. If a jet airliner could fly there at 9×10^2 kilometers per hour, about how many hours would the trip take? Round to the nearest hour.

8. (6 pts)

For $y = 6 \left(\frac{1}{3}\right)^x$, find y when $x = -1$, $x = 0$ and $x = 2$.

9. (7 pts)

Is the table x : 0, 1, 2, 3 and y : 2, 10, 50, 250 linear or exponential? Write an equation for it.

10. (8 pts)

Grace deposits \$5,000 in an account that pays 3% interest per year, compounded monthly. How much is in the account after 4 years? Round to the nearest cent.

11. (7 pts)

A farm truck costs \$36,000 and loses 15% of its value each year. Write a model for its value after t years and find its value after 3 years.

12. (6 pts)

Find the 7th term of the geometric sequence 2, -6 , 18, ...

13. (7 pts)

Samuel's family bakery sold 160 loaves of bread in its first month. Each month after that, sales were 1.5 times the month before. Write an explicit formula for the sales in month n and find the sales in month 5.

14. (8 pts)

Use the quotient rule to explain why $a^0 = 1$ and why $a^{-2} = \frac{1}{a^2}$ for $a \neq 0$.

15. **7 pts**

The graph of $y = a \cdot b^x$ (with $b > 0$) passes through $(0, 5)$ and $(2, 45)$. Write the equation, then find y when $x = 3$.

ANSWER KEY

1. $-8x^8y^7$

2. $\frac{27a^{12}b^6}{9a^5b} = 3a^7b^5$

3. $\frac{1}{25} + 1 = \frac{26}{25} = 1.04$

4. $3x^{-6}y^5 = \frac{3y^5}{x^6}$

5. (a) 5.8×10^{-5} (b) 7,400,000

6. 4×10^{12}

7. $\frac{3.84 \times 10^5}{9 \times 10^2} \approx 0.4267 \times 10^3 \approx 427$ hours

8. 18, 6, $\frac{2}{3}$

9. Exponential (constant ratio 5): $y = 2 \cdot 5^x$

10. $5000 \left(1 + \frac{0.03}{12}\right)^{48} = 5000(1.0025)^{48} \approx \$5,636.64$

11. $v = 36,000(0.85)^t$; $36,000(0.614125) = \$22,108.50$

12. $r = -3$; $a_7 = 2(-3)^6 = 1458$

13. $a_n = 160(1.5)^{n-1}$; $a_5 = 160(5.0625) = 810$ loaves

14. $\frac{a^3}{a^3} = a^{3-3} = a^0$, but a nonzero number divided by itself is 1, so $a^0 = 1$. Also $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, while dividing out three factors of a from top and bottom leaves $\frac{1}{a^2}$; so $a^{-2} = \frac{1}{a^2}$.

15. $a = 5$; $5b^2 = 45$, $b^2 = 9$, $b = 3$; $y = 5 \cdot 3^x$; at $x = 3$, $y = 135$

7.1 PROPERTIES OF EXPONENTS

OBJECTIVES

- Explain each exponent rule by writing out the repeated factors.
- Apply the product, quotient, power of a power, power of a product and power of a quotient rules.
- Simplify products and quotients of monomials completely.
- Write simplified expressions for areas and volumes with exponents.

WATCH

Algebra Basics: Laws Of Exponents - Math Antics

mathantics · 13:46

<https://www.youtube.com/watch?v=LkhPRz7Hocg>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'index laws' for the properties of exponents.

BACKUP VIDEO

Index Laws w/ Algebra

Eddie Woo · 8:22

<https://www.youtube.com/watch?v=9294eBabzVE>

Before Example 2: the exponent rules with variables, including power of a product.

KEY TERMS

power

An expression such as a^n that means the base a is used as a factor n times.

exponent

The small raised number that tells how many times the base is used as a factor; in 5^3 the exponent is 3.

base

The number or variable being multiplied by itself; in 5^3 the base is 5.

monomial

A number, a variable, or a product of a number and variables with whole-number exponents, such as 7, x , or $-4a^2b^5$.

simplified monomial

A monomial written with one coefficient, each variable appearing only once, and no powers of powers or parentheses left to remove.

CORE IDEAS

An exponent is shorthand for repeated multiplication: $x^4 = x \cdot x \cdot x \cdot x$. Every rule in this lesson comes from that one fact. If you ever forget a rule, write out the factors and count them; the rule will reappear in front of you.

Product of powers. What is $x^3 \cdot x^4$? Write it out: $(x \cdot x \cdot x)(x \cdot x \cdot x \cdot x)$. That is 3 factors of x followed by 4 more, so 7 factors in all: x^7 . In general $a^m \cdot a^n = a^{m+n}$. When you multiply powers with the **same base**, keep the base and **add** the exponents. The bases must match: $x^3 \cdot y^4$ cannot be combined.

Quotient of powers. What is $\frac{x^6}{x^2}$? Write $\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$. Each x on the bottom divides out one x on the top (since $\frac{x}{x} = 1$), leaving 4 factors: x^4 . In general $\frac{a^m}{a^n} = a^{m-n}$ for $a \neq 0$: keep the base and **subtract** the exponents. In this lesson the top exponent will always be larger; in Lesson 7.2 you will see what happens when it is not.

Power of a power. What is $(x^2)^3$? It means $x^2 \cdot x^2 \cdot x^2$, which is three groups of 2 factors, so $2 + 2 + 2 = 3 \times 2 = 6$ factors: x^6 . In general $(a^m)^n = a^{mn}$: **multiply** the exponents. Notice the difference: in $x^2 \cdot x^3$ you add because you are joining two groups; in $(x^2)^3$ you multiply because you are copying one group three times.

Power of a product and power of a quotient. $(3x)^2 = (3x)(3x) = 3 \cdot 3 \cdot x \cdot x = 9x^2$. Because multiplication can be done in any order, every factor inside the parentheses gets the exponent: $(ab)^n = a^n b^n$. The same thinking with fractions gives $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ for $b \neq 0$. Be careful: these rules are for **products and quotients only**. A sum is different, and $(a + b)^2$ is not $a^2 + b^2$ (you will see why in Unit 8).

Simplifying monomials. To simplify an expression like $(2a^3b)^4 \div (4a^5b^2)$, first remove parentheses with the power rules, then work with the coefficients (ordinary numbers) and each variable separately. Multiply or divide the coefficients as usual,

and use the product or quotient rule on each letter. A finished answer has one coefficient and each variable written once.

Watch the signs. A negative base raised to an even power is positive and to an odd power is negative: $(-2)^4 = 16$ but $(-2)^3 = -8$. Also, -3^2 means $-(3^2) = -9$, while $(-3)^2 = 9$. The exponent applies only to what is directly in front of it.

KEY FORMULAS

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$$

From history. For centuries, mathematicians wrote powers out in words or with abbreviations. René Descartes, in *La Géométrie* (1637), helped make the small raised number for a power the standard notation, writing a^3 and a^4 , though he still often wrote aa instead of a^2 . The tidy notation is what makes the exponent rules so easy to see.

Practice habit. Recite the five exponent rules until they are automatic, and say the reason with each one: "same base, multiply: add the exponents; power of a power: multiply the exponents." When unsure, write out the factors of a small example such as $x^2 \cdot x^3$ and count.

WORKED EXAMPLES

Example 1. Grace is planning a rectangular garden bed that is $3x^2$ feet wide and $5x^4$ feet long. Write a simplified expression for its area.

Show solution

1. Area of a rectangle: $A = lw = (5x^4)(3x^2)$.
2. Group the coefficients and the powers: $A = (5 \cdot 3)(x^4 \cdot x^2)$.
3. Multiply the coefficients: $5 \cdot 3 = 15$.
4. Same base, so add the exponents: $x^4 \cdot x^2 = x^{4+2} = x^6$.
5. So $A = 15x^6$ square feet.
6. Check with $x = 2$: the bed is $3 \cdot 4 = 12$ feet by $5 \cdot 16 = 80$ feet, area 960. And $15 \cdot 2^6 = 15 \cdot 64 = 960$. It matches.

Answer: $15x^6$ square feet

Example 2. Simplify $\frac{(2a^3b)^4}{4a^5b^2}$. Assume no variable equals zero.

Show solution

1. Power of a product: raise each factor in the parentheses to the 4th power:
 $(2a^3b)^4 = 2^4(a^3)^4b^4$.
2. Power of a power: $(a^3)^4 = a^{12}$, and $2^4 = 16$, so the top is $16a^{12}b^4$.
3. Now the expression is $\frac{16a^{12}b^4}{4a^5b^2}$.
4. Divide the coefficients: $16 \div 4 = 4$.
5. Quotient rule for a : $a^{12-5} = a^7$.
6. Quotient rule for b : $b^{4-2} = b^2$.
7. Result: $4a^7b^2$.
8. Check with $a = 1$, $b = 1$: original = $\frac{2^4}{4} = \frac{16}{4} = 4$, and $4 \cdot 1 \cdot 1 = 4$.

Answer: $4a^7b^2$

PRACTICE

1. Easy Simplify $x^4 \cdot x^7$.

Show answer

$$x^{11}$$

2. Easy Simplify $\frac{y^{10}}{y^4}$, where $y \neq 0$.

Show answer

$$y^6$$

3. Easy Simplify $(a^3)^5$.

Show answer

$$a^{15}$$

4. Easy Simplify $(3m)^2$.

Show answer

$$9m^2$$

5. Medium Simplify $(-2x^3y^2)(4xy^5)$.

Show answer

$$-8x^4y^7 \text{ (coefficients } -2 \cdot 4 = -8; x^{3+1}, y^{2+5})$$

6. Medium Simplify $\frac{24a^7b^3}{6a^2b}$. Assume no variable equals zero.

Show answer

$$4a^5b^2$$

7. Medium Simplify $(-3p^4)^3$.

Show answer

$$-27p^{12} \text{ (since } (-3)^3 = -27 \text{ and } (p^4)^3 = p^{12})$$

8. Medium Simplify $\left(\frac{2x^2}{y^3}\right)^3$, where $y \neq 0$.

Show answer

$$\frac{8x^6}{y^9}$$

9. **Medium** A square plot of farmland has sides $5x^2$ meters long. Write a simplified expression for its area.

Show answer

$$(5x^2)^2 = 25x^4 \text{ square meters}$$

10. **Medium** A seed company packs 2^5 seeds in each packet, 2^4 packets in each box, and 2^3 boxes in each crate. How many seeds are in one crate? Give the answer as a power of 2 and as a number.

Show answer

$$2^5 \cdot 2^4 \cdot 2^3 = 2^{12} = 4096 \text{ seeds}$$

11. **Hard** Simplify $(2a^2b)^3(3ab^4)^2$.

Show answer

$$72a^8b^{11} (8a^6b^3 \cdot 9a^2b^8)$$

12. **Hard** Simplify $\frac{(-2x^3y)^4}{8x^5y^2}$. Assume no variable equals zero.

Show answer

$$2x^7y^2 \text{ (the top is } 16x^{12}y^4\text{)}$$

13. **Hard** Henry builds a cube-shaped toy box with edges $4x^2$ centimeters long. His sister's box is also a cube, with edges twice as long, $8x^2$ centimeters. How many times as large is the volume of his sister's box?

Show answer

$$8 \text{ times } ((8x^2)^3 = 512x^6 \text{ and } (4x^2)^3 = 64x^6; 512 \div 64 = 8)$$

14. **Hard** A classmate says $x^3 \cdot x^4 = x^{12}$. Explain why this is wrong and give the correct answer.

Show answer

Writing out the factors, $x^3 \cdot x^4 = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x)$, which is 7 factors of x , so the answer is x^7 . You add exponents when multiplying powers with the same base; you multiply exponents only for a power of a power such as $(x^3)^4 = x^{12}$.

15. **Challenge** Simplify $\frac{(3x^2y^3)^2(2xy)^3}{6x^5y^7}$. Assume no variable equals zero.

Show answer

$$12x^2y^2 \text{ (the top is } 9x^4y^6 \cdot 8x^3y^3 = 72x^7y^9\text{)}$$

COMMON MISTAKES

- Multiplying exponents when you should add them: $x^3 \cdot x^4$ is x^7 , not x^{12} . Ask yourself: am I joining two groups (add) or copying one group several times (multiply)?
- Multiplying the bases: $2^3 \cdot 2^4$ is 2^7 , not 4^7 . The base stays the same; only the count of factors changes.
- Forgetting to raise the coefficient: $(3x)^2 = 9x^2$, not $3x^2$. Every factor inside the parentheses gets the exponent.
- Mixing up -3^2 and $(-3)^2$. Without parentheses the exponent applies only to 3, so $-3^2 = -9$, while $(-3)^2 = 9$.

Make it a game • Exponent Tall Tale. Parent writes three simplifications, two true and one tall tale, such as $(x^2)^3 = x^5$. Find the false one and prove it by writing out the repeated factors, so $x^2 \cdot x^4$ becomes six x's. Name the exponent rule that actually applies. Then swap: you write three claims with one tall tale and the parent hunts for it.

7.2 ZERO AND NEGATIVE EXPONENTS

OBJECTIVES

- Explain why $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$, using a pattern and the quotient rule.
- Evaluate numerical expressions with zero and negative exponents.
- Simplify expressions so that every exponent is positive.
- Use zero and negative exponents in measurement and other real situations.

WATCH

Understanding the Zero Index

Eddie Woo · 5:29

<https://www.youtube.com/watch?v=iCE6cCE9H10>

WHEN TO WATCH

Before Example 1: why any nonzero number to the zero power is 1.

Note: Eddie Woo teaches in Australia: he says 'index' for exponent.

BACKUP VIDEO

Negative Indices

Eddie Woo · 7:07

<https://www.youtube.com/watch?v=iQMyG9E-Fi0>

Before Example 2: negative exponents.

KEY TERMS

zero exponent

For any nonzero number a , $a^0 = 1$.

negative exponent

For any nonzero number a and whole number n , $a^{-n} = \frac{1}{a^n}$; a negative exponent means "one over" the positive power.

reciprocal

The number you multiply by to get 1;
the reciprocal of a is $\frac{1}{a}$, and the
reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

simplest form (with exponents)

An expression with one coefficient,
each base written once, and no zero or
negative exponents.

CORE IDEAS

So far an exponent has counted factors: 3^4 means four 3s multiplied. But "zero factors of 3" or "negative two factors of 3" make no sense as counting. Mathematicians therefore asked: what meaning for 3^0 and 3^{-2} would keep all the exponent rules working? There turns out to be only one sensible choice.

The pattern. Look down this list: $3^4 = 81$, $3^3 = 27$, $3^2 = 9$, $3^1 = 3$. Each time the exponent goes down by 1, the value is **divided by 3**. Keep the pattern going: $3^0 = 3 \div 3 = 1$, then $3^{-1} = 1 \div 3 = \frac{1}{3}$, then $3^{-2} = \frac{1}{3} \div 3 = \frac{1}{9}$, then $3^{-3} = \frac{1}{27}$. So $3^0 = 1$ and $3^{-2} = \frac{1}{3^2}$.

The quotient rule agrees. Any nonzero number divided by itself is 1, so $\frac{a^5}{a^5} = 1$. The quotient rule says $\frac{a^5}{a^5} = a^{5-5} = a^0$. For the rule to stay true, a^0 must equal 1. Now try $\frac{a^2}{a^5}$. Writing out factors, two a 's on top cancel two of the five on the bottom, leaving $\frac{1}{a^3}$. The quotient rule gives $a^{2-5} = a^{-3}$. So a^{-3} must mean $\frac{1}{a^3}$.

Why $a \neq 0$. Both arguments divide by a . Since you cannot divide by zero, 0^0 and 0^{-2} are left undefined. Whenever you see a zero or negative exponent on a variable in this course, assume the variable is not zero.

A negative exponent means reciprocal, not negative. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, a positive number. Turning it around, $\frac{1}{a^{-n}} = a^n$, so a factor with a negative exponent may move across the fraction bar if its exponent changes sign. For a fraction, $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$: flip, then square.

All the rules still work. Because the new meanings were chosen to fit the rules, the product, quotient and power rules from Lesson 7.1 hold for every integer exponent. To simplify, use the rules first, then rewrite any negative exponent as a reciprocal.

Only the factor that carries the exponent moves: in $7a^{-3}$ the exponent belongs to a , so $7a^{-3} = \frac{7}{a^3}$, not $\frac{1}{7a^3}$.

KEY FORMULAS

$$a^0 = 1 \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

$$\frac{1}{a^{-n}} = a^n \quad (a \neq 0)$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad (a \neq 0, b \neq 0)$$

Practice habit. Recite the powers of 2 from 2^{-3} to 2^{10} : $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \dots, 1024$.

Reading the list backward, from 1024 down, you divide by 2 each time, which is exactly why $2^0 = 1$ and $2^{-1} = \frac{1}{2}$.

WORKED EXAMPLES

Example 1. A carpenter's ruler is marked in sixteenths of an inch, and $\frac{1}{16}$ inch $= 2^{-4}$ inch. How many of these marks fit in a board 2^3 inches long?

Show solution

1. Divide the length by the size of one mark: $\frac{2^3}{2^{-4}}$.

2. Quotient rule: $2^{3-(-4)} = 2^{3+4}$.

3. So the number of marks is $2^7 = 128$.

4. Check: $2^3 = 8$ inches, and each inch holds 16 sixteenths, so $8 \times 16 = 128$.

Answer: $2^7 = 128$ marks

Example 2. Simplify $\frac{(2x^{-3}y^2)^{-2}}{x^4y^{-5}}$ so that all exponents are positive. Assume no variable equals zero.

Show solution

1. Power of a product: $(2x^{-3}y^2)^{-2} = 2^{-2}x^{(-3)(-2)}y^{(2)(-2)} = 2^{-2}x^6y^{-4}$.

2. The expression is now $\frac{2^{-2}x^6y^{-4}}{x^4y^{-5}}$.

3. Quotient rule for x : $x^{6-4} = x^2$.

4. Quotient rule for y : $y^{-4-(-5)} = y^1 = y$.

5. The coefficient: $2^{-2} = \frac{1}{4}$.

6. Result: $\frac{x^2y}{4}$.

7. Check with $x = 2, y = 1$: the top is $(2 \cdot \frac{1}{8})^{-2} = (\frac{1}{4})^{-2} = 16$, the bottom is $16 \cdot 1 = 16$, so the value is 1; and $\frac{4 \cdot 1}{4} = 1$.

Answer: $\frac{x^2y}{4}$

PRACTICE

1. Easy Evaluate 9^0 .

Show answer

$$1$$

2. Easy Evaluate 4^{-2} .

Show answer

$$\frac{1}{16}$$

3. Easy Rewrite x^{-5} with a positive exponent.

Show answer

$$\frac{1}{x^5}$$

4. Easy Evaluate $\frac{1}{2^{-3}}$.

Show answer

$$2^3 = 8$$

5. Medium Rewrite $7a^{-3}b^2$ with positive exponents.

Show answer

$$\frac{7b^2}{a^3} \text{ (only } a \text{ moves)}$$

6. Medium Simplify $\frac{y^3}{y^8}$ with a positive exponent.

Show answer

$$y^{-5} = \frac{1}{y^5}$$

7. Medium Evaluate $\left(\frac{2}{3}\right)^{-2}$.

Show answer

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

8. Medium Evaluate $3^{-1} + 3^{-2}$ as a single fraction.

Show answer

$$\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

9. Medium Ruth is measuring sugar with a scoop that holds 2^{-3} cup. How many scoops does she need for 2^1 cups of sugar?

Show answer

$$\frac{2^1}{2^{-3}} = 2^4 = 16 \text{ scoops}$$

10. Medium Henry's microscope makes objects appear 10^2 times as wide as they really are. A plant cell is about 10^{-3} centimeter wide. How wide does it appear under the microscope?

Show answer

$$10^{-3} \cdot 10^2 = 10^{-1} = 0.1 \text{ centimeter}$$

11. Hard Simplify $(-3x^{-2}y^4)^{-2}$ so that all exponents are positive.

Show answer

$$\frac{x^4}{9y^8} \text{ (since } (-3)^{-2} = \frac{1}{9}, x^4, y^{-8} \text{)}$$

12. **Hard** Simplify $\frac{12a^{-2}b^5}{4a^3b^{-1}}$ so that all exponents are positive.

Show answer

$$3a^{-5}b^6 = \frac{3b^6}{a^5}$$

13. **Hard** A ball is dropped from 16 feet. After n bounces it rises to $16 \cdot 2^{-n}$ feet. How high does it rise after the 3rd bounce?

Show answer

$$16 \cdot 2^{-3} = \frac{16}{8} = 2 \text{ feet}$$

14. **Hard** Samuel says $5^0 = 0$ because "zero fives is nothing." Explain why 5^0 is actually 1.

Show answer

Using the quotient rule, $\frac{5^3}{5^3} = 5^{3-3} = 5^0$, but any nonzero number divided by itself is 1, so $5^0 = 1$. The pattern agrees: $5^2 = 25$, $5^1 = 5$, and dividing by 5 again gives $5^0 = 1$.

15. **Challenge** Simplify $(x^{-2}y^3)^{-3}(x^{-1}y)^4$ so that all exponents are positive.

Show answer

$$x^6y^{-9} \cdot x^{-4}y^4 = x^2y^{-5} = \frac{x^2}{y^5}$$

COMMON MISTAKES

- Thinking a negative exponent makes the number negative: 4^{-2} is $\frac{1}{16}$, not -16 . A negative exponent means reciprocal.
- Moving the coefficient along with the variable: $7a^{-3} = \frac{7}{a^3}$, not $\frac{1}{7a^3}$. Only the factor that carries the negative exponent moves.
- Writing $a^0 = 0$. Any nonzero number to the zero power is 1; remember the pattern of dividing by the base.
- Confusing -6^0 with $(-6)^0$. The first is $-(6^0) = -1$; the second is 1.

Make it a game • Halving Staircase. On a paper strip, write $3^4 = 81$. Divide by 3 at each step and keep writing: 3^3 , 3^2 , 3^1 , then keep going into 3^0 and 3^{-1} . Say aloud what pattern forces $3^0 = 1$ and $3^{-2} = 1/9$. Then the parent flashes six cards like 5^0 , 2^{-3} and $(1/2)^{-2}$ and you answer in under five seconds each.

7.3 SCIENTIFIC NOTATION

OBJECTIVES

- Write very large and very small numbers in scientific notation and in standard form.
- Explain why moving the decimal point matches multiplying or dividing by powers of 10.
- Multiply and divide numbers in scientific notation using the exponent rules.
- Solve measurement problems from science with numbers in scientific notation.

WATCH

Math Antics - Scientific Notation

[mathantics](#) · 14:28

<https://www.youtube.com/watch?v=bXkewQ7WEI>

WHEN TO WATCH

Before Example 1.

Note: The backup is Professor Leonard (alternate tier). Math Antics is American and says 'scientific notation'.

BACKUP VIDEO

Operations with Scientific Notation (TTP Video 76)

[Professor Leonard](#) · 19:18

https://www.youtube.com/watch?v=8sspC_FxnZY

Before Example 2: multiplying and dividing numbers in scientific notation (about 19 minutes).

KEY TERMS

scientific notation

A way of writing a number as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer, such as 3.2×10^5 .

standard form

A number written out with all its digits, such as 320,000 or 0.0045.

power of 10

A number such as $10^3 = 1000$ or $10^{-2} = 0.01$; multiplying by one moves the decimal point.

E notation

How a calculator shows scientific notation: 3.2E5 means 3.2×10^5 .

CORE IDEAS

Scientists work with numbers like the mass of the earth (about 5,970,000,000,000,000,000,000 kilograms) and the length of a bacterium (about 0.000002 meter). Counting all those zeros is slow and easy to get wrong. **Scientific notation** writes such a number as a number from 1 up to (but not including) 10, times a power of 10: the earth's mass is about 5.97×10^{24} kg.

Why the decimal point moves. Our number system is built on place value, and each place is worth 10 times the place to its right. So multiplying by 10 shifts every digit one place left, which looks like moving the decimal point one place right:

$3.2 \times 10 = 32$. Multiplying by 10^5 does this five times: $3.2 \times 10^5 = 320,000$. Since $10^{-3} = \frac{1}{1000}$, multiplying by 10^{-3} divides by 10 three times and moves the point three places left: $4.5 \times 10^{-3} = 0.0045$.

Writing a number in scientific notation. Move the decimal point until exactly one nonzero digit is to its left. Count the places you moved. If the original number is 10 or more, the exponent is positive; if it is less than 1, the exponent is negative; if it is already from 1 up to 10, the exponent is 0 (for example, $4.5 = 4.5 \times 10^0$). For example, $72,000 = 7.2 \times 10^4$ (moved 4 places), and $0.00072 = 7.2 \times 10^{-4}$ (moved 4 places, and the number is small).

Multiplying and dividing. Because multiplication can be regrouped, $(3 \times 10^4)(2 \times 10^5) = (3 \times 2)(10^4 \times 10^5) = 6 \times 10^9$. Multiply the front numbers and use the product rule on the powers of 10. Division works the same way with the quotient rule:

$$\frac{8 \times 10^7}{2 \times 10^3} = 4 \times 10^4.$$

Fixing the front number. Sometimes the front number lands outside 1 to 10. Then trade a factor of 10: $20 \times 10^4 = 2 \times 10^1 \times 10^4 = 2 \times 10^5$, and $0.5 \times 10^3 = 5 \times 10^{-1} \times 10^3 = 5 \times 10^2$. A quick rule: if the front number gets smaller, the exponent gets bigger by the same number of places, and the other way around.

A scientific calculator is allowed for messy front numbers. Calculators often show answers like 6.02E23, which means 6.02×10^{23} . Always write the answer back in proper notation, with units.

KEY FORMULAS

$$a \times 10^n, \quad 1 \leq a < 10, \quad n \text{ an integer}$$

$$(a \times 10^m)(b \times 10^n) = (ab) \times 10^{m+n}$$

$$\frac{a \times 10^m}{b \times 10^n} = \frac{a}{b} \times 10^{m-n}$$

From history. Large numbers fascinated the ancient Greeks too. In *The Sand Reckoner*, Archimedes of Syracuse (third century BC) estimated how many grains of sand would fill the universe as the Greeks pictured it, and invented a way of naming numbers far larger than any Greek words allowed. His answer, in modern notation, was about 10^{63} grains.

Practice habit. Estimate first. Before multiplying $(3.1 \times 10^4)(2.9 \times 10^6)$, think "about $3 \times 3 = 9$, and $10^{4+6} = 10^{10}$," so the answer should be near 9×10^{10} . If your calculator says something else, look for a typing mistake.

WORKED EXAMPLES

Example 1. Write 0.00072 in scientific notation.

Show solution

1. Move the decimal point right until one nonzero digit is in front: 0.00072 becomes 7.2.
2. The point moved 4 places.
3. The original number is less than 1, so the exponent is negative: 10^{-4} .
4. So $0.00072 = 7.2 \times 10^{-4}$.
5. Check: 7.2×10^{-4} means move the point 4 places left: 0.00072.

Answer: 7.2×10^{-4}

Example 2. The sun is about 1.5×10^8 kilometers from the earth, and light travels about 3×10^5 kilometers per second. About how many seconds does sunlight take to reach the earth? How many minutes is that?

Show solution

1. Time = distance $\div$ rate: $t = \frac{1.5 \times 10^8}{3 \times 10^5}$.
 2. Divide the front numbers: $1.5 \div 3 = 0.5$.
 3. Quotient rule: $10^{8-5} = 10^3$.
 4. So $t = 0.5 \times 10^3$; fix the front number: $5 \times 10^2 = 500$ seconds.
 5. In minutes: $500 \div 60 \approx 8.3$ minutes.
 6. Check: $(3 \times 10^5)(5 \times 10^2) = 15 \times 10^7 = 1.5 \times 10^8$ km.
-

Answer: about $5 \times 10^2 = 500$ seconds, a little over 8 minutes

PRACTICE

1. Easy Write 4,500,000 in scientific notation.

Show answer

$$4.5 \times 10^6$$

2. Easy Write 0.0036 in scientific notation.

Show answer

$$3.6 \times 10^{-3}$$

3. Easy Write 6.02×10^5 in standard form.

Show answer

$$602,000$$

4. Easy Write 9.1×10^{-4} in standard form.

Show answer

$$0.00091$$

5. Medium The number 12×10^4 is not in scientific notation. Rewrite it correctly.

Show answer

$$1.2 \times 10^5 \text{ (since } 12 = 1.2 \times 10^1 \text{)}$$

6. **Medium** Multiply $(3 \times 10^4)(2 \times 10^5)$. Write the answer in scientific notation.

Show answer

$$6 \times 10^9$$

7. **Medium** Divide $\frac{8 \times 10^7}{2 \times 10^3}$. Write the answer in scientific notation.

Show answer

$$4 \times 10^4$$

8. **Medium** Multiply $(5 \times 10^6)(4 \times 10^{-2})$. Write the answer in scientific notation.

Show answer

$$20 \times 10^4 = 2 \times 10^5$$

9. **Medium** The mass of the earth is about 5.97×10^{24} kilograms, and the mass of the moon is about 7.35×10^{22} kilograms. About how many times as massive as the moon is the earth? Use a calculator and round to the nearest whole number.

Show answer

$$\frac{5.97}{7.35} \times 10^2 \approx 0.812 \times 10^2, \text{ about 81 times}$$

10. **Hard** Divide $\frac{6.4 \times 10^{-3}}{1.6 \times 10^{-8}}$. Write the answer in scientific notation.

Show answer

$$4 \times 10^{-3-(-8)} = 4 \times 10^5$$

11. **Hard** A certain kind of bacterium is about 2×10^{-6} meter long. How many of them, lined up end to end, would stretch 1 centimeter (10^{-2} meter)?

Show answer

$$\frac{10^{-2}}{2 \times 10^{-6}} = 0.5 \times 10^4 = 5 \times 10^3 = 5000 \text{ bacteria}$$

12. **Hard** The Miller children's sandbox holds about 3×10^3 cups of sand, and one cup of fine sand holds about 2×10^6 grains. About how many grains of sand are in the sandbox? Write the answer in scientific notation.

Show answer

$$(3 \times 10^3)(2 \times 10^6) = 6 \times 10^9 \text{ grains (about six billion)}$$

13. **Hard** Explain why 35×10^3 is not in scientific notation, and explain why 0.00045 has a negative exponent when written in scientific notation.

Show answer

In scientific notation the front number must be at least 1 and less than 10, and 35 is too big; correctly it is 3.5×10^4 . The number 0.00045 is less than 1, so it equals 4.5 divided by 10 four times, and dividing by 10^4 is multiplying by 10^{-4} : 4.5×10^{-4} .

14. **Challenge** A light-year is the distance light travels in one year. Light travels about 3×10^5 kilometers per second, and one year has about 3.15×10^7 seconds. About how many kilometers is a light-year? Write the answer in scientific notation.

Show answer

$$(3 \times 10^5)(3.15 \times 10^7) = 9.45 \times 10^{12} \text{ kilometers}$$

COMMON MISTAKES

- Using a front number that is 10 or more, or less than 1, such as 45×10^3 or 0.45×10^5 . Check that the front number has exactly one nonzero digit before the decimal point.
- Getting the sign of the exponent backwards. Big numbers (10 or more) have positive exponents; numbers between 0 and 1 have negative exponents.
- Multiplying the exponents when multiplying in scientific notation.
 $10^4 \times 10^5 = 10^9$, not 10^{20} : same base, add the exponents.
- Forgetting to adjust after multiplying: $6 \times 10^3 \times 5 \times 10^2 = 30 \times 10^5$ must be rewritten as 3×10^6 .

Make it a game • Decimal Hop. Write 93,000,000 on paper, the approximate miles from Earth to the Sun. Use a coin as the decimal point and hop it until one nonzero digit remains in front, counting the hops to find the power of 10. Then convert 0.00007, a rough width of a hair in meters. The parent calls ten numbers, and you flip between standard form and scientific notation.

7.4 EXPONENTIAL FUNCTIONS

OBJECTIVES

- Make a table of values and describe the graph of $y = a \cdot b^x$.
- Identify exponential growth ($b > 1$) and decay ($0 < b < 1$) and explain why the y-intercept is a .
- Tell whether a table shows a linear or an exponential pattern by checking differences and ratios.
- Write an exponential function from a table or from two points.

WATCH

Exponential Graphs (1 of 2: Considering the Values)

Eddie Woo · 9:58

<https://www.youtube.com/watch?v=QJKxBTdQyO8>

WHEN TO WATCH

Before Example 1: how the values of $y = b^x$ behave.

BACKUP VIDEO

Exponential Graphs (2 of 2: Plotting Points)

Eddie Woo · 10:06

<https://www.youtube.com/watch?v=FTc7-uwOzbQ>

Before Example 2: plotting an exponential graph from a table.

KEY TERMS

exponential function

A function of the form $y = a \cdot b^x$ with $a \neq 0$, $b > 0$ and $b \neq 1$; the variable is in the exponent.

exponential growth

A pattern where $b > 1$, so the outputs get larger by the same factor at each step (for $a > 0$).

growth factor (base)

The number b that each output is multiplied by when x goes up by 1.

exponential decay

A pattern where $0 < b < 1$, so the outputs shrink by the same factor at each step, getting closer and closer to 0.

common ratio

In a table with equally spaced x -values, the number you get by dividing any output by the one before it.

CORE IDEAS

A linear function **adds** the same amount each time x goes up by 1. An **exponential function multiplies** by the same amount each time. Compare $y = 2x + 3$, whose outputs for $x = 0, 1, 2, 3$ are 3, 5, 7, 9 (add 2 each time), with $y = 3 \cdot 2^x$, whose outputs are 3, 6, 12, 24 (multiply by 2 each time). Constant difference means linear; constant ratio means exponential.

Why the pattern multiplies. In $y = a \cdot b^x$, raising x by 1 gives $a \cdot b^{x+1} = a \cdot b^x \cdot b$ by the product rule. So each new output is the old output times b . That is why b is called the growth factor.

Why the y-intercept is a . The graph crosses the y-axis where $x = 0$. Then $y = a \cdot b^0 = a \cdot 1 = a$, using $b^0 = 1$ from Lesson 7.2. So a is the starting value.

Growth and decay. If $b > 1$ (and $a > 0$), multiplying by b makes the outputs grow, slowly at first and then very fast: this is **exponential growth**. If $0 < b < 1$, multiplying by b makes the outputs shrink: this is **exponential decay**. For example, $y = 80 \left(\frac{1}{2}\right)^x$ gives 80, 40, 20, 10, 5, ... We require $b > 0$ (a negative base would make the outputs flip signs back and forth) and $b \neq 1$ (then $y = a$, a horizontal line).

The graph. For $y = 2^x$: $x = -2, -1, 0, 1, 2, 3$ gives $y = \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$. The graph passes through $(0, 1)$, rises more and more steeply to the right, and to the left it gets closer and closer to the x-axis but never touches it, because each step to the left divides a positive number by 2, and dividing a positive number by 2 never gives 0. So for $a > 0$ the domain is all real numbers and the range is $y > 0$.

Exponential beats linear. Because an exponential function multiplies while a linear function only adds, a growing exponential function will eventually pass any linear function, no matter how big the linear function's head start. For example, 2^x is smaller than $50 + 20x$ for $x = 0$ to 7, but from $x = 8$ on it is larger ($2^8 = 256 > 210$).

Writing an equation from a table. If the x -values go up by 1, divide each output by the one before it. If the ratio is constant, that ratio is b , and the output at $x = 0$ is a .

KEY FORMULAS

$$y = a \cdot b^x \quad (a \neq 0, b > 0, b \neq 1)$$

growth: $b > 1$ decay: $0 < b < 1$

y-intercept: $(0, a)$

linear: constant difference exponential: constant ratio

Practice habit. When you meet a table, check both ways: subtract neighboring outputs (constant difference means linear) and divide them (constant ratio means exponential). Write the differences or ratios right under the table.

WORKED EXAMPLES

Example 1. Make a table of values for $y = 3 \cdot 2^x$ using $x = -1, 0, 1, 2, 3$, and describe the graph.

Show solution

1. $x = -1: y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2} = 1.5.$

2. $x = 0: y = 3 \cdot 2^0 = 3 \cdot 1 = 3.$

3. $x = 1: y = 3 \cdot 2 = 6.$

4. $x = 2: y = 3 \cdot 4 = 12.$

5. $x = 3: y = 3 \cdot 8 = 24.$

6. Check the ratio: $6 \div 3 = 2, 12 \div 6 = 2, 24 \div 12 = 2$, matching $b = 2$.

7. The graph crosses the y-axis at $(0, 3)$, rises faster and faster to the right (growth, since $b = 2 > 1$), and approaches the x-axis on the left without touching it.

Answer: y-values 1.5, 3, 6, 12, 24; an increasing curve through $(0, 3)$ that approaches the x-axis on the left

Example 2. Samuel drops a ball and measures how high it rises after each bounce. Bounce n : 0, 1, 2, 3 and height in inches: 81, 54, 36, 24 (bounce 0 is the starting height). Is the pattern linear or exponential? Write an equation, and predict the height after the 4th bounce.

Show solution

1. Differences: $54 - 81 = -27$, $36 - 54 = -18$. Not constant, so not linear.

2. Ratios: $\frac{54}{81} = \frac{2}{3}$, $\frac{36}{54} = \frac{2}{3}$, $\frac{24}{36} = \frac{2}{3}$. Constant, so exponential with $b = \frac{2}{3}$.

3. At $n = 0$ the height is 81, so $a = 81$.

4. Equation: $y = 81 \left(\frac{2}{3}\right)^n$.

5. Predict $n = 4$: $y = 81 \cdot \frac{16}{81} = 16$ inches.

6. Check: $24 \cdot \frac{2}{3} = 16$. It matches the pattern.

Answer: Exponential decay; $y = 81 \left(\frac{2}{3}\right)^n$; 16 inches after the 4th bounce

PRACTICE

1. Easy Evaluate $y = 5 \cdot 2^x$ when $x = 3$.

Show answer

$$5 \cdot 8 = 40$$

2. Easy What is the y-intercept of the graph of $y = 7 \cdot 3^x$?

Show answer

7, the point (0, 7), because $3^0 = 1$

3. Easy Does $y = 4(0.6)^x$ show exponential growth or decay? What is the factor?

Show answer

Decay, because $0 < 0.6 < 1$; each output is multiplied by 0.6.

4. Easy Is this table linear or exponential? x: 0, 1, 2, 3 and y: 2, 6, 18, 54.

Show answer

Exponential: the ratio is constant ($\times 3$), while the differences 4, 12, 36 are not constant.

5. **Medium** Find the y-values of $y = 2 \cdot 3^x$ for $x = -2, -1, 0, 1, 2$.

Show answer

$$\frac{2}{9}, \frac{2}{3}, 2, 6, 18$$

6. **Medium** Write an exponential function for the table x: 0, 1, 2, 3 and y: 5, 10, 20, 40.

Show answer

$$y = 5 \cdot 2^x \text{ (} a = 5, \text{ ratio } 2\text{)}$$

7. **Medium** Write an exponential function for the table x: 0, 1, 2, 3 and y: 64, 16, 4, 1.

Show answer

$$y = 64 \left(\frac{1}{4}\right)^x \text{ (} a = 64, \text{ ratio } \frac{1}{4}\text{)}$$

8. **Medium** Give the domain and range of $y = 2^x$.

Show answer

Domain: all real numbers; range: $y > 0$

9. **Medium** A lab dish starts with 300 bacteria, and the number triples every hour. Write a function for the number after t hours, and find the number after 4 hours.

Show answer

$$y = 300 \cdot 3^t; 300 \cdot 81 = 24,300 \text{ bacteria}$$

10. **Hard** A water filter lets only 25% of the dirt particles through on each pass. Water starts with 4096 particles per milliliter. Write a function for the number of particles after n passes, and find the number after 3 passes.

Show answer

$$y = 4096(0.25)^n; 4096 \cdot \frac{1}{64} = 64 \text{ particles per milliliter}$$

11. **Hard** Samuel compares two savings balances. Plan A has $50 + 20x$ dollars after x weeks, and Plan B has 2^x dollars after x weeks. After how many whole weeks does Plan B first have more money than Plan A?

Show answer

8 weeks (week 7: \$128 vs. \$190; week 8: \$256 vs. \$210)

12. **Hard** The graph of $y = a \cdot b^x$ (with $b > 0$) passes through $(0, 6)$ and $(2, 54)$. Find a and b .

Show answer

$a = 6$ (the y-intercept); $6b^2 = 54$, so $b^2 = 9$ and $b = 3$

13. **Hard** Explain why the graph of $y = a \cdot b^x$ always crosses the y-axis at $(0, a)$.

Show answer

The graph crosses the y-axis where $x = 0$. Then $y = a \cdot b^0$, and since $b^0 = 1$ for any nonzero b , $y = a$.

14. **Challenge** The graph of an exponential function $y = a \cdot b^x$ (with $b > 0$) passes through $(1, 12)$ and $(3, 108)$. Write the equation.

Show answer

Going from $x = 1$ to $x = 3$ multiplies by b^2 : $b^2 = \frac{108}{12} = 9$, so $b = 3$; then $a \cdot 3 = 12$, so $a = 4$: $y = 4 \cdot 3^x$

COMMON MISTAKES

- Multiplying a and b before applying the exponent: $5 \cdot 2^3 = 5 \cdot 8 = 40$, not 10^3 . Exponents come before multiplication.
- Checking only differences in a table. A table can fail to be linear and still be exponential; divide neighboring outputs to look for a constant ratio.
- Thinking the decay factor is the amount lost. In $y = 4(0.6)^x$ each step keeps 60% and loses 40%.
- Believing a decay graph eventually reaches zero. Multiplying a positive number by b never gives 0, so the graph only gets closer and closer to the x-axis.

Make it a game • Fold It Forty Times. Fold a sheet of paper in half over and over, counting layers after each fold: 1, 2, 4, 8. Write the table and the equation $y = 2^x$. Predict the layers after 10 folds. Then tear a sheet in half again and again, tracking the fraction left, and write $y = (1/2)^x$. Say which graph is growth and which is decay.

7.5 EXPONENTIAL GROWTH AND DECAY

OBJECTIVES

- Explain why growing by a rate r means multiplying by $1 + r$ and decaying by r means multiplying by $1 - r$.
- Write and use growth and decay models $y = a(1 + r)^t$ and $y = a(1 - r)^t$.
- Use the compound interest formula $A = P \left(1 + \frac{r}{n}\right)^{nt}$ with a calculator.
- Solve problems about population, depreciation, half-life and cooling.

WATCH

Compound Interest (1 of 2: Developing the formula)

Eddie Woo · 10:20

<https://www.youtube.com/watch?v=Ry6HNV3pw-Y>

WHEN TO WATCH

Before Example 1: compound growth, building the formula year by year.

Note: Eddie Woo teaches in Australia: he uses Australian dollars and says 'depreciation' for the loss in value.

BACKUP VIDEO

Depreciation (3 of 3: Declining Balance Method)

Eddie Woo · 9:08

https://www.youtube.com/watch?v=CPRL_gsbZeo

Before Example 2: decay, as a value that loses a fixed percent each year.

KEY TERMS

growth rate

The percent by which an amount increases each time period, written as a decimal r (5% means $r = 0.05$).

decay rate

The percent by which an amount decreases each time period, written as a decimal r .

compound interest

Interest paid on both the original money (the principal) and on the interest already earned.

depreciation

The loss in value of an item, such as a truck or tractor, as it ages.

half-life

The time it takes for half of a radioactive substance to decay, so that half the amount remains.

CORE IDEAS

Why $1 + r$. Suppose a town of 12,000 people grows by 3% this year. The new population is the old 100% plus 3% more: $12,000 + 0.03(12,000) = 12,000(1 + 0.03) = 12,000(1.03)$. Factoring out the starting amount shows that growing by 3% is the same as **multiplying by 1.03**. Next year we multiply by 1.03 again, so after t years the population is $12,000(1.03)^t$. In general, growth at rate r per period gives $y = a(1 + r)^t$.

Why $1 - r$. If a tractor loses 12% of its value each year, what remains is $100\% - 12\% = 88\%$ of last year's value: $v - 0.12v = v(1 - 0.12) = 0.88v$. After t years the value is $a(0.88)^t$. In general, decay at rate r gives $y = a(1 - r)^t$. These are the exponential functions of Lesson 7.4 with $b = 1 + r$ or $b = 1 - r$.

Growth is not adding the same amount. A 3% increase on a larger number is a larger increase. That is why the growth speeds up over time, unlike simple interest, which adds the same amount every year.

Compound interest. A bank paying 4% per year *compounded quarterly* pays $\frac{4\%}{4} = 1\%$ every quarter (every 3 months). Each quarter the balance is multiplied by $1 + 0.01$. In 5 years there are $4 \times 5 = 20$ quarters, so \$2,000 becomes $2000(1.01)^{20}$. In general, with principal P , yearly rate r , n compounding periods per year and t years, each period multiplies by $1 + \frac{r}{n}$ and there are nt periods: $A = P \left(1 + \frac{r}{n}\right)^{nt}$.

Half-life. A radioactive substance loses half its amount every half-life h . After one half-life the amount is $a \left(\frac{1}{2}\right)^1$; after two, $a \left(\frac{1}{2}\right)^2$. After t time units there have been $\frac{t}{h}$ half-lives, so $y = a \left(\frac{1}{2}\right)^{t/h}$. If $\frac{t}{h}$ is not a whole number, such as 2.5, the formula still works: type the exponent into the calculator in parentheses, as in $200(0.5)^{(20/8)}$. You will learn what fractional exponents mean in Algebra II.

Cooling. A hot drink does not cool toward 0 degrees; it cools toward the room's temperature. What decays exponentially is the **difference** between the drink's temperature and the room's. Find that difference with a decay model, then add the room temperature back.

Using a calculator. Type powers with the $\wedge$ or x^y key and use parentheses: $2000(1 + 0.04/4)^{(4 \times 5)}$. Do not round until the very end; round money to the nearest cent and people to the nearest whole person.

KEY FORMULAS

$$y = a(1 + r)^t \quad (\text{growth})$$

$$y = a(1 - r)^t \quad (\text{decay})$$

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

$$y = a \left(\frac{1}{2}\right)^{t/h} \quad (\text{half-life } h)$$

Practice habit. Estimate with the Rule of 72: money growing at r percent a year doubles in about $72 \div r$ years (at 8%, about 9 years). Use it to judge whether a calculator answer is reasonable.

WORKED EXAMPLES

Example 1. A town has 12,000 people, and its population grows 3% per year. Write a model and find the population after 5 years, to the nearest whole person.

Show solution

1. Starting amount $a = 12,000$; rate $r = 0.03$, so the growth factor is $1 + 0.03 = 1.03$.
2. Model: $y = 12,000(1.03)^t$.
3. Substitute $t = 5$: $y = 12,000(1.03)^5$.
4. Calculator: $1.03^5 \approx 1.159274$.
5. $y \approx 12,000 \times 1.159274 \approx 13,911.29$.
6. Round to a whole person: about 13,911 people.
7. Check the size: 3% of 12,000 is 360, and five years of about 360 to 400 more people each year gives roughly 1,900 more. Reasonable.

Answer: $y = 12,000(1.03)^t$; about 13,911 people

Example 2. Ruth deposits \$2,000 in a savings account that pays 4% interest per year, compounded quarterly. How much is in the account after 5 years, if she makes no other deposits or withdrawals? How much interest did she earn?

Show solution

1. Identify: $P = 2000$, $r = 0.04$, $n = 4$, $t = 5$.
2. Rate per quarter: $\frac{r}{n} = \frac{0.04}{4} = 0.01$.
3. Number of quarters: $nt = 4 \times 5 = 20$.
4. $A = 2000(1.01)^{20}$.
5. Calculator: $1.01^{20} \approx 1.220190$, so $A \approx 2440.38$.
6. Interest earned: $2440.38 - 2000 = 440.38$.
7. Check with the Rule of 72: 4% doubles money in about 18 years, so 5 years should add well under \$2,000. Reasonable.

Answer: \$2,440.38 in the account; \$440.38 in interest

PRACTICE

1. **Easy** For $y = 500(1.08)^t$, give the starting amount and the percent growth rate.

Show answer

Starting amount 500; growth rate 8% per period

2. **Easy** An amount decreases by 15% each year. By what decay factor is it multiplied each year?

Show answer

$$1 - 0.15 = 0.85$$

3. **Easy** A town of 4,000 people grows 2% per year. Write a model for its population after t years.

Show answer

$$y = 4000(1.02)^t$$

4. **Easy** Does $y = 200(0.9)^t$ show growth or decay? By what percent per period?

Show answer

Decay of 10% per period ($0.9 = 1 - 0.10$)

5. **Medium** Henry puts \$1,000 in an account that pays 5% interest compounded once a year. How much is in the account after 3 years? Round to the nearest cent.

Show answer

$$1000(1.05)^3 = 1000(1.157625) \approx \$1,157.63$$

6. **Medium** A farmer buys a tractor for \$48,000. It loses 12% of its value each year. What is it worth after 4 years? Round to the nearest cent.

Show answer

$$48,000(0.88)^4 \approx 48,000(0.599695) \approx \$28,785.38$$

7. **Medium** A radioactive substance used in hospital scans has a half-life of 6 hours. A hospital starts with 80 milligrams. How much is left after 24 hours?

Show answer

$$\frac{24}{6} = 4 \text{ half-lives: } 80 \left(\frac{1}{2}\right)^4 = 5 \text{ milligrams}$$

8. **Medium** A small town of 8,500 people is shrinking by 1.5% per year. Find its population after 6 years, to the nearest whole person.

Show answer

$$8500(0.985)^6 \approx 8500(0.913308) \approx 7763 \text{ people}$$

9. **Medium** Grace saves \$500 in an account paying 6% interest per year, compounded monthly. How much is in the account after 2 years? Round to the nearest cent.

Show answer

$$500 \left(1 + \frac{0.06}{12}\right)^{24} = 500(1.005)^{24} \approx \$563.58$$

10. **Hard** Samuel invests \$1,000 for 10 years at 6% interest per year. How much more will he have if the interest is compounded monthly instead of once a year? Round each balance to the nearest cent first.

Show answer

$$\text{Monthly: } 1000(1.005)^{120} \approx \$1,819.40; \text{ yearly: } 1000(1.06)^{10} \approx \$1,790.85; \\ \text{difference } \$28.55$$

11. **Hard** A cup of tea is poured at 190°F in a 70°F kitchen. The difference between the tea's temperature and the room's temperature shrinks by 4% each minute. What is the tea's temperature after 10 minutes, to the nearest degree?

Show answer

$$\text{Starting difference } 190 - 70 = 120; \text{ after 10 minutes } 120(0.96)^{10} \approx 79.8; \text{ tea} \\ \text{temperature } \approx 70 + 79.8 \approx 150^\circ\text{F}$$

12. **Hard** A substance has a half-life of 8 days. A lab has 200 milligrams. How much remains after 20 days? Round to the nearest tenth of a milligram.

Show answer

$$200 \left(\frac{1}{2}\right)^{20/8} = 200(0.5)^{2.5} \approx 35.4 \text{ milligrams}$$

13. **Hard** Explain why an amount that grows 5% per year is multiplied by 1.05 each year, and why after 3 years the total growth is more than 15%.

Show answer

The new amount is the old amount plus 5% of it: $A + 0.05A = A(1.05)$. After 3 years the amount is $A(1.05)^3 \approx 1.1576A$, an increase of about 15.76%, because each year's 5% is taken of a larger amount than the year before (growth on growth).

14. **Challenge** The Miller family invests \$1,000 at 8% interest compounded once a year. After how many whole years will the balance first be at least \$2,000? Use a calculator to test values.

Show answer

10 years ($1000(1.08)^9 \approx \$1,999.00$, just short; $1000(1.08)^{10} \approx \$2,158.92$)

COMMON MISTAKES

- Using the percent itself as the base, as in $y = 4000(2)^t$ for 2% growth. The factor is $1 + r = 1.02$.
- Using r instead of $1 - r$ for decay: losing 12% means multiplying by 0.88, not 0.12.
- In compound interest, forgetting to divide the rate by n or to multiply the years by n . Write $\frac{r}{n}$ and nt separately before you calculate.
- Rounding too early. Keep all calculator digits until the last step, then round to the nearest cent or whole person.

Make it a game • Coin Half-Life. Start with 64 pennies or buttons marked on one side. Shake and spill them, remove those face up, and count what remains. Repeat four rounds and compare with the model $64(1 - 0.5)^t$. Then write the decay model for a pretend bicycle that loses 15 percent of its value every year, and find its value after 3 years.

7.6 GEOMETRIC SEQUENCES

OBJECTIVES

- Identify a geometric sequence and find its common ratio.
- Write and use the explicit formula $a_n = a_1 \cdot r^{n-1}$ and the recursive form $a_n = a_{n-1} \cdot r$.
- Explain why the exponent in the explicit formula is $n - 1$.
- Compare arithmetic and geometric sequences in real situations.

WATCH

Intro to Geometric Progressions (1 of 3: Definitions)

Eddie Woo · 8:03

<https://www.youtube.com/watch?v=yo2DtZDYzKY>

WHEN TO WATCH

Before Example 1: the common ratio and the n th-term formula.

Note: Eddie Woo teaches in Australia: he says 'geometric progression (GP)' for geometric sequence.

BACKUP VIDEO

Intro to Geometric Progressions (1 of 2)

Eddie Woo · 5:17

<https://www.youtube.com/watch?v=haAsPKIFyNI>

Before the practice set: more geometric sequence examples.

KEY TERMS

geometric sequence

A sequence in which each term after the first is found by multiplying the previous term by the same nonzero number.

explicit formula

A formula that gives any term a_n directly from its position n , such as $a_n = 3 \cdot 2^{n-1}$.

common ratio

The fixed number r that each term is multiplied by to get the next term; find it by dividing any term by the one before it.

recursive formula

A rule that gives the first term and tells how to get each term from the one before it, such as $a_1 = 3$, $a_n = 2a_{n-1}$.

CORE IDEAS

In Lesson 4.5 you studied **arithmetic sequences**, where you **add** the same number each time: 5, 8, 11, 14, ... A **geometric sequence multiplies** by the same number each time: 3, 6, 12, 24, ... Here each term is twice the one before, so the **common ratio** is $r = 2$. To find r , divide any term by the term before it: $\frac{12}{6} = 2$.

The explicit formula, and why $n - 1$. Build the terms from the first one: $a_1 = a_1$, $a_2 = a_1 \cdot r$, $a_3 = a_1 \cdot r \cdot r = a_1 r^2$, $a_4 = a_1 r^3$. To reach the n th term you start at the first term and take $n - 1$ steps, multiplying by r at each step. So $a_n = a_1 \cdot r^{n-1}$. The exponent counts the steps, which is one less than the position.

The recursive form. You can also describe the sequence by its first term and the rule "multiply the previous term by r ": $a_1 = 3$, $a_n = 2 \cdot a_{n-1}$. The recursive form is good for listing terms one by one; the explicit formula is better for jumping straight to the 20th term.

Ratios can be fractions or negative. In 243, 81, 27, 9, ... the ratio is $\frac{1}{3}$, so the terms shrink. In 3, -6 , 12, -24 , ... the ratio is -2 , so the signs alternate. (Exponential functions in Lesson 7.4 required $b > 0$; a geometric sequence may have a negative ratio because its inputs are only the counting numbers 1, 2, 3, ...)

Sequences as functions. A geometric sequence with a positive common ratio (other than 1) is an exponential function whose domain is the counting numbers, just as an arithmetic sequence is a linear function on the counting numbers. Constant difference means arithmetic (linear); constant ratio means geometric (exponential). Doubling patterns, such as 1, 2, 4, 8, ..., grow surprisingly fast: the 11th term is already $2^{10} = 1024$.

KEY FORMULAS

$$r = \frac{a_n}{a_{n-1}}$$

$$a_n = a_1 \cdot r^{n-1}$$

$$a_1 \text{ given, } a_n = a_{n-1} \cdot r$$

Euclid connection. Books VIII and IX of Euclid's *Elements* study numbers "in continued proportion," which is what we call a geometric sequence, and Proposition IX.35 gives a way to find the sum of such a sequence.

Practice habit. Recite the powers of 2 up to $2^{10} = 1024$ and the powers of 3 up to $3^6 = 729$. Doubling and tripling patterns appear constantly in geometric sequences, and knowing these powers lets you check answers in your head.

WORKED EXAMPLES

Example 1. Find the common ratio and the 8th term of the geometric sequence 3, 6, 12, 24, ...

Show solution

1. Common ratio: $r = \frac{6}{3} = 2$ (check: $\frac{12}{6} = 2$, $\frac{24}{12} = 2$).
2. Explicit formula: $a_n = 3 \cdot 2^{n-1}$.
3. Substitute $n = 8$: $a_8 = 3 \cdot 2^7$.
4. $2^7 = 128$, so $a_8 = 3 \cdot 128 = 384$.
5. Check by listing: 3, 6, 12, 24, 48, 96, 192, 384. The 8th term is 384.

Answer: $r = 2$; $a_8 = 384$

Example 2. A church sets up a phone tree to share news about a service project. In round 1 the pastor calls 4 families. In each later round, every family called in the round before calls 4 new families. Write an explicit formula for the number of families called in round n , and find the round in which 1,024 families are called.

Show solution

1. Round 1: 4 families. Round 2: $4 \times 4 = 16$. Round 3: $16 \times 4 = 64$. This is geometric with $a_1 = 4$ and $r = 4$.
2. Explicit formula: $a_n = 4 \cdot 4^{n-1} = 4^n$.
3. Set $4^n = 1024$.
4. List the powers of 4: 4, 16, 64, 256, 1024. So $4^5 = 1024$.
5. Thus $n = 5$.
6. Check: $a_5 = 4 \cdot 4^4 = 4 \cdot 256 = 1024$.

Answer: $a_n = 4 \cdot 4^{n-1} = 4^n$; round 5

PRACTICE

1. Easy Find the common ratio of the geometric sequence 5, 15, 45, 135, ...

Show answer

$$r = 3$$

2. Easy Find the next two terms of the geometric sequence 80, 40, 20, ...

Show answer

$$10, 5 \text{ (ratio } \frac{1}{2} \text{)}$$

3. Easy Is the sequence 2, 6, 10, 14, ... arithmetic, geometric, or neither? Explain briefly.

Show answer

Arithmetic: it adds 4 each time; the ratios $\frac{6}{2} = 3$ and $\frac{10}{6} \neq 3$ are not constant.

4. Easy A geometric sequence has $a_1 = 2$ and $r = 5$. Find a_4 .

Show answer

$$a_4 = 2 \cdot 5^3 = 250$$

5. Medium Write an explicit formula for the geometric sequence 7, 14, 28, 56, ...

Show answer

$$a_n = 7 \cdot 2^{n-1}$$

6. Medium Find the 10th term of the geometric sequence 1, 2, 4, 8, ...

Show answer

$$a_{10} = 1 \cdot 2^9 = 512$$

7. Medium Find the common ratio and the 7th term of 3, -6, 12, -24, ...

Show answer

$$r = -2; a_7 = 3(-2)^6 = 192$$

8. Medium Find the 6th term of the geometric sequence 243, 81, 27, ...

Show answer

$$a_6 = 243 \left(\frac{1}{3}\right)^5 = \frac{243}{243} = 1$$

9. Medium Ruth plants 5 sunflowers in the first year. Each year after that she plants twice as many as the year before. How many sunflowers does she plant in year 6?

Show answer

$$a_6 = 5 \cdot 2^5 = 160 \text{ sunflowers}$$

10. **Hard** A sequence is defined by $a_1 = 4$ and $a_n = -3a_{n-1}$. List the first four terms.

Show answer

$$4, -12, 36, -108$$

11. **Hard** A ball is dropped and bounces. Its first bounce reaches 64 inches, and each bounce reaches $\frac{3}{4}$ the height of the bounce before. How high is the 4th bounce?

Show answer

$$64 \left(\frac{3}{4}\right)^3 = 64 \cdot \frac{27}{64} = 27 \text{ inches}$$

12. **Hard** Henry can be paid for weekly yard work in one of two ways. Option A: \$10 the first week and \$10 more each week after (10, 20, 30, ...). Option B: \$1 the first week, doubling each week (1, 2, 4, ...). In week 10, which option pays more for that week, and by how much?

Show answer

Option A pays $10 \cdot 10 = 100$ dollars; Option B pays $2^9 = 512$ dollars. Option B pays \$412 more.

13. **Hard** A geometric sequence has $a_2 = 12$ and $a_5 = 324$. Write its explicit formula.

Show answer

From a_2 to a_5 is 3 steps: $r^3 = \frac{324}{12} = 27$, so $r = 3$ and $a_1 = \frac{12}{3} = 4$: $a_n = 4 \cdot 3^{n-1}$

14. **Hard** Explain why the explicit formula for a geometric sequence uses r^{n-1} and not r^n .

Show answer

The first term is a_1 with no multiplying yet. Each later term takes one more multiplication by r , so the 2nd term has one factor of r , the 3rd has two, and the n th has $n - 1$. Using r^n would give $a_1 \cdot r$ for the first term, which is wrong.

15. **Challenge** Which term of the geometric sequence 3, 6, 12, 24, ... is 1536?

