Algebra II

First edition | 13 units | 67 lessons | About 10 hours of video

FIRST EDITION

Course guide

Algebra II is the study of functions and the equations they lead to. You will begin with functions and their graphs: transformations, composition and inverses. From there the course moves through systems of equations and linear programming; quadratic functions and the complex numbers; polynomials, long and synthetic division, and the Remainder and Factor Theorems; rational expressions and rational functions; radicals and rational exponents; exponential, logistic and logarithmic functions; sequences, series and proof by induction; counting, probability and the Binomial Theorem; the conic sections; matrices and vectors; and finally a short bridge to trigonometry.

The approach is simple and steady. Key results are derived, not just announced, so you understand why a rule works before you use it. Each lesson gives you two worked examples and a full practice set, so you master the skills by doing them every day. A program draws every graph in the course exactly from its equation, so what you see is true to the formula. Short history notes show how these ideas were discovered, and a daily practice habit builds the facts and working habits that make the hard problems easier.

After this course. The School Launch Trigonometry course (one semester) picks up where Unit 13 leaves off: the unit circle, trigonometric graphs and identities, and the Laws of Sines and Cosines. The School Launch Statistics course builds on the counting and probability of Unit 10, which is only a start, with data, distributions and inference. Precalculus and calculus grow from ideas met here: the rates of change and the difference quotient of Lesson 1.6, and the limits met informally in the number e and in infinite series. The functions, exponents, logarithms, the logistic function, vectors and matrices in this course are a start toward a later School Launch course on the mathematics of neural networks and modern computing, which will build linear algebra and these functions much further.

> Most videos are by Eddie Woo, a veteran classroom teacher in Australia, and Mario's Math Tutoring, a U.S. math teacher. Eddie Woo says "index laws" for exponent rules, "surds" for radicals, "algebraic fractions" for rational expressions, "simultaneous equations" for systems, "progression" for sequence, "limiting sum" for the sum of an infinite series, "factorise" for factor and "piecemeal function" for piecewise function. Each lesson notes where a video's words differ from the course.

Start with Lesson 1.1 →

Who it is for

For students in grades 10–11 (ages 15–17) who have finished Algebra I and Geometry; it prepares them for the School Launch one-semester Trigonometry course, the Statistics course, and later precalculus and calculus.

How to use this course

1. The course takes one school year: 32 weeks, 5 days a week, about 45 minutes a day (some days run a little longer).
2. Daily rhythm: start with the lesson's practice habit, read the core ideas with a pencil in hand and work through every derivation yourself, study the two worked examples, do the practice set, check your answers, and correct every mistake.
3. Lessons 1.1, 2.1, 3.1, 3.2 and 5.1 are one-day review lessons. Every other lesson takes two days: on day 1, read the teaching and examples and do the first half of the practice; on day 2, review and do the second half.
4. Each unit ends with a review day (rework the mastery checklist and missed problems) and a unit test.
5. The midterm exam (Units 1–6) comes after Unit 6, with a review day before it; the year ends with two review days and the final exam (Units 1–13).
6. A scientific calculator is allowed throughout and is required for e, logarithms and loan payments. When a problem says "by hand," put the calculator away.
7. A graphing calculator or a free online graphing tool is optional: use it to check your graphs and for the few problems that ask for it, never in place of the algebra.
8. Use graph paper for every graph you draw. The graphs in the course are drawn by a program from their exact equations, so you can trust what you read from them.

9. Students who need more time may stretch the course; steady understanding matters more than finishing on a set date.

Scope and sequence

UNIT 1 · WEEKS 1–3

Functions and Their Graphs

Review function notation and domain, then study parent functions, transformations, piecewise functions, composition, inverses and rates of change.

1.1 Functions and Function Notation: A Review

1.2 Parent Functions and Transformations

1.3 Piecewise and Absolute Value Functions

1.4 Combining and Composing Functions

1.5 Inverse Functions

1.6 Rates of Change and Function Behavior

UNIT 2 · WEEKS 3–5

Linear Systems and Linear Programming

Solve systems of two and three linear equations, graph systems of inequalities, and find the best choice with linear programming.

2.1 Systems of Two Linear Equations: A Review

2.2 Systems of Linear Inequalities

2.3 Linear Programming

2.4 Systems of Three Equations in Three Variables

UNIT 3 · WEEKS 5–7

Quadratic Functions and Complex Numbers

Work with the three forms of a quadratic, meet the complex numbers, and solve quadratic equations, inequalities and models.

3.1 Quadratic Functions and Their Three Forms

3.2 Solving Quadratic Equations: A Review

3.3 Complex Numbers

3.4 Complex Solutions and the Discriminant

3.5 Quadratic Inequalities

3.6 Quadratic Models

UNIT 4 · WEEKS 7–10

Polynomial Functions

Study end behavior, divide polynomials, use the Remainder, Factor and Rational Root Theorems, and graph polynomial functions from their zeros.

4.1 Polynomial Functions and End Behavior

4.2 Operations with Polynomials

4.3 Polynomial Long Division

4.4 Synthetic Division and the Remainder Theorem

4.5 The Factor Theorem and Factoring Polynomials

4.6 Finding Zeros of Polynomials

4.7 Graphing Polynomial Functions

UNIT 5 · WEEKS 11–13

Rational Expressions and Functions

Simplify and combine rational expressions, graph rational functions with their asymptotes and holes, and solve rational equations and variation problems.

5.1 Rational Expressions: Simplifying, Multiplying and Dividing

5.2 Adding and Subtracting Rational Expressions

5.3 Rational Functions and Asymptotes

5.4 Graphing Rational Functions

5.5 Solving Rational Equations

5.6 Variation and Rational Models

UNIT 6 · WEEKS 13–15

Radicals and Rational Exponents

Define rational exponents, simplify and combine radicals, graph radical functions, and solve radical equations.

6.1 nth Roots and Rational Exponents

6.2 Properties of Rational Exponents and Radicals

6.3 Operations with Radical Expressions

6.4 Radical Functions and Their Graphs

6.5 Solving Radical Equations

UNIT 7 · WEEKS 16–18

Exponential Functions

Graph exponential functions, meet the number e through compound interest, and build exponential and logistic models.

7.1 Exponential Functions and Their Graphs

7.2 Compound Interest and the Number e

7.3 Exponential Models

7.4 Logistic Functions

UNIT 8 · WEEKS 18–21

Logarithmic Functions

Understand logarithms as exponents, graph them, derive their properties, and use them to solve equations and read logarithmic scales.

8.1 Logarithms

8.2 Graphs of Logarithmic Functions

8.3 Properties of Logarithms

8.4 Solving Exponential Equations

8.5 Solving Logarithmic Equations

8.6 Logarithmic Scales and Models

UNIT 9 · WEEKS 21–23

Sequences and Series

Write and sum arithmetic and geometric sequences, find infinite sums, apply series to savings and loans, and prove statements by induction.

9.1 Sequences and Summation Notation

9.2 Arithmetic Sequences and Series

9.3 Geometric Sequences and Series

9.4 Infinite Geometric Series

9.5 Savings, Loans and Geometric Series

9.6 Mathematical Induction

UNIT 10 · WEEKS 24–25

Counting, Probability and the Binomial Theorem

Count with permutations and combinations, connect Pascal's triangle to the Binomial Theorem, and compute probabilities by counting.

10.1 The Counting Principle and Permutations

10.2 Combinations

10.3 Pascal's Triangle and the Binomial Theorem

10.4 Probability with Counting

UNIT 11 · WEEKS 26–28

Conic Sections

Derive and graph parabolas, circles, ellipses and hyperbolas, identify a conic from its equation, and solve nonlinear systems.

11.1 Parabolas: Focus and Directrix

11.2 Circles and Ellipses

11.3 Hyperbolas

11.4 Recognizing and Graphing Conic Sections

11.5 Nonlinear Systems

UNIT 12 · WEEKS 28–31

Matrices and Vectors

Compute with matrices, use determinants and inverses to solve systems, work with vectors and the dot product, and see matrices as transformations of the plane.

12.1 Matrices and Matrix Operations

12.2 Matrix Multiplication

12.3 Determinants and Inverse Matrices

12.4 Solving Systems with Matrices

12.5 Vectors and the Dot Product

12.6 Matrices as Transformations

UNIT 13 · WEEKS 31–32

A Bridge to Trigonometry

Review right-triangle trigonometry and take a first look at radians and the unit circle.

13.1 Right-Triangle Trigonometry: A Review

13.2 Angles, Radians and the Unit Circle: A First Look

Weekly pacing calendar

WEEK	PLAN
1	• Day 1: Lesson 1.1 Functions and Function Notation: A Review • Day 2: Lesson 1.2 Parent Functions and Transformations (day 1 of 2) • Day 3: Lesson 1.2 Parent Functions and Transformations (day 2 of 2) • Day 4: Lesson 1.3 Piecewise and Absolute Value Functions (day 1 of 2) • Day 5: Lesson 1.3 Piecewise and Absolute Value Functions (day 2 of 2) *Families on a 4-day week can combine the two days of a two-day lesson when the student is ready, or stretch the course to about 40 weeks.*
2	• Day 1: Lesson 1.4 Combining and Composing Functions (day 1 of 2) • Day 2: Lesson 1.4 Combining and Composing Functions (day 2 of 2) • Day 3: Lesson 1.5 Inverse Functions (day 1 of 2) • Day 4: Lesson 1.5 Inverse Functions (day 2 of 2) • Day 5: Lesson 1.6 Rates of Change and Function Behavior (day 1 of 2)
3	• Day 1: Lesson 1.6 Rates of Change and Function Behavior (day 2 of 2) • Day 2: Unit 1 review (mastery checklist and missed problems) • Day 3: Unit 1 test • Day 4: Lesson 2.1 Systems of Two Linear Equations: A Review • Day 5: Lesson 2.2 Systems of Linear Inequalities (day 1 of 2)
4	• Day 1: Lesson 2.2 Systems of Linear Inequalities (day 2 of 2) • Day 2: Lesson 2.3 Linear Programming (day 1 of 2) • Day 3: Lesson 2.3 Linear Programming (day 2 of 2) • Day 4: Lesson 2.4 Systems of Three Equations in Three Variables (day 1 of 2) • Day 5: Lesson 2.4 Systems of Three Equations in Three Variables (day 2 of 2)
5	• Day 1: Unit 2 review (mastery checklist and missed problems) • Day 2: Unit 2 test • Day 3: Lesson 3.1 Quadratic Functions and Their Three Forms • Day 4: Lesson 3.2 Solving Quadratic Equations: A Review • Day 5: Lesson 3.3 Complex Numbers (day 1 of 2)
6	• Day 1: Lesson 3.3 Complex Numbers (day 2 of 2) • Day 2: Lesson 3.4 Complex Solutions and the Discriminant (day 1 of 2) • Day 3: Lesson 3.4 Complex Solutions and the Discriminant (day 2 of 2) • Day 4: Lesson 3.5 Quadratic Inequalities (day 1 of 2)

WEEK	PLAN
	• Day 5: Lesson 3.5 Quadratic Inequalities (day 2 of 2)
7	• Day 1: Lesson 3.6 Quadratic Models (day 1 of 2) • Day 2: Lesson 3.6 Quadratic Models (day 2 of 2) • Day 3: Unit 3 review (mastery checklist and missed problems) • Day 4: Unit 3 test • Day 5: Lesson 4.1 Polynomial Functions and End Behavior (day 1 of 2)
8	• Day 1: Lesson 4.1 Polynomial Functions and End Behavior (day 2 of 2) • Day 2: Lesson 4.2 Operations with Polynomials (day 1 of 2) • Day 3: Lesson 4.2 Operations with Polynomials (day 2 of 2) • Day 4: Lesson 4.3 Polynomial Long Division (day 1 of 2) • Day 5: Lesson 4.3 Polynomial Long Division (day 2 of 2)
9	• Day 1: Lesson 4.4 Synthetic Division and the Remainder Theorem (day 1 of 2) • Day 2: Lesson 4.4 Synthetic Division and the Remainder Theorem (day 2 of 2) • Day 3: Lesson 4.5 The Factor Theorem and Factoring Polynomials (day 1 of 2) • Day 4: Lesson 4.5 The Factor Theorem and Factoring Polynomials (day 2 of 2) • Day 5: Lesson 4.6 Finding Zeros of Polynomials (day 1 of 2)
10	• Day 1: Lesson 4.6 Finding Zeros of Polynomials (day 2 of 2) • Day 2: Lesson 4.7 Graphing Polynomial Functions (day 1 of 2) • Day 3: Lesson 4.7 Graphing Polynomial Functions (day 2 of 2) • Day 4: Unit 4 review (mastery checklist and missed problems) • Day 5: Unit 4 test
11	• Day 1: Lesson 5.1 Rational Expressions: Simplifying, Multiplying and Dividing • Day 2: Lesson 5.2 Adding and Subtracting Rational Expressions (day 1 of 2) • Day 3: Lesson 5.2 Adding and Subtracting Rational Expressions (day 2 of 2) • Day 4: Lesson 5.3 Rational Functions and Asymptotes (day 1 of 2) • Day 5: Lesson 5.3 Rational Functions and Asymptotes (day 2 of 2)
12	• Day 1: Lesson 5.4 Graphing Rational Functions (day 1 of 2) • Day 2: Lesson 5.4 Graphing Rational Functions (day 2 of 2) • Day 3: Lesson 5.5 Solving Rational Equations (day 1 of 2) • Day 4: Lesson 5.5 Solving Rational Equations (day 2 of 2) • Day 5: Lesson 5.6 Variation and Rational Models (day 1 of 2)

WEEK	PLAN
13	• Day 1: Lesson 5.6 Variation and Rational Models (day 2 of 2) • Day 2: Unit 5 review (mastery checklist and missed problems) • Day 3: Unit 5 test • Day 4: Lesson 6.1 nth Roots and Rational Exponents (day 1 of 2) • Day 5: Lesson 6.1 nth Roots and Rational Exponents (day 2 of 2)
14	• Day 1: Lesson 6.2 Properties of Rational Exponents and Radicals (day 1 of 2) • Day 2: Lesson 6.2 Properties of Rational Exponents and Radicals (day 2 of 2) • Day 3: Lesson 6.3 Operations with Radical Expressions (day 1 of 2) • Day 4: Lesson 6.3 Operations with Radical Expressions (day 2 of 2) • Day 5: Lesson 6.4 Radical Functions and Their Graphs (day 1 of 2)
15	• Day 1: Lesson 6.4 Radical Functions and Their Graphs (day 2 of 2) • Day 2: Lesson 6.5 Solving Radical Equations (day 1 of 2) • Day 3: Lesson 6.5 Solving Radical Equations (day 2 of 2) • Day 4: Unit 6 review (mastery checklist and missed problems) • Day 5: Unit 6 test
16	• Day 1: Midterm review (Units 1–6) • Day 2: Midterm exam (Units 1–6) • Day 3: Lesson 7.1 Exponential Functions and Their Graphs (day 1 of 2) • Day 4: Lesson 7.1 Exponential Functions and Their Graphs (day 2 of 2) • Day 5: Lesson 7.2 Compound Interest and the Number e (day 1 of 2) *Midterm week. Use any spare time to correct missed problems from Units 1–6.*
17	• Day 1: Lesson 7.2 Compound Interest and the Number e (day 2 of 2) • Day 2: Lesson 7.3 Exponential Models (day 1 of 2) • Day 3: Lesson 7.3 Exponential Models (day 2 of 2) • Day 4: Lesson 7.4 Logistic Functions (day 1 of 2) • Day 5: Lesson 7.4 Logistic Functions (day 2 of 2)
18	• Day 1: Unit 7 review (mastery checklist and missed problems) • Day 2: Unit 7 test • Day 3: Lesson 8.1 Logarithms (day 1 of 2) • Day 4: Lesson 8.1 Logarithms (day 2 of 2) • Day 5: Lesson 8.2 Graphs of Logarithmic Functions (day 1 of 2)

WEEK	PLAN
19	• Day 1: Lesson 8.2 Graphs of Logarithmic Functions (day 2 of 2) • Day 2: Lesson 8.3 Properties of Logarithms (day 1 of 2) • Day 3: Lesson 8.3 Properties of Logarithms (day 2 of 2) • Day 4: Lesson 8.4 Solving Exponential Equations (day 1 of 2) • Day 5: Lesson 8.4 Solving Exponential Equations (day 2 of 2)
20	• Day 1: Lesson 8.5 Solving Logarithmic Equations (day 1 of 2) • Day 2: Lesson 8.5 Solving Logarithmic Equations (day 2 of 2) • Day 3: Lesson 8.6 Logarithmic Scales and Models (day 1 of 2) • Day 4: Lesson 8.6 Logarithmic Scales and Models (day 2 of 2) • Day 5: Unit 8 review (mastery checklist and missed problems)
21	• Day 1: Unit 8 test • Day 2: Lesson 9.1 Sequences and Summation Notation (day 1 of 2) • Day 3: Lesson 9.1 Sequences and Summation Notation (day 2 of 2) • Day 4: Lesson 9.2 Arithmetic Sequences and Series (day 1 of 2) • Day 5: Lesson 9.2 Arithmetic Sequences and Series (day 2 of 2)
22	• Day 1: Lesson 9.3 Geometric Sequences and Series (day 1 of 2) • Day 2: Lesson 9.3 Geometric Sequences and Series (day 2 of 2) • Day 3: Lesson 9.4 Infinite Geometric Series (day 1 of 2) • Day 4: Lesson 9.4 Infinite Geometric Series (day 2 of 2) • Day 5: Lesson 9.5 Savings, Loans and Geometric Series (day 1 of 2)
23	• Day 1: Lesson 9.5 Savings, Loans and Geometric Series (day 2 of 2) • Day 2: Lesson 9.6 Mathematical Induction (day 1 of 2) • Day 3: Lesson 9.6 Mathematical Induction (day 2 of 2) • Day 4: Unit 9 review (mastery checklist and missed problems) • Day 5: Unit 9 test
24	• Day 1: Lesson 10.1 The Counting Principle and Permutations (day 1 of 2) • Day 2: Lesson 10.1 The Counting Principle and Permutations (day 2 of 2) • Day 3: Lesson 10.2 Combinations (day 1 of 2) • Day 4: Lesson 10.2 Combinations (day 2 of 2) • Day 5: Lesson 10.3 Pascal's Triangle and the Binomial Theorem (day 1 of 2)
25	• Day 1: Lesson 10.3 Pascal's Triangle and the Binomial Theorem (day 2 of 2) • Day 2: Lesson 10.4 Probability with Counting (day 1 of 2)

WEEK	PLAN
	• Day 3: Lesson 10.4 Probability with Counting (day 2 of 2) • Day 4: Unit 10 review (mastery checklist and missed problems) • Day 5: Unit 10 test
26	• Day 1: Lesson 11.1 Parabolas: Focus and Directrix (day 1 of 2) • Day 2: Lesson 11.1 Parabolas: Focus and Directrix (day 2 of 2) • Day 3: Lesson 11.2 Circles and Ellipses (day 1 of 2) • Day 4: Lesson 11.2 Circles and Ellipses (day 2 of 2) • Day 5: Lesson 11.3 Hyperbolas (day 1 of 2)
27	• Day 1: Lesson 11.3 Hyperbolas (day 2 of 2) • Day 2: Lesson 11.4 Recognizing and Graphing Conic Sections (day 1 of 2) • Day 3: Lesson 11.4 Recognizing and Graphing Conic Sections (day 2 of 2) • Day 4: Lesson 11.5 Nonlinear Systems (day 1 of 2) • Day 5: Lesson 11.5 Nonlinear Systems (day 2 of 2)
28	• Day 1: Unit 11 review (mastery checklist and missed problems) • Day 2: Unit 11 test • Day 3: Lesson 12.1 Matrices and Matrix Operations (day 1 of 2) • Day 4: Lesson 12.1 Matrices and Matrix Operations (day 2 of 2) • Day 5: Lesson 12.2 Matrix Multiplication (day 1 of 2)
29	• Day 1: Lesson 12.2 Matrix Multiplication (day 2 of 2) • Day 2: Lesson 12.3 Determinants and Inverse Matrices (day 1 of 2) • Day 3: Lesson 12.3 Determinants and Inverse Matrices (day 2 of 2) • Day 4: Lesson 12.4 Solving Systems with Matrices (day 1 of 2) • Day 5: Lesson 12.4 Solving Systems with Matrices (day 2 of 2)
30	• Day 1: Lesson 12.5 Vectors and the Dot Product (day 1 of 2) • Day 2: Lesson 12.5 Vectors and the Dot Product (day 2 of 2) • Day 3: Lesson 12.6 Matrices as Transformations (day 1 of 2) • Day 4: Lesson 12.6 Matrices as Transformations (day 2 of 2) • Day 5: Unit 12 review (mastery checklist and missed problems)
31	• Day 1: Unit 12 test • Day 2: Lesson 13.1 Right-Triangle Trigonometry: A Review (day 1 of 2) • Day 3: Lesson 13.1 Right-Triangle Trigonometry: A Review (day 2 of 2) • Day 4: Lesson 13.2 Angles, Radians and the Unit Circle: A First Look (day 1 of 2)

WEEK	PLAN
	• Day 5: Lesson 13.2 Angles, Radians and the Unit Circle: A First Look (day 2 of 2)
32	• Day 1: Unit 13 review (mastery checklist and missed problems) • Day 2: Unit 13 test • Day 3: Final review (Units 1–6) • Day 4: Final review (Units 7–13) • Day 5: Final exam (Units 1–13) *Final exam week.*

Materials

Pencil and eraser

Work in pencil so corrections are easy.

A notebook

Keep all practice, corrections and tests in one place so you can review them.

Graph paper

For every graph you draw.

Ruler

For axes, lines and feasible regions.

Scientific calculator

It must have e, ln and log keys (for example, the TI-30XS MultiView).

Index cards

For the practice habits, key facts and formulas.

Optional: graphing calculator or free online graphing tool

A TI-84 or a free online grapher, for checking graphs and the few problems that ask for one.

Optional: string and two pins

For drawing ellipses in Lesson 11.2.

Grading

Grades come from daily practice, unit tests, the midterm and the final exam. The goal is mastery, so corrections and careful work count.

PART	%
Daily practice sets	**20**
Unit tests	**50**
Midterm exam	**15**
Final exam	**15**

Scale: A 90–100, B 80–89, C 70–79, D 60–69, below 60 redo

- Grade daily practice for completion and corrections: a practice set earns full credit when every problem is attempted and every mistake has been corrected.
- A student who scores below 70 on a unit test may review the missed material and take a retake test (new problems on the same lessons); record the higher score, up to 80.
- Give partial credit for correct work and sound reasoning even when the final answer has an arithmetic slip.
- Expect exact answers (fractions, radicals, logarithms, i) where a problem asks for them, and correctly rounded answers with units where a problem asks for rounding.

A NOTE TO PARENTS

You do not need to be an algebra expert to teach this course well. Every lesson explains its ideas step by step, and every practice problem, unit test and exam comes with a full answer key. Your job is to keep the work steady, check it, and ask good questions.

Help without doing the work for your student. Ask: "What is the question asking?" "What are the excluded values, or the domain?" "Did you check your answer in the original equation?" "Can you sketch it?" "Can you show me your steps?" Daily practice and correcting every mistake matter more than speed; a student who works consistently for 45 minutes a day will go much further than one who crams.

Algebra II is worth the effort. It is the foundation for trigonometry and calculus, for the sciences, engineering and the skilled trades, for business and personal finance,

and for computing. Just as important, it trains clear, honest and orderly reasoning: saying exactly what you mean, checking your work, and following an argument to its conclusion.

UNIT 1

Functions and Their Graphs

Algebra II is the study of functions: rules that turn each input into exactly one output. This unit gathers what you learned about functions in Algebra I and sharpens it into a toolkit you will use all year. You will review function notation, domain and range, then learn the six parent functions and how every member of a family is a parent that has been shifted, stretched or flipped. You will build new functions by combining and composing old ones, undo functions with inverses, and describe how a function behaves: where it rises and falls, its hilltops and valleys, its symmetry and its rates of change.

The emphasis is on understanding why the rules work. You will see why $f(x - h)$ moves a graph to the right, why the graph of an inverse is a reflection in the line $y = x$, and why even and odd functions have the symmetries they do. These ideas return in every later unit: quadratics, polynomials, rational, radical, exponential and logarithmic functions are all studied as parent functions with transformations, inverses and rates of change. The difference quotient at the end of the unit is a first look at the central idea of calculus.

PREREQUISITES

- Evaluate expressions and solve linear and quadratic equations (Algebra I).
- Use function notation, the vertical line test, and domain and range in interval notation (Algebra I).
- Find the slope of a line and write its equation (Algebra I).
- Expand products such as $(x+h)^2$ and simplify polynomial expressions (Algebra I).
- Find the vertex of a parabola $y = ax^2 + bx + c$ (Algebra I).
- Describe translations and reflections of the plane as coordinate rules (Geometry).

MASTERY CHECKLIST

- ☐ I can evaluate a function at numbers and at expressions such as $f(a+2)$ and $f(-x)$.
- ☐ I can state the domain and range of a function from a graph, an equation or a situation, in interval notation.
- ☐ I can read a graph to solve $f(x) = c$ and $f(x) > 0$.
- ☐ I can sketch the six parent functions from their key points.
- ☐ I can graph $y = a\,f(x-h)+k$ and explain why $f(bx)$ shifts the graph right.
- ☐ I can evaluate, graph and write piecewise, absolute value and step functions.
- ☐ I can add, subtract, multiply, divide and compose functions and find the domain of the result.
- ☐ I can find an inverse function, verify it by composition, and

explain why its graph is a reflection in

$y = x$.

- ☐ I can use the horizontal line test and restrict a domain so a function has an inverse.
- ☐ I can find average rates of change, intervals of increase and decrease, and relative maxima and minima.
- ☐ I can test a function for even or odd symmetry and describe its end behavior in words.
- ☐ I can simplify the difference quotient for linear and quadratic functions.

LESSONS IN THIS UNIT

1.1	**Functions and Function Notation: A Review**	45 minutes
1.2	**Parent Functions and Transformations**	2 sessions of 45 minutes
1.3	**Piecewise and Absolute Value Functions**	2 sessions of 45 minutes
1.4	**Combining and Composing Functions**	2 sessions of 45 minutes
1.5	**Inverse Functions**	2 sessions of 45 minutes
1.6	**Rates of Change and Function Behavior**	2 sessions of 45 minutes

Unit 1 Test

You have 60 minutes. A scientific calculator is allowed; graphing tools are not. Show your work for every problem: a correct answer with no work earns partial credit only. Give exact answers unless a problem tells you how to round, and write domains, ranges and solution sets in interval notation.

1. **5 pts**

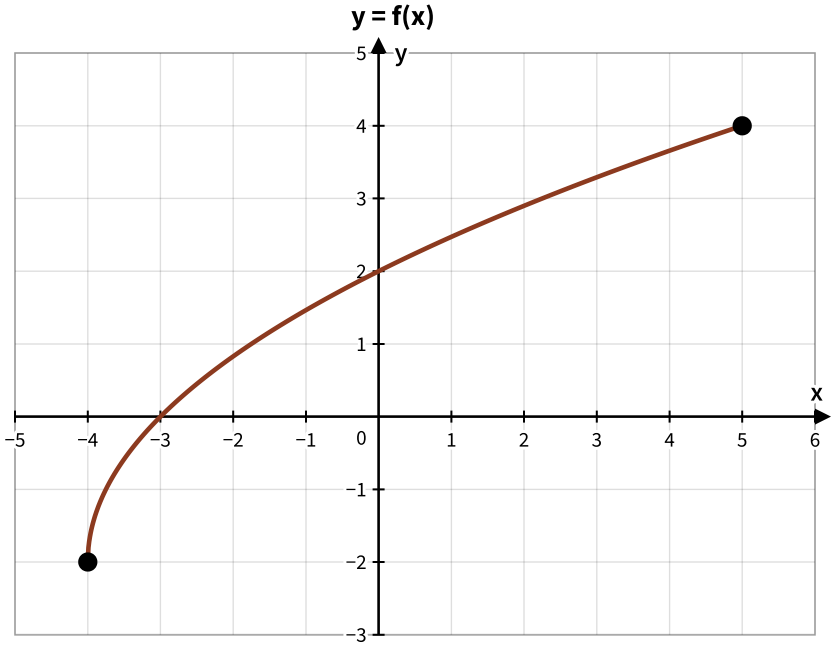
   Let $f(x) = x^2 + 3x - 2$. Find (a) $f(-4)$ and (b) $f(a - 1)$ in simplest form.

2. **6 pts**

   Find the domain of $f(x) = \dfrac{\sqrt{2x + 8}}{x - 3}$ in interval notation.

3. **6 pts**

   The graph of $y = f(x)$ is shown, including both endpoints. Give (a) the domain, (b) the range and (c) the set of x where $f(x) > 0$, all in interval notation.

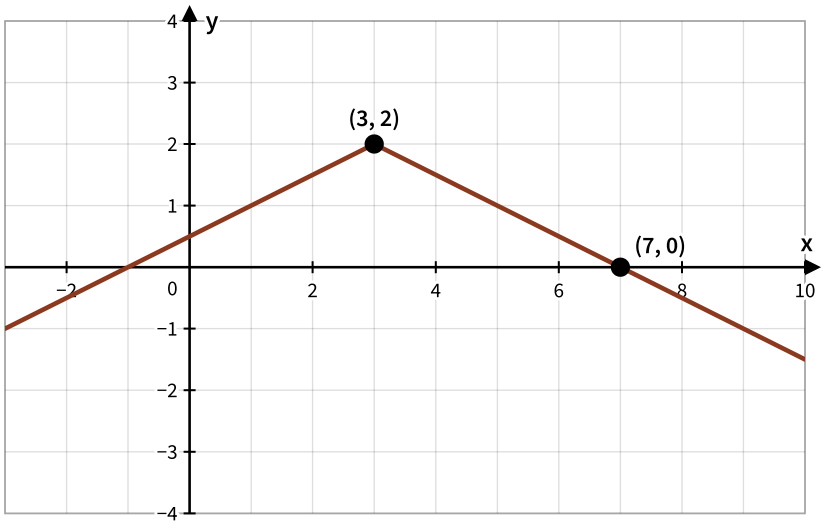


4. **7 pts**

   (a) The point $(4, -1)$ is on the graph of $y = f(x)$. Find the matching point on the graph of $y = -2f(x - 3) + 1$. (b) Write the rule for the graph of $y = \sqrt{x}$ after it is reflected in the x-axis, shifted 2 units right and shifted 6 units up.

5. **6 pts**

The graph is a transformation of $y = |x|$. Write its equation.



6. **7 pts**

A community orchard sells apples for \$1.50 a pound on purchases of up to 10 pounds. On larger purchases the first 10 pounds cost \$1.50 a pound and every pound beyond 10 costs \$1.20. (a) Write the cost $C(w)$ of w pounds as a piecewise function. (b) Find the cost of 16 pounds.