Show answer

$3 \cdot 2^{n-1} = 1536$, so $2^{n-1} = 512 = 2^9$, $n - 1 = 9$, $n = 10$: the 10th term

COMMON MISTAKES

- Using r^n instead of r^{n-1} . Test your formula with $n = 1$: it must give the first term.
- Finding the ratio upside down. Divide a term by the one **before** it: for 80, 40, 20 the ratio is $\frac{40}{80} = \frac{1}{2}$, not 2.
- Multiplying before applying the exponent: $3 \cdot 2^7 = 3 \cdot 128 = 384$, not 6^7 .
- Mixing up arithmetic and geometric sequences. Check differences (add) and ratios (multiply) before you choose a formula.

Make it a game • Add or Multiply? The parent flashes the first three terms of a sequence, mixing arithmetic and geometric ones, like 5, 9, 13 and 3, 6, 12. For each, say whether you add or multiply, name the common difference or ratio, write the explicit and recursive rule, and find the 8th term. Ten seconds per sequence. Then trade and build mystery sequences for the parent.

EXPONENTS AND EXPONENTIAL FUNCTIONS

In this unit you take a fresh look at exponents. You will learn why the rules for multiplying, dividing and raising powers work, what zero and negative exponents mean and why, and how scientific notation lets us write the mass of the earth or the size of a bacterium without a long string of zeros.

Then exponents move from the base to the variable. In an exponential function such as $y = a \cdot b^x$, each step multiplies by the same factor instead of adding the same amount. These functions describe growing towns, savings accounts earning compound interest, tractors losing value, radioactive substances with a half-life, and doubling patterns. Geometric sequences are the same idea counted in whole steps, the multiplying partner of the arithmetic sequences from Unit 4.

The exponent rules are the foundation for polynomials (Unit 8) and factoring (Unit 9), and exponential models return in Unit 13 when you choose between linear, quadratic and exponential models. From this unit on, a scientific calculator is allowed for messy arithmetic and expected for growth and decay problems.

PREREQUISITES

- Evaluate whole-number powers such as 2^5 and $(-3)^2$ (Pre-Algebra).
- Multiply and divide fractions and decimals and convert percents to decimals.
- Use the order of operations and evaluate expressions by substitution (Lesson 1.3).
- Use function notation and make tables and graphs of functions (Lessons 4.3 and 4.4).
- Write and use arithmetic sequences (Lesson 4.5) and linear functions (Unit 5).

MASTERY CHECKLIST

- ☐ I can explain and use the product, quotient and power rules for exponents.
- ☐ I can simplify products and quotients of monomials.
- ☐ I can explain why $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ and write expressions with only positive exponents.
- ☐ I can write numbers in scientific notation and in standard form.
- ☐ I can multiply and divide numbers in scientific notation and use them in science problems.
- ☐ I can make a table and describe the graph of $y = ab^x$ and tell growth from decay.
- ☐ I can tell whether a table is linear or exponential by checking differences and ratios.
- ☐ I can write and use growth and decay models $y = a(1 \pm r)^t$.
- ☐ I can use the compound interest formula with a calculator and round money to the nearest cent.
- ☐ I can solve half-life and depreciation problems.

- I can find the common ratio and any term of a geometric sequence using

$$a_n = a_1 \cdot r^{n-1}.$$

LESSONS IN THIS UNIT

7.1 Properties of Exponents	2 sessions of 45 minutes
7.2 Zero and Negative Exponents	2 sessions of 45 minutes
7.3 Scientific Notation	45 minutes
7.4 Exponential Functions	2 sessions of 45 minutes
7.5 Exponential Growth and Decay	2 sessions of 45 minutes
7.6 Geometric Sequences	45 minutes

UNIT 7 TEST

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem. Give exact answers unless a problem asks you to round; round money to the nearest cent. Write answers with exponents using positive exponents only, and assume no variable equals zero.

1. **5 pts**

Simplify $(4x^3y)(-2x^5y^6)$.

2. **6 pts**

Simplify $\frac{(3a^4b^2)^3}{9a^5b}$.

3. **5 pts**

Evaluate $5^{-2} + 5^0$.

4. **7 pts**

Simplify $\frac{15x^{-4}y^3}{5x^2y^{-2}}$ so that all exponents are positive.

5. (6 pts)

(a) Write 0.000058 in scientific notation. (b) Write 7.4×10^6 in standard form.

6. (7 pts)

Divide $\frac{4.8 \times 10^9}{1.2 \times 10^{-3}}$. Write the answer in scientific notation.

7. (8 pts)

The moon is about 3.84×10^5 kilometers from the earth. If a jet airliner could fly there at 9×10^2 kilometers per hour, about how many hours would the trip take? Round to the nearest hour.

8. (6 pts)

For $y = 6 \left(\frac{1}{3}\right)^x$, find y when $x = -1$, $x = 0$ and $x = 2$.

9. (7 pts)

Is the table x : 0, 1, 2, 3 and y : 2, 10, 50, 250 linear or exponential? Write an equation for it.

10. (8 pts)

Grace deposits \$5,000 in an account that pays 3% interest per year, compounded monthly. How much is in the account after 4 years? Round to the nearest cent.

11. (7 pts)

A farm truck costs \$36,000 and loses 15% of its value each year. Write a model for its value after t years and find its value after 3 years.

12. (6 pts)

Find the 7th term of the geometric sequence 2, -6 , 18, ...

13. (7 pts)

Samuel's family bakery sold 160 loaves of bread in its first month. Each month after that, sales were 1.5 times the month before. Write an explicit formula for the sales in month n and find the sales in month 5.

14. (8 pts)

Use the quotient rule to explain why $a^0 = 1$ and why $a^{-2} = \frac{1}{a^2}$ for $a \neq 0$.

15. **7 pts**

The graph of $y = a \cdot b^x$ (with $b > 0$) passes through $(0, 5)$ and $(2, 45)$. Write the equation, then find y when $x = 3$.

ANSWER KEY

1. $-8x^8y^7$

2. $\frac{27a^{12}b^6}{9a^5b} = 3a^7b^5$

3. $\frac{1}{25} + 1 = \frac{26}{25} = 1.04$

4. $3x^{-6}y^5 = \frac{3y^5}{x^6}$

5. (a) 5.8×10^{-5} (b) 7,400,000

6. 4×10^{12}

7. $\frac{3.84 \times 10^5}{9 \times 10^2} \approx 0.4267 \times 10^3 \approx 427$ hours

8. 18, 6, $\frac{2}{3}$

9. Exponential (constant ratio 5): $y = 2 \cdot 5^x$

10. $5000 \left(1 + \frac{0.03}{12}\right)^{48} = 5000(1.0025)^{48} \approx \$5,636.64$

11. $v = 36,000(0.85)^t$; $36,000(0.614125) = \$22,108.50$

12. $r = -3$; $a_7 = 2(-3)^6 = 1458$

13. $a_n = 160(1.5)^{n-1}$; $a_5 = 160(5.0625) = 810$ loaves

14. $\frac{a^3}{a^3} = a^{3-3} = a^0$, but a nonzero number divided by itself is 1, so $a^0 = 1$. Also $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, while dividing out three factors of a from top and bottom leaves $\frac{1}{a^2}$; so $a^{-2} = \frac{1}{a^2}$.

15. $a = 5$; $5b^2 = 45$, $b^2 = 9$, $b = 3$; $y = 5 \cdot 3^x$; at $x = 3$, $y = 135$

POLYNOMIALS

In this unit you learn to work with polynomials, the expressions built from numbers and variables with whole-number exponents, such as $3x^2 - 5x + 2$. You will name them, add and subtract them, multiply them (by a monomial, by a binomial and by larger polynomials), learn the three special products by heart, and divide a polynomial by a monomial. Every rule comes from tools you already have: combining like terms, the distributive property and the exponent rules of Unit 7.

Area pictures run through the whole unit. A rectangle cut into pieces shows why the distributive property and FOIL work, and a square cut into two squares and two rectangles shows why $(a + b)^2 = a^2 + 2ab + b^2$, exactly as Euclid proved more than two thousand years ago. These pictures also give practical uses: areas of gardens, pens, patios and quilts, and volumes of boxes.

Unit 8 sets up Unit 9 directly. Factoring is multiplying in reverse, so the faster and more accurate you are at multiplying polynomials now, the easier factoring will be. Later the special products return in completing the square and the quadratic formula (Unit 11) and in rational expressions (Unit 12).

PREREQUISITES

- Combine like terms and use the distributive property, including distributing a negative (Lesson 1.4).
- Apply the sign rules for multiplying and dividing integers.
- Use the product of powers and quotient of powers rules for exponents (Lessons 7.1 and 7.2).
- Evaluate an expression by substituting a number for the variable (Lesson 1.3).
- Find the area of a rectangle and the volume of a rectangular box.

MASTERY CHECKLIST

- ☐ I can classify a polynomial by its number of terms and find its degree and leading coefficient.
- ☐ I can write a polynomial in standard form.
- ☐ I can add polynomials and subtract them by adding the opposite of every term.
- ☐ I can multiply a polynomial by a monomial and explain it with an area picture.
- ☐ I can multiply two binomials with a box model or FOIL and explain why FOIL works.
- ☐ I can multiply a binomial by a trinomial.

- ☐ I can expand

$$(a + b)^2$$

$$(a + b)(a - b)$$

d

fro

m

m

e

m

or

y

an

d

ex

pl

ai

n

ea

ch
wi
th
a
sq
ua
re
di
ag
ra
m.

☐ I can explain why

$$(a + b)^2 \neq a^2 + b^2.$$

☐ I can use special products for mental math such as

29 and

31

31^2 .

☐ I can divide a polynomial by a monomial and check by multiplying.

☐ I can write polynomial expressions for areas, volumes and other real situations.

☐ I can check any polynomial answer by substituting a number.

LESSONS IN THIS UNIT

8.1 Polynomials and Their Terms

2 sessions of 45 minutes

8.2 Multiplying by a Monomial

45 minutes

8.3 Multiplying Binomials and Polynomials

2 sessions of 45 minutes

8.4 Special Products

2 sessions of 45 minutes

UNIT 8 TEST

Time allowed: 50 minutes. No calculator is needed or allowed on this test. Show your work on every problem, one step per line. Give all answers exactly, simplified and in standard form, and include units on word problems.

1. **5 pts**

Write $-2x + 9x^3 - 5 + x^2$ in standard form. Then give its degree and its leading coefficient.

2. **5 pts**

Add $(6x^2 - 5x + 2) + (-4x^2 + x + 7)$.

3. **6 pts**

Subtract $(3x^3 - 2x^2 + 8) - (x^3 + 4x^2 - 3x + 8)$.

4. **6 pts**

Multiply $-4x(2x^2 - 5x + 3)$.

5. **6 pts**

Multiply $3x^2y(2x - 5y)$.

6. **6 pts**

Multiply $(x - 6)(x + 2)$.

7. **7 pts**

Multiply $(4x + 3)(2x - 5)$.

8. **8 pts**

Multiply $(x - 3)(2x^2 + x - 4)$.

9. **5 pts**

Expand $(2x - 7)^2$.

10. (5 pts)

Multiply $(5x + 1)(5x - 1)$.

11. (4 pts)

Use a special product to find 39×41 mentally. Show the pattern you used.

12. (6 pts)

Divide $\frac{15x^4 - 9x^3 + 3x^2}{3x^2}$.

13. (8 pts)

Grace is making a rectangular quilt $3x$ inches wide and $(4x + 10)$ inches long. In the middle she sews on a square patch x inches on a side. Write a polynomial in standard form for the area of the quilt not covered by the patch.

14. (8 pts)

A square backyard is $(x + 4)$ meters on each side. In it lies a square flower bed $(x - 2)$ meters on each side, and the rest of the backyard is grass. Write a polynomial for the area of the grass.

15. (8 pts)

A train travels $(90x^2 + 30x)$ miles in $3x$ hours at a steady speed. Write a polynomial for its speed in miles per hour.

16. (7 pts)

Explain, with a square diagram, why $(x + 4)^2$ is not equal to $x^2 + 16$. Then give the correct expansion.

ANSWER KEY

1. $9x^3 + x^2 - 2x - 5$; degree 3; leading coefficient 9
2. $2x^2 - 4x + 9$
3. $3x^3 - 2x^2 + 8 - x^3 - 4x^2 + 3x - 8 = 2x^3 - 6x^2 + 3x$
4. $-8x^3 + 20x^2 - 12x$
5. $6x^3y - 15x^2y^2$
6. $x^2 + 2x - 6x - 12 = x^2 - 4x - 12$
7. $8x^2 - 20x + 6x - 15 = 8x^2 - 14x - 15$
8. $2x^3 + x^2 - 4x - 6x^2 - 3x + 12 = 2x^3 - 5x^2 - 7x + 12$
9. $4x^2 - 28x + 49$
10. $25x^2 - 1$
11. $(40 - 1)(40 + 1) = 1600 - 1 = 1599$
12. $5x^2 - 3x + 1$ (check: $3x^2(5x^2 - 3x + 1) = 15x^4 - 9x^3 + 3x^2$)
13. $3x(4x + 10) - x^2 = 12x^2 + 30x - x^2 = (11x^2 + 30x)$ square inches
14. $(x + 4)^2 - (x - 2)^2 = (x^2 + 8x + 16) - (x^2 - 4x + 4) = (12x + 12)$ square meters
15. Speed = $\frac{90x^2 + 30x}{3x} = (30x + 10)$ miles per hour
16. A square with side $x + 4$ splits into an x -by- x square (area x^2), a 4-by-4 square (area 16) and two x -by-4 rectangles (area $4x$ each). The expression $x^2 + 16$ leaves out the two rectangles. Correct: $(x + 4)^2 = x^2 + 8x + 16$. (At $x = 1$: $5^2 = 25$, but $1 + 16 = 17$.)

8.1 POLYNOMIALS AND THEIR TERMS

OBJECTIVES

- Identify monomials, binomials, trinomials and polynomials, and tell which expressions are not polynomials.
- Find the degree and leading coefficient of a polynomial and write it in standard form.
- Add polynomials by combining like terms.
- Subtract polynomials by adding the opposite of every term, and check by substituting a number.

WATCH

Algebra Basics: What Are Polynomials? - Math Antics

[mathantics](#) • 11:09

<https://www.youtube.com/watch?v=ffLLmV4mZwU>

WHEN TO WATCH

Before Example 1: monomials, binomials, trinomials and degree.

BACKUP VIDEO

Algebra Basics: Simplifying Polynomials - Math Antics

[mathantics](#) • 10:43

<https://www.youtube.com/watch?v=DKC74YKJpNY>

Before Example 2: adding and subtracting polynomials by combining like terms.

KEY TERMS

monomial

A number, a variable, or a product of a number and variables with whole-number exponents, such as 7, x , $-4x^2$ or $3ab^2$.

polynomial

A monomial or a sum of monomials, such as $5x^3 - 2x + 1$. Each monomial is a term of the polynomial.

binomial and trinomial

A binomial is a polynomial with two terms ($x + 4$); a trinomial is a polynomial with three terms ($x^2 - 3x + 2$).

degree

The degree of a monomial is the sum of the exponents of its variables ($-6x^2y^3$ has degree 5; a nonzero constant has degree 0). The degree of a polynomial is the greatest degree of any of its terms.

standard form

A polynomial in one variable is in standard form when its terms are written from the highest exponent to the lowest.

leading coefficient

The coefficient of the first term when a polynomial is written in standard form. In $-3x^2 + 7x + 4$ it is -3 .

CORE IDEAS

A **monomial** is one 'building block': a number, a variable, or a number times variables raised to whole-number powers. So 8 , y , $-4x^2$ and $\frac{1}{2}a^3b$ are monomials. Expressions such as $\frac{3}{x}$ (a variable in a denominator) or $\sqrt{x}$ (a variable under a root) are not monomials, because they cannot be written as a number times variables with whole-number exponents.

A **polynomial** is a monomial or a sum of monomials. We name small polynomials by their number of terms: one term is a monomial, two terms a binomial, three terms a trinomial. A subtraction counts as adding a negative term, so $5x^3 - 2x + 1$ is the sum $5x^3 + (-2x) + 1$ and has three terms.

The **degree** tells how 'big' a polynomial is. Add the exponents in each term ($4x^3y$ has degree $3 + 1 = 4$); the largest one is the degree of the polynomial. Writing a one-variable polynomial from highest power to lowest puts it in **standard form**, which makes polynomials easy to compare and line up. In $4 - 3x^2 + 7x$, standard form is $-3x^2 + 7x + 4$: degree 2, leading coefficient -3 , constant term 4.

Adding polynomials is just combining like terms, which you already know. Why can we add $3x^2 + 5x^2$ to get $8x^2$? By the distributive property, $3x^2 + 5x^2 = (3 + 5)x^2 = 8x^2$. Like terms have exactly the same variables with the same exponents, so the distributive property lets us add their coefficients. Unlike terms such as x^2 and x cannot be combined, just as 3 apples and 5 oranges are not 8 of anything.

Subtracting a polynomial means adding the opposite of *every* term. The reason is the distributive property again: $-(a - b + c) = (-1)(a - b + c) = -a + b - c$. So $(5x^2 + 2x - 3) - (2x^2 - 6x + 4)$ becomes $5x^2 + 2x - 3 - 2x^2 + 6x - 4$, which is $3x^2 + 8x - 7$. The most common error in this lesson is changing only the first sign; the minus sign in front of the parentheses belongs to every term inside.

A polynomial is a rule that gives a number for every value of x , so you can **check** any sum or difference by substituting a simple number. If $(5x^2 + 2x - 3) - (2x^2 - 6x + 4) = 3x^2 + 8x - 7$ is right, then at $x = 1$ the left side is $4 - 0 = 4$ and the right side is $3 + 8 - 7 = 4$. They match. (A match does not prove the answer, but a mismatch always means a mistake.)

KEY FORMULAS

$$ax^n + bx^n = (a + b)x^n$$

$$-(a - b + c) = -a + b - c$$

$$P - Q = P + (-Q)$$

From history. René Descartes, in *La Géométrie* (1637), helped make popular the small raised numbers for powers, such as x^3 , that we still use today. Before such notation, mathematicians wrote powers in words or abbreviations, which made long polynomials hard to read and to add.

Practice habit. Check every sum or difference by substituting $x = 1$: the value of a polynomial at $x = 1$ is just the sum of its coefficients, so the check takes seconds. If the two sides disagree, look for a sign you forgot to change.

WORKED EXAMPLES

Example 1. Add $(4x^2 - 3x + 7) + (-2x^2 + 5x - 9)$. Give the answer in standard form.

Show solution

1. Remove the parentheses (adding does not change any signs): $4x^2 - 3x + 7 - 2x^2 + 5x - 9$.
2. Group like terms: $(4x^2 - 2x^2) + (-3x + 5x) + (7 - 9)$.
3. Combine: $2x^2 + 2x - 2$.
4. Check at $x = 1$: left side $(4 - 3 + 7) + (-2 + 5 - 9) = 8 + (-6) = 2$; right side $2 + 2 - 2 = 2$. They match.

Answer: $2x^2 + 2x - 2$

Example 2. Henry has a board that is $(7x^2 - 4x + 3)$ inches long. He cuts off a piece that is $(2x^2 + 3x - 1)$ inches long. Write a polynomial in standard form for the length of the board that is left.

Show solution

1. Length left = whole board minus piece: $(7x^2 - 4x + 3) - (2x^2 + 3x - 1)$.
2. Add the opposite of every term of the piece: $7x^2 - 4x + 3 - 2x^2 - 3x + 1$.
3. Group like terms: $(7x^2 - 2x^2) + (-4x - 3x) + (3 + 1)$.
4. Combine: $5x^2 - 7x + 4$.
5. Check at $x = 2$: board $28 - 8 + 3 = 23$ inches, piece $8 + 6 - 1 = 13$ inches, so 10 inches are left; and $5(4) - 7(2) + 4 = 20 - 14 + 4 = 10$. They match.

Answer: $(5x^2 - 7x + 4)$ inches

PRACTICE

1. Easy Find the degree of the monomial $-6x^2y^3$.

Show answer

Degree 5 (because $2 + 3 = 5$)

2. Easy Write $4 - 3x^2 + 7x$ in standard form and name its leading coefficient.

Show answer

$$-3x^2 + 7x + 4; \text{ leading coefficient } -3$$

3. Easy Add $(3x + 5) + (6x - 8)$.

Show answer

$$9x - 3$$

4. Easy Add $(2x^2 - 4x + 1) + (x^2 + 7x - 6)$.

Show answer

$$3x^2 + 3x - 5$$

5. Medium Subtract $(5x^2 + 2x - 3) - (2x^2 - 6x + 4)$.

Show answer

$$5x^2 + 2x - 3 - 2x^2 + 6x - 4 = 3x^2 + 8x - 7$$

6. Medium Subtract $(4a^3 - a + 9) - (a^3 + 2a^2 - 5a + 9)$.

Show answer

$$4a^3 - a + 9 - a^3 - 2a^2 + 5a - 9 = 3a^3 - 2a^2 + 4a$$

7. Medium Is $\frac{3}{x} + 2$ a polynomial? Answer yes or no and give the reason.

Show answer

No. $\frac{3}{x}$ has the variable in a denominator, so it is not a monomial (it cannot be written as a number times x to a whole-number power).

8. Medium A triangular flower bed has sides of $(2x + 3)$ feet, $(x^2 + 1)$ feet and $(3x - 2)$ feet. Write a polynomial in standard form for its perimeter.

Show answer

$$(x^2 + 5x + 2) \text{ feet}$$

9. **Medium** Mark wrote $(6x^2 - 2x + 1) - (4x^2 + 3x - 5) = 2x^2 + x - 4$. Explain his mistake and give the correct answer.

Show answer

Mark subtracted only the first term, $4x^2$. The minus sign applies to every term, so $3x$ becomes $-3x$ and -5 becomes $+5$: $6x^2 - 2x + 1 - 4x^2 - 3x + 5 = 2x^2 - 5x + 6$.

10. **Hard** Simplify $(2x^2y - 3xy + y^2) - (5x^2y + xy - 4y^2)$.

Show answer

$$-3x^2y - 4xy + 5y^2$$

11. **Hard** Grace sells loaves of bread at a church bake sale. Her income from selling x loaves is $5x$ dollars, and her cost is $(0.02x^2 + x + 6)$ dollars. (a) Write a polynomial for her profit (income minus cost). (b) Find her profit if she sells 50 loaves.

Show answer

(a) $5x - (0.02x^2 + x + 6) = -0.02x^2 + 4x - 6$ dollars; (b) $-0.02(2500) + 200 - 6 = -50 + 200 - 6 = 144$, so \$144

12. **Hard** The perimeter of a triangular pasture is $(6x^2 + 2x + 9)$ yards. Two of its sides measure $(2x^2 + x + 3)$ yards and $(x^2 - 4x + 2)$ yards. Write a polynomial for the third side.

Show answer

The two sides total $3x^2 - 3x + 5$; third side $= (6x^2 + 2x + 9) - (3x^2 - 3x + 5) = (3x^2 + 5x + 4)$ yards

13. **Hard** What polynomial must be added to $3x^2 - 5x + 2$ to get $7x^2 + x - 4$?

Show answer

$$(7x^2 + x - 4) - (3x^2 - 5x + 2) = 4x^2 + 6x - 6$$

14. **Challenge** Find the numbers a and b that make this true for every x : $(ax^2 + 3x - 2) + (4x^2 + bx + 7) = 9x^2 - 2x + 5$. Give a , then b .

Show answer

$a + 4 = 9$, so $a = 5$; $3 + b = -2$, so $b = -5$ (and $-2 + 7 = 5$ checks)

COMMON MISTAKES

- Changing only the first sign when subtracting, as in $-(4x^2 + 3x - 5) = -4x^2 + 3x - 5$. The minus sign multiplies every term: $-4x^2 - 3x + 5$. Rewrite the subtraction as addition of the opposite before combining.
- Combining unlike terms, such as $3x^2 + 2x = 5x^3$. Only terms with the same variables and the same exponents can be combined, and the exponent never changes when you add like terms.
- Finding the degree of a term like $4x^3y$ as 3 instead of 4. Add the exponents of all the variables in the term (y has exponent 1).
- Naming the leading coefficient before writing the polynomial in standard form. In $4 - 3x^2 + 7x$ the leading coefficient is -3 , not 4; reorder the terms first.

Make it a game • Polynomial Name Tags. Write eight expressions on sticky notes, including a few impostors like $3x^2 + 1$ and $\sqrt{x} + 4$. Race the clock to sort them into Polynomial and Not a Polynomial. For each polynomial, write its name tag: monomial, binomial or trinomial, its degree and its leading coefficient, then rewrite it in standard form. Count your correct tags.

8.2 MULTIPLYING BY A MONOMIAL

OBJECTIVES

- Multiply a polynomial by a monomial using the distributive property and the product of powers rule.
- Explain the product with an area (rectangle) picture.
- Write and simplify polynomial expressions for areas and volumes with algebraic sides.
- Simplify expressions and solve equations that involve multiplying by a monomial.

WATCH

Expanding Algebraic Expressions

Eddie Woo · 13:15

<https://www.youtube.com/watch?v=aI9fAobtplA>

WHEN TO WATCH

Before Example 1: multiplying a polynomial by a monomial.

Note: Eddie Woo teaches in Australia: he says 'expanding' for multiplying out and 'pronomeral' for variable.

BACKUP VIDEO

Multiplying Algebraic Terms

Eddie Woo · 11:18

<https://www.youtube.com/watch?v=myeerS0v1Tg>

Before the Core ideas: multiplying monomials with the product of powers rule.

KEY TERMS

distributive property

For any numbers a , b and c ,
 $a(b + c) = ab + ac$. The factor outside
 the parentheses multiplies every term
 inside.

product of powers rule

When you multiply powers with the
 same base, add the exponents: $x^m \cdot$
 $x^n = x^{m+n}$ (from Lesson 7.1).

area model

A rectangle split into smaller rectangles, used to picture a product.
The area of the whole equals the sum of the areas of the parts.

CORE IDEAS

To multiply a polynomial by a monomial, use the **distributive property**: the monomial multiplies *every* term inside the parentheses. For example, $4x(x + 3) = 4x \cdot x + 4x \cdot 3 = 4x^2 + 12x$. A good count to remember: if there are three terms inside, there are three products in the answer (before combining).

Each product of two monomials uses two things you already know. Multiply the coefficients, and multiply the powers of the same variable by **adding exponents**. So $-3x^2 \cdot 4x^3 = (-3 \cdot 4)(x^2 \cdot x^3) = -12x^5$. Why add exponents? $x^2 \cdot x^3 = (x \cdot x)(x \cdot x \cdot x)$, which is 5 factors of x . Watch the signs: a negative monomial outside changes the sign of every term inside.

Why the distributive property is true: the area picture. Draw a rectangle x units tall and $x + 3$ units long. Split the long side into a piece of length x and a piece of length 3. The rectangle is now a square of area $x \cdot x = x^2$ next to a rectangle of area $x \cdot 3 = 3x$. The whole area is $x(x + 3)$, and it is also $x^2 + 3x$, so the two expressions must be equal. The same cutting works for any number of pieces.

Areas and volumes with algebraic sides are polynomials. A pen x meters wide and $(2x + 3)$ meters long has area $x(2x + 3) = 2x^2 + 3x$ square meters. A box x by $2x$ by $(x + 5)$ inches has volume $2x^2(x + 5) = 2x^3 + 10x^2$ cubic inches. When a problem subtracts one area from another, multiply out each product first, then subtract by adding the opposite of every term, as in Lesson 8.1.

As always, you can **check** a product by substituting a number. If $-3x^2(4x^3 - 2x + 5) = -12x^5 + 6x^3 - 15x^2$, then at $x = 1$ the left side is $-3(4 - 2 + 5) = -21$ and the right side is $-12 + 6 - 15 = -21$.

KEY FORMULAS

$$a(b + c) = ab + ac$$

$$x^m \cdot x^n = x^{m+n}$$

$$ax^m(bx^n + c) = abx^{m+n} + acx^m$$

Euclid connection. Euclid's *Elements*, Book II, Proposition 1, proves the distributive property with pictures: a rectangle built on one line and a line cut into pieces equals the sum of the rectangles built on that line and each piece, just like our area model.

Practice habit. Draw a small arrow from the monomial to each term inside the parentheses before you multiply, and count: the number of products must equal the number of terms inside. Then check by substituting $x = 1$.

WORKED EXAMPLES

Example 1. Multiply $-3x^2(4x^3 - 2x + 5)$.

Show solution

1. Distribute $-3x^2$ to each of the three terms: $(-3x^2)(4x^3) + (-3x^2)(-2x) + (-3x^2)(5)$.
2. First product: $(-3)(4) = -12$ and $x^2 \cdot x^3 = x^5$, giving $-12x^5$.
3. Second product: $(-3)(-2) = 6$ and $x^2 \cdot x = x^3$, giving $6x^3$.
4. Third product: $(-3)(5) = -15$, giving $-15x^2$.
5. Result: $-12x^5 + 6x^3 - 15x^2$.
6. Check at $x = 1$: $-3(4 - 2 + 5) = -21$ and $-12 + 6 - 15 = -21$. They match.

Answer: $-12x^5 + 6x^3 - 15x^2$

Example 2. The Miller family's vegetable garden is a rectangle $2x$ feet wide and $(3x + 4)$ feet long. A rectangular tool shed x feet wide and $(x + 2)$ feet long sits inside the garden. Write a polynomial in standard form for the area of the garden not covered by the shed.

Show solution

1. Garden area: $2x(3x + 4) = 6x^2 + 8x$ square feet.
2. Shed area: $x(x + 2) = x^2 + 2x$ square feet.
3. Uncovered area: $(6x^2 + 8x) - (x^2 + 2x) = 6x^2 + 8x - x^2 - 2x$.
4. Combine like terms: $5x^2 + 6x$.
5. Check with $x = 2$: garden $4 \times 10 = 40$, shed $2 \times 4 = 8$, so $40 - 8 = 32$; and $5(4) + 6(2) = 20 + 12 = 32$. They match.

Answer: $(5x^2 + 6x)$ square feet

PRACTICE

1. Easy Multiply $4x(x + 3)$.

Show answer

$$4x^2 + 12x$$

2. Easy Multiply $-2y(5y - 7)$.

Show answer

$$-10y^2 + 14y$$

3. Easy Multiply $3a^2(2a^2 - a + 6)$.

Show answer

$$6a^4 - 3a^3 + 18a^2$$

4. Easy Multiply $x^3(x^2 + 4x - 1)$.

Show answer

$$x^5 + 4x^4 - x^3$$

5. Medium Multiply $-5x^2(2x^3 - 3x + 4)$.

Show answer

$$-10x^5 + 15x^3 - 20x^2$$

6. Medium Multiply $2ab(3a^2 - ab + 4b)$.

Show answer

$$6a^3b - 2a^2b^2 + 8ab^2$$

7. Medium A rectangular chicken run is x meters wide and $(2x + 3)$ meters long. Write a polynomial for its area.

Show answer

$$x(2x + 3) = (2x^2 + 3x) \text{ square meters}$$

8. Medium Explain with an area picture why $x(x + 3) = x^2 + 3x$.

Show answer

Draw a rectangle x tall and $x + 3$ long, so its area is $x(x + 3)$. Cut the long side into pieces x and 3 . The rectangle splits into an x -by- x square (area x^2) and an x -by- 3 rectangle (area $3x$). The same area written two ways gives $x(x + 3) = x^2 + 3x$.

9. Medium Simplify $3x(x - 4) + 2x(5x + 1)$.

Show answer

$$3x^2 - 12x + 10x^2 + 2x = 13x^2 - 10x$$

10. **Hard** Simplify $4x(x^2 - 2x + 3) - x(3x^2 + x - 5)$.

Show answer

$$4x^3 - 8x^2 + 12x - 3x^3 - x^2 + 5x = x^3 - 9x^2 + 17x$$

11. **Hard** Solve $2x(x + 3) - 2x^2 = 18$.

Show answer

$$2x^2 + 6x - 2x^2 = 18, \text{ so } 6x = 18 \text{ and } x = 3 \text{ (check: } 6(6) - 18 = 18)$$

12. **Hard** Henry builds a wooden toolbox shaped like a rectangular box. It is x inches wide, $2x$ inches long and $(x + 5)$ inches tall. Write a polynomial in standard form for its volume.

Show answer

$$x \cdot 2x \cdot (x + 5) = 2x^2(x + 5) = (2x^3 + 10x^2) \text{ cubic inches}$$

13. **Hard** A farmer plants $3x$ rows of corn with $(2x + 7)$ plants in each row. (a) Write a polynomial for the total number of plants. (b) How many plants are there when $x = 5$?

Show answer

(a) $3x(2x + 7) = 6x^2 + 21x$; (b) $6(25) + 21(5) = 150 + 105 = 255$ plants (15 rows of 17)

14. **Challenge** Find the monomial M that makes this true for every x : $M(3x^2 - 2x + 5) = -12x^4 + 8x^3 - 20x^2$.

Show answer

$-4x^2$ (check: $-4x^2 \cdot 3x^2 = -12x^4$, $-4x^2 \cdot (-2x) = 8x^3$, $-4x^2 \cdot 5 = -20x^2$)

15. **Challenge** One rectangular plot of land is x yards wide and $(x + 5)$ yards long. A second plot is x yards wide and $(x - 3)$ yards long. The first plot has 40 square yards more area than the second. Find x .

Show answer

$x(x + 5) - x(x - 3) = 40$, so $x^2 + 5x - x^2 + 3x = 40$, $8x = 40$, $x = 5$ yards (check: $50 - 10 = 40$)

COMMON MISTAKES

- Multiplying only the first term: $4x(x + 3) = 4x^2 + 3$. The monomial multiplies every term inside; draw an arrow to each term.
- Multiplying exponents instead of adding them: $x^2 \cdot x^3 = x^6$. Write out the factors once ($x \cdot x$ times $x \cdot x \cdot x$) to see that there are 5 factors, so x^5 .
- Losing a sign with a negative monomial: $-2y(5y - 7) = -10y^2 - 14y$. A negative times a negative is positive, so the answer is $-10y^2 + 14y$.
- Forgetting parentheses when subtracting a product, as in $4x(x^2 - 2x + 3) - x(3x^2 + x - 5)$. Multiply out $x(3x^2 + x - 5)$ in parentheses first, then subtract every term.

Make it a game • Draw the Area. The parent calls a product such as $2x(x + 5)$. Draw a rectangle with sides $2x$ and $x + 5$, split it into two parts, label each area, and write the expansion. Then reverse: the parent draws a rectangle with the labeled parts $3x^2$ and $12x$, and you write the product. Three correct areas in a row win.

8.3 MULTIPLYING BINOMIALS AND POLYNOMIALS

OBJECTIVES

- Multiply two binomials using the area (box) model and the distributive property.
- Use FOIL as a memory aid for multiplying two binomials, and explain why it works.
- Multiply a binomial by a trinomial and simplify the result.
- Write polynomial expressions for areas and volumes in real situations.

WATCH

Binomial Products (1 of 2: Visualising the expansion)

Eddie Woo · 8:21

<https://www.youtube.com/watch?v=1Mw3Ff7KRuU>

WHEN TO WATCH

Before Example 1: the area model for multiplying two binomials.

Note: The backup is Professor Leonard (alternate tier). Eddie Woo teaches in Australia: he says 'binomial products' and 'expanding'.

BACKUP VIDEO

Distribution of Binomials (TTP Video 69)

Professor Leonard · 17:58

https://www.youtube.com/watch?v=_7auuhBwe9M

Before the practice set: FOIL and distributing binomials (about 18 minutes).

KEY TERMS

box model

A rectangle divided into a grid, with the terms of one polynomial along the top and the terms of the other down the side. Each cell holds the product of its row term and column term, and the sum of all cells is the product.

FOIL

A memory aid for multiplying two binomials: multiply the **F**irst terms, the **O**uter terms, the **I**nner terms and the **L**ast terms, then add. It works only for two binomials.

product

The result of multiplying. The product of two polynomials is found by multiplying every term of one by every term of the other and combining like terms.

CORE IDEAS

To multiply two binomials, picture a rectangle whose sides are the two binomials. For $(2x + 3)(x + 5)$, cut the top side into $2x$ and 3 and the left side into x and 5 . The rectangle splits into four smaller rectangles with areas $2x \cdot x = 2x^2$, $3 \cdot x = 3x$, $2x \cdot 5 = 10x$ and $3 \cdot 5 = 15$. Adding them gives $2x^2 + 13x + 15$. This is the **box model**: write the terms along the top and side of a grid, fill each cell with a product, then add the cells and combine like terms.

Why it works: the box model is the distributive property used twice. Treat the second binomial as one quantity and distribute it: $(a + b)(c + d) = a(c + d) + b(c + d)$. Then distribute again: $= ac + ad + bc + bd$. Every term of the first binomial meets every term of the second exactly once, so there are $2 \times 2 = 4$ products.

FOIL is a handy way to remember those four products for two binomials: First (ac), Outer (ad), Inner (bc), Last (bd). For $(x + 4)(x - 7)$: F is x^2 , O is $-7x$, I is $4x$, L is -28 , so the product is $x^2 - 3x - 28$. Keep each sign with its term. Remember that FOIL is only a memory aid for two binomials; the real rule is *every term times every term*.

The same rule handles bigger polynomials. A binomial times a trinomial has $2 \times 3 = 6$ products. For $(2x - 3)(x^2 + 4x - 5)$, a box with 2 rows and 3 columns gives $2x^3$, $8x^2$, $-10x$, $-3x^2$, $-12x$ and 15 . Combining like terms: $2x^3 + 5x^2 - 22x + 15$. Notice the degrees add: degree 1 times degree 2 gives degree 3.

Check a product by substituting a small number such as $x = 1$ or $x = 2$ into both the original and your answer. For $(2x - 3)(x^2 + 4x - 5)$ at $x = 2$: $(1)(4 + 8 - 5) = 7$, and $2(8) + 5(4) - 22(2) + 15 = 16 + 20 - 44 + 15 = 7$. They match.

KEY FORMULAS

$$(a + b)(c + d) = ac + ad + bc + bd$$

$$(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$$

Practice habit. Count the products before you combine: a polynomial with m terms times one with n terms gives $m \times n$ products. Then check your answer by substituting $x = 2$ into both the problem and the answer.

WORKED EXAMPLES

Example 1. Samuel's rectangular garden is $(2x + 3)$ feet long and $(x + 5)$ feet wide. Write a polynomial in standard form for its area.

Show solution

1. Area = length times width: $(2x + 3)(x + 5)$.
2. First: $2x \cdot x = 2x^2$.
3. Outer: $2x \cdot 5 = 10x$.
4. Inner: $3 \cdot x = 3x$.
5. Last: $3 \cdot 5 = 15$.
6. Add and combine like terms: $2x^2 + 10x + 3x + 15 = 2x^2 + 13x + 15$.
7. Check at $x = 1$: $(5)(6) = 30$ and $2 + 13 + 15 = 30$. They match.

Answer: $(2x^2 + 13x + 15)$ square feet

Example 2. Multiply $(2x - 3)(x^2 + 4x - 5)$.

Show solution

1. Multiply $2x$ by each term of the trinomial: $2x^3 + 8x^2 - 10x$.
2. Multiply -3 by each term of the trinomial: $-3x^2 - 12x + 15$.
3. Add the six products: $2x^3 + 8x^2 - 10x - 3x^2 - 12x + 15$.
4. Combine like terms: $2x^3 + 5x^2 - 22x + 15$.
5. Check at $x = 2$: $(4 - 3)(4 + 8 - 5) = 1 \times 7 = 7$ and $16 + 20 - 44 + 15 = 7$. They match.

Answer: $2x^3 + 5x^2 - 22x + 15$

PRACTICE

1. Easy Multiply $(x + 3)(x + 5)$.

Show answer

$$x^2 + 5x + 3x + 15 = x^2 + 8x + 15$$

2. Easy Multiply $(x - 4)(x + 6)$.

Show answer

$$x^2 + 6x - 4x - 24 = x^2 + 2x - 24$$

3. Easy Multiply $(x - 2)(x - 9)$.

Show answer

$$x^2 - 9x - 2x + 18 = x^2 - 11x + 18$$

4. Easy Multiply $(a + 1)(a - 8)$.

Show answer

$$a^2 - 8a + a - 8 = a^2 - 7a - 8$$

5. Medium Multiply $(2x + 1)(x + 4)$.

Show answer

$$2x^2 + 8x + x + 4 = 2x^2 + 9x + 4$$

6. Medium Multiply $(3x - 2)(2x + 5)$.

Show answer

$$6x^2 + 15x - 4x - 10 = 6x^2 + 11x - 10$$

7. Medium Multiply $(4x - 3)(x - 2)$.

Show answer

$$4x^2 - 8x - 3x + 6 = 4x^2 - 11x + 6$$

8. Medium Grace puts an 8-inch by 10-inch photo in a frame that is x inches wide on every side, so the outside of the frame measures $(8 + 2x)$ inches by $(10 + 2x)$ inches. Write a polynomial in standard form for the area covered by the photo and frame together.

Show answer

$$(8 + 2x)(10 + 2x) = 80 + 16x + 20x + 4x^2 = (4x^2 + 36x + 80) \text{ square inches}$$

9. **Medium** Leah says $(x + 3)(x + 5) = x^2 + 15$. Use a box model to explain what she left out.

Show answer

A box with x and 3 on top and x and 5 on the side has four cells: x^2 , $3x$, $5x$ and 15. Leah kept only the first and last cells and forgot the two middle products, $3x + 5x = 8x$. The correct product is $x^2 + 8x + 15$.

10. **Hard** Multiply $(x + 2)(x^2 - 3x + 4)$.

Show answer

$$x^3 - 3x^2 + 4x + 2x^2 - 6x + 8 = x^3 - x^2 - 2x + 8$$

11. **Hard** Multiply $(2x - 1)(3x^2 + x - 4)$.

Show answer

$$6x^3 + 2x^2 - 8x - 3x^2 - x + 4 = 6x^3 - x^2 - 9x + 4$$

12. **Hard** Simplify $(x + 5)(x - 2) - (x - 3)(x + 1)$.

Show answer

$$(x^2 + 3x - 10) - (x^2 - 2x - 3) = 5x - 7$$

13. **Hard** Henry makes an open-top tray from a 12-inch by 16-inch sheet of tin by cutting a square x inches on a side from each corner and folding up the sides. The tray is x inches tall, $(12 - 2x)$ inches wide and $(16 - 2x)$ inches long. Write a polynomial in standard form for its volume.

Show answer

$$(12 - 2x)(16 - 2x) = 4x^2 - 56x + 192, \text{ so } V = x(4x^2 - 56x + 192) = (4x^3 - 56x^2 + 192x) \text{ cubic inches}$$

14. **Challenge** A rectangular garden bed is x feet wide and $(x + 3)$ feet long. If both the width and the length are increased by 2 feet, the area grows by 26 square feet. Find the original width x .

Show answer

$$(x + 2)(x + 5) - x(x + 3) = 26, \text{ so } x^2 + 7x + 10 - x^2 - 3x = 26, 4x + 10 = 26, \\ x = 4 \text{ feet (check: } 6 \times 9 - 4 \times 7 = 54 - 28 = 26)$$

15. **Challenge** Multiply $(x^2 + 2x - 1)(x^2 - x + 3)$.

Show answer

Nine products: $x^4 - x^3 + 3x^2 + 2x^3 - 2x^2 + 6x - x^2 + x - 3 = x^4 + x^3 + 7x - 3$ (the x^2 terms cancel)

COMMON MISTAKES

- Multiplying only first times first and last times last: $(x + 3)(x + 5) = x^2 + 15$. Every term must multiply every term; for two binomials there are four products.
- Dropping a negative sign in the middle terms, as in $(x - 4)(x + 6) = x^2 - 2x - 24$. Keep each sign attached to its term: the outer product is $+6x$ and the inner product is $-4x$, so the middle term is $+2x$.
- Trying to use FOIL on a binomial times a trinomial and missing products. Use a box with 2 rows and 3 columns (6 cells) or distribute each term of the binomial in turn.
- Forgetting to combine like terms at the end, or combining unlike ones such as x^3 and x^2 . Line up terms by degree before you add.

Make it a game • Box Versus FOIL. Race a partner on the same four binomial products, such as $(x + 3)(x - 5)$. One of you uses the area box, the other uses FOIL. Compare answers after each problem. If they disagree, hunt together for the slip, and explain which step of the distributive property each method hides. Swap methods halfway through.

8.4 SPECIAL PRODUCTS

OBJECTIVES

- Expand $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$ quickly using the special product patterns.
- Explain each pattern with a square diagram and explain why $(a + b)^2 \neq a^2 + b^2$.
- Use the patterns for mental math, such as 31^2 and 29×31 .
- Apply special products to area problems.

WATCH

Binomial Products (2 of 2: Perfect squares)

Eddie Woo · 10:57

<https://www.youtube.com/watch?v=2CebIaO9fe0>

WHEN TO WATCH

Before Example 1: the square of a binomial and the difference of two squares.

Note: Eddie Woo teaches in Australia: he says 'perfect squares' for squares of binomials.

BACKUP VIDEO

Difference of Squares and Cubes

Eddie Woo · 8:17

<https://www.youtube.com/watch?v=OREp1uM6iW8>

After Example 1: a picture proof of the difference of two squares. It also shows cubes, which this course does not cover.

KEY TERMS

square of a binomial

A product such as $(a + b)^2 = (a + b)(a + b)$ or $(a - b)^2 = (a - b)(a - b)$, a binomial multiplied by itself.

perfect-square trinomial

The three-term result of squaring a binomial, such as $x^2 + 10x + 25 = (x + 5)^2$.

difference of two squares

An expression of the form $a^2 - b^2$. It is the product $(a + b)(a - b)$ of a sum and a difference of the same two terms.

special products

The three patterns $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$ and $(a + b)(a - b) = a^2 - b^2$, which are worth knowing by heart.

CORE IDEAS

Some products come up so often that it pays to know them by heart. Each is just the multiplication from Lesson 8.3: $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$. In words: **the square of the first, plus twice the product, plus the square of the last.**

The square picture. Draw a square whose side is $a + b$, cutting each side into a piece a and a piece b . The big square splits into four parts: an a -by- a square (area a^2), a b -by- b square (area b^2) and two a -by- b rectangles (area ab each). So $(a + b)^2 = a^2 + 2ab + b^2$. This picture also shows **why** $(a + b)^2 \neq a^2 + b^2$: the two squares a^2 and b^2 do not fill the big square; the two rectangles, $2ab$, are missing. Try numbers: $(3 + 4)^2 = 49$, but $3^2 + 4^2 = 25$, and the difference is $2 \cdot 3 \cdot 4 = 24$.

Squaring a difference works the same way with a minus sign: $(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$. The last term is $+b^2$ because $(-b)(-b) = b^2$. For example, $(3x - 5)^2 = (3x)^2 - 2(3x)(5) + 5^2 = 9x^2 - 30x + 25$. Notice that a is the whole first term, $3x$, so it is squared to $9x^2$, not $3x^2$.

Sum times difference. In $(a + b)(a - b) = a^2 - ab + ab - b^2$, the two middle terms are opposites and cancel, leaving $a^2 - b^2$. A picture: from an a -by- a square, cut away a b -by- b corner square. The L-shaped piece left has area $a^2 - b^2$. Cut the L into two rectangles, one a by $(a - b)$ and one b by $(a - b)$. Turn the smaller one and set it against the end of the larger one, matching their $(a - b)$ sides: together they form a rectangle $a + b$ long and $a - b$ wide. So $(a + b)(a - b) = a^2 - b^2$.

Mental math. The patterns make some hard-looking arithmetic easy. $31^2 = (30 + 1)^2 = 900 + 60 + 1 = 961$. $29^2 = (30 - 1)^2 = 900 - 60 + 1 = 841$. $29 \times 31 = (30 - 1)(30 + 1) = 900 - 1 = 899$. Look for a round number in the middle.

KEY FORMULAS

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Euclid connection. Euclid's *Elements*, Book II, Proposition 4, proves the square of a sum with exactly our picture: if a line is cut into two parts, the square on the whole line equals the squares on the two parts plus twice the rectangle formed by the parts, that is, $(a + b)^2 = a^2 + b^2 + 2ab$.

Practice habit. Recite the three special products until they are automatic: $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, $(a + b)(a - b) = a^2 - b^2$. Practice with mental math such as $29 \times 31 = 30^2 - 1 = 899$ and $41^2 = 40^2 + 80 + 1 = 1681$.

WORKED EXAMPLES

Example 1. Expand $(3x - 5)^2$.

Show solution

1. Use $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 3x$ and $b = 5$.
2. Square the first: $(3x)^2 = 9x^2$.
3. Twice the product: $2(3x)(5) = 30x$, subtracted.
4. Square the last: $5^2 = 25$.
5. Result: $9x^2 - 30x + 25$.
6. Check at $x = 1$: $(3 - 5)^2 = (-2)^2 = 4$ and $9 - 30 + 25 = 4$. They match.

Answer: $9x^2 - 30x + 25$

Example 2. Henry's square patio is x feet on each side. He rebuilds it as a rectangle that is 3 feet longer and 3 feet narrower than before. Write a polynomial for the new area, and tell how it compares with the old area. Then check with $x = 20$.

Show solution

1. New length $x + 3$, new width $x - 3$; new area $(x + 3)(x - 3)$.
2. Use $(a + b)(a - b) = a^2 - b^2$: new area $= x^2 - 9$ square feet.
3. Old area $= x^2$, so the new patio is always 9 square feet smaller.
4. Check with $x = 20$: old area $20^2 = 400$; new area $23 \times 17 = 391$; and $400 - 9 = 391$. They match.

Answer: New area $(x^2 - 9)$ square feet, which is 9 square feet less than the old area

PRACTICE

1. Easy Expand $(x + 6)^2$.

Show answer

$$x^2 + 12x + 36$$

2. Easy Expand $(y - 4)^2$.

Show answer

$$y^2 - 8y + 16$$

3. Easy Multiply $(x + 7)(x - 7)$.

Show answer

$$x^2 - 49$$

4. Easy Use $(a + b)^2$ to find 31^2 mentally.

Show answer

$$(30 + 1)^2 = 900 + 60 + 1 = 961$$

5. Medium Expand $(2x + 3)^2$.

Show answer

$$4x^2 + 12x + 9$$

6. Medium Expand $(5a - 2)^2$.

Show answer

$$25a^2 - 20a + 4$$

7. Medium Multiply $(3x + 4y)(3x - 4y)$.

Show answer

$$9x^2 - 16y^2$$

8. Medium Use a special product to find 48×52 mentally.

Show answer

$$(50 - 2)(50 + 2) = 2500 - 4 = 2496$$

9. Medium Caleb says $(x + 5)^2 = x^2 + 25$. Use a square diagram to explain why he is wrong.

Show answer

A square with side $x + 5$ splits into an x -by- x square (x^2), a 5-by-5 square (25) and two x -by-5 rectangles ($5x$ each). Caleb left out the two rectangles. The correct expansion is $x^2 + 10x + 25$. (At $x = 1$: $6^2 = 36$, not 26.)

10. **Hard** A square sheep pen is x feet on each side. The farmer enlarges it by adding 5 feet to each side, keeping it square. Write a polynomial for the number of square feet added.

Show answer

$$(x + 5)^2 - x^2 = x^2 + 10x + 25 - x^2 = (10x + 25) \text{ square feet}$$

11. **Hard** Simplify $(x + 3)^2 - (x - 3)^2$.

Show answer

$$(x^2 + 6x + 9) - (x^2 - 6x + 9) = 12x$$

12. **Hard** The floor of a church fellowship hall is a square 95 feet on each side. Use $(a - b)^2$ to find its area mentally.

Show answer

$$(100 - 5)^2 = 10000 - 1000 + 25 = 9025 \text{ square feet}$$

13. **Hard** Expand $(x^2 - 3)^2$.

Show answer

$$(x^2)^2 - 2(x^2)(3) + 9 = x^4 - 6x^2 + 9$$

14. **Challenge** Ruth has a square garden. If she made one side 4 meters longer and the other side 4 meters shorter, the rectangle would have an area of 84 square meters. How long is each side of her square garden?

Show answer

$$(x + 4)(x - 4) = x^2 - 16 = 84, \text{ so } x^2 = 100 \text{ and } x = 10 \text{ meters (check: } 14 \times 6 = 84)$$

15. **Challenge** Find the numbers a and c so that $(x + a)^2 = x^2 + 10x + c$ for every x . Give a , then c .

Show answer

$$(x + a)^2 = x^2 + 2ax + a^2, \text{ so } 2a = 10, a = 5, \text{ and } c = a^2 = 25$$

COMMON MISTAKES

- Writing $(a + b)^2 = a^2 + b^2$. The square diagram shows two rectangles of area ab are missing; always include the middle term $2ab$.
- Forgetting to square the coefficient: $(3x)^2 = 3x^2$. The whole first term is squared, so $(3x)^2 = 9x^2$.
- Writing the last term of $(a - b)^2$ as $-b^2$. Since $(-b)(-b) = +b^2$, the last term is always positive: $(y - 4)^2 = y^2 - 8y + 16$.
- Using $(a + b)(a - b) = a^2 - b^2$ when the two binomials do not have the same terms, as in $(x + 3)(x - 5)$. That pattern needs a sum and a difference of the same two terms; otherwise multiply all four products.

Make it a game • Cut the Square. Cut a paper square with sides $a + b$, using $a = 5$ and $b = 2$. Cut it into four pieces and label the areas a^2 , ab , ab and b^2 . Show that $(a + b)^2$ is not $a^2 + b^2$ by comparing 49 with 29. Then the parent calls special products and you expand each in 15 seconds, including $(x + 4)(x - 4)$.

8.5 DIVIDING POLYNOMIALS BY MONOMIALS

OBJECTIVES

- Divide a polynomial by a monomial by dividing each term, using the quotient of powers rule.
- Explain why dividing each term works and why it is the reverse of distributing.
- Check a quotient by multiplying it by the divisor.
- Solve area, volume and sharing problems by dividing a polynomial by a monomial.

WATCH

Dividing Polynomials by Monomials (TTP Video 77)

Professor Leonard · 15:40

<https://www.youtube.com/watch?v=AAHAojm1acg>

WHEN TO WATCH

Before Example 1.

Note: Alternate-tier primary (Professor Leonard), about 16 minutes. It matches the lesson exactly. Eddie Woo teaches in Australia: the backup says 'pronumeral' for variable.

BACKUP VIDEO

Products & Factors (1 of 2: Multiplying and Dividing Terms with Pronumerals)

Eddie Woo · 7:13

<https://www.youtube.com/watch?v=tPBQbW5XYqc>

Before the Core ideas: dividing single terms with variables.

KEY TERMS

dividend, divisor, quotient

In $\frac{12x^2 + 8x}{4x} = 3x + 2$, the dividend is $12x^2 + 8x$ (what is divided), the divisor is $4x$ (what you divide by), and the quotient is $3x + 2$ (the result).

quotient of powers rule

When you divide powers with the same base, subtract the exponents: $\frac{x^m}{x^n} = x^{m-n}$ for $x \neq 0$ (from Lessons 7.1 and 7.2).

reciprocal

The number you multiply by to get 1.

The reciprocal of c is $\frac{1}{c}$, and dividing by c is the same as multiplying by $\frac{1}{c}$.

CORE IDEAS

To divide a polynomial by a monomial, **divide every term** of the polynomial by the monomial: $\frac{a + b + c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}$. For each term, divide the coefficients and

subtract the exponents of matching variables. For example, $\frac{12x^4}{4x^2} = 3x^2$, because $12 \div 4 = 3$ and $x^{4-2} = x^2$. Throughout this lesson we assume the divisor is not zero.

Why it works. Dividing by d is the same as multiplying by its reciprocal $\frac{1}{d}$. So $\frac{a + b}{d} = \frac{1}{d}(a + b) = \frac{a}{d} + \frac{b}{d}$ by the distributive property. It is the same reason that $\frac{60 + 9}{3} = 20 + 3 = 23$, which is how you divide 69 by 3 in your head.

Division undoes multiplication. Multiplying by a monomial as in Lesson 8.2 gives $4x^2(3x^2 - 2x + 1) = 12x^4 - 8x^3 + 4x^2$. Dividing reverses that: $\frac{12x^4 - 8x^3 + 4x^2}{4x^2} = 3x^2 - 2x + 1$. That gives the best way to **check** any quotient: multiply the quotient by the divisor and you must get the dividend back.

Divide **every** term, not just the first. In $\frac{6x + 8}{2}$, the 2 divides both terms, giving $3x + 4$, never $3x + 8$. Also, you may not 'cancel' a number or variable that appears in only one term: $\frac{x + 6}{x}$ is $1 + \frac{6}{x}$, not 6. You can only divide out a factor of the *whole* numerator, and you will study this more in Unit 12.

Sometimes a term does not divide evenly. For example, $\frac{6x^3 + 4x^2 - 3x}{2x^2} = 3x + 2 - \frac{3}{2x}$. The answer is correct, but it is not a polynomial, because it has a variable in a denominator (or, using Lesson 7.2, a negative exponent). Dividing by a binomial such as $x + 2$ needs polynomial long division, which you will learn in Algebra II.

KEY FORMULAS

$$\frac{a + b + c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}, \quad d \neq 0$$

$$\frac{x^m}{x^n} = x^{m-n}, \quad x \neq 0$$

$$\text{quotient} \times \text{divisor} = \text{dividend}$$

Practice habit. Check every division by multiplying: (quotient) $\times$ (divisor) must give back the dividend term by term. Write the check on the line right after your answer.

WORKED EXAMPLES

Example 1. Divide $\frac{12x^4 - 8x^3 + 4x^2}{4x^2}$.

Show solution

1. Divide each term by $4x^2$: $\frac{12x^4}{4x^2} - \frac{8x^3}{4x^2} + \frac{4x^2}{4x^2}$.
2. First term: $12 \div 4 = 3$ and $x^{4-2} = x^2$, giving $3x^2$.
3. Second term: $8 \div 4 = 2$ and $x^{3-2} = x$, giving $-2x$.
4. Third term: $\frac{4x^2}{4x^2} = 1$.
5. Quotient: $3x^2 - 2x + 1$.
6. Check: $4x^2(3x^2 - 2x + 1) = 12x^4 - 8x^3 + 4x^2$. It matches the dividend.

Answer: $3x^2 - 2x + 1$

Example 2. A rectangular pasture has an area of $(6x^3 + 9x^2 - 15x)$ square meters. Its width is $3x$ meters. Write a polynomial for its length.

Show solution

1. Length = area divided by width: $\frac{6x^3 + 9x^2 - 15x}{3x}$.
2. Divide each term: $\frac{6x^3}{3x} = 2x^2$, $\frac{9x^2}{3x} = 3x$, $\frac{-15x}{3x} = -5$.
3. Length: $2x^2 + 3x - 5$ meters.
4. Check by multiplying: $3x(2x^2 + 3x - 5) = 6x^3 + 9x^2 - 15x$. It matches.
5. Check with $x = 2$: area $48 + 36 - 30 = 54$, width 6, length $8 + 6 - 5 = 9$, and $6 \times 9 = 54$.

Answer: $(2x^2 + 3x - 5)$ meters

PRACTICE

1. Easy Divide $\frac{10x + 15}{5}$.

Show answer

$$2x + 3$$

2. Easy Divide $\frac{8x^2 - 6x}{2x}$.

Show answer

$$4x - 3$$

3. Easy Divide $\frac{9y^3 + 12y^2}{3y}$.

Show answer

$$3y^2 + 4y$$

4. Easy Divide $\frac{14a^4 - 21a^2}{7a^2}$.

Show answer

$$2a^2 - 3$$

5. Medium Divide $\frac{20x^5 - 15x^3 + 10x^2}{5x^2}$.

Show answer

$$4x^3 - 3x + 2$$

6. Medium Divide $\frac{18x^3 - 12x^2 + 6x}{-6x}$.

Show answer

$$-3x^2 + 2x - 1 \text{ (check: } -6x(-3x^2 + 2x - 1) = 18x^3 - 12x^2 + 6x \text{)}$$

7. Medium Divide $\frac{12a^3b^2 - 8a^2b + 4ab}{4ab}$.

Show answer

$$3a^2b - 2a + 1$$

8. Medium The Miller family's rectangular hayfield has an area of $(8x^2 + 12x)$ square yards. Its width is $4x$ yards. Write a polynomial for its length.

Show answer

$$\frac{8x^2 + 12x}{4x} = (2x + 3) \text{ yards}$$

9. Medium Ava wrote $\frac{6x + 8}{2} = 3x + 8$. Explain her mistake and give the correct quotient.

Show answer

Ava divided only the first term by 2. Dividing by 2 means multiplying the whole numerator by $\frac{1}{2}$, so every term must be divided: $\frac{6x}{2} + \frac{8}{2} = 3x + 4$.
Check: $2(3x + 4) = 6x + 8$.

10. Hard Divide $\frac{-24x^6 + 16x^4 - 8x^3}{-8x^3}$.

Show answer

$$3x^3 - 2x + 1 \text{ (check: } -8x^3(3x^3 - 2x + 1) = -24x^6 + 16x^4 - 8x^3 \text{)}$$

11. Hard Henry builds a rectangular planter box with a volume of $(12x^3 + 30x^2)$ cubic inches. Its base is $2x$ inches by $3x$ inches. Write a polynomial for its height.

Show answer

$$\text{Base area} = 6x^2; \text{ height} = \frac{12x^3 + 30x^2}{6x^2} = (2x + 5) \text{ inches}$$

12. **Hard** A farmer harvests $(36x^2 + 24x)$ bushels of apples from $12x$ trees, and each tree gives the same amount. Write a polynomial for the number of bushels per tree.