7. **5 pts**

Let $g(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$. (a) Find $g(7)$ and $g(-3)$. (b) For which x is $\lfloor x \rfloor = -2$? Use interval notation.

8. **7 pts**

Let $f(x) = x^2 - 2x$ and $g(x) = x + 3$. Find and simplify (a) $f(g(x))$ and (b) $g(f(x))$.

9. **6 pts**

Let $f(x) = \dfrac{1}{x - 4}$ and $g(x) = \sqrt{x}$. Write $f(g(x))$ and give its domain in interval notation.

10. **7 pts**

A family furniture shop prices a chair by marking up its wholesale cost x by 40%, $m(x) = 1.4x$, and then adding a \$20 delivery charge, $d(x) = x + 20$. (a) Write the customer's price $d(m(x))$. (b) Find the price of a chair with a wholesale cost of \$150. (c) Write $m(d(x))$; what would this order of steps do to the delivery charge?

11. **7 pts**

Find $f^{-1}(x)$ for $f(x) = \dfrac{4x + 1}{3}$, and verify that $f(f^{-1}(x)) = x$.

12. **7 pts**

Let $f(x) = (x - 2)^2 + 1$ with domain $x \geq 2$. Find (a) $f^{-1}(x)$ and (b) the domain of f^{-1}.

13. **7 pts**

On a flight to visit family, a plane's distance from its destination is $d = 1500 - 450t$ miles, t hours after it reaches cruising speed. (a) Solve for t to write time as a function of distance. (b) How long after reaching cruising speed are 600 miles left?

14. **6 pts**

Let $f(x) = x^3 - x$. (a) Find the average rate of change of f from $x = -1$ to $x = 2$. (b) Is f even, odd or neither? Show the test.

15. **6 pts**

Find and simplify the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for $f(x) = 3x^2 - x$.

16. **5 pts**

Prove that if f and g are both odd functions, then their product $p(x) = f(x)\,g(x)$ is an even function.

ANSWER KEY

1. (a) $16 - 12 - 2 = 2$; (b) $a^2 - 2a + 1 + 3a - 3 - 2 = a^2 + a - 4$
2. $2x + 8 \geq 0$ gives $x \geq -4$, and $x \neq 3$: $[-4, 3) \cup (3, \infty)$
3. (a) $[-4, 5]$; (b) $[-2, 4]$; (c) $(-3, 5]$
4. (a) $(4 + 3,\ -2(-1) + 1) = (7, 3)$; (b) $y = -\sqrt{x - 2} + 6$
5. Vertex $(3, 2)$: $y = a|x - 3| + 2$. Through $(7, 0)$: $0 = 4a + 2$, so $a = -\frac{1}{2}$. $y = -\frac{1}{2}|x - 3| + 2$
6. (a) $C(w) = \begin{cases} 1.5w, & 0 \leq w \leq 10 \\ 15 + 1.2(w - 10) = 1.2w + 3, & w > 10 \end{cases}$ (b) $1.2(16) + 3 = \$22.20$
7. (a) $\lfloor 3.5 \rfloor = 3$ and $\lfloor -1.5 \rfloor = -2$; (b) $[-2, -1)$
8. (a) $(x + 3)^2 - 2(x + 3) = x^2 + 4x + 3$; (b) $x^2 - 2x + 3$
9. $f(g(x)) = \dfrac{1}{\sqrt{x} - 4}$. Need $x \geq 0$ for g and $\sqrt{x} \neq 4$, so $x \neq 16$: $[0, 16) \cup (16, \infty)$
10. (a) $1.4x + 20$; (b) $1.4(150) + 20 = \$230$; (c) $1.4(x + 20) = 1.4x + 28$: the delivery charge would be marked up too, from \$20 to \$28
11. $x = \dfrac{4y + 1}{3}$, $3x - 1 = 4y$, so $f^{-1}(x) = \dfrac{3x - 1}{4}$. Check: $f\left(\dfrac{3x - 1}{4}\right) = \dfrac{(3x - 1) + 1}{3} = x$ ✓
12. (a) $x = (y - 2)^2 + 1$, $y - 2 = \sqrt{x - 1}$ (positive, since $y \geq 2$), so $f^{-1}(x) = 2 + \sqrt{x - 1}$; (b) the range of f: $[1, \infty)$
13. (a) $t = \dfrac{1500 - d}{450}$; (b) $t = \dfrac{900}{450} = 2$ hours
14. (a) $f(-1) = 0$ and $f(2) = 6$, so $\dfrac{6 - 0}{2 - (-1)} = 2$; (b) $f(-x) = -x^3 + x = -f(x)$: odd
15. $f(x + h) - f(x) = 6xh + 3h^2 - h$, so the quotient is $6x + 3h - 1$
16. $p(-x) = f(-x)\,g(-x) = (-f(x))(-g(x)) = f(x)\,g(x) = p(x)$, so p is even.

1.1 Functions and Function Notation: A Review

OBJECTIVES

- Decide whether a relation given as a list, table, equation or graph is a function.
- Evaluate a function at numbers and at expressions such as $f(a+2)$ and $f(-x)$.
- State the domain and range of a function from a graph, an equation or a situation, in interval notation.
- Read a graph to find where $f(x)$ equals a given value and where $f(x) > 0$.

WATCH

Function Notation - What is it? And How to Evaluate Functions

Mario's Math Tutoring · 11:38

https://www.youtube.com/watch?v=yAYAStx4CpQ

> **WHEN TO WATCH**
> Before Example 1: what f(x) notation means and how to evaluate a function.

BACKUP VIDEO

Finding the Domain and Range From a Graph (Interval Notation)

Mario's Math Tutoring · 5:52

https://www.youtube.com/watch?v=IZWVlEB0IbU

Before Example 2: reading domain and range from a graph in interval notation.

KEY TERMS

relation
Any set of ordered pairs (input, output).

function
A relation in which every input has exactly one output.

domain
The set of all inputs a function accepts.

range
The set of all outputs a function actually produces.

function notation

Writing $f(x)$ for the output of the function f when the input is x. It is read "f of x" and does not mean f times x.

interval notation

A way to write a set of numbers by its endpoints: a bracket [ ] includes an endpoint, a parenthesis ( ) leaves it out, and ∞ always gets a parenthesis.

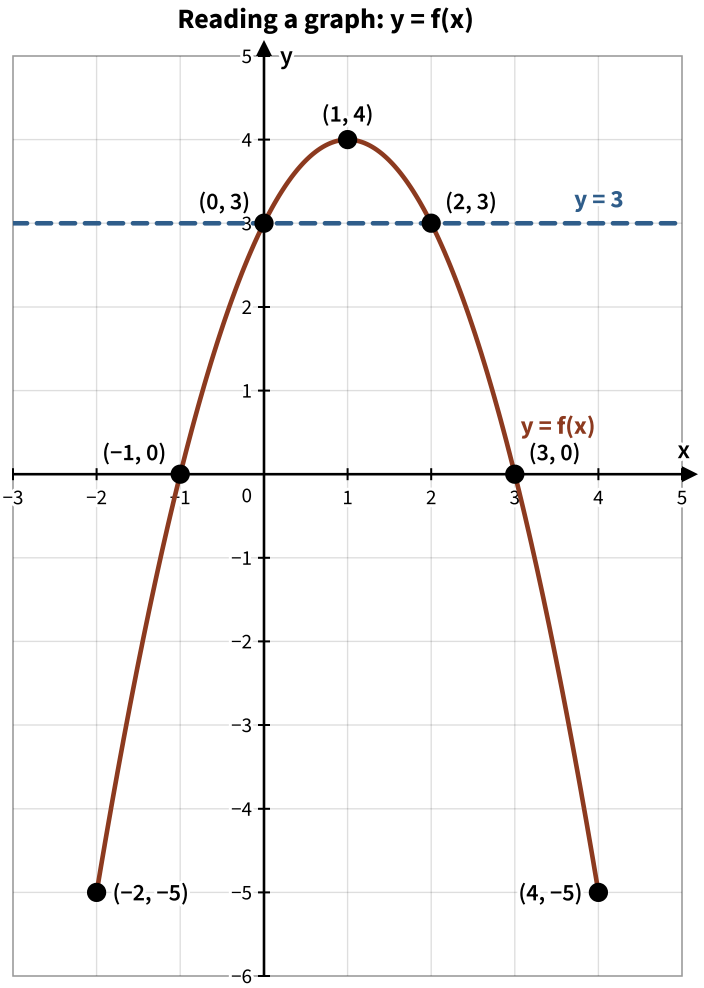
CORE IDEAS

Algebra II is a course about functions. A **relation** is any pairing of inputs with outputs, and a **function** is a relation that gives exactly one output for each input. The pairs $(1, 5), (2, 7), (3, 5)$ form a function (two inputs may share an output), but $(4, 1), (4, 6)$ do not, because the input 4 would have two outputs. A function is a dependable rule: tell it the input and it tells you one answer.

In **function notation**, f is the name of the rule, x is the input and $f(x)$ is the output. To evaluate, replace every x with the input, in parentheses. If $f(x) = x^2 - 3x$, then $f(5) = 25 - 15 = 10$. The input can be an expression: $f(a + 2) = (a + 2)^2 - 3(a + 2) = a^2 + 4a + 4 - 3a - 6 = a^2 + a - 2$, and $f(-x) = (-x)^2 - 3(-x) = x^2 + 3x$. Notice that $f(a + 2)$ is not $f(a) + 2$: the whole input goes through the rule.

The **vertical line test** says a graph is the graph of a function exactly when no vertical line crosses it more than once. Here is why: every point on the vertical line $x = a$ has input a, so if the line meets the graph twice, the input a has two different outputs.

The **domain** of a function given by an equation is every real number unless something forbids it. For now, two things forbid an input (later units will add a few more): dividing by zero and taking the square root of a negative number. For $g(x) = \sqrt{x - 4}$ we need $x - 4 \geq 0$, so the domain is $[4, \infty)$. For $h(x) = \dfrac{1}{x + 2}$ we need $x \neq -2$, so the domain is $(-\infty, -2) \cup (-2, \infty)$. In a situation, the domain is the set of inputs that make sense: a length cannot be negative, and a bakery cannot sell 3.7 loaves.



On a graph, the domain is the shadow of the graph on the x-axis and the range is its shadow on the y-axis. The graph above has domain $[-2, 4]$ and range $[-5, 4]$. To solve $f(x) = 3$ from a graph, go to height 3 and read the inputs where the graph is at that height: here $x = 0$ and $x = 2$. To solve $f(x) > 0$, find where the graph is above the x-axis: here on $(-1, 3)$. The endpoints -1 and 3 are left out because there $f(x) = 0$, which is not greater than 0.

KEY FORMULAS

$$f(x) = x^2 - 3x \quad \Longrightarrow \quad f(a+2) = (a+2)^2 - 3(a+2)$$

Domain: no division by 0, no square root of a negative number

From history. The Swiss mathematician Leonhard Euler introduced the notation $f(x)$ in 1734. He wrote more mathematics than almost anyone in history, and much of the notation you use in this course comes from him.

Practice habit. When you substitute into a function, first copy the rule with empty parentheses wherever x appears, then put the whole input inside each pair. This one habit prevents most sign errors with $f(-x)$ and $f(a+2)$.

WORKED EXAMPLES

Example 1. Let $f(x) = 2x^2 - x + 5$. Find (a) $f(-3)$, (b) $f(a+1)$ in simplest form, and (c) $f(-x)$.

Show solution

1. (a) Replace x with (-3): $f(-3) = 2(-3)^2 - (-3) + 5$.
2. $= 2(9) + 3 + 5 = 26$.
3. (b) Replace x with $(a+1)$: $f(a+1) = 2(a+1)^2 - (a+1) + 5$.
4. Expand the square: $2(a^2 + 2a + 1) - a - 1 + 5$.
5. Distribute: $2a^2 + 4a + 2 - a - 1 + 5$.
6. Combine like terms: $2a^2 + 3a + 6$.
7. Check with $a = 0$: the formula gives 6, and $f(1) = 2 - 1 + 5 = 6$ ✓.
8. (c) Replace x with $(-x)$: $f(-x) = 2(-x)^2 - (-x) + 5 = 2x^2 + x + 5$.

Answer: (a) 26; (b) $2a^2 + 3a + 6$; (c) $2x^2 + x + 5$

Example 2. Ruth has 60 feet of fencing to build a rectangular pen for goats against the side of a long barn, so fencing is needed on only three sides. Let x be the width in feet (each of the two sides that meet the barn). The area is $A(x) = x(60 - 2x)$. (a) What is the domain of A in this situation? (b) Find $A(10)$ and explain what it means. (c) What is the range of A?

Show solution

1. (a) Both x and the side along the barn, $60 - 2x$, must be positive lengths.
2. $x > 0$ and $60 - 2x > 0$, so $x < 30$. Domain: $(0, 30)$.
3. (b) $A(10) = 10(60 - 20) = 10(40) = 400$. A pen 10 feet wide has an area of 400 square feet.
4. (c) Write $A(x) = -2x^2 + 60x$. This parabola opens down, so its highest point is the vertex.
5. The vertex is at $x = -\dfrac{b}{2a} = -\dfrac{60}{2(-2)} = 15$, and $A(15) = 15(30) = 450$.
6. As x moves toward 0 or 30, the area shrinks toward 0 but never reaches it, since those widths are not in the domain.
7. Range: $(0, 450]$ square feet.

Answer: (a) $(0, 30)$; (b) $A(10) = 400$: a 10-foot width gives 400 square feet; (c) $(0, 450]$

PRACTICE

1. Easy Is the relation $\{(1, 4), (2, 5), (3, 4), (1, 6)\}$ a function? Explain in one sentence.

   Show answer

   No. The input 1 has two outputs, 4 and 6.

2. Easy Let $g(x) = 5 - 3x$. Find (a) $g(4)$ and (b) $g(-2)$.

   Show answer

   (a) $g(4) = 5 - 12 = -7$; (b) $g(-2) = 5 + 6 = 11$

3. Easy Find the domain of $f(x) = \sqrt{x + 6}$ in interval notation.

   Show answer

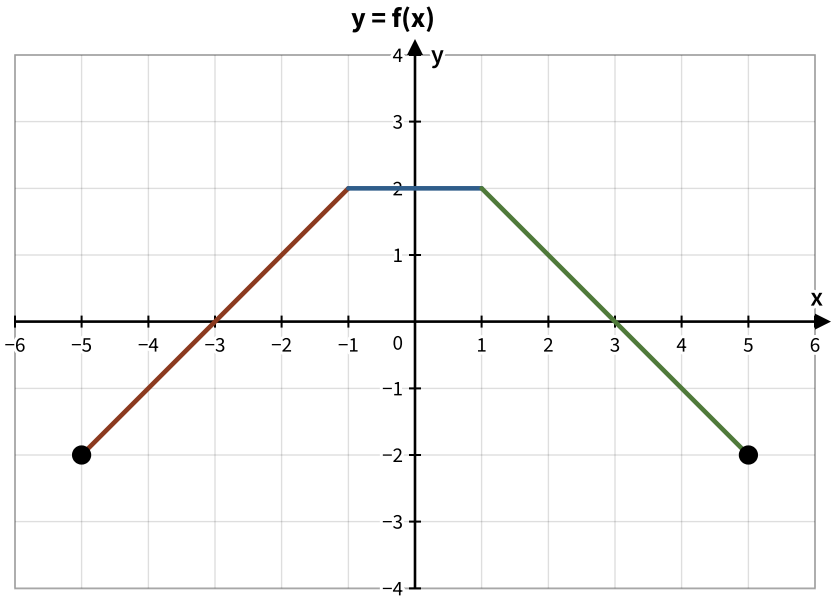
   $x + 6 \geq 0$, so $x \geq -6$: $[-6, \infty)$

4. **Easy** A freight train travels at a steady 60 miles per hour for a 5-hour trip. Its distance from the station after t hours is $d(t) = 60t$, for $0 \le t \le 5$. Give (a) the domain and (b) the range of d in interval notation.

Show answer

(a) $[0, 5]$ hours; (b) $[0, 300]$ miles, since $d(5) = 300$

5. **Medium** The graph shows $y = f(x)$, including both endpoints. Give (a) the domain and (b) the range of f in interval notation.



Show answer

(a) $[-5, 5]$; (b) $[-2, 2]$

6. **Medium** Let $f(x) = x^2 - 4x$. Find $f(a + 2)$ in simplest form.

Show answer

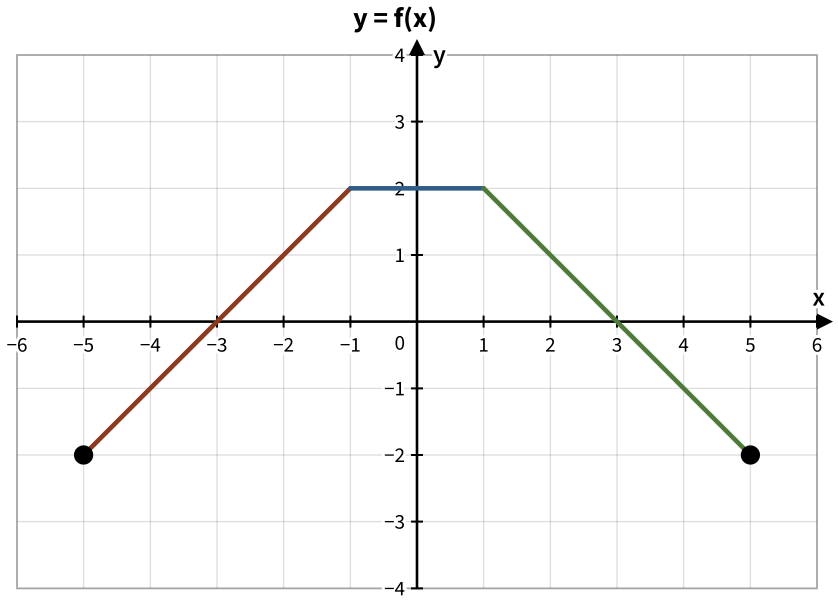
$(a+2)^2 - 4(a+2) = a^2 + 4a + 4 - 4a - 8 = a^2 - 4$

7. **Medium** Let $h(x) = x^3 - 2x$. Find $h(-x)$ in simplest form. How does it compare with $h(x)$?

Show answer

$h(-x) = (-x)^3 - 2(-x) = -x^3 + 2x$, which is exactly $-h(x)$.

8. **Medium** Use the graph of $y = f(x)$ in Problem 5. (a) For which x is $f(x) = 1$? (b) For which x is $f(x) > 0$? Write (b) in interval notation.



Show answer

(a) $x = -2$ and $x = 2$; (b) the graph is above the x-axis between its zeros -3 and 3: $(-3, 3)$

9. **Medium** Grace's bakery has daily costs of \$40 plus \$2.50 for each loaf baked, so $C(n) = 40 + 2.5n$ dollars for n loaves. (a) Find $C(24)$. (b) How many loaves give a daily cost of \$115?

Show answer

(a) $C(24) = 40 + 60 = \$100$; (b) $40 + 2.5n = 115$, so $2.5n = 75$ and $n = 30$ loaves

10. **Hard** Find the domain of $f(x) = \dfrac{\sqrt{10 - 2x}}{x + 1}$ in interval notation.

Show answer

Need $10 - 2x \geq 0$, so $x \leq 5$, and $x + 1 \neq 0$, so $x \neq -1$: $(-\infty, -1) \cup (-1, 5]$

11. Hard Henry throws a ball upward from the edge of a 64-foot cliff. Its height above the ground below after t seconds is $h(t) = -16t^2 + 48t + 64$ feet, until it lands. (a) When does the ball land? (b) Give the domain of h in this situation. (c) Give the range of h. (Hint: find the vertex.)

Show answer

(a) $-16(t^2 - 3t - 4) = -16(t - 4)(t + 1) = 0$, so $t = 4$ s; (b) $[0, 4]$; (c) the vertex is at $t = 1.5$ with $h(1.5) = -36 + 72 + 64 = 100$, so the range is $[0, 100]$ feet

12. Hard Let $f(x) = x^2 - 3x$. Find and simplify $f(x + 1) - f(x)$.

Show answer

$$(x + 1)^2 - 3(x + 1) - (x^2 - 3x) = x^2 + 2x + 1 - 3x - 3 - x^2 + 3x = 2x - 2$$

13. Challenge Explain, using both the vertical line test and algebra, why the circle $x^2 + y^2 = 25$ is not the graph of a function of x.

Show answer

The vertical line $x = 3$ crosses the circle twice. Algebraically, $x = 3$ gives $9 + y^2 = 25$, so $y^2 = 16$ and $y = 4$ or $y = -4$. The input 3 has two outputs, $(3, 4)$ and $(3, -4)$, so the relation is not a function.

COMMON MISTAKES

- Treating $f(a + 2)$ as $f(a) + 2$. The whole expression $a + 2$ is the input: replace every x with $(a + 2)$ and then simplify.
- Dropping parentheses in $f(-x)$, so that $-x^2$ appears where $(-x)^2 = x^2$ belongs. Always write the input in parentheses first.
- Reading the range from the x-axis. The domain is about inputs (left to right); the range is about outputs (bottom to top).
- Writing a bracket next to ∞, as in $[4, \infty]$. Infinity is not a number you reach, so it always gets a parenthesis: $[4, \infty)$.

> **Make it a game · Verdict and Evaluate.** The parent presents five relations: a list of pairs, a table, an equation like $x^2 + y^2 = 1$, and two quick graph sketches. In 60 seconds, rule each one a function or not, using a pencil as a vertical line sweeping across the graphs. Then evaluate $f(x) = x^2 - 3x$ at $a + 2$ and at $-x$. Correct verdicts earn one point each.

1.2 Parent Functions and Transformations

OBJECTIVES

- Graph the parent functions $y = x$, $y = x^2$, $y = |x|$, $y = \sqrt{x}$, $y = x^3$ and $y = \dfrac{1}{x}$ from their key points.
- Explain why $y = f(x - h)$ moves a graph h units to the right.
- Describe and graph $y = a\,f(x - h) + k$ and $y = f(bx)$ by tracking key points.
- Write the rule for a graph described in words or shown on a grid.

WATCH

Modifying Graphs by Shift, Reflection & Stretch

Eddie Woo · 10:23

https://www.youtube.com/watch?v=EdhsfRedaMw

> **WHEN TO WATCH**
> Before Example 1: shifts, reflections and stretches of a graph.

Note: Eddie Woo teaches in Australia: he says 'dilation' for a stretch.

BACKUP VIDEO

Parent Function Graphs Transformations (Shift, Stretch, Reflect)

Mario's Math Tutoring · 8:57

https://www.youtube.com/watch?v=vP7z9PfGBQw

Before the practice set: the same transformations applied to each parent function.

KEY TERMS

parent function

The simplest function of a family, such as $y = x^2$ for all parabolas. Every other member is a transformation of it.

transformation

A change that moves or reshapes a graph: a translation (slide), a reflection (flip), or a stretch or compression.

translation

A slide of the whole graph up, down, left or right without changing its shape.

reflection

A flip of the graph across a line. $y = -f(x)$ flips across the x-axis and $y = f(-x)$ flips across the y-axis.

vertical stretch or compression

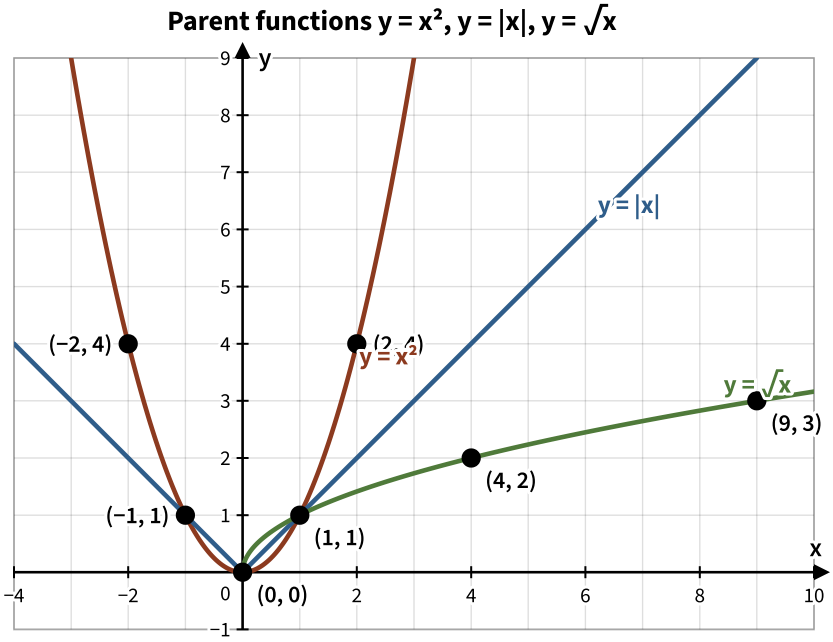
Multiplying every output by a positive number a: a stretch away from the x-axis if $a > 1$, a compression toward it if $0 < a < 1$.

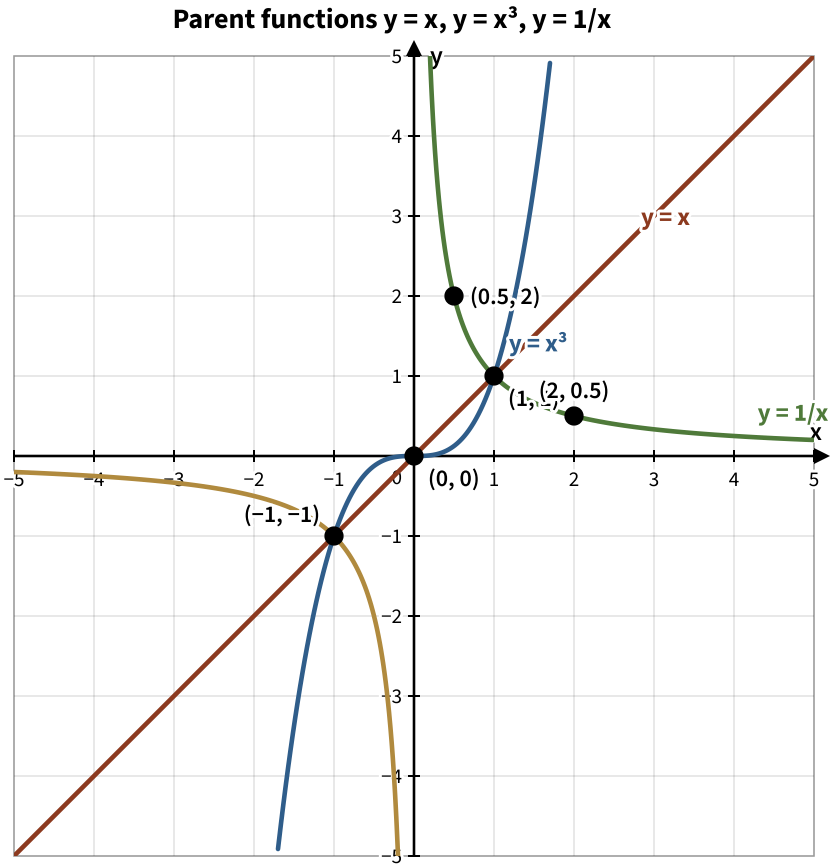
horizontal stretch or compression

Replacing x with bx ($b > 0$): a compression toward the y-axis by the factor $\frac{1}{b}$ if $b > 1$, a stretch if $0 < b < 1$.

CORE IDEAS

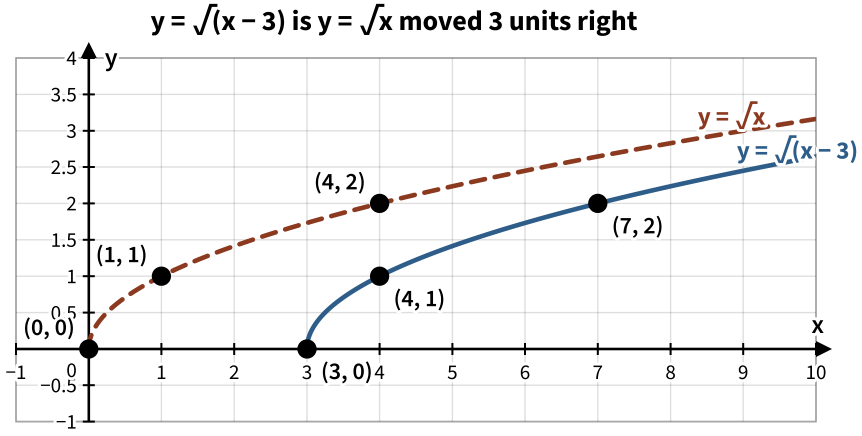
A few **parent functions** come up again and again, and you should know each by its key points. $y = x$: $(-1, -1), (0, 0), (1, 1)$. $y = x^2$: $(0, 0), (\pm 1, 1), (\pm 2, 4)$. $y = |x|$: a V with vertex $(0, 0)$ through $(\pm 1, 1)$. $y = \sqrt{x}$: $(0, 0), (1, 1), (4, 2), (9, 3)$, with domain and range $[0, \infty)$. $y = x^3$: $(-1, -1), (0, 0), (1, 1), (2, 8)$. $y = \frac{1}{x}$: $(1, 1), (-1, -1), (2, \frac{1}{2}), (\frac{1}{2}, 2)$, with $x \neq 0$; its graph gets closer and closer to the lines $x = 0$ and $y = 0$, which are called its **asymptotes**.





Changes **outside** the function act on the outputs, so they move the graph vertically, exactly as you would expect. In $y = f(x) + k$ every output goes up by k, so the graph moves up k (down if $k < 0$). In $y = a\,f(x)$ every output is multiplied by a: the point (c, d) becomes (c, ad). If $|a| > 1$ the graph is stretched away from the x-axis, if $0 < |a| < 1$ it is compressed toward it, and if $a < 0$ it is also reflected in the x-axis.

Changes **inside** the function act on the inputs, and they seem backward at first. Let $g(x) = f(x - h)$. Then $g(c + h) = f(c + h - h) = f(c)$. So g produces the output $f(c)$ only when its input is h **larger** than c: every point (c, d) on the graph of f becomes $(c + h, d)$, and the graph moves h units to the right. For example, $\sqrt{x}$ starts at $x = 0$, but $\sqrt{x - 3}$ cannot start until $x - 3 = 0$, that is, at $x = 3$. So $y = (x + 4)^2 = (x - (-4))^2$ moves left 4.



The same reasoning explains $y = f(bx)$ for any $b \neq 0$. If $g(x) = f(bx)$, then $g\left(\frac{c}{b}\right) = f(c)$, so (c, d) becomes $\left(\frac{c}{b}, d\right)$: every input is divided by b. With $b = 2$ the graph is squeezed halfway toward the y-axis. With $b = -1$, (c, d) becomes $(-c, d)$, so $y = f(-x)$ is the reflection in the y-axis.

To graph $y = a\,f(x - h) + k$, send each key point (c, d) of the parent to $(c + h,\ ad + k)$. The order matters for the outputs: multiply by a first (stretch or reflect), then add k (shift), just as the order of operations says. For instance, $y = 2\sqrt{x + 1} - 3$ sends $(4, 2)$ on $y = \sqrt{x}$ to $(4 - 1,\ 2 \cdot 2 - 3) = (3, 1)$, and its starting point is $(-1, -3)$. To write a rule from a description, start with the parent and build the formula one step at a time in the order the steps are done.

KEY FORMULAS

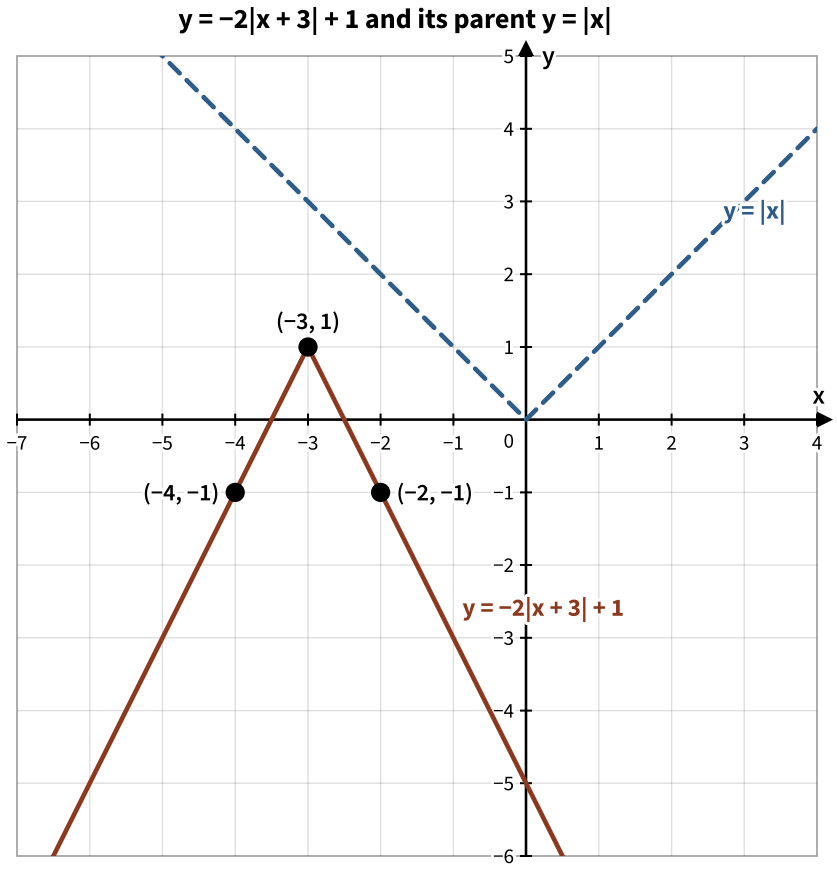
$$y = a\,f(x - h) + k: \quad (c, d) \to (c + h,\ ad + k)$$

$$y = f(bx): \quad (c, d) \to \left(\frac{c}{b},\ d\right)$$

$y = -f(x)$: reflect in the x-axis $\qquad$ $y = f(-x)$: reflect in the y-axis

Practice habit. Know the six parent functions and their key points cold, so you can sketch any of them in a few seconds: x^2 through $(\pm 1, 1), (\pm 2, 4)$; $\sqrt{x}$ through $(1, 1), (4, 2), (9, 3)$; x^3 through $(\pm 1, \pm 1), (2, 8)$; $\frac{1}{x}$ through $(1, 1), (-1, -1)$.

Example 1. Describe how the graph of $y = -2|x + 3| + 1$ is obtained from $y = |x|$. Find the images of the key points $(-1, 1)$, $(0, 0)$ and $(1, 1)$, and sketch the graph.



Show solution

1. Match the form $y = a\,f(x - h) + k$: $x + 3 = x - (-3)$, so $h = -3$, $a = -2$, $k = 1$.
2. $h = -3$: shift left 3. $a = -2$: stretch vertically by a factor of 2 and reflect in the x-axis. $k = 1$: shift up 1.
3. Rule for points: $(c, d) \to (c - 3,\ -2d + 1)$.
4. $(0, 0) \to (-3,\ 0 + 1) = (-3, 1)$, the new vertex.
5. $(-1, 1) \to (-4,\ -2 + 1) = (-4, -1)$.
6. $(1, 1) \to (-2,\ -2 + 1) = (-2, -1)$.

7. Check one point in the equation: at $x = -4$, $-2|-4+3|+1 = -2(1)+1 = -1$ ✓.
8. Plot the three points and draw a V that opens downward from $(-3, 1)$.

Answer: Left 3, vertical stretch by 2, reflection in the x-axis, up 1. Key points $(-4, -1)$, $(-3, 1)$, $(-2, -1)$; vertex $(-3, 1)$.

Example 2. Samuel drops a stone from the edge of a 144-foot cliff above a lake at time $t = 0$. Its height is $h_1(t) = 144 - 16t^2$ feet after t seconds. Ruth drops an identical stone from the same spot exactly 2 seconds later. (a) Write Ruth's stone's height $h_2(t)$, using the same clock t. (b) When does each stone hit the water?

Show solution

1. (a) Ruth's stone does at time t what Samuel's did 2 seconds earlier, at time $t - 2$.
2. So $h_2(t) = h_1(t-2) = 144 - 16(t-2)^2$ for $t \geq 2$: the graph of h_1 shifted 2 units right.
3. (b) Samuel: $144 - 16t^2 = 0$, so $t^2 = 9$ and $t = 3$ (time cannot be negative).
4. Ruth: $144 - 16(t-2)^2 = 0$, so $(t-2)^2 = 9$, $t - 2 = 3$ and $t = 5$.
5. Check: $h_2(5) = 144 - 16(3)^2 = 144 - 144 = 0$ ✓. Ruth's stone lands 2 seconds after Samuel's, as it should.
6. This is why $f(x - h)$ moves a graph right: the second stone needs a clock reading 2 seconds larger to reach each height.

Answer: (a) $h_2(t) = 144 - 16(t-2)^2$; (b) Samuel's stone at $t = 3$ s, Ruth's at $t = 5$ s

PRACTICE

1. Easy Give (a) the domain and (b) the range of $y = \sqrt{x}$ in interval notation.

Show answer

(a) $[0, \infty)$; (b) $[0, \infty)$

2. **Easy** Describe how $y = (x-4)^2 + 2$ is obtained from $y = x^2$, and give its vertex.

Show answer

Shift right 4 and up 2; vertex $(4, 2)$

3. **Easy** Write the rule for the graph of $y = |x|$ shifted 5 units left and 3 units down.

Show answer

$y = |x + 5| - 3$

4. **Easy** A small bakery's daily cost is $C(n) = 2n + 30$ dollars for n dozen rolls. The rent goes up by \$15 a day. Write the new cost function and describe the transformation of the graph.

Show answer

$C(n) + 15 = 2n + 45$; the graph shifts up 15

5. **Medium** The point $(3, -2)$ is on the graph of $y = f(x)$. Find the matching point on the graph of (a) $y = f(x + 1) - 4$ and (b) $y = 3f(x)$.

Show answer

(a) left 1, down 4: $(2, -6)$; (b) outputs times 3: $(3, -6)$

6. **Medium** Let $g(x) = -\frac{1}{2}\sqrt{x - 1} + 3$. Describe the transformations of $y = \sqrt{x}$, then give (a) the domain and (b) the range of g in interval notation.

Show answer

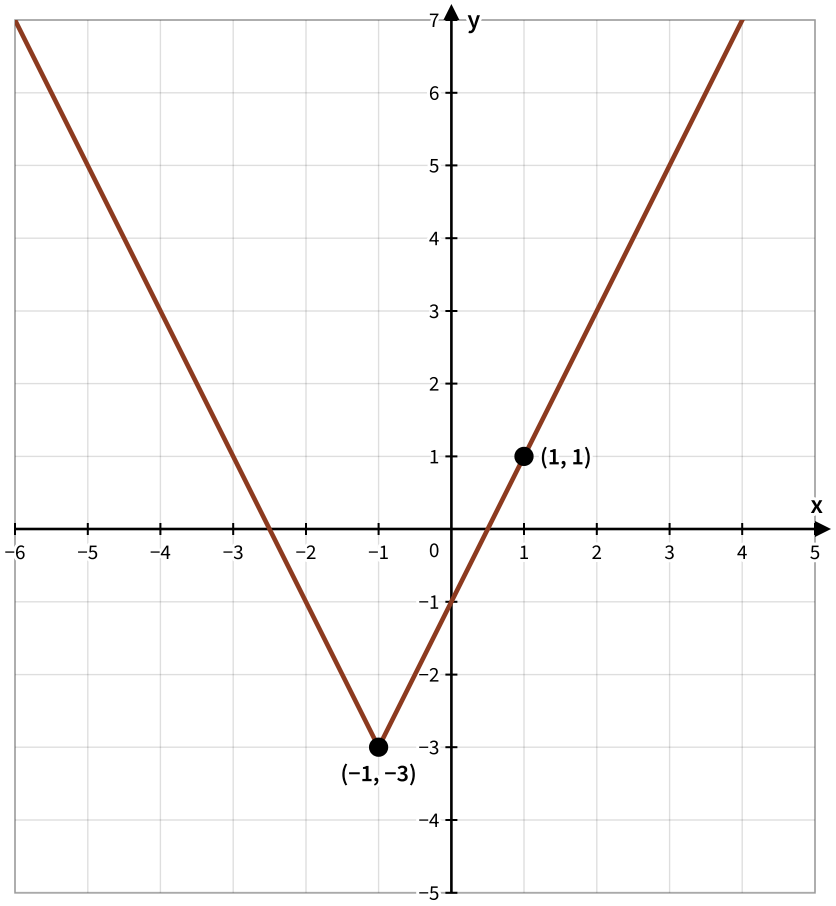
Right 1, vertical compression by $\frac{1}{2}$, reflection in the x-axis, up 3; the graph starts at $(1, 3)$ and goes down. (a) $[1, \infty)$; (b) $(-\infty, 3]$

7. **Medium** Describe how $y = \dfrac{1}{x - 2} + 1$ is obtained from $y = \dfrac{1}{x}$ and give the equations of its two asymptotes.

Show answer

Shift right 2 and up 1; asymptotes $x = 2$ and $y = 1$

8. **Medium** The graph is a transformation of $y = |x|$. Write its equation.



Show answer

Vertex $(-1, -3)$ gives $y = a|x + 1| - 3$; the point $(1, 1)$ gives $1 = 2a - 3$, so $a = 2$: $y = 2|x + 1| - 3$

9. **Medium** A freight train leaves the station at 9:00 AM and travels at 60 miles per hour, so its distance is $d(t) = 60t$ miles, where t is hours after 9:00 AM. A second train leaves at 10:30 AM at the same speed on a parallel track. (a) Write the second train's distance as a function of the same t. (b) How far has the second train gone at noon?

Show answer

(a) It leaves 1.5 hours later, so $d_2(t) = 60(t - 1.5)$ for $t \geq 1.5$; (b) noon is $t = 3$: $60(1.5) = 90$ miles

10. Hard The stream of a garden fountain follows $h(x) = -0.5x^2 + 8$, where x is the horizontal distance in feet from the stream's highest point and h is the height in feet. Henry rebuilds it so the stream has the same shape, but its highest point is 3 feet farther right and 1 foot higher. (a) Write the new rule. (b) How high is the new stream at $x = 5$?

Show answer

(a) $h(x) = -0.5(x - 3)^2 + 9$; (b) $-0.5(2)^2 + 9 = -2 + 9 = 7$ feet

11. Hard Start with $f(x) = \sqrt{x}$. (a) Shift the graph 3 units right, then reflect it in the y-axis. Write the rule. (b) Reflect the graph in the y-axis first, then shift it 3 units right. Write the rule. Are the results the same?

Show answer

(a) $\sqrt{x - 3}$, then replace x with $-x$: $y = \sqrt{-x - 3}$ (domain $x \le -3$); (b) $\sqrt{-x}$, then replace x with $x - 3$: $y = \sqrt{-(x - 3)} = \sqrt{3 - x}$ (domain $x \le 3$). They are different, so order matters.

12. Hard Explain why the graph of $y = (x + 4)^2$ is the graph of $y = x^2$ shifted 4 units **left**, even though the rule has a plus sign. Use the vertex in your explanation.

Show answer

The parabola $y = x^2$ has its lowest output, 0, at $x = 0$. For $y = (x + 4)^2$ the output is 0 when $x + 4 = 0$, that is, at $x = -4$. In general the new function gives the output c^2 at the input $c - 4$, which is 4 less than c, so every point moves 4 units left.

13. Challenge The graph of $y = \sqrt{9x}$ is a horizontal compression of $y = \sqrt{x}$ by the factor $\frac{1}{9}$. Show that it is also a vertical stretch $y = a\sqrt{x}$ for $x \ge 0$, and find a.

Show answer

$\sqrt{9x} = \sqrt{9}\,\sqrt{x} = 3\sqrt{x}$, so $a = 3$.

COMMON MISTAKES

- Moving $y = f(x - h)$ the wrong way. Ask where the new graph has the parent's key point: $\sqrt{x - 3}$ starts where $x - 3 = 0$, at $x = 3$, so it moves right.
- Adding k before multiplying by a. For $y = a\,f(x - h) + k$ the new output is $ad + k$, not $a(d + k)$: stretch or reflect first, then shift.

- Confusing $-f(x)$ with $f(-x)$. A minus sign outside flips outputs (across the x-axis); a minus sign inside flips inputs (across the y-axis).
- Forgetting how a shift changes the domain. $y = \sqrt{x+2}$ has domain $[-2, \infty)$, not $[0, \infty)$.

> **Make it a game · Shift Charades.** Act out the six parent functions with your body: a diagonal arm for y = x, arms in a U for x^2, a V for |x|, one curving arm for √x, an S-shape for x^3, and two corner curves for 1/x. The parent then calls shifts such as 'f(x - 3)'. Step three paces right and explain why the graph moves right.

1.3 Piecewise and Absolute Value Functions

OBJECTIVES

- Evaluate and graph piecewise functions, using open and closed endpoints correctly.
- Write $|x|$ and $y = a|x - h| + k$ as piecewise functions and graph them as transformations of $y = |x|$.
- Evaluate and graph step functions such as the greatest integer function and fee schedules.
- Write a piecewise rule from a graph or a price list.

WATCH

Piecemeal Functions (1 of 2: Constructing the graph)

Eddie Woo · 9:39

https://www.youtube.com/watch?v=FQqv-1zBiCA

> **WHEN TO WATCH**
> Before Example 1: building the graph of a piecewise function.

Note: Eddie Woo teaches in Australia: he says 'piecemeal function' for piecewise function. Absolute value graphs are taught from the Core ideas.

BACKUP VIDEO

Greatest Integer Function (Step Function) Evaluating and Graphing

Mario's Math Tutoring · 7:20

https://www.youtube.com/watch?v=otrRndq2H9I

After Example 2: the greatest-integer (step) function.

KEY TERMS

piecewise function
A function defined by different formulas on different parts of its domain.

closed and open endpoints
A solid dot shows that an endpoint belongs to a piece of the graph; a hollow dot shows that it does not.

absolute value

$|x|$ is the distance from x to 0 on the number line, so it is never negative.

step function

A piecewise function that is constant on each piece, so its graph looks like a staircase.

greatest integer function

$\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. For example, $\lfloor 2.7 \rfloor = 2$ and $\lfloor -2.3 \rfloor = -3$.

CORE IDEAS

A **piecewise function** uses different rules on different intervals. To evaluate it, first find which condition the input satisfies, then use only that rule. Each input must satisfy exactly one condition; otherwise one input could get two outputs and the relation would not be a function. For example, a shop might charge $8 a bale for small hay orders and less per bale for large ones, with the change at 20 bales.

To graph a piecewise function, graph each rule only over its own interval. At an endpoint, use a closed dot if the endpoint belongs to that piece ($\leq$ or $\geq$) and an open dot if it does not ($<$ or $>$). The pieces may meet, or the graph may jump; for each input, though, there is only one solid point above or below it.

Absolute value is a piecewise function in disguise. If $x \geq 0$, its distance from 0 is x itself. If $x < 0$, its distance from 0 is $-x$, which is positive because x is negative (for example, $|-5| = -(-5) = 5$). So

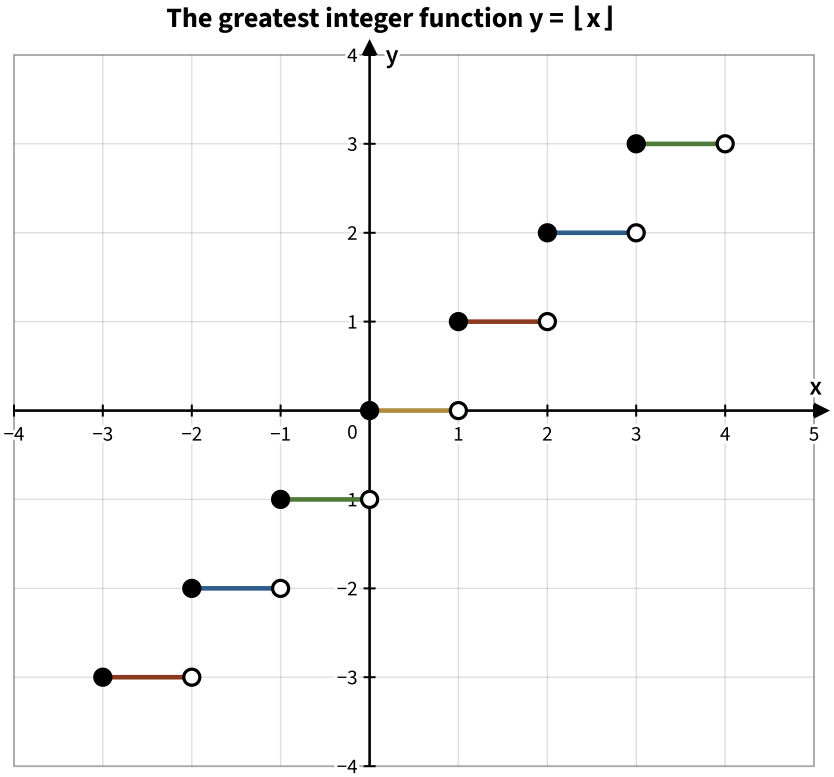
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

This is why the graph of $y = |x|$ is a V made of the line $y = x$ on the right and the line $y = -x$ on the left.

By Lesson 1.2, $y = a|x - h| + k$ is $y = |x|$ shifted so the vertex is at (h, k), stretched by $|a|$, and opening down if $a < 0$. Writing it piecewise shows the two slopes: for $x \geq h$, $|x - h| = x - h$, so $y = a(x - h) + k$, a line of slope a; for $x < h$, $|x - h| = -(x - h)$, so $y = -a(x - h) + k$, a line of slope $-a$.

A **step function** is constant on each piece. The most important one is the **greatest integer function** $\lfloor x \rfloor$, which rounds down to an integer: $\lfloor 3 \rfloor = 3$, $\lfloor 3.9 \rfloor = 3$, $\lfloor -0.5 \rfloor =$

-1. On $[n, n+1)$ its value is n, so each step has a closed dot on the left and an open dot on the right. Fees that charge "per hour or part of an hour" or by weight class are step functions too, but they round **up**, so their steps are open on the left and closed on the right.



To write a piecewise rule from a graph or a price list, treat each piece separately. For a line segment, find its slope from two points and then its equation; for a flat step, the rule is a constant. Then write the interval for each piece, taking care with the endpoints: a closed dot or the words "up to and including" mean $\le$, and an open dot or "more than" mean $<$ or $>$.

KEY FORMULAS

$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

$$y = a|x - h| + k: \ \text{vertex } (h, k), \ \text{slope } a \text{ to the right and } -a \text{ to the left}$$

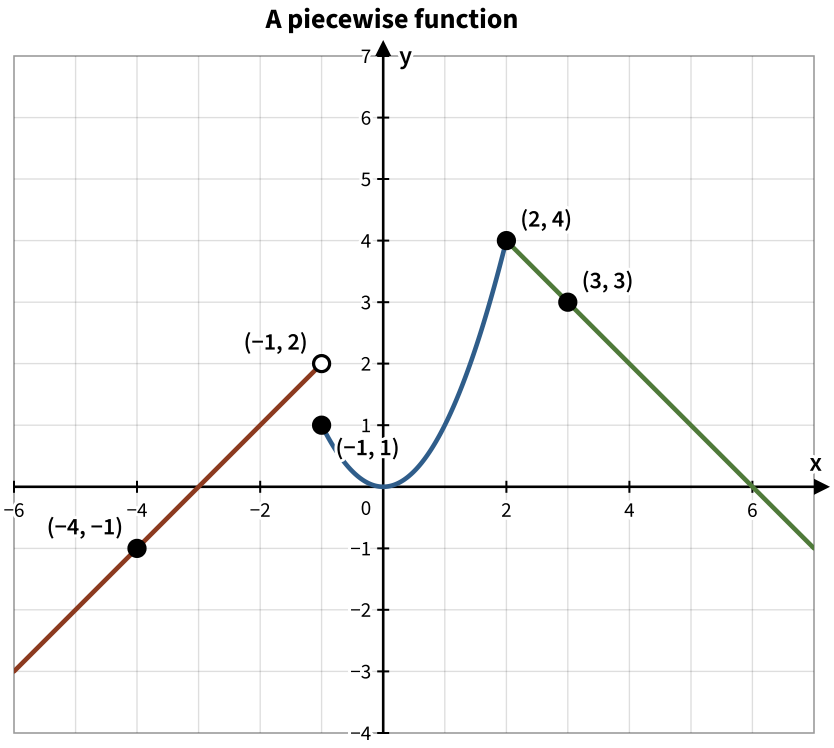
$$\lfloor x \rfloor = n \quad \text{exactly when} \quad n \le x < n + 1 \ (n \text{ an integer})$$

> **Practice habit.** Before evaluating or graphing a piecewise function, write the intervals on a number line and mark each endpoint as open or closed. Decide which piece an input belongs to before you compute anything.

Example 1. Let

$$f(x) = \begin{cases} x + 3, & x < -1 \\ x^2, & -1 \le x \le 2 \\ 6 - x, & x > 2. \end{cases}$$

Find $f(-4)$, $f(-1)$ and $f(3)$, and graph f.



Show solution

1. $-4 < -1$, so use the first rule: $f(-4) = -4 + 3 = -1$.
2. -1 satisfies $-1 \le x \le 2$, so use the second rule: $f(-1) = (-1)^2 = 1$.
3. $3 > 2$, so use the third rule: $f(3) = 6 - 3 = 3$.
4. Graph $y = x + 3$ for $x < -1$. At $x = -1$ it would reach 2, but -1 is not in this piece: open dot at $(-1, 2)$.
5. Graph $y = x^2$ from $x = -1$ to $x = 2$ with closed dots at $(-1, 1)$ and $(2, 4)$.
6. Graph $y = 6 - x$ for $x > 2$. It starts at $6 - 2 = 4$, the same point $(2, 4)$, so the graph does not jump there; it does jump at $x = -1$.

Answer: $f(-4) = -1$, $f(-1) = 1$, $f(3) = 3$

Example 2. A farm co-op sells hay for \$8 a bale on orders of up to 20 bales. On larger orders the first 20 bales cost \$8 each and every bale beyond 20 costs \$6. (a) Write the cost $C(n)$ of n bales as a piecewise function. (b) Find the cost of 30 bales.

Show solution

1. Up to 20 bales: $C(n) = 8n$ for $0 \le n \le 20$.
2. More than 20: the first 20 cost $8(20) = 160$ and the other $n - 20$ cost \$6 each.
3. $C(n) = 160 + 6(n - 20) = 6n + 40$ for $n > 20$.
4. Check where the pieces meet: $8(20) = 160$ and $6(20) + 40 = 160$, so there is no jump at 20 ✓.
5. (b) $30 > 20$, so $C(30) = 6(30) + 40 = 220$.
6. Check by thinking: 20 bales at \$8 is \$160, plus 10 bales at \$6 is \$60, for \$220 ✓.

Answer: (a) $C(n) = \begin{cases} 8n, & 0 \le n \le 20 \\ 6n + 40, & n > 20 \end{cases}$ (b) \$220

PRACTICE

1. Easy Let $f(x) = \begin{cases} 3x - 1, & x < 2 \\ x^2 + 1, & x \ge 2. \end{cases}$ Find $f(0)$, $f(2)$ and $f(5)$.

   Show answer

   $f(0) = -1$; $f(2) = 4 + 1 = 5$; $f(5) = 26$

2. Easy Evaluate (a) $\lfloor 4.8 \rfloor$, (b) $\lfloor -1.2 \rfloor$, (c) $\lfloor 6 \rfloor$.

   Show answer

   (a) 4; (b) -2 (the greatest integer not more than -1.2); (c) 6

3. Easy Write $y = |x - 3|$ as a piecewise function without absolute value bars.

   Show answer

   $y = \begin{cases} x - 3, & x \ge 3 \\ 3 - x, & x < 3 \end{cases}$

4. **Easy** Give the vertex of $y = -3|x + 2| + 5$ and say whether the graph opens up or down.

Show answer

Vertex $(-2, 5)$; opens down because $a = -3 < 0$

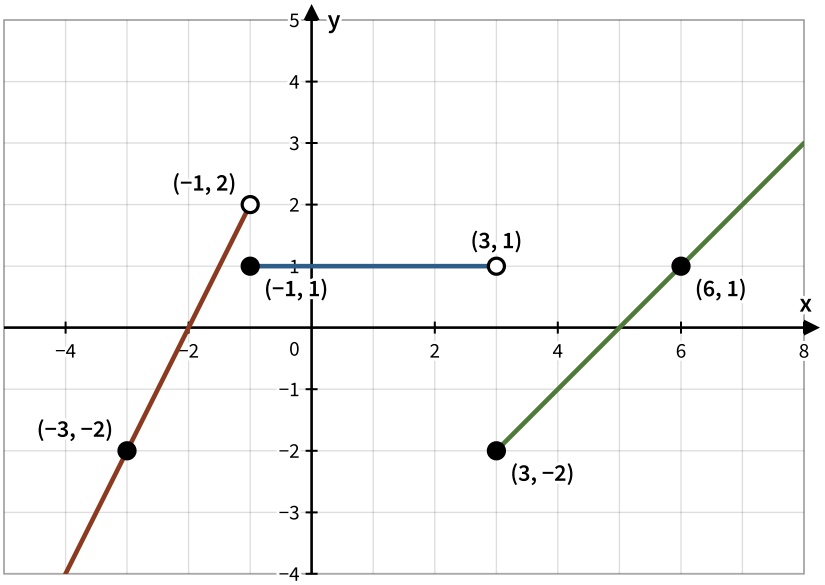
5. **Medium** A small business ships orders using the rates in the table. What does it cost to ship (a) a 3-pound package and (b) a 3.2-pound package?

Package weight w (pounds)	Shipping cost (dollars)
0 < w ≤ 1	5.00
1 < w ≤ 3	8.50
3 < w ≤ 10	14.00

Show answer

(a) $1 < 3 \le 3$: \$8.50; (b) $3 < 3.2 \le 10$: \$14.00

6. **Medium** Write a piecewise rule for the function graphed.



Show answer

Left piece through $(-3,-2)$ and $(-1,2)$ has slope 2: $y = 2x + 4$. $f(x) =$

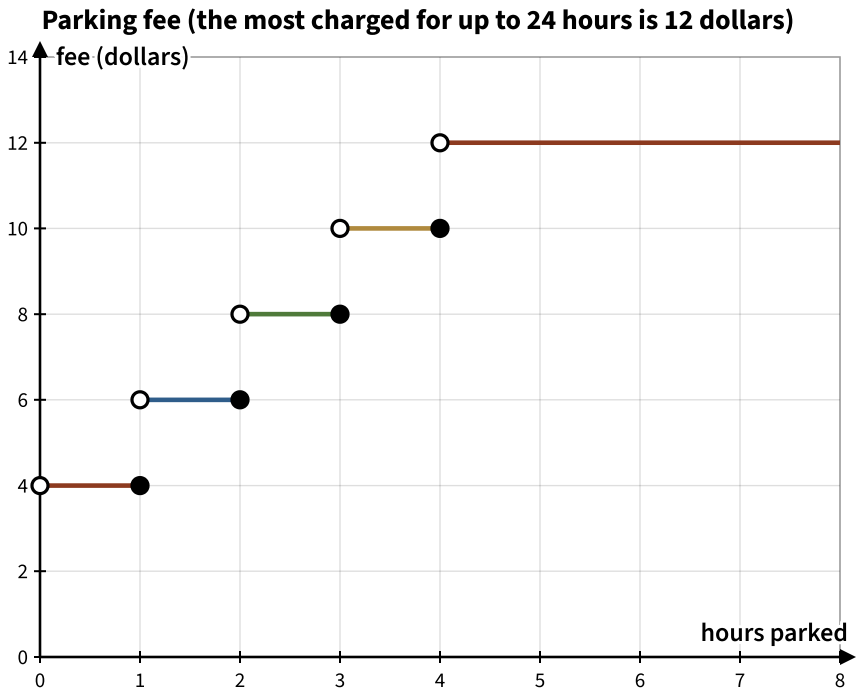
$$\begin{cases} 2x+4, & x < -1 \\ 1, & -1 \le x < 3 \\ x - 5, & x \ge 3 \end{cases}$$

7. **Medium** Let $g(x) = 2|x - 1| - 6$. Find (a) the x-intercepts and (b) the range of g in interval notation.

Show answer

(a) $2|x-1| = 6$, $|x-1| = 3$, so $x = -2$ or $x = 4$; (b) the vertex is $(1, -6)$ and the graph opens up: $[-6, \infty)$

8. Medium The graph shows the fee at a train-station parking garage. (a) What is the fee for 2.5 hours? (b) What is the fee for 6 hours? (c) What is the longest time you can park for \$8?



Show answer

(a) \$8; (b) \$12; (c) 3 hours (the step at \$8 is closed at its right end, $t = 3$)

9. Medium Write the equation of the form $y = a|x - h| + k$ whose graph has vertex $(3, -1)$ and passes through $(5, 3)$.

Show answer

$y = a|x - 3| - 1$; $3 = a(2) - 1$, so $a = 2$: $y = 2|x - 3| - 1$

10. Hard A church youth group rents a van for a service trip at \$45 for each day or part of a day. (a) What is the cost for 2.25 days? (b) Ruth suggests the formula $C(t) = 45(\lfloor t \rfloor + 1)$. Explain why it gives the wrong cost when $t = 2$.

Show answer

(a) 2.25 days counts as 3 days: $3 \times 45 = \$135$. (b) $C(2) = 45(2 + 1) = \$135$, but exactly 2 days should cost $2 \times 45 = \$90$. The formula rounds up correctly only when t is not a whole number.