Show answer

$$\frac{36x^2 + 24x}{12x} = 3x + 2 \text{ bushels per tree}$$

13. **Hard** Simplify $\frac{15x^3 - 10x^2}{5x} - \frac{8x^3 + 4x^2}{4x}$.

Show answer

$$(3x^2 - 2x) - (2x^2 + x) = x^2 - 3x$$

14. **Challenge** A polynomial divided by $3x^2$ gives the quotient $4x^2 - x + 5$. What is the polynomial?

Show answer

$$\text{Multiply: } 3x^2(4x^2 - x + 5) = 12x^4 - 3x^3 + 15x^2$$

15. **Challenge** Divide $\frac{6x^3 + 4x^2 - 3x}{2x^2}$. Write the answer with positive exponents, and tell whether the quotient is a polynomial.

Show answer

$$3x + 2 - \frac{3}{2x}; \text{ it is not a polynomial, because the last term has } x \text{ in the denominator}$$

COMMON MISTAKES

- Dividing only the first term: $\frac{6x + 8}{2} = 3x + 8$. The divisor divides every term of the numerator; draw a short fraction bar under each term if it helps.
- Dividing the exponents instead of subtracting them: $\frac{x^6}{x^3} = x^2$. Write out the factors once to see that three of the six x 's divide out, leaving x^3 .
- Losing signs when the divisor is negative. Divide the sign as well: $\frac{-8x^3}{-8x^3} = 1$ and $\frac{16x^4}{-8x^3} = -2x$. Check by multiplying back.
- Forgetting that a term equal to the divisor becomes 1, not 0: $\frac{4x^2}{4x^2} = 1$. Multiplying back catches this mistake right away.

Make it a game • Build It, Then Break It. First, multiply a monomial by a polynomial, such as $3x(2x^2 - x + 4)$, and write the product on a card. Hand the parent only the product and the monomial $3x$. The parent divides each term to recover the original, then multiplies back to check. Trade roles three times and try to include a term that cancels completely.

POLYNOMIALS

In this unit you learn to work with polynomials, the expressions built from numbers and variables with whole-number exponents, such as $3x^2 - 5x + 2$. You will name them, add and subtract them, multiply them (by a monomial, by a binomial and by larger polynomials), learn the three special products by heart, and divide a polynomial by a monomial. Every rule comes from tools you already have: combining like terms, the distributive property and the exponent rules of Unit 7.

Area pictures run through the whole unit. A rectangle cut into pieces shows why the distributive property and FOIL work, and a square cut into two squares and two rectangles shows why $(a + b)^2 = a^2 + 2ab + b^2$, exactly as Euclid proved more than two thousand years ago. These pictures also give practical uses: areas of gardens, pens, patios and quilts, and volumes of boxes.

Unit 8 sets up Unit 9 directly. Factoring is multiplying in reverse, so the faster and more accurate you are at multiplying polynomials now, the easier factoring will be. Later the special products return in completing the square and the quadratic formula (Unit 11) and in rational expressions (Unit 12).

PREREQUISITES

- Combine like terms and use the distributive property, including distributing a negative (Lesson 1.4).
- Apply the sign rules for multiplying and dividing integers.
- Use the product of powers and quotient of powers rules for exponents (Lessons 7.1 and 7.2).
- Evaluate an expression by substituting a number for the variable (Lesson 1.3).
- Find the area of a rectangle and the volume of a rectangular box.

MASTERY CHECKLIST

- ☐ I can classify a polynomial by its number of terms and find its degree and leading coefficient.
- ☐ I can write a polynomial in standard form.
- ☐ I can add polynomials and subtract them by adding the opposite of every term.
- ☐ I can multiply a polynomial by a monomial and explain it with an area picture.
- ☐ I can multiply two binomials with a box model or FOIL and explain why FOIL works.
- ☐ I can multiply a binomial by a trinomial.

- ☐ I can expand

$$(a + b)^2$$

$$(a + b)(a - b)$$

d

fro

m

m

e

m

or

y

an

d

ex

pl

ai

n

ea

ch
wi
th
a
sq
ua
re
di
ag
ra
m.

☐ I can explain why

$$(a + b)^2 \neq a^2 + b^2.$$

☐ I can use special products for mental math such as

29 and

31

31^2 .

☐ I can divide a polynomial by a monomial and check by multiplying.

☐ I can write polynomial expressions for areas, volumes and other real situations.

☐ I can check any polynomial answer by substituting a number.

LESSONS IN THIS UNIT

8.1 Polynomials and Their Terms

2 sessions of 45 minutes

8.2 Multiplying by a Monomial

45 minutes

8.3 Multiplying Binomials and Polynomials

2 sessions of 45 minutes

8.4 Special Products

2 sessions of 45 minutes

UNIT 8 TEST

Time allowed: 50 minutes. No calculator is needed or allowed on this test. Show your work on every problem, one step per line. Give all answers exactly, simplified and in standard form, and include units on word problems.

1. **5 pts**

Write $-2x + 9x^3 - 5 + x^2$ in standard form. Then give its degree and its leading coefficient.

2. **5 pts**

Add $(6x^2 - 5x + 2) + (-4x^2 + x + 7)$.

3. **6 pts**

Subtract $(3x^3 - 2x^2 + 8) - (x^3 + 4x^2 - 3x + 8)$.

4. **6 pts**

Multiply $-4x(2x^2 - 5x + 3)$.

5. **6 pts**

Multiply $3x^2y(2x - 5y)$.

6. **6 pts**

Multiply $(x - 6)(x + 2)$.

7. **7 pts**

Multiply $(4x + 3)(2x - 5)$.

8. **8 pts**

Multiply $(x - 3)(2x^2 + x - 4)$.

9. **5 pts**

Expand $(2x - 7)^2$.

10. (5 pts)

Multiply $(5x + 1)(5x - 1)$.

11. (4 pts)

Use a special product to find 39×41 mentally. Show the pattern you used.

12. (6 pts)

Divide $\frac{15x^4 - 9x^3 + 3x^2}{3x^2}$.

13. (8 pts)

Grace is making a rectangular quilt $3x$ inches wide and $(4x + 10)$ inches long. In the middle she sews on a square patch x inches on a side. Write a polynomial in standard form for the area of the quilt not covered by the patch.

14. (8 pts)

A square backyard is $(x + 4)$ meters on each side. In it lies a square flower bed $(x - 2)$ meters on each side, and the rest of the backyard is grass. Write a polynomial for the area of the grass.

15. (8 pts)

A train travels $(90x^2 + 30x)$ miles in $3x$ hours at a steady speed. Write a polynomial for its speed in miles per hour.

16. (7 pts)

Explain, with a square diagram, why $(x + 4)^2$ is not equal to $x^2 + 16$. Then give the correct expansion.

ANSWER KEY

1. $9x^3 + x^2 - 2x - 5$; degree 3; leading coefficient 9
2. $2x^2 - 4x + 9$
3. $3x^3 - 2x^2 + 8 - x^3 - 4x^2 + 3x - 8 = 2x^3 - 6x^2 + 3x$
4. $-8x^3 + 20x^2 - 12x$
5. $6x^3y - 15x^2y^2$
6. $x^2 + 2x - 6x - 12 = x^2 - 4x - 12$
7. $8x^2 - 20x + 6x - 15 = 8x^2 - 14x - 15$
8. $2x^3 + x^2 - 4x - 6x^2 - 3x + 12 = 2x^3 - 5x^2 - 7x + 12$
9. $4x^2 - 28x + 49$
10. $25x^2 - 1$
11. $(40 - 1)(40 + 1) = 1600 - 1 = 1599$
12. $5x^2 - 3x + 1$ (check: $3x^2(5x^2 - 3x + 1) = 15x^4 - 9x^3 + 3x^2$)
13. $3x(4x + 10) - x^2 = 12x^2 + 30x - x^2 = (11x^2 + 30x)$ square inches
14. $(x + 4)^2 - (x - 2)^2 = (x^2 + 8x + 16) - (x^2 - 4x + 4) = (12x + 12)$ square meters
15. Speed = $\frac{90x^2 + 30x}{3x} = (30x + 10)$ miles per hour
16. A square with side $x + 4$ splits into an x -by- x square (area x^2), a 4-by-4 square (area 16) and two x -by-4 rectangles (area $4x$ each). The expression $x^2 + 16$ leaves out the two rectangles. Correct: $(x + 4)^2 = x^2 + 8x + 16$. (At $x = 1$: $5^2 = 25$, but $1 + 16 = 17$.)

FACTORING

In Unit 8 you multiplied polynomials. In this unit you learn to run that process backward: to **factor** a polynomial is to write it as a product. You will take out the greatest common factor, factor trinomials such as $x^2 + bx + c$ and $ax^2 + bx + c$, recognize the special products (difference of squares and perfect-square trinomials), and factor four-term polynomials by grouping. Along the way you will learn a dependable strategy for factoring completely: GCF first, count the terms, check each factor, and multiply to check.

Why does factoring matter? Because a product is much easier to reason about than a sum. The zero-product property, if $ab = 0$ then $a = 0$ or $b = 0$, lets you solve quadratic equations by factoring, and that answers real questions: the size of a garden with a given area, or when a thrown ball comes back to the ground.

Factoring is used constantly in the rest of the course. In Unit 10 you will factor out perfect squares to simplify radicals; in Unit 11 you will solve quadratic equations, find the zeros of parabolas and complete the square; and in Unit 12 you will simplify rational expressions by factoring numerators and denominators. Time spent making factoring quick and accurate now pays off all year and in Algebra II.

PREREQUISITES

- Find the GCF of whole numbers using prime factorization (Pre-Algebra).
- Use the exponent rules for products and quotients of powers (Lesson 7.1).
- Multiply a polynomial by a monomial (Lesson 8.2).
- Multiply two binomials with the box model or FOIL (Lesson 8.3).
- Know the special products $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$ (Lesson 8.4).
- Solve one-step and two-step linear equations and check by substitution (Unit 2).

MASTERY CHECKLIST

- ☐ I can find the GCF of monomials and factor it out of a polynomial, including a negative GCF.
- ☐ I can factor out a common binomial factor such as $(x + 5)$.
- ☐ I can factor $x^2 + bx + c$ by finding two numbers whose product is c and whose sum is b , and explain why this works.
- ☐ I can recognize a prime trinomial.
- ☐ I can factor $ax^2 + bx + c$ using the ac (grouping) method or trial and error.
- ☐ I can factor a difference of two squares and a perfect-square trinomial, and explain why $x^2 - 9$ does not factor.
- ☐ I can factor a four-term polynomial by grouping.
- ☐ I can follow a factoring strategy to factor a polynomial completely and check my answer by multiplying.
- ☐ I can explain why the zero-product property is true and use it to solve equations by factoring.
- ☐ I can solve area, number and falling-object problems by

factoring and reject answers that make no sense.

LESSONS IN THIS UNIT

9.1	Factoring Out the Greatest Common Factor	2 sessions of 45 minutes
9.2	Factoring $x^2 + bx + c$	2 sessions of 45 minutes
9.3	Factoring $ax^2 + bx + c$	2 sessions of 45 minutes
9.4	Factoring Special Products	2 sessions of 45 minutes
9.5	Factoring by Grouping and Factoring Completely	2 sessions of 45 minutes
9.6	Solving Equations by Factoring	2 sessions of 45 minutes

UNIT 9 TEST

Time allowed: 60 minutes. No calculator. Show your work for every problem. Factor completely unless told otherwise, and write "prime" for a polynomial that does not factor over the integers. Give exact answers (fractions, not decimals) and include units in word problems. Check your factoring by multiplying and your solutions by substituting.

1. **5 pts**
Factor $15x^4 - 10x^3 + 25x^2$.
2. **5 pts**
Factor out a negative GCF: $-6y^2 + 21y$.
3. **6 pts**
Factor $2x(x + 9) - 7(x + 9)$.
4. **6 pts**
Factor $x^2 + 9x + 14$.

5. (6 pts)

Factor $x^2 - 2x - 35$.

6. (7 pts)

Factor $3x^2 + 11x + 10$.

7. (7 pts)

Factor $8x^2 - 10x - 3$.

8. (6 pts)

Factor $64x^2 - 9$.

9. (6 pts)

Factor $9x^2 - 42x + 49$.

10. (5 pts)

A square flower bed in front of the church has area $4x^2 + 20x + 25$ square feet.
Find an expression for the length of one side.

11. (7 pts)

Factor completely: $x^3 + 4x^2 - 9x - 36$.

12. (7 pts)

Factor completely: $5x^3 - 45x$.

13. (7 pts)

Solve $x^2 + 4x = 21$.

14. (8 pts)

A rectangular rug is 4 feet longer than it is wide, and its area is 45 square feet.
Find its width and length.

15. (8 pts)

From a platform 48 feet high, Samuel throws a ball straight up at 32 feet per second. Its height after t seconds is $h = -16t^2 + 32t + 48$ feet. After how many seconds does the ball hit the ground?

16. **4 pts**

To solve $x^2 = 5x$, Henry divided both sides by x and got $x = 5$. Explain what went wrong and give the complete solution.

ANSWER KEY

1. $5x^2(3x^2 - 2x + 5)$
2. $-3y(2y - 7)$
3. $(x + 9)(2x - 7)$
4. $(x + 2)(x + 7)$
5. $(x - 7)(x + 5)$
6. $ac = 30$, numbers 5 and 6: $x(3x + 5) + 2(3x + 5) = (3x + 5)(x + 2)$
7. $ac = -24$, numbers -12 and 2 : $4x(2x - 3) + 1(2x - 3) = (2x - 3)(4x + 1)$
8. $(8x + 3)(8x - 3)$
9. $2 \cdot 3x \cdot 7 = 42x$, so $(3x - 7)^2$
10. $4x^2 + 20x + 25 = (2x + 5)^2$, so each side is $2x + 5$ feet.
11. $x^2(x + 4) - 9(x + 4) = (x + 4)(x^2 - 9) = (x + 4)(x + 3)(x - 3)$
12. $5x(x^2 - 9) = 5x(x + 3)(x - 3)$
13. $x^2 + 4x - 21 = 0$, $(x + 7)(x - 3) = 0$, so $x = -7$ or $x = 3$
14. $w(w + 4) = 45$, $w^2 + 4w - 45 = 0$, $(w + 9)(w - 5) = 0$; reject $w = -9$. Width 5 feet, length 9 feet.
15. $-16t^2 + 32t + 48 = 0$, $-16(t^2 - 2t - 3) = 0$, $-16(t - 3)(t + 1) = 0$; reject $t = -1$. It hits the ground after 3 seconds.
16. Dividing by x assumes $x \neq 0$, so it throws away the solution $x = 0$. Instead write $x^2 - 5x = 0$, factor $x(x - 5) = 0$, and use the zero-product property: $x = 0$ or $x = 5$.

9.1 FACTORING OUT THE GREATEST COMMON FACTOR

OBJECTIVES

- Find the greatest common factor (GCF) of two or more monomials.
- Factor the GCF out of a polynomial and check the result by distributing.
- Factor out a negative GCF when the leading coefficient is negative.
- Factor out a common binomial factor such as $(x + 5)$.

WATCH

Factorising Algebraic Expressions (1 of 2: All Positives)

Eddie Woo · 8:28

<https://www.youtube.com/watch?v=O9A496z9Xeg>

WHEN TO WATCH

Before Example 1: taking out the greatest common factor.

Note: Eddie Woo teaches in Australia: he says 'factorise' and 'highest common factor'. Neither video factors out a common binomial (objective 4); see Lesson 9.5.

BACKUP VIDEO

Factorising Algebraic Expressions (2 of 2: With Negatives)

Eddie Woo · 7:05

<https://www.youtube.com/watch?v=lqtmY5h45W8>

Before Example 2: taking out a negative common factor.

KEY TERMS

factor (verb)

To write an expression as a product.
Factoring is the reverse of multiplying out: $6x + 9$ factored is $3(2x + 3)$.

factor (noun)

One of the expressions being multiplied in a product. In $3(2x + 3)$, the factors are 3 and $2x + 3$.

greatest common factor (GCF) of monomials

The largest monomial that divides every one of the monomials evenly: the GCF of the coefficients times each shared variable raised to its smallest exponent.

common binomial factor

A binomial, such as $(x - 4)$, that appears as a factor in every term of an expression, so it can be factored out just like a number.

factored completely

Written as a product in which no factor (other than a single number) can be factored any further.

CORE IDEAS

In Unit 8 you multiplied: $3x(2x + 5) = 6x^2 + 15x$. **Factoring** runs that film backward. You start with $6x^2 + 15x$ and ask, "What was multiplied to get this?" The distributive property $a(b + c) = ab + ac$ read from right to left, $ab + ac = a(b + c)$, is the whole idea of this lesson.

To find the GCF of monomials, handle the numbers and the variables separately. For $18x^3$ and $24x^2$: the GCF of 18 and 24 is 6. For the variables, write out the factors: $x^3 = x \cdot x \cdot x$ and $x^2 = x \cdot x$. They share only two x 's, so the common part is x^2 . That is **why** you always take the *smallest* exponent of each shared variable: you cannot take out more copies of x than the term with the fewest copies has. A variable that is missing from even one term is not part of the GCF. So the GCF is $6x^2$.

To factor out the GCF, divide each term by the GCF and write the quotients inside parentheses. For $12x^3 - 18x^2 + 30x$, the GCF is $6x$, and $12x^3 \div 6x = 2x^2$, $-18x^2 \div 6x = -3x$, $30x \div 6x = 5$. So $12x^3 - 18x^2 + 30x = 6x(2x^2 - 3x + 5)$. Check by distributing: $6x \cdot 2x^2 - 6x \cdot 3x + 6x \cdot 5 = 12x^3 - 18x^2 + 30x$. The number of terms inside the parentheses always equals the number of terms you started with; if a term divides to 1, you must write the 1: $4x^2 + 4x = 4x(x + 1)$.

When the leading coefficient is negative, it is usually neater to factor out a **negative** GCF. Dividing every term by a negative number flips every sign: $-4x^2 + 10x = -2x(2x - 5)$. Check: $-2x \cdot 2x = -4x^2$ and $-2x \cdot (-5) = 10x$. This is very common in the falling-object formula $h = -16t^2 + vt$, where we take out $-16t$.

A GCF does not have to be a monomial. In $x(x + 5) + 3(x + 5)$ both terms contain the factor $(x + 5)$. Think of $(x + 5)$ as one block, like a single letter B :

$xB + 3B = B(x + 3)$. So $x(x + 5) + 3(x + 5) = (x + 5)(x + 3)$. Watch for binomials that are *opposites*, such as $(4 - x)$ and $(x - 4)$: since $4 - x = -(x - 4)$, you can rewrite one to match the other. This idea of a common binomial factor is the key step in factoring by grouping later in this unit.

From now on, the first question to ask about any polynomial is, "Is there a GCF?" Factoring out the GCF first makes every later factoring step easier, and a polynomial is not factored completely while a common factor remains.

KEY FORMULAS

$$ab + ac = a(b + c)$$

GCF of x^m and x^n is $x^{\text{smaller of } m, n}$

$$-a - b = -(a + b)$$

Euclid connection. Euclid's Elements, Book VII, Proposition 2, gives a method for finding the greatest common measure of two whole numbers by repeated subtraction; today we call it the Euclidean algorithm, and it still finds the GCF of large numbers quickly.

Practice habit. Check every factoring by multiplying it back out. Distributing the GCF takes ten seconds and catches a missing 1 or a wrong sign every time.

WORKED EXAMPLES

Example 1. Factor $12x^3 - 18x^2 + 30x$.

Show solution

1. GCF of the coefficients 12, 18 and 30 is 6.
2. Every term has x ; the smallest exponent is 1, so the variable part is x .
3. The GCF is $6x$.
4. Divide each term by $6x$: $12x^3 \div 6x = 2x^2$, $-18x^2 \div 6x = -3x$, $30x \div 6x = 5$.
5. Write the product: $6x(2x^2 - 3x + 5)$.
6. Check: $6x(2x^2 - 3x + 5) = 12x^3 - 18x^2 + 30x$. ✓

Answer: $6x(2x^2 - 3x + 5)$

Example 2. Grace tosses a ball straight up from the ground at 48 feet per second. Its height in feet after t seconds is $h = -16t^2 + 48t$. Factor the right side by taking out a negative GCF, then use the factored form to find the height after 1 second.

Show solution

1. GCF of 16 and 48 is 16; both terms contain t ; the leading coefficient is negative, so take out $-16t$.
2. Divide: $-16t^2 \div (-16t) = t$ and $48t \div (-16t) = -3$.
3. So $h = -16t(t - 3)$.
4. Check: $-16t \cdot t = -16t^2$ and $-16t \cdot (-3) = 48t$. ✓
5. At $t = 1$: $h = -16(1)(1 - 3) = -16(-2) = 32$.
6. Check with the original: $-16(1)^2 + 48(1) = -16 + 48 = 32$. ✓

Answer: $h = -16t(t - 3)$; after 1 second the ball is 32 feet high.

PRACTICE

1. Easy Find the GCF of $18x^3$ and $24x^2$.

Show answer

$$6x^2$$

2. Easy Factor $5x + 20$.

Show answer

$$5(x + 4)$$

3. Easy Factor $7y^2 - 14y$.

Show answer

$$7y(y - 2)$$

4. Easy Factor $9a^3 + 6a^2$.

Show answer

$$3a^2(3a + 2)$$

5. Medium Factor $8x^4 - 12x^3 + 20x^2$.

Show answer

$$\text{GCF } 4x^2: 4x^2(2x^2 - 3x + 5)$$

6. Medium Factor $6x^2y^3 + 15xy^2$.

Show answer

$$\text{GCF } 3xy^2: 3xy^2(2xy + 5)$$

7. Medium Factor out a negative GCF: $-4x^2 + 10x$.

Show answer

$$-2x(2x - 5)$$

8. Medium The Miller family buys 12 bags of chicken feed at p dollars each and 12 bales of straw at q dollars each. Their total cost is $12p + 12q$ dollars. Write the total cost in factored form.

Show answer

$$12(p + q) \text{ dollars: 12 times the cost of one bag and one bale together.}$$

9. Medium Factor $x(x + 5) + 3(x + 5)$.

Show answer

$$(x + 5)(x + 3)$$

10. Medium Ruth factored $6x^2 + 9x$ as $3(2x^2 + 3x)$. Explain why her answer is not factored completely, and finish the job.

Show answer

The terms $2x^2$ and $3x$ still share a factor of x , so 3 was not the greatest common factor. The GCF is $3x$: $6x^2 + 9x = 3x(2x + 3)$.

11. Hard Factor $2y(y - 7) - 5(y - 7)$.

Show answer

$$(y - 7)(2y - 5)$$

12. Hard A rectangular vegetable bed has area $4x^2 + 12x$ square meters. Its width is $4x$ meters. Find an expression for its length.

Show answer

$$4x^2 + 12x = 4x(x + 3), \text{ so the length is } x + 3 \text{ meters.}$$

13. **Hard** A model rocket is launched straight up from the ground at 96 feet per second, so its height in feet after t seconds is $h = -16t^2 + 96t$. (a) Factor the right side by taking out $-16t$. (b) Use the factored form to find the height after 2 seconds.

Show answer

(a) $h = -16t(t - 6)$. (b) $-16(2)(2 - 6) = -32(-4) = 128$ feet.

14. **Challenge** Factor $3x(x - 4) + 5(4 - x)$.

Show answer

Rewrite $5(4 - x) = -5(x - 4)$, so the expression is $3x(x - 4) - 5(x - 4) = (x - 4)(3x - 5)$.

COMMON MISTAKES

- Dropping the 1: $4x^2 + 4x$ is $4x(x + 1)$, not $4x(x)$. Count terms: the parentheses must hold as many terms as you started with.
- Taking out a common factor that is not the greatest, such as $2x$ from $8x^3 + 12x^2$. Use the GCF of the coefficients and the smallest exponent of each shared variable: $4x^2(2x + 3)$.
- Forgetting to flip every sign when factoring out a negative:
 $-6x^2 + 9x = -3x(2x - 3)$, not $-3x(2x + 3)$. Distribute back to check.
- Treating $(x - 4)$ and $(4 - x)$ as the same factor. They are opposites:
 $4 - x = -(x - 4)$.

Make it a game • Biggest Piece Out. Beat the clock. The parent writes ten polynomials such as $12x^3 + 18x^2$, $6x^2 - 9x$, and $x(x + 2) + 3(x + 2)$. In 90 seconds, factor out the greatest common factor from as many as you can. Check each answer by distributing. Two points for a fully factored answer, one if the GCF is not yet the biggest.

9.2 FACTORING $x^2 + bx + c$

OBJECTIVES

- Explain why factoring $x^2 + bx + c$ means finding two numbers whose product is c and whose sum is b .
- Factor trinomials of the form $x^2 + bx + c$, using the signs of b and c to guide the search.
- Recognize a prime trinomial that cannot be factored over the integers.
- Check every factorization by multiplying the binomials back out.

WATCH

Factorising Monic Quadratic Expressions (1 of 2: Symbolically)

Eddie Woo · 8:54

<https://www.youtube.com/watch?v=PSM1xqzJD9U>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: 'monic' means the x^2 coefficient is 1.

BACKUP VIDEO

Factorising Monic Quadratic Expressions (2 of 2: Visually)

Eddie Woo · 10:04

<https://www.youtube.com/watch?v=1rwJq4zYXiM>

After Example 1: the same factoring shown with area tiles.

KEY TERMS

trinomial

A polynomial with three terms, such as $x^2 + 7x + 12$.

quadratic trinomial

A trinomial of degree 2. In this lesson the leading coefficient is 1, so it looks like $x^2 + bx + c$.

factor pair

Two integers whose product is a given number. The factor pairs of 12 are 1×12 , 2×6 , 3×4 and the same pairs with both numbers negative.

prime polynomial

A polynomial (other than a single number) that cannot be written as a product of polynomials of lower degree with integer coefficients, such as $x^2 + 4x + 6$.

CORE IDEAS

Start with multiplication, which you already know. Multiply $(x + p)(x + q)$ with the box model or FOIL: $x \cdot x + qx + px + pq = x^2 + (p + q)x + pq$. Look at what happened: the middle coefficient is the **sum** $p + q$ and the constant is the **product** pq . For example, $(x + 3)(x + 4) = x^2 + 7x + 12$, and indeed $3 + 4 = 7$ and $3 \times 4 = 12$.

Factoring reverses that. To factor $x^2 + bx + c$, you need two numbers p and q with $p \cdot q = c$ and $p + q = b$. Then $x^2 + bx + c = (x + p)(x + q)$. **Why it works:** multiplying $(x + p)(x + q)$ gives back exactly $x^2 + (p + q)x + pq = x^2 + bx + c$. This is "reverse FOIL." The search is short: list the factor pairs of c and find the pair that adds to b .

The signs of b and c tell you where to look. If c is positive, p and q have the **same** sign (a product is positive only when the signs match); then they share the sign of b , since two positives add to a positive and two negatives add to a negative. So $x^2 + 7x + 12 = (x + 3)(x + 4)$ and $x^2 - 7x + 12 = (x - 3)(x - 4)$. If c is negative, p and q have **opposite** signs, and the number with the larger absolute value takes the sign of b : for $x^2 + 2x - 15$, the pair 5 and -3 works because $5 \times (-3) = -15$ and $5 + (-3) = 2$.

A short table keeps you organized. For $x^2 - 9x + 20$: $c = 20 > 0$ and $b < 0$, so both numbers are negative. Pairs: $-1, -20$ (sum -21); $-2, -10$ (sum -12); $-4, -5$ (sum -9). Found it: $x^2 - 9x + 20 = (x - 4)(x - 5)$. The order of the factors does not matter, because multiplication is commutative.

Sometimes no pair works. For $x^2 + 4x + 6$ the pairs of 6 are 1, 6 (sum 7) and 2, 3 (sum 5), and the negative pairs have negative sums; none adds to 4. Since the list of factor pairs is finite and you checked them all, you know for certain the trinomial does not factor over the integers. It is **prime**. Writing "prime" is a correct, complete answer.

The same idea works with two variables. In $x^2 - 4xy - 21y^2$ you need two terms whose product is $-21y^2$ and whose sum is $-4y$: they are $-7y$ and $3y$, so it factors as

$(x - 7y)(x + 3y)$. And always look for a GCF first: $3x^2 + 6x - 72 = 3(x^2 + 2x - 24) = 3(x + 6)(x - 4)$.

KEY FORMULAS

$$(x + p)(x + q) = x^2 + (p + q)x + pq$$

$$x^2 + bx + c = (x + p)(x + q) \text{ when } p + q = b \text{ and } pq = c$$

Practice habit. Before you factor, say the sign pattern out loud: "c positive, b negative, so both numbers are negative." Then list factor pairs of c in order, smallest first, so you never skip one.

WORKED EXAMPLES

Example 1. Factor $x^2 - 9x + 20$.

Show solution

1. We need $p \cdot q = 20$ and $p + q = -9$.
2. $c = 20$ is positive and $b = -9$ is negative, so both numbers are negative.
3. Try pairs: $-1, -20$ (sum -21); $-2, -10$ (sum -12); $-4, -5$ (sum -9). ✓
4. Write $(x - 4)(x - 5)$.
5. Check: $(x - 4)(x - 5) = x^2 - 5x - 4x + 20 = x^2 - 9x + 20$. ✓

Answer: $(x - 4)(x - 5)$

Example 2. Samuel is building a rectangular pen for his goats. Its area is $x^2 + 3x - 28$ square feet. Factor the expression to find expressions for the length and width of the pen.

Show solution

1. We need $p \cdot q = -28$ and $p + q = 3$.
2. $c = -28$ is negative, so the signs are opposite; $b = 3$ is positive, so the larger number is positive.
3. Try pairs: 28, -1 (sum 27); 14, -2 (sum 12); 7, -4 (sum 3). ✓
4. So $x^2 + 3x - 28 = (x + 7)(x - 4)$.
5. Check: $(x + 7)(x - 4) = x^2 - 4x + 7x - 28 = x^2 + 3x - 28$. ✓
6. The longer side is $x + 7$ feet and the shorter side is $x - 4$ feet.

Answer: $(x + 7)(x - 4)$: length $x + 7$ feet, width $x - 4$ feet.

PRACTICE

1. Easy Factor $x^2 + 7x + 12$.

Show answer

$$(x + 3)(x + 4)$$

2. Easy Factor $x^2 + 8x + 15$.

Show answer

$$(x + 3)(x + 5)$$

3. Easy Factor $x^2 - 6x + 8$.

Show answer

$$(x - 2)(x - 4)$$

4. Easy Factor $x^2 + 2x - 15$.

Show answer

$$(x + 5)(x - 3)$$

5. Medium Factor $x^2 - x - 30$.

Show answer

$$-6 \times 5 = -30 \text{ and } -6 + 5 = -1: (x - 6)(x + 5)$$

6. Medium Factor $x^2 - 13x + 36$.

Show answer

$$(x - 4)(x - 9)$$

7. Medium Factor $y^2 + 4y - 45$.

Show answer

$$(y + 9)(y - 5)$$

8. Medium Factor $x^2 + 4x + 6$, or write "prime" if it does not factor over the integers.

Show answer

Prime: the pairs of 6 add to 7 or 5 (or -7 , -5), never 4.

9. Medium A rectangular herb garden has area $x^2 + 11x + 24$ square feet. Factor to find expressions for its length and width.

Show answer

$$(x + 3)(x + 8): \text{length } x + 8 \text{ feet, width } x + 3 \text{ feet.}$$

10. Medium Explain why, when you factor $x^2 - 10x + 21$, both numbers you are looking for must be negative.

Show answer

Their product is $+21$, so they have the same sign. Their sum is -10 , and two positive numbers cannot add to a negative number, so both must be negative (-3 and -7).

11. Hard Factor $x^2 - 4xy - 21y^2$.

Show answer

$$-7y \cdot 3y = -21y^2 \text{ and } -7y + 3y = -4y: (x - 7y)(x + 3y)$$

12. Hard Factor completely: $3x^2 + 6x - 72$.

Show answer

$$3(x^2 + 2x - 24) = 3(x + 6)(x - 4)$$

13. **Hard** The floor of a rectangular hay loft has area $x^2 + 2x - 48$ square feet. One side is $x - 6$ feet long. Find an expression for the other side.

Show answer

$x^2 + 2x - 48 = (x - 6)(x + 8)$, so the other side is $x + 8$ feet.

14. **Challenge** Find every integer b for which $x^2 + bx + 12$ can be factored over the integers. List them in increasing order.

Show answer

b must be the sum of a factor pair of 12: $1 + 12 = 13$, $2 + 6 = 8$, $3 + 4 = 7$, and the negatives. So $b = -13, -8, -7, 7, 8, 13$.

15. **Challenge** A rectangular field has area $x^2 + 20x + 96$ square yards. Factor to find its dimensions, then find how many yards of fence go around the field when $x = 30$.

Show answer

$(x + 8)(x + 12)$; when $x = 30$ the field is 38 yards by 42 yards, so the fence is $2(38 + 42) = 160$ yards.

COMMON MISTAKES

- Finding numbers with the right product but the wrong sum, such as writing $x^2 + 2x - 15 = (x - 5)(x + 3)$. That gives $-2x$ in the middle. Always multiply back out to check the middle term.
- Getting the signs backwards when c is negative. The number with the larger absolute value takes the sign of b .
- Giving up too soon and calling a trinomial prime. List every factor pair of c before deciding.
- Forgetting to factor out a GCF first, which makes the numbers large and the search harder.

Make it a game • Trinomial Time Trial. Ten trinomials, five minutes. The parent lists them, such as $x^2 + 7x + 12$, $x^2 - 3x - 10$ and $x^2 + 3x + 5$. For each, say the two numbers whose product is c and whose sum is b , then write the factors. One is prime: you must list all factor pairs before declaring it so. Record your score for next time.

9.3 FACTORING $AX^2 + BX + C$

OBJECTIVES

- Factor trinomials $ax^2 + bx + c$ with $a \neq 1$ using the ac (grouping) method.
- Explain why the ac method works by multiplying two general binomials.
- Use trial and error with the outer and inner products as an alternative method.
- Factor out a GCF (or -1) first and check every answer by multiplying.

WATCH

Factorising Non-Monic Quadratic Expressions (2 of 2: Without a common numerical factor)

Eddie Woo · 10:42

<https://www.youtube.com/watch?v=NqmMefQM4yg>

WHEN TO WATCH

Before Example 1: factoring $ax^2 + bx + c$.

Note: Eddie Woo teaches in Australia: 'non-monic' means the x^2 coefficient is not 1.

BACKUP VIDEO

Factorising Non-Monic Quadratics: 4 Methods

Eddie Woo · 7:08

<https://www.youtube.com/watch?v=vt491JmFgP0>

After Example 2: four methods compared, including the ac (grouping) method and trial and error.

KEY TERMS

leading coefficient

The coefficient of the highest-degree term. In $6x^2 + 11x + 4$ it is $a = 6$.

ac method

A way to factor $ax^2 + bx + c$: find two numbers whose product is $a \cdot c$ and whose sum is b , use them to split the middle term into two terms, then factor by grouping.

splitting the middle term

Rewriting bx as a sum of two terms, such as $11x = 3x + 8x$, so the trinomial becomes four terms that can be grouped.

factoring by grouping

Factoring the first two terms and the last two terms separately so that a common binomial factor appears, then factoring that binomial out.

CORE IDEAS

When the leading coefficient is not 1, the shortcut from 9.2 is not enough, because the x -terms of the two binomials carry coefficients too. Look at a general product: $(px + q)(rx + s) = prx^2 + psx + qrx + qs = prx^2 + (ps + qr)x + qs$. So $a = pr$, $c = qs$, and the middle coefficient b is the sum of two pieces, ps and qr .

Why the ac method works. Multiply those two pieces: $(ps)(qr) = (pr)(qs) = a \cdot c$. So the middle term bx always splits into two terms whose coefficients have product ac and sum b . If we find those two numbers, we have recovered the two hidden pieces of the FOIL, and grouping undoes the multiplication. For $6x^2 + 11x + 4$: $ac = 24$, and $3 + 8 = 11$, $3 \times 8 = 24$.

The steps: (1) Factor out any GCF. (2) Find two numbers with product ac and sum b . (3) Split the middle term: $6x^2 + 3x + 8x + 4$. (4) Group and factor each pair: $3x(2x + 1) + 4(2x + 1)$. (5) Factor out the common binomial: $(2x + 1)(3x + 4)$. (6) Check: $(2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$. If the two parentheses after step 4 do not match, recheck your arithmetic; when ac and b are found correctly they always match.

Trial and error is a good alternative when the numbers are small. Write $(_x + _)(_x + _)$, choose factors of a for the first terms and factors of c for the last terms, and test whether outer plus inner gives bx . For $3x^2 + 7x + 2$: try $(3x + 1)(x + 2)$; outer $6x$ plus inner x gives $7x$. ✓ The sign rules from 9.2 still help: if $c > 0$, both last terms have the sign of b ; if $c < 0$, they have opposite signs.

Always factor out a GCF first. In $12x^2 + 2x - 4$, every term is even, so start with $2(6x^2 + x - 2)$. The numbers inside are smaller, and if you skip this step your binomials will still hide a common factor, so the answer would not be factored completely. If a is negative, factor out -1 (or a negative GCF) so the trinomial inside has a positive leading coefficient: $-2x^2 + 5x + 3 = -(2x^2 - 5x - 3)$.

If no pair of integers has product ac and sum b , the trinomial is prime. For $2x^2 + 3x + 4$, $ac = 8$, and the pairs 1, 8 and 2, 4 add to 9 and 6 (negative pairs give negative sums), never 3. So it is prime.

KEY FORMULAS

$$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$$

$$ax^2 + bx + c : \text{ find } m, n \text{ with } mn = ac, m + n = b; \text{ then } ax^2 + mx + nx + c$$

Practice habit. Factor out the GCF first, every time, before you do anything else. Then write the product ac and the sum b at the top of your work so you always know what you are looking for.

WORKED EXAMPLES

Example 1. Ruth is sewing a rectangular rug with area $2x^2 + 7x + 3$ square feet. Factor to find expressions for its length and width.

Show solution

1. There is no GCF. Here $a = 2$, $b = 7$, $c = 3$, so $ac = 6$.
2. Two numbers with product 6 and sum 7: 1 and 6.
3. Split the middle term: $2x^2 + x + 6x + 3$.
4. Group: $x(2x + 1) + 3(2x + 1)$.
5. Factor out $(2x + 1)$: $(2x + 1)(x + 3)$.
6. Check: $(2x + 1)(x + 3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$. ✓

Answer: $(2x + 1)(x + 3)$: the sides are $2x + 1$ feet and $x + 3$ feet.

Example 2. Factor completely: $12x^2 + 2x - 4$.

Show solution

1. GCF first: $12x^2 + 2x - 4 = 2(6x^2 + x - 2)$.
 2. Inside: $a = 6$, $b = 1$, $c = -2$, so $ac = -12$.
 3. Two numbers with product -12 and sum 1: 4 and -3 .
 4. Split: $6x^2 + 4x - 3x - 2$.
 5. Group: $2x(3x + 2) - 1(3x + 2)$.
 6. Factor out $(3x + 2)$: $(3x + 2)(2x - 1)$.
 7. Answer: $2(3x + 2)(2x - 1)$.
 8. Check: $(3x + 2)(2x - 1) = 6x^2 - 3x + 4x - 2 = 6x^2 + x - 2$, and $2(6x^2 + x - 2) = 12x^2 + 2x - 4$. ✓
-

Answer: $2(3x + 2)(2x - 1)$

PRACTICE

1. Easy Factor $2x^2 + 5x + 3$.

Show answer

$$(2x + 3)(x + 1)$$

2. Easy Factor $3x^2 + 7x + 2$.

Show answer

$$(3x + 1)(x + 2)$$

3. Easy Factor $5x^2 - 11x + 2$.

Show answer

$$(5x - 1)(x - 2)$$

4. Easy Factor $2x^2 - x - 6$.

Show answer

$$ac = -12, \text{ numbers } -4 \text{ and } 3: (2x + 3)(x - 2)$$

5. Medium Factor $6x^2 + 13x + 6$.

Show answer

$$ac = 36, \text{ numbers } 4 \text{ and } 9: (2x + 3)(3x + 2)$$

6. Medium Factor $4x^2 - 4x - 15$.

Show answer

$$ac = -60, \text{ numbers } -10 \text{ and } 6: (2x - 5)(2x + 3)$$

7. Medium Factor $3x^2 - 10x + 8$.

Show answer

$$ac = 24, \text{ numbers } -4 \text{ and } -6: (3x - 4)(x - 2)$$

8. Medium Factor $10x^2 + 11x - 6$.

Show answer

$$ac = -60, \text{ numbers } 15 \text{ and } -4: 5x(2x + 3) - 2(2x + 3) = (2x + 3)(5x - 2)$$

9. Medium Grace's family is making a rectangular quilt with area $3x^2 + 14x + 8$ square feet. The width of the quilt is $x + 4$ feet. Find an expression for its length.

Show answer

$$3x^2 + 14x + 8 = (3x + 2)(x + 4), \text{ so the length is } 3x + 2 \text{ feet.}$$

10. Medium Henry wrote $4x^2 + 10x + 4 = (2x + 1)(2x + 4)$. Multiply to show his product is correct, then explain why his answer is not factored completely and finish it.

Show answer

$$(2x + 1)(2x + 4) = 4x^2 + 8x + 2x + 4 = 4x^2 + 10x + 4, \text{ so the product is right.}$$

But $2x + 4$ still has a common factor of 2. He should have taken out the GCF 2 first: $4x^2 + 10x + 4 = 2(2x^2 + 5x + 2) = 2(2x + 1)(x + 2)$.

11. Hard Factor completely: $6x^2 + 3x - 30$.

Show answer

$$3(2x^2 + x - 10) = 3(2x + 5)(x - 2)$$

12. Hard Factor completely: $-2x^2 + 5x + 3$.

Show answer

$$-(2x^2 - 5x - 3) = -(2x + 1)(x - 3)$$

13. **Hard** Factor $8x^2 - 14xy + 3y^2$.

Show answer

$$ac = 24, \text{ split } -14xy = -12xy - 2xy: 4x(2x - 3y) - y(2x - 3y) = (4x - y)(2x - 3y)$$

14. **Hard** A rectangular garden bed beside the church has area $4x^2 + 16x + 15$ square feet. Factor to find expressions for its length and width.

Show answer

$$ac = 60, \text{ numbers 6 and 10: } (2x + 3)(2x + 5); \text{ length } 2x + 5 \text{ feet, width } 2x + 3 \text{ feet.}$$

15. **Challenge** Samuel tosses a stone upward at 8 feet per second from the top of a 48-foot cliff, so its height in feet after t seconds is $h = -16t^2 + 8t + 48$. Factor the right side completely.

Show answer

$$\text{Take out } -8: -8(2t^2 - t - 6). \text{ Then } ac = -12, \text{ numbers } -4 \text{ and } 3: 2t^2 - t - 6 = (2t + 3)(t - 2). \text{ So } h = -8(2t + 3)(t - 2).$$

COMMON MISTAKES

- Looking for two numbers whose product is c instead of ac . When $a \neq 1$, the product you need is $a \cdot c$.
- Skipping the GCF, which leaves a common factor hiding inside a binomial such as $(2x + 4)$. Take out the GCF first and check each binomial at the end.
- Sign errors when grouping: in $6x^2 + 4x - 3x - 2$, the second group is $-1(3x + 2)$, not $-1(3x - 2)$. Factoring out a negative flips the sign of each term inside.
- Not checking. Multiply your binomials back out and compare all three terms, especially the middle one.

Make it a game • Teach the Teddy. Teach a stuffed animal, or a sibling, how to factor $6x^2 + 7x + 2$ with the ac method. Show finding $ac = 12$, picking the pair 3 and 4, splitting the middle term, and factoring by grouping. The parent plays confused student and asks 'why does that work?' Multiply the binomials back to show the proof.

9.4 FACTORING SPECIAL PRODUCTS

OBJECTIVES

- Recognize and factor a difference of two squares, $a^2 - b^2 = (a + b)(a - b)$.
- Recognize and factor perfect-square trinomials, $a^2 \pm 2ab + b^2 = (a \pm b)^2$.
- Explain why a sum of two squares such as $x^2 + 9$ does not factor over the integers.
- Use special products to factor completely and to do mental arithmetic.

WATCH

Visual Proof for Difference of Squares $a^2 - b^2$

Eddie Woo · 5:40

<https://www.youtube.com/watch?v=wIPGEVCOzJ4>

WHEN TO WATCH

Before Example 1: why $a^2 - b^2 = (a + b)(a - b)$.

Note: Neither video factors perfect-square trinomials. Use the Lesson 8.4 primary video in reverse, with Example 2.

BACKUP VIDEO

Factorisation (3 of 4: Applying difference of squares)

Eddie Woo · 10:01

<https://www.youtube.com/watch?v=KnPcgFFeLGo>

Optional challenge after Example 2: a harder expression factored with the difference of squares.

KEY TERMS

perfect square

A number or monomial that is the square of another: $49 = 7^2$, $9x^2 = (3x)^2$, $x^4 = (x^2)^2$.

perfect-square trinomial

A trinomial that is the square of a binomial: $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$.

difference of two squares

A binomial of the form $a^2 - b^2$, one perfect square minus another, such as $x^2 - 36$.

sum of two squares

A binomial of the form $a^2 + b^2$, such as $x^2 + 9$. It does not factor over the integers.

CORE IDEAS

In 8.4 you learned three special products. Factoring simply reads them from right to left: $a^2 - b^2 = (a + b)(a - b)$, $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$. Recognizing these patterns saves the work of searching for factor pairs.

Why $a^2 - b^2 = (a + b)(a - b)$. Multiply: $(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$; the middle terms cancel. Picture it too: from a square of side a , cut away a small square of side b from one corner. The L-shaped piece left has area $a^2 - b^2$. Slice the L into two rectangles and slide one beside the other: together they form one rectangle that is $a + b$ long and $a - b$ wide. So $x^2 - 36 = (x + 6)(x - 6)$ and $9x^2 - 4 = (3x + 2)(3x - 2)$.

Recognizing a perfect-square trinomial. Ask three questions: Is the first term a perfect square, a^2 ? Is the last term a perfect square, b^2 , with a plus sign? Is the middle term exactly $2ab$ (or $-2ab$)? For $25x^2 - 40x + 16$: $25x^2 = (5x)^2$, $16 = 4^2$, and $2 \cdot 5x \cdot 4 = 40x$. Yes, so $25x^2 - 40x + 16 = (5x - 4)^2$, using the sign of the middle term. If the middle term is wrong, as in $x^2 + 12x + 16$ (we would need $8x$), it is not a perfect square, and you factor it another way or call it prime.

Why a sum of two squares does not factor. Suppose $x^2 + 9 = (x + p)(x + q)$ for integers p and q . Multiplying out gives $x^2 + (p + q)x + pq$. There is no x -term in $x^2 + 9$, so $p + q = 0$, meaning $q = -p$. Then $pq = -p^2$, which is zero or negative, so it can never equal $+9$. Therefore $x^2 + 9$ is prime. The same argument works for any $x^2 + b^2$ with $b \neq 0$. Notice the difference: $x^2 - 9$ factors, $x^2 + 9$ does not.

Special products often hide after a GCF or inside another factor, so keep going until nothing more factors. For example, $3x^2 - 75 = 3(x^2 - 25) = 3(x + 5)(x - 5)$, and $x^4 - 16 = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2)$; the factor $x^2 + 4$ is a sum of squares and stops there.

Difference of squares is also a mental-math tool: $53^2 - 47^2 = (53 + 47)(53 - 47) = 100 \times 6 = 600$, far faster than squaring each number.

KEY FORMULAS

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

$x^2 + b^2$ ($b \neq 0$) is prime over the integers

Euclid connection. Euclid's Elements II.4 shows with a square diagram that the square on a whole line equals the squares on its two parts plus twice the rectangle on the parts, which is our $(a + b)^2 = a^2 + 2ab + b^2$. Reading that picture backward is exactly what factoring a perfect-square trinomial does.

Practice habit. Recite the perfect squares from 1^2 to 20^2 and the three special products until they are automatic. You cannot spot $81x^2 - 49$ as a difference of squares unless 81 and 49 jump out at you as squares.

WORKED EXAMPLES

Example 1. Factor $49x^2 - 25$.

Show solution

1. Two terms with a minus sign between them: check for a difference of squares.
2. $49x^2 = (7x)^2$ and $25 = 5^2$, so $a = 7x$ and $b = 5$.
3. Use $a^2 - b^2 = (a + b)(a - b)$: $(7x + 5)(7x - 5)$.
4. Check: $(7x + 5)(7x - 5) = 49x^2 - 35x + 35x - 25 = 49x^2 - 25$. ✓

Answer: $(7x + 5)(7x - 5)$

Example 2. The Miller family is laying a square stone patio with area $4x^2 - 28x + 49$ square feet. Find an expression for the length of one side.

Show solution

1. First term: $4x^2 = (2x)^2$, so $a = 2x$.
2. Last term: $49 = 7^2$ with a plus sign, so $b = 7$.
3. Middle term test: $2ab = 2 \cdot 2x \cdot 7 = 28x$, and the middle term is $-28x$. ✓
4. So it is a perfect-square trinomial: $4x^2 - 28x + 49 = (2x - 7)^2$.
5. Check: $(2x - 7)^2 = 4x^2 - 14x - 14x + 49 = 4x^2 - 28x + 49$. ✓
6. A square with area $(2x - 7)^2$ has side $2x - 7$.

Answer: Each side is $2x - 7$ feet, since $4x^2 - 28x + 49 = (2x - 7)^2$.

PRACTICE

1. Easy Factor $x^2 - 36$.

Show answer

$$(x + 6)(x - 6)$$

2. Easy Factor $x^2 + 10x + 25$.

Show answer

$$(x + 5)^2$$

3. Easy Factor $y^2 - 14y + 49$.

Show answer

$$(y - 7)^2$$

4. Easy Factor $9x^2 - 4$.

Show answer

$$(3x + 2)(3x - 2)$$

5. Medium Factor $16a^2 - 81b^2$.

Show answer

$$(4a + 9b)(4a - 9b)$$

6. Medium Factor $4x^2 + 12x + 9$.

Show answer

$$(2x)^2 + 2(2x)(3) + 3^2 = (2x + 3)^2$$

7. Medium Factor $x^2 + 16$, or write "prime" if it does not factor over the integers.

Show answer

Prime (a sum of two squares).

8. Medium Factor $25x^2 - 40x + 16$.

Show answer

$$2 \cdot 5x \cdot 4 = 40x, \text{ so } (5x - 4)^2$$

9. Medium Henry needs $53^2 - 47^2$ for a woodworking calculation and has no calculator. Use the difference of squares to find it mentally.

Show answer

$$(53 + 47)(53 - 47) = 100 \times 6 = 600$$

10. Medium Explain why $x^2 - 9$ can be factored but $x^2 + 9$ cannot be factored over the integers.

Show answer

$x^2 - 9 = (x + 3)(x - 3)$ because the middle terms $3x$ and $-3x$ cancel. For $x^2 + 9 = (x + p)(x + q)$ we would need $p + q = 0$ (no x -term), so $q = -p$ and $pq = -p^2$, which is never positive. It cannot equal $+9$, so $x^2 + 9$ is prime.

11. Hard Factor completely: $3x^2 - 75$.

Show answer

$$3(x^2 - 25) = 3(x + 5)(x - 5)$$

12. Hard Factor completely: $x^4 - 16$.

Show answer

$$(x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2)$$

13. **Hard** A square courtyard is x feet on each side. In it sits a square fountain 4 feet on each side, and the rest of the courtyard is paved. The paved area is $x^2 - 16$ square feet. Factor this expression.

Show answer

$$(x + 4)(x - 4) \text{ square feet}$$

14. **Hard** A square vegetable garden has area $36x^2 + 60x + 25$ square feet. Find an expression for the length of one side.

Show answer

$$36x^2 + 60x + 25 = (6x + 5)^2 \text{ since } 2 \cdot 6x \cdot 5 = 60x; \text{ each side is } 6x + 5 \text{ feet.}$$

15. **Challenge** A square picture frame measures $x + 3$ inches on each outside edge, and the square opening for the picture is 5 inches on each side. The area of the wood is $(x + 3)^2 - 25$ square inches. Factor this expression completely.

Show answer

$$\text{Difference of squares with } a = x + 3, b = 5: (x + 3 + 5)(x + 3 - 5) = (x + 8)(x - 2). \text{ Check: } (x + 3)^2 - 25 = x^2 + 6x - 16 = (x + 8)(x - 2).$$

COMMON MISTAKES

- Factoring a sum of squares as if it were a difference: $x^2 + 25 \neq (x + 5)(x - 5)$ and $\neq (x + 5)^2$. A sum of two squares is prime.
- Calling a trinomial a perfect square because the first and last terms are squares, without checking the middle term. $x^2 + 12x + 16$ is not $(x + 4)^2$, because $(x + 4)^2 = x^2 + 8x + 16$.
- Writing $(a - b)^2$ as $a^2 - b^2$. Squaring a binomial always makes a middle term $\pm 2ab$; use the square picture from 8.4 to remember.
- Stopping too early: $x^4 - 16 = (x^2 + 4)(x^2 - 4)$ is not finished, because $x^2 - 4$ is another difference of squares.

Make it a game • Special or Ordinary. The parent shows ten binomials and trinomials on cards, such as $x^2 - 49$, $4x^2 + 12x + 9$ and $x^2 + 9$. Shout 'difference of squares', 'perfect-square trinomial' or 'neither', then factor. Beware $x^2 + 9$, because a sum of two squares will not factor over the integers. A wrong shout costs a point, so check before speaking.

9.5 FACTORING BY GROUPING AND FACTORING COMPLETELY

OBJECTIVES

- Factor four-term polynomials by grouping, rearranging terms when needed.
- Apply a step-by-step factoring strategy: GCF first, count the terms, then check each factor.
- Factor polynomials completely, combining the GCF, trinomial, special-product and grouping methods.
- Explain why a factorization is or is not complete.

WATCH

Factorising Algebraic Expressions (1 of 3: Grouping)

Eddie Woo · 8:15

<https://www.youtube.com/watch?v=TArekc3kS64>

WHEN TO WATCH

Before Example 1: factoring four terms by grouping.

Note: Eddie Woo teaches in Australia: he says 'factorise'. The factoring-completely checklist is taught from the Core ideas.

BACKUP VIDEO

Factorising Further Algebraic Binomials

Eddie Woo · 6:26

<https://www.youtube.com/watch?v=90n1Crk8xXw>

Before the practice set: more expressions that need several factoring steps.

KEY TERMS

factoring by grouping

Splitting a polynomial (usually four terms) into two groups, factoring the GCF out of each group, and then factoring out the binomial the groups share.

factored completely

Written as a product in which every factor except a single number is prime: none of them can be factored further over the integers.

factoring strategy

An ordered checklist for deciding which factoring method to use, starting with the GCF and then looking at the number of terms.

CORE IDEAS

Factoring by grouping uses the common-binomial idea from 9.1. Take

$x^3 - 5x^2 + 4x - 20$. Group the first two terms and the last two:

$(x^3 - 5x^2) + (4x - 20)$. Factor each group: $x^2(x - 5) + 4(x - 5)$. Now both pieces share the block $(x - 5)$, so by the distributive property $x^2B + 4B = B(x^2 + 4)$ we get $(x - 5)(x^2 + 4)$. Grouping works **because** it is the distributive property used twice, once inside each group and once for the shared binomial.

If the two binomials do not match, try two fixes. First, when the third term is negative, factor a negative out of the second group so the signs line up: $x^3 + 2x^2 - 9x - 18 = x^2(x + 2) - 9(x + 2)$. Second, if that still fails, rearrange the terms and group again: $x^3 - 6 + 3x - 2x^2$ becomes $x^3 - 2x^2 + 3x - 6 = x^2(x - 2) + 3(x - 2) = (x - 2)(x^2 + 3)$. Grouping is also exactly the last stage of the ac method in 9.3.

Here is a **factoring strategy** that handles every polynomial in this course. (1) Factor out the GCF (use a negative GCF if the leading coefficient is negative). (2) Count the terms. Two terms: is it a difference of squares? (A sum of squares such as $x^2 + 9$, with no GCF, is prime.) Three terms: is it a perfect-square trinomial? If not, use the $x^2 + bx + c$ method or the ac method. Four terms: try grouping. (3) Check every factor you wrote to see whether it can be factored again. (4) Check the whole answer by multiplying.

Step 3 is what makes a factorization **complete**. In $x^3 + 2x^2 - 9x - 18 = (x + 2)(x^2 - 9)$, the factor $x^2 - 9$ is a difference of squares, so the complete answer is $(x + 2)(x + 3)(x - 3)$. Likewise $3x^3 - 12x^2 - 36x = 3x(x^2 - 4x - 12) = 3x(x - 6)(x + 2)$. A factorization can multiply back correctly and still be incomplete, so a correct product is not enough: every factor must be prime.

Why take the GCF first? If you skip it, the common factor does not disappear; it hides inside the binomials. For example, $(2x + 6)(2x - 6)$ multiplies to $4x^2 - 36$, but both binomials still contain a 2. Taking out the GCF first gives $4(x^2 - 9) = 4(x + 3)(x - 3)$ cleanly, with smaller numbers to work with.

KEY FORMULAS

$$ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$$

Practice habit. Work the strategy in the same order every time and say it to yourself: "GCF first. Count the terms. Check each factor. Multiply to check." When you finish, circle each factor and ask, "Can this one factor again?"

WORKED EXAMPLES

Example 1. Factor $x^3 - 5x^2 + 4x - 20$.

Show solution

1. No GCF for all four terms. Four terms, so try grouping.
2. Group: $(x^3 - 5x^2) + (4x - 20)$.
3. Factor each group: $x^2(x - 5) + 4(x - 5)$.
4. Factor out the common binomial: $(x - 5)(x^2 + 4)$.
5. Check each factor: $x - 5$ is prime; $x^2 + 4$ is a sum of squares, prime.
6. Check by multiplying: $(x - 5)(x^2 + 4) = x^3 + 4x - 5x^2 - 20 = x^3 - 5x^2 + 4x - 20$. ✓

Answer: $(x - 5)(x^2 + 4)$

Example 2. A carpenter builds a wooden storage box whose volume is $x^3 + 2x^2 - 9x - 18$ cubic inches. Factor completely to find expressions for the three dimensions of the box.

Show solution

1. No GCF. Four terms, so try grouping.
2. Group: $(x^3 + 2x^2) + (-9x - 18)$.
3. Factor each group, taking -9 from the second: $x^2(x + 2) - 9(x + 2)$.
4. Factor out $(x + 2)$: $(x + 2)(x^2 - 9)$.
5. Check each factor: $x^2 - 9$ is a difference of squares, $(x + 3)(x - 3)$.
6. Complete answer: $(x + 2)(x + 3)(x - 3)$.
7. Check: $(x + 3)(x - 3) = x^2 - 9$, and $(x + 2)(x^2 - 9) = x^3 - 9x + 2x^2 - 18 = x^3 + 2x^2 - 9x - 18$. ✓

Answer: $(x + 2)(x + 3)(x - 3)$: the dimensions are $x + 2$, $x + 3$ and $x - 3$ inches.

PRACTICE

1. Easy Factor by grouping: $x^3 + 2x^2 + 3x + 6$.

Show answer

$$x^2(x + 2) + 3(x + 2) = (x + 2)(x^2 + 3)$$

2. Easy Factor by grouping: $ab + 3a + 2b + 6$.

Show answer

$$a(b + 3) + 2(b + 3) = (b + 3)(a + 2)$$

3. Easy Factor completely: $5x^2 - 20$.

Show answer

$$5(x^2 - 4) = 5(x + 2)(x - 2)$$

4. Easy Factor completely: $2x^2 + 12x + 18$.

Show answer

$$2(x^2 + 6x + 9) = 2(x + 3)^2$$

5. Medium Factor by grouping: $x^3 - 4x^2 + 3x - 12$.

Show answer

$$x^2(x - 4) + 3(x - 4) = (x - 4)(x^2 + 3)$$

6. Medium Factor completely: $x^3 - 9x$.

Show answer

$$x(x^2 - 9) = x(x + 3)(x - 3)$$

7. Medium The area of a rectangular pasture on the Miller farm is $3x^2 + 15x + 18$ square yards. Factor this expression completely.

Show answer

$$3(x^2 + 5x + 6) = 3(x + 2)(x + 3)$$

8. Medium Factor by grouping: $6x^3 + 9x^2 - 4x - 6$.

Show answer

$$3x^2(2x + 3) - 2(2x + 3) = (2x + 3)(3x^2 - 2)$$

9. Medium Factor completely: $3x^3 - 12x^2 - 36x$.

Show answer

$$3x(x^2 - 4x - 12) = 3x(x - 6)(x + 2)$$

10. Medium Ruth factored $4x^2 - 36$ as $(2x + 6)(2x - 6)$. Her product is correct. Explain why her answer is not factored completely, and give the complete factorization.

Show answer

Both $2x + 6$ and $2x - 6$ still contain a common factor of 2, so they can be factored again. Taking out the GCF 4 first gives $4(x^2 - 9) = 4(x + 3)(x - 3)$, which is complete.