11. Hard Solve $|2x - 5| = x + 1$ by writing $|2x - 5|$ piecewise and solving on each piece. Check your answers.

Show answer

If $x \geq 2.5$: $2x - 5 = x + 1$, $x = 6$ ✓ ($6 \geq 2.5$). If $x < 2.5$: $5 - 2x = x + 1$, $x = \frac{4}{3}$ ✓ ($\frac{4}{3} < 2.5$). Check: $|7| = 7$ and $\left|-\frac{7}{3}\right| = \frac{7}{3}$ ✓. Solutions $x = \frac{4}{3}$ and $x = 6$.

12. Hard Find the value of k so that the graph of $f(x) = \begin{cases} x^2 - 1, & x \leq 2 \\ kx + 1, & x > 2 \end{cases}$ has no jump at $x = 2$.

Show answer

The pieces must meet: $2^2 - 1 = 2k + 1$, so $3 = 2k + 1$ and $k = 1$.

13. Challenge Explain why, for $a > 0$, the graph of $y = a|x - h| + k$ is made of two rays with slopes a and $-a$ that meet at (h, k).

Show answer

For $x \geq h$, $x - h \geq 0$, so $|x - h| = x - h$ and $y = a(x - h) + k$: a line of slope a through (h, k). For $x < h$, $x - h < 0$, so $|x - h| = -(x - h)$ and $y = -a(x - h) + k$: a line of slope $-a$ through (h, k). The two rays meet at (h, k), where $y = k$.

14. Challenge Solve $\lfloor 2x + 1 \rfloor = 4$. Write the answer in interval notation.

Show answer

$4 \leq 2x + 1 < 5$, so $3 \leq 2x < 4$ and $1.5 \leq x < 2$: $[1.5, 2)$

COMMON MISTAKES

- Using the wrong piece at a boundary. At $x = -1$ in a rule with "$x < -1$" and "$-1 \leq x$", only the second piece applies. Check the inequality signs before computing.
- Drawing both endpoints closed where two pieces meet with a jump. Only one piece can own the endpoint; the other gets an open dot.
- Rounding negative numbers the wrong way with $\lfloor x \rfloor$. It always rounds down, toward the left on the number line: $\lfloor -2.3 \rfloor = -3$, not -2.
- Using the greatest integer function for fees charged "per hour or part of an hour." Those fees round up, so a whole number of hours must not be charged for an extra hour.

Make it a game · Which Rule Applies. Write a piecewise function such as $f(x) = x + 2$ for $x < 1$ and $f(x) = 5$ for $x \geq 1$. The parent calls inputs, including the border value 1. Pick the right rule, evaluate, and say whether the endpoint is open or closed. Then solve the parking meter step function: 2 pretend dollars for each hour or part of an hour.

1.4 Combining and Composing Functions

OBJECTIVES

- Find the sum, difference, product and quotient of two functions and state their domains.
- Evaluate and simplify a composition $f(g(x))$ from formulas, tables and graphs, and find its domain.
- Explain why $f(g(x))$ and $g(f(x))$ are usually different.
- Break a function into simpler steps and use compositions to model discounts, unit conversions and growing areas.

WATCH

How to do Composition of Functions

Mario's Math Tutoring · 7:07

https://www.youtube.com/watch?v=qPUAkMILgSY

> **WHEN TO WATCH**
> Before Example 1: finding f(g(x)) and g(f(x)).

Note: Neither video covers sums, products or quotients of functions and their domains. Teach those from the Core ideas.

BACKUP VIDEO

Decomposing Functions Using Composition of Functions

Mario's Math Tutoring · 5:15

https://www.youtube.com/watch?v=TYhGWDi5HsU

After Example 1: breaking a function into a composition of two simpler ones.

KEY TERMS

sum, difference, product and quotient of functions

New functions made by combining outputs: $(f+g)(x) = f(x) + g(x)$, $(f-g)(x) = f(x) - g(x)$, $(fg)(x) = f(x) \cdot g(x)$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

composition

The function $(f \circ g)(x) = f(g(x))$: apply g first, then apply f to the result. Read "f of g of x."

inner and outer function

In $f(g(x))$, g is the inner function (it acts first) and f is the outer function (it acts second).

decomposition

Writing a complicated function as a composition of simpler ones.

CORE IDEAS

Functions can be combined with ordinary arithmetic on their outputs. If $f(x) = \sqrt{x}$ and $g(x) = x - 2$, then $(f + g)(x) = \sqrt{x} + x - 2$ and $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x - 2}$. An input works for $f + g$, $f - g$ or fg only if it works for **both** f and g, so the domain is the overlap of the two domains. A quotient also needs $g(x) \neq 0$. Here the domain of $\frac{f}{g}$ is $[0, 2) \cup (2, \infty)$.

Composition chains two functions like two machines on an assembly line: x goes into g, and the output $g(x)$ goes straight into f. So $(f \circ g)(x) = f(g(x))$ means "do g first, then f," even though f is written first. To find a formula, put the whole expression $g(x)$ into f wherever f has an x. If $f(x) = x^2$ and $g(x) = x + 3$, then $f(g(x)) = (x + 3)^2$.

Order matters. With the same f and g, $g(f(x)) = x^2 + 3$. The first adds 3 and then squares; the second squares and then adds 3. At $x = 1$ they give 16 and 4. Doing two steps in the opposite order usually gives a different result, just as putting on socks and then shoes is not the same as shoes and then socks. So $f(g(x)) \neq g(f(x))$ in general, although for some special pairs they agree.

The **domain of a composition** $f(g(x))$ contains each x that (1) is in the domain of g and (2) gives an output $g(x)$ that is in the domain of f. For $f(x) = \sqrt{x}$ and $g(x) = x - 4$, we need $x - 4 \geq 0$, so $f(g(x)) = \sqrt{x - 4}$ has domain $[4, \infty)$. Find the domain **before** simplifying: if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $f(g(x)) = (\sqrt{x})^2 = x$, but only for $x \geq 0$, because g cannot accept negative inputs.

To **decompose** a function, look for an expression sitting inside another operation. In $h(x) = (2x + 1)^3$, the inside step is $g(x) = 2x + 1$ and the outside step is $f(x) = x^3$, so $h(x) = f(g(x))$. Seeing functions as chains of simple steps is a skill you will use for inverses in the next lesson and throughout calculus.

Many real situations are compositions. Converting hours to kilometers and then kilometers to miles, applying a discount and then a coupon, or finding the area of a

circle whose radius grows with time are all chains. For a ripple whose radius grows 3 inches per second, $r(t) = 3t$ and $A(r) = \pi r^2$, so the area at time t is $A(r(t)) = \pi(3t)^2 = 9\pi t^2$ square inches. Keep track of units: the output units of the inner function must match the input units of the outer function.

KEY FORMULAS

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)\,g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$$

$$(f \circ g)(x) = f(g(x)) \quad \text{(apply } g \text{ first)}$$

$$\text{Domain of } f(g(x)) : \ x \text{ in the domain of } g \text{ and } g(x) \text{ in the domain of } f$$

Practice habit. Write one step per line when composing: first write $f(\ \ )$ with the whole inner expression in the parentheses, then simplify on the next line. Find the domain before you simplify.

WORKED EXAMPLES

Example 1. Let $f(x) = 2x - 5$ and $g(x) = x^2 + 1$. Find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$ and (c) $(f \circ g)(3)$.

Show solution

1. (a) Put $g(x)$ into f: $f(g(x)) = 2(x^2 + 1) - 5$.
2. $= 2x^2 + 2 - 5 = 2x^2 - 3$.
3. (b) Put $f(x)$ into g: $g(f(x)) = (2x - 5)^2 + 1$.
4. $= 4x^2 - 20x + 25 + 1 = 4x^2 - 20x + 26$.
5. The two compositions are different, as we expect.
6. (c) From the formula: $(f \circ g)(3) = 2(9) - 3 = 15$.
7. Check step by step: $g(3) = 10$, then $f(10) = 20 - 5 = 15$ ✓.

Answer: (a) $2x^2 - 3$; (b) $4x^2 - 20x + 26$; (c) 15

Example 2. Samuel wants a table saw priced at \$250. The hardware store is having a 20% off sale, and Samuel also has a \$15-off coupon. Let $D(x) = 0.8x$ (the sale) and $C(x) = x - 15$ (the coupon). (a) Write $D(C(x))$ and $C(D(x))$ and say what each means. (b) Find the price both ways for $x = 250$. Which order is better for Samuel?

Show solution

1. (a) $D(C(x)) = 0.8(x - 15) = 0.8x - 12$: the coupon is taken off first, then 20% off the rest.
2. $C(D(x)) = 0.8x - 15$: 20% off first, then the coupon.
3. (b) $D(C(250)) = 0.8(250) - 12 = 200 - 12 = 188$.
4. Check: $C(250) = 235$, and $0.8(235) = 188$ ✓.
5. $C(D(250)) = 0.8(250) - 15 = 200 - 15 = 185$.
6. Taking the coupon last saves the full \$15; taking it first lets the sale shrink the coupon to $0.8(15) = 12$ dollars.

Answer: (a) $D(C(x)) = 0.8x - 12$, $C(D(x)) = 0.8x - 15$; (b) \$188 and \$185: sale first, then coupon, saves \$3 more

PRACTICE

1. Easy Let $f(x) = 3x + 2$ and $g(x) = x - 4$. Find (a) $(f + g)(x)$ and (b) $(fg)(x)$.

   Show answer

   (a) $4x - 2$; (b) $(3x + 2)(x - 4) = 3x^2 - 10x - 8$

2. Easy Let $f(x) = x^2$ and $g(x) = x + 5$. Find (a) $f(g(-3))$ and (b) $g(f(-3))$.

   Show answer

   (a) $g(-3) = 2$, $f(2) = 4$; (b) $f(-3) = 9$, $g(9) = 14$

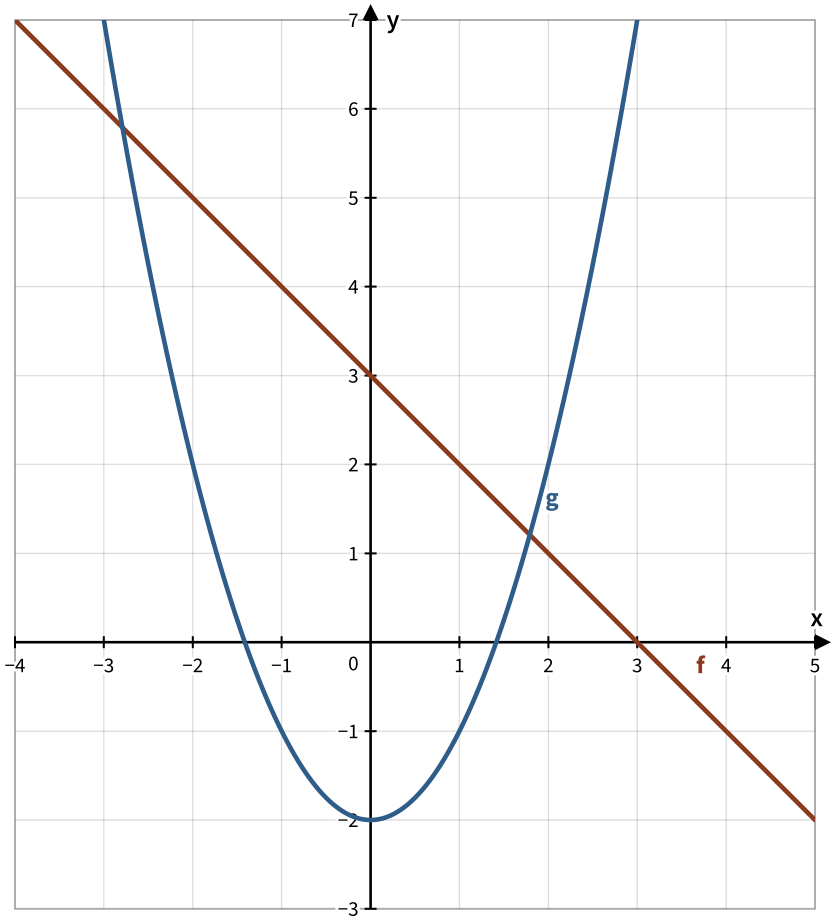
3. **Easy** Use the table to find (a) $f(g(1))$ and (b) $g(f(1))$.

x	f(x)	g(x)
0	2	4
1	4	3
2	0	1
3	1	0
4	3	2

Show answer

(a) $g(1) = 3$, $f(3) = 1$; (b) $f(1) = 4$, $g(4) = 2$

4. **Easy** Use the graphs of f and g to find (a) $f(g(2))$ and (b) $g(f(4))$.



Show answer

(a) $g(2) = 2$, $f(2) = 1$; (b) $f(4) = -1$, $g(-1) = -1$

5. **Medium** Let $f(x) = \sqrt{x}$ and $g(x) = x - 9$. Find (a) $f(g(x))$ and (b) its domain in interval notation.

Show answer

(a) $\sqrt{x - 9}$; (b) $x - 9 \geq 0$: $[9, \infty)$

6. **Medium** Let $f(x) = x + 2$ and $g(x) = x^2 - 9$. Write $\left(\frac{f}{g}\right)(x)$ and give its domain in interval notation.

Show answer

$\frac{x + 2}{x^2 - 9}$; $x^2 - 9 \neq 0$ means $x \neq \pm 3$: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

7. **Medium** Find functions f and g, neither equal to x, so that $h(x) = \sqrt{5x - 1}$ is $f(g(x))$.

   Show answer

   One answer: inner $g(x) = 5x - 1$, outer $f(x) = \sqrt{x}$. Check: $f(g(x)) = \sqrt{5x - 1}$ ✓. (Other answers are possible.)

8. **Medium** A passenger train travels at 80 kilometers per hour, so it covers $d(t) = 80t$ kilometers in t hours. One kilometer is about 0.621 mile, so $m(d) = 0.621d$ converts kilometers to miles. (a) Write $m(d(t))$. (b) About how many miles does the train cover in 2.5 hours?

   Show answer

   (a) $m(d(t)) = 0.621(80t) = 49.68t$; (b) $49.68(2.5) = 124.2$ miles

9. **Medium** A stone dropped into a still farm pond makes a circular ripple whose radius grows 2 feet per second, so $r(t) = 2t$. The area of a circle is $A(r) = \pi r^2$. (a) Write $A(r(t))$. (b) Find the area inside the ripple after 3 seconds, to the nearest tenth of a square foot.

   Show answer

   (a) $\pi(2t)^2 = 4\pi t^2$; (b) $4\pi(9) = 36\pi \approx 113.1$ square feet

10. **Hard** For a bake sale, a church kitchen bakes 6 dozen cookies an hour, so $n(h) = 6h$ dozen in h hours. Ingredients cost \$4 per dozen plus \$10 for supplies, so $c(n) = 4n + 10$ dollars. (a) Write the cost as a function of baking time, $c(n(h))$. (b) What does a 3-hour baking session cost?

    Show answer

    (a) $c(n(h)) = 4(6h) + 10 = 24h + 10$; (b) $24(3) + 10 = \$82$

11. **Hard** Let $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x - 1}$. Find (a) $f(g(x))$ and (b) its domain in interval notation.

    Show answer

    (a) $\dfrac{1}{\sqrt{x - 1}}$; (b) g needs $x \geq 1$, and f needs $g(x) \neq 0$, so $x \neq 1$: $(1, \infty)$

12. Hard Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. (a) Simplify $f(g(x))$. (b) Give the domain of $f(g(x))$. (c) Explain why the domain is not all real numbers, even though the simplified formula is defined everywhere.

Show answer

(a) $(\sqrt{x})^2 = x$; (b) $[0, \infty)$; (c) the input must first go through g, and $\sqrt{x}$ is not a real number when $x < 0$. Simplifying does not remove that first step.

13. Challenge Let $f(x) = 2x + 1$. Find a function g so that $f(g(x)) = 6x - 7$.

Show answer

$2g(x) + 1 = 6x - 7$, so $g(x) = 3x - 4$. Check: $2(3x - 4) + 1 = 6x - 7$ ✓

14. Challenge Let $f(x) = 2x + 1$ and $g(x) = 3x + b$. Find the value of b for which $f(g(x)) = g(f(x))$ for every x.

Show answer

$f(g(x)) = 6x + 2b + 1$ and $g(f(x)) = 6x + 3 + b$. Equal when $2b + 1 = 3 + b$, so $b = 2$.

COMMON MISTAKES

- Doing the outer function first. In $f(g(x))$ the function written closest to x, here g, acts first.
- Multiplying instead of composing. $f(g(x))$ is not $f(x) \cdot g(x)$: substitute the whole expression $g(x)$ for x in f.
- Finding the domain of a composition only from its simplified formula. Check that x works in the inner function first.
- Forgetting to exclude the zeros of the denominator in $\frac{f}{g}$. Solve $g(x) = 0$ and remove those inputs.

> **Make it a game · Two-Machine Assembly Line.** Be two machines. The parent is g(x) = x^2 and you are f(x) = 2x + 3. The parent says an input, performs g, and hands the result to you for f. Write the composition f(g(3)), then swap order and show g(f(3)) differs. Finally, the parent names h(x) = $(3x - 1)^2$ and you call out the inner and outer functions.

1.5 Inverse Functions

OBJECTIVES

- Find the inverse of a function by swapping x and y and solving for y.
- Explain why the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.
- Verify that two functions are inverses using $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
- Use the horizontal line test, and restrict a domain when needed, to decide whether a function has an inverse.

WATCH

Intro to Inverse Functions (1 of 4: Overview & basic examples)

Eddie Woo · 9:57

https://www.youtube.com/watch?v=i7yA-Tqt6Zs

> **WHEN TO WATCH**
> Before Example 1: what an inverse function is and how to find one.

BACKUP VIDEO

Investigating Inverse Functions (3 of 3: Visualising the Horizontal Line Test)

Eddie Woo · 11:31

https://www.youtube.com/watch?v=1sBG-yd3RO4

After Example 2: why the horizontal line test tells you whether an inverse is a function.

KEY TERMS

inverse function

The function f^{-1} that undoes f: if $f(a) = b$, then $f^{-1}(b) = a$. Read "f inverse."

one-to-one function

A function in which different inputs always give different outputs. Only one-to-one functions have inverse functions.

horizontal line test

A function is one-to-one exactly when no horizontal line crosses its graph more than once.

restricted domain

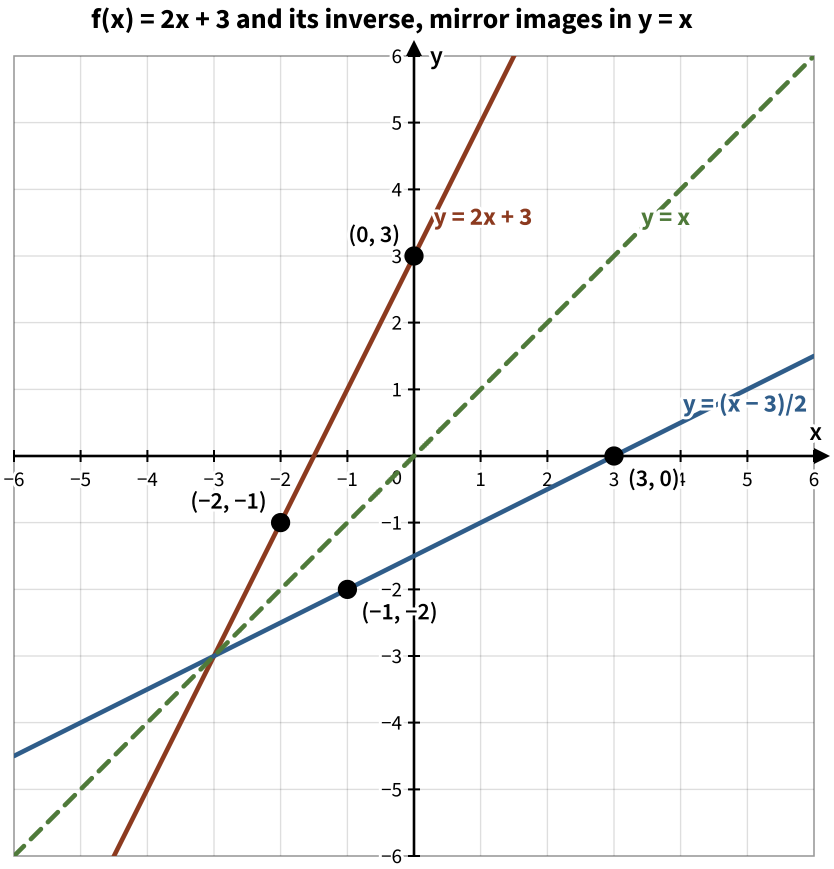
A smaller domain chosen for a function so that it becomes one-to-one, such as $y = x^2$ for $x \geq 0$ only.

CORE IDEAS

An **inverse function** undoes what a function does. The function $f(x) = 2x + 3$ says "double, then add 3." To undo it, reverse the steps in reverse order: subtract 3, then halve. So $f^{-1}(x) = \dfrac{x - 3}{2}$. Check: $f(4) = 11$ and $f^{-1}(11) = \dfrac{8}{2} = 4$. The inverse sends every output back to the input it came from.

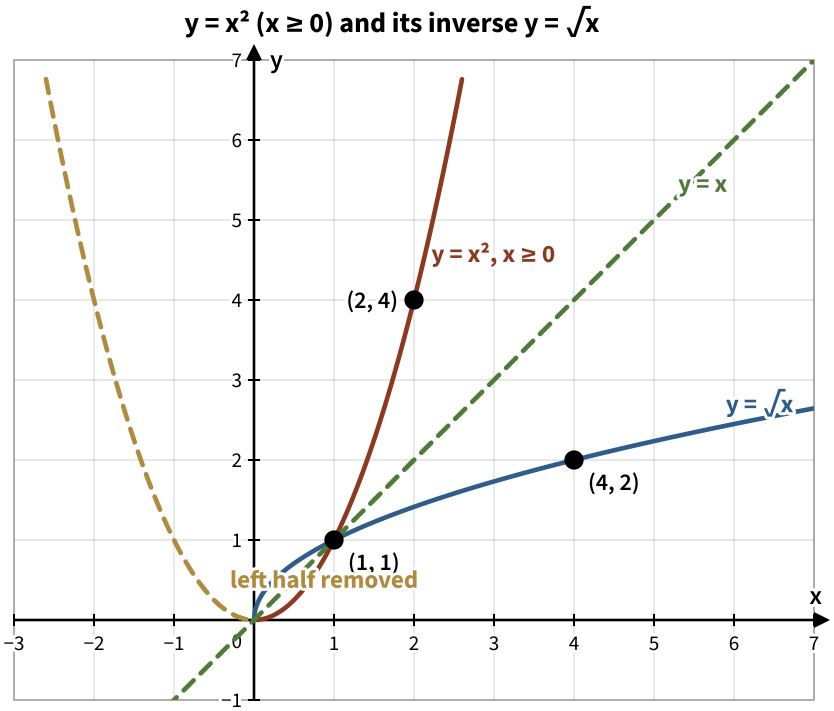
To find an inverse algebraically, write $y = f(x)$, **swap** x and y, and solve for y. Swapping is exactly what an inverse does: the inputs of f become the outputs of f^{-1} and the outputs become the inputs. For $y = 2x + 3$: swap to get $x = 2y + 3$, so $x - 3 = 2y$ and $y = \dfrac{x - 3}{2}$. This also shows that the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Because the inverse swaps coordinates, every point (a, b) on the graph of f becomes the point (b, a) on the graph of f^{-1}. These two points are mirror images in the line $y = x$. Here is why: the midpoint of the segment joining them is $\left(\dfrac{a + b}{2}, \dfrac{a + b}{2}\right)$, which lies on $y = x$; and the slope of the segment is $\dfrac{a - b}{b - a} = -1$ (for $a \neq b$), so the segment is perpendicular to $y = x$, which has slope 1. A segment cut in half at a right angle by a line is exactly what a reflection in that line produces. So the whole graph of f^{-1} is the reflection of the graph of f in $y = x$.



To **verify** that f and g are inverses, compose them both ways. If $f(g(x)) = x$ and $g(f(x)) = x$ for every x in the right domains, then each undoes the other. Checking both orders matters; one order alone can be fooled when domains are restricted.

Not every function has an inverse. For $f(x) = x^2$, both $f(2)$ and $f(-2)$ equal 4, so an inverse would not know whether to send 4 back to 2 or to -2. A function has an inverse exactly when it is **one-to-one**, and on a graph that means no horizontal line crosses it twice (the **horizontal line test**). We can fix $y = x^2$ by restricting its domain to $x \geq 0$; then its inverse is $y = \sqrt{x}$, and the two graphs are mirror images in $y = x$. In the same way, $f(x) = x^2 - 3$ with $x \geq 0$ has inverse $f^{-1}(x) = \sqrt{x + 3}$: we take the positive square root because the outputs of f^{-1} must be the original inputs, which were not negative.



A warning about notation: the -1 in $f^{-1}(x)$ is not an exponent. For $f(x) = 2x + 3$, the inverse is $\dfrac{x-3}{2}$, but the reciprocal is $\dfrac{1}{f(x)} = \dfrac{1}{2x+3}$, a completely different function. Inverses appear whenever you need to reverse a conversion: from Fahrenheit back to Celsius, from euros back to dollars, or from a height back to the time it took to fall.

KEY FORMULAS

$$f(a) = b \iff f^{-1}(b) = a$$

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

$$(a, b) \text{ on } f \iff (b, a) \text{ on } f^{-1} : \text{ reflection in } y = x$$

$$\text{domain of } f^{-1} = \text{range of } f, \qquad \text{range of } f^{-1} = \text{domain of } f$$

Practice habit. Check every inverse you find by composing: compute $f(f^{-1}(x))$ and make sure it simplifies to x. A quick numeric check works too: if $f(4) = 11$, then your $f^{-1}(11)$ must be 4.

WORKED EXAMPLES

Example 1. Find the inverse of $f(x) = 5 - 2x$ and verify your answer by composition.

Show solution

1. Write $y = 5 - 2x$.
2. Swap x and y: $x = 5 - 2y$.
3. Solve for y: $2y = 5 - x$, so $y = \dfrac{5 - x}{2}$.
4. So $f^{-1}(x) = \dfrac{5 - x}{2}$.
5. Verify: $f(f^{-1}(x)) = 5 - 2\left(\dfrac{5 - x}{2}\right) = 5 - (5 - x) = x$ ✓.
6. Verify: $f^{-1}(f(x)) = \dfrac{5 - (5 - 2x)}{2} = \dfrac{2x}{2} = x$ ✓.

Answer: $f^{-1}(x) = \dfrac{5 - x}{2}$

Example 2. Celsius and Fahrenheit temperatures are related by $F = 1.8C + 32$. (a) Find the inverse function, giving C in terms of F. (b) Grace's grandmother's pie recipe says to bake at 350°F. What is that in degrees Celsius, to the nearest tenth?

Show solution

1. (a) Here the letters already say which is input and which is output, so we solve for C instead of swapping letters.
2. $F - 32 = 1.8C$, so $C = \dfrac{F - 32}{1.8}$.
3. Check with water's boiling point: $F = 212$ gives $C = \dfrac{180}{1.8} = 100$ ✓.
4. (b) $C = \dfrac{350 - 32}{1.8} = \dfrac{318}{1.8} = 176.666\ldots$
5. Round: 176.7°C.
6. Check: $1.8(176.67) + 32 \approx 318 + 32 = 350$ ✓.

Answer: (a) $C = \dfrac{F - 32}{1.8}$; (b) about 176.7°C

PRACTICE

1. **Easy** Find the inverse of the function $\{(1,4),(2,7),(3,10)\}$.

   Show answer

   $\{(4,1),(7,2),(10,3)\}$

2. **Easy** A shuttle from the train station charges \$3 plus \$2.50 per mile, so the fare for m miles is $F = 2.5m + 3$. (a) Solve for m to get the inverse function. (b) How long was a ride that cost \$23?

   Show answer

   (a) $m = \dfrac{F-3}{2.5}$; (b) $m = \dfrac{20}{2.5} = 8$ miles

3. **Easy** Find $f^{-1}(x)$ for $f(x) = 3x + 12$.

   Show answer

   $x = 3y + 12$, so $f^{-1}(x) = \dfrac{x-12}{3}$

4. **Easy** A one-to-one function has $f(5) = -2$. Find $f^{-1}(-2)$ and name a point on the graph of f^{-1}.

   Show answer

   $f^{-1}(-2) = 5$; the point $(-2, 5)$

5. **Medium** Find $f^{-1}(x)$ for $f(x) = \dfrac{2x-1}{3}$.

   Show answer

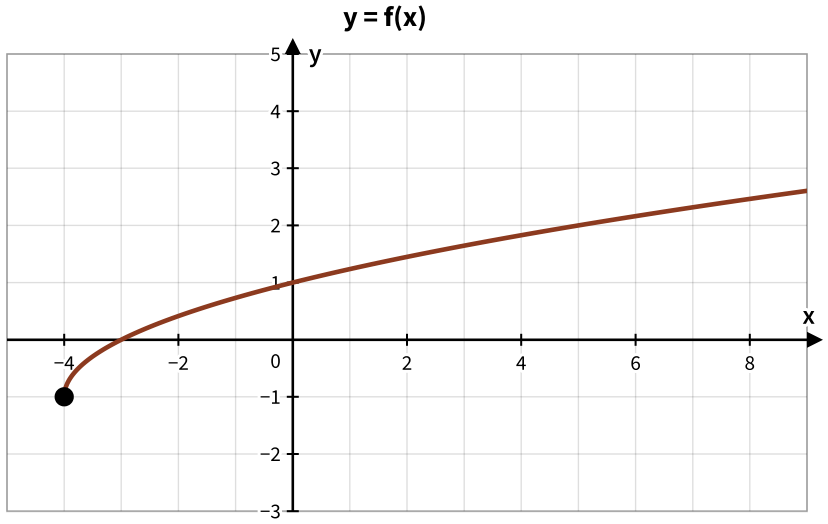
   $x = \dfrac{2y-1}{3}$, $3x = 2y - 1$, so $f^{-1}(x) = \dfrac{3x+1}{2}$

6. **Medium** Use composition to decide whether $f(x) = 4x - 1$ and $g(x) = \dfrac{x+1}{4}$ are inverses.

   Show answer

   Yes. $f(g(x)) = (x+1) - 1 = x$ and $g(f(x)) = \dfrac{4x}{4} = x$.

7. **Medium** The graph shows a one-to-one function f; it starts at the solid dot and continues to the right. Find (a) $f^{-1}(2)$, (b) the domain of f^{-1} and (c) the range of f^{-1}, in interval notation.



Show answer

(a) $f(5) = 2$, so $f^{-1}(2) = 5$; (b) the range of f: $[-1, \infty)$; (c) the domain of f: $[-4, \infty)$

8. **Medium** Before a family trip to Canada, Henry's bank gives 1.35 Canadian dollars for each U.S. dollar and subtracts a flat fee of 3 Canadian dollars, so $c(u) = 1.35u - 3$. (a) Find the inverse function $u(c)$. (b) How many U.S. dollars must Henry exchange to receive 267 Canadian dollars?

Show answer

(a) $u = \dfrac{c + 3}{1.35}$; (b) $u = \dfrac{270}{1.35} = 200$ U.S. dollars

9. **Medium** Is $f(x) = x^2 - 4$ (with domain all real numbers) one-to-one? Give a reason.

Show answer

No. $f(-1) = f(1) = -3$, so a horizontal line $y = -3$ crosses the graph twice.

10. **Hard** Let $f(x) = x^2 + 2$ with domain $x \geq 0$. Find (a) $f^{-1}(x)$ and (b) the domain of f^{-1}.

    Show answer

    (a) $x = y^2 + 2$, $y^2 = x - 2$, and $y \geq 0$, so $f^{-1}(x) = \sqrt{x - 2}$; (b) the range of f: $[2, \infty)$

11. **Hard** Let $f(x) = \dfrac{2}{x - 1}$. (a) Find $f^{-1}(x)$. (b) Find $\dfrac{1}{f(x)}$. (c) Are they the same function?

    Show answer

    (a) $x = \dfrac{2}{y - 1}$, $y - 1 = \dfrac{2}{x}$, so $f^{-1}(x) = \dfrac{2}{x} + 1$; (b) $\dfrac{x - 1}{2}$; (c) no

12. **Hard** Ruth drops a stone from a 256-foot bridge tower, so its height after t seconds is $h = 256 - 16t^2$ feet, for $0 \leq t \leq 4$. (a) Solve for t to get time as a function of height. (b) When is the stone 112 feet above the ground?

    Show answer

    (a) $16t^2 = 256 - h$, and $t \geq 0$, so $t = \dfrac{\sqrt{256 - h}}{4}$; (b) $t = \dfrac{\sqrt{144}}{4} = 3$ seconds. Check: $256 - 16(9) = 112$ ✓

13. **Challenge** Prove that for $a \neq b$ the points (a, b) and (b, a) are reflections of each other in the line $y = x$.

    Show answer

    The midpoint is $\left(\dfrac{a + b}{2}, \dfrac{a + b}{2}\right)$, which lies on $y = x$ because its coordinates are equal. The slope of the segment is $\dfrac{a - b}{b - a} = -1$, and $y = x$ has slope 1; since $(-1)(1) = -1$, the segment is perpendicular to $y = x$. So $y = x$ is the perpendicular bisector of the segment, which means each point is the reflection of the other.

14. **Challenge** Find $f^{-1}(x)$ for $f(x) = \dfrac{x + 1}{x - 1}$. What do you notice?

    Show answer

    $x = \dfrac{y + 1}{y - 1}$, $xy - x = y + 1$, $y(x - 1) = x + 1$, so $f^{-1}(x) = \dfrac{x + 1}{x - 1}$. The function is its own inverse, so its graph is symmetric about $y = x$.

COMMON MISTAKES

- Reading $f^{-1}(x)$ as $\frac{1}{f(x)}$. The -1 means "inverse," not "reciprocal."
- Undoing the steps in the wrong order. Undo the last step first: for $f(x) = 2x + 3$, subtract 3 before dividing by 2.
- Taking both signs of a square root when the domain was restricted. If f uses $x \geq 0$, the inverse must give outputs ≥ 0, so use only the positive root.
- Verifying only one composition. Check both $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

> **Make it a game · Backward Detective.** The parent thinks of a number, applies f(x) = 3x - 4 in secret, and says only the result. Run the machine backward in your head to find the original number. Then write the inverse by swapping x and y, and test it on three secrets. Last round: use the horizontal line test to decide if $y = x^2$ has an inverse.

1.6 Rates of Change and Function Behavior

OBJECTIVES

- Compute an average rate of change and interpret it as the slope of a secant line, with units.
- Identify intervals where a function is increasing, decreasing or constant, and its relative maxima and minima, from graphs and tables.
- Test a function algebraically for even or odd symmetry and describe its end behavior in words.
- Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for linear and quadratic functions.

WATCH

Intervals where the Function is Increasing, Decreasing, or Constant

Mario's Math Tutoring · 8:26

https://www.youtube.com/watch?v=4Nd5AWwOqkI

> **WHEN TO WATCH**
> Before Example 1: intervals where a function is increasing, decreasing or constant.

Note: Neither video does the average rate of change or the difference quotient. Teach those from the Core ideas. Eddie Woo teaches in Australia: the backup uses 'odd and even' the same way this course does.

BACKUP VIDEO

Odd & Even Functions (1 of 2: Introductory Examples)

Eddie Woo · 6:53

https://www.youtube.com/watch?v=8Ss_-kGyp8c

After Example 1: even and odd functions, for the symmetry part of the Core ideas.

KEY TERMS

average rate of change

For a function f from $x = a$ to $x = b$, the change in output divided by the change in input: $\dfrac{f(b) - f(a)}{b - a}$.

secant line

A line through two points of a graph. Its slope is the average rate of change between those points.

increasing, decreasing

f is increasing on an interval if its graph rises from left to right there (larger inputs give larger outputs), and decreasing if it falls.

relative maximum and minimum

A relative maximum is an output larger than every nearby output (a hilltop); a relative minimum is smaller than every nearby output (a valley).

even and odd functions

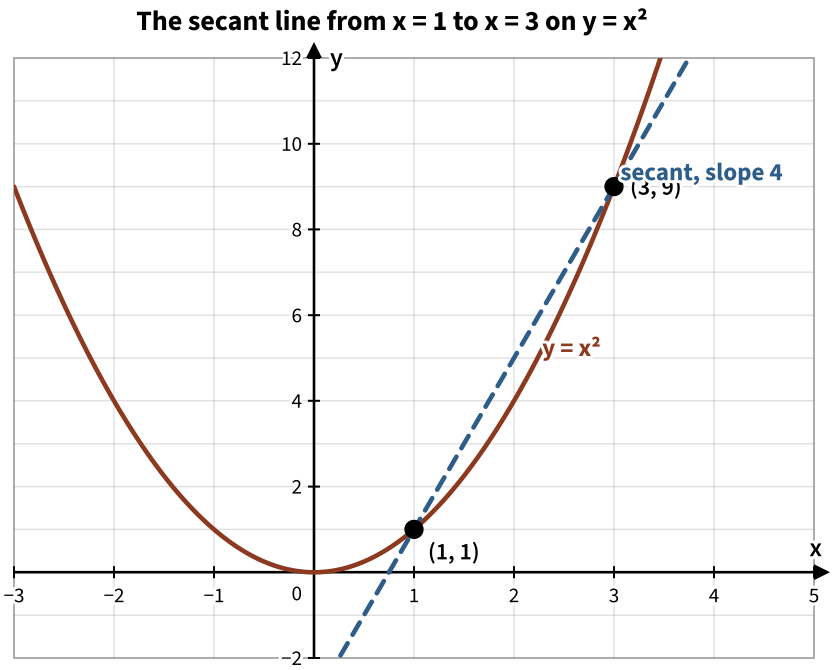
f is even if $f(-x) = f(x)$ for every x in its domain, and odd if $f(-x) = -f(x)$.

end behavior

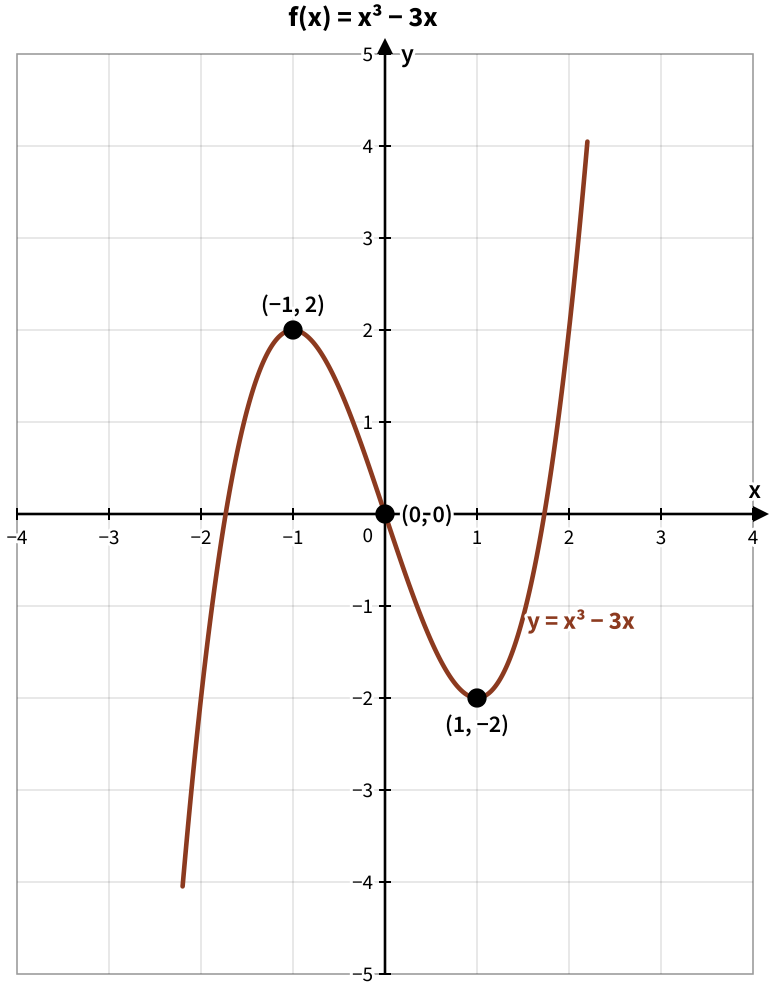
What the outputs of a function do as x grows without bound to the right ($x \to \infty$) and to the left ($x \to -\infty$).

CORE IDEAS

The **average rate of change** of f from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$: how much the output changed for each unit of input, on average. On the graph it is the slope of the **secant line** through $(a, f(a))$ and $(b, f(b))$, since slope is rise over run. For $f(x) = x^2$ from $x = 1$ to $x = 3$, the rate is $\dfrac{9 - 1}{3 - 1} = 4$. Its units are output units per input unit, such as miles per hour or dollars per year.



A function is **increasing** on an interval when its graph rises as you move left to right, **decreasing** when it falls, and **constant** when it is flat. Describe these intervals with x-values, using open intervals. A **relative maximum** is a hilltop, an output higher than all nearby outputs; a **relative minimum** is a valley. For $f(x) = x^3 - 3x$ below, f increases on $(-\infty, -1)$, decreases on $(-1, 1)$ and increases on $(1, \infty)$; it has a relative maximum of 2 at $x = -1$ and a relative minimum of -2 at $x = 1$. A graphing tool can locate hilltops and valleys that do not fall on grid lines; from a table you can only say where one **appears** to be.



Some graphs have symmetry we can test with algebra. If $f(-x) = f(x)$ for every x, then whenever (a, b) is on the graph, so is $(-a, b)$: the graph is its own mirror image in the y-axis, and f is **even** (like x^2 and $|x|$). If $f(-x) = -f(x)$, then whenever (a, b) is on the graph, so is $(-a, -b)$: turning the graph $180°$ about the origin leaves it unchanged, and f is **odd** (like x^3 and $\frac{1}{x}$). For $f(x) = x^3 - 3x$: $f(-x) = -x^3 + 3x = -(x^3 - 3x) = -f(x)$, so it is odd, as the graph shows. Most functions, such as $x^2 + x$, are neither.

End behavior describes the far left and far right of a graph. For $f(x) = x^3 - 3x$: as $x \to \infty$, $f(x) \to \infty$ (the graph rises to the right), and as $x \to -\infty$, $f(x) \to -\infty$ (it falls to the left). For $y = -x^2$ the graph falls on both ends. Unit 4 explains how to predict end behavior from a formula.

The **difference quotient** $\dfrac{f(x+h)-f(x)}{h}$ is the average rate of change from x to $x+h$, where $h \neq 0$ is the length of the step. For a line $f(x)=mx+b$ it is $\dfrac{m(x+h)+b-mx-b}{h}=\dfrac{mh}{h}=m$: a line changes at the same rate everywhere. For $f(x)=x^2$ it is $\dfrac{x^2+2xh+h^2-x^2}{h}=\dfrac{2xh+h^2}{h}=2x+h$. When the step h is made very small, this gets close to $2x$, the rate of change of x^2 at the single point x. That idea, a rate at an instant, is where calculus begins.

KEY FORMULAS

$$\text{average rate of change}=\frac{f(b)-f(a)}{b-a}$$

even: $f(-x)=f(x)$ (symmetric in the y-axis) odd: $f(-x)=-f(x)$ (symmetric about the origin)

$$\text{difference quotient}=\frac{f(x+h)-f(x)}{h},\quad h\neq 0$$

Practice habit. Sketch before you compute: a quick sketch tells you whether a rate of change should be positive or negative and whether a function looks even, odd or neither, so you can catch a wrong sign at once.

WORKED EXAMPLES

Example 1. Let $f(x)=x^2-4x+1$. (a) Find the average rate of change of f from $x=1$ to $x=4$. (b) Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

Show solution

1. (a) $f(1)=1-4+1=-2$ and $f(4)=16-16+1=1$.
2. Average rate: $\dfrac{1-(-2)}{4-1}=\dfrac{3}{3}=1$.
3. (b) $f(x+h)=(x+h)^2-4(x+h)+1=x^2+2xh+h^2-4x-4h+1$.
4. Subtract $f(x)=x^2-4x+1$: $f(x+h)-f(x)=2xh+h^2-4h$.
5. Divide by h (each term has a factor h): $\dfrac{h(2x+h-4)}{h}=2x+h-4$.
6. Check against (a): from $x=1$ with step $h=3$, $2(1)+3-4=1$ ✓.

Answer: (a) 1; (b) $2x+h-4$

Example 2. Henry throws a baseball straight up from a height of 5 feet at 64 feet per second, so its height is $h(t) = -16t^2 + 64t + 5$ feet after t seconds. (a) Find the average velocity (average rate of change of height) from $t = 0$ to $t = 1$. (b) Find the average velocity from $t = 1$ to $t = 3$ and explain the result. (c) Find the relative maximum and say on what interval the height is increasing.

Show solution

1. (a) $h(0) = 5$ and $h(1) = -16 + 64 + 5 = 53$.
2. $\dfrac{53 - 5}{1 - 0} = 48$ feet per second: on average the ball rose 48 feet each second.
3. (b) $h(3) = -144 + 192 + 5 = 53$.
4. $\dfrac{53 - 53}{3 - 1} = 0$ feet per second: the ball went up and came back to the same height, so its overall change was zero.
5. (c) The vertex of the parabola is at $t = -\dfrac{64}{2(-16)} = 2$, and $h(2) = -64 + 128 + 5 = 69$.
6. The relative (and overall) maximum is 69 feet at $t = 2$ seconds. The height increases on $(0, 2)$ and decreases after that until the ball lands.

Answer: (a) 48 ft/s; (b) 0 ft/s, because $h(1) = h(3) = 53$; (c) maximum height 69 feet at $t = 2$; increasing on $(0, 2)$

PRACTICE

1. Easy Find the average rate of change of $f(x) = 3x^2$ from $x = 1$ to $x = 3$.

   Show answer

   $$\frac{27 - 3}{3 - 1} = 12$$

2. Easy Is $f(x) = x^4 + 2x^2$ even, odd or neither? Show the test.

   Show answer

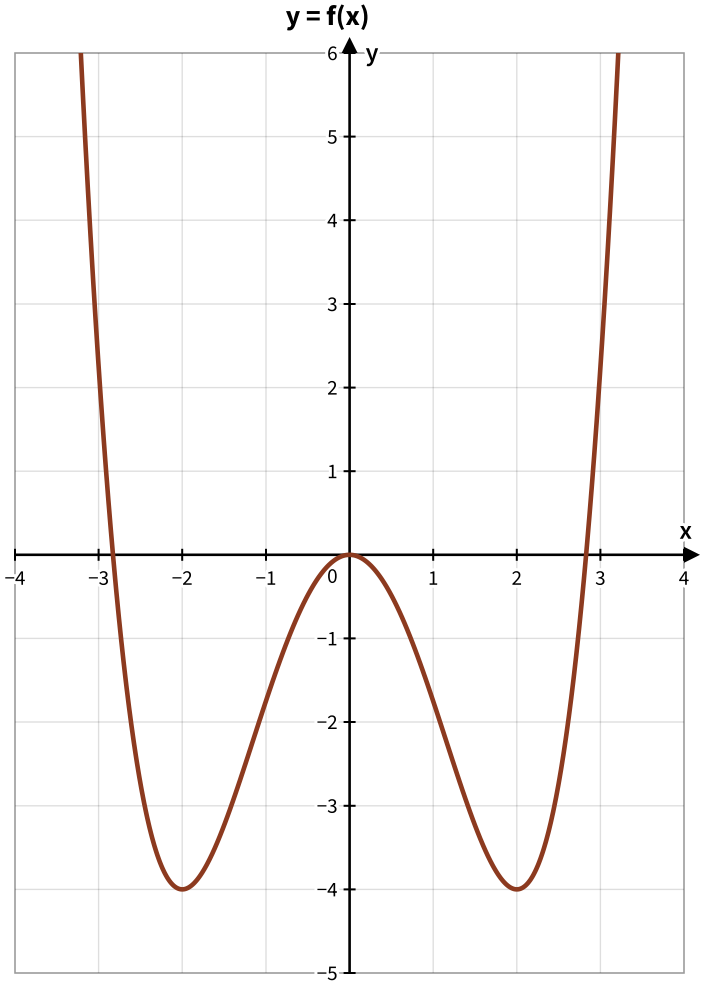
   $f(-x) = (-x)^4 + 2(-x)^2 = x^4 + 2x^2 = f(x)$: even

3. Easy Is $g(x) = x^3 - 5x$ even, odd or neither? Show the test.

   Show answer

   $g(-x) = -x^3 + 5x = -(x^3 - 5x) = -g(x)$: odd

4. **Easy** The graph of $y = f(x)$ is shown; it keeps rising beyond the window on both sides. Give each relative maximum and relative minimum as a point (x, y).



Show answer

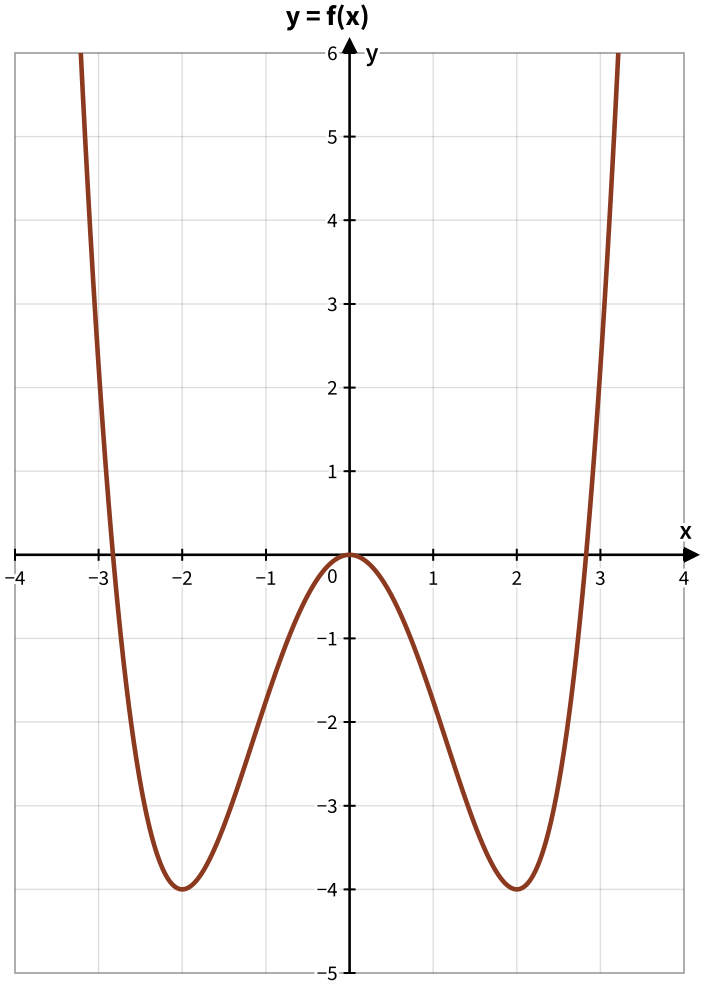
Relative maximum at $(0, 0)$; relative minimum at $(-2, -4)$ and at $(2, -4)$

5. **Medium** Is $h(x) = x^2 + 3x$ even, odd or neither? Show the test.

Show answer

$h(-x) = x^2 - 3x$, which equals neither $h(x)$ nor $-h(x) = -x^2 - 3x$: neither

6. **Medium** Use the graph in Problem 4. (a) On what intervals is f increasing? (b) On what intervals is it decreasing? (c) Does the graph look even, odd or neither?



Show answer

(a) increasing on $(-2, 0)$ and on $(2, \infty)$; (b) decreasing on $(-\infty, -2)$ and on $(0, 2)$; (c) even: it is symmetric in the y-axis

7. **Medium** Grace and her father hike a mountain trail and record their elevation every hour (see the table). Find the average rate of change of elevation (a) from $t = 0$ to $t = 3$ and (b) from $t = 3$ to $t = 6$, with units.

Hours on the trail, t	Elevation (feet)
0	1200
1	1650
2	2100
3	2400
4	2250
5	2600
6	2100

Show answer

(a) $\dfrac{2400 - 1200}{3} = 400$ feet per hour; (b) $\dfrac{2100 - 2400}{3} = -100$ feet per hour (on average they descended)

8. **Medium** Describe the end behavior of $f(x) = -2x^3 + x$ in words, using a few large positive and negative values of x to decide.

Show answer

$f(10) = -1990$ and $f(-10) = 1990$. As $x \to \infty$, $f(x) \to -\infty$ (the graph falls to the right); as $x \to -\infty$, $f(x) \to \infty$ (it rises to the left).

9. **Medium** The Millers put \$500 in a savings account earning 4% interest compounded once a year, so the balance is $B(t) = 500(1.04)^t$ dollars after t years. Find the average rate of change of the balance over the first 2 years, to the nearest cent per year.

Show answer

$B(2) = 500(1.0816) = 540.80$, so $\dfrac{540.80 - 500}{2} = \$20.40$ per year

10. Hard Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=2x^2+3$.

Show answer

$$\frac{2x^2+4xh+2h^2+3-2x^2-3}{h}=\frac{4xh+2h^2}{h}=4x+2h$$

11. Hard Find and simplify the difference quotient for $f(x)=-x^2+5x$.

Show answer

$f(x+h)-f(x)=-2xh-h^2+5h$, so the quotient is $-2x-h+5$

12. Hard A stone dropped from a high bridge falls $d(t)=16t^2$ feet in t seconds. (a) Find and simplify its average speed from $t=1$ to $t=1+h$. (b) Evaluate it for $h=0.1$. (c) What number does the average speed approach as h gets very small?

Show answer

(a) $\dfrac{16(1+h)^2-16}{h}=\dfrac{32h+16h^2}{h}=32+16h$ ft/s; (b) 33.6 ft/s; (c) 32 ft/s, the speed at the instant $t=1$

13. Challenge Prove that if f is an odd function and 0 is in its domain, then $f(0)=0$.

Show answer

Because f is odd, $f(-x)=-f(x)$ for every x in the domain. Put $x=0$: $f(0)=f(-0)=-f(0)$. Adding $f(0)$ to both sides gives $2f(0)=0$, so $f(0)=0$.

14. Challenge For what value of k is $f(x)=x^3+kx^2+x$ an odd function?

Show answer

$f(-x)=-x^3+kx^2-x$ and $-f(x)=-x^3-kx^2-x$. These are equal for every x only if $kx^2=-kx^2$, that is, $k=0$.

COMMON MISTAKES

- Subtracting in opposite orders on top and bottom, as in $\dfrac{f(b)-f(a)}{a-b}$. Keep the same order: later minus earlier in both.
- Giving increasing or decreasing intervals with y-values. The intervals describe where on the x-axis the graph rises or falls.
- Calling a function odd because its formula has an odd power somewhere. Always test $f(-x)$: x^3+1 is neither even nor odd.

- Expanding $f(x+h)$ as $f(x)+h$. Substitute $(x+h)$ for every x, expand carefully, and check that every remaining term has a factor of h before you divide.

> **Make it a game · Walk and Table.** Mark a 10-foot path with tape. Walk it away and back while the parent calls out distances at the 2, 4, 6 and 8 second marks, using a clock with a second hand. Make a table, then compute the average rate of change for each interval, with units. Name the interval where you were fastest, and where the distance decreased.

UNIT 1

Functions and Their Graphs

Algebra II is the study of functions: rules that turn each input into exactly one output. This unit gathers what you learned about functions in Algebra I and sharpens it into a toolkit you will use all year. You will review function notation, domain and range, then learn the six parent functions and how every member of a family is a parent that has been shifted, stretched or flipped. You will build new functions by combining and composing old ones, undo functions with inverses, and describe how a function behaves: where it rises and falls, its hilltops and valleys, its symmetry and its rates of change.

The emphasis is on understanding why the rules work. You will see why $f(x - h)$ moves a graph to the right, why the graph of an inverse is a reflection in the line $y = x$, and why even and odd functions have the symmetries they do. These ideas return in every later unit: quadratics, polynomials, rational, radical, exponential and logarithmic functions are all studied as parent functions with transformations, inverses and rates of change. The difference quotient at the end of the unit is a first look at the central idea of calculus.

PREREQUISITES

- Evaluate expressions and solve linear and quadratic equations (Algebra I).
- Use function notation, the vertical line test, and domain and range in interval notation (Algebra I).
- Find the slope of a line and write its equation (Algebra I).
- Expand products such as $(x+h)^2$ and simplify polynomial expressions (Algebra I).
- Find the vertex of a parabola $y = ax^2 + bx + c$ (Algebra I).
- Describe translations and reflections of the plane as coordinate rules (Geometry).

MASTERY CHECKLIST

- ☐ I can evaluate a function at numbers and at expressions such as $f(a+2)$ and $f(-x)$.
- ☐ I can state the domain and range of a function from a graph, an equation or a situation, in interval notation.
- ☐ I can read a graph to solve $f(x) = c$ and $f(x) > 0$.
- ☐ I can sketch the six parent functions from their key points.
- ☐ I can graph $y = a\,f(x-h)+k$ and explain why $f(bx)$ shifts the graph right.
- ☐ I can evaluate, graph and write piecewise, absolute value and step functions.
- ☐ I can add, subtract, multiply, divide and compose functions and find the domain of the result.
- ☐ I can find an inverse function, verify it by composition, and

explain why its graph is a reflection in

$y = x$.

- ☐ I can use the horizontal line test and restrict a domain so a function has an inverse.
- ☐ I can find average rates of change, intervals of increase and decrease, and relative maxima and minima.
- ☐ I can test a function for even or odd symmetry and describe its end behavior in words.
- ☐ I can simplify the difference quotient for linear and quadratic functions.

LESSONS IN THIS UNIT

1.1	**Functions and Function Notation: A Review**	45 minutes
1.2	**Parent Functions and Transformations**	2 sessions of 45 minutes
1.3	**Piecewise and Absolute Value Functions**	2 sessions of 45 minutes
1.4	**Combining and Composing Functions**	2 sessions of 45 minutes
1.5	**Inverse Functions**	2 sessions of 45 minutes
1.6	**Rates of Change and Function Behavior**	2 sessions of 45 minutes

Unit 1 Test

You have 60 minutes. A scientific calculator is allowed; graphing tools are not. Show your work for every problem: a correct answer with no work earns partial credit only. Give exact answers unless a problem tells you how to round, and write domains, ranges and solution sets in interval notation.

1. **5 pts**

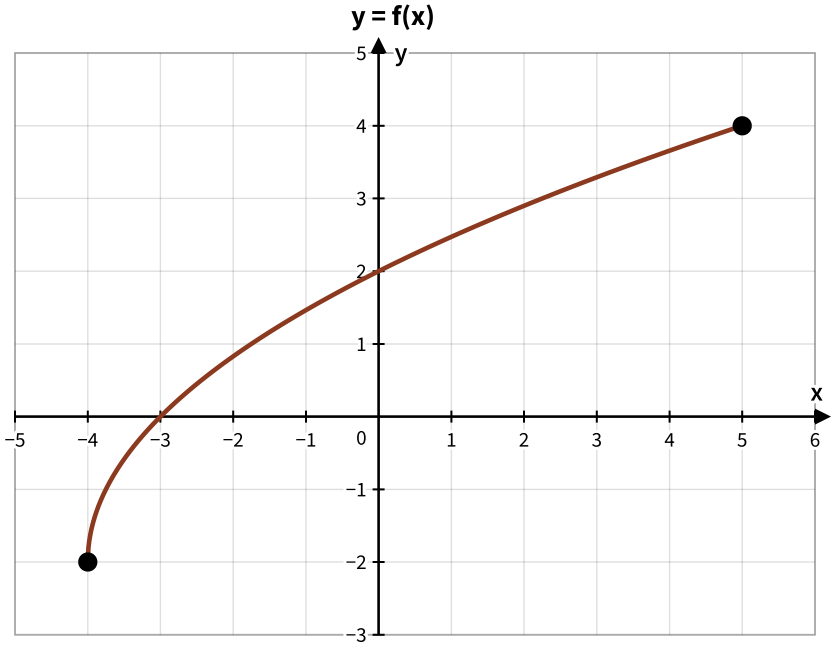
   Let $f(x) = x^2 + 3x - 2$. Find (a) $f(-4)$ and (b) $f(a - 1)$ in simplest form.

2. **6 pts**

   Find the domain of $f(x) = \dfrac{\sqrt{2x + 8}}{x - 3}$ in interval notation.

3. **6 pts**

   The graph of $y = f(x)$ is shown, including both endpoints. Give (a) the domain, (b) the range and (c) the set of x where $f(x) > 0$, all in interval notation.

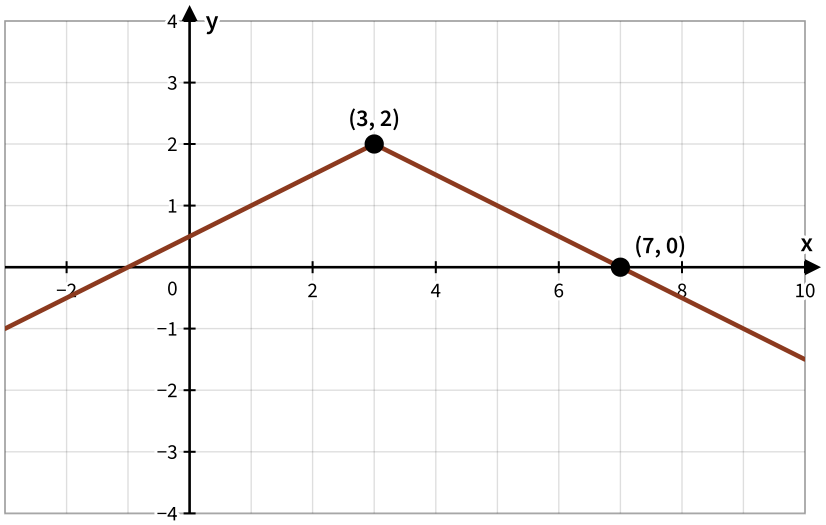


4. **7 pts**

   (a) The point $(4, -1)$ is on the graph of $y = f(x)$. Find the matching point on the graph of $y = -2f(x - 3) + 1$. (b) Write the rule for the graph of $y = \sqrt{x}$ after it is reflected in the x-axis, shifted 2 units right and shifted 6 units up.