11. Hard Factor completely: $x^3 + 5x^2 - 4x - 20$.

Show answer

$$x^2(x + 5) - 4(x + 5) = (x + 5)(x^2 - 4) = (x + 5)(x + 2)(x - 2)$$

12. Hard Factor completely: $12x^2 - 22x - 20$.

Show answer

$$2(6x^2 - 11x - 10); ac = -60, \text{ numbers } -15 \text{ and } 4: 2(3x + 2)(2x - 5)$$

13. **Hard** A rectangular water tank for livestock has volume $2x^3 + 10x^2 + 12x$ cubic feet, and its height is $2x$ feet. Factor completely and find expressions for the length and width of the tank.

Show answer

$2x(x^2 + 5x + 6) = 2x(x + 2)(x + 3)$. The height is $2x$ feet, so the length and width are $x + 3$ feet and $x + 2$ feet.

14. **Hard** A shipping box for a church's food drive has volume $x^3 + 3x^2 - x - 3$ cubic feet. Factor completely to find expressions for its three dimensions.

Show answer

$x^2(x + 3) - 1(x + 3) = (x + 3)(x^2 - 1) = (x + 3)(x + 1)(x - 1)$: dimensions $x + 3$, $x + 1$ and $x - 1$ feet.

15. **Challenge** Factor $x^3 - 6 + 3x - 2x^2$.

Show answer

Rearrange in standard form: $x^3 - 2x^2 + 3x - 6 = x^2(x - 2) + 3(x - 2) = (x - 2)(x^2 + 3)$

COMMON MISTAKES

- Stopping after the first step: $(x + 2)(x^2 - 9)$ is not complete. Check every factor for a difference of squares or a factorable trinomial.
- Sign errors in the second group: $-9x - 18 = -9(x + 2)$, not $-9(x - 2)$. When you factor out a negative, every sign inside flips.
- Writing the answer as a sum, such as $x^2(x - 5) + 4(x - 5)$. That is not factored yet; you must factor out the common binomial to get a product.
- Skipping the GCF, which leaves common factors hidden inside the binomials. Always start with the GCF.

Make it a game • Strategy Stepping Stones. Place four sheets in a line on the floor: GCF First, Count the Terms, Special Patterns, Check. Hand the parent a polynomial like $2x^3 + 6x^2 + 8x + 24$. Walk it through every stone, saying what you do, until it is factored completely. Do four polynomials. The parent may stop you if a stone is skipped.

9.6 SOLVING EQUATIONS BY FACTORING

OBJECTIVES

- State the zero-product property and explain why it is true.
- Solve quadratic and other polynomial equations by getting 0 on one side, factoring, and setting each factor equal to 0.
- Check solutions by substitution and reject answers that make no sense in a word problem.
- Solve area, number and falling-object problems using $h = -16t^2 + vt + s$ in feet.

WATCH

Solving Quadratic Equations (2 of 2: Using factorisation)

Eddie Woo · 8:26

https://www.youtube.com/watch?v=_B77Mwtldpl

WHEN TO WATCH

Before Example 1: the zero-product property.

Note: Eddie Woo teaches in Australia: he says 'null factor law' for the zero-product property. Falling-object problems in feet are taught from the Core ideas.

BACKUP VIDEO

Problem Solving with Quadratic Equations (1 of 2: Geometry example)

Eddie Woo · 9:12

https://www.youtube.com/watch?v=mjFIVwGGU_A

Before Example 2: writing and solving a quadratic equation from an area problem.

KEY TERMS

quadratic equation

An equation that can be written as $ax^2 + bx + c = 0$ with $a \neq 0$, such as $x^2 - 5x - 14 = 0$.

standard form of a quadratic equation

The form $ax^2 + bx + c = 0$, with every term on one side and 0 on the other.

zero-product property

If the product of two numbers is 0,
then at least one of the numbers is 0: if
 $ab = 0$, then $a = 0$ or $b = 0$.

root (solution)

A value of the variable that makes the
equation true. A quadratic equation
can have two, one or no real roots.

CORE IDEAS

Up to now you have solved equations by undoing operations. That does not work on $x^2 - 5x - 14 = 0$, because x appears both squared and to the first power. Factoring gives a new way in: $x^2 - 5x - 14 = (x - 7)(x + 2)$, so the equation says $(x - 7)(x + 2) = 0$, a product that equals zero.

The zero-product property: if $ab = 0$, then $a = 0$ or $b = 0$ (or both). **Why it is true:** suppose $ab = 0$ and a is not 0. Then we may divide both sides by a : $\frac{ab}{a} = \frac{0}{a}$, so $b = 0$. So either a is 0, or else b must be 0. There is no way for two nonzero numbers to multiply to zero. So from $(x - 7)(x + 2) = 0$ we know $x - 7 = 0$ or $x + 2 = 0$, giving $x = 7$ or $x = -2$. Both check: $49 - 35 - 14 = 0$ and $4 + 10 - 14 = 0$.

Zero is special here. If $(x - 2)(x + 3) = 6$, you cannot say $x - 2 = 6$ or $x + 3 = 6$, because many pairs of numbers multiply to 6 (1×6 , 2×3 , $\frac{1}{2} \times 12$, ...). Only a product of **zero** forces one factor to be zero. So the first step is always to move every term to one side and get **0 on the other side**.

The method: (1) Write the equation in standard form, $ax^2 + bx + c = 0$. (2) Factor completely. (3) Set each factor equal to 0 and solve. (4) Check each answer in the original equation. Never divide both sides by a variable: dividing $x^2 = 5x$ by x gives only $x = 5$ and loses the solution $x = 0$. Instead write $x^2 - 5x = 0$, so $x(x - 5) = 0$ and $x = 0$ or $x = 5$. If both factors are the same, as in $(x - 5)^2 = 0$, there is just one solution, $x = 5$.

Falling and thrown objects. An object launched straight up at v feet per second from a height of s feet has height $h = -16t^2 + vt + s$ feet after t seconds. It hits the ground when $h = 0$. Solving gives times; a negative time does not fit the situation (it is before the throw), so we reject it. Always ask whether each solution makes sense: a length or a time cannot be negative.

KEY FORMULAS

$$ab = 0 \implies a = 0 \text{ or } b = 0$$

$$h = -16t^2 + vt + s \quad (\text{feet, seconds})$$

Practice habit. Check by substituting. Put each solution back into the original equation, not the factored one, so you also catch a factoring mistake.

WORKED EXAMPLES

Example 1. Solve $x^2 - 5x - 14 = 0$.

Show solution

1. The equation already has 0 on one side.
2. Factor: two numbers with product -14 and sum -5 are -7 and 2 , so $(x - 7)(x + 2) = 0$.
3. Zero-product property: $x - 7 = 0$ or $x + 2 = 0$.
4. So $x = 7$ or $x = -2$.
5. Check $x = 7$: $49 - 35 - 14 = 0$. ✓
6. Check $x = -2$: $4 + 10 - 14 = 0$. ✓

Answer: $x = -2$ or $x = 7$

Example 2. From a rooftop 32 feet high, Henry tosses a ball straight up at 16 feet per second. Its height after t seconds is $h = -16t^2 + 16t + 32$ feet. How many seconds until the ball hits the ground?

Show solution

1. The ball hits the ground when $h = 0$: $-16t^2 + 16t + 32 = 0$.
2. Factor out the GCF -16 : $-16(t^2 - t - 2) = 0$.
3. Factor the trinomial: $-16(t - 2)(t + 1) = 0$.
4. -16 is never 0, so $t - 2 = 0$ or $t + 1 = 0$, giving $t = 2$ or $t = -1$.
5. A time before the throw makes no sense, so reject $t = -1$.
6. Check $t = 2$: $-16(4) + 16(2) + 32 = -64 + 32 + 32 = 0$. ✓

Answer: The ball hits the ground after 2 seconds.

PRACTICE

1. Easy Solve $(x - 3)(x + 8) = 0$.

Show answer

$$x = -8 \text{ or } x = 3$$

2. Easy Solve $x(2x - 5) = 0$.

Show answer

$$x = 0 \text{ or } 2x = 5, \text{ so } x = 0 \text{ or } x = \frac{5}{2}$$

3. Easy Solve $x^2 + 7x + 10 = 0$.

Show answer

$$(x + 2)(x + 5) = 0, \text{ so } x = -5 \text{ or } x = -2$$

4. Easy Solve $x^2 - 49 = 0$.

Show answer

$$(x + 7)(x - 7) = 0, \text{ so } x = -7 \text{ or } x = 7$$

5. Medium Solve $x^2 - 3x = 18$.

Show answer

$$x^2 - 3x - 18 = 0, (x - 6)(x + 3) = 0, \text{ so } x = -3 \text{ or } x = 6$$

6. Medium Solve $2x^2 + 5x - 3 = 0$.

Show answer

$$(2x - 1)(x + 3) = 0, \text{ so } x = -3 \text{ or } x = \frac{1}{2}$$

7. Medium Solve $x^2 = 10x - 25$.

Show answer

$$x^2 - 10x + 25 = 0, (x - 5)^2 = 0, \text{ so } x = 5 \text{ (one solution)}$$

8. Medium The square of a number is 12 more than the number. Find all such numbers.

Show answer

$$n^2 = n + 12, \text{ so } n^2 - n - 12 = 0, (n - 4)(n + 3) = 0: \text{ the number is } -3 \text{ or } 4.$$

9. **Medium** Grace solved $(x - 2)(x + 3) = 6$ by writing $x - 2 = 6$ or $x + 3 = 6$. Explain her mistake, then solve the equation correctly.

Show answer

The zero-product property works only when the product is 0; many pairs of numbers multiply to 6. Multiply out and get 0 on one side: $x^2 + x - 6 = 6$, so $x^2 + x - 12 = 0$, $(x + 4)(x - 3) = 0$, and $x = -4$ or $x = 3$.

10. **Hard** A rectangular garden is 3 feet longer than it is wide, and its area is 40 square feet. Find the width and the length.

Show answer

$w(w + 3) = 40$, $w^2 + 3w - 40 = 0$, $(w + 8)(w - 5) = 0$; reject $w = -8$. Width 5 feet, length 8 feet.

11. **Hard** Solve $6x^2 + 7x = 3$.

Show answer

$6x^2 + 7x - 3 = 0$, $(2x + 3)(3x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = \frac{1}{3}$

12. **Hard** A bale of hay is dropped from a loft door 64 feet above the ground, so its height after t seconds is $h = -16t^2 + 64$ feet. How long does it take to reach the ground?

Show answer

$-16t^2 + 64 = 0$, $-16(t^2 - 4) = 0$, $-16(t + 2)(t - 2) = 0$; reject $t = -2$. It lands after 2 seconds.

13. **Hard** Ruth kicks a ball straight up from the ground at 48 feet per second, so its height after t seconds is $h = -16t^2 + 48t$ feet. At what times is the ball 32 feet high?

Show answer

$-16t^2 + 48t = 32$, so $-16t^2 + 48t - 32 = 0$, $-16(t - 1)(t - 2) = 0$: at 1 second (going up) and at 2 seconds (coming down).

14. **Challenge** Solve $x^3 - 4x^2 - 21x = 0$.

Show answer

$x(x^2 - 4x - 21) = 0$, $x(x - 7)(x + 3) = 0$, so $x = -3$, 0 or 7

15. **Challenge** The ages of two Miller children, in years, are consecutive whole numbers, and the product of their ages is 56. How old are the two children?

Show answer

$n(n + 1) = 56$, $n^2 + n - 56 = 0$, $(n + 8)(n - 7) = 0$; reject $n = -8$. The children are 7 and 8 years old.

COMMON MISTAKES

- Using the zero-product property when the product is not zero, as in $(x - 2)(x + 3) = 6$. Get 0 on one side first, then factor.
- Dividing both sides by x , which throws away the solution $x = 0$. Factor out x instead.
- Stopping at the factored form, or giving the numbers in the factors instead of the solutions: $(x + 2) = 0$ means $x = -2$, not $x = 2$.
- Keeping answers that make no sense in a word problem, such as a negative width or a negative time. Solve, then check each answer against the situation.

Make it a game • Crack the Code. Each equation is a lock whose code is its two solutions. The parent writes four on cards, such as $x^2 - 5x - 14 = 0$ and $x^2 = 9x$. Get zero on one side, factor, and set each factor equal to zero. Say the zero-product property aloud, then substitute each solution to test the lock. Make up two locks for the parent.

FACTORING

In Unit 8 you multiplied polynomials. In this unit you learn to run that process backward: to **factor** a polynomial is to write it as a product. You will take out the greatest common factor, factor trinomials such as $x^2 + bx + c$ and $ax^2 + bx + c$, recognize the special products (difference of squares and perfect-square trinomials), and factor four-term polynomials by grouping. Along the way you will learn a dependable strategy for factoring completely: GCF first, count the terms, check each factor, and multiply to check.

Why does factoring matter? Because a product is much easier to reason about than a sum. The zero-product property, if $ab = 0$ then $a = 0$ or $b = 0$, lets you solve quadratic equations by factoring, and that answers real questions: the size of a garden with a given area, or when a thrown ball comes back to the ground.

Factoring is used constantly in the rest of the course. In Unit 10 you will factor out perfect squares to simplify radicals; in Unit 11 you will solve quadratic equations, find the zeros of parabolas and complete the square; and in Unit 12 you will simplify rational expressions by factoring numerators and denominators. Time spent making factoring quick and accurate now pays off all year and in Algebra II.

PREREQUISITES

- Find the GCF of whole numbers using prime factorization (Pre-Algebra).
- Use the exponent rules for products and quotients of powers (Lesson 7.1).
- Multiply a polynomial by a monomial (Lesson 8.2).
- Multiply two binomials with the box model or FOIL (Lesson 8.3).
- Know the special products $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$ (Lesson 8.4).
- Solve one-step and two-step linear equations and check by substitution (Unit 2).

MASTERY CHECKLIST

- ☐ I can find the GCF of monomials and factor it out of a polynomial, including a negative GCF.
- ☐ I can factor out a common binomial factor such as $(x + 5)$.
- ☐ I can factor $x^2 + bx + c$ by finding two numbers whose product is c and whose sum is b , and explain why this works.
- ☐ I can recognize a prime trinomial.
- ☐ I can factor $ax^2 + bx + c$ using the ac (grouping) method or trial and error.
- ☐ I can factor a difference of two squares and a perfect-square trinomial, and explain why $x^2 - 9$ does not factor.
- ☐ I can factor a four-term polynomial by grouping.
- ☐ I can follow a factoring strategy to factor a polynomial completely and check my answer by multiplying.
- ☐ I can explain why the zero-product property is true and use it to solve equations by factoring.
- ☐ I can solve area, number and falling-object problems by

factoring and reject answers that make no sense.

LESSONS IN THIS UNIT

9.1	Factoring Out the Greatest Common Factor	2 sessions of 45 minutes
9.2	Factoring $x^2 + bx + c$	2 sessions of 45 minutes
9.3	Factoring $ax^2 + bx + c$	2 sessions of 45 minutes
9.4	Factoring Special Products	2 sessions of 45 minutes
9.5	Factoring by Grouping and Factoring Completely	2 sessions of 45 minutes
9.6	Solving Equations by Factoring	2 sessions of 45 minutes

UNIT 9 TEST

Time allowed: 60 minutes. No calculator. Show your work for every problem. Factor completely unless told otherwise, and write "prime" for a polynomial that does not factor over the integers. Give exact answers (fractions, not decimals) and include units in word problems. Check your factoring by multiplying and your solutions by substituting.

1. 5 pts

Factor $15x^4 - 10x^3 + 25x^2$.

2. 5 pts

Factor out a negative GCF: $-6y^2 + 21y$.

3. 6 pts

Factor $2x(x + 9) - 7(x + 9)$.

4. 6 pts

Factor $x^2 + 9x + 14$.

5. (6 pts)

Factor $x^2 - 2x - 35$.

6. (7 pts)

Factor $3x^2 + 11x + 10$.

7. (7 pts)

Factor $8x^2 - 10x - 3$.

8. (6 pts)

Factor $64x^2 - 9$.

9. (6 pts)

Factor $9x^2 - 42x + 49$.

10. (5 pts)

A square flower bed in front of the church has area $4x^2 + 20x + 25$ square feet.
Find an expression for the length of one side.

11. (7 pts)

Factor completely: $x^3 + 4x^2 - 9x - 36$.

12. (7 pts)

Factor completely: $5x^3 - 45x$.

13. (7 pts)

Solve $x^2 + 4x = 21$.

14. (8 pts)

A rectangular rug is 4 feet longer than it is wide, and its area is 45 square feet.
Find its width and length.

15. (8 pts)

From a platform 48 feet high, Samuel throws a ball straight up at 32 feet per second. Its height after t seconds is $h = -16t^2 + 32t + 48$ feet. After how many seconds does the ball hit the ground?

16. **4 pts**

To solve $x^2 = 5x$, Henry divided both sides by x and got $x = 5$. Explain what went wrong and give the complete solution.

ANSWER KEY

1. $5x^2(3x^2 - 2x + 5)$
2. $-3y(2y - 7)$
3. $(x + 9)(2x - 7)$
4. $(x + 2)(x + 7)$
5. $(x - 7)(x + 5)$
6. $ac = 30$, numbers 5 and 6: $x(3x + 5) + 2(3x + 5) = (3x + 5)(x + 2)$
7. $ac = -24$, numbers -12 and 2 : $4x(2x - 3) + 1(2x - 3) = (2x - 3)(4x + 1)$
8. $(8x + 3)(8x - 3)$
9. $2 \cdot 3x \cdot 7 = 42x$, so $(3x - 7)^2$
10. $4x^2 + 20x + 25 = (2x + 5)^2$, so each side is $2x + 5$ feet.
11. $x^2(x + 4) - 9(x + 4) = (x + 4)(x^2 - 9) = (x + 4)(x + 3)(x - 3)$
12. $5x(x^2 - 9) = 5x(x + 3)(x - 3)$
13. $x^2 + 4x - 21 = 0$, $(x + 7)(x - 3) = 0$, so $x = -7$ or $x = 3$
14. $w(w + 4) = 45$, $w^2 + 4w - 45 = 0$, $(w + 9)(w - 5) = 0$; reject $w = -9$. Width 5 feet, length 9 feet.
15. $-16t^2 + 32t + 48 = 0$, $-16(t^2 - 2t - 3) = 0$, $-16(t - 3)(t + 1) = 0$; reject $t = -1$. It hits the ground after 3 seconds.
16. Dividing by x assumes $x \neq 0$, so it throws away the solution $x = 0$. Instead write $x^2 - 5x = 0$, factor $x(x - 5) = 0$, and use the zero-product property: $x = 0$ or $x = 5$.

RADICALS

In this unit you learn to work with square roots exactly instead of only as calculator decimals. You will write a root such as $\sqrt{72}$ in simplest radical form ($6\sqrt{2}$), handle roots of fractions and rationalize denominators, and add, subtract and multiply radical expressions using the same like-terms and distributive ideas you used for polynomials. Along the way you will see why each rule is true by going back to the meaning of a square root.

Next you will solve equations with the variable under a square root, learning why squaring both sides can produce false (extraneous) answers and why every answer must be checked. The unit ends with the Pythagorean theorem and its converse from Euclid's *Elements*, used to derive the distance and midpoint formulas on the coordinate plane, with problems from carpentry, ladders and maps.

These skills are needed right away in Unit 11, where quadratic equations often have answers such as $-2 \pm \sqrt{3}$, and again in Geometry, where the Pythagorean theorem, distance and midpoint appear constantly. Exact radical answers are part of the common language of all later mathematics.

PREREQUISITES

- Perfect squares to 20^2 and perfect cubes to 10^3 , and estimating square roots (Pre-Algebra)
- The exponent rules, especially the power of a product and power of a power (Lesson 7.1)
- Combining like terms and multiplying binomials, including the special products $(a + b)^2$ and $(a + b)(a - b)$ (Lessons 1.4, 8.3 and 8.4)
- Solving quadratic equations by factoring and the zero-product property (Lesson 9.6)
- Plotting points on the coordinate plane (Lesson 4.1) and the Pythagorean theorem (Pre-Algebra)

MASTERY CHECKLIST

- ☐ I can explain why $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and use it to write square roots in simplest radical form.
- ☐ I can simplify square roots of variable expressions with even exponents and find cube roots of perfect cubes.
- ☐ I can simplify the square root of a fraction and rationalize a one-term denominator, and I can explain why we do it.
- ☐ I can add and subtract radicals by simplifying first and combining like radicals.
- ☐ I can multiply radical expressions, including binomials and conjugates such as

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

- ☐ I can solve a radical equation by isolating the radical, squaring both sides and checking every answer.
- ☐ I can explain why squaring both sides can create extraneous solutions and reject them.
- ☐ I can use the Pythagorean theorem to find a missing side and its converse to test for a right angle, as in squaring a corner with 3-4-5.

- ☐ I can derive the distance formula from the Pythagorean theorem and use it.
- ☐ I can find the midpoint of a segment by averaging the coordinates.

LESSONS IN THIS UNIT

10.1 Simplifying Square Roots	2 sessions of 45 minutes
10.2 Quotients and Rationalizing Denominators	2 sessions of 45 minutes
10.3 Operations with Radical Expressions	2 sessions of 45 minutes
10.4 Solving Radical Equations	2 sessions of 45 minutes
10.5 The Pythagorean Theorem, Distance and Midpoint	2 sessions of 45 minutes

UNIT 10 TEST

You have 60 minutes. Show your work on every problem. Give exact answers in simplest radical form unless a problem asks you to round. A scientific calculator may be used only for decimal approximations when a problem asks for them. Check every solution of a radical equation in the original equation and state any extraneous solutions you reject. Assume all variables under radicals are nonnegative.

1. **5 pts**
Simplify $\sqrt{98}$.
2. **4 pts**
Find $\sqrt[3]{-125}$.
3. **6 pts**
Simplify $\sqrt{63x^6}$.

4. (6 pts)

Simplify $\frac{14}{\sqrt{7}}$.

5. (7 pts)

Simplify $\sqrt{\frac{11}{18}}$.

6. (6 pts)

Simplify $\sqrt{45} - \sqrt{20} + \sqrt{5}$.

7. (7 pts)

Multiply and simplify $(3 - \sqrt{2})(5 + 2\sqrt{2})$.

8. (5 pts)

Multiply $(\sqrt{10} + 3)(\sqrt{10} - 3)$.

9. (6 pts)

Solve $\sqrt{4x - 3} = 7$.

10. (8 pts)

Solve $x = \sqrt{5x + 6}$.

11. (7 pts)

Grace drops a pebble into an old stone well, and it takes 2.5 seconds to reach the water. Use $t = \frac{\sqrt{h}}{4}$, where h is the drop in feet, to find how far below the top of the well the water is.

12. (6 pts)

Find the exact distance between $(-1, -2)$ and $(5, 1)$ in simplest radical form.

13. (5 pts)

Find the midpoint of the segment joining $(-7, 2)$ and $(3, 10)$.

14. (8 pts)

A 12-foot ladder leans against the side of a house, with its foot 4 feet from the wall on level ground. How high up the wall does it reach? Give the exact answer in simplest radical form and an approximation to the nearest tenth of a foot.

15. 7 pts

Samuel is building a deck. He marks 9 feet along the house wall and 12 feet along the edge of the deck frame, and the diagonal between the marks measures 15.5 feet. (a) Is the corner a right angle? (b) What should the diagonal measure, in feet, if the corner is square?

16. 7 pts

Henry writes $\sqrt{16 + 9} = \sqrt{16} + \sqrt{9} = 7$. Explain his error, give the correct value, and state the rule he may have been thinking of.

ANSWER KEY

1. $\sqrt{49 \cdot 2} = 7\sqrt{2}$
2. -5 , because $(-5)^3 = -125$
3. $\sqrt{9 \cdot 7 \cdot x^6} = 3x^3\sqrt{7}$
4. $\frac{14\sqrt{7}}{7} = 2\sqrt{7}$
5. $\frac{\sqrt{11}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{22}}{6}$
6. $3\sqrt{5} - 2\sqrt{5} + \sqrt{5} = 2\sqrt{5}$
7. $15 + 6\sqrt{2} - 5\sqrt{2} - 2 \cdot 2 = 11 + \sqrt{2}$
8. $10 - 9 = 1$
9. $4x - 3 = 49$, so $x = 13$ (check: $\sqrt{49} = 7$)
10. $x^2 = 5x + 6$, $(x - 6)(x + 1) = 0$; $x = 6$ checks ($\sqrt{36} = 6$), but $x = -1$ is extraneous ($\sqrt{1} = 1 \neq -1$). So $x = 6$.
11. $2.5 = \frac{\sqrt{h}}{4}$, $\sqrt{h} = 10$, so $h = 100$ feet
12. $\sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$
13. $\left(\frac{-7 + 3}{2}, \frac{2 + 10}{2}\right) = (-2, 6)$
14. $h^2 = 144 - 16 = 128$, so $h = \sqrt{128} = 8\sqrt{2} \approx 11.3$ feet
15. (a) No. (b) $\sqrt{9^2 + 12^2} = \sqrt{225} = 15$ feet; since 15.5 is too long, the angle is a bit more than 90° .
16. A square root does not split over addition. The correct value is $\sqrt{16 + 9} = \sqrt{25} = 5$. The true rule is for products: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, which works because $(\sqrt{a} \cdot \sqrt{b})^2 = ab$; there is no such rule for sums.

10.1 SIMPLIFYING SQUARE ROOTS

OBJECTIVES

- Explain why $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ using the meaning of a square root.
- Simplify square roots of whole numbers into simplest radical form using perfect-square factors.
- Simplify square roots of variable expressions with even exponents, assuming the variables are nonnegative.
- Find cube roots of perfect cubes, including negative numbers and simple monomials.

WATCH

Math Antic - Simplifying Square Roots

[mathantics](#) • 12:01

<https://www.youtube.com/watch?v=2mejAHKMBiM>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Simplifying Square Roots | Math with Mr. J

[Math with Mr. J](#) • 7:44

<https://www.youtube.com/watch?v=GjiA0nGYILM>

Before the practice set. For cube roots (objective 4), see Mr. J's 'Square Roots and Cube Roots' ([youtube.com/watch?v=Sjpkp8-0t1s](https://www.youtube.com/watch?v=Sjpkp8-0t1s)).

KEY TERMS

radical

An expression with a root symbol, such as $\sqrt{50}$ or $\sqrt[3]{8}$. The symbol $\sqrt{\quad}$ itself is called the radical sign.

radicand

The number or expression under the radical sign. In $\sqrt{50}$ the radicand is 50.

principal square root

The nonnegative square root of a number. Both 5 and -5 square to 25, but $\sqrt{25}$ means only the principal root, 5.

perfect square

A number that is the square of a whole number (1, 4, 9, 16, 25, ...), or a monomial such as x^6 that is the square of another monomial ($x^6 = (x^3)^2$).

simplest radical form

A square root is in simplest radical form when the radicand has no perfect-square factor other than 1 (more rules are added in Lesson 10.2).

cube root

A number whose cube is the given number. $\sqrt[3]{64} = 4$ because $4^3 = 64$, and $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$.

CORE IDEAS

In Pre-Algebra you learned that $\sqrt{49} = 7$ because $7^2 = 49$, and that a number like $\sqrt{50}$ is not a whole number (it is a little more than 7). A calculator gives $\sqrt{50} \approx 7.0711$, but that is only an approximation. In algebra we often want the exact value, written as neatly as possible. This lesson shows how to write $\sqrt{50}$ exactly as $5\sqrt{2}$.

The key fact is the **product property**: for $a \geq 0$ and $b \geq 0$, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. Why is it true? The symbol $\sqrt{ab}$ means the nonnegative number whose square is ab . Now square the right side: $(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 = a \cdot b$, using the power-of-a-product rule from Unit 7. So $\sqrt{a} \cdot \sqrt{b}$ is a nonnegative number whose square is ab , which means it is exactly $\sqrt{ab}$. You can test it: $\sqrt{4 \cdot 9} = \sqrt{36} = 6$ and $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$.

To simplify, split the radicand into a perfect square times what is left, then take the square root of the perfect square: $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. Use the **largest** perfect-square factor you can find. If you use a smaller one, you are not finished: $\sqrt{72} = \sqrt{4 \cdot 18} = 2\sqrt{18}$, but 18 still contains 9, so $2\sqrt{18} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$. Going straight to $72 = 36 \cdot 2$ gives $6\sqrt{2}$ in one step. If you cannot spot a factor, write the prime factorization and pair up equal primes: $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$; each pair comes out as one factor, so $\sqrt{72} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$.

Variables work the same way. In this unit we assume every variable under a square root is nonnegative. Since $(x^3)^2 = x^6$, we get $\sqrt{x^6} = x^3$: to take the square root of a

power with an even exponent, **halve the exponent**. So $\sqrt{x^{10}} = x^5$ and $\sqrt{a^2b^8} = ab^4$. A mixed example: $\sqrt{75x^4} = \sqrt{25 \cdot 3 \cdot x^4} = 5x^2\sqrt{3}$.

Be careful: the product property works for **products**, not sums. $\sqrt{9 + 16} = \sqrt{25} = 5$, but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$. There is no rule that splits a square root over addition.

A **cube root** undoes cubing. $\sqrt[3]{125} = 5$ because $5^3 = 125$. Unlike square roots, cube roots of negative numbers are real: $(-4)^3 = -64$, so $\sqrt[3]{-64} = -4$. (There is no real square root of -64 , because every real number squared is 0 or positive.) For variables, divide the exponent by 3: $\sqrt[3]{x^6} = x^2$ because $(x^2)^3 = x^6$. Knowing the perfect cubes 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000 makes cube roots quick.

KEY FORMULAS

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad (a \geq 0, b \geq 0)$$

$$\sqrt{x^{2n}} = x^n \quad (x \geq 0)$$

$$\sqrt[3]{a^3} = a \quad \text{for every real number } a$$

From history. People have worked with square roots for a very long time. A small Babylonian clay tablet from nearly 4,000 years ago, now kept at Yale University, shows a square with its diagonals drawn and gives the value of $\sqrt{2}$ (the diagonal of a square with side 1) correct to about six decimal places, written in their base-60 number system.

Practice habit. Recite the perfect squares up to $20^2 = 400$ and the perfect cubes up to $10^3 = 1000$ until they are automatic. When you simplify a square root, ask "What is the largest perfect square that divides this number?" and check your answer by squaring it.

WORKED EXAMPLES

Example 1. Simplify $\sqrt{72x^4y^2}$. Assume all variables are nonnegative.

Show solution

1. Find the largest perfect-square factor of 72: $72 = 36 \cdot 2$.
2. The variable parts are already perfect squares: $x^4 = (x^2)^2$ and $y^2 = (y)^2$.
3. Use the product property: $\sqrt{72x^4y^2} = \sqrt{36} \cdot \sqrt{x^4} \cdot \sqrt{y^2} \cdot \sqrt{2}$.
4. Take each square root: $= 6 \cdot x^2 \cdot y \cdot \sqrt{2} = 6x^2y\sqrt{2}$.
5. Check by squaring: $(6x^2y\sqrt{2})^2 = 36x^4y^2 \cdot 2 = 72x^4y^2$. It matches.

Answer: $6x^2y\sqrt{2}$

Example 2. Ruth wants a square brick patio with an area of 162 square feet. Find the exact side length in simplest radical form, then approximate it to the nearest tenth of a foot.

Show solution

1. The area of a square is s^2 , so $s^2 = 162$ and $s = \sqrt{162}$ (a length is positive, so we use the principal root).
2. Find the largest perfect-square factor: $162 = 81 \cdot 2$.
3. Simplify: $\sqrt{162} = \sqrt{81} \cdot \sqrt{2} = 9\sqrt{2}$.
4. Approximate: $9\sqrt{2} \approx 9 \times 1.41421 \approx 12.728$, which rounds to 12.7.
5. Check: $12.7^2 = 161.29$, very close to 162, so the answer is reasonable.

Answer: $9\sqrt{2}$ feet, about 12.7 feet

PRACTICE

1. Easy Simplify $\sqrt{50}$.

Show answer

$$\sqrt{25 \cdot 2} = 5\sqrt{2}$$

2. Easy Simplify $\sqrt{72}$.

Show answer

$$\sqrt{36 \cdot 2} = 6\sqrt{2}$$

3. Easy Find $\sqrt[3]{-64}$.

Show answer

$$-4, \text{ because } (-4)^3 = -64$$

4. Easy Simplify $\sqrt{x^{10}}$. Assume $x \geq 0$.

Show answer

$$x^5, \text{ because } (x^5)^2 = x^{10}$$

5. Medium Simplify $\sqrt{180}$.

Show answer

$$\sqrt{36 \cdot 5} = 6\sqrt{5}$$

6. Medium Simplify $3\sqrt{48}$.

Show answer

$$3\sqrt{16 \cdot 3} = 3 \cdot 4\sqrt{3} = 12\sqrt{3}$$

7. Medium Simplify $\sqrt{75x^4}$. Assume $x \geq 0$.

Show answer

$$\sqrt{25 \cdot 3 \cdot x^4} = 5x^2\sqrt{3}$$

8. Medium Simplify $\sqrt{98a^2b^6}$. Assume all variables are nonnegative.

Show answer

$$\sqrt{49 \cdot 2 \cdot a^2 \cdot b^6} = 7ab^3\sqrt{2}$$

9. Medium Simplify $\sqrt[3]{8x^6}$.

Show answer

$$2x^2, \text{ because } (2x^2)^3 = 8x^6$$

10. Hard Grace is planting a square vegetable garden with an area of 200 square feet. Find the exact length of one side in simplest radical form.

Show answer

$$\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2} \text{ feet (about 14.1 feet)}$$

11. **Hard** Samuel is building a cube-shaped wooden toy chest that will hold 3375 cubic inches. How long is each edge of the chest?

Show answer

$$\sqrt[3]{3375} = 15 \text{ inches, because } 15^3 = 3375$$

12. **Hard** Henry simplified $\sqrt{48}$ and wrote $2\sqrt{12}$. Is his answer in simplest radical form? Explain, and finish the problem if needed.

Show answer

No. The radicand 12 still has the perfect-square factor 4, so $2\sqrt{12} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$. Henry used the perfect square 4 instead of the largest one, 16; using $48 = 16 \cdot 3$ gives $4\sqrt{3}$ in one step.

13. **Hard** Simplify $\sqrt{500m^8n^2}$. Assume all variables are nonnegative.

Show answer

$$\sqrt{100 \cdot 5 \cdot m^8 \cdot n^2} = 10m^4n\sqrt{5}$$

14. **Challenge** Find the smallest positive whole number n that makes $\sqrt{12n}$ a whole number.

Show answer

$$n = 3. \text{ Since } 12 = 4 \cdot 3, \text{ the factor 3 needs a partner: } \sqrt{12 \cdot 3} = \sqrt{36} = 6.$$

15. **Challenge** The Miller family has a rectangular quilt that is 50 inches by 36 inches. Ruth wants to sew a square quilt with exactly the same area. Find the side length of the square quilt in simplest radical form, then approximate it to the nearest tenth of an inch.

Show answer

$$\text{Area} = 50 \times 36 = 1800 \text{ square inches; side} = \sqrt{1800} = \sqrt{900 \cdot 2} = 30\sqrt{2} \approx 42.4 \text{ inches}$$

COMMON MISTAKES

- Stopping before the radical is fully simplified, as in $\sqrt{72} = 2\sqrt{18}$. Always check whether the new radicand still has a perfect-square factor, or start with the largest perfect square (here 36).
- Splitting a square root over addition: $\sqrt{9 + 16}$ is 5, not $3 + 4 = 7$. The product property works only for multiplication.

- Taking the square root of the exponent instead of halving it: $\sqrt{x^{16}} = x^8$, not x^4 . Check by squaring: $(x^8)^2 = x^{16}$.
- Thinking a negative number has no cube root. $\sqrt[3]{-27} = -3$ because $(-3)^3 = -27$; only square roots of negatives are not real.

Make it a game • Perfect Squares Sprint. Round one: list perfect squares from 1 to 400 from memory in 60 seconds. Count them. Round two: the parent calls radicals, such as $\sqrt{72}$, $\sqrt{50}$ and $\sqrt{200}$. Find the largest perfect-square factor and give the simplest radical form. Explain aloud why $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. Next day, try to beat your list.

10.2 QUOTIENTS AND RATIONALIZING DENOMINATORS

OBJECTIVES

- Explain why $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ using the meaning of a square root.
- Simplify square roots of fractions with the quotient property.
- Rationalize a denominator that is a single square root term, and explain why we do it.
- Decide whether a radical expression is in simplest radical form using the three rules.

WATCH

Multiplying & Dividing Surds

Eddie Woo · 8:46

<https://www.youtube.com/watch?v=hUYBrufdUNw>

WHEN TO WATCH

Before Example 1: dividing square roots and the quotient property.

Note: Eddie Woo teaches in Australia: he says 'surd' for an irrational square root and 'rationalising' for rationalizing.

BACKUP VIDEO

Y10 Review Questions (Rationalising denominators)

Eddie Woo · 6:15

<https://www.youtube.com/watch?v=Mqmv9JMjCBg>

Before Example 2: rationalizing a single square-root denominator.

KEY TERMS

quotient property of square roots

For $a \geq 0$ and $b > 0$, the square root of a fraction equals the square root of the top over the square root of the bottom:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

rationalizing the denominator

Rewriting a fraction so that no radical is left in the denominator, usually by multiplying the top and bottom by the same square root.

simplest radical form

A radical expression is in simplest radical form when (1) no radicand has a perfect-square factor other than 1, (2) no radicand is a fraction, and (3) no denominator contains a radical.

CORE IDEAS

In Lesson 10.1 you split the square root of a product. Square roots of quotients split the same way: for $a \geq 0$ and $b > 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. Why? Square the right side:

$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}$. So $\frac{\sqrt{a}}{\sqrt{b}}$ is a nonnegative number whose square is $\frac{a}{b}$, and that is exactly what $\sqrt{\frac{a}{b}}$ means. For example, $\sqrt{\frac{9}{25}} = \frac{3}{5}$, and indeed $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

When the bottom is a perfect square, the quotient property finishes the job: $\sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{\sqrt{16}} = \frac{\sqrt{7}}{4}$. When it is not, you get a radical in the denominator, such as $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}}$.

To **rationalize the denominator**, multiply the fraction by $\frac{\sqrt{5}}{\sqrt{5}}$. This fraction equals 1, so the value does not change, but the bottom becomes $\sqrt{5} \cdot \sqrt{5} = 5$, a rational number (that is where the name comes from): $\frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$. In general, $\sqrt{b} \cdot \sqrt{b} = b$ by the definition of a square root.

Why bother? First, it makes estimating by hand easy. To find $\frac{1}{\sqrt{2}}$ without a calculator you would have to divide 1 by 1.41421..., a long and messy division. But $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and $1.41421 \div 2 \approx 0.7071$ takes a moment. Before calculators this saved a great deal of work. Second, it gives everyone one standard way to write an answer, so it is easy to tell whether two answers agree: $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ look different but are the same number.

Simplify any radicals first; it keeps the numbers small. For $\frac{10}{\sqrt{8}}$, write $\sqrt{8} = 2\sqrt{2}$, so $\frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$. Finally, check the three rules of simplest radical form: no perfect-square factors under a radical, no fractions under a radical, no radicals in a denominator.

One warning: you may only cancel numbers that are both **outside** the radical. In $\frac{6\sqrt{5}}{3}$ the 6 and the 3 are both outside, so it equals $2\sqrt{5}$. But in $\frac{\sqrt{6}}{2}$ the 6 is inside the radical and cannot be divided by 2; $\frac{\sqrt{6}}{2}$ is already simplest.

KEY FORMULAS

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (a \geq 0, b > 0)$$

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

$$\sqrt{b} \cdot \sqrt{b} = b \quad (b \geq 0)$$

Practice habit. Memorize three decimals: $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, $\sqrt{5} \approx 2.236$. After you rationalize, use them to estimate both the original and the final form; if the two estimates agree, your work is probably right.

WORKED EXAMPLES

Example 1. Simplify $\frac{12}{\sqrt{6}}$.

Show solution

1. The denominator $\sqrt{6}$ has no perfect-square factor, so rationalize right away.
2. Multiply by $\frac{\sqrt{6}}{\sqrt{6}}$, which equals 1: $\frac{12}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{12\sqrt{6}}{6}$.
3. The 12 and the 6 are both outside the radical, so divide: $\frac{12\sqrt{6}}{6} = 2\sqrt{6}$.
4. Check with decimals: $12 \div 2.4495 \approx 4.899$ and $2 \times 2.4495 = 4.899$. They agree.

Answer: $2\sqrt{6}$

Example 2. For the church picnic, Grace is painting a rectangular banner with an area of 30 square feet. Its height is $\sqrt{6}$ feet. Find the exact width in simplest radical form, then approximate it to the nearest tenth of a foot.

Show solution

1. Area = height $\times$ width, so width = $\frac{30}{\sqrt{6}}$.

2. Rationalize: $\frac{30}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{30\sqrt{6}}{6}$.

3. Divide the outside numbers: $\frac{30\sqrt{6}}{6} = 5\sqrt{6}$.

4. Approximate: $5\sqrt{6} \approx 5 \times 2.44949 \approx 12.247$, which rounds to 12.2 feet.

5. Check: $\sqrt{6} \cdot 5\sqrt{6} = 5 \cdot 6 = 30$ square feet. Correct.

Answer: $5\sqrt{6}$ feet, about 12.2 feet

PRACTICE

1. Easy Simplify $\sqrt{\frac{25}{49}}$.

Show answer

$$\frac{5}{7}$$

2. Easy Simplify $\sqrt{\frac{7}{16}}$.

Show answer

$$\frac{\sqrt{7}}{4}$$

3. Easy Rationalize the denominator: $\frac{1}{\sqrt{2}}$.

Show answer

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

4. Easy Simplify $\frac{6}{\sqrt{3}}$.

Show answer

$$\frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

5. Medium Simplify $\sqrt{\frac{18}{25}}$.

Show answer

$$\frac{\sqrt{18}}{5} = \frac{3\sqrt{2}}{5}$$

6. Medium Simplify $\frac{5}{\sqrt{10}}$.

Show answer

$$\frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$$

7. Medium Simplify $\sqrt{\frac{3}{5}}$.

Show answer

$$\frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$$

8. Medium Simplify $\sqrt{\frac{20}{x^4}}$. Assume $x > 0$.

Show answer

$$\frac{\sqrt{20}}{\sqrt{x^4}} = \frac{2\sqrt{5}}{x^2}$$

9. Medium Simplify $\sqrt{\frac{5}{12}}$.

Show answer

$$\frac{\sqrt{5}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{6}$$

10. Hard By the Pythagorean theorem, the diagonal of a square with side s is $s\sqrt{2}$. Henry is laying square floor tiles whose diagonal is 10 inches, so $s = \frac{10}{\sqrt{2}}$. Write the side length in simplest radical form.

Show answer

$$\frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \text{ inches (about 7.1 inches)}$$

11. **Hard** The men of the church are building a long, narrow rectangular flower bed beside the walkway. It must have an area of 40 square yards and a width of $\sqrt{5}$ yards. Find the exact length in simplest radical form.

Show answer

$$\text{Length} = \frac{40}{\sqrt{5}} = \frac{40\sqrt{5}}{5} = 8\sqrt{5} \text{ yards (about 17.9 yards)}$$

12. **Hard** Explain why $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$ are the same number, and why the second form is easier to estimate without a calculator.

Show answer

Multiplying $\frac{1}{\sqrt{3}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$ multiplies it by 1, so its value does not change, and the bottom becomes $\sqrt{3} \cdot \sqrt{3} = 3$. To estimate $\frac{\sqrt{3}}{3}$ you divide 1.732 by 3, which is easy (about 0.577); to estimate $\frac{1}{\sqrt{3}}$ you would have to divide 1 by the long decimal 1.732....

13. **Hard** Rachel drops a smooth stone from a footbridge 50 feet above a creek. The time in seconds for it to reach the water is $t = \sqrt{\frac{h}{16}}$, where h is the height in feet. Find the time exactly in simplest radical form, then to the nearest tenth of a second.

Show answer

$$t = \sqrt{\frac{50}{16}} = \frac{\sqrt{50}}{4} = \frac{5\sqrt{2}}{4} \approx 1.8 \text{ seconds}$$

14. **Challenge** Simplify $\frac{4\sqrt{3}}{\sqrt{8}}$.

Show answer

$$\sqrt{8} = 2\sqrt{2}, \text{ so } \frac{4\sqrt{3}}{2\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{6}}{2} = \sqrt{6}$$

15. **Challenge** Simplify $\frac{\sqrt{45x^3}}{\sqrt{5x}}$. Assume $x > 0$.

Show answer

$$\sqrt{\frac{45x^3}{5x}} = \sqrt{9x^2} = 3x$$

COMMON MISTAKES

- Multiplying only the denominator by $\sqrt{b}$. That changes the value of the fraction. Multiply the top and the bottom by the same square root, so you are multiplying by 1.
- Writing $\sqrt{5} \cdot \sqrt{5} = 25$. A square root times itself gives back the radicand: $\sqrt{5} \cdot \sqrt{5} = 5$.
- Cancelling a number inside a radical with a number outside, as in $\frac{\sqrt{6}}{2} = \sqrt{3}$. That is false ($\sqrt{6} \div 2 \approx 1.22$, while $\sqrt{3} \approx 1.73$). Only cancel numbers that are both outside the radical.
- Forgetting to simplify at the end. After rationalizing $\frac{12}{\sqrt{6}}$ to $\frac{12\sqrt{6}}{6}$, divide the 12 by 6 to get $2\sqrt{6}$.

Make it a game • Rationalize Race. Six radical fractions, one timer. The parent writes problems such as $\sqrt{(9/16)}$, $\sqrt{(5/12)}$ and $5/\sqrt{3}$. Simplify each, multiplying by a 'fancy one' such as $\sqrt{3}/\sqrt{3}$ to clear a root from the denominator. Add ten seconds to your time for each answer that is not in simplest radical form. Beat your best time on a second set.

10.3 OPERATIONS WITH RADICAL EXPRESSIONS

OBJECTIVES

- Add and subtract radicals by simplifying first and combining like radicals.
- Multiply radical expressions, including a monomial times a binomial and a binomial times a binomial.
- Explain why $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$ and use conjugates to get a rational product.
- Solve area and perimeter problems whose lengths contain radicals.

WATCH

Adding & Subtracting Surds

Eddie Woo · 11:37

<https://www.youtube.com/watch?v=gUN6Z0uR-P0>

WHEN TO WATCH

Before Example 1: adding and subtracting like radicals.

Note: Eddie Woo teaches in Australia: he says 'surd' for an irrational square root.

BACKUP VIDEO

Rationalising Binomial Denominators

Eddie Woo · 8:15

https://www.youtube.com/watch?v=altB_OEHu08

Before Example 2: multiplying by a conjugate.

KEY TERMS

like radicals

Radicals with the same index and the same radicand, such as $3\sqrt{2}$ and $-7\sqrt{2}$. Only like radicals can be combined by adding or subtracting.

unlike radicals

Radicals with different radicands (after simplifying), such as $\sqrt{2}$ and $\sqrt{3}$. Their sum cannot be written as a single term.

conjugates

Two binomials that are the same except for the sign in the middle, such as $5 + \sqrt{3}$ and $5 - \sqrt{3}$. Their product has no radical in it.

CORE IDEAS

Adding radicals is combining like terms again. Just as $3x + 5x = 8x$, we have $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. The reason is the distributive property: $3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2}$. Think of $\sqrt{2}$ as a unit, like "feet": 3 of them plus 5 of them is 8 of them.

Unlike radicals do not combine, just as $2x + 3y$ does not. In particular, $\sqrt{2} + \sqrt{3}$ is **not** $\sqrt{5}$: a quick estimate shows $1.414 + 1.732 = 3.146$, while $\sqrt{5} \approx 2.236$. But always simplify first, because radicals that look unlike may turn out to be like: $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$.

Multiplying uses the product property from Lesson 10.1 read backward: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Multiply the outside numbers together and the radicands together, then simplify: $(2\sqrt{3})(5\sqrt{6}) = 10\sqrt{18} = 10 \cdot 3\sqrt{2} = 30\sqrt{2}$. Notice the difference from adding: you may multiply $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ even though you cannot add $\sqrt{2} + \sqrt{3}$.

Expressions with more than one term multiply exactly like the polynomials of Unit 8: distribute, or use the box model or FOIL. For example, $(2 + \sqrt{3})(4 - \sqrt{3}) = 8 - 2\sqrt{3} + 4\sqrt{3} - \sqrt{3} \cdot \sqrt{3} = 8 + 2\sqrt{3} - 3 = 5 + 2\sqrt{3}$. The special product $(a + b)^2 = a^2 + 2ab + b^2$ still holds, so $(\sqrt{7} + 1)^2 = 7 + 2\sqrt{7} + 1 = 8 + 2\sqrt{7}$, not $7 + 1$.

Conjugates such as $a + \sqrt{b}$ and $a - \sqrt{b}$ are special. They fit the pattern $(a + b)(a - b) = a^2 - b^2$ from Lesson 8.4, so $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - (\sqrt{b})^2 = a^2 - b$. Multiplying it out shows why: the middle terms $-a\sqrt{b}$ and $+a\sqrt{b}$ cancel, and $\sqrt{b} \cdot \sqrt{b} = b$. For example, $(5 + \sqrt{3})(5 - \sqrt{3}) = 25 - 3 = 22$, a whole number with no radical.

Conjugates let you rationalize a denominator with two terms. To simplify $\frac{4}{3 - \sqrt{5}}$, multiply the top and bottom by the conjugate $3 + \sqrt{5}$: the bottom becomes $9 - 5 = 4$, so the fraction is $\frac{4(3 + \sqrt{5})}{4} = 3 + \sqrt{5}$. As in Lesson 10.2, you are multiplying by a form of 1, so the value never changes.

KEY FORMULAS

$$a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$$

$$a\sqrt{x} \cdot b\sqrt{y} = ab\sqrt{xy} \quad (x \geq 0, y \geq 0)$$

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

Practice habit. Simplify every radical before you try to add, subtract or combine: write one simplified radical per step, then gather the like ones. Afterward, estimate with $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$ to check that your answer has the right size.

WORKED EXAMPLES

Example 1. Simplify $\sqrt{50} + \sqrt{18} - \sqrt{8}$.

Show solution

1. Simplify each radical: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.
2. $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$.
3. $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$.
4. Now all three are like radicals: $5\sqrt{2} + 3\sqrt{2} - 2\sqrt{2} = (5 + 3 - 2)\sqrt{2} = 6\sqrt{2}$.
5. Check with decimals: $7.071 + 4.243 - 2.828 = 8.486$ and $6 \times 1.414 = 8.484$.
They agree (the tiny difference is rounding).

Answer: $6\sqrt{2}$

Example 2. Henry builds a rectangular raised garden bed that is $(4 + \sqrt{2})$ feet long and $(4 - \sqrt{2})$ feet wide. Find its exact area in square feet.

Show solution

1. Area = length $\times$ width = $(4 + \sqrt{2})(4 - \sqrt{2})$.
2. Multiply (FOIL): $16 - 4\sqrt{2} + 4\sqrt{2} - \sqrt{2} \cdot \sqrt{2}$.
3. The middle terms cancel, and $\sqrt{2} \cdot \sqrt{2} = 2$: $16 - 2 = 14$.
4. This matches the conjugate pattern $a^2 - b = 4^2 - 2 = 14$.
5. Check with decimals: $5.414 \times 2.586 \approx 14.0$. Correct.

Answer: 14 square feet

PRACTICE

1. Easy Simplify $4\sqrt{7} + 9\sqrt{7}$.

Show answer

$$13\sqrt{7}$$

2. Easy Simplify $8\sqrt{3} - 5\sqrt{3} + \sqrt{3}$.

Show answer

$$(8 - 5 + 1)\sqrt{3} = 4\sqrt{3}$$

3. Easy Multiply $\sqrt{5} \cdot \sqrt{20}$.

Show answer

$$\sqrt{100} = 10$$

4. Easy Multiply $(3\sqrt{2})(4\sqrt{5})$.

Show answer

$$12\sqrt{10}$$

5. Medium Simplify $\sqrt{12} + \sqrt{75}$.

Show answer

$$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$$

6. Medium Simplify $3\sqrt{8} - \sqrt{32} + \sqrt{2}$.

Show answer

$$6\sqrt{2} - 4\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$

7. Medium Multiply and simplify $\sqrt{3}(\sqrt{6} + 4)$.

Show answer

$$\sqrt{18} + 4\sqrt{3} = 3\sqrt{2} + 4\sqrt{3}$$

8. Medium Multiply $(2 + \sqrt{5})(3 + \sqrt{5})$.

Show answer

$$6 + 2\sqrt{5} + 3\sqrt{5} + 5 = 11 + 5\sqrt{5}$$

9. Medium Multiply the conjugates $(6 + \sqrt{2})(6 - \sqrt{2})$.

Show answer

$$36 - 2 = 34$$

10. **Hard** Expand and simplify $(\sqrt{7} - 3)^2$.

Show answer

$$(\sqrt{7})^2 - 2 \cdot 3\sqrt{7} + 9 = 7 - 6\sqrt{7} + 9 = 16 - 6\sqrt{7}$$

11. **Hard** Ruth's rectangular chicken run measures $\sqrt{200}$ feet by $\sqrt{128}$ feet. How many feet of fencing go all the way around it? Give the exact answer in simplest radical form.

Show answer

Sides are $10\sqrt{2}$ and $8\sqrt{2}$; perimeter = $2(10\sqrt{2} + 8\sqrt{2}) = 36\sqrt{2}$ feet (about 50.9 feet)

12. **Hard** Samuel is building a square dining table whose sides are $(2 + \sqrt{3})$ feet long. Find the exact area of the tabletop in square feet.

Show answer

$$(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} \text{ square feet (about 13.9 square feet)}$$

13. **Hard** Explain why $\sqrt{2} + \sqrt{8}$ can be written as a single term, but $\sqrt{2} + \sqrt{3}$ cannot.

Show answer

Simplifying gives $\sqrt{8} = 2\sqrt{2}$, so $\sqrt{2} + \sqrt{8} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$: they are like radicals. $\sqrt{2}$ and $\sqrt{3}$ are already simplified and have different radicands, so they are unlike, just as $x + y$ cannot be combined. (And the sum is not $\sqrt{5}$: $1.414 + 1.732 \approx 3.146$, but $\sqrt{5} \approx 2.236$.)

14. **Challenge** Grace is designing a rectangular stained-glass window with an area of 8 square feet. Its width is $(1 + \sqrt{3})$ feet. Find the exact height. Rationalize the denominator using the conjugate.

Show answer

$$\text{Height} = \frac{8}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{8(1 - \sqrt{3})}{1 - 3} = -4(1 - \sqrt{3}) = 4\sqrt{3} - 4 \text{ feet (about 2.93 feet)}$$

15. **Challenge** Simplify $\frac{4}{3 - \sqrt{5}}$ by rationalizing the denominator.

Show answer

$$\text{Multiply by } \frac{3 + \sqrt{5}}{3 + \sqrt{5}}: \frac{4(3 + \sqrt{5})}{9 - 5} = \frac{4(3 + \sqrt{5})}{4} = 3 + \sqrt{5}$$

COMMON MISTAKES

- Adding radicands: $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$. Only like radicals combine, and you add the numbers in front, keeping the radical: $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$.
- Not simplifying before combining. $\sqrt{12} + \sqrt{27}$ looks like it cannot be combined, but it is $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$.
- Squaring a binomial term by term: $(\sqrt{7} + 1)^2 \neq 7 + 1$. Write it as $(\sqrt{7} + 1)(\sqrt{7} + 1)$ and include the middle term $2\sqrt{7}$.
- Writing $\sqrt{b} \cdot \sqrt{b} = b^2$ or $\sqrt{b^2}$ left unfinished. A square root times itself is just the radicand: $\sqrt{2} \cdot \sqrt{2} = 2$.

Make it a game • Radical Memory Match. Make twelve cards with radicals like $\sqrt{12}$, $\sqrt{3}$, $\sqrt{18}$, $\sqrt{50}$, $\sqrt{8}$ and $\sqrt{2}$. Turn them face down. Flip two at a time: if they become like radicals once simplified, keep the pair and say their sum, for example $\sqrt{18} + \sqrt{50} = 8\sqrt{2}$. Unlike pairs go back. Finish with a card that multiplies a monomial by a binomial.

10.4 SOLVING RADICAL EQUATIONS

OBJECTIVES

- Solve radical equations by isolating the radical and squaring both sides.
- Check every answer in the original equation and reject extraneous solutions.
- Explain why squaring both sides of an equation can create extraneous solutions.
- Use pendulum and falling-object formulas to solve real problems.

WATCH

Solving Equations: What happens when the pronumeral has a square root ($\sqrt{a}$)?

Eddie Woo · 7:05

<https://www.youtube.com/watch?v=hKnAYXh1PmM>

WHEN TO WATCH

Before Example 2: squaring both sides, and why $\sqrt{x} = -6.25$ has no solution.

Note: Weak fit: neither video checks for extraneous solutions after squaring (objectives 2 and 3). Teach that from Example 2 and the Core ideas.

BACKUP VIDEO

Y10 Review Questions (Solving an equation with surds)

Eddie Woo · 11:24

<https://www.youtube.com/watch?v=DvKlCKAiMcE>

Before the practice set: a review question solving an equation with square roots.

KEY TERMS

radical equation

An equation with the variable inside a radical, such as $\sqrt{x+4} = 6$.

squaring both sides

Raising both sides of an equation to the second power to remove a square root, since $(\sqrt{a})^2 = a$.

extraneous solution

A number produced by the solving steps that does **not** make the original equation true. It must be thrown out.

CORE IDEAS

A **radical equation** has the variable under a square root. To undo a square root, we square it: $(\sqrt{a})^2 = a$. So the plan is (1) isolate the radical on one side, (2) square both sides, (3) solve the equation that is left, and (4) check every answer in the original equation. For example, to solve $\sqrt{x+4} = 6$, square both sides: $x+4 = 36$, so $x = 32$. Check: $\sqrt{36} = 6$.

Isolate first. In $\sqrt{x} + 3 = 8$, subtract 3 to get $\sqrt{x} = 5$, then square: $x = 25$. If you squared too early you would have to square the whole left side, $(\sqrt{x} + 3)^2 = x + 6\sqrt{x} + 9$, and the radical would still be there.

Why must we check? Squaring both sides is not like adding or multiplying both sides by the same number; it can change the solutions. If $a = b$, then certainly $a^2 = b^2$. But the reverse fails: $a^2 = b^2$ only tells you $a = b$ **or** $a = -b$, because, for example, 3^2 and $(-3)^2$ are both 9. So when you square, the new equation is true both for the solutions you want and for solutions of the "opposite" equation. Those extra numbers are called **extraneous solutions**.

Here is the simplest case. The equation $\sqrt{x} = -3$ has no solution, because $\sqrt{x}$ means the principal (nonnegative) square root and can never be negative. But squaring both sides gives $x = 9$. Checking shows $\sqrt{9} = 3$, not -3 , so 9 is extraneous. Squaring turned $\sqrt{x} = -3$ into the same equation you get from $\sqrt{x} = 3$, and the 9 belongs to that other equation.

Extraneous solutions often appear when the variable is on both sides. Solve $x = \sqrt{x+6}$: squaring gives $x^2 = x+6$, so $x^2 - x - 6 = 0$ and $(x-3)(x+2) = 0$ by factoring (Unit 9). The possible answers are 3 and -2 . Checking: $\sqrt{3+6} = 3$ works, but $\sqrt{-2+6} = 2$, not -2 , so -2 is extraneous. The only solution is $x = 3$.

Radical equations appear in science. A stone dropped from a height of h feet takes $t = \frac{\sqrt{h}}{4}$ seconds to reach the ground (this comes from $h = 16t^2$). A pendulum of length L feet takes $T = 2\pi\sqrt{\frac{L}{32}}$ seconds to swing over and back. Solving these for h or L is solving a radical equation.

KEY FORMULAS

If $\sqrt{a} = b$, then $a = b^2$ (then check!)

$$t = \frac{\sqrt{h}}{4} \quad (\text{dropped object; } h \text{ in feet, } t \text{ in seconds})$$

$$T = 2\pi\sqrt{\frac{L}{32}} \quad (\text{pendulum; } L \text{ in feet, } T \text{ in seconds})$$

Practice habit. Check by substituting, every time, in the **original** equation. Write the check as its own line, such as "Check $x = -2$: $\sqrt{4} = 2 \neq -2$, reject." A radical-equation answer without a check is not finished.

WORKED EXAMPLES

Example 1. Samuel drops a stone from a cliff ledge into a lake below. It takes 3 seconds to hit the water. Use $t = \frac{\sqrt{h}}{4}$ to find the height h of the ledge in feet.

Show solution

1. Substitute $t = 3$: $3 = \frac{\sqrt{h}}{4}$.
 2. Isolate the radical by multiplying both sides by 4: $\sqrt{h} = 12$.
 3. Square both sides: $h = 144$.
 4. Check: $\frac{\sqrt{144}}{4} = \frac{12}{4} = 3$ seconds. Correct.
-

Answer: 144 feet

Example 2. Solve $\sqrt{x+7} = x-5$.

Show solution

1. The radical is already isolated. Square both sides: $x+7 = (x-5)^2$.
2. Expand the right side (it is a binomial squared, so keep the middle term):
 $x+7 = x^2 - 10x + 25$.
3. Get 0 on one side: $0 = x^2 - 11x + 18$.
4. Factor: $0 = (x-9)(x-2)$, so $x = 9$ or $x = 2$.
5. Check $x = 9$: $\sqrt{16} = 4$ and $9 - 5 = 4$. True.
6. Check $x = 2$: $\sqrt{9} = 3$ but $2 - 5 = -3$. False, so 2 is extraneous.