5. **6 pts**

The graph is a transformation of $y = |x|$. Write its equation.



6. **7 pts**

A community orchard sells apples for \$1.50 a pound on purchases of up to 10 pounds. On larger purchases the first 10 pounds cost \$1.50 a pound and every pound beyond 10 costs \$1.20. (a) Write the cost $C(w)$ of w pounds as a piecewise function. (b) Find the cost of 16 pounds.

7. **5 pts**

Let $g(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$. (a) Find $g(7)$ and $g(-3)$. (b) For which x is $\lfloor x \rfloor = -2$? Use interval notation.

8. **7 pts**

Let $f(x) = x^2 - 2x$ and $g(x) = x + 3$. Find and simplify (a) $f(g(x))$ and (b) $g(f(x))$.

9. **6 pts**

Let $f(x) = \dfrac{1}{x - 4}$ and $g(x) = \sqrt{x}$. Write $f(g(x))$ and give its domain in interval notation.

10. **7 pts**

A family furniture shop prices a chair by marking up its wholesale cost x by 40%, $m(x) = 1.4x$, and then adding a \$20 delivery charge, $d(x) = x + 20$. (a) Write the customer's price $d(m(x))$. (b) Find the price of a chair with a wholesale cost of \$150. (c) Write $m(d(x))$; what would this order of steps do to the delivery charge?

11. **7 pts**

Find $f^{-1}(x)$ for $f(x) = \dfrac{4x + 1}{3}$, and verify that $f(f^{-1}(x)) = x$.

12. **7 pts**

Let $f(x) = (x - 2)^2 + 1$ with domain $x \geq 2$. Find (a) $f^{-1}(x)$ and (b) the domain of f^{-1}.

13. **7 pts**

On a flight to visit family, a plane's distance from its destination is $d = 1500 - 450t$ miles, t hours after it reaches cruising speed. (a) Solve for t to write time as a function of distance. (b) How long after reaching cruising speed are 600 miles left?

14. **6 pts**

Let $f(x) = x^3 - x$. (a) Find the average rate of change of f from $x = -1$ to $x = 2$. (b) Is f even, odd or neither? Show the test.

15. **6 pts**

Find and simplify the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for $f(x) = 3x^2 - x$.

16. **5 pts**

Prove that if f and g are both odd functions, then their product $p(x) = f(x)\,g(x)$ is an even function.

ANSWER KEY

1. (a) $16 - 12 - 2 = 2$; (b) $a^2 - 2a + 1 + 3a - 3 - 2 = a^2 + a - 4$
2. $2x + 8 \geq 0$ gives $x \geq -4$, and $x \neq 3$: $[-4, 3) \cup (3, \infty)$
3. (a) $[-4, 5]$; (b) $[-2, 4]$; (c) $(-3, 5]$
4. (a) $(4 + 3,\ -2(-1) + 1) = (7, 3)$; (b) $y = -\sqrt{x - 2} + 6$
5. Vertex $(3, 2)$: $y = a|x - 3| + 2$. Through $(7, 0)$: $0 = 4a + 2$, so $a = -\frac{1}{2}$. $y = -\frac{1}{2}|x - 3| + 2$
6. (a) $C(w) = \begin{cases} 1.5w, & 0 \leq w \leq 10 \\ 15 + 1.2(w - 10) = 1.2w + 3, & w > 10 \end{cases}$ (b) $1.2(16) + 3 = \$22.20$
7. (a) $\lfloor 3.5 \rfloor = 3$ and $\lfloor -1.5 \rfloor = -2$; (b) $[-2, -1)$
8. (a) $(x + 3)^2 - 2(x + 3) = x^2 + 4x + 3$; (b) $x^2 - 2x + 3$
9. $f(g(x)) = \dfrac{1}{\sqrt{x} - 4}$. Need $x \geq 0$ for g and $\sqrt{x} \neq 4$, so $x \neq 16$: $[0, 16) \cup (16, \infty)$
10. (a) $1.4x + 20$; (b) $1.4(150) + 20 = \$230$; (c) $1.4(x + 20) = 1.4x + 28$: the delivery charge would be marked up too, from \$20 to \$28
11. $x = \dfrac{4y + 1}{3}$, $3x - 1 = 4y$, so $f^{-1}(x) = \dfrac{3x - 1}{4}$. Check: $f\left(\dfrac{3x - 1}{4}\right) = \dfrac{(3x - 1) + 1}{3} = x$ ✓
12. (a) $x = (y - 2)^2 + 1$, $y - 2 = \sqrt{x - 1}$ (positive, since $y \geq 2$), so $f^{-1}(x) = 2 + \sqrt{x - 1}$; (b) the range of f: $[1, \infty)$
13. (a) $t = \dfrac{1500 - d}{450}$; (b) $t = \dfrac{900}{450} = 2$ hours
14. (a) $f(-1) = 0$ and $f(2) = 6$, so $\dfrac{6 - 0}{2 - (-1)} = 2$; (b) $f(-x) = -x^3 + x = -f(x)$: odd
15. $f(x + h) - f(x) = 6xh + 3h^2 - h$, so the quotient is $6x + 3h - 1$
16. $p(-x) = f(-x)\,g(-x) = (-f(x))(-g(x)) = f(x)\,g(x) = p(x)$, so p is even.

UNIT 2

Linear Systems and Linear Programming

Many real decisions involve several unknowns and several conditions at once: how many tickets of each kind were sold, how much of each grain goes into a feed mix, how many acres of each crop to plant. This unit begins with a quick review of systems of two linear equations from Algebra I, with attention to why substitution and elimination work and how the algebra reveals one solution, no solution or infinitely many.

Next you graph systems of linear inequalities and find the corners of their solution regions, and then use them for linear programming: choosing the best plan when land, money and time are limited. You will see why the best value of a linear objective always occurs at a corner, an idea that grew into one of the most widely used tools of modern planning. The unit ends with systems of three equations in three variables, including how to find the parabola through three points.

These skills come back often: three-variable systems are used to build quadratic models in Unit 3, and in Unit 12 you will solve the same systems with matrices, using row operations that are exactly the elimination steps you learn here.

PREREQUISITES

- Solve linear equations in one variable, including equations with fractions.
- Graph a line from slope-intercept or standard form, and know that parallel lines have equal slopes.
- Solve a system of two linear equations by graphing, substitution or elimination (Algebra I).
- Graph a linear inequality in two variables with a solid or dashed boundary (Algebra I).
- Evaluate functions and expressions such as $y = ax^2 + bx + c$ at given values (Lessons 1.1–1.2).

MASTERY CHECKLIST

- ☐ I can solve a system of two linear equations by graphing, substitution or elimination, and choose an efficient method.
- ☐ I can explain why elimination works and tell from the algebra whether a system has one, no or infinitely many solutions.
- ☐ I can write and solve a two-variable system for a ticket, mixture, rate or cost problem.
- ☐ I can graph the solution region of a system of linear inequalities using solid or dashed boundaries and test points.
- ☐ I can find the vertices of a solution region and tell whether the region is bounded or unbounded.
- ☐ I can write constraints and an objective function from a word problem.
- ☐ I can explain why a linear objective function reaches its maximum or minimum at a vertex, and use the corner-point method.
- ☐ I can recognize when an unbounded region has no maximum and when two corners tie.
- ☐ I can solve a system of three linear equations in three variables by elimination and check the solution in all three equations.
- ☐ I can find the parabola

$y =$ through three points and solve
ax^2 three-variable word problems.
$bx +$
c

LESSONS IN THIS UNIT

2.1	**Systems of Two Linear Equations: A Review**	45 minutes
2.2	**Systems of Linear Inequalities**	2 sessions of 45 minutes
2.3	**Linear Programming**	2 sessions of 45 minutes
2.4	**Systems of Three Equations in Three Variables**	2 sessions of 45 minutes

Unit 2 test

You have 60 minutes. A scientific calculator is allowed, but no graphing calculator. Show your work on every problem: write your variables, your equations or inequalities, and each step on its own line. Check solutions of systems in the original equations. Give exact answers unless a problem says how to round, and include units in answers to word problems.

1. **5 pts**
   Solve by substitution: $x = 2y + 1$ and $3x - 4y = 9$.

2. **6 pts**
   Solve by elimination: $5x + 2y = 4$ and $3x - 5y = 21$.

3. **6 pts**
   Without graphing, decide how many solutions each system has. (a) $4x - 6y = 8$ and $-2x + 3y = -4$. (b) $3x + y = 2$ and $6x + 2y = 5$.

4. **7 pts**
   A hardware store mixes grass seed that sells for $6 per pound with seed that sells for $10 per pound to make 40 pounds of a blend that sells for $7 per pound. How many pounds of each kind go into the blend?

5. **5 pts**

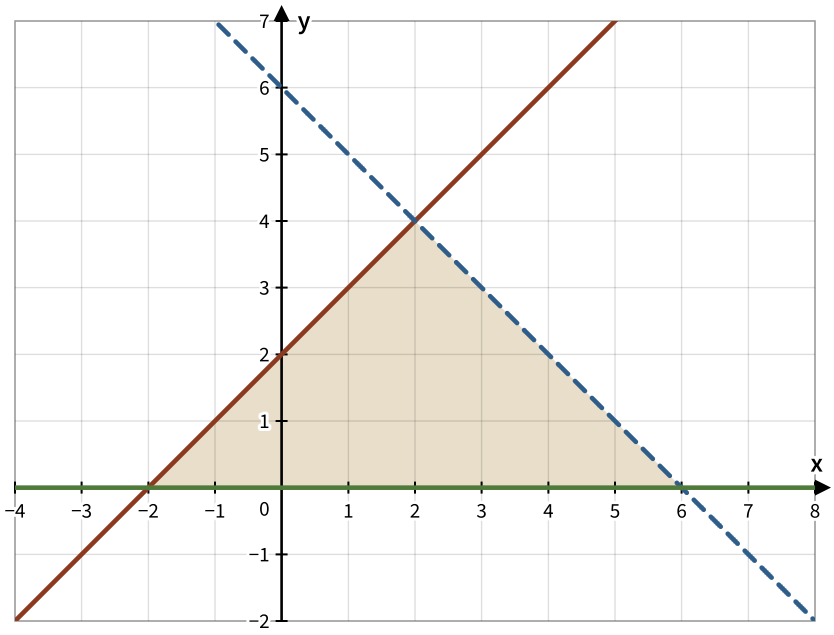
Is $(1, 2)$ a solution of the system $y \geq 3x - 2$ and $2x + y < 4$? Explain.

6. **8 pts**

Graph the system $x \geq 0$, $y \geq 2$, $x + 2y \leq 12$, $2x + y \leq 12$, and find all the vertices of the region.

7. **7 pts**

Write the system of three inequalities whose solution region is shaded. Read the boundary lines from the grid and note which one is dashed.



8. **10 pts**

Henry's woodshop builds chairs and benches. A chair needs 1 hour of cutting and 2 hours of finishing; a bench needs 2 hours of cutting and 1 hour of finishing. Each week there are at most 16 hours for cutting and 14 hours for finishing. The profit is \$30 per chair and \$50 per bench. How many chairs and benches should be built each week to make the most profit, and what is that profit?

9. **8 pts**

   Find the minimum value of $C = 3x + 2y$ subject to $x \geq 0$, $y \geq 0$, $x + y \geq 5$, $2x + y \geq 8$, and the point where it occurs. Does C have a maximum on this region?

10. **6 pts**

    A classmate says the maximum of $P = 2x + 3y$ over a feasible region might be at a point in the middle of the region, away from every edge. Explain why this cannot happen.

11. **8 pts**

    Solve: $x + 2y + z = 0$, $2x - y + 3z = 11$, $x + y - 2z = -1$.

12. **5 pts**

    How many solutions does the system $x + 2y - z = 3$, $2x + 4y - 2z = 6$, $x - y + z = 0$ have? Explain briefly.

13. **9 pts**

    Find the equation of the parabola $y = ax^2 + bx + c$ that passes through $(-1, 2)$, $(1, 6)$ and $(3, 2)$.

14. **10 pts**

    The Miller family buys 20 young trees for a new orchard: apple trees at \$30 each, pear trees at \$35 each and cherry trees at \$40 each. They spend \$690 and buy twice as many apple trees as cherry trees. How many of each kind do they buy?

ANSWER KEY

1. $3(2y+1)-4y=9$, so $2y=6$, $y=3$ and $x=7$: $(7,3)$.
2. Multiply by 5 and by 2: $25x+10y=20$ and $6x-10y=42$; adding gives $31x=62$, so $x=2$ and $y=-3$: $(2,-3)$.
3. (a) Multiplying the second equation by -2 gives the first, so adding gives $0=0$: infinitely many solutions. (b) Doubling the first gives $6x+2y=4$, and subtracting gives $0=1$: no solution.
4. $x+y=40$ and $6x+10y=280$; $6x+10(40-x)=280$ gives $4x=120$. 30 pounds of the \$6 seed and 10 pounds of the \$10 seed.
5. No: $2\geq 1$ is true, but $2(1)+2=4<4$ is false (the point is on the dashed boundary).
6. $(0,2)$, $(5,2)$, $(4,4)$, $(0,6)$. For $(4,4)$: subtracting $x+2y=12$ from $2(2x+y)=24$ gives $3x=12$.
7. $y\leq x+2$, $x+y<6$, $y\geq 0$.
8. Constraints $x+2y\leq 16$, $2x+y\leq 14$, $x,y\geq 0$; vertices $(0,0)$, $(7,0)$, $(4,6)$, $(0,8)$ give \$0, \$210, \$420, \$400. Build 4 chairs and 6 benches for \$420.
9. Vertices $(0,8)$, $(3,2)$, $(5,0)$ give $C=16,13,15$; the minimum is 13 at $(3,2)$. There is no maximum, because the region is unbounded and C grows without limit.
10. At a point in the middle of the region there is room to move a little in every direction and stay in the region. Moving a little to the right (or up) increases $2x+3y$, so that point cannot give the largest value. The maximum must be on the boundary, and by sliding the parallel level lines $2x+3y=c$ outward, the last point they touch is a corner.
11. (2) − 2 × (1): $-5y+z=11$; (1) − (3): $y+3z=1$. Then $y=1-3z$ gives $16z=16$, so $z=1$, $y=-2$, $x=3$: $(3,-2,1)$.
12. The second equation is twice the first, so it is the same plane; subtracting gives $0=0$. That plane meets the third plane in a line, so there are infinitely many solutions.
13. $a-b+c=2$, $a+b+c=6$, $9a+3b+c=2$. Subtracting the first two gives $b=2$; then $a+c=4$ and $9a+c=-4$, so $a=-1$, $c=5$: $y=-x^2+2x+5$.
14. $a+p+c=20$, $30a+35p+40c=690$, $a=2c$. Then $p=20-3c$ and $60c+700-105c+40c=690$, so $c=2$: 4 apple, 14 pear and 2 cherry trees. Check: $120+490+80=690$ ✓.

2.1 Systems of Two Linear Equations: A Review

OBJECTIVES

- Solve a system of two linear equations by graphing, substitution or elimination, and choose the most efficient method.
- Explain why substitution and elimination produce the solution of a system.
- Recognize from the algebra whether a system has one solution, no solution or infinitely many solutions, and describe what each looks like on a graph.
- Write and solve a system of two equations from a word problem.

WATCH

Review Questions (Simultaneous Equations)

Eddie Woo · 9:08

https://www.youtube.com/watch?v=KNXnY1EDKV0

> **WHEN TO WATCH**
> Before Example 1: solving two equations by substitution and by elimination.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations.

BACKUP VIDEO

Consistent Independent, Dependent and Inconsistent

Mario's Math Tutoring · 5:16

https://www.youtube.com/watch?v=K5MmwciV1BA

After Example 1: one solution, no solution or infinitely many solutions.

KEY TERMS

system of linear equations

Two or more linear equations in the same variables that must be true at the same time.

solution of a system

An ordered pair (x, y) that makes every equation in the system true. On a graph it is a point that lies on every line.

consistent system

A system with at least one solution (exactly one, or infinitely many).

inconsistent system

A system with no solution; for two lines this means the lines are parallel.

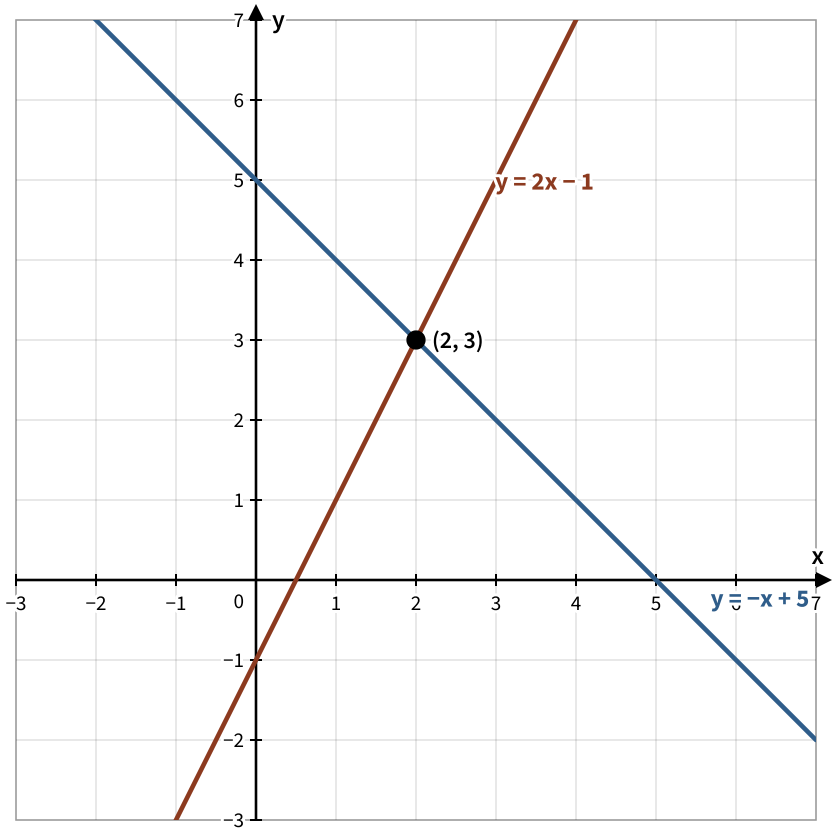
dependent system

A system whose equations describe the same line, so it has infinitely many solutions.

CORE IDEAS

You solved systems of two linear equations in Algebra I. A **solution** of a system is an ordered pair that makes both equations true at once. Each equation's graph is a line made of all the points that satisfy it, so a point that satisfies both must lie on both lines: the solution is where the lines meet. For example, the lines $y = 2x - 1$ and $y = -x + 5$ cross at $(2, 3)$, and indeed $2(2) - 1 = 3$ and $-2 + 5 = 3$.

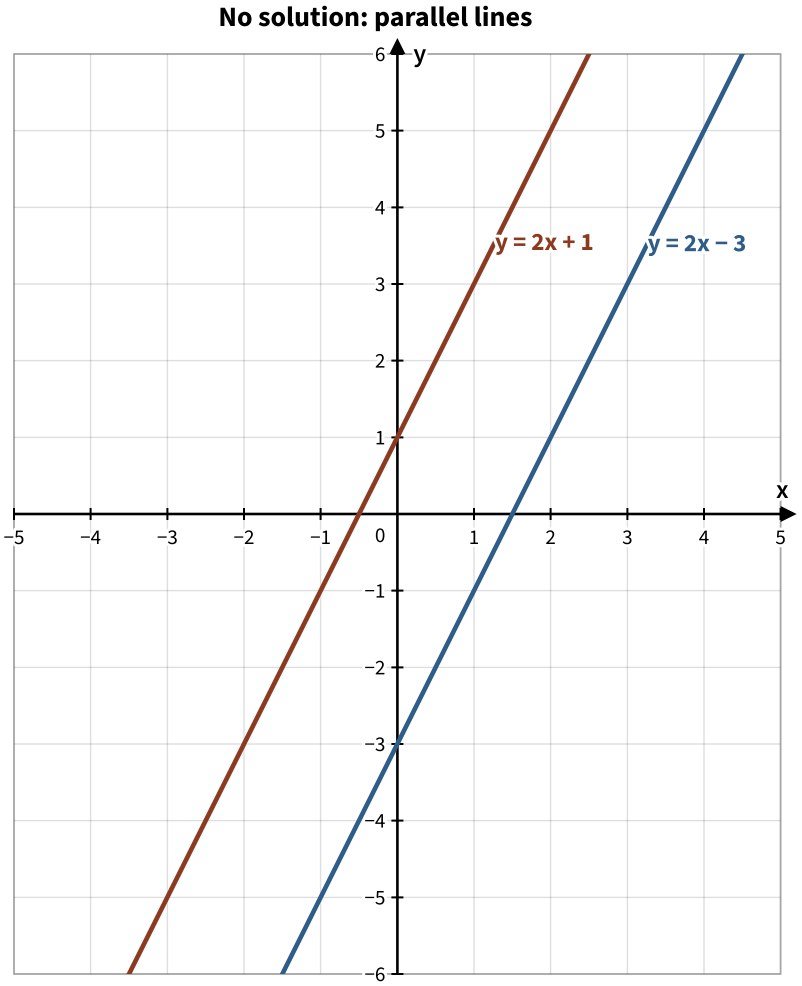
One solution: the lines cross



Substitution works because at the solution, the two sides of an equation such as $y = 2x - 1$ are the same number. So wherever y appears in the other equation, you may replace it with $2x - 1$. That leaves one equation in one variable, which you already know how to solve. Then substitute the value you found back into either equation to get the other variable.

Elimination works because if $A = B$ and $C = D$ are both true, then $A + C = B + D$ is also true: you are adding equal amounts to both sides. Multiplying an equation by a nonzero number does not change which points satisfy it, so you may first multiply one or both equations so that the coefficients of x (or of y) are opposites. Adding the equations then eliminates that variable.

Sometimes both variables disappear at once. That happens exactly when the left sides are multiples of each other, which means the two lines have the same slope. If what remains is false, such as $0 = 7$, no point can satisfy both equations: the lines are parallel and the system has **no solution**. If what remains is always true, such as $0 = 0$, the two equations describe the same line and every point on it is a solution: **infinitely many solutions**.



Choosing a method: graphing gives a picture and a good estimate, but it is exact only when the crossing point lands on grid lines. Substitution is quickest when one

variable already has a coefficient of 1 or -1 (as in $y = 3x - 2$ or $x - y = 3$). Elimination is quickest when the equations are in standard form $Ax + By = C$. Whatever method you use, check your pair in **both original equations.**

KEY FORMULAS

$$\text{different slopes} \;\Rightarrow\; \text{exactly one solution (lines cross)}$$

$$\text{same slope, different intercepts} \;\Rightarrow\; \text{no solution (parallel lines; algebra gives a false statement such as } 0 = 7)$$

$$\text{same slope, same intercept} \;\Rightarrow\; \text{infinitely many solutions (same line; algebra gives } 0 = 0)$$

Practice habit. Check every solution in both ORIGINAL equations, not in an equation you rewrote along the way. A mistake made while rewriting will pass a check in the rewritten equation but fail in the original.

WORKED EXAMPLES

Example 1. Solve by elimination: $3x + 2y = 7$ and $4x - 3y = 15$.

Show solution

1. Choose to eliminate y. The coefficients are 2 and -3; make them 6 and -6.
2. Multiply the first equation by 3: $9x + 6y = 21$.
3. Multiply the second equation by 2: $8x - 6y = 30$.
4. Add the two new equations: $17x = 51$, so $x = 3$.
5. Substitute into the first equation: $3(3) + 2y = 7$, so $2y = -2$ and $y = -1$.
6. Check in the first original equation: $3(3) + 2(-1) = 9 - 2 = 7$ ✓.
7. Check in the second original equation: $4(3) - 3(-1) = 12 + 3 = 15$ ✓.

Answer: $(3, -1)$

Example 2. The church choir sold 240 tickets to its spring concert. Adult tickets cost \$8 and student tickets cost \$5, and the choir took in \$1,650. How many of each kind of ticket were sold?

Show solution

1. Let a be the number of adult tickets and s the number of student tickets.
2. Count of tickets: $a + s = 240$.
3. Money collected: $8a + 5s = 1650$.
4. The first equation gives $s = 240 - a$, so substitution is quick.
5. Substitute: $8a + 5(240 - a) = 1650$, so $8a + 1200 - 5a = 1650$.
6. Simplify: $3a = 450$, so $a = 150$.
7. Then $s = 240 - 150 = 90$.
8. Check: $150 + 90 = 240$ ✓ and $8(150) + 5(90) = 1200 + 450 = 1650$ ✓.

Answer: 150 adult tickets and 90 student tickets

PRACTICE

1. Easy Is $(2, -1)$ a solution of the system $3x + y = 5$ and $x - 2y = 4$? Show your check.

   Show answer

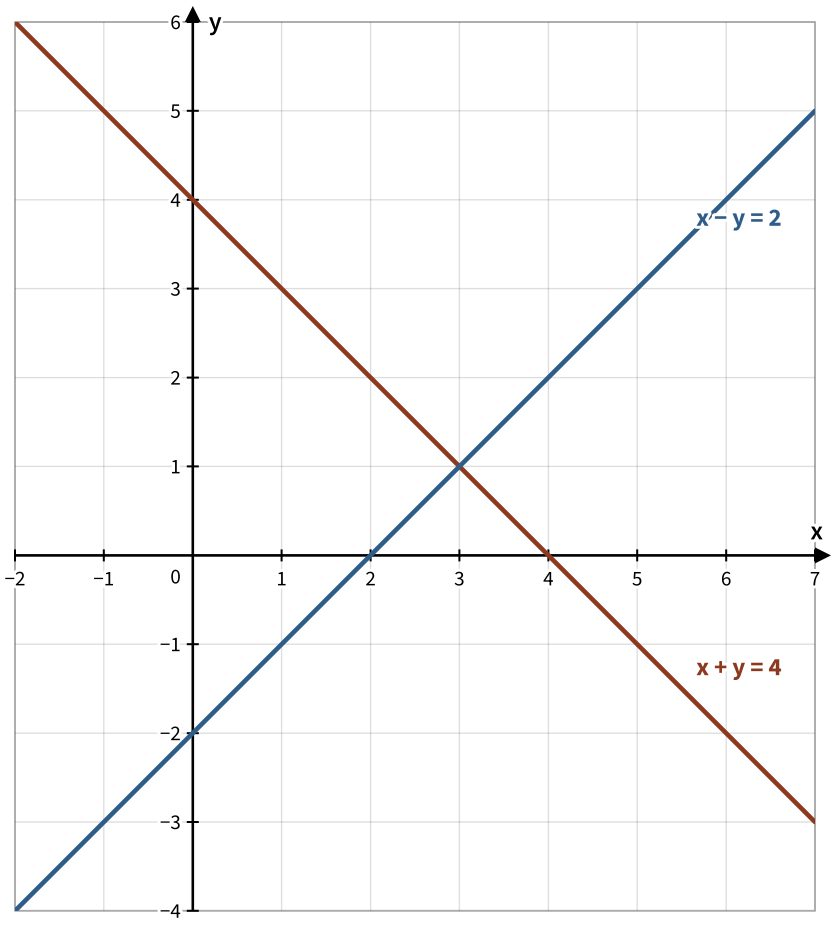
   Yes: $3(2) + (-1) = 5$ ✓ and $2 - 2(-1) = 4$ ✓.

2. Easy Solve by substitution: $y = 3x - 2$ and $2x + y = 13$.

   Show answer

   $2x + (3x - 2) = 13$, so $5x = 15$, $x = 3$ and $y = 7$: $(3, 7)$.

3. Easy The graph shows the lines $x + y = 4$ and $x - y = 2$. Read the solution of the system from the graph, then check it in both equations.



Show answer

$(3, 1)$; check: $3 + 1 = 4$ ✓ and $3 - 1 = 2$ ✓.

4. Easy Solve by elimination: $2x + 3y = 1$ and $2x - y = 13$.

Show answer

Subtract: $4y = -12$, so $y = -3$; then $2x + 3 = 13$, so $x = 5$: $(5, -3)$.

5. Medium Solve by elimination: $4x + 3y = 10$ and $5x - 2y = 24$.

Show answer

Multiply by 2 and by 3: $8x + 6y = 20$ and $15x - 6y = 72$; add: $23x = 92$, so $x = 4$; then $16 + 3y = 10$, so $y = -2$: $(4, -2)$.

6. Medium Use algebra to decide how many solutions the system $6x - 4y = 10$ and $3x - 2y = 5$ has. Describe the graph.

Show answer

Multiply the second equation by -2: $-6x + 4y = -10$; adding gives $0 = 0$. Infinitely many solutions: both equations describe the same line.

7. Medium Use algebra to decide how many solutions the system $-2x + 6y = 7$ and $x - 3y = 4$ has. Describe the graph.

Show answer

Multiply the second equation by 2: $2x - 6y = 8$; adding gives $0 = 15$, which is false. No solution: the lines are parallel.

8. Medium Solve: $\frac{x}{2} + \frac{y}{3} = 4$ and $x - y = 3$.

Show answer

Multiply the first equation by 6: $3x + 2y = 24$. With $x = y + 3$: $3y + 9 + 2y = 24$, so $y = 3$ and $x = 6$: $(6, 3)$. Check: $3 + 1 = 4$ ✓.

9. Medium Grace is making 30 pounds of trail mix for a church youth retreat. She mixes nuts that cost \$4 per pound with dried fruit that costs \$7 per pound, and she wants the mix to cost \$5 per pound. How many pounds of nuts and how many pounds of dried fruit should she use?

Show answer

$n + f = 30$ and $4n + 7f = 150$; $4n + 7(30 - n) = 150$ gives $3n = 60$. So 20 pounds of nuts and 10 pounds of dried fruit.

10. Hard A boat travels 36 miles downstream in 2 hours. The return trip upstream takes 3 hours. Find the speed of the boat in still water and the speed of the current, in miles per hour.

Show answer

Downstream: $b + c = 18$; upstream: $b - c = 12$. Adding gives $2b = 30$. The boat goes 15 mph and the current is 3 mph.

11. Hard For what value of k does the system $kx + 4y = 8$ and $3x + 2y = 5$ have no solution?

Show answer

The lines must have the same slope: $-\frac{k}{4} = -\frac{3}{2}$, so $k = 6$. Then the first equation is $3x + 2y = 4$, which is parallel to $3x + 2y = 5$, so there is no solution: $k = 6$.

12. Hard Henry needs to rent a garden tiller. Store A charges \$25 plus \$8 per hour. Store B charges \$10 plus \$11 per hour. After how many hours do the two stores charge the same amount, and what is that amount?

Show answer

$25 + 8h = 10 + 11h$ gives $15 = 3h$, so $h = 5$. Both charge \$65 for 5 hours.

13. Challenge Explain why, when elimination turns a system of two linear equations into a false statement such as $0 = 7$, the two lines must be parallel.

Show answer

Both variables disappeared, so after the multiplying, the left sides were exact opposites; that means the original equations had proportional x- and y-coefficients, so the lines have the same slope. The false statement $0 = 7$ shows no pair (x, y) satisfies both, so the lines share no point. Two lines with the same slope and no common point are parallel.

14. Challenge The point $(3, 2)$ is a solution of both $ax + by = 16$ and $ax - by = 8$. Find a and b.

Show answer

$3a + 2b = 16$ and $3a - 2b = 8$; adding gives $6a = 24$, so $a = 4$, and then $2b = 4$, so $b = 2$.

COMMON MISTAKES

- Multiplying only the x- and y-terms of an equation and forgetting the constant. Multiply every term on both sides: $2(4x + 3y = 10)$ is $8x + 6y = 20$.
- Stopping after finding one variable. A solution of a system is an ordered pair; substitute back to find the other coordinate.
- Reading $0 = 0$ as "$x = 0$" or "no solution." A true statement with no variables means every point on the line works: infinitely many solutions.

- Sign errors when subtracting equations. If you subtract, put the whole second equation in parentheses, or multiply it by -1 and add instead.

> **Make it a game · Method Matchmaker.** The parent writes three systems, such as y = 2x - 1 with 3x + y = 9, and 2x + 3y = 12 with 4x - 3y = 6. For each, name the most efficient method in five seconds: graphing, substitution or elimination. Then solve it. Include one inconsistent and one dependent system, and say how you can tell. Score two points per match.

2.2 Systems of Linear Inequalities

OBJECTIVES

- Graph the solution region of a system of two to four linear inequalities, using solid or dashed boundaries and test points.
- Explain why a single test point decides which side of a boundary line to shade.
- Find the vertices of a solution region by solving pairs of boundary equations, and tell whether the region is bounded or unbounded.
- Write a system of inequalities from constraints described in words.

WATCH

Combining Regions (1 of 4: From Lines to Planes)

Eddie Woo · 9:55

https://www.youtube.com/watch?v=VOObcBQdJd4

> **WHEN TO WATCH**
> Before Example 1: from a boundary line to a shaded half-plane.

Note: Eddie Woo teaches in Australia: he says 'region' for the solution set of an inequality.

BACKUP VIDEO

Graphing Systems of Linear Inequalities (Made Easy!)

Mario's Math Tutoring · 10:40

https://www.youtube.com/watch?v=htscKOfe7yk

Before Example 2: graphing a system of inequalities and its overlap.

KEY TERMS

boundary line

The line you get by replacing the inequality sign with an equals sign. It separates the points that satisfy the inequality from those that do not.

half-plane

One of the two regions a line divides the plane into. The solution of one linear inequality is a half-plane, with or without its boundary line.

test point

A point not on the boundary line that you substitute into the inequality to find out which half-plane is the solution.

solution region

The set of all points that satisfy every inequality in a system; on a graph it is the overlap of the half-planes.

vertex (corner point)

A point where two boundary lines meet on the edge of the solution region.

bounded region

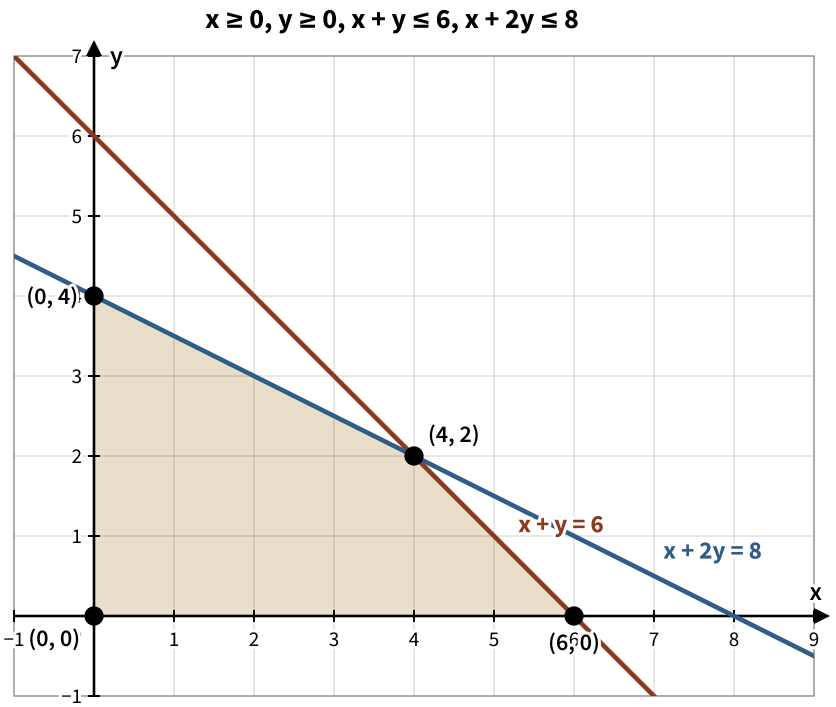
A region that can be enclosed in a circle; a region that goes on forever in some direction is unbounded.

CORE IDEAS

Recall from Algebra I how to graph one linear inequality. First draw the **boundary line**. Draw it solid for $\leq$ or $\geq$ (points on the line are solutions) and dashed for $<$ or $>$ (points on the line are not). The line splits the plane into two **half-planes**, and the solutions fill exactly one of them.

Why can one test point decide the whole side? Write a non-vertical boundary as $y = mx + b$. Pick any x. The point on the line above that x has y-coordinate exactly $mx + b$; every point straight above it has a larger y, and every point straight below has a smaller y. So the points above the line are exactly the points with $y > mx + b$, and the points below are exactly those with $y < mx + b$. All the points on one side behave alike, so testing one point (often $(0, 0)$, if it is not on the line) tells you about all of them.

A **system** of linear inequalities asks for the points that satisfy every inequality at once. Graph each inequality on the same axes; the **solution region** is where all the shadings overlap. For example, the system $x \geq 0$, $y \geq 0$, $x + y \leq 6$ and $x + 2y \leq 8$ has the four-sided region shown below.



The **vertices** (corner points) of a region are where two boundary lines meet. You find each one exactly by solving the two boundary equations as a system, as in Lesson 2.1. Above, $x + y = 6$ and $x + 2y = 8$ give $y = 2$ (subtract the first from the second) and then $x = 4$, so $(4, 2)$ is a vertex. Be careful: not every crossing of two boundary lines is a corner. The lines $x + 2y = 8$ and $y = 0$ meet at $(8, 0)$, but $8 + 0 > 6$, so that point breaks the inequality $x + y \leq 6$ and lies outside the region. A crossing is a vertex only if it satisfies **all** the inequalities.

A region is **bounded** if it is closed in on every side, like the one above, and **unbounded** if it runs off forever in some direction. For example, $x \geq 0$, $y \geq 0$, $x + y \geq 4$ is unbounded: it contains $(100, 100)$ and points even farther out. In the next lesson this difference will matter.

Real limits on what you can do are called **constraints**, and they translate directly into inequalities. "At most" and "no more than" mean $\leq$; "at least" and "no fewer than" mean $\geq$. Quantities such as acres, hours or numbers of loaves cannot be negative, so you usually include $x \geq 0$ and $y \geq 0$. Always say what each variable stands for, with units.

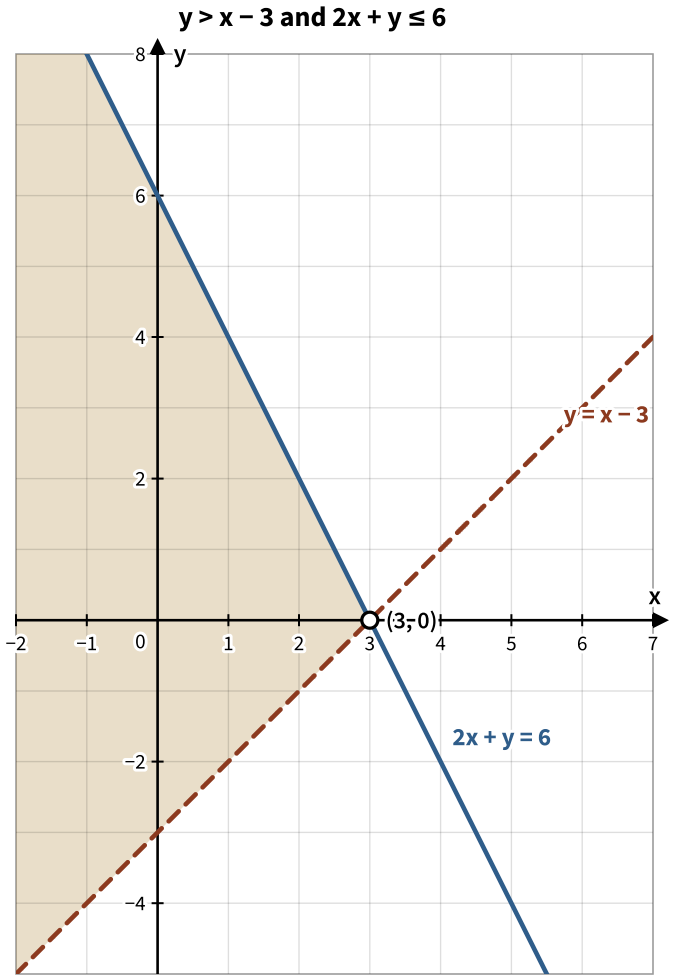
KEY FORMULAS

$y > mx + b$ or $y \geq mx + b$: shade above the line $\qquad$ $y < mx + b$ or $y \leq mx + b$: shade below the line

$\leq,\ \geq$: solid boundary $\qquad$ $<,\ >$: dashed boundary

Practice habit. Sketch before you compute. A quick sketch of the boundary lines shows which pairs of lines actually meet at corners of the region, so you solve only the systems you need and catch crossings that lie outside the region.

Example 1. Graph the system $y > x - 3$ and $2x + y \leq 6$. Find the corner point of the region and decide whether the corner belongs to the solution set.



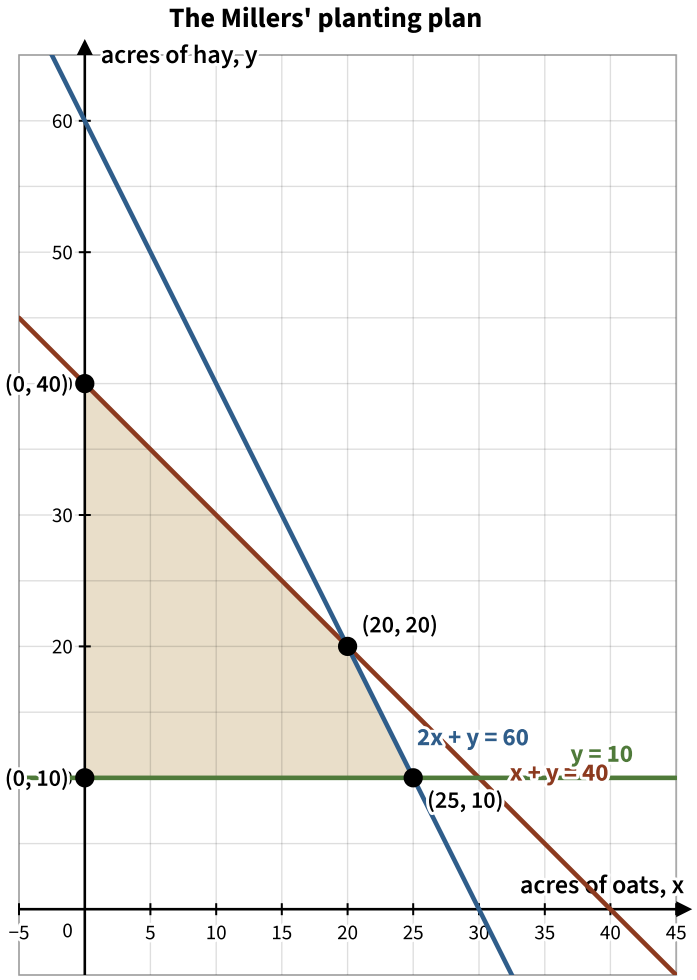
Show solution

1. First boundary: $y = x - 3$, with slope 1 and y-intercept -3. The sign is $>$, so draw it dashed and shade above it.
2. Second boundary: $2x + y = 6$, or $y = -2x + 6$. The sign is $\leq$, so draw it solid.
3. Test $(0, 0)$ in the second inequality: $2(0) + 0 = 0 \leq 6$ is true, so shade the side containing the origin (below the line).
4. The solution region is the overlap: above the dashed line and below the solid line, opening to the left. It is unbounded.
5. Corner: solve $x - 3 = -2x + 6$, so $3x = 9$, $x = 3$ and $y = 0$.

6. Check: $3 - 3 = 0$ ✓ and $2(3) + 0 = 6$ ✓, so $(3, 0)$ is on both lines.
7. Since $(3, 0)$ lies on the dashed line, $0 > 3 - 3$ is false, so the corner is not a solution. It is drawn as an open dot.

Answer: The corner is $(3, 0)$; it is not in the solution set because it lies on the dashed boundary.

Example 2. The Miller family has at most 40 acres to plant in oats and hay. Each acre of oats needs 2 hours of work per week and each acre of hay needs 1 hour, and the family can give at most 60 hours per week. They want at least 10 acres of hay for their cattle. Write a system of inequalities, graph it, and find the vertices of the region.



Show solution

1. Let x be the acres of oats and y the acres of hay.
2. Land: $x + y \leq 40$. Work: $2x + y \leq 60$. Hay: $y \geq 10$. Acres cannot be negative: $x \geq 0$.
3. Corner of $x = 0$ and $y = 10$: $(0, 10)$.
4. Corner of $y = 10$ and $2x + y = 60$: $2x = 50$, so $(25, 10)$. Check land: $25 + 10 = 35 \leq 40$ ✓.

5. Corner of $x + y = 40$ and $2x + y = 60$: subtract to get $x = 20$, then $y = 20$, so $(20, 20)$.
6. Corner of $x = 0$ and $x + y = 40$: $(0, 40)$. Check work: $0 + 40 = 40 \leq 60$ ✓.
7. Crossings that are NOT corners: $x + y = 40$ meets $y = 10$ at $(30, 10)$, but $2(30) + 10 = 70 > 60$; $2x + y = 60$ meets $x = 0$ at $(0, 60)$, but $0 + 60 > 40$.
8. The region is the bounded four-sided figure shown.

Answer: $x + y \leq 40$, $2x + y \leq 60$, $y \geq 10$, $x \geq 0$; vertices $(0, 10)$, $(25, 10)$, $(20, 20)$, $(0, 40)$.

PRACTICE

1. Easy Is $(2, 1)$ a solution of the system $y \leq 2x - 1$ and $x + y > 2$?

Show answer

Yes: $1 \leq 3$ ✓ and $3 > 2$ ✓.

2. Easy Is $(-1, 4)$ a solution of the system $y > x + 3$ and $y \leq -x + 2$?

Show answer

No: $4 > 2$ is true, but $4 \leq 3$ is false.

3. Easy For the inequality $3x - 2y < 6$: (a) Should the boundary line be drawn solid? (b) Is the origin in the solution region?

Show answer

(a) No, dashed, because the sign is $<$. (b) Yes: $3(0) - 2(0) = 0 < 6$.

4. Easy Find the point where the boundary lines of $y \leq x + 1$ and $y \leq -x + 5$ meet.

Show answer

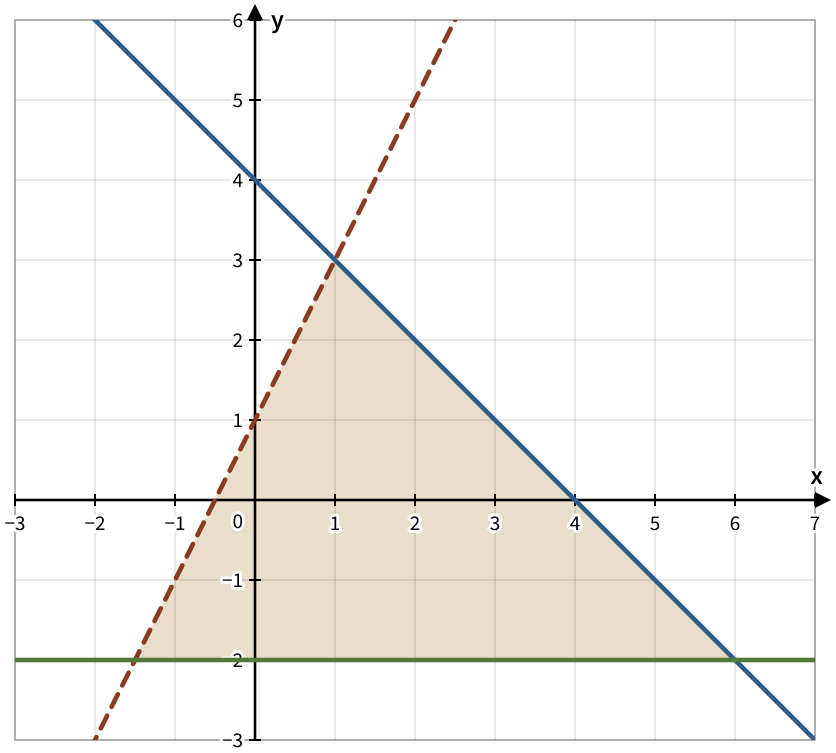
$x + 1 = -x + 5$, so $x = 2$ and $y = 3$: $(2, 3)$.

5. Medium Graph the system $x \geq 0$, $y \geq 0$, $2x + 3y \leq 12$ and list the vertices of the region.

Show answer

A triangle with vertices $(0, 0)$, $(6, 0)$ and $(0, 4)$.

6. **Medium** Write the system of three inequalities whose solution region is shaded. Read the boundary lines from the grid and note which one is dashed.



Show answer

$y < 2x + 1$, $y \le -x + 4$, $y \ge -2$.

7. **Medium** Find all the vertices of the region $x \ge 0$, $y \ge 0$, $x + y \le 10$, $3x + y \le 18$.

Show answer

$(0, 0)$, $(6, 0)$, $(4, 6)$, $(0, 10)$. For $(4, 6)$: subtracting $x + y = 10$ from $3x + y = 18$ gives $2x = 8$.

8. **Medium** For the system $x \ge 0$, $y \ge 0$, $x + y \ge 4$: (a) list the vertices; (b) is the region bounded?

Show answer

(a) $(0, 4)$ and $(4, 0)$. (b) No; the region extends forever up and to the right.

9. Medium Ruth has \$60 to spend on tomato plants at \$3 each and pepper plants at \$4 each for her garden. She wants at least 5 of each kind. Let x be the number of tomato plants and y the number of pepper plants. (a) Write a system of inequalities. (b) Can she buy 10 tomato plants and 8 pepper plants?

Show answer

(a) $3x + 4y \le 60$, $x \ge 5$, $y \ge 5$. (b) No: $3(10) + 4(8) = 62 > 60$.

10. Hard Henry feeds his goats a daily mix of x pounds of alfalfa and y pounds of grain. Each pound of alfalfa gives 2 units of protein and each pound of grain gives 3 units. The goats need at least 12 units of protein, and the total feed must be at most 10 pounds. Write the system (including $x \ge 0$, $y \ge 0$) and find the vertices of the region.

Show answer

$2x + 3y \ge 12$, $x + y \le 10$, $x \ge 0$, $y \ge 0$; vertices $(6, 0)$, $(10, 0)$, $(0, 10)$, $(0, 4)$. (The two slanted lines meet at $(18, -8)$, outside the region.)

11. Hard Find the three vertices of the triangle given by $y \le 2x + 2$, $y \le -x + 8$ and $y \ge \dfrac{1}{2}x - 1$.

Show answer

$2x + 2 = -x + 8$ gives $(2, 6)$; $2x + 2 = \frac{1}{2}x - 1$ gives $(-2, -2)$; $-x + 8 = \frac{1}{2}x - 1$ gives $(6, 2)$.

12. Hard Samuel mows lawns for \$15 per hour and tutors for \$20 per hour. He can work at most 15 hours a week and wants to earn at least \$240. Let x be the hours mowing and y the hours tutoring. (a) Write the system of inequalities. (b) Find the vertices of the region.

Show answer

(a) $x \ge 0$, $y \ge 0$, $x + y \le 15$, $15x + 20y \ge 240$. (b) $(0, 12)$, $(12, 3)$, $(0, 15)$. For $(12, 3)$: $15x + 20(15 - x) = 240$ gives $x = 12$.

13. **Challenge** Explain why the system $y > 2x + 3$ and $y < 2x - 1$ has no solution.

Show answer

The boundary lines both have slope 2, so they are parallel, and $y = 2x + 3$ is always 4 units above $y = 2x - 1$. A solution would need $2x + 3 < y < 2x - 1$, so $2x + 3 < 2x - 1$, which means $3 < -1$, which is false. No point is above the higher line and below the lower one.

14. **Challenge** Explain why every point above the line $y = mx + b$ satisfies $y > mx + b$, so that one test point is enough to decide which side to shade.

Show answer

Take a point (x_0, y_0) above the line. The point of the line with the same x-coordinate is $(x_0, mx_0 + b)$, and "above" means y_0 is larger: $y_0 > mx_0 + b$. The same reasoning shows every point below has $y_0 < mx_0 + b$. So all points on one side satisfy the same inequality, and testing any one of them tells you which side is the solution.

COMMON MISTAKES

- Using a solid line for $<$ or $>$. Decide solid or dashed from the sign before you draw: only $\leq$ and $\geq$ include the line.
- Shading "above" for $Ax + By \leq C$ without checking. When the inequality is not solved for y (or you divided by a negative), use a test point such as $(0, 0)$ instead of guessing.
- Listing every crossing of two boundary lines as a vertex. Check each crossing in all the other inequalities; if it fails one, it is outside the region.
- Forgetting $x \geq 0$ and $y \geq 0$ in word problems. Counts, acres and hours cannot be negative, so those constraints belong in the system.

> **Make it a game · Mystery Region.** The parent secretly draws a triangle on graph paper, such as one with corners (0, 0), (6, 0) and (0, 4). You may ask eight yes-or-no questions of the form 'Is (2, 1) inside?' Then write the three inequalities that describe the region, with correct boundaries and the test point that proves each. Swap roles and hide a region of your own.

2.3 Linear Programming

OBJECTIVES

- Identify the constraints, feasible region and objective function in a linear programming problem written in words.
- Explain, using parallel level lines, why a linear objective function reaches its maximum or minimum over a polygonal region at a vertex.
- Use the corner-point method to find the maximum or minimum value of an objective function and where it occurs.
- Recognize when an unbounded region has no maximum (or no minimum) and when two corners tie.

WATCH

Linear Programming

Mario's Math Tutoring · 8:10

https://www.youtube.com/watch?v=reKV1lRn_uw

> **WHEN TO WATCH**
> Before Example 1: the feasible region and testing its corners.

BACKUP VIDEO

Linear Programming (Optimization) 2 Examples Minimize & Maximize

Mario's Math Tutoring · 15:07

https://www.youtube.com/watch?v=Y7e7DCsDUMY

Before the practice set: two word problems, one maximizing and one minimizing.

KEY TERMS

linear programming

A method for finding the best (largest or smallest) value of a linear expression when the variables must satisfy a system of linear inequalities.

constraint

One of the inequalities that limits the choices, such as the land, money or hours available.

feasible region

The solution region of the system of constraints: every point in it is a choice that is actually possible.

objective function

The linear expression, such as profit $P = 150x + 100y$ or cost $C = 0.30x + 0.20y$, that you want to maximize or minimize.

level line

The set of points where the objective function equals a fixed number c, such as $3x + 4y = 24$. Every point on it gives the same value.

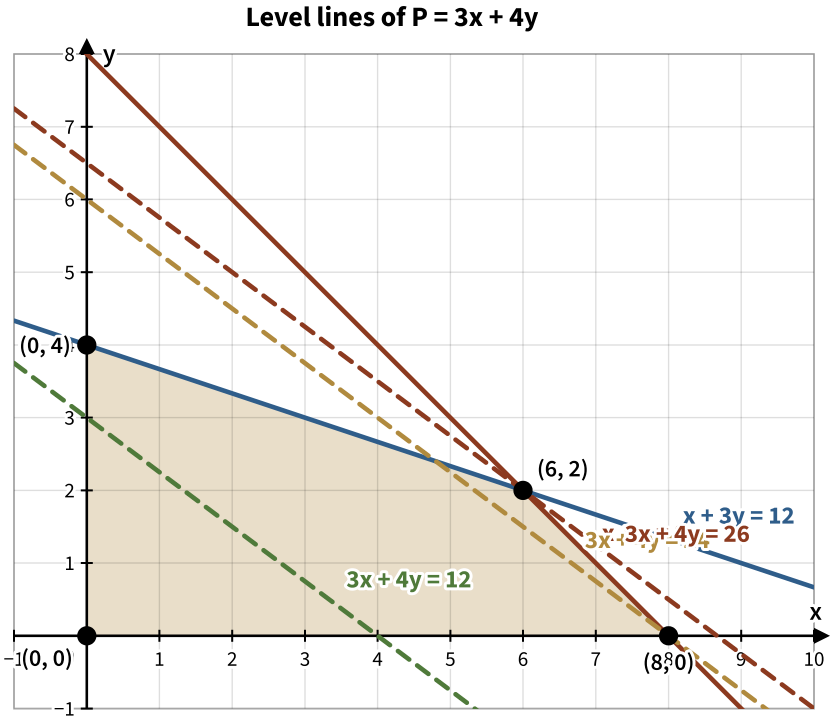
corner-point method

Find every vertex of the feasible region, evaluate the objective function at each, and pick the largest or smallest value.

CORE IDEAS

Farmers, bakers and builders often must decide how much of two things to make when land, money, time or materials are limited. In a **linear programming** problem, the limits are linear inequalities called **constraints**; their solution region (Lesson 2.2) is the **feasible region**, the set of all possible choices. The quantity to make as large or as small as possible is a linear **objective function** such as $P = 3x + 4y$.

A feasible region usually contains infinitely many points, so you cannot test them all. Here is the key fact that saves us: *if a linear objective function has a maximum or minimum over a polygonal feasible region, it occurs at a vertex.* To see why, look at the **level lines** $3x + 4y = c$. Each one is the set of points where P equals c. Solving for y gives $y = -\frac{3}{4}x + \frac{c}{4}$, so every level line has the same slope $-\frac{3}{4}$: they are all parallel, and a larger c just slides the line farther up and to the right.



In the picture the feasible region is $x \geq 0$, $y \geq 0$, $x + y \leq 8$, $x + 3y \leq 12$. The level line $3x + 4y = 12$ cuts through the region, so some feasible points give $P = 12$. Slide the line outward: $3x + 4y = 24$ still cuts through. Keep sliding, and the line is about to leave the region. Because the edges of the region are straight, the last point it touches is a corner, here $(6, 2)$, on the line $3x + 4y = 26$. Any larger c misses the region entirely, so the maximum is $P = 26$. The same argument, sliding the other way, shows the minimum is at a corner too (here $(0, 0)$, with $P = 0$).

This gives the **corner-point method**: (1) define the variables; (2) write the constraints and the objective function; (3) graph the feasible region; (4) find every vertex by solving pairs of boundary equations; (5) evaluate the objective at each vertex; (6) the largest value is the maximum and the smallest is the minimum. Answer the question in words, with units.

Two special cases. If the last level line lies along an edge of the region instead of touching a single corner, the two corners at the ends of that edge give the same value, and so does every point of the edge between them: there are many best choices. If the feasible region is **unbounded**, the level lines may be able to slide outward forever while still touching the region; then there is **no maximum**. A minimum can still exist, and when it does it is again at a corner. Cost problems with

"at least" constraints often look like this: you can always spend more, but there is a cheapest choice.

KEY FORMULAS

$$\text{objective } P = ax + by; \quad \text{level lines } ax + by = c \text{ all have slope } -\frac{a}{b}$$

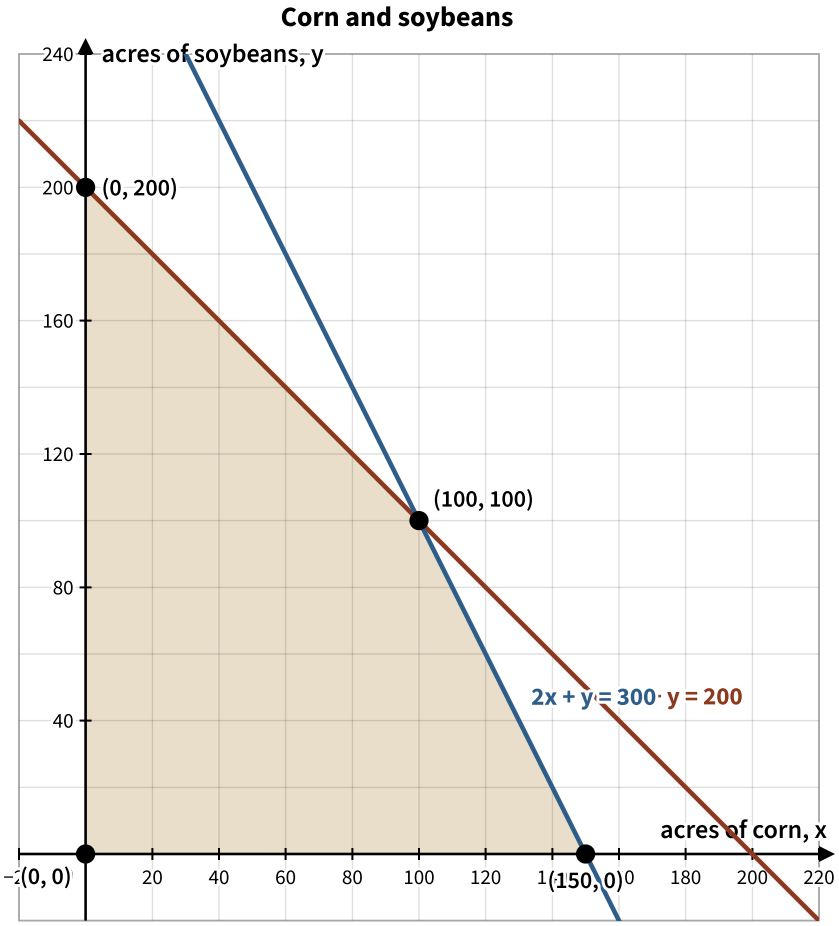
bounded region: the maximum and the minimum both occur at vertices

unbounded region: check whether the objective can grow (or shrink) without limit

From history. In 1939 the mathematician Leonid Kantorovich showed how to plan the best use of machines and materials with methods of this kind. In 1947 George Dantzig devised the simplex method, which travels from corner to corner of the feasible region, improving the objective at each step until no neighboring corner is better. Problems with thousands of variables, far too many to graph, are solved this way today.