Answer: $x = 9$ ($x = 2$ is extraneous)

PRACTICE

1. Easy Solve $\sqrt{x} = 7$.

Show answer

$$x = 49$$

2. Easy Solve $\sqrt{x} + 3 = 8$.

Show answer

$$\sqrt{x} = 5, \text{ so } x = 25$$

3. Easy Solve $\sqrt{x-4} = 6$.

Show answer

$$x - 4 = 36, \text{ so } x = 40$$

4. Easy Solve $\sqrt{x} = -5$.

Show answer

No solution. A principal square root is never negative; squaring gives 25, but $\sqrt{25} = 5 \neq -5$.

5. Medium Solve $3\sqrt{x} = 12$.

Show answer

$$\sqrt{x} = 4, \text{ so } x = 16$$

6. Medium Solve $\sqrt{3x + 1} = 5$.

Show answer

$$3x + 1 = 25, \text{ so } x = 8$$

7. Medium Solve $\sqrt{2x - 1} + 4 = 7$.

Show answer

$$\sqrt{2x - 1} = 3, 2x - 1 = 9, \text{ so } x = 5$$

8. Medium Solve $\sqrt{5x - 6} = \sqrt{x + 10}$.

Show answer

$$5x - 6 = x + 10, 4x = 16, \text{ so } x = 4 \text{ (check: } \sqrt{14} = \sqrt{14}\text{)}$$

9. Medium Ruth drops a tennis ball from the top of a hayloft, and it takes 1.5 seconds to reach the ground. Use $t = \frac{\sqrt{h}}{4}$ to find the height h in feet.

Show answer

$$1.5 = \frac{\sqrt{h}}{4}, \sqrt{h} = 6, \text{ so } h = 36 \text{ feet}$$

10. Hard Solve $x = \sqrt{2x + 8}$.

Show answer

$$x^2 = 2x + 8, (x - 4)(x + 2) = 0; x = 4 \text{ checks } (\sqrt{16} = 4), \text{ but } x = -2 \text{ is extraneous } (\sqrt{4} = 2 \neq -2). \text{ So } x = 4.$$

11. Hard A grandfather clock keeps time with a pendulum that takes 2 seconds to swing over and back. Use $T = 2\pi\sqrt{\frac{L}{32}}$ to find the length L of the pendulum in feet. Use a calculator and round to the nearest hundredth.

Show answer

$$\sqrt{\frac{L}{32}} = \frac{2}{2\pi} = \frac{1}{\pi}, \text{ so } \frac{L}{32} = \frac{1}{\pi^2} \text{ and } L = \frac{32}{\pi^2} \approx 3.24 \text{ feet}$$

12. **Hard** Henry solved $\sqrt{x} + 6 = 2$ by writing $\sqrt{x} = -4$, then $x = 16$. Explain what went wrong and give the correct answer. Why can squaring both sides create a false answer?

Show answer

The steps are fine until the end, but 16 does not check: $\sqrt{16} + 6 = 10 \neq 2$. The equation has no solution, because $\sqrt{x}$ can never equal -4 . Squaring changes $a = b$ into $a^2 = b^2$, which is also true when $a = -b$; so squaring $\sqrt{x} = -4$ gives the same result as squaring $\sqrt{x} = 4$, and the 16 belongs to that other equation. That is why every answer must be checked.

13. **Hard** Solve $\sqrt{x+5} + 1 = x$.

Show answer

$\sqrt{x+5} = x-1$, $x+5 = x^2 - 2x + 1$, $(x-4)(x+1) = 0$; $x = 4$ checks ($3 + 1 = 4$), but $x = -1$ is extraneous ($2 + 1 = 3 \neq -1$). So $x = 4$.

14. **Challenge** On a clear day, the distance d in miles you can see to the horizon from a height of h feet above calm water is about $d = \sqrt{1.5h}$. How high up the mast of a sailing ship must Grace climb to see 15 miles?

Show answer

$15 = \sqrt{1.5h}$, $225 = 1.5h$, so $h = 150$ feet

15. **Challenge** Solve $\sqrt{x+3} = x+1$.

Show answer

$x+3 = x^2 + 2x + 1$, $x^2 + x - 2 = 0$, $(x+2)(x-1) = 0$; $x = 1$ checks ($\sqrt{4} = 2$), but $x = -2$ is extraneous ($\sqrt{1} = 1 \neq -1$). So $x = 1$.

COMMON MISTAKES

- Squaring before isolating the radical. In $\sqrt{x} + 3 = 8$, first subtract 3; squaring $\sqrt{x} + 3$ as it stands does not remove the radical.
- Squaring a binomial term by term: $(x-5)^2$ is $x^2 - 10x + 25$, not $x^2 - 25$. Write it as $(x-5)(x-5)$ if you are unsure.
- Skipping the check. Squaring can add extraneous solutions, so substitute every answer into the original equation and reject any that fail.
- Giving $x = 25$ as the answer to $\sqrt{x} = -5$. A principal square root is never negative, so an isolated radical equal to a negative number means no solution.

Make it a game • Spot the Impostor. The parent writes four radical equations, such as $\sqrt{x + 6} = x$, and gives each solving steps with candidate answers. You decide which answers are true and which are impostors by checking them in the original equation. One point for each true answer, two for catching an extraneous solution and explaining why squaring let it in.

10.5 THE PYTHAGOREAN THEOREM, DISTANCE AND MIDPOINT

OBJECTIVES

- Use the Pythagorean theorem to find a missing side of a right triangle, giving exact answers in simplest radical form.
- Use the converse of the Pythagorean theorem to decide whether a triangle is a right triangle.
- Derive the distance formula from the Pythagorean theorem and use it to find the distance between two points.
- Find the midpoint of a segment and explain why the formula averages the coordinates.

WATCH

Deriving the Distance Formula

Eddie Woo · 10:26

<https://www.youtube.com/watch?v=99MRLjoj5Tk>

WHEN TO WATCH

Before the distance example: deriving the distance formula from the Pythagorean theorem.

Note: Eddie Woo teaches in Australia: he says 'Pythagoras' theorem'. For a Pythagorean review, see Math Antics ([youtube.com/watch?v=WqhlG3Vakw8](https://www.youtube.com/watch?v=WqhlG3Vakw8)); for the converse, Eddie Woo ([youtube.com/watch?v=XIM3UToQPZs](https://www.youtube.com/watch?v=XIM3UToQPZs)).

BACKUP VIDEO

Finding the Midpoint (2 of 3: Between 2 Random Points)

Eddie Woo · 4:28

<https://www.youtube.com/watch?v=K0I5iqPE0fY>

Before the midpoint example.

KEY TERMS

hypotenuse

The longest side of a right triangle, the side across from the right angle. The other two sides are the legs.

converse of the Pythagorean theorem

If the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle, with the right angle across from side c .

distance formula

The formula for the length of the segment between two points: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Pythagorean theorem

In a right triangle with legs a and b and hypotenuse c , $a^2 + b^2 = c^2$.

Pythagorean triple

Three whole numbers that fit $a^2 + b^2 = c^2$, such as 3, 4, 5 or 5, 12, 13.

midpoint

The point exactly halfway between two points, dividing the segment into two equal parts.

CORE IDEAS

You met the **Pythagorean theorem** in Pre-Algebra: in a right triangle, $a^2 + b^2 = c^2$, where c is the hypotenuse. Read as a picture, it says that if you build a square on each side of a right triangle, the areas of the two smaller squares add up to the area of the largest square. Now that you can simplify radicals, you can give exact answers. If the legs are 4 and 8, then $c^2 = 16 + 64 = 80$, so $c = \sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$. To find a leg, subtract: if $c = 26$ and $a = 10$, then $b^2 = 676 - 100 = 576$ and $b = 24$.

The **converse** runs the other way: if $a^2 + b^2 = c^2$ for the three sides of a triangle, the triangle must have a right angle. Why? Draw a second triangle that does have a right angle, with legs a and b . By the theorem its hypotenuse is $\sqrt{a^2 + b^2} = c$. Now the two triangles have the same three side lengths, and a triangle's shape is fixed once its three sides are fixed (think of three sticks nailed together: they cannot wobble). So the first triangle is a copy of the second, and it has a right angle too. To test a triangle, check whether the two shorter sides squared add up to the longest side squared: 7, 24, 25 works because $49 + 576 = 625$.

Carpenters use the converse every day to **square a corner**. To check that two walls meet at a right angle, mark 3 feet along one wall and 4 feet along the other. If the corner is square, the diagonal between the marks is exactly 5 feet, because

$3^2 + 4^2 = 5^2$. For larger work they use multiples such as 6-8-10 or 9-12-15. If the diagonal is too long, the angle is more than 90° ; if too short, less.

The **distance formula** is the Pythagorean theorem on the coordinate plane. Take two points (x_1, y_1) and (x_2, y_2) . Draw a horizontal segment from the first point and a vertical segment from the second until they meet at the corner (x_2, y_1) . This makes a right triangle whose hypotenuse is the segment you want. The horizontal leg has length $|x_2 - x_1|$ and the vertical leg has length $|y_2 - y_1|$. By the Pythagorean theorem, $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, so $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The absolute value bars can be dropped because squaring makes every difference nonnegative anyway, which also means it does not matter which point you call the first.

The **midpoint** of a segment is found by averaging: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$. Why the average? On a number line, the point halfway from x_1 to x_2 is x_1 plus half the gap: $x_1 + \frac{x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{x_1 + x_2}{2}$. Going halfway along the segment means going halfway across and halfway up, so you do this once for the x -coordinates and once for the y -coordinates. For $(2, 7)$ and $(8, -3)$ the midpoint is $\left(\frac{10}{2}, \frac{4}{2} \right) = (5, 2)$.

KEY FORMULAS

$$a^2 + b^2 = c^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

From history. The theorem carries the name of Pythagoras, but the relationship was known and used long before him: Babylonian clay tablets from nearly 4,000 years ago work with right triangles and with $\sqrt{2}$, the diagonal of a unit square. The distance formula needed coordinates, which came from René Descartes's *La Géométrie* (1637).

Euclid connection. Euclid's *Elements* (about 300 BC) closes Book I with this theorem: Proposition I.47 proves that the squares on the two legs of a right triangle together equal the square on the hypotenuse, using actual square areas, and Proposition I.48 proves the converse, the fact a carpenter relies on when squaring a corner with 3-4-5.

Practice habit. Draw a quick sketch for every problem: label the legs and the hypotenuse, or for two points draw the right triangle with its horizontal and vertical legs. Memorize the triples 3-4-5, 5-12-13, 8-15-17 and 7-24-25 so you can spot them instantly.

WORKED EXAMPLES

Example 1. Find the distance between $(-2, 3)$ and $(4, -5)$, and find the midpoint of the segment joining them.

Show solution

1. Horizontal change: $x_2 - x_1 = 4 - (-2) = 6$.
2. Vertical change: $y_2 - y_1 = -5 - 3 = -8$.
3. Distance: $d = \sqrt{6^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.
4. Midpoint: $\left(\frac{-2 + 4}{2}, \frac{3 + (-5)}{2}\right) = \left(\frac{2}{2}, \frac{-2}{2}\right) = (1, -1)$.
5. Check: the distance from $(-2, 3)$ to $(1, -1)$ is $\sqrt{3^2 + (-4)^2} = 5$, exactly half of 10.

Answer: Distance 10; midpoint $(1, -1)$

Example 2. Henry leans a 20-foot ladder against the wall of a barn, with the foot of the ladder 6 feet from the wall on level ground. How high up the wall does the ladder reach? Give the exact answer in simplest radical form and an approximation to the nearest tenth of a foot.

Show solution

1. Sketch: the wall and the ground form the right angle; the ladder is the hypotenuse, $c = 20$, and the ground leg is $a = 6$.
2. Pythagorean theorem: $6^2 + h^2 = 20^2$, so $36 + h^2 = 400$.
3. Subtract 36: $h^2 = 364$, so $h = \sqrt{364}$ (a height is positive).
4. Simplify: $364 = 4 \cdot 91$, and 91 has no perfect-square factor, so $h = 2\sqrt{91}$.
5. Approximate: $2\sqrt{91} \approx 2 \times 9.5394 \approx 19.08$, which rounds to 19.1 feet.
6. Check: $6^2 + 19.08^2 \approx 36 + 364.0 = 400 = 20^2$.

Answer: $2\sqrt{91}$ feet, about 19.1 feet

PRACTICE

1. **Easy** A right triangle has legs of 9 and 12. Find the hypotenuse.

Show answer

$$c^2 = 81 + 144 = 225, \text{ so } c = 15$$

2. **Easy** A right triangle has a hypotenuse of 26 and one leg of 10. Find the other leg.

Show answer

$$b^2 = 676 - 100 = 576, \text{ so } b = 24$$

3. **Easy** Find the distance between $(1, 2)$ and $(4, 6)$.

Show answer

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

4. **Easy** Find the midpoint of the segment joining $(2, 7)$ and $(8, -3)$.

Show answer

$$\left(\frac{2 + 8}{2}, \frac{7 + (-3)}{2} \right) = (5, 2)$$

5. **Medium** A right triangle has legs of 4 and 8. Find the exact length of the hypotenuse in simplest radical form.

Show answer

$$c = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}$$

6. **Medium** A triangle has sides 7, 24 and 25. Is it a right triangle?

Show answer

$$\text{Yes: } 7^2 + 24^2 = 49 + 576 = 625 = 25^2$$

7. **Medium** A triangle has sides 6, 9 and 11. Is it a right triangle?

Show answer

$$\text{No: } 6^2 + 9^2 = 36 + 81 = 117, \text{ but } 11^2 = 121$$

8. **Medium** Find the exact distance between $(-3, 1)$ and $(2, -4)$ in simplest radical form.

Show answer

$$\sqrt{5^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

9. **Medium** Find the midpoint of the segment joining $(-5, 4)$ and $(3, -2)$.

Show answer

$$\left(\frac{-5 + 3}{2}, \frac{4 + (-2)}{2} \right) = (-1, 1)$$

10. **Hard** Samuel is laying out the corner of a garden shed. He marks a point 6 feet along one side and a point 8 feet along the other side. What must the diagonal between the two marks measure, in feet, for the corner to be a right angle?

Show answer

$$\sqrt{6^2 + 8^2} = \sqrt{100} = 10 \text{ feet (the 3-4-5 triangle doubled)}$$

11. **Hard** On a map with a grid measured in miles, the Miller farm is at $(2, 1)$ and the church is at $(8, 9)$. How far apart are they in a straight line?

Show answer

$$\sqrt{6^2 + 8^2} = \sqrt{100} = 10 \text{ miles}$$

12. **Hard** On the same kind of map grid, Grace lives at $(-4, 3)$ and Ruth lives at $(6, -1)$. They agree to meet at the point exactly halfway between their houses. Where do they meet?

Show answer

$$\left(\frac{-4 + 6}{2}, \frac{3 + (-1)}{2} \right) = (1, 1)$$

13. **Hard** Explain how the distance formula comes from the Pythagorean theorem. Why does it not matter which point you call (x_1, y_1) ?

Show answer

Draw a horizontal leg and a vertical leg from the two points to make a right triangle whose hypotenuse is the segment. The legs have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$, so $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ and d is the square root. Swapping the points only changes the sign of each difference, and squaring makes the sign disappear, so the distance is the same.

14. **Challenge** A square hay field is 120 yards on each side. Ruth walks from one corner to the opposite corner straight across the diagonal instead of along two edges. How many yards shorter is her walk? Round to the nearest tenth of a yard.

Show answer

Diagonal = $\sqrt{120^2 + 120^2} = 120\sqrt{2} \approx 169.7$ yards; along the edges is 240 yards; she saves $240 - 120\sqrt{2} \approx 70.3$ yards

15. **Challenge** The midpoint of segment AB is $M(3, -1)$, and one endpoint is $A(7, 4)$. Find the other endpoint B .

Show answer

$\frac{7+x}{2} = 3$ gives $x = -1$; $\frac{4+y}{2} = -1$ gives $y = -6$. So $B = (-1, -6)$.

COMMON MISTAKES

- Adding when you should subtract. When the hypotenuse is known and a leg is missing, compute $c^2 - a^2$, not $c^2 + a^2$. A quick check: the hypotenuse must be the longest side.
- Testing the converse with the wrong side as c . Always square the two shorter sides and compare their sum with the square of the longest side.
- Sign errors with negative coordinates. Write $4 - (-2) = 6$ with parentheses, and remember that $(-8)^2 = 64$, not -64 .
- Subtracting in the midpoint formula or forgetting to divide by 2. The midpoint averages: add the coordinates, then divide by 2.

Make it a game • Corner Check Hunt. Hunt for rectangles: a book, a table, a window. Measure the two sides, calculate the diagonal with the Pythagorean theorem as an exact radical and a decimal, then measure it to check. Use the 3-4-5 string trick to test whether a corner is square, which uses the converse. Closest prediction wins the round.

RADICALS

In this unit you learn to work with square roots exactly instead of only as calculator decimals. You will write a root such as $\sqrt{72}$ in simplest radical form ($6\sqrt{2}$), handle roots of fractions and rationalize denominators, and add, subtract and multiply radical expressions using the same like-terms and distributive ideas you used for polynomials. Along the way you will see why each rule is true by going back to the meaning of a square root.

Next you will solve equations with the variable under a square root, learning why squaring both sides can produce false (extraneous) answers and why every answer must be checked. The unit ends with the Pythagorean theorem and its converse from Euclid's *Elements*, used to derive the distance and midpoint formulas on the coordinate plane, with problems from carpentry, ladders and maps.

These skills are needed right away in Unit 11, where quadratic equations often have answers such as $-2 \pm \sqrt{3}$, and again in Geometry, where the Pythagorean theorem, distance and midpoint appear constantly. Exact radical answers are part of the common language of all later mathematics.

PREREQUISITES

- Perfect squares to 20^2 and perfect cubes to 10^3 , and estimating square roots (Pre-Algebra)
- The exponent rules, especially the power of a product and power of a power (Lesson 7.1)
- Combining like terms and multiplying binomials, including the special products $(a + b)^2$ and $(a + b)(a - b)$ (Lessons 1.4, 8.3 and 8.4)
- Solving quadratic equations by factoring and the zero-product property (Lesson 9.6)
- Plotting points on the coordinate plane (Lesson 4.1) and the Pythagorean theorem (Pre-Algebra)

MASTERY CHECKLIST

- ☐ I can explain why $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and use it to write square roots in simplest radical form.
- ☐ I can simplify square roots of variable expressions with even exponents and find cube roots of perfect cubes.
- ☐ I can simplify the square root of a fraction and rationalize a one-term denominator, and I can explain why we do it.
- ☐ I can add and subtract radicals by simplifying first and combining like radicals.
- ☐ I can multiply radical expressions, including binomials and conjugates such as

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

- ☐ I can solve a radical equation by isolating the radical, squaring both sides and checking every answer.
- ☐ I can explain why squaring both sides can create extraneous solutions and reject them.
- ☐ I can use the Pythagorean theorem to find a missing side and its converse to test for a right angle, as in squaring a corner with 3-4-5.

- ☐ I can derive the distance formula from the Pythagorean theorem and use it.
- ☐ I can find the midpoint of a segment by averaging the coordinates.

LESSONS IN THIS UNIT

10.1 Simplifying Square Roots	2 sessions of 45 minutes
10.2 Quotients and Rationalizing Denominators	2 sessions of 45 minutes
10.3 Operations with Radical Expressions	2 sessions of 45 minutes
10.4 Solving Radical Equations	2 sessions of 45 minutes
10.5 The Pythagorean Theorem, Distance and Midpoint	2 sessions of 45 minutes

UNIT 10 TEST

You have 60 minutes. Show your work on every problem. Give exact answers in simplest radical form unless a problem asks you to round. A scientific calculator may be used only for decimal approximations when a problem asks for them. Check every solution of a radical equation in the original equation and state any extraneous solutions you reject. Assume all variables under radicals are nonnegative.

1. **5 pts**
Simplify $\sqrt{98}$.
2. **4 pts**
Find $\sqrt[3]{-125}$.
3. **6 pts**
Simplify $\sqrt{63x^6}$.

4. (6 pts)

Simplify $\frac{14}{\sqrt{7}}$.

5. (7 pts)

Simplify $\sqrt{\frac{11}{18}}$.

6. (6 pts)

Simplify $\sqrt{45} - \sqrt{20} + \sqrt{5}$.

7. (7 pts)

Multiply and simplify $(3 - \sqrt{2})(5 + 2\sqrt{2})$.

8. (5 pts)

Multiply $(\sqrt{10} + 3)(\sqrt{10} - 3)$.

9. (6 pts)

Solve $\sqrt{4x - 3} = 7$.

10. (8 pts)

Solve $x = \sqrt{5x + 6}$.

11. (7 pts)

Grace drops a pebble into an old stone well, and it takes 2.5 seconds to reach the water. Use $t = \frac{\sqrt{h}}{4}$, where h is the drop in feet, to find how far below the top of the well the water is.

12. (6 pts)

Find the exact distance between $(-1, -2)$ and $(5, 1)$ in simplest radical form.

13. (5 pts)

Find the midpoint of the segment joining $(-7, 2)$ and $(3, 10)$.

14. (8 pts)

A 12-foot ladder leans against the side of a house, with its foot 4 feet from the wall on level ground. How high up the wall does it reach? Give the exact answer in simplest radical form and an approximation to the nearest tenth of a foot.

15. (7 pts)

Samuel is building a deck. He marks 9 feet along the house wall and 12 feet along the edge of the deck frame, and the diagonal between the marks measures 15.5 feet. (a) Is the corner a right angle? (b) What should the diagonal measure, in feet, if the corner is square?

16. (7 pts)

Henry writes $\sqrt{16 + 9} = \sqrt{16} + \sqrt{9} = 7$. Explain his error, give the correct value, and state the rule he may have been thinking of.

ANSWER KEY

1. $\sqrt{49 \cdot 2} = 7\sqrt{2}$
2. -5 , because $(-5)^3 = -125$
3. $\sqrt{9 \cdot 7 \cdot x^6} = 3x^3\sqrt{7}$
4. $\frac{14\sqrt{7}}{7} = 2\sqrt{7}$
5. $\frac{\sqrt{11}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{22}}{6}$
6. $3\sqrt{5} - 2\sqrt{5} + \sqrt{5} = 2\sqrt{5}$
7. $15 + 6\sqrt{2} - 5\sqrt{2} - 2 \cdot 2 = 11 + \sqrt{2}$
8. $10 - 9 = 1$
9. $4x - 3 = 49$, so $x = 13$ (check: $\sqrt{49} = 7$)
10. $x^2 = 5x + 6$, $(x - 6)(x + 1) = 0$; $x = 6$ checks ($\sqrt{36} = 6$), but $x = -1$ is extraneous ($\sqrt{1} = 1 \neq -1$). So $x = 6$.
11. $2.5 = \frac{\sqrt{h}}{4}$, $\sqrt{h} = 10$, so $h = 100$ feet
12. $\sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$
13. $\left(\frac{-7 + 3}{2}, \frac{2 + 10}{2}\right) = (-2, 6)$
14. $h^2 = 144 - 16 = 128$, so $h = \sqrt{128} = 8\sqrt{2} \approx 11.3$ feet
15. (a) No. (b) $\sqrt{9^2 + 12^2} = \sqrt{225} = 15$ feet; since 15.5 is too long, the angle is a bit more than 90° .
16. A square root does not split over addition. The correct value is $\sqrt{16 + 9} = \sqrt{25} = 5$. The true rule is for products: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, which works because $(\sqrt{a} \cdot \sqrt{b})^2 = ab$; there is no such rule for sums.

UNIT 11

QUADRATIC FUNCTIONS AND EQUATIONS

In this unit you meet the parabola, the curve of a thrown ball, a stream of water from a fountain, and many arches and dishes. You will graph quadratic functions, find their vertex and axis of symmetry, and use the vertex to answer "greatest" and "least" questions, such as how to fence the largest garden or which price brings the most profit.

Then you will learn to solve any quadratic equation. In Unit 9 you solved some by factoring; now you add square roots, completing the square (the method al-Khwarizmi drew with squares and rectangles around 820), and the quadratic formula, which you will derive yourself. The discriminant tells you ahead of time how many real solutions to expect, and vertex form ties the graph and the algebra together.

This unit draws on factoring (Unit 9), simplifying radicals (Unit 10), special products (Lesson 8.4) and functions (Unit 4). It prepares you for Lesson 13.3, where you will tell quadratic patterns apart from linear and exponential ones, and for the deeper study of functions in Algebra II.

PREREQUISITES

- Graph points and functions on the coordinate plane and describe domain and range (Units 4 and 5).
- Expand $(x + d)^2$ and recognize perfect-square trinomials (Lessons 8.4 and 9.4).
- Factor trinomials and solve equations with the zero-product property (Lessons 9.2, 9.3 and 9.6).
- Simplify square roots and write answers in simplest radical form (Lessons 10.1 and 10.2).
- Substitute negative numbers and fractions into expressions correctly, using parentheses (Lesson 1.3).

MASTERY CHECKLIST

- ☐ I can graph $y = ax^2 + c$ and explain how a and c change the parabola.
- ☐ I can find the axis of symmetry $x = -\frac{b}{2a}$ and the vertex of $y = ax^2 + bx + c$ and explain why the formula works.
- ☐ I can use the vertex to solve maximum and minimum problems about area, height and profit.
- ☐ I can solve a quadratic equation by graphing and explain why it can have two, one or no real solutions.
- ☐ I can solve $x^2 = k$ and $(x - h)^2 = k$ with square roots, remembering $\pm$.
- ☐ I can solve a quadratic equation by completing the square, both when $a = 1$ and when $a \neq 1$.
- ☐ I can derive the quadratic formula and use it to give exact and decimal solutions.

- ☐ I can use the discriminant to tell how many real solutions an equation has and how many x -intercepts a parabola has.
- ☐ I can choose an efficient method for solving a quadratic equation.
- ☐ I can convert between standard form and vertex form and read the vertex from $y = a(x - h)^2 + k$.
- ☐ I can use quadratic models of falling and thrown objects and interpret the vertex and zeros in context.

LESSONS IN THIS UNIT

11.1 Graphing $y = ax^2$	2 sessions of 45 minutes
11.2 Graphing $y = ax^2 + bx + c$	2 sessions of 45 minutes
11.3 Solving Quadratic Equations by Graphing and Square Roots 2 sessions of 45 minutes	
11.4 Completing the Square	2 sessions of 45 minutes
11.5 The Quadratic Formula	2 sessions of 45 minutes
11.6 The Discriminant and Choosing a Method	2 sessions of 45 minutes
11.7 Vertex Form and Quadratic Models	2 sessions of 45 minutes

UNIT 11 TEST

You have 60 minutes. A scientific calculator is allowed, but show all of your algebra; a correct answer without work earns only partial credit. Give exact answers (in

simplest radical form) unless a problem asks you to round; when it does, round as directed. Include units in answers to word problems.

1. (6 pts)

For $y = -3x^2 + 4$, give the vertex and the maximum value.

2. (6 pts)

Find the axis of symmetry and the vertex of $y = x^2 + 8x + 10$.

3. (6 pts)

Solve $5x^2 = 245$.

4. (6 pts)

Solve $(x - 2)^2 = 12$. Give answers in simplest radical form.

5. (7 pts)

Solve $x^2 + 12x + 11 = 0$ by completing the square.

6. (7 pts)

Solve $x^2 - 4x - 3 = 0$ by completing the square. Give exact answers.

7. (7 pts)

Solve $2x^2 - 3x - 4 = 0$ using the quadratic formula. Give exact answers.

8. (6 pts)

Find the discriminant of $3x^2 + 6x + 3 = 0$ and tell how many real solutions the equation has.

9. (7 pts)

Write $y = 2x^2 + 8x + 1$ in vertex form and give the vertex.

10. (8 pts)

Samuel has 120 feet of fencing to make a rectangular pasture along a straight riverbank; the river side needs no fence. Let x be the length of each side perpendicular to the river. What dimensions give the largest area, and what is that area?

11. **8 pts**

Ruth drops an apple from a footbridge 81 feet above a pond, so its height after t seconds is $h = -16t^2 + 81$. How long does the apple take to reach the pond?

12. **8 pts**

Henry throws a ball upward from a height of 6 feet at 24 feet per second, so $h = -16t^2 + 24t + 6$. When does the ball hit the ground? Round to the nearest tenth of a second.

13. **8 pts**

Without solving, explain in two different ways how you know that the graph of $y = x^2 + 2x + 7$ never touches the x -axis.

14. **10 pts**

A garden fountain shoots a stream of water in a parabola. The water leaves the nozzle at ground level, at $(0, 0)$, and reaches its greatest height of 8 feet when it is 4 feet away horizontally. (a) Write the equation of the stream in vertex form, with x and y in feet. (b) How far from the nozzle does the water land?

ANSWER KEY

1. Vertex $(0, 4)$; maximum value 4
2. $x = -\frac{8}{2} = -4$; $y = 16 - 32 + 10 = -6$; vertex $(-4, -6)$
3. $x^2 = 49$, so $x = \pm 7$
4. $x - 2 = \pm\sqrt{12} = \pm 2\sqrt{3}$, so $x = 2 \pm 2\sqrt{3}$
5. $x^2 + 12x = -11$; $(x + 6)^2 = 25$; $x + 6 = \pm 5$, so $x = -1$ or $x = -11$
6. $x^2 - 4x = 3$; $(x - 2)^2 = 7$; $x = 2 \pm \sqrt{7}$
7. $b^2 - 4ac = 9 + 32 = 41$; $x = \frac{3 \pm \sqrt{41}}{4}$
8. $D = 36 - 36 = 0$; 1 real solution
9. $y = 2(x^2 + 4x + 4) - 8 + 1 = 2(x + 2)^2 - 7$; vertex $(-2, -7)$
10. $A = x(120 - 2x) = -2x^2 + 120x$; $x = -\frac{120}{2(-2)} = 30$; the other side is $120 - 60 = 60$. The pasture is 30 feet by 60 feet, with area 1,800 square feet.
11. $16t^2 = 81$, $t^2 = \frac{81}{16}$, $t = \frac{9}{4} = 2.25$ seconds (reject the negative root)
12. Divide $0 = -16t^2 + 24t + 6$ by -2 : $8t^2 - 12t - 3 = 0$; $t = \frac{12 \pm \sqrt{240}}{16} \approx 1.7$ seconds (reject the negative root)
13. The discriminant is $2^2 - 4(1)(7) = 4 - 28 = -24 < 0$, so $x^2 + 2x + 7 = 0$ has no real solutions. Also, the vertex is at $x = -1$, $y = 1 - 2 + 7 = 6$; the parabola opens up, so its lowest point is 6 units above the x -axis.
14. (a) Vertex $(4, 8)$: $0 = a(0 - 4)^2 + 8$, so $16a = -8$ and $a = -\frac{1}{2}$: $y = -\frac{1}{2}(x - 4)^2 + 8$.
(b) $(x - 4)^2 = 16$, so $x = 0$ or $x = 8$; the water lands 8 feet from the nozzle.

11.1 GRAPHING $y = ax^2$

OBJECTIVES

- Graph $y = ax^2$ and $y = ax^2 + c$ from a symmetric table of values.
- Identify the vertex, the axis of symmetry and the direction a parabola opens.
- Explain how the value of a changes the width of a parabola and how c shifts it up or down.
- Find the minimum or maximum value of $y = ax^2 + c$ and use it in simple real situations.

WATCH

Modifications of x^2 (2 of 2: Parabolas through the Origin)

Eddie Woo · 12:08

<https://www.youtube.com/watch?v=rqFOOzkPnTs>

WHEN TO WATCH

Before Example 1: how a changes the width and direction of $y = ax^2$.

Note: Eddie Woo teaches in Australia: he says 'turning point' for vertex.

BACKUP VIDEO

Introduction to Parabolas (3 of 3: Drawing & Shifting)

Eddie Woo · 7:36

<https://www.youtube.com/watch?v=tHYrg9w5gk4>

Before Example 2: shifting a parabola up or down with $+c$.

KEY TERMS

quadratic function

A function that can be written as $y = ax^2 + bx + c$ with $a \neq 0$; the highest power of x is 2.

parabola

The U-shaped curve that is the graph of a quadratic function. It opens up or down.

vertex

The turning point of a parabola: its lowest point if it opens up, its highest point if it opens down.

axis of symmetry

The vertical line through the vertex that cuts the parabola into two mirror-image halves.

minimum value / maximum value

The smallest (or largest) y -value a function ever reaches; for a parabola it is the y -coordinate of the vertex.

CORE IDEAS

A **quadratic function** is one whose highest power of x is 2, such as $y = x^2$ or $y = 3x^2 - 5x + 1$. Start with the simplest one, $y = x^2$. A table gives x : -3, -2, -1, 0, 1, 2, 3 and y : 9, 4, 1, 0, 1, 4, 9. Plot these points and join them with a smooth curve (not straight segments), and you get a U-shaped curve called a **parabola**.

Look at the table: $x = 2$ and $x = -2$ both give 4, and $x = 3$ and $x = -3$ both give 9. This happens because a number and its opposite have the same square: $(-x)^2 = x^2$. So every point (x, y) on the graph has a mirror partner $(-x, y)$, and the graph is symmetric about the y -axis. That mirror line, $x = 0$, is the **axis of symmetry**, and the turning point $(0, 0)$ is the **vertex**.

In $y = ax^2$, every y -value of $y = x^2$ is multiplied by a . If $a = 3$, the point $(2, 4)$ becomes $(2, 12)$: the curve climbs three times as fast, so it looks narrower. If $a = \frac{1}{2}$, the point $(2, 4)$ becomes $(2, 2)$ and the curve looks wider. If a is negative, every y -value changes sign, so the graph flips upside down across the x -axis and opens downward. In short: $a > 0$ opens up, $a < 0$ opens down, and the larger $|a|$ is, the narrower the parabola.

In $y = ax^2 + c$, we add c to every y -value of $y = ax^2$, so the whole graph slides up c units (down if c is negative). The shape and the axis $x = 0$ do not change; only the vertex moves, from $(0, 0)$ to $(0, c)$.

Why is the vertex the lowest or highest point? Since $x^2 \geq 0$ for every real x , when $a > 0$ we have $ax^2 \geq 0$, so $y = ax^2 + c \geq c$. The smallest possible y is c , and it happens only when $x = 0$. So the vertex is the lowest point and c is the **minimum**

value. When $a < 0$, $ax^2 \leq 0$, so $y \leq c$: the vertex is the highest point and c is the **maximum value**.

This also tells you the domain and range. You may square any real number, so the domain is all real numbers. The range is $y \geq c$ when the parabola opens up and $y \leq c$ when it opens down. In real situations, like the height of a falling stone, the domain is often only the part that makes sense (for example $t \geq 0$).

KEY FORMULAS

$$y = ax^2 + c : \quad \text{vertex } (0, c), \quad \text{axis of symmetry } x = 0$$

$a > 0$: opens up, minimum value c $a < 0$: opens down, maximum value c

$$(-x)^2 = x^2 \quad (\text{the reason for the symmetry})$$

Practice habit. Recite the perfect squares from 1^2 to 20^2 until they are automatic. Then build every parabola table with x -values that are symmetric about the axis (such as $-2, -1, 0, 1, 2$), so the y -values come in matching pairs and a mistake stands out at once.

WORKED EXAMPLES

Example 1. Graph $y = -2x^2 + 3$. State the vertex and the maximum value.

Show solution

1. Here $a = -2$ and $c = 3$. Since $a < 0$, the parabola opens down; since $|a| = 2 > 1$, it is narrower than $y = x^2$.
2. $x = 0$: $y = -2(0)^2 + 3 = 3$.
3. $x = 1$: $y = -2(1)^2 + 3 = 1$. $x = -1$: $y = -2(-1)^2 + 3 = -2(1) + 3 = 1$.
4. $x = 2$: $y = -2(2)^2 + 3 = -8 + 3 = -5$. $x = -2$ gives -5 as well.
5. Plot $(-2, -5)$, $(-1, 1)$, $(0, 3)$, $(1, 1)$, $(2, -5)$ and draw a smooth curve through them.
6. The vertex $(0, 3)$ is the highest point, so the maximum value is 3.

Answer: A parabola opening down with vertex $(0, 3)$ and axis $x = 0$, passing through $(\pm 1, 1)$ and $(\pm 2, -5)$. The maximum value is 3.

Example 2. Samuel drops a stone from a bridge 64 feet above a creek. Its height in feet after t seconds is $h = -16t^2 + 64$. Make a table for $t = 0, 0.5, 1, 1.5, 2$ and use it to find when the stone reaches the water.

Show solution

1. $t = 0: h = -16(0)^2 + 64 = 64.$
 2. $t = 0.5: h = -16(0.25) + 64 = -4 + 64 = 60.$
 3. $t = 1: h = -16(1) + 64 = 48.$
 4. $t = 1.5: h = -16(2.25) + 64 = -36 + 64 = 28.$
 5. $t = 2: h = -16(4) + 64 = -64 + 64 = 0.$
 6. The graph is the right half of a downward parabola with vertex $(0, 64)$; only $t \geq 0$ makes sense here.
 7. Height 0 means the water's surface, so the stone reaches the water at $t = 2$.
Check: $-16(2)^2 + 64 = 0.$
-

Answer: The stone reaches the water after 2 seconds.

PRACTICE

1. Easy Complete the table for $y = 2x^2$ with $x: -2, -1, 0, 1, 2$.

Show answer

$y: 8, 2, 0, 2, 8$

2. Easy What is the maximum value of $y = -x^2 + 5$?

Show answer

The parabola opens down with vertex $(0, 5)$, so the maximum value is 5.

3. Easy Give the vertex and the axis of symmetry of $y = 4x^2 - 9$.

Show answer

Vertex $(0, -9)$; axis of symmetry $x = 0$.

4. Easy List these parabolas in order from widest to narrowest: $y = 5x^2$, $y = 0.5x^2$, $y = -2x^2$.

Show answer

$y = 0.5x^2$, then $y = -2x^2$, then $y = 5x^2$ (compare $|a|: 0.5, 2, 5$).

5. Medium The point $(3, y)$ lies on the graph of $y = -3x^2 + 2$. Find y .

Show answer

$$y = -3(9) + 2 = -25$$

6. Medium Make a table for $y = x^2 - 4$ using $x = -3$ to 3 and graph it. At which x -values does the graph cross the x -axis?

Show answer

y : 5, 0, -3, -4, -3, 0, 5; it crosses the x -axis at $x = -2$ and $x = 2$.

7. Medium A parabola has the form $y = ax^2 + c$, has its vertex at $(0, 2)$, and passes through $(1, 5)$. Write its equation.

Show answer

$$c = 2 \text{ and } a(1)^2 + 2 = 5, \text{ so } a = 3: y = 3x^2 + 2$$

8. Medium Write the range of $y = -3x^2 + 6$ as an inequality.

Show answer

It opens down with maximum 6, so the range is $y \leq 6$.

9. Medium The Miller family builds a garden arch shaped like the parabola $y = -0.5x^2 + 8$, where x is the distance in feet from the center and y is the height in feet. How high is the arch 2 feet from the center?

Show answer

$$y = -0.5(4) + 8 = 6 \text{ feet}$$

10. **Hard** Ruth drops a ball from a window 100 feet above the ground. Its height after t seconds is $h = -16t^2 + 100$. How high is the ball after 1.5 seconds?

Show answer

$$h = -16(2.25) + 100 = -36 + 100 = 64 \text{ feet}$$

11. **Hard** A parabola of the form $y = ax^2 + c$ passes through $(2, 3)$ and $(4, 15)$. Find its equation.

Show answer

$$4a + c = 3 \text{ and } 16a + c = 15; \text{ subtracting gives } 12a = 12, \text{ so } a = 1 \text{ and } c = -1:$$
$$y = x^2 - 1$$

12. **Hard** The cross-section of a small satellite dish follows $y = \frac{1}{16}x^2$, where x and y are in inches and the vertex is the center of the dish. The dish is 16 inches across, so its edge is at $x = 8$. How deep is the dish?

Show answer

$$y = \frac{64}{16} = 4 \text{ inches}$$

13. **Challenge** A parabola $y = ax^2 + c$ has its vertex at $(0, 12)$ and crosses the x -axis at $x = 2$ and $x = -2$. Find its equation.

Show answer

$$c = 12 \text{ and } a(2)^2 + 12 = 0, \text{ so } 4a = -12 \text{ and } a = -3: y = -3x^2 + 12$$

14. **Challenge** Explain why the graph of $y = ax^2 + c$ is symmetric about the y -axis, and why its vertex is the lowest point when $a > 0$.

Show answer

Because $(-x)^2 = x^2$, the inputs x and $-x$ give the same y , so each point (x, y) has a mirror point $(-x, y)$. When $a > 0$, $ax^2 \geq 0$, so $y \geq c$, and $y = c$ only at $x = 0$; so $(0, c)$ is the lowest point.

COMMON MISTAKES

- Squaring a negative wrongly. In $y = -2x^2$ at $x = -3$, square first: $-2(-3)^2 = -2(9) = -18$, not $(-2 \cdot -3)^2 = 36$. Put a substituted negative in parentheses and square it before multiplying by a .
- Thinking a larger a makes a wider parabola. It is the opposite: a larger $|a|$ makes y grow faster, so the graph is narrower. Test $x = 1$ to see how high the curve climbs.
- Moving the vertex sideways for c . In $y = x^2 + 5$, the 5 moves the graph up 5, so the vertex is $(0, 5)$, not $(5, 0)$.
- Joining the points with straight segments or a pointed bottom. A parabola is a smooth curve that rounds off at the vertex.

Make it a game • Parabola Charades. Act out parabolas with your arms. The parent calls $y = x^2$ (arms up, wide U), $y = -3x^2$ (arms down, narrow), $y = \frac{1}{4}x^2 + 2$ (wide, shifted up). Strike the pose, then name the vertex and axis of symmetry. Then fill a symmetric table of values for one equation in 60 seconds. Charade correct and table correct earn two points.

11.2 GRAPHING $Y = AX^2 + BX + C$

OBJECTIVES

- Explain why the axis of symmetry of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.
- Find the vertex, axis of symmetry and y -intercept of a parabola and graph it from a short table.
- Solve maximum and minimum problems such as fencing a garden or setting a price for the greatest profit.

WATCH

Parabolas: Axis of Symmetry & Vertex

Eddie Woo · 8:03

<https://www.youtube.com/watch?v=2-GNTieFQsE>

WHEN TO WATCH

Before Example 1: the axis of symmetry $x = -b/2a$ and the vertex.

Note: Eddie Woo teaches in Australia: he says 'turning point' for vertex.

BACKUP VIDEO

Maximum/Minimum with Quadratics (1 of 2: Axis of symmetry)

Eddie Woo · 8:30

<https://www.youtube.com/watch?v=MfbnM7OC4cM>

Before Example 2: a maximum or minimum problem.

KEY TERMS

standard form of a quadratic function

$y = ax^2 + bx + c$ with $a \neq 0$.

y-intercept

The point where the graph crosses the y -axis; for $y = ax^2 + bx + c$ it is $(0, c)$.

mirror point

The point on the other side of the axis of symmetry at the same height as a given point and the same distance from the axis.

optimize

To find the greatest or least possible value of a quantity, such as the largest area or the highest profit.

CORE IDEAS

When $b \neq 0$, the parabola $y = ax^2 + bx + c$ is no longer centered on the y -axis. It still opens up when $a > 0$ and down when $a < 0$, and it is still symmetric about a vertical line through its vertex. The whole job is to find that line.

Here is why the axis is $x = -\frac{b}{2a}$. Put $x = 0$: $y = c$, so $(0, c)$ is on the graph. Which other point has the same height c ? Solve $ax^2 + bx + c = c$, which is $ax^2 + bx = 0$, or $x(ax + b) = 0$. By the zero-product property, $x = 0$ or $x = -\frac{b}{a}$. These two points sit at the same height, so they are mirror images, and the axis of symmetry runs exactly halfway between them: $x = \frac{0 + (-\frac{b}{a})}{2} = -\frac{b}{2a}$.

The same halfway idea works with x -intercepts. For $y = x^2 - 6x + 8 = (x - 2)(x - 4)$, the x -intercepts are 2 and 4, and the axis is halfway, at $x = 3$. The formula agrees: $-\frac{-6}{2(1)} = 3$. The formula is better because it works even when the parabola never crosses the x -axis.

To graph $y = ax^2 + bx + c$: (1) find the axis $x = -\frac{b}{2a}$; (2) substitute that x to get the y -coordinate of the vertex; (3) plot the y -intercept $(0, c)$ and its mirror point on the other side of the axis; (4) add one or two more points from a table, reflect them, and draw a smooth curve.

The vertex answers "greatest" and "least" questions. If $a < 0$ the vertex is the highest point, so its y -value is the **maximum**; if $a > 0$ it is the lowest point, the **minimum**. The x -value of the vertex tells you *where* the best value happens (the best width, the best price, the best time); the y -value tells you *what* the best value is. Read the question carefully to see which one it asks for.

KEY FORMULAS

$$\text{axis of symmetry: } x = -\frac{b}{2a}$$

$$\text{vertex: } \left(-\frac{b}{2a}, a \left(-\frac{b}{2a} \right)^2 + b \left(-\frac{b}{2a} \right) + c \right)$$

$$y\text{-intercept: } (0, c)$$

Practice habit. Write one step per line: first $x = -\frac{b}{2a}$, then the substitution in parentheses, then the vertex as an ordered pair. Check the vertex by testing one point on each side of it: the two y -values should match when the points are the same distance from the axis.

WORKED EXAMPLES

Example 1. Graph $y = x^2 - 4x + 3$. Give the vertex, the axis of symmetry and the minimum value.

Show solution

1. $a = 1, b = -4, c = 3$. Since $a > 0$, it opens up.
2. Axis: $x = -\frac{-4}{2(1)} = 2$.
3. Vertex y : $y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$, so the vertex is $(2, -1)$.
4. y -intercept $(0, 3)$; its mirror point is 2 units on the other side of $x = 2$, at $(4, 3)$.
5. $x = 1$: $y = 1 - 4 + 3 = 0$, so $(1, 0)$; its mirror point is $(3, 0)$.
6. Plot $(0, 3), (1, 0), (2, -1), (3, 0), (4, 3)$ and draw a smooth U through them.
7. The vertex is the lowest point, so the minimum value is -1 .

Answer: Parabola opening up, vertex $(2, -1)$, axis $x = 2$, y -intercept 3, x -intercepts 1 and 3; minimum value -1 .

Example 2. Grace has 60 feet of fencing for a rectangular garden along the side of a barn. The barn wall is one side, so she fences only the other three sides. Let x be the length of each side perpendicular to the barn. What dimensions give the largest area, and what is that area?

Show solution

1. The side parallel to the barn uses the rest of the fence: $60 - 2x$ feet.
2. Area: $A = x(60 - 2x) = -2x^2 + 60x$.
3. $a = -2 < 0$, so the vertex is a maximum.
4. Axis: $x = -\frac{60}{2(-2)} = 15$.
5. Side parallel to the barn: $60 - 2(15) = 30$ feet.
6. Area: $A = 15 \times 30 = 450$ square feet.
7. Check with a nearby width: $x = 14$ gives $14 \times 32 = 448$, and $x = 16$ gives $16 \times 28 = 448$, both smaller.

Answer: 15 feet by 30 feet (the 30-foot side along the barn), for a largest area of 450 square feet.

PRACTICE

1. Easy Find the axis of symmetry of $y = x^2 + 6x + 2$.

Show answer

$$x = -\frac{6}{2(1)} = -3$$

2. Easy Find the axis of symmetry of $y = 2x^2 - 8x + 1$.

Show answer

$$x = -\frac{-8}{2(2)} = 2$$

3. Easy What is the y -intercept of $y = 3x^2 - 5x + 7$?

Show answer

7 (the point $(0, 7)$)

4. Easy Find the vertex of $y = x^2 + 2x - 8$.

Show answer

$$x = -1; y = 1 - 2 - 8 = -9; \text{vertex } (-1, -9)$$

5. Medium Find the vertex of $y = -x^2 + 6x - 5$.

Show answer

$$x = -\frac{6}{2(-1)} = 3; y = -9 + 18 - 5 = 4; \text{vertex } (3, 4)$$

6. Medium Find the vertex of $y = 2x^2 + 12x + 13$.

Show answer

$$x = -\frac{12}{4} = -3; y = 2(9) - 36 + 13 = -5; \text{vertex } (-3, -5)$$

7. Medium Graph $y = x^2 - 2x - 3$ and give its vertex.

Show answer

Vertex $(1, -4)$; parabola opening up through $(0, -3)$ and its mirror $(2, -3)$, with x -intercepts -1 and 3 .

8. Medium Find the minimum value of $y = 3x^2 - 6x + 10$.

Show answer

$$x = 1; y = 3 - 6 + 10 = 7; \text{minimum value } 7$$

9. Medium Henry throws a ball upward. Its height in feet after t seconds is $h = -16t^2 + 48t + 4$. When does the ball reach its highest point, and how high is it then?

Show answer

$$t = -\frac{48}{2(-16)} = 1.5 \text{ seconds}; h = -16(2.25) + 72 + 4 = 40 \text{ feet}$$

10. **Hard** A church youth group sells muffins at a bake sale. If they charge x dollars per muffin, their profit is $P = -40x^2 + 200x - 100$ dollars. What price gives the greatest profit, and what is that profit?

Show answer

$$x = -\frac{200}{2(-40)} = 2.5; P = -40(6.25) + 500 - 100 = 150. \text{ Charge } \$2.50 \text{ for a profit of } \$150.$$

11. **Hard** Samuel has 80 feet of fencing to make a rectangular pen for his goats, fenced on all four sides. What dimensions give the greatest area, and what is that area?

Show answer

Width x , length $40 - x$; $A = -x^2 + 40x$; $x = 20$. The pen is 20 feet by 20 feet with area 400 square feet.

12. **Hard** Find the vertex of $y = -\frac{1}{2}x^2 + 4x - 1$.

Show answer

$$x = -\frac{4}{2(-\frac{1}{2})} = 4; y = -8 + 16 - 1 = 7; \text{vertex } (4, 7)$$

13. **Challenge** The parabola $y = x^2 + bx + 5$ has axis of symmetry $x = 3$. Find b and the vertex.

Show answer

$$-\frac{b}{2} = 3, \text{ so } b = -6; y = 9 - 18 + 5 = -4; \text{vertex } (3, -4)$$

14. **Challenge** Explain why the axis of symmetry of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$. (Hint: find the two points on the graph with height c .)

Show answer

Setting $ax^2 + bx + c = c$ gives $x(ax + b) = 0$, so $x = 0$ or $x = -\frac{b}{a}$. These two points have the same height, so they are mirror images, and the axis lies halfway between them, at $x = -\frac{b}{2a}$.

COMMON MISTAKES

- Dropping the negative sign in $-\frac{b}{2a}$. When b is negative, $-b$ is positive: for $y = x^2 - 4x + 3$, the axis is $x = 2$, not -2 . Write $-\frac{(-4)}{2(1)}$ with parentheses.
- Forgetting the 2 in $2a$. The axis is $-\frac{b}{2a}$, not $-\frac{b}{a}$; $-\frac{b}{a}$ is the x -coordinate of the mirror point of the y -intercept, not the axis.
- Giving only the x -value when the question asks for the maximum or minimum value. The best value is the y -coordinate of the vertex; the x -coordinate tells where it happens.
- Substituting a negative x without parentheses: $(-3)^2 = 9$, but $-3^2 = -9$.

Make it a game • Vertex Duel. The parent calls quadratics such as $y = x^2 - 6x + 5$ or $y = -2x^2 + 8x + 1$. Before computing, both players guess where the vertex lies. Then use $x = -b/(2a)$ to find the axis of symmetry, the vertex, the y-intercept and the mirror point. Closest guess scores one point, correct work scores two. First to ten wins.

11.3 SOLVING QUADRATIC EQUATIONS BY GRAPHING AND SQUARE ROOTS

OBJECTIVES

- Explain why the solutions of $ax^2 + bx + c = 0$ are the x -intercepts of $y = ax^2 + bx + c$, and read them from a graph or table.
- Explain why a quadratic equation can have two, one or no real solutions.
- Solve equations of the form $x^2 = k$ and $(x - h)^2 = k$ using $\pm$ square roots, in simplest radical form when needed.
- Solve falling-object problems using $h = -16t^2 + s$.

WATCH

Solving Simple Quadratic Equations

Eddie Woo · 10:54

https://www.youtube.com/watch?v=ETD_hTyYOwc

WHEN TO WATCH

Before Example 1: solving $x^2 = k$ with $\pm$ square roots.

BACKUP VIDEO

Graphing Quadratics Equations (3 of 6: x & y-intercepts)

Eddie Woo · 7:57

https://www.youtube.com/watch?v=X_2qm7n76lw

Before the Core ideas: solutions as the x -intercepts of the graph.

KEY TERMS

quadratic equation

An equation that can be written as $ax^2 + bx + c = 0$ with $a \neq 0$.

x-intercept

A point where a graph crosses or touches the x -axis; its y -coordinate is 0.

root (zero)

A solution of $ax^2 + bx + c = 0$; it is also an x -value where the graph of $y = ax^2 + bx + c$ has height 0.

plus-or-minus ($\pm$)

A short way to write two numbers at once: ± 5 means 5 or -5 .

CORE IDEAS

Solving $ax^2 + bx + c = 0$ means finding the x -values that make the expression equal 0. On the graph of $y = ax^2 + bx + c$, those are exactly the points where the height y is 0, which are the x -intercepts. So you can solve a quadratic equation by graphing (or by looking in a table for $y = 0$). Graphing is quick and gives a picture, but unless the intercepts land on grid lines it only gives estimates.

A parabola can cross the x -axis twice, touch it once at its vertex, or miss it completely. So a quadratic equation has two real solutions, one real solution, or no real solution. For example, $y = x^2 - 4$ (vertex $(0, -4)$, opening up) crosses twice; $y = x^2$ touches once; $y = x^2 + 4$ (vertex $(0, 4)$, opening up) never reaches the axis.

The square-root method. To solve $x^2 = 25$, ask what numbers have a square of 25. Both $5^2 = 25$ and $(-5)^2 = 25$, so $x = \pm 5$. In general, if $k > 0$, then $x^2 = k$ has two solutions, $x = \sqrt{k}$ and $x = -\sqrt{k}$, because a number and its opposite have the same square. If $k = 0$, the only solution is $x = 0$. If $k < 0$, there is no real solution, because the square of a real number is never negative.

Isolate the square first. In $3x^2 - 12 = 15$, add 12 and divide by 3 to get $x^2 = 9$, and only then take square roots: $x = \pm 3$. The same idea works when a whole binomial is squared. In $(x - 2)^2 = 49$, the quantity $x - 2$ must be 7 or -7 , so $x - 2 = \pm 7$ and $x = 2 \pm 7$, giving $x = 9$ or $x = -5$. In general, $(x - h)^2 = k$ with $k \geq 0$ gives $x = h \pm \sqrt{k}$. Write any irrational answer in simplest radical form, and give a decimal only when the problem asks for one.

Falling objects. An object dropped (not thrown) from a height of s feet has height $h = -16t^2 + s$ after t seconds. It reaches the ground when $h = 0$, so solve $-16t^2 + s = 0$, or $t^2 = \frac{s}{16}$. The negative square root is a real solution of the equation, but a negative time makes no sense here, so we reject it.

KEY FORMULAS

$$x^2 = k \ (k > 0) \Rightarrow x = \pm\sqrt{k}$$

$$(x - h)^2 = k \ (k \geq 0) \Rightarrow x = h \pm \sqrt{k}$$

$$\text{dropped object: } h = -16t^2 + s \quad (\text{feet, seconds})$$

Practice habit. Every time you take a square root to solve an equation, write $\pm$ in the very same step, then check both answers by substituting them into the original equation.

WORKED EXAMPLES

Example 1. Solve $2(x - 3)^2 = 50$.

Show solution

1. Divide both sides by 2: $(x - 3)^2 = 25$.
2. Take square roots, with $\pm$: $x - 3 = \pm 5$.
3. Add 3: $x = 3 + 5 = 8$ or $x = 3 - 5 = -2$.
4. Check $x = 8$: $2(8 - 3)^2 = 2(25) = 50$. Correct.
5. Check $x = -2$: $2(-2 - 3)^2 = 2(-5)^2 = 2(25) = 50$. Correct.

Answer: $x = -2$ or $x = 8$

Example 2. A hiker drops a pebble from the edge of a cliff 144 feet above a river. Its height after t seconds is $h = -16t^2 + 144$. How long does the pebble take to reach the river?

Show solution

1. The pebble reaches the river when $h = 0$: $-16t^2 + 144 = 0$.
2. Add $16t^2$ to both sides: $144 = 16t^2$.
3. Divide by 16: $t^2 = 9$.
4. Take square roots: $t = \pm 3$.
5. Time cannot be negative, so reject $t = -3$; $t = 3$.
6. Check: $-16(3)^2 + 144 = -144 + 144 = 0$.

Answer: 3 seconds

PRACTICE

1. Easy Solve $x^2 = 49$.

Show answer

$$x = \pm 7$$

2. Easy Solve $x^2 - 81 = 0$.

Show answer

$$x^2 = 81, \text{ so } x = \pm 9$$

3. Easy Solve $x^2 = -16$.

Show answer

No real solution; no real number squared is negative.

4. Easy A table for $y = x^2 + 2x - 3$ gives x: -4, -3, -2, -1, 0, 1, 2 and y: 5, 0, -3, -4, -3, 0, 5. What are the solutions of $x^2 + 2x - 3 = 0$?

Show answer

The y -values are 0 at $x = -3$ and $x = 1$.

5. Medium Solve $4x^2 = 100$.

Show answer

$$x^2 = 25, \text{ so } x = \pm 5$$

6. Medium Solve $3x^2 - 7 = 41$.

Show answer

$$3x^2 = 48, x^2 = 16, \text{ so } x = \pm 4$$

7. Medium Solve $x^2 = 18$. Give the answers in simplest radical form.

Show answer

$$x = \pm\sqrt{18} = \pm 3\sqrt{2}$$

8. Medium Solve $(x + 4)^2 = 36$.

Show answer

$$x + 4 = \pm 6, \text{ so } x = 2 \text{ or } x = -10$$

9. Medium Solve $(x - 5)^2 = 7$. Give exact answers.

Show answer

$$x - 5 = \pm\sqrt{7}, \text{ so } x = 5 \pm \sqrt{7}$$

10. **Hard** The parabola $y = x^2 - 4x + 7$ opens up and has its vertex at $(2, 3)$. How many real solutions does $x^2 - 4x + 7 = 0$ have?

Show answer

0. The lowest point of the graph is 3 units above the x -axis, so the graph never reaches the axis.

11. **Hard** The Miller family wants a square vegetable plot with an area of 200 square feet. How long should each side be, to the nearest tenth of a foot?

Show answer

$$s^2 = 200, s = \sqrt{200} = 10\sqrt{2} \approx 14.1 \text{ feet (a negative length is rejected)}$$

12. **Hard** A carpenter's hammer slips off a roof 36 feet above the ground. Its height after t seconds is $h = -16t^2 + 36$. How long until it hits the ground?

Show answer

$$16t^2 = 36, t^2 = 2.25, t = 1.5 \text{ seconds}$$

13. **Hard** Grace drops a water balloon from a window 50 feet above the lawn, so $h = -16t^2 + 50$. Using a calculator, find how long it takes to reach the lawn, to the nearest tenth of a second.

Show answer

$$t^2 = \frac{50}{16} = 3.125, t = \sqrt{3.125} \approx 1.8 \text{ seconds}$$

14. **Challenge** Solve $2(3x - 1)^2 - 5 = 45$.

Show answer

$$(3x - 1)^2 = 25, \text{ so } 3x - 1 = \pm 5: 3x = 6 \text{ or } 3x = -4, \text{ giving } x = 2 \text{ or } x = -\frac{4}{3}$$

15. **Challenge** Explain why $x^2 = 9$ has two real solutions but $x^2 = -9$ has none. Describe what each looks like on a graph.

Show answer

Both 3^2 and $(-3)^2$ equal 9, so $x^2 = 9$ has the solutions 3 and -3 . But every real number squared is 0 or positive, so nothing squares to -9 . On a graph, $y = x^2 - 9$ crosses the x -axis at ± 3 , while $y = x^2 + 9$ has its lowest point at $(0, 9)$ and never reaches the axis.

COMMON MISTAKES

- Forgetting the negative solution. $x^2 = 49$ means $x = 7$ or $x = -7$. Write $\pm$ the moment you take the square root.
- Taking the square root before isolating the square. In $3x^2 - 7 = 41$, you cannot take $\sqrt{41}$; first get $x^2 = 16$ alone.
- Getting the sign of h wrong. $(x + 4)^2 = 36$ gives $x = -4 \pm 6$, not 4 ± 6 ; subtract 4 from both sides as a separate step.
- Keeping a negative time or length. In a real situation, check whether each solution makes sense and reject the ones that do not.

Make it a game • Slide the Line. Draw $y = x^2 - 4$ on graph paper. Lay a pencil flat as the horizontal line $y = k$ and slide it up and down. Call how many solutions $x^2 - 4 = k$ has: two, one or none. Find the k values where the number changes. Then solve $x^2 = 25$ by square roots, remembering the plus-or-minus.

11.4 COMPLETING THE SQUARE

OBJECTIVES

- Explain, with the square picture, why adding $\left(\frac{b}{2}\right)^2$ turns $x^2 + bx$ into a perfect-square trinomial.
- Solve quadratic equations with $a = 1$ by completing the square, including odd values of b .
- Solve quadratic equations with $a \neq 1$ by first dividing every term by a .
- Use completing the square to solve area and motion problems.

WATCH

Quadratic Equations (2 of : Example of completing the square)

Eddie Woo · 6:57

https://www.youtube.com/watch?v=ql_f_ok57NQ

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Quadratic Equations (3 of : Visually Completing the Square)

Eddie Woo · 8:24

<https://www.youtube.com/watch?v=VEY1oKJ4p-A>

Before the Core ideas: the square picture that explains completing the square.

KEY TERMS

perfect-square trinomial

A trinomial that equals the square of a binomial, such as

$$x^2 + 6x + 9 = (x + 3)^2.$$

completing the square

Adding the number $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ so that it becomes a perfect-square trinomial $\left(x + \frac{b}{2}\right)^2$.

al-jabr

Arabic for "restoring"; from the title of al-Khwarizmi's book, it gave us the word algebra.

CORE IDEAS

Recall the special product $(x + d)^2 = x^2 + 2dx + d^2$. Compare it with $x^2 + bx$: the middle coefficient b matches $2d$, so $d = \frac{b}{2}$. The missing piece is $d^2 = \left(\frac{b}{2}\right)^2$.

. So: **take half of b , square it, and add it.** For example, $x^2 + 6x$: half of 6 is 3, $3^2 = 9$, and $x^2 + 6x + 9 = (x + 3)^2$.

Here is the picture. Draw a square with side x (area x^2). Split the bx into two equal rectangles, each x by $\frac{b}{2}$, and lay one along the right side of the square and one along the bottom. The shape is almost a big square with side $x + \frac{b}{2}$, but one small corner is empty. That corner is a square $\frac{b}{2}$ by $\frac{b}{2}$, with area $\left(\frac{b}{2}\right)^2$. Filling in that corner literally completes the square. This is why we add $\left(\frac{b}{2}\right)^2$, not b^2 or $\frac{b}{2}$.

To solve an equation, add the same number to **both** sides so the balance is kept. For $x^2 + 6x = 7$: add 9 to both sides, $x^2 + 6x + 9 = 16$, so $(x + 3)^2 = 16$. Now use square roots: $x + 3 = \pm 4$, so $x = 1$ or $x = -7$. Completing the square turns any quadratic into the $(x - h)^2 = k$ type you solved in Lesson 11.3.

Steps when $a = 1$: (1) move the constant term to the right side; (2) add $\left(\frac{b}{2}\right)^2$ to both sides; (3) write the left side as $\left(x + \frac{b}{2}\right)^2$; (4) take square roots with $\pm$; (5) solve and check. If b is odd, half of b is a fraction, and that is fine: for $x^2 + 3x = 10$, add $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ to get $\left(x + \frac{3}{2}\right)^2 = \frac{49}{4}$, so $x + \frac{3}{2} = \pm \frac{7}{2}$, and $x = 2$ or $x = -5$.

When $a \neq 1$, the picture needs the x^2 part to be a single square, so first divide **every** term by a . For $2x^2 + 8x - 10 = 0$, dividing by 2 gives $x^2 + 4x - 5 = 0$. Then $x^2 + 4x = 5$, add 4 to both sides, $(x + 2)^2 = 9$, $x + 2 = \pm 3$, so $x = 1$ or $x = -5$.

KEY FORMULAS

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

$$x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$$

From history. Around 820, al-Khwarizmi of Baghdad wrote a book on al-jabr and al-muqabala, and one of his examples was "a square and ten roots equal thirty-nine," which we write $x^2 + 10x = 39$. He drew a square for x^2 , split the ten roots into two rectangles of 5 by x , and noticed a missing corner square of $5 \times 5 = 25$. Adding 25 makes a whole square of area $39 + 25 = 64$, so its side $x + 5$ is 8 and $x = 3$. He worked only with positive answers, so he did not list the second solution, -13 .

Euclid connection. Euclid's Elements, Book II, Proposition 4, shows with a square divided into two squares and two equal rectangles that the square on a whole line equals the squares on its two parts plus twice the rectangle formed by the parts, which is $(a + b)^2 = a^2 + 2ab + b^2$; completing the square uses this same picture in reverse.

Practice habit. Say it as you work: "half it, square it, add it to both sides." Then check that your trinomial really is a perfect square by multiplying $\left(x + \frac{b}{2}\right)^2$ back out.

WORKED EXAMPLES

Example 1. Solve al-Khwarizmi's equation $x^2 + 10x = 39$ by completing the square.

Show solution

1. Half of 10 is 5, and $5^2 = 25$.
2. Add 25 to both sides: $x^2 + 10x + 25 = 39 + 25 = 64$.
3. Write the left side as a square: $(x + 5)^2 = 64$.
4. Take square roots: $x + 5 = \pm 8$.
5. Subtract 5: $x = 3$ or $x = -13$.
6. Check $x = 3$: $9 + 30 = 39$. Correct.
7. Check $x = -13$: $169 - 130 = 39$. Correct.

Answer: $x = -13$ or $x = 3$

Example 2. Samuel is building a rectangular chicken run. Its length is 2 feet more than twice its width, and its area is 84 square feet. Use completing the square to find the width and length.

Show solution

1. Let w be the width; the length is $2w + 2$. Area: $w(2w + 2) = 84$, so $2w^2 + 2w = 84$.
2. Divide every term by 2: $w^2 + w = 42$.
3. Half of 1 is $\frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Add it to both sides: $w^2 + w + \frac{1}{4} = \frac{168}{4} + \frac{1}{4} = \frac{169}{4}$.
4. Write as a square: $\left(w + \frac{1}{2}\right)^2 = \frac{169}{4}$.
5. Take square roots: $w + \frac{1}{2} = \pm \frac{13}{2}$, so $w = 6$ or $w = -7$.
6. A width cannot be negative, so $w = 6$ feet; the length is $2(6) + 2 = 14$ feet.
7. Check: $6 \times 14 = 84$ square feet, and 14 is 2 more than twice 6.

Answer: The run is 6 feet wide and 14 feet long.

PRACTICE

1. Easy What number must be added to $x^2 + 12x$ to make a perfect-square trinomial?

Show answer

$$\left(\frac{12}{2}\right)^2 = 36, \text{ giving } (x + 6)^2$$

2. Easy What number must be added to $x^2 - 14x$ to make a perfect-square trinomial?

Show answer

$$(-7)^2 = 49, \text{ giving } (x - 7)^2$$

3. Easy What number must be added to $x^2 + 5x$ to make a perfect-square trinomial?

Show answer

$$\left(\frac{5}{2}\right)^2 = \frac{25}{4}, \text{ giving } \left(x + \frac{5}{2}\right)^2$$

4. Easy Solve $x^2 + 4x = 12$ by completing the square.

Show answer

$$(x + 2)^2 = 16, x + 2 = \pm 4, \text{ so } x = 2 \text{ or } x = -6$$

5. Medium Solve $x^2 - 6x = 16$ by completing the square.

Show answer

$$(x - 3)^2 = 25, x - 3 = \pm 5, \text{ so } x = 8 \text{ or } x = -2$$

6. Medium Solve $x^2 + 8x + 3 = 0$ by completing the square. Give exact answers.

Show answer

$$x^2 + 8x = -3, (x + 4)^2 = 13, \text{ so } x = -4 \pm \sqrt{13}$$

7. Medium Solve $x^2 - 2x - 7 = 0$ by completing the square. Give answers in simplest radical form.

Show answer

$$x^2 - 2x = 7, (x - 1)^2 = 8, \text{ so } x = 1 \pm 2\sqrt{2}$$

8. Medium Solve $x^2 + 3x - 10 = 0$ by completing the square.

Show answer

$$x^2 + 3x = 10, \left(x + \frac{3}{2}\right)^2 = \frac{49}{4}, x = -\frac{3}{2} \pm \frac{7}{2}, \text{ so } x = 2 \text{ or } x = -5$$

9. Medium Solve $3x^2 - 18x + 15 = 0$ by completing the square.

Show answer

$$\text{Divide by 3: } x^2 - 6x = -5, (x - 3)^2 = 4, \text{ so } x = 1 \text{ or } x = 5$$

10. **Hard** Solve $2x^2 + 4x - 9 = 0$ by completing the square. Give answers in simplest radical form.

Show answer

$$x^2 + 2x = \frac{9}{2}, (x + 1)^2 = \frac{11}{2}, x = -1 \pm \sqrt{\frac{11}{2}} = -1 \pm \frac{\sqrt{22}}{2}$$

11. **Hard** Ruth's rectangular herb garden is 6 feet longer than it is wide and has an area of 40 square feet. Use completing the square to find its width and length.

Show answer

$$w^2 + 6w = 40, (w + 3)^2 = 49, w = 4 \text{ (reject } -10\text{)}. \text{ Width 4 feet, length 10 feet.}$$

12. **Hard** Henry kicks a ball straight up from the ground. Its height in feet after t seconds is $h = -16t^2 + 64t$. Use completing the square to find when the ball is 48 feet high.

Show answer

$$-16t^2 + 64t = 48; \text{ divide by } -16: t^2 - 4t = -3, (t - 2)^2 = 1, \text{ so } t = 1 \text{ or } t = 3. \text{ It is 48 feet high at 1 second (going up) and at 3 seconds (coming down).}$$

13. **Hard** A rectangular banner for the church picnic is 3 feet longer than it is wide, and its area is 20 square feet. Use completing the square and a calculator to find the width to the nearest tenth of a foot.

Show answer

$$w^2 + 3w = 20, \left(w + \frac{3}{2}\right)^2 = \frac{89}{4}, w = \frac{-3 + \sqrt{89}}{2} \approx 3.2 \text{ feet}$$

14. **Challenge** Solve $4x^2 - 12x + 1 = 0$ by completing the square. Give exact answers.

Show answer

Divide by 4: $x^2 - 3x = -\frac{1}{4}$; add $\frac{9}{4}$: $\left(x - \frac{3}{2}\right)^2 = 2$, so $x = \frac{3}{2} \pm \sqrt{2}$

15. **Challenge** Using the square-and-rectangles picture, explain why we add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$, and not b^2 or $\frac{b}{2}$.

Show answer

Draw an x -by- x square and split bx into two rectangles, each x by $\frac{b}{2}$, placed along two sides. The only gap left in the big square of side $x + \frac{b}{2}$ is a corner square that is $\frac{b}{2}$ by $\frac{b}{2}$, so its area $\left(\frac{b}{2}\right)^2$ is exactly what must be added.

COMMON MISTAKES

- Adding $\left(\frac{b}{2}\right)^2$ to only one side. That breaks the balance; whatever you add on the left, add on the right.
- Completing the square before dividing by a . In $2x^2 + 4x - 9 = 0$, divide every term by 2 first, so the x^2 term has coefficient 1.
- Adding b^2 or $\frac{b}{2}$ instead of $\left(\frac{b}{2}\right)^2$. Halve first, then square.
- Forgetting the $\pm$ when taking the square root, which loses one of the two solutions.

Make it a game • Tile Puzzle. Cut paper tiles: one big square for x^2 , strips for x , and small unit squares. For $x^2 + 6x$, arrange the tiles into a square with a corner missing and figure out how many unit squares complete it. Write $(x + 3)^2 = x^2 + 6x + 9$. Then use the same idea to solve $x^2 + 6x = 7$, and try one with odd b .

11.5 THE QUADRATIC FORMULA

OBJECTIVES

- Derive the quadratic formula by completing the square on $ax^2 + bx + c = 0$.
- Identify a , b and c after writing an equation in standard form, and solve it with the quadratic formula.
- Give solutions exactly in simplest radical form and as decimal approximations when asked.
- Use the quadratic formula to solve area, motion and business problems.

WATCH

The Quadratic Formula (Derivation)

Eddie Woo · 9:37

<https://www.youtube.com/watch?v=38TysPMjU7E>

WHEN TO WATCH

Before Example 1: identifying a , b and c , and where the formula comes from.

BACKUP VIDEO

Quadratic Equations (4 of : Deriving the quadratic formula)

Eddie Woo · 7:22

<https://www.youtube.com/watch?v=je0U5ktiA78>

After Example 2: a second derivation by completing the square.

KEY TERMS

standard form of a quadratic equation

$ax^2 + bx + c = 0$ with $a \neq 0$: everything on one side, 0 on the other, terms in order of decreasing power.

quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives the solutions of every quadratic equation $ax^2 + bx + c = 0$.

coefficient

The number multiplying a variable term; in $3x^2 - 5x + 1$, the coefficients are $a = 3$ and $b = -5$, and the constant is $c = 1$.

CORE IDEAS

Completing the square works on every quadratic equation, so let us do it once and for all with letters instead of numbers. The result is a formula you can use on any equation. Start with $ax^2 + bx + c = 0$, where $a \neq 0$.

Step 1: subtract c from both sides: $ax^2 + bx = -c$. Step 2: divide every term by a so the x^2 term has coefficient 1: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Step 3: half of $\frac{b}{a}$ is $\frac{b}{2a}$, and its square is $\frac{b^2}{4a^2}$. Add it to both sides: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$. Step 4: the left side is now a perfect square, $\left(x + \frac{b}{2a}\right)^2$. On the right, use the common denominator $4a^2$: $-\frac{c}{a} = -\frac{4ac}{4a^2}$, so the right side is $\frac{b^2 - 4ac}{4a^2}$. Now $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$.

Step 5: take square roots of both sides, with $\pm$: $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. (The square root of $4a^2$ is $2a$ or $-2a$, but because of the $\pm$ sign we get the same two numbers either way.) Step 6: subtract $\frac{b}{2a}$: $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, and over one denominator this is the **quadratic formula**: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Notice that $-\frac{b}{2a}$, the axis of symmetry from Lesson 11.2, sits right inside the formula. The two solutions are the same distance to the left and right of the axis, exactly as the symmetry of the parabola says they should be.

Using the formula: (1) write the equation in standard form, with 0 on one side; (2) write down a , b and c with their signs; (3) substitute, using parentheses around negatives; (4) simplify the part under the square root first; (5) simplify the radical and reduce the fraction, or use a calculator for a decimal if the problem asks. In a real situation, check that each answer makes sense.

KEY FORMULAS

$$\text{If } ax^2 + bx + c = 0 \text{ and } a \neq 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{thrown object: } h = -16t^2 + vt + s \quad (\text{feet, seconds})$$

From history. Babylonian scribes solved problems that we would write as quadratic equations on clay tablets nearly 4,000 years ago, giving their methods as step-by-step recipes with particular numbers. A single formula that covers every case had to wait for symbolic algebra: François Viète (1591) used letters for known as well as unknown quantities, and René Descartes (1637) wrote equations much as we do today.

Practice habit. Recite the quadratic formula until it is automatic: " x equals negative b , plus or minus the square root of b squared minus $4ac$, all over $2a$." Before substituting, always write the line " $a = _, b = _, c = _$ " with signs.

WORKED EXAMPLES

Example 1. Solve $2x^2 + 3x - 2 = 0$ using the quadratic formula.

Show solution

1. $a = 2, b = 3, c = -2.$

2. $b^2 - 4ac = 3^2 - 4(2)(-2) = 9 + 16 = 25.$

3. $x = \frac{-3 \pm \sqrt{25}}{2(2)} = \frac{-3 \pm 5}{4}.$

4. $x = \frac{-3 + 5}{4} = \frac{1}{2}$ or $x = \frac{-3 - 5}{4} = -2.$

5. Check $x = \frac{1}{2}$: $2\left(\frac{1}{2}\right) + \frac{3}{2} - 2 = \frac{1}{2} + \frac{3}{2} - 2 = 0.$ Correct.

6. Check $x = -2$: $2(4) - 6 - 2 = 0.$ Correct.

Answer: $x = -2$ or $x = \frac{1}{2}$

Example 2. Henry throws a baseball straight up from a height of 6 feet at 40 feet per second, so its height after t seconds is $h = -16t^2 + 40t + 6$. When does the ball hit the ground? Give the exact answer and then round to the nearest tenth of a second.

Show solution

1. The ball hits the ground when $h = 0$: $-16t^2 + 40t + 6 = 0$.
2. Divide every term by -2 to get smaller numbers: $8t^2 - 20t - 3 = 0$. So $a = 8$, $b = -20$, $c = -3$.
3. $b^2 - 4ac = (-20)^2 - 4(8)(-3) = 400 + 96 = 496$.
4. $\sqrt{496} = \sqrt{16 \cdot 31} = 4\sqrt{31}$.
5. $t = \frac{20 \pm 4\sqrt{31}}{16} = \frac{5 \pm \sqrt{31}}{4}$.
6. $\sqrt{31} \approx 5.568$, so $t \approx \frac{10.568}{4} \approx 2.64$ or $t \approx \frac{-0.568}{4} \approx -0.14$.
7. Reject the negative time. Check: $-16(2.64)^2 + 40(2.64) + 6 \approx -111.5 + 105.6 + 6 \approx 0$.

Answer: $t = \frac{5 + \sqrt{31}}{4} \approx 2.6$ seconds

PRACTICE

1. Easy Give a , b and c for $3x^2 - 5x + 1 = 0$.

Show answer

$$a = 3, b = -5, c = 1$$

2. Easy Write $x^2 = 4x + 7$ in standard form and give a , b and c .

Show answer

$$x^2 - 4x - 7 = 0: a = 1, b = -4, c = -7$$

3. Easy Solve $x^2 + 5x + 6 = 0$ using the quadratic formula.

Show answer

$$b^2 - 4ac = 25 - 24 = 1; x = \frac{-5 \pm 1}{2}, \text{ so } x = -2 \text{ or } x = -3$$

4. Easy Solve $x^2 - 2x - 15 = 0$ using the quadratic formula.

Show answer

$$b^2 - 4ac = 4 + 60 = 64; x = \frac{2 \pm 8}{2}, \text{ so } x = 5 \text{ or } x = -3$$

5. Medium Solve $2x^2 - 7x + 3 = 0$ using the quadratic formula.

Show answer

$$b^2 - 4ac = 49 - 24 = 25; x = \frac{7 \pm 5}{4}, \text{ so } x = 3 \text{ or } x = \frac{1}{2}$$

6. Medium Solve $x^2 + 4x - 1 = 0$. Give answers in simplest radical form.

Show answer

$$b^2 - 4ac = 16 + 4 = 20; x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

7. Medium Solve $x^2 - 6x + 4 = 0$. Give decimal answers to the nearest hundredth.

Show answer

$$x = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}, \text{ so } x \approx 0.76 \text{ or } x \approx 5.24$$

8. Medium Solve $3x^2 + 2x - 2 = 0$. Give answers in simplest radical form.

Show answer

$$b^2 - 4ac = 4 + 24 = 28; x = \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}$$

9. Medium Solve $4x^2 + 4x + 1 = 0$ using the quadratic formula.

Show answer

$$b^2 - 4ac = 16 - 16 = 0; x = \frac{-4}{8} = -\frac{1}{2} \text{ (one solution)}$$

10. Hard Solve $5x^2 = 3x + 1$. Give exact answers.

Show answer

$$5x^2 - 3x - 1 = 0; b^2 - 4ac = 9 + 20 = 29; x = \frac{3 \pm \sqrt{29}}{10}$$

11. Hard A rectangular vegetable garden is 5 feet longer than it is wide, and its area is 120 square feet. Find the width to the nearest tenth of a foot.

Show answer

$$w^2 + 5w - 120 = 0; w = \frac{-5 + \sqrt{505}}{2} \approx 8.7 \text{ feet (reject the negative root)}$$

12. **Hard** Grace tosses a ball upward from a height of 5 feet at 30 feet per second, so $h = -16t^2 + 30t + 5$. When does the ball hit the ground, to the nearest tenth of a second?

Show answer

$$16t^2 - 30t - 5 = 0; t = \frac{30 + \sqrt{1220}}{32} \approx 2.0 \text{ seconds}$$

13. **Hard** Samuel sells birdhouses from his woodshop. When he makes x birdhouses in a week, his profit is $P = -2x^2 + 40x - 150$ dollars. For what numbers of birdhouses does he exactly break even ($P = 0$)?

Show answer

$$\text{Divide by } -2: x^2 - 20x + 75 = 0; x = \frac{20 \pm \sqrt{100}}{2}, \text{ so 5 or 15 birdhouses}$$

14. **Challenge** Solve $(x + 1)(x - 3) = 2$. Give answers in simplest radical form.

Show answer

$$x^2 - 2x - 3 = 2, \text{ so } x^2 - 2x - 5 = 0; x = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$$

15. **Challenge** In the derivation of the quadratic formula, why do we divide by a near the start, and where does the $\pm$ come from?

Show answer

We divide by a so the x^2 term has coefficient 1; only then does adding the square of half the x -coefficient make a perfect square. The $\pm$ comes from taking the square root of both sides of $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$, because a positive number has two square roots, one positive and one negative.

COMMON MISTAKES

- Using the formula before the equation is in standard form. For $x^2 = 4x + 7$, first rewrite it as $x^2 - 4x - 7 = 0$, so $b = -4$ and $c = -7$.
- Sign errors with negative b or c . If $b = -20$, then $-b = 20$ and $b^2 = (-20)^2 = 400$. If c is negative, $-4ac$ becomes positive. Use parentheses when substituting.
- Dividing only part of the numerator by $2a$. The fraction bar runs under the whole $-b \pm \sqrt{b^2 - 4ac}$; to simplify $\frac{-4 \pm 2\sqrt{5}}{2}$, divide both terms by 2 to get $-2 \pm \sqrt{5}$.

- Leaving an answer like $\frac{-2 \pm \sqrt{28}}{6}$ unsimplified. Simplify the radical, then reduce by a factor common to every term.

Make it a game • Formula From Memory. Write the quadratic formula from memory, three times, as fast as you can, and have the parent check each against the book. Then solve $2x^2 - 3x - 5 = 0$. Say a, b and c before you substitute, evaluate the discriminant, then write both solutions. Score one point for each flawless line. Beat your time tomorrow.

11.6 THE DISCRIMINANT AND CHOOSING A METHOD

OBJECTIVES

- Compute the discriminant $b^2 - 4ac$ and use it to tell how many real solutions a quadratic equation has.
- Explain how the discriminant is linked to the number of x -intercepts of a parabola.
- Choose an efficient method (square roots, factoring, completing the square or the quadratic formula) and solve.
- Use the discriminant to decide whether a real situation is possible, such as a ball reaching a certain height.

WATCH

The Discriminant

Eddie Woo · 7:39

<https://www.youtube.com/watch?v=9f8aFn7LaMY>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Quadratic Equations (1 of 4: Overview of methods)

Eddie Woo · 4:58

<https://www.youtube.com/watch?v=TCZRQP2MdYM>

Before Example 2: choosing a method for solving a quadratic.

KEY TERMS

discriminant

The expression $b^2 - 4ac$ under the square root in the quadratic formula; its sign tells how many real solutions $ax^2 + bx + c = 0$ has.

real solution

A solution that is a real number, one you can place on the number line.

double root

The single solution of a quadratic equation whose discriminant is 0; the parabola touches the x -axis at its vertex.

CORE IDEAS

Look again at the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Everything depends on the number under the square root, $b^2 - 4ac$, called the **discriminant** (it "discriminates", or tells apart, the cases). Many books write it as D .

If $D > 0$, then $\sqrt{D}$ is a positive number, and adding it and subtracting it give two different answers: **two real solutions**. If $D = 0$, then $\pm\sqrt{0}$ adds and subtracts nothing, so both answers are $-\frac{b}{2a}$: **one real solution**. If $D < 0$, we would need the square root of a negative number, and no real number squared is negative: **no real solutions**.

Link to the graph. The solutions of $ax^2 + bx + c = 0$ are the x -intercepts of $y = ax^2 + bx + c$. So $D > 0$ means the parabola crosses the x -axis twice; $D = 0$ means its vertex sits exactly on the x -axis; $D < 0$ means the whole parabola is above the axis (if it opens up) or below it (if it opens down). Remember: a negative discriminant means *no real* solutions, not *negative* solutions.

A bonus: when a , b and c are integers and D is a perfect square (like 1, 16 or 121), $\sqrt{D}$ is a whole number, the solutions are rational, and the trinomial can be factored. If D is positive but not a perfect square, the solutions are irrational and the trinomial does not factor over the integers.

Choosing a method. **Square roots**: when there is no x term, as in $5x^2 - 80 = 0$, or the equation looks like $(x - h)^2 = k$. **Factoring**: when the trinomial factors quickly, as in $x^2 + 3x - 28 = 0$. **Completing the square**: when $a = 1$ and b is even, as in $x^2 + 6x - 2 = 0$. **Quadratic formula**: always works, and is the best choice when the numbers are not friendly. Whatever you choose, the solutions must be the same, so you can check one method with another.

KEY FORMULAS

$$D = b^2 - 4ac$$

$D > 0$: two real solutions $D = 0$: one real solution $D < 0$: no real solutions

Practice habit. Before you solve any quadratic, compute the discriminant first. It tells you how many answers to expect and whether factoring will work, and it catches a lot of sign errors before they spread.

WORKED EXAMPLES

Example 1. Find the discriminant of $3x^2 - 4x + 2 = 0$ and tell how many real solutions the equation has.

Show solution

1. $a = 3, b = -4, c = 2$.
2. $D = (-4)^2 - 4(3)(2) = 16 - 24 = -8$.
3. $D < 0$, so there are no real solutions.
4. On a graph, $y = 3x^2 - 4x + 2$ opens up and its vertex, at $x = \frac{4}{6} = \frac{2}{3}$, has $y = \frac{4}{3} - \frac{8}{3} + 2 = \frac{2}{3} > 0$, so it never reaches the x -axis.

Answer: $D = -8$; 0 real solutions

Example 2. Ruth tosses a ball straight up from a height of 4 feet at 32 feet per second, so its height after t seconds is $h = -16t^2 + 32t + 4$. Does the ball ever reach a height of 25 feet?

Show solution

1. Set $h = 25$: $-16t^2 + 32t + 4 = 25$.
2. Standard form: $-16t^2 + 32t - 21 = 0$. So $a = -16, b = 32, c = -21$.
3. $D = 32^2 - 4(-16)(-21) = 1024 - 1344 = -320$.
4. $D < 0$, so there is no time at which $h = 25$.
5. Check with the vertex: $t = -\frac{32}{2(-16)} = 1$ and $h = -16 + 32 + 4 = 20$. The ball's greatest height is only 20 feet.

Answer: No. The discriminant is -320 , and the ball's maximum height is 20 feet.

PRACTICE

1. **Easy** Find the discriminant of $x^2 + 6x + 5 = 0$.

Show answer

$$D = 36 - 20 = 16$$

2. **Easy** How many real solutions does $x^2 - 4x + 4 = 0$ have?

Show answer

$$D = 16 - 16 = 0, \text{ so 1 real solution}$$

3. **Easy** How many real solutions does $2x^2 + x + 3 = 0$ have?

Show answer

$$D = 1 - 24 = -23 < 0, \text{ so 0 real solutions}$$

4. **Easy** How many real solutions does $x^2 - 5x - 2 = 0$ have?

Show answer

$$D = 25 + 8 = 33 > 0, \text{ so 2 real solutions}$$

5. **Medium** How many x -intercepts does the graph of $y = -x^2 + 2x - 3$ have?

Show answer

$$D = 4 - 4(-1)(-3) = 4 - 12 = -8 < 0, \text{ so 0 } x\text{-intercepts}$$

6. **Medium** How many x -intercepts does the graph of $y = 4x^2 - 12x + 9$ have?

Show answer

$$D = 144 - 144 = 0, \text{ so 1 (the vertex touches the axis at } x = 1.5)$$

7. **Medium** Find the value of k for which $x^2 + 8x + k = 0$ has exactly one real solution.

Show answer

$$D = 64 - 4k = 0, \text{ so } k = 16$$

8. **Medium** Choose an efficient method and solve $5x^2 - 80 = 0$.

Show answer

$$\text{Square roots: } x^2 = 16, \text{ so } x = \pm 4$$

9. **Medium** Choose an efficient method and solve $x^2 + 3x - 28 = 0$.

Show answer

Factoring: $(x + 7)(x - 4) = 0$, so $x = -7$ or $x = 4$

10. **Hard** Choose an efficient method and solve $3x^2 - 5x - 1 = 0$. Give exact answers.

Show answer

Quadratic formula: $D = 25 + 12 = 37$, $x = \frac{5 \pm \sqrt{37}}{6}$

11. **Hard** Grace has 40 feet of fencing for a rectangular flower bed fenced on all four sides. Can she enclose 96 square feet? If so, give the dimensions.

Show answer

$w(20 - w) = 96$ gives $w^2 - 20w + 96 = 0$; $D = 400 - 384 = 16 > 0$, so yes:
 $w = \frac{20 \pm 4}{2}$, an 8-foot by 12-foot bed.

12. **Hard** A model rocket is launched from the ground with $h = -16t^2 + 96t$. Does it reach 128 feet? If so, at what times is it at that height?

Show answer

$-16t^2 + 96t - 128 = 0$ gives $t^2 - 6t + 8 = 0$; $D = 36 - 32 = 4 > 0$, so yes, at $t = 2$ seconds and $t = 4$ seconds.

13. **Hard** The Miller family's farm stand earns a profit of $P = -10x^2 + 80x - 120$ dollars per day when baskets of peaches cost x dollars each. Is there a price that gives a daily profit of \$50?

Show answer

$-10x^2 + 80x - 170 = 0$ gives $x^2 - 8x + 17 = 0$; $D = 64 - 68 = -4 < 0$, so no.
(The greatest profit is \$40, at \$4 per basket.)

14. **Challenge** For what values of c does $x^2 - 6x + c = 0$ have two real solutions?

Show answer

$D = 36 - 4c > 0$, so $c < 9$

15. **Challenge** Explain, using the quadratic formula, why the sign of $b^2 - 4ac$ tells how many real solutions $ax^2 + bx + c = 0$ has.

Show answer

The formula adds and subtracts $\sqrt{b^2 - 4ac}$. If it is positive, adding and subtracting it give two different answers. If it is 0, both give $-\frac{b}{2a}$, so there is one answer. If it is negative, there is no real square root, so there is no real answer.

COMMON MISTAKES

- Sign errors in $-4ac$. When a and c have opposite signs, $-4ac$ is positive: for $x^2 - 5x - 2$, $D = 25 - 4(1)(-2) = 25 + 8 = 33$.
- Finding D before the equation is in standard form. Move every term to one side first, as in the ball problem, where $= 25$ became $-16t^2 + 32t - 21 = 0$.
- Thinking $D < 0$ means the solutions are negative. It means there are no real solutions at all; the parabola does not touch the x -axis.
- Using $-b^2$ for b^2 . b^2 is never negative: for $b = -4$, $b^2 = (-4)^2 = 16$.

Make it a game • Predict the Crossings. The parent calls a quadratic such as $x^2 - 6x + 9 = 0$ or $x^2 + 2x + 5 = 0$. Before graphing, compute $b^2 - 4ac$ and predict how many times the parabola crosses the x -axis: two, one or none. Then pick the easiest method from factoring, square roots or the formula and solve it. Sketch to check your prediction.

11.7 VERTEX FORM AND QUADRATIC MODELS

OBJECTIVES

- Read the vertex, axis of symmetry and direction of opening from $y = a(x - h)^2 + k$ and explain why the vertex is (h, k) .
- Convert a quadratic from standard form to vertex form by completing the square.
- Write a quadratic in vertex form from its vertex and one other point.
- Use quadratic models of thrown objects, areas and profit, and interpret the vertex and zeros in context.

WATCH

Equation of a Parabola - Vertex Form (Completing the square)

Eddie Woo · 10:00

<https://www.youtube.com/watch?v=6QF-7OFMWRC>

WHEN TO WATCH

Before Example 1: converting to vertex form by completing the square.

Note: Eddie Woo teaches in Australia: he says 'turning point' for vertex. For thrown-object and profit models, rewatch the Lesson 11.2 backup.

BACKUP VIDEO

Applications of Completing the Square (1 of 2: Finding a parabola's vertex)

Eddie Woo · 8:34

<https://www.youtube.com/watch?v=ebPeW93LI-U>

Before the practice set: finding the vertex by completing the square.

KEY TERMS

vertex form

$y = a(x - h)^2 + k$ with $a \neq 0$; the vertex is (h, k) .

zeros of a function

The input values that make the output 0; for a quadratic they are the x -intercepts of its graph.

quadratic model

A quadratic function used to describe a real situation, such as the height of a ball over time.

CORE IDEAS

A quadratic in **vertex form** looks like $y = a(x - h)^2 + k$. Why is the vertex (h, k) ? The square $(x - h)^2$ is never negative, and it equals 0 only when $x = h$. If $a > 0$, then $a(x - h)^2 \geq 0$, so $y \geq k$, with $y = k$ only at $x = h$: the lowest point is (h, k) . If $a < 0$, then $y \leq k$ and (h, k) is the highest point. The axis of symmetry is $x = h$, and a still controls the direction and width, just as in Lesson 11.1.

Another way to see it: vertex form is the graph of $y = ax^2$ slid h units right and k units up. Watch the sign inside the parentheses. $y = (x - 3)^2$ has its vertex at $x = 3$, because that is where $x - 3 = 0$. And $y = (x + 3)^2 = (x - (-3))^2$ has its vertex at $x = -3$.

To convert from standard form, complete the square on the right side, but instead of adding a number to both sides, **add it and subtract it** on the same side (adding 0 changes nothing). For $y = x^2 + 6x + 5$: half of 6 is 3, and $3^2 = 9$, so $y = (x^2 + 6x + 9) - 9 + 5 = (x + 3)^2 - 4$. The vertex is $(-3, -4)$. Check with Lesson 11.2: $x = -\frac{6}{2} = -3$ and $y = 9 - 18 + 5 = -4$.

When $a \neq 1$, factor a out of the x^2 and x terms only. For $y = 2x^2 - 8x + 3$: $y = 2(x^2 - 4x) + 3$. Inside, add 4 to complete the square; but that 4 is multiplied by 2, so you have really added 8, and you must subtract 8 outside: $y = 2(x^2 - 4x + 4) - 8 + 3 = 2(x - 2)^2 - 5$. Vertex $(2, -5)$.

To write an equation from a graph, put the vertex into $y = a(x - h)^2 + k$ and use one more point to find a . If the vertex is $(2, 5)$ and the graph passes through $(0, 1)$, then $1 = a(0 - 2)^2 + 5$, so $4a = -4$ and $a = -1$: $y = -(x - 2)^2 + 5$.

In a model, every feature means something. For the height of a ball, the vertex gives the greatest height and when it happens, the y -intercept is the starting height, and the positive zero is when the ball lands. For area or profit, the vertex gives the best size or best price and the greatest area or profit, and the zeros are where the area or profit is 0 (for profit, the break-even points).

KEY FORMULAS

$$y = a(x - h)^2 + k : \quad \text{vertex } (h, k), \quad \text{axis } x = h$$

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Practice habit. Know the three forms of a quadratic by heart: standard $y = ax^2 + bx + c$ (shows the y -intercept), vertex $y = a(x - h)^2 + k$ (shows the vertex), and factored $y = a(x - r)(x - s)$ (shows the zeros). After converting to vertex form, check the vertex with $x = -\frac{b}{2a}$.

WORKED EXAMPLES

Example 1. Write $y = x^2 - 10x + 21$ in vertex form and give the vertex.

Show solution

1. Half of -10 is -5 , and $(-5)^2 = 25$.
2. Add and subtract 25: $y = (x^2 - 10x + 25) - 25 + 21$.
3. Write the square: $y = (x - 5)^2 - 4$.
4. The vertex is $(5, -4)$.
5. Check: $x = -\frac{-10}{2(1)} = 5$ and $y = 25 - 50 + 21 = -4$. Correct.

Answer: $y = (x - 5)^2 - 4$; vertex $(5, -4)$

Example 2. A football is kicked from a height of 3 feet with an upward speed of 64 feet per second, so its height after t seconds is $h = -16t^2 + 64t + 3$. Write h in vertex form. Then find the greatest height, when it happens, and when the ball lands (to the nearest tenth of a second).

Show solution

1. Factor -16 from the first two terms: $h = -16(t^2 - 4t) + 3$.
2. Half of -4 is -2 , and $(-2)^2 = 4$. Adding 4 inside really adds $-16 \times 4 = -64$, so add 64 outside: $h = -16(t^2 - 4t + 4) + 64 + 3$.
3. Vertex form: $h = -16(t - 2)^2 + 67$.
4. The vertex is $(2, 67)$: the greatest height is 67 feet, reached at 2 seconds.
5. It lands when $h = 0$: $16(t - 2)^2 = 67$, so $(t - 2)^2 = \frac{67}{16}$.
6. $t - 2 = \pm \frac{\sqrt{67}}{4}$. Take the positive root: $t = 2 + \frac{\sqrt{67}}{4} \approx 2 + 2.046 = 4.046$.
7. Check: $h = -16(4.046)^2 + 64(4.046) + 3 \approx -261.9 + 258.9 + 3 \approx 0$.

Answer: $h = -16(t - 2)^2 + 67$. The ball reaches a greatest height of 67 feet at 2 seconds and lands after about 4.0 seconds.

PRACTICE

1. Easy Give the vertex of $y = (x - 3)^2 + 2$.

Show answer

$$(3, 2)$$

2. Easy Give the vertex of $y = -2(x + 1)^2 + 7$.

Show answer

$$(-1, 7)$$

3. Easy Give the axis of symmetry of $y = 5(x - 4)^2 - 6$.

Show answer

$$x = 4$$

4. Easy What is the maximum value of $y = -(x - 2)^2 + 9$?

Show answer

$a < 0$, so the vertex $(2, 9)$ is the highest point; the maximum value is 9.

5. Medium Write $y = x^2 + 4x + 1$ in vertex form.

Show answer

$$y = (x^2 + 4x + 4) - 4 + 1 = (x + 2)^2 - 3$$

6. Medium Write $y = x^2 - 6x + 11$ in vertex form.

Show answer

$$y = (x^2 - 6x + 9) - 9 + 11 = (x - 3)^2 + 2$$

7. Medium Write $y = 3x^2 + 12x + 5$ in vertex form.

Show answer

$$y = 3(x^2 + 4x + 4) - 12 + 5 = 3(x + 2)^2 - 7$$

8. Medium A parabola has vertex $(1, -4)$ and passes through $(3, 4)$. Write its equation in vertex form.

Show answer

$$4 = a(3 - 1)^2 - 4, \text{ so } 4a = 8 \text{ and } a = 2: y = 2(x - 1)^2 - 4$$

9. Medium Find the x -intercepts of $y = (x - 1)^2 - 9$.

Show answer

$$(x - 1)^2 = 9, x - 1 = \pm 3, \text{ so } x = -2 \text{ or } x = 4$$

10. Hard Write $y = -x^2 + 8x - 10$ in vertex form.

Show answer

$$y = -(x^2 - 8x + 16) + 16 - 10 = -(x - 4)^2 + 6$$

11. Hard Henry has 100 feet of fencing for a rectangular sheep pen along a stone wall; the wall needs no fence. If each of the two sides perpendicular to the wall is x feet long, the side parallel to the wall is $100 - 2x$ feet and the area in square feet is $A = -2x^2 + 100x$. Write A in vertex form, then give the length x that gives the greatest area and that area.

Show answer

$$A = -2(x^2 - 50x + 625) + 1250 = -2(x - 25)^2 + 1250; x = 25 \text{ feet gives the greatest area, 1,250 square feet (the pen is 25 feet by 50 feet).}$$

12. **Hard** A stone bridge has an arch shaped like a parabola. The arch is 24 feet wide at the water and 12 feet high at its center. Put the left foot of the arch at $(0, 0)$ and the right foot at $(24, 0)$, with x and y in feet. Write the equation of the arch in vertex form.

Show answer

$$\text{Vertex } (12, 12): 0 = a(0 - 12)^2 + 12, \text{ so } 144a = -12 \text{ and } a = -\frac{1}{12}; y = -\frac{1}{12}(x - 12)^2 + 12$$

13. **Hard** A church youth group sells pies. If they charge x dollars per pie, their profit is $P = -5x^2 + 60x - 100$ dollars. Write P in vertex form, then give the best price and the greatest profit.

Show answer

$$P = -5(x^2 - 12x + 36) + 180 - 100 = -5(x - 6)^2 + 80; \text{ charge \$6 per pie for a profit of \$80.}$$

14. **Challenge** A ball is thrown upward from 5 feet, and its height after t seconds is $h = -16t^2 + 32t + 5$. (a) Write h in vertex form. (b) When does the ball land, to the nearest tenth of a second?

Show answer

$$\text{(a) } h = -16(t^2 - 2t + 1) + 16 + 5 = -16(t - 1)^2 + 21. \text{ (b) } (t - 1)^2 = \frac{21}{16}, \text{ so } t = 1 + \frac{\sqrt{21}}{4} \approx 2.1 \text{ seconds.}$$

15. **Challenge** Explain why $y = a(x - h)^2 + k$ with $a < 0$ has a maximum value of k , reached at $x = h$.

Show answer

$$(x - h)^2 \geq 0 \text{ for every } x, \text{ and multiplying by a negative } a \text{ makes } a(x - h)^2 \leq 0. \text{ So } y = a(x - h)^2 + k \leq k. \text{ The square is 0 only when } x = h, \text{ so } y \text{ reaches } k \text{ there and nowhere is it larger.}$$

COMMON MISTAKES

- Reading the sign of h wrong. In $y = (x + 2)^2 - 3$, the vertex is $(-2, -3)$, not $(2, -3)$; ask what value of x makes the square 0.
- Forgetting to multiply by a when balancing. In $y = 3(x^2 + 4x + 4) + 5$, the 4 inside is really $3 \times 4 = 12$, so you must subtract 12 outside, not 4.

- Adding the completing number to only one side or not subtracting it back. When converting $y = \dots$, add and subtract on the same side so the function does not change.
- Confusing which part of the vertex answers the question. For a thrown ball, the t -value of the vertex is *when* it is highest and the h -value is *how high*.

Make it a game • Slide the Vertex. Put a button on graph paper at the origin. The parent calls $y = (x - 3)^2 + 5$, and you slide the button to the vertex. Then they change it to $y = -2(x + 1)^2 - 4$, and you slide again and say which way it opens. Finally, turn $y = x^2 - 4x + 7$ into vertex form by completing the square.

QUADRATIC FUNCTIONS AND EQUATIONS

In this unit you meet the parabola, the curve of a thrown ball, a stream of water from a fountain, and many arches and dishes. You will graph quadratic functions, find their vertex and axis of symmetry, and use the vertex to answer "greatest" and "least" questions, such as how to fence the largest garden or which price brings the most profit.

Then you will learn to solve any quadratic equation. In Unit 9 you solved some by factoring; now you add square roots, completing the square (the method al-Khwarizmi drew with squares and rectangles around 820), and the quadratic formula, which you will derive yourself. The discriminant tells you ahead of time how many real solutions to expect, and vertex form ties the graph and the algebra together.

This unit draws on factoring (Unit 9), simplifying radicals (Unit 10), special products (Lesson 8.4) and functions (Unit 4). It prepares you for Lesson 13.3, where you will tell quadratic patterns apart from linear and exponential ones, and for the deeper study of functions in Algebra II.

PREREQUISITES

- Graph points and functions on the coordinate plane and describe domain and range (Units 4 and 5).
- Expand $(x + d)^2$ and recognize perfect-square trinomials (Lessons 8.4 and 9.4).
- Factor trinomials and solve equations with the zero-product property (Lessons 9.2, 9.3 and 9.6).
- Simplify square roots and write answers in simplest radical form (Lessons 10.1 and 10.2).
- Substitute negative numbers and fractions into expressions correctly, using parentheses (Lesson 1.3).

MASTERY CHECKLIST

- ☐ I can graph $y = ax^2 + c$ and explain how a and c change the parabola.
- ☐ I can find the axis of symmetry $x = -\frac{b}{2a}$ and the vertex of $y = ax^2 + bx + c$ and explain why the formula works.
- ☐ I can use the vertex to solve maximum and minimum problems about area, height and profit.
- ☐ I can solve a quadratic equation by graphing and explain why it can have two, one or no real solutions.
- ☐ I can solve $x^2 = k$ and $(x - h)^2 = k$ with square roots, remembering $\pm$.
- ☐ I can solve a quadratic equation by completing the square, both when $a = 1$ and when $a \neq 1$.
- ☐ I can derive the quadratic formula and use it to give exact and decimal solutions.

- ☐ I can use the discriminant to tell how many real solutions an equation has and how many x -intercepts a parabola has.
- ☐ I can choose an efficient method for solving a quadratic equation.
- ☐ I can convert between standard form and vertex form and read the vertex from $y = a(x - h)^2 + k$.
- ☐ I can use quadratic models of falling and thrown objects and interpret the vertex and zeros in context.

LESSONS IN THIS UNIT

11.1 Graphing $y = ax^2$	2 sessions of 45 minutes
11.2 Graphing $y = ax^2 + bx + c$	2 sessions of 45 minutes
11.3 Solving Quadratic Equations by Graphing and Square Roots 2 sessions of 45 minutes	
11.4 Completing the Square	2 sessions of 45 minutes
11.5 The Quadratic Formula	2 sessions of 45 minutes
11.6 The Discriminant and Choosing a Method	2 sessions of 45 minutes
11.7 Vertex Form and Quadratic Models	2 sessions of 45 minutes

UNIT 11 TEST

You have 60 minutes. A scientific calculator is allowed, but show all of your algebra; a correct answer without work earns only partial credit. Give exact answers (in

simplest radical form) unless a problem asks you to round; when it does, round as directed. Include units in answers to word problems.

1. (6 pts)

For $y = -3x^2 + 4$, give the vertex and the maximum value.

2. (6 pts)

Find the axis of symmetry and the vertex of $y = x^2 + 8x + 10$.

3. (6 pts)

Solve $5x^2 = 245$.

4. (6 pts)

Solve $(x - 2)^2 = 12$. Give answers in simplest radical form.

5. (7 pts)

Solve $x^2 + 12x + 11 = 0$ by completing the square.

6. (7 pts)

Solve $x^2 - 4x - 3 = 0$ by completing the square. Give exact answers.

7. (7 pts)

Solve $2x^2 - 3x - 4 = 0$ using the quadratic formula. Give exact answers.

8. (6 pts)

Find the discriminant of $3x^2 + 6x + 3 = 0$ and tell how many real solutions the equation has.

9. (7 pts)

Write $y = 2x^2 + 8x + 1$ in vertex form and give the vertex.

10. (8 pts)

Samuel has 120 feet of fencing to make a rectangular pasture along a straight riverbank; the river side needs no fence. Let x be the length of each side perpendicular to the river. What dimensions give the largest area, and what is that area?

11. **8 pts**

Ruth drops an apple from a footbridge 81 feet above a pond, so its height after t seconds is $h = -16t^2 + 81$. How long does the apple take to reach the pond?

12. **8 pts**

Henry throws a ball upward from a height of 6 feet at 24 feet per second, so $h = -16t^2 + 24t + 6$. When does the ball hit the ground? Round to the nearest tenth of a second.

13. **8 pts**

Without solving, explain in two different ways how you know that the graph of $y = x^2 + 2x + 7$ never touches the x -axis.

14. **10 pts**

A garden fountain shoots a stream of water in a parabola. The water leaves the nozzle at ground level, at $(0, 0)$, and reaches its greatest height of 8 feet when it is 4 feet away horizontally. (a) Write the equation of the stream in vertex form, with x and y in feet. (b) How far from the nozzle does the water land?

ANSWER KEY

1. Vertex $(0, 4)$; maximum value 4
2. $x = -\frac{8}{2} = -4$; $y = 16 - 32 + 10 = -6$; vertex $(-4, -6)$
3. $x^2 = 49$, so $x = \pm 7$
4. $x - 2 = \pm\sqrt{12} = \pm 2\sqrt{3}$, so $x = 2 \pm 2\sqrt{3}$
5. $x^2 + 12x = -11$; $(x + 6)^2 = 25$; $x + 6 = \pm 5$, so $x = -1$ or $x = -11$
6. $x^2 - 4x = 3$; $(x - 2)^2 = 7$; $x = 2 \pm \sqrt{7}$
7. $b^2 - 4ac = 9 + 32 = 41$; $x = \frac{3 \pm \sqrt{41}}{4}$
8. $D = 36 - 36 = 0$; 1 real solution
9. $y = 2(x^2 + 4x + 4) - 8 + 1 = 2(x + 2)^2 - 7$; vertex $(-2, -7)$
10. $A = x(120 - 2x) = -2x^2 + 120x$; $x = -\frac{120}{2(-2)} = 30$; the other side is $120 - 60 = 60$. The pasture is 30 feet by 60 feet, with area 1,800 square feet.
11. $16t^2 = 81$, $t^2 = \frac{81}{16}$, $t = \frac{9}{4} = 2.25$ seconds (reject the negative root)
12. Divide $0 = -16t^2 + 24t + 6$ by -2 : $8t^2 - 12t - 3 = 0$; $t = \frac{12 \pm \sqrt{240}}{16} \approx 1.7$ seconds (reject the negative root)
13. The discriminant is $2^2 - 4(1)(7) = 4 - 28 = -24 < 0$, so $x^2 + 2x + 7 = 0$ has no real solutions. Also, the vertex is at $x = -1$, $y = 1 - 2 + 7 = 6$; the parabola opens up, so its lowest point is 6 units above the x -axis.
14. (a) Vertex $(4, 8)$: $0 = a(0 - 4)^2 + 8$, so $16a = -8$ and $a = -\frac{1}{2}$: $y = -\frac{1}{2}(x - 4)^2 + 8$.
(b) $(x - 4)^2 = 16$, so $x = 0$ or $x = 8$; the water lands 8 feet from the nozzle.

RATIONAL EXPRESSIONS: AN INTRODUCTION

This unit is about fractions whose numerators and denominators are polynomials, called rational expressions. It begins with inverse variation, $y = \frac{k}{x}$, where one quantity goes down as the other goes up: driving faster shortens a trip, and more workers finish a job sooner. Then you will simplify, multiply, divide, add and subtract rational expressions, and finally solve equations that contain them.

Nothing in this unit is truly new. Every rule is a rule you already know for ordinary fractions, and almost every step depends on the factoring you learned in Unit 9. The key new idea is care about division by zero: excluded values, and extraneous solutions that appear when both sides of an equation are multiplied by an expression that can be zero.

These skills lead straight into Algebra II, where rational functions, their graphs and more complicated rational equations are studied in depth. They also help in physics, chemistry and everyday problems about rates, work and travel.

PREREQUISITES

- Add, subtract, multiply and divide numerical fractions, including finding a least common denominator.
- Factor GCFs, trinomials and differences of squares (Lessons 9.1–9.5).
- Solve equations by clearing fractions (Lesson 2.4) and quadratic equations by factoring (Lesson 9.6).
- Use direct variation $y = kx$ (Lesson 5.2) and the formula $d = rt$.
- Apply the properties of exponents to simplify monomials (Lesson 7.1).

MASTERY CHECKLIST

- ☐ I can tell whether a table or situation shows inverse variation by checking that xy is constant.
- ☐ I can find k , write $y = \frac{k}{x}$, and solve gear, travel and worker problems.
- ☐ I can describe the graph of $y = \frac{k}{x}$ and explain why it never touches the axes.
- ☐ I can find the excluded values of a rational expression.
- ☐ I can simplify a rational expression by factoring and dividing out common factors, and explain why terms cannot be cancelled.
- ☐ I can recognize opposites and use $\frac{a-b}{b-a} = -1$.
- ☐ I can multiply and divide rational expressions, and explain why dividing means multiplying by the reciprocal.
- ☐ I can find the LCD of monomial and binomial denominators and add or subtract rational expressions.
- ☐ I can use parentheses correctly when subtracting a numerator with more than one term.

- ☐ I can solve rational equations and proportions and reject extraneous solutions.
- ☐ I can write and solve rational equations for work and travel problems.

LESSONS IN THIS UNIT

12.1 Inverse Variation	45 minutes
12.2 Simplifying Rational Expressions	2 sessions of 45 minutes
12.3 Multiplying and Dividing Rational Expressions	2 sessions of 45 minutes
12.4 Adding and Subtracting Rational Expressions	2 sessions of 45 minutes
12.5 Solving Rational Equations	2 sessions of 45 minutes

UNIT 12 TEST

You have 60 minutes. A scientific calculator is allowed, but every problem can be done by hand. Show your work on every problem. Give exact answers in simplest form unless a problem says otherwise, state excluded values when asked, and include units in the answers to word problems. Check the solutions of every equation in the original equation.

1. **5 pts**
 y varies inversely with x , and $y = 9$ when $x = 4$. Find y when $x = 12$.
2. **5 pts**
 Six volunteers can pack the food boxes for a community food drive in 4 hours. Working at the same rate, how long would 8 volunteers take?
3. **5 pts**
 The table shows x : 2, 4, 5, 10 and y : 15, 7.5, 6, 3. (a) Does it show inverse variation? (b) If so, what is k ?

4. (5 pts)

Find the excluded values of $\frac{x+4}{x^2-6x}$.

5. (7 pts)

Simplify $\frac{x^2-49}{x^2+4x-21}$.

6. (7 pts)

Simplify $\frac{15-5x}{x^2-9}$.

7. (7 pts)

A student simplified $\frac{x+6}{x+2}$ to 3 by crossing out the x 's and dividing 6 by 2.
Explain the error, and use $x = 1$ to show the answer is wrong.

8. (7 pts)

Multiply and simplify: $\frac{x^2+3x}{4} \cdot \frac{8}{x^2-9}$.

9. (7 pts)

Divide and simplify: $\frac{x^2-x-6}{x^2-4} \div \frac{x-3}{2x}$.

10. (6 pts)

Add: $\frac{3}{2x} + \frac{5}{6x^2}$.

11. (8 pts)

Subtract and simplify: $\frac{5}{x-1} - \frac{3}{x+1}$.

12. (7 pts)

Solve $\frac{6}{x} - \frac{1}{2} = \frac{1}{x}$.

13. (8 pts)

Solve $\frac{x}{x+3} + 1 = \frac{-3}{x+3}$.

14. (8 pts)

A father can split the winter's firewood alone in 6 hours. His son can do it alone in 12 hours. How long will it take them working together?

15. **8 pts**

A small plane flies 300 miles with the wind in the same time it flies 200 miles against the wind. The wind blows at 25 miles per hour. What is the plane's speed in still air?

ANSWER KEY

1. $k = 36$, so $y = \frac{36}{12} = 3$

2. $6 \cdot 4 = 24$ worker-hours, so $\frac{24}{8} = 3$ hours

3. (a) Yes, every product xy is 30. (b) $k = 30$

4. $x(x - 6) = 0$, so $x = 0$ and $x = 6$

5. $\frac{(x - 7)(x + 7)}{(x + 7)(x - 3)} = \frac{x - 7}{x - 3}$, with $x \neq -7, 3$

6. $\frac{-5(x - 3)}{(x - 3)(x + 3)} = -\frac{5}{x + 3}$, with $x \neq 3, -3$

7. Model answer: x , 6 and 2 are terms (added), not factors of the whole numerator and denominator, and only common factors can be divided out. With $x = 1$, $\frac{1 + 6}{1 + 2} = \frac{7}{3}$, not 3. The expression is already in simplest form.

8. $\frac{x(x + 3)}{4} \cdot \frac{8}{(x - 3)(x + 3)} = \frac{2x}{x - 3}$

9. $\frac{(x - 3)(x + 2)}{(x - 2)(x + 2)} \cdot \frac{2x}{x - 3} = \frac{2x}{x - 2}$

10. LCD $6x^2$: $\frac{9x}{6x^2} + \frac{5}{6x^2} = \frac{9x + 5}{6x^2}$

11. $\frac{5(x + 1) - 3(x - 1)}{(x - 1)(x + 1)} = \frac{2x + 8}{(x - 1)(x + 1)} = \frac{2(x + 4)}{(x - 1)(x + 1)}$

12. Multiply by $2x$: $12 - x = 2$, so $x = 10$

13. Multiply by $x + 3$: $x + x + 3 = -3$, so $x = -3$. But -3 is excluded, so there is no solution.

14. $\frac{t}{6} + \frac{t}{12} = 1$; multiply by 12: $2t + t = 12$, so $t = 4$ hours

15. $\frac{300}{r + 25} = \frac{200}{r - 25}$, so $300r - 7500 = 200r + 5000$, $100r = 12500$ and $r = 125$ miles per hour

12.1 INVERSE VARIATION

OBJECTIVES

- Recognize inverse variation from an equation, a table or a word situation by checking that the product xy is constant.
- Find the constant of variation k and write an equation $y = \frac{k}{x}$.
- Describe the graph of $y = \frac{k}{x}$ in words and explain why it never touches either axis.
- Solve problems about gears, travel time and workers using inverse variation.

WATCH

Inverse Variation (2 of 4: The painter problem)

Eddie Woo · 11:10

<https://www.youtube.com/watch?v=3lz4GpNSMj4>

WHEN TO WATCH

Before Example 1: a workers-and-time inverse variation problem.

BACKUP VIDEO

Inverse Variation (4 of 4: Graphing principles)

Eddie Woo · 5:22

https://www.youtube.com/watch?v=4P_9LKxY2ns

Before Example 2: the graph of $y = k/x$ and why it never touches the axes.

KEY TERMS

inverse variation

A relationship between two quantities that can be written $y = \frac{k}{x}$ with $k \neq 0$.

When one quantity is multiplied by a number, the other is divided by that same number.

constant of variation

The fixed number k in $y = \frac{k}{x}$. For inverse variation it equals the product xy of any matching pair of values.

direct variation

Review from Lesson 5.2: a relationship $y = kx$, where the ratio $\frac{y}{x}$ stays constant and the graph is a line through the origin.

hyperbola

The shape of the graph of $y = \frac{k}{x}$: two separate curved pieces, called branches, that get closer and closer to the axes without ever touching them.

CORE IDEAS

In Lesson 5.2 you met direct variation, $y = kx$: when x doubles, y doubles. Many real situations work the other way. If the Miller family drives 240 miles to visit their grandparents, driving faster makes the trip shorter. Since distance = rate $\times$ time, we have $rt = 240$, so $t = \frac{240}{r}$. At 40 miles per hour the trip takes 6 hours; at 60 miles per hour it takes 4 hours; at 80 miles per hour it takes 3 hours. When the speed is doubled from 40 to 80, the time is cut in half from 6 to 3.

We say y **varies inversely with** x when $y = \frac{k}{x}$ for some nonzero constant k . Multiply both sides by x and you get $xy = k$. So the test for inverse variation is simple: **the product of each pair is the same number**. (For direct variation the *ratio* $\frac{y}{x}$ is the same; for inverse variation the *product* xy is the same.) For example, the table x : 1, 2, 3, 6 and y : 12, 6, 4, 2 shows inverse variation, because $1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 6 \cdot 2 = 12$, so $k = 12$ and $y = \frac{12}{x}$.

Why does multiplying x by a number divide y by that number? If x is replaced by $3x$, the new y is $\frac{k}{3x} = \frac{1}{3} \cdot \frac{k}{x}$, which is one third of the old y . The same argument works for any factor: double x and y is halved, multiply x by 10 and y is divided by 10. That is why the product never changes: whatever x gains, y gives back.

To solve an inverse variation problem, first find k by multiplying a known pair: $k = xy$. Then write $y = \frac{k}{x}$ and substitute the new value. Because every pair has the same product, you can also write $x_1y_1 = x_2y_2$ and solve for the missing number in one step.

The graph of $y = \frac{k}{x}$ is a **hyperbola**. Picture $y = \frac{12}{x}$: for positive x it passes through (1, 12), (2, 6), (3, 4), (4, 3), (6, 2) and (12, 1), falling quickly at first and then flattening out. For negative x the values are the opposites, such as (−2, −6) and (−6, −2).

When $k > 0$ the two branches lie in Quadrants I and III; when $k < 0$ they lie in Quadrants II and IV.

Why does the graph never touch the axes? It cannot cross the y -axis because that would need $x = 0$, and $\frac{k}{0}$ is undefined: you can never divide by zero. It cannot cross the x -axis because that would need $y = 0$, but a fraction equals zero only when its numerator is zero, and k is not zero. Instead, as x gets farther from 0 (in either direction), y gets closer and closer to 0, and as x gets close to 0, y gets farther and farther from 0 (very large positive on one side, very large negative on the other).

Inverse variation shows up wherever a fixed total is shared. Two meshing gears: (number of teeth) $\times$ (turns per minute) is the same for both gears, because the same number of teeth pass the meeting point each minute. Workers and time: if a job takes a fixed number of worker-hours, then (workers) $\times$ (hours) is constant, assuming everyone works at the same rate. Travel: (speed) $\times$ (time) equals the fixed distance.

KEY FORMULAS

$$y = \frac{k}{x}, \quad k \neq 0$$

$$xy = k$$

$$x_1y_1 = x_2y_2$$

From history. Archimedes of Syracuse (about 250 BC) proved the law of the lever: two weights balance when their distances from the balance point are inversely proportional to the weights. In modern terms, weight $\times$ distance is the same on both sides. That is the rule that makes a heavier person sit closer to the middle of a seesaw.

Practice habit. Multiply the pair. Whenever you see a table or a word problem, multiply each x by its y ; if the products all match, it is inverse variation and the product is k . Then check your final answer by multiplying it with its partner value.

WORKED EXAMPLES

Example 1. y varies inversely with x , and $y = 6$ when $x = 4$. Write an equation for the relationship, then find y when $x = 3$.

Show solution

1. Find the constant: $k = xy = 4 \cdot 6 = 24$.
2. Write the equation: $y = \frac{24}{x}$.
3. Substitute $x = 3$: $y = \frac{24}{3} = 8$.
4. Check: the products match, $3 \cdot 8 = 24$ and $4 \cdot 6 = 24$. As x went down, y went up, as it should.

Answer: $y = \frac{24}{x}$; when $x = 3$, $y = 8$

Example 2. On Henry's grandfather's old farm machine, a gear with 36 teeth turns at 40 revolutions per minute. It meshes with a smaller gear that has 24 teeth. How fast does the smaller gear turn?

Show solution

1. For meshing gears, teeth $\times$ revolutions per minute is the same for both gears, so this is inverse variation.
2. Find the constant: $k = 36 \cdot 40 = 1440$ teeth pass the meeting point each minute.
3. Let r be the speed of the small gear: $24r = 1440$.
4. Divide by 24: $r = \frac{1440}{24} = 60$.
5. Check: $24 \cdot 60 = 1440$. The smaller gear turns faster, which makes sense.

Answer: 60 revolutions per minute

PRACTICE

1. Easy Does the table show inverse variation? x : 1, 2, 4, 8 and y : 16, 8, 4, 2.

Show answer

Yes; every product xy equals 16.

2. **Easy** Does the table show inverse variation? x : 1, 2, 3, 4 and y : 12, 10, 8, 6.

Show answer

No; the products are 12, 20, 24, 24, which are not all equal.

3. **Easy** y varies inversely with x , and $y = 3$ when $x = 8$. Find the constant of variation k .

Show answer

$$k = 8 \cdot 3 = 24$$

4. **Easy** If $y = \frac{30}{x}$, find y when $x = 5$.

Show answer

$$y = \frac{30}{5} = 6$$

5. **Medium** y varies inversely with x , and $y = 10$ when $x = 3$. Write an equation for the relationship, then find y when $x = 6$.

Show answer

$$k = 30, \text{ so } y = \frac{30}{x}; \text{ when } x = 6, y = 5$$

6. **Medium** y varies inversely with x , and $y = -4$ when $x = 5$. Find x when $y = 2$.

Show answer

$$k = 5(-4) = -20, \text{ so } 2x = -20 \text{ and } x = -10$$

7. **Medium** y varies inversely with x , and $y = 1.5$ when $x = 8$. Find y when $x = 3$.

Show answer

$$k = 8(1.5) = 12, \text{ so } y = \frac{12}{3} = 4$$

8. **Medium** Grace rides her bike to her aunt's farm. At 12 miles per hour the trip takes 3 hours. How long would the same trip take at 9 miles per hour?

Show answer

The distance is $12 \cdot 3 = 36$ miles, so the time is $\frac{36}{9} = 4$ hours.

9. **Medium** Four volunteers can stack the hay in the church's barn in 6 hours. Working at the same rate, how long would 3 volunteers take?

Show answer

The job takes $4 \cdot 6 = 24$ worker-hours, so 3 volunteers take $\frac{24}{3} = 8$ hours.

10. **Medium** y varies inversely with x . Explain why tripling x makes y one third as large.

Show answer

Model answer: $y = \frac{k}{x}$. If x becomes $3x$, the new value is $\frac{k}{3x} = \frac{1}{3} \cdot \frac{k}{x}$, which is one third of the old y . The product xy stays equal to k .

11. **Hard** A gear with 48 teeth turns at 25 revolutions per minute. It meshes with a gear that has 20 teeth. How many revolutions per minute does the smaller gear make?

Show answer

$48 \cdot 25 = 1200$, so $20r = 1200$ and $r = 60$ revolutions per minute

12. **Hard** Henry weighs 120 pounds and sits 5 feet from the center of a seesaw. His sister Ruth weighs 80 pounds. How far from the center must Ruth sit for the seesaw to balance? (Weight $\times$ distance must be the same on both sides.)

Show answer

$120 \cdot 5 = 600$, so $80d = 600$ and $d = 7.5$ feet

13. **Hard** Describe the graph of $y = \frac{6}{x}$. Give the points with $x = 1, 2, 3, 6$ and $x = -1, -2, -3, -6$, and say where the graph is and is not.

Show answer

A hyperbola with two branches. One branch is in Quadrant I through $(1, 6), (2, 3), (3, 2), (6, 1)$; the other is in Quadrant III through $(-1, -6), (-2, -3), (-3, -2), (-6, -1)$. It never touches the y -axis ($x = 0$ is not allowed) or the x -axis (y is never 0).

14. **Challenge** y varies inversely with x . When x increases by 2, y decreases from 12 to 8. Find the constant of variation k .

Show answer

Let the first x be x . Then $12x = 8(x + 2)$, so $4x = 16$ and $x = 4$. So $k = 4 \cdot 12 = 48$ (check: $6 \cdot 8 = 48$).

COMMON MISTAKES

- Mixing up direct and inverse variation. For direct variation the ratio $\frac{y}{x}$ is constant; for inverse variation the product xy is constant. Multiply a pair to check before you write an equation.
- Setting up a proportion $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ for an inverse problem. Use products instead:
 $x_1y_1 = x_2y_2$.
- Getting an answer that goes the wrong way, such as more workers taking more time. Ask whether the answer makes sense: in inverse variation, when one quantity goes up the other goes down.
- Drawing the graph crossing an axis. x can never be 0 and y can never be 0, so the branches only get closer and closer to the axes.

Make it a game • Deal the Beans. Count out 24 beans. Deal them equally among 1, 2, 3, 4, 6 and 8 friends and fill in a table of friends x and beans each y . Check that xy stays 24, write $y = 24/x$, and sketch the hyperbola. Then the parent reads tables, and you shout 'inverse', 'direct' or 'neither' by testing xy and y/x .

12.2 SIMPLIFYING RATIONAL EXPRESSIONS

OBJECTIVES

- Find the excluded values of a rational expression.
- Simplify a rational expression by factoring the numerator and denominator and dividing out common factors.
- Explain why only common factors, never terms, may be divided out.
- Recognize opposite binomials and use $\frac{a-b}{b-a} = -1$.

WATCH

Simplifying Algebraic Expressions and Fractions & Outlining Common Errors with Simplification

Eddie Woo · 11:34

<https://www.youtube.com/watch?v=PHiV1t3v4BM>

WHEN TO WATCH

Before Example 1: why only common factors, never terms, can be canceled.

Note: Eddie Woo teaches in Australia: he says 'algebraic fractions' for rational expressions.

BACKUP VIDEO

Algebraic Fractions (3 of 3: Denominators & restricted domains)

Eddie Woo · 5:48

<https://www.youtube.com/watch?v=fs6YeGZ3QYM>

Before the Core ideas: excluded values (restricted domains).

KEY TERMS

rational expression

A fraction whose numerator and denominator are polynomials, such as $\frac{x+2}{x^2-9}$. The denominator cannot be zero.

excluded value

A value of the variable that makes the denominator equal to zero. The expression is undefined there, so that value is not allowed.

simplest form

A rational expression is in simplest form when its numerator and denominator have no common factor other than 1 (or -1).

opposites

Two expressions whose sum is zero, such as $x - 5$ and $5 - x$. Each one equals -1 times the other.

CORE IDEAS

A **rational expression** is a fraction made of polynomials, like $\frac{3}{x}$, $\frac{x+1}{x-4}$ or $\frac{x^2-9}{x^2+5x+6}$. The word *rational* comes from *ratio*. Everything you know about fractions carries over, because a rational expression turns into an ordinary fraction as soon as you put in a number for x .

Since you can never divide by zero, the first job is to find the **excluded values**: set the denominator equal to zero and solve. For $\frac{x+2}{x^2-9}$, factor the denominator: $(x-3)(x+3) = 0$ gives $x = 3$ or $x = -3$. Those two values are not allowed. Always find excluded values from the *original* denominator, before anything is divided out, because the original expression is still undefined there.

To simplify $\frac{12}{18}$ you write $\frac{2 \cdot 6}{3 \cdot 6}$ and divide out the 6 to get $\frac{2}{3}$. Why is that allowed?

Because $\frac{2 \cdot 6}{3 \cdot 6} = \frac{2}{3} \cdot \frac{6}{6} = \frac{2}{3} \cdot 1$. The same reasoning works with polynomials: $\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b}$ whenever $c \neq 0$. So simplifying means: **factor the numerator and denominator completely, then divide out factors they share**. For example,

$$\frac{x^2+5x+6}{x^2-9} = \frac{(x+2)(x+3)}{(x-3)(x+3)} = \frac{x+2}{x-3}, \text{ with } x \neq 3 \text{ and } x \neq -3.$$

Only factors cancel, never terms. A factor is something that is *multiplied*; a term is something that is *added*. In $\frac{x+3}{3}$ the 3 on top is a term, not a factor, so you may not cross it out to get x . Test it with a number: when $x = 6$, $\frac{6+3}{3} = 3$, not 6. The argument above used $\frac{c}{c} = 1$, which only appears when c multiplies the whole numerator and the whole denominator. With plain numbers, $\frac{5+3}{3} = \frac{8}{3}$, not 5. If you want to cancel something, first rewrite the numerator and denominator as products.

Watch for **opposites**. Since $-(a-b) = -a+b = b-a$, the binomial $b-a$ is -1 times $a-b$. So $\frac{a-b}{b-a} = \frac{a-b}{-(a-b)} = -1$ whenever $a \neq b$. For example, $\frac{x-5}{5-x} = -1$ for

$x \neq 5$. To use this in a larger problem, factor -1 out of one binomial: $\frac{12 - 3x}{x^2 - 16} = \frac{-3(x - 4)}{(x - 4)(x + 4)} = \frac{-3}{x + 4}$, with $x \neq 4$ and $x \neq -4$.

A step-by-step plan: (1) factor the numerator and denominator completely (GCF first, as in Unit 9); (2) write down the excluded values from the denominator; (3) divide out common factors, including opposite pairs; (4) write the result and the restrictions. The simplified expression equals the original for every allowed value of x ; it is the same expression in a cleaner form.

KEY FORMULAS

$$\frac{ac}{bc} = \frac{a}{b}, \quad b \neq 0, c \neq 0$$

$$b - a = -(a - b)$$

$$\frac{a - b}{b - a} = -1, \quad a \neq b$$

Practice habit. Factor first, cancel second. Never cross anything out until both the numerator and denominator are written as products, and test your simplified answer by substituting an easy number such as $x = 1$ or $x = 2$ into the original and into your result.

WORKED EXAMPLES

Example 1. Grace's rectangular vegetable garden has an area of $(x^2 + 7x + 12)$ square feet and a width of $(x + 3)$ feet. Write the length as a simplified expression.

Show solution

1. Length = area $\div$ width: $\frac{x^2 + 7x + 12}{x + 3}$.

2. Factor the numerator: $x^2 + 7x + 12 = (x + 3)(x + 4)$.

3. Rewrite: $\frac{(x + 3)(x + 4)}{x + 3}$, with $x \neq -3$.

4. Divide out the common factor $x + 3$: the length is $x + 4$.

5. Check with $x = 2$: area $4 + 14 + 12 = 30$, width 5, and $30 \div 5 = 6 = 2 + 4$.

Answer: $x + 4$ feet

Example 2. Simplify $\frac{12 - 3x}{x^2 - 16}$ and state the excluded values.

Show solution

1. Factor the numerator: $12 - 3x = -3(x - 4)$.
2. Factor the denominator: $x^2 - 16 = (x - 4)(x + 4)$.
3. Excluded values: $(x - 4)(x + 4) = 0$ gives $x = 4$ and $x = -4$.
4. Rewrite: $\frac{-3(x - 4)}{(x - 4)(x + 4)}$.
5. Divide out $x - 4$: $\frac{-3}{x + 4}$.
6. Check with $x = 0$: original $\frac{12}{-16} = -\frac{3}{4}$; result $\frac{-3}{4}$. They match.

Answer: $-\frac{3}{x + 4}$, with $x \neq 4$ and $x \neq -4$

PRACTICE

1. Easy Find the excluded value of $\frac{5}{x - 7}$.

Show answer

$$x - 7 = 0, \text{ so } x = 7$$

2. Easy Find the excluded values of $\frac{x + 1}{x^2 - 4x}$.

Show answer

$$x(x - 4) = 0, \text{ so } x = 0 \text{ and } x = 4$$

3. Easy Simplify $\frac{12x^3}{18x}$. Assume $x \neq 0$.

Show answer

$$\frac{2x^2}{3}$$

4. Easy Simplify $\frac{4x + 8}{4}$.

Show answer

$$\frac{4(x + 2)}{4} = x + 2$$

5. Medium Simplify $\frac{3x - 6}{x^2 - 4}$.

Show answer

$$\frac{3(x - 2)}{(x - 2)(x + 2)} = \frac{3}{x + 2}, \text{ with } x \neq 2, -2$$

6. Medium Simplify $\frac{x - 8}{8 - x}$.

Show answer

$8 - x = -(x - 8)$, so the expression equals -1 (for $x \neq 8$)

7. Medium Simplify $\frac{x^2 - 2x - 15}{x^2 + x - 6}$ and state the excluded values.

Show answer

$$\frac{(x - 5)(x + 3)}{(x + 3)(x - 2)} = \frac{x - 5}{x - 2}, \text{ with } x \neq -3 \text{ and } x \neq 2$$

8. Medium Simplify $\frac{10 - 5x}{x^2 - 4x + 4}$.

Show answer

$$\frac{-5(x - 2)}{(x - 2)^2} = -\frac{5}{x - 2}, \text{ with } x \neq 2$$

9. Medium Samuel simplified $\frac{x + 5}{5}$ to x by crossing out the 5s. Explain why this is wrong.

Show answer

Model answer: The 5 in the numerator is a term (it is added), not a factor of the whole numerator, so it cannot be divided out. Only factors cancel. Test $x = 10$: $\frac{10 + 5}{5} = 3$, not 10. The expression is already in simplest form (or can be written $\frac{5x}{5} + 1$).

10. **Hard** A rectangular sign for the church bake sale has an area of $(2x^2 + 7x + 3)$ square inches and a length of $(2x + 1)$ inches. Write the width as a simplified expression.

Show answer

$$\frac{(2x + 1)(x + 3)}{2x + 1} = x + 3 \text{ inches}$$

11. **Hard** Henry's family drives $(x^2 + 9x + 20)$ miles at an average speed of $(x + 5)$ miles per hour. Write the travel time as a simplified expression.

Show answer

$$\text{Time} = \text{distance} \div \text{speed} = \frac{(x + 4)(x + 5)}{x + 5} = x + 4 \text{ hours}$$

12. **Hard** A wooden storage box built by Ruth's father has a volume of $(x^3 + 5x^2 + 6x)$ cubic inches. Its base has an area of $(x^2 + 3x)$ square inches. Write the height of the box as a simplified expression.

Show answer

$$\text{Height} = \text{volume} \div \text{base area} = \frac{x(x + 2)(x + 3)}{x(x + 3)} = x + 2 \text{ inches}$$

13. **Hard** Simplify $\frac{2x^2 - 5x - 3}{9 - x^2}$.

Show answer

$$\frac{(2x + 1)(x - 3)}{(3 - x)(3 + x)} = \frac{(2x + 1)(x - 3)}{-(x - 3)(x + 3)} = -\frac{2x + 1}{x + 3}, \text{ with } x \neq 3, -3$$

14. **Challenge** Simplify $\frac{x^3 - 9x}{x^3 + x^2 - 6x}$ and state all excluded values.

Show answer

$$\frac{x(x - 3)(x + 3)}{x(x + 3)(x - 2)} = \frac{x - 3}{x - 2}, \text{ with } x \neq -3, x \neq 0 \text{ and } x \neq 2$$

COMMON MISTAKES

- Cancelling terms instead of factors, as in $\frac{x + 6}{x + 2} = \frac{6}{2} = 3$. Factor first, and cancel only expressions that multiply the entire numerator and the entire denominator. A quick number test (try $x = 1$) will catch this mistake.
- Finding excluded values from the simplified expression. In $\frac{(x + 2)(x + 3)}{(x - 3)(x + 3)}$, $x = -3$ is still excluded even though $x + 3$ is gone after simplifying. Always use the original denominator.
- Treating $x - 5$ and $5 - x$ as the same. They are opposites: $\frac{x - 5}{5 - x} = -1$, not 1. Factor out -1 to make them match before cancelling.
- Stopping before factoring completely. Take out the GCF first, then look for a trinomial or a difference of squares, so that no common factor is missed.

Make it a game • Illegal Cancel Court. The parent plays a dramatic judge. Present three claims of cancelling, such as $(x + 3)/3 = x + 1$, which is illegal, and $(x^2 - 9)/(x - 3) = x + 3$, which is legal after factoring. Argue each case. Then simplify one rational expression properly, naming the excluded values before you cancel anything.

12.3 MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS

OBJECTIVES

- Multiply rational expressions by factoring first and dividing out common factors.
- Divide rational expressions by multiplying by the reciprocal of the divisor, and explain why this works.
- State the excluded values of a product or quotient, including values that make the divisor zero.
- Use multiplication of fractions to convert units and to solve area problems.

WATCH

Algebraic Fractions (2 of 2: Multiplication/Division)

Eddie Woo · 7:10

<https://www.youtube.com/watch?v=D3lbeq42p8w>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'algebraic fractions' for rational expressions.

BACKUP VIDEO

Products & Factors (2 of 2: Multiplying and Dividing Algebraic Fractions)

Eddie Woo · 5:46

<https://www.youtube.com/watch?v=ubuWAXeEAqw>

Before the practice set: more multiplying and dividing.

KEY TERMS

reciprocal

The number or expression you multiply by to get 1. The reciprocal of $\frac{c}{d}$ is $\frac{d}{c}$, because $\frac{c}{d} \cdot \frac{d}{c} = 1$.

divisor

The expression you are dividing by. In $\frac{a}{b} \div \frac{c}{d}$ the divisor is $\frac{c}{d}$.

conversion factor

A fraction equal to 1 that changes units, such as $\frac{5280 \text{ ft}}{1 \text{ mi}}$ or $\frac{1 \text{ h}}{60 \text{ min}}$.

CORE IDEAS

Rational expressions are multiplied exactly like fractions: multiply the numerators and multiply the denominators, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. For example, $\frac{2}{3} \cdot \frac{9}{10} = \frac{18}{30} = \frac{3}{5}$. The rule holds for any numbers, so it holds for expressions that stand for numbers.

The smart way is to **factor first and divide out common factors before multiplying**.

Since the whole product is one big fraction, $\frac{ac}{bd}$, any factor on top of either fraction can cancel with a matching factor on the bottom of either fraction. For example, $\frac{x^2 - 9}{4x} \cdot \frac{8x^2}{x + 3} = \frac{(x - 3)(x + 3)}{4x} \cdot \frac{8x^2}{x + 3}$. Divide out $x + 3$ and $4x$ (since $8x^2 = 4x \cdot 2x$) to get $2x(x - 3)$. Multiplying everything out first would give large polynomials that you would then have to factor all over again.

To divide, **multiply by the reciprocal of the divisor**: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. Why? A division can be written as a fraction, $\frac{a/b}{c/d}$. Multiply its top and bottom by $\frac{d}{c}$ (that is multiplying by 1, so nothing changes). The bottom becomes $\frac{c}{d} \cdot \frac{d}{c} = 1$, and the top becomes $\frac{a}{b} \cdot \frac{d}{c}$. Dividing by 1 changes nothing, so the quotient equals $\frac{a}{b} \cdot \frac{d}{c}$. Flip only the divisor (the second fraction), never the first.

Excluded values: any value that makes an original denominator zero is excluded. In a division there is one more: the divisor itself cannot be zero, so any value that makes the **numerator of the divisor** zero is also excluded. (After flipping, that numerator lands in the denominator.) In $\frac{x}{x + 1} \div \frac{x - 3}{x + 2}$, the excluded values are $x = -1$, $x = -2$ and $x = 3$.

The same idea is behind converting units. Units cancel like factors. To change 6 miles per hour into feet per minute, multiply by conversion factors that each equal 1: $\frac{6 \text{ mi}}{1 \text{ h}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ h}}{60 \text{ min}} = \frac{6 \times 5280}{60} \frac{\text{ft}}{\text{min}} = 528 \text{ feet per minute}$. Miles cancel with miles and hours with hours, just as $x + 3$ cancelled with $x + 3$ above.

A plan for every problem: (1) rewrite a division as multiplication by the reciprocal; (2) factor every numerator and denominator completely; (3) note the excluded values; (4) divide out common factors; (5) multiply what is left. You may leave the answer in factored form, such as $2x(x - 3)$.

KEY FORMULAS

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad b \neq 0, d \neq 0$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}, \quad b \neq 0, c \neq 0, d \neq 0$$

Practice habit. Write one step per line: flip the divisor, factor everything, cross out matching factors with a single slash, and then multiply what remains. Keeping each move on its own line makes it easy to spot a sign error or a factor that was cancelled twice.

WORKED EXAMPLES

Example 1. A rectangular quilt panel has length $\frac{x^2 - 4}{3x}$ feet and width $\frac{6x}{x + 2}$ feet. Write its area as a simplified expression.

Show solution

1. Area = length $\times$ width: $\frac{x^2 - 4}{3x} \cdot \frac{6x}{x + 2}$.

2. Factor: $\frac{(x - 2)(x + 2)}{3x} \cdot \frac{6x}{x + 2}$.

3. Divide out $x + 2$, and divide $6x$ by $3x$ to leave 2.

4. What remains: $2(x - 2)$.

5. Check with $x = 4$: length $\frac{12}{12} = 1$, width $\frac{24}{6} = 4$, area 4; and $2(4 - 2) = 4$.

Answer: $2(x - 2)$ square feet, or $2x - 4$ square feet

Example 2. Divide and simplify: $\frac{x^2 + 2x - 15}{x^2 - 4} \div \frac{x + 5}{x + 2}$. State the excluded values.

Show solution

1. Multiply by the reciprocal of the divisor: $\frac{x^2 + 2x - 15}{x^2 - 4} \cdot \frac{x + 2}{x + 5}$.

2. Factor: $\frac{(x + 5)(x - 3)}{(x - 2)(x + 2)} \cdot \frac{x + 2}{x + 5}$.

3. Excluded values: $x = 2$ and $x = -2$ (original denominators) and $x = -5$ (the divisor's numerator).

4. Divide out $x + 5$ and $x + 2$: $\frac{x - 3}{x - 2}$.

5. Check with $x = 0$: original $\frac{-15}{-4} \div \frac{5}{2} = 3.75 \div 2.5 = 1.5$; result $\frac{-3}{-2} = 1.5$.

Answer: $\frac{x - 3}{x - 2}$, with $x \neq -5, x \neq -2, x \neq 2$

PRACTICE

1. Easy Multiply and simplify: $\frac{3}{x} \cdot \frac{x^2}{12}$.

Show answer

$$\frac{3x^2}{12x} = \frac{x}{4}$$

2. Easy Multiply and simplify: $\frac{5a^2}{2b} \cdot \frac{4b^3}{15a}$.

Show answer

$$\frac{20a^2b^3}{30ab} = \frac{2ab^2}{3}$$

3. Easy Divide and simplify: $\frac{6x}{7} \div \frac{3x^2}{14}$.

Show answer

$$\frac{6x}{7} \cdot \frac{14}{3x^2} = \frac{84x}{21x^2} = \frac{4}{x}$$

4. Easy Multiply and simplify: $\frac{x + 2}{5} \cdot \frac{10}{x + 2}$.

Show answer

2 (for $x \neq -2$)

5. Medium Multiply and simplify: $\frac{x^2 - 1}{x} \cdot \frac{x^2}{x + 1}$.

Show answer

$$\frac{(x - 1)(x + 1)}{x} \cdot \frac{x^2}{x + 1} = x(x - 1)$$

6. Medium Multiply and simplify: $\frac{2x + 6}{x - 4} \cdot \frac{x^2 - 16}{x^2 + 3x}$.

Show answer

$$\frac{2(x + 3)}{x - 4} \cdot \frac{(x - 4)(x + 4)}{x(x + 3)} = \frac{2(x + 4)}{x}$$

7. Medium Divide and simplify: $\frac{x^2 - 25}{3x} \div \frac{x - 5}{6x^2}$.

Show answer

$$\frac{(x - 5)(x + 5)}{3x} \cdot \frac{6x^2}{x - 5} = 2x(x + 5)$$

8. Medium Multiply and simplify: $\frac{x^2 + 5x + 6}{x^2 - x - 12} \cdot \frac{x - 4}{x + 2}$.

Show answer

$$\frac{(x + 2)(x + 3)}{(x - 4)(x + 3)} \cdot \frac{x - 4}{x + 2} = 1$$

9. Medium Explain why dividing by $\frac{c}{d}$ gives the same result as multiplying by $\frac{d}{c}$.

Show answer

Model answer: Write the division as $\frac{a/b}{c/d}$ and multiply top and bottom by $\frac{d}{c}$, which is multiplying by 1. The bottom becomes $\frac{c}{d} \cdot \frac{d}{c} = 1$, so the result is just the top, $\frac{a}{b} \cdot \frac{d}{c}$.

10. Medium Samuel's tractor moves across the field at 4 miles per hour. How many feet per minute is that? (1 mile = 5280 feet.)

Show answer

$$\frac{4 \text{ mi}}{1 \text{ h}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ h}}{60 \text{ min}} = 352 \text{ feet per minute}$$

11. **Hard** Divide and simplify: $\frac{x^2 - 4x}{x^2 - 9} \div \frac{x^2 - 16}{x^2 + 7x + 12}$.

Show answer

$$\frac{x(x-4)}{(x-3)(x+3)} \cdot \frac{(x+3)(x+4)}{(x-4)(x+4)} = \frac{x}{x-3}, \text{ with } x \neq -4, -3, 3, 4$$

12. **Hard** A triangular wooden brace has a base of $\frac{x^2 - 1}{x}$ inches and a height of $\frac{4x}{x+1}$ inches. Write its area as a simplified expression. (Area of a triangle = $\frac{1}{2} \cdot \text{base} \cdot \text{height}$.)

Show answer

$$\frac{1}{2} \cdot \frac{(x-1)(x+1)}{x} \cdot \frac{4x}{x+1} = 2(x-1) \text{ square inches}$$

13. **Hard** A rectangular brick patio for the church courtyard has an area of $\frac{x^2 + 5x}{x-2}$ square meters and a length of $\frac{x^2 - 25}{x^2 - 4}$ meters. Write the width as a simplified expression.

Show answer

$$\text{Width} = \text{area} \div \text{length} = \frac{x(x+5)}{x-2} \cdot \frac{(x-2)(x+2)}{(x-5)(x+5)} = \frac{x(x+2)}{x-5} \text{ meters}$$

14. **Challenge** Divide and simplify: $\frac{x^3 - 4x}{x^2 + 4x + 4} \div \frac{x^2 - 2x}{3x + 6}$. State the excluded values.

Show answer

$$\frac{x(x-2)(x+2)}{(x+2)^2} \cdot \frac{3(x+2)}{x(x-2)} = 3, \text{ with } x \neq -2, x \neq 0, x \neq 2$$

COMMON MISTAKES

- Flipping the wrong fraction, or both. Only the divisor (the fraction after the $\div$ sign) is replaced by its reciprocal.
- Cancelling before flipping in a division. Rewrite the division as a multiplication first; only then may a factor on top cancel with a factor on the bottom.
- Multiplying everything out before simplifying. Factor first and divide out common factors; the work stays small and the answer comes out in simplest form.

- Forgetting the extra excluded values in a division. A value that makes the divisor's numerator zero is also excluded, because you cannot divide by zero.

Make it a game • Factor-Cancel Relay. Play as a team. You factor every numerator and denominator on the page, then pass the paper to the parent, who cancels common factors and multiplies. For division problems, say 'flip the second' and explain it as multiplying by the reciprocal, a conversion-factor fancy one. Swap roles after each problem for four problems.

12.4 ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

OBJECTIVES

- Add and subtract rational expressions with like denominators, using parentheses when subtracting a numerator with more than one term.
- Find the least common denominator (LCD) of monomial and simple binomial denominators.
- Add and subtract rational expressions with unlike denominators and simplify the result.
- Write sums of rational expressions for travel and work situations.

WATCH

Algebraic Fractions (1 of 2: Addition & Subtraction)

Eddie Woo · 9:46

<https://www.youtube.com/watch?v=Mk-p6ZNvIJ4>

WHEN TO WATCH

Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'algebraic fractions' for rational expressions.

BACKUP VIDEO

Algebraic Fractions (1 of 2: Addition/Subtraction)

Eddie Woo · 6:36

<https://www.youtube.com/watch?v=iv7bnl1I93A>

Before Example 2: unlike denominators.

KEY TERMS

like denominators

Denominators that are exactly the same, as in $\frac{3}{x+1}$ and $\frac{5}{x+1}$.

least common denominator (LCD)

The simplest expression that every denominator divides into. Build it from each different factor, using each factor the greatest number of times it appears in any one denominator.

equivalent expressions

Expressions that have the same value for every allowed value of the variable, such as $\frac{2}{x}$ and $\frac{2(x+3)}{x(x+3)}$ (for $x \neq -3$).

CORE IDEAS

With like denominators, add or subtract the numerators and keep the denominator:

$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$. Why? Dividing by c is the same as multiplying by $\frac{1}{c}$, so $\frac{a}{c} + \frac{b}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = (a+b) \cdot \frac{1}{c}$ by the distributive property. Two sevenths plus three sevenths is five sevenths for the same reason.

When you subtract, the whole second numerator is subtracted, so put it in

parentheses: $\frac{3x+1}{x-2} - \frac{x+5}{x-2} = \frac{3x+1-(x+5)}{x-2} = \frac{2x-4}{x-2}$. Then simplify: $\frac{2(x-2)}{x-2} = 2$. Without the parentheses you would write $3x+1-x+5$ and get the wrong answer. Always finish by factoring the numerator to see whether anything cancels.

With unlike denominators, first find the **least common denominator**. Factor each denominator, then take every different factor the greatest number of times it appears in any one denominator. For $6x^2$ and $4x$: $6x^2 = 2 \cdot 3 \cdot x \cdot x$ and $4x = 2 \cdot 2 \cdot x$, so the LCD is $2 \cdot 2 \cdot 3 \cdot x \cdot x = 12x^2$. For x and $x+3$, which share no factor, the LCD is the product $x(x+3)$. For x^2-9 and $x+3$, factor first: $x^2-9 = (x-3)(x+3)$ already contains $x+3$, so the LCD is $(x-3)(x+3)$.

Next rewrite each fraction with the LCD by multiplying its top and bottom by the factors it is missing. This is allowed because something like $\frac{x+3}{x+3}$ equals 1, and multiplying by 1 does not change a value. For example, $\frac{2}{x} + \frac{3}{x+1} = \frac{2(x+1)}{x(x+1)} + \frac{3x}{x(x+1)} = \frac{2x+2+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$.

This explains a mistake to avoid: $\frac{1}{x} + \frac{1}{y}$ is **not** $\frac{2}{x+y}$. Adding the denominators would be like saying $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$, which is only one half. Fractions are added by counting equal-size pieces, so the pieces (the denominators) must match first.

Sums of rational expressions come up naturally. If a trip has two legs, each leg's time is $\frac{\text{distance}}{\text{speed}}$, and the total time is the sum. If Ruth can do a job alone in x hours, she does $\frac{1}{x}$ of it each hour; working together, two people's hourly fractions add. You will solve equations like these in Lesson 12.5.

KEY FORMULAS

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad b \neq 0, d \neq 0$$

Practice habit. Bracket before you subtract: the moment you see a minus sign in front of a fraction, rewrite its numerator in parentheses, then distribute the minus to every term. Finish every problem by factoring the numerator to see whether the answer simplifies.

WORKED EXAMPLES

Example 1. Add: $\frac{5}{6x} + \frac{3}{4x^2}$.

Show solution

1. Factor the denominators: $6x = 2 \cdot 3 \cdot x$ and $4x^2 = 2 \cdot 2 \cdot x \cdot x$.

2. LCD: $2 \cdot 2 \cdot 3 \cdot x \cdot x = 12x^2$.

3. Rewrite: $\frac{5 \cdot 2x}{6x \cdot 2x} + \frac{3 \cdot 3}{4x^2 \cdot 3} = \frac{10x}{12x^2} + \frac{9}{12x^2}$.

4. Add the numerators: $\frac{10x+9}{12x^2}$.

5. The numerator $10x+9$ has no factor in common with $12x^2$, so this is simplest form.

6. Check with $x = 1$: $\frac{5}{6} + \frac{3}{4} = \frac{10}{12} + \frac{9}{12} = \frac{19}{12}$, and $\frac{10+9}{12} = \frac{19}{12}$.

Answer: $\frac{10x+9}{12x^2}$

Example 2. Henry drives 30 miles to town at x miles per hour and comes home on a faster road at $(x + 10)$ miles per hour. Write his total driving time as a single simplified rational expression.

Show solution

1. Time = distance $\div$ speed, so the total time is $\frac{30}{x} + \frac{30}{x + 10}$.

2. The LCD is $x(x + 10)$.

3. Rewrite: $\frac{30(x + 10)}{x(x + 10)} + \frac{30x}{x(x + 10)}$.

4. Add the numerators: $\frac{30x + 300 + 30x}{x(x + 10)} = \frac{60x + 300}{x(x + 10)}$.

5. Factor the numerator: $\frac{60(x + 5)}{x(x + 10)}$. No factor cancels.

6. Check with $x = 50$: $\frac{30}{50} + \frac{30}{60} = 0.6 + 0.5 = 1.1$ hours, and $\frac{60 \cdot 55}{50 \cdot 60} = \frac{3300}{3000} = 1.1$ hours.

Answer: $\frac{60(x + 5)}{x(x + 10)}$ hours

PRACTICE

1. Easy Add: $\frac{3}{x} + \frac{7}{x}$.

Show answer

$$\frac{10}{x}$$

2. Easy Add and simplify: $\frac{2x}{x + 4} + \frac{8}{x + 4}$.

Show answer

$$\frac{2x + 8}{x + 4} = \frac{2(x + 4)}{x + 4} = 2$$

3. Easy Find the LCD of $\frac{1}{6x^2}$ and $\frac{1}{9x^3}$.

Show answer

$$18x^3$$

4. Easy Subtract: $\frac{x}{4} - \frac{x}{6}$.

Show answer

$$\frac{3x}{12} - \frac{2x}{12} = \frac{x}{12}$$

5. Medium Subtract and simplify: $\frac{5x+2}{x-3} - \frac{2x+11}{x-3}$.

Show answer

$$\frac{5x+2-(2x+11)}{x-3} = \frac{3x-9}{x-3} = 3$$

6. Medium Subtract: $\frac{7}{2a} - \frac{5}{3a^2}$.

Show answer

$$\text{LCD } 6a^2: \frac{21a}{6a^2} - \frac{10}{6a^2} = \frac{21a-10}{6a^2}$$

7. Medium Add: $\frac{2}{x} + \frac{3}{x+1}$.

Show answer

$$\frac{2(x+1)+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$$

8. Medium Samuel walks 2 miles at x miles per hour and then jogs 3 miles at $2x$ miles per hour. Write his total time as a single rational expression.

Show answer

$$\frac{2}{x} + \frac{3}{2x} = \frac{4}{2x} + \frac{3}{2x} = \frac{7}{2x} \text{ hours}$$

9. Medium Subtract: $\frac{4}{x-2} - \frac{1}{x+3}$.

Show answer

$$\frac{4(x+3)-(x-2)}{(x-2)(x+3)} = \frac{3x+14}{(x-2)(x+3)}$$

10. **Medium** Explain why $\frac{3}{x} + \frac{2}{y}$ is not equal to $\frac{5}{x+y}$. Use $x = 1$ and $y = 1$ in your explanation.

Show answer

Model answer: Fractions can only be added after they have a common denominator; you cannot add the denominators. With $x = y = 1$, $\frac{3}{1} + \frac{2}{1} = 5$, but $\frac{5}{1+1} = 2.5$. The correct sum is $\frac{3y+2x}{xy}$.

11. **Hard** Add and simplify: $\frac{x}{x^2-9} + \frac{2}{x+3}$.

Show answer

$$\text{LCD } (x-3)(x+3): \frac{x+2(x-3)}{(x-3)(x+3)} = \frac{3x-6}{(x-3)(x+3)} = \frac{3(x-2)}{(x-3)(x+3)}$$

12. **Hard** Add: $\frac{3}{x-4} + \frac{5}{4-x}$.

Show answer

$$\frac{5}{4-x} = -\frac{5}{x-4}, \text{ so the sum is } \frac{3-5}{x-4} = -\frac{2}{x-4}$$

13. **Hard** Ruth can paint the fence around the garden alone in x hours. Her sister Grace takes $(x+2)$ hours alone. Write the fraction of the fence they paint together in one hour as a single simplified expression.

Show answer

$$\frac{1}{x} + \frac{1}{x+2} = \frac{x+2+x}{x(x+2)} = \frac{2x+2}{x(x+2)}, \text{ or } \frac{2(x+1)}{x(x+2)}$$

14. **Hard** A fishing boat travels 12 miles downstream at $(r+3)$ miles per hour and then 12 miles back upstream at $(r-3)$ miles per hour. Write the total time for the round trip as a single simplified expression.

Show answer

$$\frac{12}{r+3} + \frac{12}{r-3} = \frac{12(r-3) + 12(r+3)}{(r+3)(r-3)} = \frac{24r}{(r+3)(r-3)} \text{ hours}$$

15. **Challenge** Subtract and simplify: $\frac{x}{x-2} - \frac{8}{x^2-4}$.

Show answer

$$\frac{x(x+2)-8}{(x-2)(x+2)} = \frac{x^2+2x-8}{(x-2)(x+2)} = \frac{(x+4)(x-2)}{(x-2)(x+2)} = \frac{x+4}{x+2}$$

COMMON MISTAKES

- Adding denominators, as in $\frac{1}{x} + \frac{1}{3} = \frac{2}{x+3}$. Find a common denominator first; only the numerators are added.
- Forgetting parentheses when subtracting. In $\frac{4(x+3) - (x-2)}{(x-2)(x+3)}$, write $-(x-2) = -x+2$, not $-x-2$; the minus sign must reach every term of the second numerator.
- Multiplying only the denominator by the missing factor. To keep the value the same, multiply the numerator and the denominator by the same factor.
- Stopping too soon. After combining, factor the numerator; it may share a factor with the denominator, as in $\frac{3x-9}{x-3} = 3$.

Make it a game • LCD Lightning. Write denominators on slips: x , x^2 , $3x$, $x+1$, $x-2$ and x^2-1 . Draw two slips and shout the LCD within five seconds. Then add them as fractions, with numerators of your choice. For a challenge, subtract instead, using parentheses on the second numerator. Score a point for a correct LCD and another for the final answer.

12.5 SOLVING RATIONAL EQUATIONS

OBJECTIVES

- Solve rational equations by multiplying both sides by the least common denominator.
- Identify extraneous solutions by checking each answer against the excluded values, and explain why they appear.
- Solve proportions that contain variable expressions.
- Write and solve rational equations for work and travel problems.

WATCH

Equations with Algebraic Fractions

Eddie Woo · 9:20

<https://www.youtube.com/watch?v=UpjDFA97AaE>

WHEN TO WATCH

Before Example 1.

Note: Neither video stresses extraneous solutions (objective 2). Teach that from Example 2 and the Core ideas.

BACKUP VIDEO

Solving Equations w/ Algebraic Fractions (Example 1 of 3)

Eddie Woo · 8:38

<https://www.youtube.com/watch?v=KXqFei5mE4I>

Before the practice set: a second worked equation.

KEY TERMS

rational equation

An equation that contains at least one rational expression with the variable in a denominator, such as $\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$.

extraneous solution

A number that solves the simpler equation you got by clearing fractions but does not solve the original equation, usually because it makes a denominator zero.

proportion

An equation stating that two ratios are equal, such as $\frac{x+2}{5} = \frac{x-1}{2}$.

work rate

The fraction of a job done in one unit of time. Someone who finishes a job in 6 hours alone has a work rate of $\frac{1}{6}$ of the job per hour.

CORE IDEAS

In Lesson 2.4 you cleared fractions from an equation by multiplying both sides by the LCD. Rational equations are solved the same way. First list the excluded values, the numbers that make a denominator zero. Then multiply **every term on both sides** by the LCD. The fractions disappear and you are left with a linear or quadratic equation you already know how to solve. For example, in $\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$ the LCD is $3x$; multiplying gives $15 - x = 3$, so $x = 12$. Check: $\frac{5}{12} - \frac{1}{12} = \frac{1}{12}$.

Why must you check? Multiplying both sides of an equation by a nonzero number keeps exactly the same solutions. But the LCD contains x , and for some value of x it equals zero. Multiplying both sides by zero turns any equation into $0 = 0$, which is true, so the cleared equation can pick up a solution the original never had. That new number is an **extraneous solution**. It is always one of the excluded values, so compare each answer with the excluded values and throw out any match. Substituting into the original equation is the surest check.

Here is an example with no solution. Solve $\frac{x}{x-2} + 1 = \frac{2}{x-2}$. The excluded value is $x = 2$. Multiply by $x - 2$: $x + (x - 2) = 2$, so $2x = 4$ and $x = 2$. But 2 is excluded: putting it back in gives $\frac{2}{0}$, which is undefined. The equation has **no solution**.

A **proportion** is a rational equation with one fraction on each side. Cross multiplying, $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$, is simply multiplying both sides by bd . So $\frac{3}{x-1} = \frac{2}{x+4}$ becomes $3(x+4) = 2(x-1)$, so $3x + 12 = 2x - 2$ and $x = -14$. Since -14 is not excluded and $\frac{3}{-15} = \frac{2}{-10} = -\frac{1}{5}$, the answer checks.

Work problems. If Samuel can paint the barn alone in 6 hours, he paints $\frac{1}{6}$ of it each hour, so in t hours he paints $\frac{t}{6}$ of it. If Grace takes 9 hours alone, in t hours she

paints $\frac{t}{9}$. Working together until the job is done, their parts add up to one whole barn: $\frac{t}{6} + \frac{t}{9} = 1$. In general, (rate) $\times$ (time) = (part of the job done), and the parts add to 1.

Travel problems. Since $d = rt$, the time for a trip is $t = \frac{d}{r}$. When two trips take the same time, set their times equal; when one trip takes longer, the difference of the times is known. With a river current of c , a boat whose speed in still water is r goes $r + c$ downstream and $r - c$ upstream. Always ask whether your answer is reasonable: a speed or a time cannot be negative.

KEY FORMULAS

$$\begin{aligned}\text{work rate} &= \frac{1}{\text{time to do the job alone}} \\ \frac{t}{a} + \frac{t}{b} &= 1 \\ t &= \frac{d}{r}\end{aligned}$$

Practice habit. Check by substituting, every time. Before you solve, write the excluded values beside the equation; after you solve, cross out any answer on that list and substitute the rest into the original equation.

WORKED EXAMPLES

Example 1. Solve $\frac{x^2}{x-3} - 8 = \frac{9}{x-3}$.

Show solution

1. Excluded value: $x - 3 = 0$, so $x \neq 3$.
2. Multiply every term by $x - 3$: $x^2 - 8(x - 3) = 9$.
3. Distribute: $x^2 - 8x + 24 = 9$.
4. Get 0 on one side: $x^2 - 8x + 15 = 0$.
5. Factor: $(x - 3)(x - 5) = 0$, so $x = 3$ or $x = 5$.
6. $x = 3$ is excluded, so it is extraneous.
7. Check $x = 5$: $\frac{25}{2} - 8 = 12.5 - 8 = 4.5$ and $\frac{9}{2} = 4.5$. It works.

Answer: $x = 5$ ($x = 3$ is extraneous)

Example 2. Samuel can paint the family's barn alone in 6 hours. His sister Grace can paint it alone in 9 hours. How long will it take them working together? Give the answer in hours, then in hours and minutes.

Show solution

1. Let t be the time together in hours. Samuel paints $\frac{t}{6}$ of the barn and Grace paints $\frac{t}{9}$.
2. The parts make one whole barn: $\frac{t}{6} + \frac{t}{9} = 1$.
3. Multiply both sides by the LCD, 18: $3t + 2t = 18$.
4. Combine: $5t = 18$, so $t = 3.6$.
5. Convert: 0.6 hour = $0.6 \times 60 = 36$ minutes.
6. Check: $\frac{3.6}{6} + \frac{3.6}{9} = 0.6 + 0.4 = 1$. Also, the time together is less than either time alone, as it should be.

Answer: 3.6 hours, which is 3 hours 36 minutes

PRACTICE

1. Easy Solve $\frac{3}{x} = \frac{6}{10}$.

Show answer

$$6x = 30, \text{ so } x = 5$$

2. Easy Solve $\frac{8}{x} + 2 = 6$.

Show answer

$$\frac{8}{x} = 4, \text{ so } 8 = 4x \text{ and } x = 2$$

3. Easy Solve $\frac{x+2}{5} = \frac{x-1}{2}$.

Show answer

$$2(x+2) = 5(x-1), \text{ so } 2x+4 = 5x-5, 9 = 3x \text{ and } x = 3$$

4. Easy Solve $\frac{1}{x} + \frac{1}{2x} = \frac{1}{4}$.

Show answer

$$\text{Multiply by } 4x: 4 + 2 = x, \text{ so } x = 6$$

5. Medium Solve $\frac{3}{x-1} = \frac{2}{x+4}$.

Show answer

$3(x+4) = 2(x-1)$, so $3x + 12 = 2x - 2$ and $x = -14$

6. Medium Solve $\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$.

Show answer

Multiply by $3x$: $15 - x = 3$, so $x = 12$

7. Medium Solve $\frac{x}{x-2} + 1 = \frac{2}{x-2}$.

Show answer

Multiply by $x-2$: $x + x - 2 = 2$, so $x = 2$. But $x = 2$ is excluded, so there is no solution.

8. Medium Solve $x + \frac{6}{x} = 5$.

Show answer

Multiply by x : $x^2 + 6 = 5x$, so $x^2 - 5x + 6 = 0$, $(x-2)(x-3) = 0$; $x = 2$ or $x = 3$

9. Medium Ruth uses 5 cups of flour to bake 2 loaves of bread. How many cups of flour does she need to bake 9 loaves for the church potluck?

Show answer

$\frac{5}{2} = \frac{f}{9}$, so $2f = 45$ and $f = 22.5$ cups

10. Medium Henry can mow the church lawn alone in 3 hours. His brother can mow it alone in 6 hours. How long will it take them working together?

Show answer

$\frac{t}{3} + \frac{t}{6} = 1$; multiply by 6: $2t + t = 6$, so $t = 2$ hours

11. **Hard** A riverboat travels 24 miles downstream in the same time it takes to travel 16 miles upstream. The current flows at 2 miles per hour. What is the boat's speed in still water?

Show answer

$\frac{24}{r+2} = \frac{16}{r-2}$, so $24r - 48 = 16r + 32$, $8r = 80$ and $r = 10$ miles per hour (each trip takes 2 hours)

12. **Hard** Solve $\frac{x}{x-2} + \frac{1}{x} = \frac{4}{x(x-2)}$.

Show answer

Excluded: 0 and 2. Multiply by $x(x-2)$: $x^2 + x - 2 = 4$, so $x^2 + x - 6 = 0$ and $(x+3)(x-2) = 0$. $x = 2$ is extraneous, so $x = -3$.

13. **Hard** Grace and her father can paint a barn together in 4 days. Her father alone could do it in 6 days. How many days would Grace take alone?

Show answer

$\frac{4}{6} + \frac{4}{g} = 1$, so $\frac{4}{g} = \frac{1}{3}$ and $g = 12$ days

14. **Hard** Explain why you must check the answers to a rational equation even when you made no arithmetic mistakes.

Show answer

Model answer: To clear the fractions you multiply both sides by the LCD, which contains the variable. For an excluded value the LCD equals zero, and multiplying by zero can make a false equation true. So the cleared equation may have a solution that makes a denominator of the original equation zero. That extraneous solution must be thrown out.

15. **Challenge** Henry drives 120 miles to visit his grandparents. On the way home he drives 10 miles per hour faster, and the trip takes 24 minutes less. How fast did he drive on the way there?

Show answer

24 minutes = $\frac{2}{5}$ hour. $\frac{120}{r} - \frac{120}{r+10} = \frac{2}{5}$; multiply by $5r(r+10)$:
 $6000 = 2r^2 + 20r$, so $r^2 + 10r - 3000 = 0$, $(r+60)(r-50) = 0$. A speed is positive, so $r = 50$ miles per hour.

COMMON MISTAKES

- Multiplying only some terms by the LCD. Every term on both sides, including whole numbers like the 8 in $\frac{x^2}{x-3} - 8$, must be multiplied.
- Keeping an extraneous solution. Write the excluded values first and reject any answer that matches one of them.
- Adding the times in a work problem instead of the rates. Two people do not take $6 + 9 = 15$ hours together; add the parts of the job, $\frac{t}{6} + \frac{t}{9} = 1$. The time together must be less than either time alone.
- Accepting an answer that makes no sense in the story, such as a negative speed. Check the answer in the words of the problem.

Make it a game • Work Rate Experiment. Time yourself moving 30 beans from one bowl to another with one hand, and time the parent as well. Predict the combined time using $1/a + 1/b = 1/t$. Then both work together and check against the clock. Afterwards, solve $x/(x-3) = 3/(x-3) + 2$ and check the answer against the excluded value.

RATIONAL EXPRESSIONS: AN INTRODUCTION

This unit is about fractions whose numerators and denominators are polynomials, called rational expressions. It begins with inverse variation, $y = \frac{k}{x}$, where one quantity goes down as the other goes up: driving faster shortens a trip, and more workers finish a job sooner. Then you will simplify, multiply, divide, add and subtract rational expressions, and finally solve equations that contain them.

Nothing in this unit is truly new. Every rule is a rule you already know for ordinary fractions, and almost every step depends on the factoring you learned in Unit 9. The key new idea is care about division by zero: excluded values, and extraneous solutions that appear when both sides of an equation are multiplied by an expression that can be zero.

These skills lead straight into Algebra II, where rational functions, their graphs and more complicated rational equations are studied in depth. They also help in physics, chemistry and everyday problems about rates, work and travel.

PREREQUISITES

- Add, subtract, multiply and divide numerical fractions, including finding a least common denominator.
- Factor GCFs, trinomials and differences of squares (Lessons 9.1–9.5).
- Solve equations by clearing fractions (Lesson 2.4) and quadratic equations by factoring (Lesson 9.6).
- Use direct variation $y = kx$ (Lesson 5.2) and the formula $d = rt$.
- Apply the properties of exponents to simplify monomials (Lesson 7.1).

MASTERY CHECKLIST

- ☐ I can tell whether a table or situation shows inverse variation by checking that xy is constant.
- ☐ I can find k , write $y = \frac{k}{x}$, and solve gear, travel and worker problems.
- ☐ I can describe the graph of $y = \frac{k}{x}$ and explain why it never touches the axes.
- ☐ I can find the excluded values of a rational expression.
- ☐ I can simplify a rational expression by factoring and dividing out common factors, and explain why terms cannot be cancelled.
- ☐ I can recognize opposites and use $\frac{a-b}{b-a} = -1$.
- ☐ I can multiply and divide rational expressions, and explain why dividing means multiplying by the reciprocal.
- ☐ I can find the LCD of monomial and binomial denominators and add or subtract rational expressions.
- ☐ I can use parentheses correctly when subtracting a numerator with more than one term.

- ☐ I can solve rational equations and proportions and reject extraneous solutions.
- ☐ I can write and solve rational equations for work and travel problems.

LESSONS IN THIS UNIT

12.1 Inverse Variation

45 minutes

12.2 Simplifying Rational Expressions

2 sessions of 45 minutes

12.3 Multiplying and Dividing Rational Expressions

2 sessions of 45 minutes

12.4 Adding and Subtracting Rational Expressions

2 sessions of 45 minutes

12.5 Solving Rational Equations

2 sessions of 45 minutes

UNIT 12 TEST

You have 60 minutes. A scientific calculator is allowed, but every problem can be done by hand. Show your work on every problem. Give exact answers in simplest form unless a problem says otherwise, state excluded values when asked, and include units in the answers to word problems. Check the solutions of every equation in the original equation.

1. **5 pts**
 y varies inversely with x , and $y = 9$ when $x = 4$. Find y when $x = 12$.
2. **5 pts**
 Six volunteers can pack the food boxes for a community food drive in 4 hours. Working at the same rate, how long would 8 volunteers take?
3. **5 pts**
 The table shows x : 2, 4, 5, 10 and y : 15, 7.5, 6, 3. (a) Does it show inverse variation? (b) If so, what is k ?

4. (5 pts)

Find the excluded values of $\frac{x+4}{x^2-6x}$.

5. (7 pts)

Simplify $\frac{x^2-49}{x^2+4x-21}$.

6. (7 pts)

Simplify $\frac{15-5x}{x^2-9}$.

7. (7 pts)

A student simplified $\frac{x+6}{x+2}$ to 3 by crossing out the x 's and dividing 6 by 2.
Explain the error, and use $x = 1$ to show the answer is wrong.

8. (7 pts)

Multiply and simplify: $\frac{x^2+3x}{4} \cdot \frac{8}{x^2-9}$.

9. (7 pts)

Divide and simplify: $\frac{x^2-x-6}{x^2-4} \div \frac{x-3}{2x}$.

10. (6 pts)

Add: $\frac{3}{2x} + \frac{5}{6x^2}$.

11. (8 pts)

Subtract and simplify: $\frac{5}{x-1} - \frac{3}{x+1}$.

12. (7 pts)

Solve $\frac{6}{x} - \frac{1}{2} = \frac{1}{x}$.

13. (8 pts)

Solve $\frac{x}{x+3} + 1 = \frac{-3}{x+3}$.

14. (8 pts)

A father can split the winter's firewood alone in 6 hours. His son can do it alone in 12 hours. How long will it take them working together?

15. **8 pts**

A small plane flies 300 miles with the wind in the same time it flies 200 miles against the wind. The wind blows at 25 miles per hour. What is the plane's speed in still air?

ANSWER KEY

1. $k = 36$, so $y = \frac{36}{12} = 3$

2. $6 \cdot 4 = 24$ worker-hours, so $\frac{24}{8} = 3$ hours

3. (a) Yes, every product xy is 30. (b) $k = 30$

4. $x(x - 6) = 0$, so $x = 0$ and $x = 6$

5. $\frac{(x - 7)(x + 7)}{(x + 7)(x - 3)} = \frac{x - 7}{x - 3}$, with $x \neq -7, 3$

6. $\frac{-5(x - 3)}{(x - 3)(x + 3)} = -\frac{5}{x + 3}$, with $x \neq 3, -3$

7. Model answer: x , 6 and 2 are terms (added), not factors of the whole numerator and denominator, and only common factors can be divided out. With $x = 1$, $\frac{1 + 6}{1 + 2} = \frac{7}{3}$, not 3. The expression is already in simplest form.

8. $\frac{x(x + 3)}{4} \cdot \frac{8}{(x - 3)(x + 3)} = \frac{2x}{x - 3}$

9. $\frac{(x - 3)(x + 2)}{(x - 2)(x + 2)} \cdot \frac{2x}{x - 3} = \frac{2x}{x - 2}$

10. LCD $6x^2$: $\frac{9x}{6x^2} + \frac{5}{6x^2} = \frac{9x + 5}{6x^2}$

11. $\frac{5(x + 1) - 3(x - 1)}{(x - 1)(x + 1)} = \frac{2x + 8}{(x - 1)(x + 1)} = \frac{2(x + 4)}{(x - 1)(x + 1)}$

12. Multiply by $2x$: $12 - x = 2$, so $x = 10$

13. Multiply by $x + 3$: $x + x + 3 = -3$, so $x = -3$. But -3 is excluded, so there is no solution.

14. $\frac{t}{6} + \frac{t}{12} = 1$; multiply by 12: $2t + t = 12$, so $t = 4$ hours

15. $\frac{300}{r + 25} = \frac{200}{r - 25}$, so $300r - 7500 = 200r + 5000$, $100r = 12500$ and $r = 125$ miles per hour

UNIT 13

DATA AND STATISTICS: SCATTER PLOTS AND LINES OF FIT

In this last unit, algebra meets real measurements. You will plot paired data on scatter plots, describe the correlation you see, and learn the most important caution in statistics: correlation does not prove causation. Then you will summarize a trend with a line of fit, write its equation, interpret its slope and intercept, make predictions, and judge the fit with residuals, the idea behind the least-squares method published by Legendre in 1805.

The unit closes with a review of the whole course. Linear, exponential and quadratic functions each leave a fingerprint in a table: constant differences, constant ratios or constant second differences. Learning to recognize them, and to explain why each pattern appears, ties together Units 5, 7 and 11 and prepares you for the modeling you will do in Algebra II, in science, and in everyday decisions on the farm, in a business or in the household budget.

PREREQUISITES

- Plot ordered pairs on the coordinate plane (Lesson 4.1).
- Find slope with the slope formula and write equations in slope-intercept and point-slope form (Lessons 5.1, 5.3 and 5.4).
- Interpret slope and y -intercept in a linear model (Lesson 5.7).
- Evaluate and recognize exponential functions $y = a \cdot b^x$ (Lessons 7.4 and 7.5).
- Recognize quadratic functions and their graphs (Lessons 11.1 and 11.2).
- Solve a system of two linear equations (Unit 6) and find a mean (Pre-Algebra).

MASTERY CHECKLIST

- ☐ I can make a scatter plot from paired data with the independent variable on the x -axis.
- ☐ I can describe a correlation as positive, negative or none, and as strong or weak.
- ☐ I can identify an outlier and describe how it affects the data.
- ☐ I can explain why correlation does not prove causation and name a possible lurking variable.
- ☐ I can draw a line of fit and write its equation from two points on the line.
- ☐ I can interpret the slope and y -intercept of a line of fit with units.
- ☐ I can make predictions and tell interpolation from extrapolation, and explain why extrapolation is risky.
- ☐ I can compute residuals and explain what the correlation coefficient r tells me.
- ☐ I can use first differences, second differences and ratios to decide whether data are linear, quadratic or exponential.
- ☐ I can write a linear, exponential or quadratic model from a table and check it by substitution.

LESSONS IN THIS UNIT

13.1 Scatter Plots and Correlation

45 minutes

13.2 Lines of Best Fit

2 sessions of 45 minutes

13.3 Choosing a Model: Linear, Exponential or Quadratic

45 minutes

UNIT 13 TEST

You have 60 minutes. A scientific calculator is allowed; a graphing calculator may be used only to check your work. Show your work for every problem, including difference tables and the steps for finding an equation. Give exact answers unless a problem asks you to round, and include units in answers to word problems.

1. (5 pts)

As a candle burns, the number of hours it has burned and the height of the candle are recorded. Is the correlation positive, negative or none?

2. (5 pts)

Which point is the outlier: (1, 4), (2, 7), (3, 10), (4, 25), (5, 16), (6, 19)?

3. (6 pts)

A study of many towns finds a strong positive correlation between the number of churches in a town and the number of restaurants. Does building a church cause new restaurants to open? Explain, and name a likely lurking variable.

4. (6 pts)

Which correlation coefficient shows the strongest correlation: 0.52, -0.88 , 0.10 or -0.35 ?

5. (7 pts)

A line of fit passes through (3, 20) and (7, 32). Write its equation in slope-intercept form.

6. **7 pts**

During a church food drive, the line of fit $y = 12x + 40$ models the total pounds of food collected, y , after x days. Predict the total after 9 days.

7. **7 pts**

A line of fit is $y = 4x + 3$. Find the residual for the data point (5, 21).

8. **8 pts**

A line of fit for the height of an apple tree is $y = 1.5x + 4$, where x is years since planting and y is height in feet. What do the slope and the y -intercept mean?

9. **8 pts**

Using data from a puppy's first 6 months, Ruth finds the line of fit $w = 3.5m + 6$, where w is weight in pounds and m is age in months. Use it to predict the dog's weight at 120 months, and explain why the prediction is not reasonable.

10. **7 pts**

Decide whether the table is linear, exponential or quadratic, and write a model:
 x : 0, 1, 2, 3, 4 and y : 1, 4, 16, 64, 256.

11. **7 pts**

Decide whether the table is linear, exponential or quadratic, and write a model:
 x : 0, 1, 2, 3, 4 and y : 5, 6, 9, 14, 21.

12. **8 pts**

Ruth makes candles to sell. Her total cost for making candles is: 0 candles: \$25, 10 candles: \$55, 20 candles: \$85, 30 candles: \$115. Choose a model and find the cost of making 50 candles.

13. **9 pts**

Henry's savings account balance (rounded to the cent) is: year 0: \$500.00, year 1: \$525.00, year 2: \$551.25, year 3: \$578.81. Choose a model and find the balance after 10 years, to the nearest cent.

14. 10 pts

Samuel tossed a ball upward from a porch and recorded its height: time t in seconds: 0, 1, 2, 3, 4 and height h in feet: 6, 70, 102, 102, 70. Choose a model and write an equation for h in terms of t .

ANSWER KEY

1. Negative: the longer the candle burns, the shorter it gets.
2. (4, 25); the others go up by 3 each step, so it should be about (4, 13).
3. No. The population of the town is a lurking variable: larger towns have more people, so they have both more churches and more restaurants. Correlation does not prove causation.
4. $r = -0.88$, because it is closest to -1 .
5. $m = \frac{32 - 20}{7 - 3} = 3$; $y - 20 = 3(x - 3)$, so $y = 3x + 11$.
6. $y = 12(9) + 40 = 148$ pounds
7. Predicted $4(5) + 3 = 23$; residual $= 21 - 23 = -2$.
8. Slope 1.5: the tree grows about 1.5 feet per year. y -intercept 4: the tree was about 4 feet tall when it was planted.
9. $w = 3.5(120) + 6 = 426$ pounds. This is extrapolation far beyond the data; a dog stops growing after it is grown, so the straight-line pattern from the first 6 months does not continue.
10. Ratios are all 4, so it is exponential: $y = 4^x$.
11. First differences 1, 3, 5, 7; second differences all 2, so $a = 1$; $c = 5$; from (1, 6): $1 + b + 5 = 6$, so $b = 0$. Model: $y = x^2 + 5$.
12. Cost rises \$30 for every 10 candles, so it is linear with slope 3: $y = 3x + 25$; $y = 3(50) + 25 = 175$, so \$175.
13. Ratios are 1.05, so $y = 500 \cdot (1.05)^x$; $y = 500 \cdot (1.05)^{10} \approx 814.45$, so \$814.45.
14. First differences 64, 32, 0, -32 ; second differences all -32 , so $a = -16$; $c = 6$; from (1, 70): $-16 + b + 6 = 70$, so $b = 80$. Model: $h = -16t^2 + 80t + 6$.

13.1 SCATTER PLOTS AND CORRELATION

OBJECTIVES

- Make a scatter plot from a table of paired data, putting the independent variable on the horizontal axis.
- Describe the correlation shown by a scatter plot as positive, negative or none, and as strong or weak.
- Identify outliers and explain how they affect the pattern.
- Explain why a correlation between two quantities does not prove that one causes the other.

WATCH

Scatter Plots

Eddie Woo · 4:20

<https://www.youtube.com/watch?v=38bp26lxYnc>

WHEN TO WATCH

Before Example 1: scatter plots and bivariate data.

Note: Eddie Woo teaches in Australia: 'bivariate data' means paired data. Correlation versus causation (objective 4) is taught from the Core ideas.

BACKUP VIDEO

Introduction to Bivariate Data (2 of 2: Correlation)

Eddie Woo · 7:12

<https://www.youtube.com/watch?v=cfDfyzE5G8I>

Before Example 2: independent and dependent variables and describing correlation.

KEY TERMS

paired data

Data that come in pairs of numbers measured on the same thing, such as the hours a student studied and that student's quiz score. Each pair becomes an ordered pair (x, y) .

scatter plot

A graph of paired data in which each pair is plotted as a single point. The points are not connected.

correlation

A relationship between two variables in which one tends to change in a regular way as the other changes. It is positive if y tends to rise as x rises, negative if y tends to fall as x rises, and none if there is no clear trend.

strong and weak correlation

A correlation is strong when the points lie close to a line and weak when they are widely scattered around it.

outlier

A data point that lies far away from the pattern formed by the other points.

lurking variable

A hidden third quantity that affects both variables and can make them look related even though neither one causes the other.

CORE IDEAS

Many questions in daily life compare two measurements: the temperature and the amount of firewood burned, the number of hens and the pounds of feed they eat, the hours of practice and the number of mistakes. When you measure both quantities many times, you get **paired data**, and each pair can be written as an ordered pair (x, y) . Put the independent variable (the one you think of as the input, often the one that comes first or that you control) on the horizontal x -axis and the dependent variable on the vertical y -axis. Plotting every pair as a dot, without connecting the dots, gives a **scatter plot**.

Look first at the **direction** of the cloud of points. If the points tend to rise as you move from left to right, the correlation is **positive**: larger x goes with larger y . If they tend to fall from left to right, the correlation is **negative**: larger x goes with smaller y . This is the same idea as the sign of a slope from Unit 5. A positive correlation looks like a line with positive slope that has been smudged; a negative correlation looks like a smudged line with negative slope. If the points form a shapeless blob with no upward or downward trend, there is **no correlation**.

Next look at the **strength**. Imagine laying a pencil along the cloud of points. If the points hug the pencil closely, the correlation is **strong**; if many points sit far above and below it, the correlation is **weak**. If every point lies exactly on one line, the correlation is perfect, which almost never happens with real measurements. Note

that a strong negative correlation is just as strong as a strong positive one; the sign tells direction, not strength.

An **outlier** is a point far from the pattern of the others. When you find one, first ask whether it is a mistake (a misread scale, a number copied wrong). If it is a real measurement, keep it and mention it, because it may tell you something important, such as a day when the stove door was left open. An outlier can make a strong correlation look weak, so it is good practice to describe the pattern both with and without it.

The most important warning in statistics: **correlation does not prove causation**. Two quantities can rise and fall together for three different reasons. (1) One really does cause the other: colder weather makes a family burn more firewood. (2) A **lurking variable** drives both: in summer, ice cream sales and sunburns both go up, but ice cream does not cause sunburn; hot sunny weather causes both. (3) Coincidence: with enough data, some unrelated things will appear to move together by chance. To show that one thing causes another, scientists use careful experiments that change one quantity while holding everything else the same. A scatter plot alone can only show that two things are related, not why.

Practice habit. Before you describe any scatter plot, label both axes with the quantity and its unit, then say the direction (positive, negative, none) and the strength (strong, weak) in one sentence. If you see a point that does not fit, name it as an ordered pair.

WORKED EXAMPLES

Example 1. Grace recorded how many hours she studied for each of six quizzes and her score: (1, 70), (2, 76), (3, 79), (4, 85), (5, 88), (6, 94). Make a scatter plot and describe the correlation.

Show solution

1. The hours studied come first and Grace controls them, so hours go on the x -axis (0 to 6) and quiz score goes on the y -axis (say 60 to 100).
2. Plot the six points as separate dots; do not connect them.
3. Direction: reading left to right, the scores rise from 70 to 94, so the correlation is positive.
4. Strength: each extra hour adds between 3 and 6 points, so the dots lie close to a rising line. The correlation is strong.
5. Caution: the data show that more study goes with higher scores for Grace; they do not by themselves prove the extra hour caused each extra point, though it is a sensible explanation.

Answer: Strong positive correlation: more hours of study go with higher quiz scores.

Example 2. The Miller family recorded the outdoor temperature in °F and the pounds of firewood they burned that day: (10, 62), (20, 54), (30, 45), (40, 60), (50, 28), (60, 20). Which point is the outlier?

Show solution

1. Temperature goes on the x -axis and pounds of firewood on the y -axis.
2. Follow the pattern: from 10°F to 30°F the wood drops 62, 54, 45, about 8 or 9 pounds for every 10 degrees.
3. From 30°F to 50°F the wood drops from 45 to 28, about 8 or 9 pounds per 10 degrees again, so at 40°F we would expect about 37 pounds.
4. The point (40, 60) is far above that, so it is the outlier. (Perhaps a door was left open that day, or the number was written down wrong.)
5. Without the outlier, the other five points fall steadily from left to right and lie close to a line: a strong negative correlation.

Answer: (40, 60) is the outlier; the other points show a strong negative correlation.

PRACTICE

1. **Easy** As the outside temperature rises, the number of logs a family burns in its woodstove each day goes down. Is this a positive correlation, a negative correlation or no correlation?

Show answer

Negative correlation: as one quantity increases, the other decreases.

2. **Easy** Describe the correlation (direction and strength) of the points (1, 3), (2, 5), (3, 6), (4, 8), (5, 9), (6, 11).

Show answer

Strong positive correlation: y rises steadily as x rises and the points lie close to a line.

3. **Easy** Which point is the outlier: (1, 10), (2, 12), (3, 14), (4, 30), (5, 18), (6, 20)?

Show answer

(4, 30); the pattern goes up by 2 each time, so it should be about (4, 16).

4. **Easy** Would you expect a positive correlation, a negative correlation or no correlation between a student's shoe size and the number of letters in the student's first name?

Show answer

No correlation: there is no reason for one to rise or fall with the other.

5. **Medium** Henry recorded the number of hens in several flocks and the pounds of feed each flock eats per week: hens: 10, 15, 20, 25, 30 and feed: 16, 23, 31, 38, 46. The data show a strong positive correlation. On average, how many more pounds of feed per week does each additional 5 hens need?

Show answer

From 10 to 30 hens is four steps of 5 hens, and the feed rises $46 - 16 = 30$ pounds; $30 \div 4 = 7.5$ pounds per week for each 5 more hens.

6. **Medium** At a church winter fair, Ruth recorded the outdoor temperature in °F and the cups of hot cocoa sold that hour: temperature: 20, 30, 40, 50, 60 and cups: 95, 80, 62, 50, 31. Describe the correlation.

Show answer

Strong negative correlation: as the temperature rises, fewer cups of cocoa are sold, and the points lie close to a falling line.

7. **Medium** Set A: (1, 2), (2, 4), (3, 6), (4, 8), (5, 10). Set B: (1, 2), (2, 7), (3, 3), (4, 9), (5, 6). Both show a positive correlation. Which set shows the stronger correlation, and why?

Show answer

Set A; its points lie exactly on the line $y = 2x$, while the points of Set B zigzag far above and below any line.

8. **Medium** Ruth recorded the minutes she practiced piano each day and the number of mistakes she made in her piece: (10, 7), (15, 6), (20, 5), (25, 4), (30, 28). The last point is an outlier (she was ill that day). Find the mean number of mistakes with the outlier and without it.

Show answer

With: $\frac{7 + 6 + 5 + 4 + 28}{5} = \frac{50}{5} = 10$ mistakes; without: $\frac{22}{4} = 5.5$ mistakes.

9. **Medium** In one town, the daily number of ice cream cones sold and the daily number of sunburns have a strong positive correlation. Does eating ice cream cause sunburn? Explain.

Show answer

No. A lurking variable, hot sunny weather, causes both: on sunny days more people buy ice cream and more people get sunburned. Correlation does not prove causation.

10. **Hard** The Henderson family drove to their grandparents' house. They recorded hours driven and miles still to go: hours: 0, 1, 2, 3, 4 and miles left: 300, 245, 190, 140, 80. On average, by how many miles did the distance left drop each hour?

Show answer

$300 - 80 = 220$ miles in 4 hours; $220 \div 4 = 55$ miles each hour (a strong negative correlation).

11. **Hard** A study of house fires finds a strong positive correlation between the number of fire trucks sent to a fire and the dollar amount of damage. Does sending more trucks cause more damage? Name a likely lurking variable.

Show answer

No. The size of the fire is a lurking variable: a bigger fire both brings more trucks and does more damage.

12. **Hard** Grace sells loaves of bread at a farmers' market. At different prices she sold: \$3: 40 loaves, \$4: 34 loaves, \$5: 28 loaves, \$6: 22 loaves, \$7: 16 loaves. Price and loaves sold have a strong negative correlation. Find her total income (price times loaves) at each price. At which price is her income greatest?

Show answer

Incomes: \$120, \$136, \$140, \$132, \$112. The greatest income is at \$5 per loaf. (Income rises and then falls, a curved pattern rather than a line.)

13. **Challenge** Five of these points lie exactly on the line $y = 2x + 1$ and one does not: (1, 3), (2, 5), (3, 7), (4, 15), (5, 11), (6, 13). Which point is the outlier, and what y -value would put it exactly on the line?

Show answer

(4, 15) is the outlier; $2(4) + 1 = 9$, so $y = 9$ would fit.

COMMON MISTAKES

- Calling a falling pattern "no correlation." A falling pattern is a negative correlation; "no correlation" means there is no upward or downward trend at all.
- Thinking a negative correlation is weaker than a positive one. The sign gives the direction; how closely the points hug a line gives the strength.
- Concluding that one variable causes the other just because they are correlated. Always ask whether a lurking variable or coincidence could explain the pattern.
- Connecting the dots or putting the variables on the wrong axes. Plot each pair as a separate dot, with the input (independent variable) on the x -axis.

Make it a game • Bookshelf Data Hunt. Choose five books of different sizes. Record pages as x and thickness in centimeters as y . Plot the points on graph paper, with the independent variable horizontal. Describe the correlation as positive, negative or none, and strong or weak. Circle any outlier and guess why it differs. Predict the thickness of a book with 300 pages.

13.2 LINES OF BEST FIT

OBJECTIVES

- Draw a line of fit by eye on a scatter plot and write its equation from two points on the line.
- Interpret the slope and y -intercept of a line of fit in the words of the situation.
- Use a line of fit to make predictions, telling interpolation from extrapolation and explaining why extrapolation is risky.
- Compute residuals and use them to judge how well a line fits the data.

WATCH

Line of Best Fit (2 of 2: Finding the equation from 2 points).

Eddie Woo · 10:20

<https://www.youtube.com/watch?v=QJLULB5soIA>

WHEN TO WATCH

Before Example 1: writing the equation of a line of fit from two points.

Note: Neither video computes residuals (objective 4). Teach that from the Core ideas. Eddie Woo's exam review ([youtube.com/watch?v=JCgLnTE4wo](https://www.youtube.com/watch?v=JCgLnTE4wo)) covers interpolation and extrapolation.

BACKUP VIDEO

Line of Best Fit (1 of 2: Overview).

Eddie Woo · 12:19

<https://www.youtube.com/watch?v=jFRpakGjrT0>

Before Example 2: drawing a line of fit by eye and predicting.

KEY TERMS

line of fit

A line drawn through the middle of the points of a scatter plot that shows the trend of the data. It does not have to pass through any of the data points.

line of best fit

The line that fits the data best by a definite rule. The usual rule is least squares: the line that makes the sum of the squared residuals as small as possible.

interpolation

Using a model to predict a value inside the range of the data you collected.

extrapolation

Using a model to predict a value outside the range of the data you collected. It is riskier because the pattern may not continue.

residual

The actual y -value minus the y -value predicted by the line. A positive residual means the point is above the line; a negative residual means it is below.

correlation coefficient r

A number between -1 and 1 that a calculator reports to measure how closely data fit a line. Its sign gives the direction of the correlation, and values near -1 or 1 mean a strong correlation.

CORE IDEAS

When a scatter plot shows a fairly strong linear trend, we can summarize the whole cloud of points with one line, called a **line of fit**. To draw one by eye, lay a straightedge along the cloud so that it follows the direction of the points and has about as many points above it as below it, with the points on both sides spread along its whole length. The line does not need to pass through any data point; it is a summary of the trend, not a connect-the-dots drawing.

To write the equation of your line, choose two points that lie **on the line you drew** (points where it crosses grid lines are easiest to read). Then use the tools from Unit 5: find the slope with $m = \frac{y_2 - y_1}{x_2 - x_1}$, put one point into point-slope form $y - y_1 = m(x - x_1)$, and solve for y to get slope-intercept form $y = mx + b$. Because the line is drawn by eye, two students may get slightly different equations; both are acceptable if each line follows the trend.

The numbers in $y = mx + b$ have meanings. The **slope** m is the predicted change in y for each increase of 1 in x , because moving 1 unit right along a line always changes y by m . The **y -intercept** b is the predicted value of y when $x = 0$, because $y = m(0) + b = b$. Always state these in the units of the problem, for example "the sunflower grows about 5.4 inches per week." Sometimes $x = 0$ is far from the data, and then b has no sensible real meaning.

Predicting with the line is easy: substitute the x -value and compute y . A prediction inside the range of the data is **interpolation** and is usually trustworthy when the

correlation is strong. A prediction outside the range is **extrapolation**. Why is extrapolation risky? The line only describes what happened where we measured. Nothing guarantees the same rate continues forever: a sunflower stops growing, a cup of cocoa cannot cool below room temperature, a child does not grow 2 inches a year until age 50. The farther you extrapolate, the less you should trust the answer.

To judge a line, look at the **residuals**: $\text{residual} = \text{actual } y - \text{predicted } y$. A residual is the vertical distance from a point to the line, with a sign: positive if the point is above the line, negative if below. For a good line the residuals are small, and the positive and negative ones roughly balance. But many different lines can make the residuals add up to about zero, so mathematicians use a sharper rule: square each residual and make the total of the squares as small as possible. Squaring does two useful things: it makes every value positive or zero, so misses above and below cannot cancel, and it makes large misses count much more than small ones. The line that wins by this rule is the **least-squares line**, the line of best fit.

You do not need to find the least-squares line by hand. A graphing calculator or spreadsheet does it with **linear regression**: you enter the data and it reports $y = ax + b$ (calculators often use a for the slope) along with the correlation coefficient r . The value of r is always between -1 and 1 . A positive r means positive correlation, a negative r means negative correlation, values near 1 or -1 mean the points lie close to a line, and values near 0 mean little or no linear pattern. Use the calculator to check your work, but always understand what the slope, the intercept and r mean.

KEY FORMULAS

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = m(x - x_1)$$

$$\text{residual} = y_{\text{actual}} - y_{\text{predicted}}$$

$$-1 \leq r \leq 1$$

From history. The French mathematician Adrien-Marie Legendre published the method of least squares in 1805, in a work on finding the orbits of comets. The German mathematician Carl Friedrich Gauss also used the method in his work on the orbits of heavenly bodies. Ever since, least squares has been one of the most widely used tools in science for fitting a line or curve to measurements that are not perfectly exact.

Practice habit. Check your line before you use it: substitute both of the points you chose into your equation, and make sure about as many data points lie above the line as below it. Then say what the slope means, with units, in one sentence.

WORKED EXAMPLES

Example 1. A line of fit drawn on a scatter plot passes through the points (2, 30) and (6, 50). Write its equation in slope-intercept form.

Show solution

1. Slope: $m = \frac{50 - 30}{6 - 2} = \frac{20}{4} = 5.$

2. Point-slope form with (2, 30): $y - 30 = 5(x - 2).$

3. Distribute: $y - 30 = 5x - 10.$

4. Add 30 to both sides: $y = 5x + 20.$

5. Check with the other point: $5(6) + 20 = 30 + 20 = 50. \checkmark$

Answer: $y = 5x + 20$

Example 2. Henry measured the height of his sunflower each week: week: 1, 2, 3, 4, 5, 6 and height in inches: 7, 12, 19, 23, 30, 34. He draws a line of fit through (1, 7) and (6, 34). Write the equation of the line, interpret the slope, and use the line to predict the height in week 8.

Show solution

1. Slope: $m = \frac{34 - 7}{6 - 1} = \frac{27}{5} = 5.4$.
2. Point-slope form: $y - 7 = 5.4(x - 1)$, so $y - 7 = 5.4x - 5.4$.
3. Add 7: $y = 5.4x + 1.6$.
4. Check the fit: the line predicts 12.4, 17.8, 23.2 and 28.6 for weeks 2 to 5, so the residuals are -0.4 , 1.2 , -0.2 and 1.4 ; points lie on both sides of the line and all are close. Here the line through the first and last points happens to follow the whole cloud well, but always check this; often such a line does not.
5. Slope: the sunflower grows about 5.4 inches per week.
6. Week 8: $y = 5.4(8) + 1.6 = 43.2 + 1.6 = 44.8$ inches.
7. Week 8 is outside the data (weeks 1 to 6), so this is extrapolation. It is reasonable two weeks out, but the sunflower will not keep growing 5.4 inches a week forever.

Answer: $y = 5.4x + 1.6$; about 44.8 inches in week 8 (an extrapolation).

PRACTICE

1. Easy A line of fit for a set of data is $y = 3x + 10$. Predict y when $x = 7$.

Show answer

$$y = 3(7) + 10 = 31$$

2. Easy A line of fit passes through (0, 4) and (5, 14). Write its equation in slope-intercept form.

Show answer

$$m = \frac{14 - 4}{5 - 0} = 2 \text{ and } b = 4, \text{ so } y = 2x + 4.$$

3. **Easy** A line of fit predicts that a family of 5 will eat 15 rolls at the church potluck, but the family actually eats 18 rolls. What is the residual?

Show answer

$18 - 15 = 3$ rolls (the point is above the line).

4. **Easy** Samuel measured the height of his corn plants every week from week 1 to week 10. Is using his line of fit to predict the height at week 6 interpolation or extrapolation?

Show answer

Interpolation, because week 6 is inside the range of the data (weeks 1 to 10).

5. **Medium** A line of fit passes through (2, 9) and (8, 27). Write its equation in slope-intercept form.

Show answer

$$m = \frac{27 - 9}{8 - 2} = \frac{18}{6} = 3; y - 9 = 3(x - 2), \text{ so } y = 3x + 3.$$

6. **Medium** Samuel's line of fit for his lemonade stand is $y = 0.75x - 5$, where x is the daily high temperature in °F and y is the number of cups sold. Predict the number of cups sold on an 80°F day.

Show answer

$$y = 0.75(80) - 5 = 60 - 5 = 55 \text{ cups}$$

7. **Medium** Ruth's line of fit for the height of her tomato plants is $y = 1.2x + 8$, where x is the number of days since she planted them in the garden and y is the height in inches. What do the slope and the y -intercept mean?

Show answer

Slope 1.2: the plants grow about 1.2 inches per day. y -intercept 8: the plants were about 8 inches tall on the day they were planted ($x = 0$).

8. **Medium** A line of fit is $y = 2x + 5$. Find the residual for the data point (6, 15). Is the point above or below the line?

Show answer

Predicted: $2(6) + 5 = 17$; residual = $15 - 17 = -2$, so the point is below the line.

9. **Medium** Mr. Miller recorded the acres he plowed and the gallons of diesel his tractor used. His line of fit passes through (2, 7) and (10, 31), where x is acres and y is gallons. Use the line to predict how many gallons he will use to plow 6 acres.

Show answer

$$m = \frac{31 - 7}{10 - 2} = 3, \text{ so } y = 3x + 1; y = 3(6) + 1 = 19 \text{ gallons.}$$

10. **Hard** Grace measured the temperature of a mug of cocoa as it cooled. Her line of fit passes through (2, 150) and (10, 110), where x is minutes and y is degrees Fahrenheit. Write the equation of the line in slope-intercept form.

Show answer

$$m = \frac{110 - 150}{10 - 2} = \frac{-40}{8} = -5; y - 150 = -5(x - 2), \text{ so } y = -5x + 160.$$

11. **Hard** Grace's data in Problem 10 were collected from 2 to 10 minutes. Use $y = -5x + 160$ to predict the temperature of the cocoa after 40 minutes, and explain why the prediction is not reasonable.

Show answer

$y = -5(40) + 160 = -40^\circ\text{F}$. This is extrapolation far beyond the data. The cocoa cannot cool below the temperature of the room, and it cools more slowly as it nears room temperature, so the straight-line pattern does not continue.

12. **Hard** A calculator reports the correlation coefficient r for four data sets: 0.41, -0.93 , 0.78 and -0.12 . Which value shows the strongest correlation?

Show answer

$r = -0.93$; it is closest to -1 (strong negative correlation). The sign gives the direction, not the strength.

13. **Hard** For the line $y = 2x + 1$ and the data points (1, 4), (2, 4), (3, 8), (4, 8), find the four residuals in order.

Show answer

Predicted: 3, 5, 7, 9. Residuals: $4 - 3 = 1$, $4 - 5 = -1$, $8 - 7 = 1$, $8 - 9 = -1$. They balance (sum 0).

14. **Challenge** For the data $(1, 3)$, $(2, 6)$, $(3, 6)$, $(4, 9)$, Henry suggests Line A: $y = 2x + 1$ and Ruth suggests Line B: $y = 3x$. Find the sum of the squared residuals for Line A and for Line B.

Show answer

Line A residuals: 0, 1, -1 , 0, so the sum of squares is $0 + 1 + 1 + 0 = 2$. Line B residuals: 0, 0, -3 , -3 , so the sum of squares is $0 + 0 + 9 + 9 = 18$. Line A fits better by the least-squares rule.

COMMON MISTAKES

- Drawing a line that just connects the first and last data points without checking that it follows the middle of the whole cloud, or a line that passes through as many points as possible. A line of fit follows the trend of all the points, with points on both sides.
- Using two data points that are not on your line to write its equation. Choose two points that lie on the line you actually drew.
- Computing the residual backward. It is actual minus predicted, so a point above the line has a positive residual.
- Trusting an extrapolation far beyond the data. Check whether the answer makes sense in the real situation before you believe it.

Make it a game • Spaghetti Line of Fit. Make a scatter plot of five points on graph paper and lay a strand of uncooked spaghetti where it seems to fit best. Pick two points on the strand and write its equation. Say in a sentence what the slope and y-intercept mean in the situation. Predict one value inside the data and one beyond it.

13.3 CHOOSING A MODEL: LINEAR, EXPONENTIAL OR QUADRATIC

OBJECTIVES

- Use first differences, second differences and ratios to decide whether a table of values is linear, quadratic or exponential.
- Explain why each family of functions leaves its own pattern in a table.
- Write a linear, exponential or quadratic model from a table and use it to make predictions.
- Compare the three families of functions studied in this course.

WATCH

Comparing Linear & Non-Linear Functions

Eddie Woo · 9:04

https://www.youtube.com/watch?v=8um1l_3ONck

WHEN TO WATCH

Before Example 1: telling linear from non-linear patterns in tables.

Note: Weak fit: neither video uses second differences to spot a quadratic (objective 1). Teach that from Example 2 and the Core ideas. Eddie Woo teaches in Australia: the backup uses Australian dollars and says 'arithmetic' and 'geometric' sequence for the two kinds of growth.

BACKUP VIDEO

Compound Interest (1 of 2: Context - comparing two wages)

Eddie Woo · 7:44

<https://www.youtube.com/watch?v=ccB4Y6LWIEo>

After Example 1: two salaries compared year by year in a table. One rises by the same amount each year (constant differences, linear); the other rises by the same percent (constant ratio, exponential).

KEY TERMS

first differences

The changes in y found by subtracting each y -value from the next one, when the x -values go up by equal steps.

second differences

The differences of the first differences: subtract each first difference from the next one.

ratios

The numbers found by dividing each y -value by the one before it, when the x -values go up by equal steps.

model

A function, such as a line, an exponential function or a parabola, chosen to describe a real situation so that we can understand it and make predictions.

CORE IDEAS

This year you have studied three families of functions: **linear** functions $y = mx + b$ (Unit 5), **exponential** functions $y = a \cdot b^x$ (Unit 7) and **quadratic** functions $y = ax^2 + bx + c$ (Unit 11). When you are given a table of data, how do you know which family to use? Each family leaves its own fingerprint in the table, as long as the x -values go up by equal steps. Write the y -values in a row, then find the first differences, the second differences and the ratios.

Linear: constant first differences. Why? If $y = mx + b$ and x goes up by 1, the new y is $m(x + 1) + b = mx + m + b$, which is exactly m more than the old y . So the y -values go up (or down) by the same amount m every step. Repeated adding means a line. For example, the y -values 5, 9, 13, 17 for $x = 0, 1, 2, 3$ have first differences 4, 4, 4, so $m = 4$; the y -value at $x = 0$ is $b = 5$, giving $y = 4x + 5$.

Exponential: constant ratios. Why? If $y = a \cdot b^x$ and x goes up by 1, the new y is $a \cdot b^{x+1} = (a \cdot b^x) \cdot b$, which is the old y times b . So each y -value is the one before multiplied by the same number b . Repeated multiplying means exponential. For example, the y -values 3, 6, 12, 24 for $x = 0, 1, 2, 3$ have ratios 2, 2, 2, so $b = 2$; the y -value at $x = 0$ is $a = 3$, giving $y = 3 \cdot 2^x$. A ratio greater than 1 is growth; a ratio between 0 and 1 is decay.

Quadratic: constant second differences. Why? If $y = ax^2 + bx + c$ and x goes up by 1, the first difference is $a(x + 1)^2 + b(x + 1) + c - (ax^2 + bx + c)$. Expanding $(x + 1)^2 = x^2 + 2x + 1$ and cancelling gives $2ax + a + b$. That expression is itself linear in x with slope $2a$, so the first differences go up by $2a$ each step: the **second differences are all $2a$** . For example, 0, 1, 4, 9, 16 (from $y = x^2$) has first differences 1, 3, 5, 7 and second differences 2, 2, 2, and indeed $2a = 2$ gives $a = 1$.

To write a quadratic model from a table with $x = 0, 1, 2, \dots$: find a from the second difference ($a = \text{second difference} \div 2$), read c as the y -value at $x = 0$, then substitute

one more point to find b . If the table does not start at $x = 0$, use two points and solve the resulting system for b and c , as you learned in Unit 6.

Real data are rarely perfect, so look for differences or ratios that are **nearly** constant, and let the situation guide you. Steady amounts added each week (savings, a constant speed) suggest a line. A fixed percent of growth or loss (interest, depreciation, a spreading plant) suggests an exponential. Falling objects and areas with a fixed perimeter suggest a quadratic. Remember the shapes too: a line is straight, an exponential curve rises ever more steeply (or levels off toward zero in decay), and a parabola turns around at its vertex. Over the long run exponential growth outruns any linear or quadratic function, because it keeps multiplying by the same factor, while a linear function adds the same amount each step and a quadratic adds amounts that grow by only a fixed amount each step.

KEY FORMULAS

Linear: $y = mx + b$ (constant first differences, equal to m when x steps by 1)

Exponential: $y = a \cdot b^x$ (constant ratios, equal to b when x steps by 1)

Quadratic: $y = ax^2 + bx + c$ (constant second differences, equal to $2a$ when x steps by 1)

Practice habit. Build a difference table every time: write the y -values in a row, the first differences in the row below, the second differences below that, and the ratios off to the side. Then check your final model by substituting at least two x -values from the table.

WORKED EXAMPLES

Example 1. Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 3, 6, 12, 24, 48.

Show solution

1. First differences: $6 - 3 = 3$, $12 - 6 = 6$, $24 - 12 = 12$, $48 - 24 = 24$. Not constant, so not linear.
2. Second differences: 3, 6, 12. Not constant, so not quadratic.
3. Ratios: $\frac{6}{3} = 2$, $\frac{12}{6} = 2$, $\frac{24}{12} = 2$, $\frac{48}{24} = 2$. Constant, so exponential with $b = 2$.
4. At $x = 0$, $y = 3$, so $a = 3$.
5. Model: $y = 3 \cdot 2^x$.
6. Check: $x = 4$ gives $3 \cdot 2^4 = 3 \cdot 16 = 48$. ✓

Answer: Exponential: $y = 3 \cdot 2^x$

Example 2. Grace has 20 feet of fencing to make a rectangular rabbit pen. She lists the width x in feet and the area y in square feet: x : 1, 2, 3, 4, 5 and y : 9, 16, 21, 24, 25. Decide which kind of model fits and write its equation.

Show solution

1. First differences: 7, 5, 3, 1. Not constant, so not linear.
2. Ratios: $\frac{16}{9}$, $\frac{21}{16}$, ... are not equal, so not exponential.
3. Second differences: $5 - 7 = -2$, $3 - 5 = -2$, $1 - 3 = -2$. Constant, so quadratic.
4. $2a = -2$, so $a = -1$ and $y = -x^2 + bx + c$.
5. Use (1, 9): $-1 + b + c = 9$, so $b + c = 10$.
6. Use (2, 16): $-4 + 2b + c = 16$, so $2b + c = 20$.
7. Subtract the first equation from the second: $b = 10$; then $c = 0$.
8. Model: $y = -x^2 + 10x$. This makes sense: the length is $10 - x$, so the area is $x(10 - x)$.
9. Check: $x = 5$ gives $-25 + 50 = 25$ ✓ and $x = 3$ gives $-9 + 30 = 21$ ✓.

Answer: Quadratic: $y = -x^2 + 10x$

PRACTICE

1. **Easy** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 5, 9, 13, 17, 21.

Show answer

First differences are all 4, so it is linear: $y = 4x + 5$.

2. **Easy** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3 and y : 2, 6, 18, 54.

Show answer

Ratios are all 3, so it is exponential: $y = 2 \cdot 3^x$.

3. **Easy** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 0, 1, 4, 9, 16.

Show answer

First differences 1, 3, 5, 7; second differences all 2, so it is quadratic with $a = 1$: $y = x^2$.

4. **Easy** Is the table linear, exponential or quadratic? x : 1, 2, 3, 4, 5 and y : 7, 10, 15, 22, 31.

Show answer

Quadratic: first differences 3, 5, 7, 9 and second differences all 2.

5. **Medium** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 80, 40, 20, 10, 5.

Show answer

Ratios are all $\frac{1}{2}$, so it is exponential decay: $y = 80 \cdot \left(\frac{1}{2}\right)^x$.

6. **Medium** Samuel's savings at the end of each week are: week 0: \$15, week 1: \$22, week 2: \$29, week 3: \$36. Choose a model and use it to find his savings after 10 weeks if the pattern continues.

Show answer

First differences are all 7, so $y = 7x + 15$; $y = 7(10) + 15 = 85$, so \$85.

7. **Medium** Ruth planted mint, which spreads quickly. The area it covers in square feet is: year 0: 5, year 1: 10, year 2: 20, year 3: 40. Choose a model and use it to predict the area in year 5 if the pattern continues.

Show answer

Ratios are all 2, so $y = 5 \cdot 2^x$; $y = 5 \cdot 2^5 = 5 \cdot 32 = 160$ square feet.

8. **Medium** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 1, 3, 9, 19, 33.

Show answer

First differences 2, 6, 10, 14; second differences all 4, so $a = 2$; $c = 1$; from $(1, 3)$: $2 + b + 1 = 3$, so $b = 0$. Model: $y = 2x^2 + 1$.

9. **Medium** Explain why a table whose x -values go up by 1 and whose y -values all have the same ratio must come from an exponential function rather than a linear one.

Show answer

Having the same ratio means each y -value is the one before multiplied by the same number b . Starting from a at $x = 0$, after x steps we have multiplied by b a total of x times, so $y = a \cdot b^x$. A linear function adds the same amount each step instead of multiplying, so (unless $b = 1$) its ratios are not constant.

10. **Hard** Decide whether the table is linear, exponential or quadratic, and write a model: x : 1, 2, 3, 4, 5 and y : 3, 12, 48, 192, 768.

Show answer

Ratios are all 4, so $y = a \cdot 4^x$. Using $(1, 3)$: $4a = 3$, so $a = 0.75$. Model: $y = 0.75 \cdot 4^x$.

11. **Hard** Henry dropped a stone from a high bridge and recorded its height above the water: time t in seconds: 0, 1, 2, 3 and height h in feet: 144, 128, 80, 0. Choose a model and write an equation for h in terms of t .

Show answer

First differences $-16, -48, -80$; second differences all -32 , so $a = -16$; $c = 144$; from $(1, 128)$: $-16 + b + 144 = 128$, so $b = 0$. Model: $h = -16t^2 + 144$.

12. **Hard** Decide whether the table is linear, exponential or quadratic, and write a model: x : 0, 1, 2, 3, 4 and y : 4, 3, 4, 7, 12.

Show answer

First differences $-1, 1, 3, 5$; second differences all 2, so $a = 1$; $c = 4$; from $(1, 3)$: $1 + b + 4 = 3$, so $b = -2$. Model: $y = x^2 - 2x + 4$.

13. **Hard** A farm truck's value is: year 0: \$30,000, year 1: \$24,000, year 2: \$19,200, year 3: \$15,360. Choose a model and use a calculator to find the truck's value after 5 years, to the nearest cent.

Show answer

Ratios are all 0.8, so $y = 30000 \cdot (0.8)^x$; $y = 30000 \cdot (0.8)^5 = 30000 \times 0.32768 = 9830.40$, so \$9,830.40.

14. **Challenge** Grandfather offers Henry two savings plans. Plan A: after x weeks Henry has $50x$ dollars. Plan B: after x weeks Henry has 2^x dollars (\$2 after week 1, \$4 after week 2, doubling each week). After which whole week does Plan B first give more money than Plan A?

Show answer

Week 8: $2^8 = 256$ and $50(8) = 400$, so A is ahead. Week 9: $2^9 = 512$ and $50(9) = 450$, so B is ahead. Plan B first gives more after week 9.

COMMON MISTAKES

- Using differences or ratios when the x -values are not equally spaced. Check the x -values first; the patterns only work when x goes up by equal steps.
- Stopping at the first differences. If they are not constant, also check the second differences and the ratios before deciding.
- Thinking a quadratic's second difference equals a . When x steps by 1, the second difference is $2a$, so divide by 2.
- Reading a in $y = a \cdot b^x$ from the first row of the table when that row is not $x = 0$. Substitute a point to find a instead.

Make it a game • Pattern Detective. The parent secretly picks a rule: linear ($y = 2x + 1$), quadratic ($y = x^2$) or exponential ($y = 3 \cdot 2^x$), and reads you a table for $x = 0$ to 4. Compute first differences, second differences and ratios to name the family, then predict the next value. Fastest correct detective in 90 seconds wins. Trade roles and write your own table.

UNIT 13

DATA AND STATISTICS: SCATTER PLOTS AND LINES OF FIT

In this last unit, algebra meets real measurements. You will plot paired data on scatter plots, describe the correlation you see, and learn the most important caution in statistics: correlation does not prove causation. Then you will summarize a trend with a line of fit, write its equation, interpret its slope and intercept, make predictions, and judge the fit with residuals, the idea behind the least-squares method published by Legendre in 1805.

The unit closes with a review of the whole course. Linear, exponential and quadratic functions each leave a fingerprint in a table: constant differences, constant ratios or constant second differences. Learning to recognize them, and to explain why each pattern appears, ties together Units 5, 7 and 11 and prepares you for the modeling you will do in Algebra II, in science, and in everyday decisions on the farm, in a business or in the household budget.

PREREQUISITES

- Plot ordered pairs on the coordinate plane (Lesson 4.1).
- Find slope with the slope formula and write equations in slope-intercept and point-slope form (Lessons 5.1, 5.3 and 5.4).
- Interpret slope and y -intercept in a linear model (Lesson 5.7).
- Evaluate and recognize exponential functions $y = a \cdot b^x$ (Lessons 7.4 and 7.5).
- Recognize quadratic functions and their graphs (Lessons 11.1 and 11.2).
- Solve a system of two linear equations (Unit 6) and find a mean (Pre-Algebra).

MASTERY CHECKLIST

- ☐ I can make a scatter plot from paired data with the independent variable on the x -axis.
- ☐ I can describe a correlation as positive, negative or none, and as strong or weak.
- ☐ I can identify an outlier and describe how it affects the data.
- ☐ I can explain why correlation does not prove causation and name a possible lurking variable.
- ☐ I can draw a line of fit and write its equation from two points on the line.
- ☐ I can interpret the slope and y -intercept of a line of fit with units.
- ☐ I can make predictions and tell interpolation from extrapolation, and explain why extrapolation is risky.
- ☐ I can compute residuals and explain what the correlation coefficient r tells me.
- ☐ I can use first differences, second differences and ratios to decide whether data are linear, quadratic or exponential.
- ☐ I can write a linear, exponential or quadratic model from a table and check it by substitution.

LESSONS IN THIS UNIT

13.1 Scatter Plots and Correlation

45 minutes

13.2 Lines of Best Fit

2 sessions of 45 minutes

13.3 Choosing a Model: Linear, Exponential or Quadratic

45 minutes

UNIT 13 TEST

You have 60 minutes. A scientific calculator is allowed; a graphing calculator may be used only to check your work. Show your work for every problem, including difference tables and the steps for finding an equation. Give exact answers unless a problem asks you to round, and include units in answers to word problems.

1. (5 pts)

As a candle burns, the number of hours it has burned and the height of the candle are recorded. Is the correlation positive, negative or none?

2. (5 pts)

Which point is the outlier: (1, 4), (2, 7), (3, 10), (4, 25), (5, 16), (6, 19)?

3. (6 pts)

A study of many towns finds a strong positive correlation between the number of churches in a town and the number of restaurants. Does building a church cause new restaurants to open? Explain, and name a likely lurking variable.

4. (6 pts)

Which correlation coefficient shows the strongest correlation: 0.52, -0.88 , 0.10 or -0.35 ?

5. (7 pts)

A line of fit passes through (3, 20) and (7, 32). Write its equation in slope-intercept form.

6. **7 pts**

During a church food drive, the line of fit $y = 12x + 40$ models the total pounds of food collected, y , after x days. Predict the total after 9 days.

7. **7 pts**

A line of fit is $y = 4x + 3$. Find the residual for the data point $(5, 21)$.

8. **8 pts**

A line of fit for the height of an apple tree is $y = 1.5x + 4$, where x is years since planting and y is height in feet. What do the slope and the y -intercept mean?

9. **8 pts**

Using data from a puppy's first 6 months, Ruth finds the line of fit $w = 3.5m + 6$, where w is weight in pounds and m is age in months. Use it to predict the dog's weight at 120 months, and explain why the prediction is not reasonable.

10. **7 pts**

Decide whether the table is linear, exponential or quadratic, and write a model:
 x : 0, 1, 2, 3, 4 and y : 1, 4, 16, 64, 256.

11. **7 pts**

Decide whether the table is linear, exponential or quadratic, and write a model:
 x : 0, 1, 2, 3, 4 and y : 5, 6, 9, 14, 21.

12. **8 pts**

Ruth makes candles to sell. Her total cost for making candles is: 0 candles: \$25, 10 candles: \$55, 20 candles: \$85, 30 candles: \$115. Choose a model and find the cost of making 50 candles.

13. **9 pts**

Henry's savings account balance (rounded to the cent) is: year 0: \$500.00, year 1: \$525.00, year 2: \$551.25, year 3: \$578.81. Choose a model and find the balance after 10 years, to the nearest cent.

14. 10 pts

Samuel tossed a ball upward from a porch and recorded its height: time t in seconds: 0, 1, 2, 3, 4 and height h in feet: 6, 70, 102, 102, 70. Choose a model and write an equation for h in terms of t .

ANSWER KEY

1. Negative: the longer the candle burns, the shorter it gets.
2. (4, 25); the others go up by 3 each step, so it should be about (4, 13).
3. No. The population of the town is a lurking variable: larger towns have more people, so they have both more churches and more restaurants. Correlation does not prove causation.
4. $r = -0.88$, because it is closest to -1 .
5. $m = \frac{32 - 20}{7 - 3} = 3$; $y - 20 = 3(x - 3)$, so $y = 3x + 11$.
6. $y = 12(9) + 40 = 148$ pounds
7. Predicted $4(5) + 3 = 23$; residual $= 21 - 23 = -2$.
8. Slope 1.5: the tree grows about 1.5 feet per year. y -intercept 4: the tree was about 4 feet tall when it was planted.
9. $w = 3.5(120) + 6 = 426$ pounds. This is extrapolation far beyond the data; a dog stops growing after it is grown, so the straight-line pattern from the first 6 months does not continue.
10. Ratios are all 4, so it is exponential: $y = 4^x$.
11. First differences 1, 3, 5, 7; second differences all 2, so $a = 1$; $c = 5$; from (1, 6): $1 + b + 5 = 6$, so $b = 0$. Model: $y = x^2 + 5$.
12. Cost rises \$30 for every 10 candles, so it is linear with slope 3: $y = 3x + 25$; $y = 3(50) + 25 = 175$, so \$175.
13. Ratios are 1.05, so $y = 500 \cdot (1.05)^x$; $y = 500 \cdot (1.05)^{10} \approx 814.45$, so \$814.45.
14. First differences 64, 32, 0, -32 ; second differences all -32 , so $a = -16$; $c = 6$; from (1, 70): $-16 + b + 6 = 70$, so $b = 80$. Model: $h = -16t^2 + 80t + 6$.

MIDTERM EXAM

Covers Units 1–6. Time: 90 minutes.

You have 90 minutes. No calculator is allowed on this exam. Show all your work clearly, one step per line; correct work earns partial credit even if the final answer has a small error. Give exact answers (fractions, not rounded decimals) unless a problem asks for rounding. Check solutions by substituting where you can. Use graph paper for any graph or sketch. The exam has 25 problems worth a total of 100 points.

1. **4 pts** Lesson 1.1

Order these numbers from least to greatest: $\sqrt{10}$, 3.1, $-\frac{7}{2}$, -3.4 , π .

2. **3 pts** Lesson 1.3

Evaluate $a^2 - 3ab + b$ when $a = -2$ and $b = 5$.

3. **4 pts** Lesson 1.4

Simplify $4(3x - 2) - 2(5x - 6)$.

4. **4 pts** Lesson 2.3

Solve $7x - 5 = 3(2x + 1) + x$. If there is no solution or every real number is a solution, say so.

5. **4 pts** Lesson 2.4

Solve $\frac{x}{3} + \frac{x-1}{4} = 5$.

6. **4 pts** Lesson 2.5

Solve $7x + 2y = 10$ for y .

7. **5 pts** Lesson 2.6

Two trains leave the same station at the same time and travel in opposite directions. One train is 15 miles per hour faster than the other. After 3 hours they are 345 miles apart. How fast is the slower train?

8. **4 pts** Lesson 3.2

Solve $5 - 3x \geq 2x + 20$.

9. (4 pts) Lesson 3.3

Solve $-7 < 2x - 3 \leq 9$. Write the answer as an inequality and in interval notation.

10. (4 pts) Lesson 3.4

Solve $|2x + 1| - 4 = 5$.

11. (4 pts) Lesson 3.5

Henry is cutting shelf boards that should be 48 inches long. A board is acceptable if its length L is within $\frac{1}{8}$ inch of 48 inches. Write an absolute value inequality for the acceptable lengths and solve it.

12. (4 pts) Lesson 4.2

Explain why a graph is not the graph of a function if some vertical line crosses it at two points.

13. (3 pts) Lesson 4.3

If $f(x) = 2x^2 - 3x + 1$, find $f(-2)$.

14. (4 pts) Lesson 4.4

Find the domain and range of $f(x) = \sqrt{x - 4}$.

15. (4 pts) Lesson 4.5

The arithmetic sequence 7, 11, 15, 19, ... continues. (a) Write an explicit formula for a_n . (b) Find a_{40} .

16. (3 pts) Lesson 5.1

Find the slope of the line through $(-3, 5)$ and $(3, 1)$.

17. (4 pts) Lesson 5.4

Write an equation in slope-intercept form for the line through $(2, -1)$ and $(6, 7)$.

18. (4 pts) Lesson 5.5

Find the x -intercept and the y -intercept of $4x - 6y = 24$.

19. (4 pts) Lesson 5.6

Write an equation in slope-intercept form for the line through $(1, 4)$ that is perpendicular to $y = -\frac{1}{3}x + 2$.

20. (5 pts) Lesson 5.7

A small-engine repair shop charges a fixed fee plus an hourly rate. A 2-hour repair costs \$110 and a 5-hour repair costs \$215. (a) Write an equation for the cost C of an h -hour repair. (b) How much does a 3.5-hour repair cost?

21. (4 pts) Lesson 6.2

Solve by substitution: $y = 3x - 7$ and $2x + 5y = 16$.

22. (4 pts) Lesson 6.4

Solve by elimination: $3x + 4y = 10$ and $5x - 6y = 23$.

23. (5 pts) Lesson 6.5

For a church spaghetti supper, adult tickets cost \$8 and children's tickets cost \$5. The church sold 150 tickets and collected \$1,020. How many adult tickets and how many children's tickets were sold?

24. (4 pts) Lesson 6.6

Samuel has at most \$60 to spend on seed. Corn seed costs \$4 per bag and bean seed costs \$6 per bag, and he needs at least 10 bags in all. Let x be the number of bags of corn seed and y the number of bags of bean seed. (a) Write an inequality for the budget. (b) Write an inequality for the number of bags. (c) Can he buy 6 bags of corn seed and 5 bags of bean seed?

25. (4 pts) Lesson 6.1

Challenge: For what value of k does the system $2x - 3y = 7$ and $kx + 6y = 5$ have no solution?

ANSWER KEY

1. $-\frac{7}{2}, -3.4, 3.1, \pi, \sqrt{10}$ (since $-\frac{7}{2} = -3.5$, $\pi \approx 3.14$ and $\sqrt{10} \approx 3.16$)
2. $(-2)^2 - 3(-2)(5) + 5 = 4 + 30 + 5 = 39$
3. $12x - 8 - 10x + 12 = 2x + 4$
4. The right side is $7x + 3$, so $7x - 5 = 7x + 3$ gives $-5 = 3$, which is false: no solution.
5. Multiply by 12: $4x + 3(x - 1) = 60$, so $7x - 3 = 60$ and $x = 9$. Check: $3 + 2 = 5$.
6. $2y = -7x + 10$, so $y = -\frac{7}{2}x + 5$
7. Let r be the slower speed: $3r + 3(r + 15) = 345$, so $6r + 45 = 345$ and $r = 50$. The slower train travels 50 miles per hour (the faster one 65).
8. $-5x \geq 15$; dividing by -5 reverses the sign: $x \leq -3$
9. Add 3: $-4 < 2x \leq 12$; divide by 2: $-2 < x \leq 6$, or $(-2, 6]$
10. $|2x + 1| = 9$, so $2x + 1 = 9$ or $2x + 1 = -9$: $x = 4$ or $x = -5$
11. $|L - 48| \leq \frac{1}{8}$, so $47\frac{7}{8} \leq L \leq 48\frac{1}{8}$ inches ($47.875 \leq L \leq 48.125$)
12. Every point on a vertical line has the same x -value. If the line crosses the graph at two points, then that one input x is paired with two different outputs y . A function must give exactly one output for each input, so the graph is not a function.
13. $f(-2) = 2(4) - 3(-2) + 1 = 8 + 6 + 1 = 15$
14. Domain: $x \geq 4$ (we cannot take the square root of a negative number). Range: $y \geq 0$.
15. (a) $a_n = 7 + (n - 1) \cdot 4 = 4n + 3$ (b) $a_{40} = 4(40) + 3 = 163$
16. $m = \frac{1 - 5}{3 - (-3)} = \frac{-4}{6} = -\frac{2}{3}$
17. $m = \frac{8}{4} = 2$; $y + 1 = 2(x - 2)$, so $y = 2x - 5$
18. x -intercept 6 (the point $(6, 0)$); y -intercept -4 (the point $(0, -4)$)
19. The perpendicular slope is 3 (since $-\frac{1}{3} \cdot 3 = -1$); $y - 4 = 3(x - 1)$, so $y = 3x + 1$
20. (a) $m = \frac{215 - 110}{5 - 2} = 35$ and $b = 110 - 70 = 40$, so $C = 35h + 40$ (b) $C = 35(3.5) + 40 = \$162.50$

21. $2x + 5(3x - 7) = 16$, so $17x = 51$, $x = 3$, $y = 2$: the solution is $(3, 2)$. Check:
 $6 + 10 = 16$.
22. Multiply by 3 and 2: $9x + 12y = 30$ and $10x - 12y = 46$; add: $19x = 76$, $x = 4$;
then $12 + 4y = 10$, $y = -\frac{1}{2}$. Solution $\left(4, -\frac{1}{2}\right)$.
23. $a + c = 150$ and $8a + 5c = 1020$; substituting $c = 150 - a$ gives $3a + 750 = 1020$,
so $a = 90$ and $c = 60$: 90 adult tickets and 60 children's tickets. Check:
 $720 + 300 = 1020$.
24. (a) $4x + 6y \leq 60$ (b) $x + y \geq 10$ (also $x \geq 0$, $y \geq 0$) (c) Yes: $4(6) + 6(5) = 54 \leq 60$
and $6 + 5 = 11 \geq 10$.
25. No solution means parallel lines: equal slopes, different intercepts. The first
slope is $\frac{2}{3}$ and the second is $-\frac{k}{6}$, so $-\frac{k}{6} = \frac{2}{3}$ and $k = -4$. Then the second
equation is $-4x + 6y = 5$, or $2x - 3y = -\frac{5}{2}$, a different line from $2x - 3y = 7$.

FINAL EXAM

Covers Units 1–13. Time: 2 hours.

You have 2 hours. A scientific calculator is allowed, except on problems marked "by hand", where you must show the work without a calculator. Show all your work clearly, one step per line; correct work earns partial credit even if the final answer has a small error. Give exact answers (fractions or simplest radical form) unless a problem asks for rounding. Check solutions by substituting where you can. Use graph paper for any graph or sketch. The exam has 30 problems worth a total of 100 points.

1. **3 pts** Lesson 1.4

Simplify $2(3x - 4) - (5x - 9) + x$.

2. **3 pts** Lesson 2.6

Henry has \$240 saved and adds \$35 each week. After how many weeks will he have \$660 saved?

3. **3 pts** Lesson 3.5

Solve $|2x + 5| > 9$.

4. **3 pts** Lesson 4.3

If $f(x) = 5 - 3x$, solve $f(x) = -16$.

5. **3 pts** Lesson 5.6

Write an equation in slope-intercept form for the line through $(-2, 3)$ that is parallel to $2x + 5y = 10$.

6. **3 pts** Lesson 6.3

Solve by elimination: $4x + 3y = 6$ and $2x - 3y = 12$.

7. **4 pts** Lesson 6.5

The Miller farm mixes a feed that is 10% protein with a feed that is 25% protein to make 300 pounds of feed that is 15% protein. How many pounds of each feed do they use?

8. **3 pts** Lesson 7.2

By hand: simplify $\frac{6a^{-2}b^5}{3a^4b^{-1}}$ so that all exponents are positive.

9. **3 pts** Lesson 7.3

By hand: multiply $(4.8 \times 10^7)(2.5 \times 10^{-3})$ and write the answer in scientific notation.

10. **4 pts** Lesson 7.5

Grace deposits \$2,500 in a savings account that pays 4% interest compounded quarterly. How much is in the account after 6 years? Round to the nearest cent.

11. **3 pts** Lesson 7.6

Find the 8th term of the geometric sequence $3, -6, 12, -24, \dots$

12. **3 pts** Lesson 8.1

Subtract: $(5x^2 - 3x + 8) - (2x^2 + 4x - 6)$.

13. **3 pts** Lesson 8.3

Multiply: $(3x + 2)(x^2 - 2x + 4)$.

14. **4 pts** Lesson 8.4

A student writes $(x + 7)^2 = x^2 + 49$. Explain why this is wrong, using a square diagram, and give the correct product.

15. **3 pts** Lesson 9.3

Factor $6x^2 + 7x - 20$.

16. **4 pts** Lesson 9.5

Factor completely: $3x^3 - 12x^2 - 3x + 12$.

17. **4 pts** Lesson 9.6

Samuel throws a ball upward from a roof 24 feet high with an initial speed of 40 feet per second. Its height after t seconds is $h = -16t^2 + 40t + 24$ feet. By factoring, find when the ball hits the ground.

18. **3 pts** Lesson 10.3

By hand: simplify $3\sqrt{20} - \sqrt{45} + \sqrt{80}$.

19. **3 pts** Lesson 10.4

Solve $\sqrt{2x + 7} = x + 2$. Check for extraneous solutions.

20. **3 pts** Lesson 10.5

On a map with a grid measured in miles, the Millers' farmhouse is at $(-3, 2)$ and the church is at $(5, -4)$. How far apart are they in a straight line?

21. **3 pts** Lesson 11.2

For $y = 2x^2 - 12x + 7$, find the vertex and the axis of symmetry, and say whether the vertex is a maximum or a minimum.

22. **4 pts** Lesson 11.4

Solve $x^2 + 6x - 11 = 0$ by completing the square. Give exact answers in simplest radical form.

23. **4 pts** Lesson 11.5

Use the quadratic formula to solve $2x^2 - 7x + 2 = 0$. Give the exact solutions, then round each to the nearest hundredth.

24. **4 pts** Lesson 11.6

Challenge: For what values of k does $x^2 + kx + 9 = 0$ have exactly one real solution?

25. **4 pts** Lesson 11.7

Challenge: Ruth has 120 feet of fencing to build a rectangular garden against the long wall of a barn, so fence is needed on only three sides. What is the greatest area she can enclose?

26. **3 pts** Lesson 12.2

Simplify $\frac{x^2 - 9}{x^2 + x - 6}$ and state the excluded values.

27. **3 pts** Lesson 12.4

Subtract: $\frac{5}{x - 2} - \frac{3}{x + 1}$.

28. **4 pts** Lesson 12.5

Grace can paint the fence around the church yard in 6 hours, and her brother Samuel can paint it in 12 hours. How long will it take them working together?

29. **3 pts** Lesson 13.2

Henry measures a young sunflower and draws a line of fit on a scatter plot of day x and height y in centimeters. The line passes through $(2, 15)$ and $(10, 55)$.

(a) Write the equation of the line of fit. (b) Use it to predict the height on day 14.

30. **3 pts** Lesson 13.3

A table gives x : 0, 1, 2, 3, 4 and y : 3, 6, 12, 24, 48. Decide whether the data are linear, quadratic or exponential, and write an equation for y .

ANSWER KEY

1. $6x - 8 - 5x + 9 + x = 2x + 1$
2. $240 + 35w = 660$, so $35w = 420$ and $w = 12$: 12 weeks
3. $2x + 5 > 9$ or $2x + 5 < -9$, so $x > 2$ or $x < -7$
4. $5 - 3x = -16$, so $-3x = -21$ and $x = 7$
5. The slope of $2x + 5y = 10$ is $-\frac{2}{5}$; $y - 3 = -\frac{2}{5}(x + 2)$, so $y = -\frac{2}{5}x + \frac{11}{5}$
6. Add: $6x = 18$, $x = 3$; then $12 + 3y = 6$, $y = -2$. Solution $(3, -2)$.
7. $x + y = 300$ and $0.10x + 0.25y = 45$; substituting $y = 300 - x$ gives
 $-0.15x + 75 = 45$, so $x = 200$ and $y = 100$: 200 pounds of the 10% feed and 100 pounds of the 25% feed
8. $2a^{-6}b^6 = \frac{2b^6}{a^6}$
9. $12 \times 10^4 = 1.2 \times 10^5$
10. $A = 2500 \left(1 + \frac{0.04}{4}\right)^{4 \cdot 6} = 2500(1.01)^{24} \approx \$3,174.34$
11. $r = -2$, so $a_8 = 3(-2)^7 = 3(-128) = -384$
12. $5x^2 - 3x + 8 - 2x^2 - 4x + 6 = 3x^2 - 7x + 14$
13. $3x^3 - 6x^2 + 12x + 2x^2 - 4x + 8 = 3x^3 - 4x^2 + 8x + 8$
14. A square with side $x + 7$ splits into four pieces: an x by x square (x^2), a 7 by 7 square (49), and two x by 7 rectangles ($7x$ each). The answer $x^2 + 49$ leaves out the two rectangles. The correct product is $(x + 7)^2 = x^2 + 14x + 49$.
15. $ac = -120$; use 15 and -8 : $6x^2 + 15x - 8x - 20 = 3x(2x + 5) - 4(2x + 5) = (3x - 4)(2x + 5)$
16. $3(x^3 - 4x^2 - x + 4) = 3[x^2(x - 4) - (x - 4)] = 3(x - 4)(x^2 - 1) = 3(x - 4)(x - 1)(x + 1)$
17. $-16t^2 + 40t + 24 = 0$; divide by -8 : $2t^2 - 5t - 3 = 0$, so $(2t + 1)(t - 3) = 0$ and $t = -\frac{1}{2}$ or $t = 3$. Time cannot be negative, so the ball hits the ground after 3 seconds.
18. $6\sqrt{5} - 3\sqrt{5} + 4\sqrt{5} = 7\sqrt{5}$
19. Square: $2x + 7 = x^2 + 4x + 4$, so $x^2 + 2x - 3 = 0$ and $(x + 3)(x - 1) = 0$. Check: $x = 1$ gives $\sqrt{9} = 3$ ✓; $x = -3$ gives $\sqrt{1} = 1 \neq -1$, extraneous. Solution: $x = 1$.
20. $d = \sqrt{(5 - (-3))^2 + (-4 - 2)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$ miles

21. $x = -\frac{-12}{2(2)} = 3$ and $y = 18 - 36 + 7 = -11$: vertex $(3, -11)$, axis $x = 3$; since $a = 2 > 0$ the parabola opens up, so -11 is a minimum.
22. $x^2 + 6x + 9 = 11 + 9$, so $(x + 3)^2 = 20$, $x + 3 = \pm 2\sqrt{5}$ and $x = -3 \pm 2\sqrt{5}$
23. $x = \frac{7 \pm \sqrt{49 - 16}}{4} = \frac{7 \pm \sqrt{33}}{4}$, so $x \approx 0.31$ or $x \approx 3.19$
24. Exactly one real solution when the discriminant is 0: $k^2 - 4(1)(9) = 0$, so $k^2 = 36$ and $k = -6$ or $k = 6$
25. Let x be each side perpendicular to the barn; the side parallel is $120 - 2x$, so $A = x(120 - 2x) = -2x^2 + 120x$. The vertex is at $x = -\frac{120}{2(-2)} = 30$, giving a 30 ft by 60 ft garden with area 1,800 square feet.
26. $\frac{(x - 3)(x + 3)}{(x + 3)(x - 2)} = \frac{x - 3}{x - 2}$, with $x \neq -3$ and $x \neq 2$
27. $\frac{5(x + 1) - 3(x - 2)}{(x - 2)(x + 1)} = \frac{2x + 11}{(x - 2)(x + 1)}$
28. $\frac{t}{6} + \frac{t}{12} = 1$; multiply by 12: $2t + t = 12$, so $t = 4$ hours
29. (a) $m = \frac{55 - 15}{10 - 2} = 5$ and $b = 15 - 10 = 5$, so $y = 5x + 5$ (b) $y = 5(14) + 5 = 75$ centimeters
30. Each y -value is 2 times the one before (a constant ratio), so the data are exponential: $y = 3 \cdot 2^x$