Practice habit. Make a corner table. List every vertex in one column and the value of the objective function in the next, one row per vertex, so you can see the largest and smallest values at a glance and none gets skipped.

Example 1. Samuel has at most 200 acres to plant in corn and soybeans. Seed and fertilizer cost \$400 per acre of corn and \$200 per acre of soybeans, and he can spend at most \$60,000. He expects a profit of \$150 per acre of corn and \$100 per acre of soybeans. How many acres of each should he plant to make the most profit, and what is that profit?



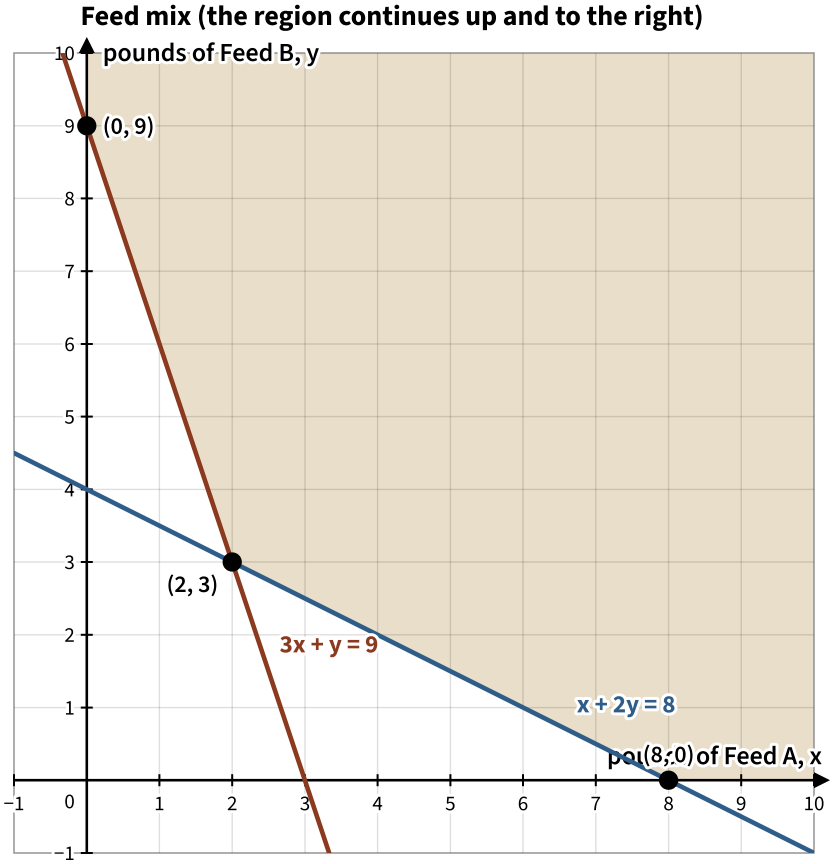
Show solution

1. Let x be the acres of corn and y the acres of soybeans.
2. Constraints: $x + y \le 200$; $400x + 200y \le 60000$, which divides by 200 to give $2x + y \le 300$; $x \ge 0$; $y \ge 0$.
3. Objective: maximize $P = 150x + 100y$.
4. Vertices: $(0, 0)$; $(150, 0)$ from $2x + y = 300$ with $y = 0$; $(0, 200)$ from $x + y = 200$ with $x = 0$.

5. The last vertex: subtract $x + y = 200$ from $2x + y = 300$ to get $x = 100$, so $y = 100$.
6. Corner table: $P(0, 0) = 0$; $P(150, 0) = 22500$; $P(100, 100) = 15000 + 10000 = 25000$; $P(0, 200) = 20000$.
7. The largest value is 25,000 at $(100, 100)$. Check the constraints: $100 + 100 = 200$ ✓ and $400(100) + 200(100) = 60000$ ✓.

Answer: 100 acres of corn and 100 acres of soybeans, for a profit of $25,000.

Example 2. A farmer mixes two feeds for a daily ration. Each pound of Feed A has 3 units of protein and 1 unit of fat and costs \$0.30. Each pound of Feed B has 1 unit of protein and 2 units of fat and costs \$0.20. The ration needs at least 9 units of protein and at least 8 units of fat. How many pounds of each feed give the cheapest ration, and what does it cost? Is there a most expensive ration?



Show solution

1. Let x be the pounds of Feed A and y the pounds of Feed B.
2. Constraints: protein $3x + y \geq 9$; fat $x + 2y \geq 8$; $x \geq 0$; $y \geq 0$.
3. Objective: minimize $C = 0.30x + 0.20y$.
4. Vertex on the y-axis: $3x + y = 9$ with $x = 0$ gives $(0, 9)$; check fat: $0 + 18 \geq 8$ ✓.
5. Vertex on the x-axis: $x + 2y = 8$ with $y = 0$ gives $(8, 0)$; check protein: $24 \geq 9$ ✓.
6. Middle vertex: from $y = 9 - 3x$, substitute into $x + 2y = 8$: $x + 18 - 6x = 8$, so $x = 2$ and $y = 3$.

7. Corner table: at $(0, 9)$ the cost is $\$1.80$; at $(2, 3)$ it is $\$0.60 + \$0.60 = \$1.20$; at $(8, 0)$ it is $\$2.40$.
8. The region is unbounded up and to the right; the level lines of C can slide outward forever, so there is no maximum cost, but the minimum is at the corner $(2, 3)$.

Answer: 2 pounds of Feed A and 3 pounds of Feed B, for $\$1.20$; there is no maximum cost because the feasible region is unbounded.

PRACTICE

1. Easy A feasible region has vertices $(0, 0)$, $(4, 0)$, $(3, 3)$ and $(0, 5)$. Find the maximum value of $P = 5x + 2y$ and the vertex where it occurs.

   Show answer

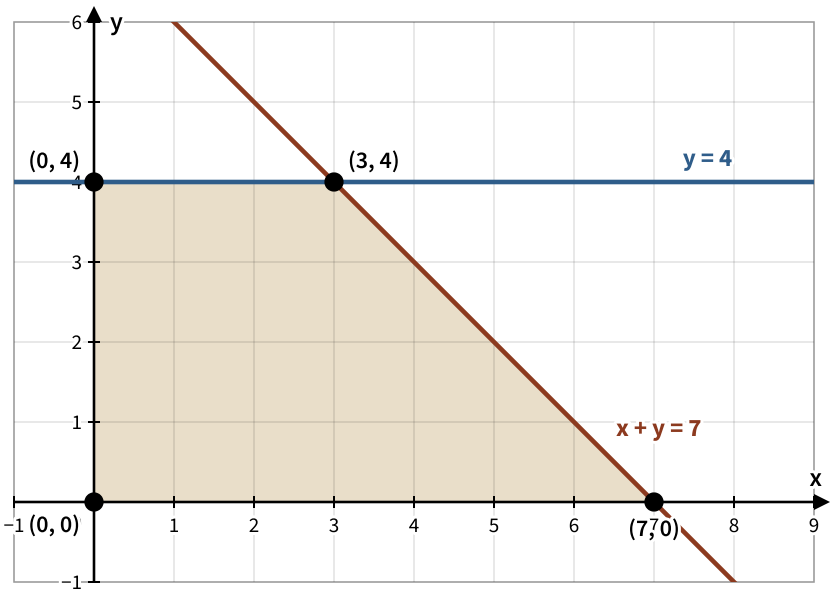
   P is 0, 20, 21, 10 at the four vertices; the maximum is 21 at $(3, 3)$.

2. Easy A feasible region has vertices $(1, 1)$, $(5, 1)$, $(4, 4)$ and $(1, 6)$. Find the maximum and the minimum of $P = 3x + 2y$ and where each occurs.

   Show answer

   P is 5, 17, 20, 15; maximum 20 at $(4, 4)$, minimum 5 at $(1, 1)$.

3. **Easy** The feasible region is shaded. Find the maximum value of $P = 2x + 5y$ and the vertex where it occurs.



Show answer

$P(0,0) = 0$, $P(7,0) = 14$, $P(3,4) = 26$, $P(0,4) = 20$; the maximum is 26 at $(3,4)$.

4. **Easy** What is the common slope of all the level lines $2x + 3y = c$ of the objective function $P = 2x + 3y$?

Show answer

$y = -\frac{2}{3}x + \frac{c}{3}$, so the slope is $-\frac{2}{3}$ for every c.

5. **Medium** Maximize $P = 4x + 3y$ subject to $x \geq 0$, $y \geq 0$, $x + y \leq 9$, $2x + y \leq 14$.

Show answer

Vertices $(0,0)$, $(7,0)$, $(5,4)$, $(0,9)$ give $P = 0, 28, 32, 27$; the maximum is 32 at $(5,4)$.

6. **Medium** Minimize $C = 5x + 4y$ subject to $x \geq 0$, $y \geq 0$, $x + y \geq 6$, $x + 3y \geq 12$.

Show answer

Vertices $(0,6)$, $(3,3)$, $(12,0)$ give $C = 24, 27, 60$; the minimum is 24 at $(0,6)$.

7. **Medium** Over the feasible region of Problem 6, does $P = x + y$ have a maximum value? Explain.

Show answer

No. The region is unbounded: for example (n, n) is feasible for every $n \geq 3$, and there $P = 2n$, which grows without limit.

8. **Medium** Grace's bakery makes bread loaves and pies. Each loaf takes 10 minutes of preparation and each pie takes 20 minutes, and she has at most 480 minutes. Her oven can bake at most 36 items a day. She makes a profit of \$3 per loaf and \$5 per pie. How many of each should she make to maximize profit, and what is the maximum profit?

Show answer

Constraints $x + 2y \leq 48$, $x + y \leq 36$, $x, y \geq 0$; vertices $(0, 0)$, $(36, 0)$, $(24, 12)$, $(0, 24)$ give $P = 0, 108, 132, 120$. Make 24 loaves and 12 pies for \$132.

9. **Hard** Maximize $P = 6x + 3y$ subject to $x \geq 0$, $y \geq 0$, $x + y \leq 9$, $2x + y \leq 14$. Find the maximum value and describe every point where it occurs.

Show answer

$P(0, 0) = 0$, $P(7, 0) = 42$, $P(5, 4) = 42$, $P(0, 9) = 27$. The maximum 42 occurs at $(7, 0)$, at $(5, 4)$ and at every point of the edge between them, because $P = 3(2x + y)$ and that edge lies on $2x + y = 14$.

10. **Hard** A church youth group builds birdhouses and planter boxes for a fundraiser. A birdhouse uses 3 board-feet of lumber and 2 hours of work; a planter uses 4 board-feet and 1 hour. They have 120 board-feet and 60 hours, and they want at least 10 planters. A birdhouse raises \$12 and a planter raises \$10. How many of each should they build to raise the most, and how much is that?

Show answer

Constraints $3x + 4y \leq 120$, $2x + y \leq 60$, $y \geq 10$, $x \geq 0$; vertices $(0, 10)$, $(25, 10)$, $(24, 12)$, $(0, 30)$ give \$100, \$400, \$408, \$300. Build 24 birdhouses and 12 planters to raise \$408.

11. Hard Ruth needs at least 10 pounds of nitrogen and at least 15 pounds of phosphorus for her garden. A bag of Brand A has 2 pounds of nitrogen and 1 pound of phosphorus and costs \$8. A bag of Brand B has 1 pound of nitrogen and 3 pounds of phosphorus and costs \$12. How many bags of each should she buy to spend the least, and what is the cost?

Show answer

Constraints $2x + y \geq 10$, $x + 3y \geq 15$, $x, y \geq 0$; vertices $(0, 10)$, $(3, 4)$, $(15, 0)$ give \$120, \$72, \$120. Buy 3 bags of A and 4 bags of B for \$72.

12. Challenge Explain, using level lines, why the maximum of $P = ax + by$ over a bounded feasible region with straight edges occurs at a vertex.

Show answer

The level lines $ax + by = c$ all have the same slope, so they are parallel, and increasing c slides the line in one direction. The maximum is the largest c whose line still touches the region. At that moment the whole region lies on one side of the line, so the line touches only the boundary. Because the edges are straight, the line touches either a single corner or a whole edge, and an edge ends in corners. Either way a vertex gives the maximum value.

13. Challenge Over the region $x \geq 0$, $y \geq 0$, $x + y \leq 9$, $2x + y \leq 14$, find the value of k for which $P = kx + y$ has the same value at the vertices $(7, 0)$ and $(5, 4)$.

Show answer

$P(7, 0) = 7k$ and $P(5, 4) = 5k + 4$. Setting $7k = 5k + 4$ gives $k = 2$; then the level lines are parallel to the edge $2x + y = 14$.

COMMON MISTAKES

- Testing only the vertices you can see easily, such as the intercepts. Use a sketch to find every corner, including the ones where two slanted lines meet, and solve for them exactly.
- Evaluating the objective at a crossing of two boundary lines that is outside the feasible region. Check every candidate corner in all the constraints first.
- Claiming a maximum on an unbounded region. If the region runs off forever in a direction where the objective grows, there is no maximum; say so.
- Answering with only a number. State the choice (how many acres, loaves or bags of each) and the best value, with units.

Make it a game · Corner Shop Profit. Pretend you bake cookies and loaves with 12 cups of flour and 10 eggs available. List the constraints, shade the feasible region on graph paper, and mark its corners. Choose a profit rule such as 3x + 5y in pretend dollars. Slide a pencil parallel to a level line across the region and see which corner it touches last.

2.4 Systems of Three Equations in Three Variables

OBJECTIVES

- Describe the graph of a linear equation in three variables and the ways three planes can meet.
- Solve a system of three linear equations in three variables by elimination and back-substitution, and check the solution in all three equations.
- Recognize systems with no solution or infinitely many solutions from the algebra.
- Write and solve three-variable systems from word problems, including finding the parabola $y = ax^2 + bx + c$ through three points.

WATCH

Systems of 3 Linear Equations (1 of 2: Introduction & approach)

Eddie Woo · 7:57

https://www.youtube.com/watch?v=8xvtRvuPxc0

> **WHEN TO WATCH**
> Before Example 1: the plan for eliminating one variable at a time.

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system of equations.

BACKUP VIDEO

Solve a System of Equations with 3 Variables and 3 Equations

Mario's Math Tutoring · 9:42

https://www.youtube.com/watch?v=455-25-P1rU

After Example 1: a second full worked solution of a 3-by-3 system.

KEY TERMS

linear equation in three variables

An equation that can be written $ax + by + cz = d$, where a, b, c are not all zero. Its graph in space is a flat plane.

ordered triple

A point (x, y, z) in space, written in the order x, y, z. A solution of a three-variable system is an ordered triple.

plane

A flat surface that extends forever in every direction, like an endless sheet of paper or the floor of a room continued without end.

back-substitution

After finding the last variable, substituting it into earlier equations, one at a time, to find the other variables.

CORE IDEAS

An equation such as $x + y + z = 6$ has three variables, so a solution is an **ordered triple** (x, y, z), such as $(1, 2, 3)$. Points with three coordinates live in space, not in the flat plane, and the graph of a linear equation in three variables is a **plane**: a flat surface. You will not need to draw these; it is enough to picture them.

A solution of a system of three such equations is a point that lies on all three planes. Think of a room. The floor and two walls meet at a single corner: **one solution**. The floor and the ceiling never meet, so no point is on both: **no solution** (three planes can also fail to share a point when each pair meets in a line and the three lines are parallel, like the three sides of a tent). Three pages of an open book meet along the spine, a whole line of points: **infinitely many solutions**.

To solve by **elimination**, reduce three equations to two, then two to one. Number the equations (1), (2), (3). Choose one variable to eliminate. Use one pair of equations, say (1) and (2), to eliminate it, and then a *different* pair, say (1) and (3) or (2) and (3), to eliminate the same variable. Now you have two equations in two variables, which you solve as in Lesson 2.1. Finally, back-substitute into one of the original equations to find the third variable.

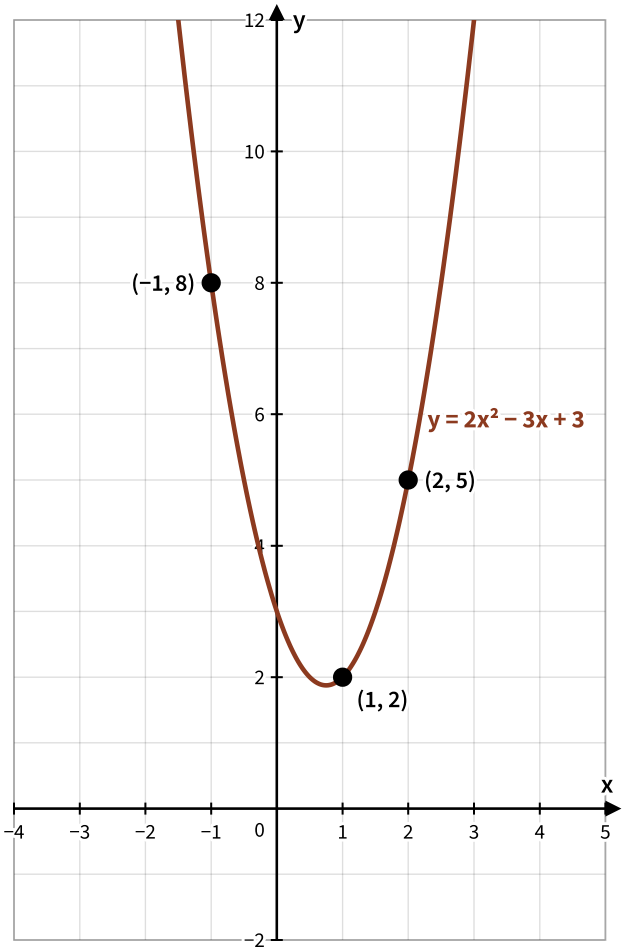
Why does this work? It is the same reason as in Lesson 2.1: adding equal quantities to both sides, or multiplying an equation by a nonzero number, never loses a solution of the system. Why two *different* pairs? Each original equation must be used, or else the new two-variable system leaves one of the planes out, and you would be solving a different problem. Because a slip in any step gives a wrong triple, always **check the triple in all three original equations**.

Special cases show up just as they did with two variables. If, while eliminating, you reach a false statement such as $0 = 3$, the planes share no point: **no solution**. If an equation becomes $0 = 0$, one of the equations gave no new information (for example, it was the sum of the other two). Keep going with the remaining equations: if no false statement appears, the system has **infinitely many solutions**, usually all

the points of a line where the planes meet; if a false statement does appear, there is still no solution.

A useful application: there is exactly one parabola $y = ax^2 + bx + c$ through three points with different x-coordinates (not all on one line). Substituting each point gives an equation. The unknowns are a, b and c; the coordinates are known numbers, so each equation is *linear* in a, b, c, even though the curve is not a line. For the points $(-1, 8)$, $(1, 2)$ and $(2, 5)$ you get $a - b + c = 8$, $a + b + c = 2$ and $4a + 2b + c = 5$. Subtracting the second from the first gives $-2b = 6$, so $b = -3$; then $a + c = 5$ and $4a + c = 11$ give $a = 2$, $c = 3$. The parabola is $y = 2x^2 - 3x + 3$.

The parabola through three points



KEY FORMULAS

$$ax + by + cz = d \quad \text{(a plane in space)}$$

three equations $\rightarrow$ two equations in two variables $\rightarrow$ one variable $\rightarrow$ back-substitute $\rightarrow$ check all three

$$y = ax^2 + bx + c \text{ through } (x_1, y_1): \quad ax_1^2 + bx_1 + c = y_1$$

Practice habit. Label your equations (1), (2), (3) and write which ones you combine on every line, such as "(1) + (2)". One step per line makes it easy to find a slip, and the final check in all three original equations catches any that remain.

WORKED EXAMPLES

Example 1. Solve the system: (1) $x + y + z = 6$, (2) $2x - y + z = 3$, (3) $x + 2y - z = 2$.

Show solution

1. Eliminate z, since it has coefficients 1 and -1.
2. (1) + (3): $2x + 3y = 8$. Call this (4).
3. (2) + (3): $3x + y = 5$. Call this (5). This pair is different from the first, so all three equations have been used.
4. From (5), $y = 5 - 3x$. Substitute into (4): $2x + 3(5 - 3x) = 8$, so $2x + 15 - 9x = 8$ and $-7x = -7$.
5. So $x = 1$ and $y = 5 - 3(1) = 2$.
6. Back-substitute into (1): $1 + 2 + z = 6$, so $z = 3$.
7. Check (1): $1 + 2 + 3 = 6$ ✓. Check (2): $2 - 2 + 3 = 3$ ✓. Check (3): $1 + 4 - 3 = 2$ ✓.

Answer: $(1, 2, 3)$

Example 2. The community orchestra sold 300 tickets to its fall concert: adult tickets at \$12, student tickets at \$8 and child tickets at \$5. Ticket sales came to \$2,820, and there were as many adult tickets as student and child tickets combined. How many of each kind were sold?

Show solution

1. Let a, s and c be the numbers of adult, student and child tickets.
2. (1) $a + s + c = 300$; (2) $12a + 8s + 5c = 2820$; (3) $a = s + c$, or $a - s - c = 0$.
3. (1) + (3): $2a = 300$, so $a = 150$. (Both s and c were eliminated at once.)
4. Then (1) gives $s + c = 150$, and (2) gives $8s + 5c = 2820 - 1800 = 1020$.
5. Substitute $c = 150 - s$: $8s + 750 - 5s = 1020$, so $3s = 270$ and $s = 90$.
6. Then $c = 150 - 90 = 60$.
7. Check (1): $150 + 90 + 60 = 300$ ✓. Check (2): $1800 + 720 + 300 = 2820$ ✓. Check (3): $150 = 90 + 60$ ✓.

Answer: 150 adult, 90 student and 60 child tickets

PRACTICE

1. Easy Is $(2, -1, 3)$ a solution of the system $x + y + z = 4$, $2x - y + z = 8$, $x + 3y - z = -4$?

   Show answer

   Yes: $2 - 1 + 3 = 4$ ✓, $4 + 1 + 3 = 8$ ✓, $2 - 3 - 3 = -4$ ✓.

2. Easy Is $(1, 1, 1)$ a solution of the system $x + y + z = 3$, $x - y + 2z = 2$, $3x + y - z = 4$?

   Show answer

   No: the first two check, but $3 + 1 - 1 = 3 \neq 4$.

3. Easy Solve by back-substitution: $x + y + z = 10$, $y + z = 7$, $z = 2$.

   Show answer

   $y = 7 - 2 = 5$, then $x = 10 - 5 - 2 = 3$: $(3, 5, 2)$.

4. **Easy** Add the equations $x + y + z = 6$ and $x - y + 2z = 7$. Which variable is eliminated, and what equation results?

Show answer

y is eliminated, leaving $2x + 3z = 13$.

5. **Medium** Solve: $x + y + z = 4$, $2x + y - z = 0$, $x - 2y + z = 7$.

Show answer

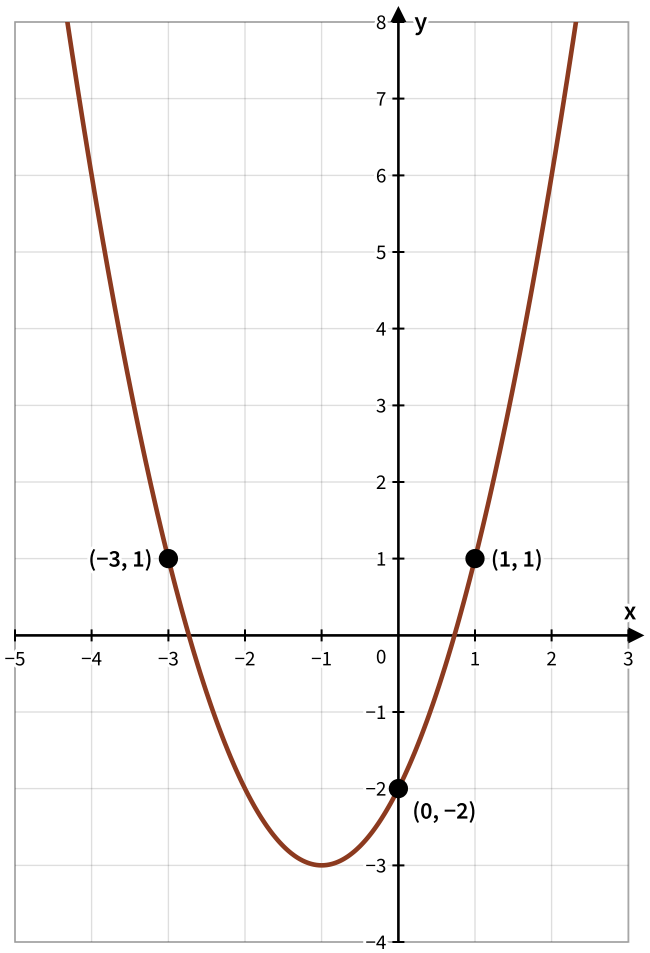
(1) + (2): $3x + 2y = 4$; (2) + (3): $3x - y = 7$; subtracting gives $3y = -3$, so $y = -1$, $x = 2$, $z = 3$: $(2, -1, 3)$.

6. **Medium** Solve: $2x + y + z = 4$, $x - y + 2z = 5$, $3x + 2y - z = -3$.

Show answer

(1) + (3): $5x + 3y = 1$; $2 \times (1) - (2)$: $3x + 3y = 3$; subtracting gives $2x = -2$, so $x = -1$, $y = 2$, and $z = 4 - 2(-1) - 2 = 4$: $(-1, 2, 4)$.

7. **Medium** Find the equation $y = ax^2 + bx + c$ of the parabola through the three labeled points.



Show answer

$c = -2$; $a + b = 3$ and $9a - 3b = 3$, so $3a - b = 1$; adding gives $4a = 4$.
$y = x^2 + 2x - 2$.

8. **Medium** Show that the system $x + y + z = 5$, $2x + 2y + 2z = 7$, $x - y + z = 1$ has no solution.

Show answer

$2 \times (1)$ gives $2x + 2y + 2z = 10$; subtracting (2) gives $0 = 3$, which is false. No solution (the first two planes are parallel).

9. Medium Grace's jar for the church food pantry holds 30 coins, all nickels, dimes and quarters, worth $3.30 in all. There are twice as many dimes as quarters. How many of each coin are there?

Show answer

$n + d + q = 30$, $5n + 10d + 25q = 330$, $d = 2q$. Then $n = 30 - 3q$ and $150 - 15q + 45q = 330$, so $q = 6$: 12 nickels, 12 dimes, 6 quarters.

10. Hard A farmer mixes 200 pounds of feed from oats at $0.20 per pound, corn at $0.15 per pound and barley at $0.25 per pound. The mix costs $37 in all, and the weight of corn equals the weight of oats and barley together. How many pounds of oats, corn and barley does he use?

Show answer

$o + c + b = 200$, $0.20o + 0.15c + 0.25b = 37$, $c = o + b$. Then $2c = 200$, so $c = 100$; $o + b = 100$ and $0.20o + 0.25b = 22$ give $o = 60$, $b = 40$. Oats 60 lb, corn 100 lb, barley 40 lb.

11. Hard Henry throws a ball straight up. The table gives its height at three times. Find the model $h = at^2 + bt + c$ (feet and seconds), and say what b and c mean.

time t (seconds)	height h (feet)
0.5	25
1	37
2	37

Show answer

$0.25a + 0.5b + c = 25$, $a + b + c = 37$, $4a + 2b + c = 37$. Then $3a + b = 0$ and $3a + 2b = 48$, so $b = 48$, $a = -16$, $c = 5$: $h = -16t^2 + 48t + 5$. The ball left his hand 5 feet above the ground at 48 feet per second.

12. Hard How many solutions does the system $x + y + z = 4$, $x - y + z = 2$, $2x + 2z = 6$ have? Explain.

Show answer

(1) + (2) gives $2x + 2z = 6$, which is exactly (3), so (3) adds no new information and subtracting gives $0 = 0$. There are infinitely many solutions: the three planes share a line.

13. Challenge Explain why finding the parabola $y = ax^2 + bx + c$ through three given points leads to a system of LINEAR equations, even though the parabola is not a line.

Show answer

The unknowns are a, b and c, not x and y. Each point (x_1, y_1) has known coordinates, so x_1^2 and x_1 are just numbers. The equation $x_1^2 \cdot a + x_1 \cdot b + c = y_1$ is a number times a plus a number times b plus c equals a number, which is linear in a, b, c.

14. Challenge For what value of k does the system $x + y + z = 1$, $x + 2y + 3z = 2$, $2x + 3y + kz = 4$ have no solution?

Show answer

(2) − (1): $y + 2z = 1$. (3) − 2 × (1): $y + (k - 2)z = 2$. Subtracting: $(k - 4)z = 1$. When $k = 4$ this says $0 = 1$, so there is no solution: $k = 4$.

COMMON MISTAKES

- Eliminating a variable from (1) and (2) and then eliminating a DIFFERENT variable from another pair. Both new equations must be missing the same variable, or you still have three unknowns.
- Using the same pair of equations twice, so that one equation is never used. Make sure each of (1), (2), (3) appears in at least one of your two combinations.
- Checking the answer in only one or two equations. A triple can satisfy two of the equations and still be wrong; check all three original equations.
- Sign slips when subtracting equations. Write the subtraction with parentheses or multiply by -1 and add.

Make it a game · Three Planes, Three Clues. First, use three index cards as planes and show the parent ways they can meet: at one point, along a line, or not at all. Then the parent secretly chooses an ordered triple and writes three equations that it satisfies, such as $x + y + z = 6$. Use elimination and back-substitution to find it, and check in all three.

UNIT 2

Linear Systems and Linear Programming

Many real decisions involve several unknowns and several conditions at once: how many tickets of each kind were sold, how much of each grain goes into a feed mix, how many acres of each crop to plant. This unit begins with a quick review of systems of two linear equations from Algebra I, with attention to why substitution and elimination work and how the algebra reveals one solution, no solution or infinitely many.

Next you graph systems of linear inequalities and find the corners of their solution regions, and then use them for linear programming: choosing the best plan when land, money and time are limited. You will see why the best value of a linear objective always occurs at a corner, an idea that grew into one of the most widely used tools of modern planning. The unit ends with systems of three equations in three variables, including how to find the parabola through three points.

These skills come back often: three-variable systems are used to build quadratic models in Unit 3, and in Unit 12 you will solve the same systems with matrices, using row operations that are exactly the elimination steps you learn here.

PREREQUISITES

- Solve linear equations in one variable, including equations with fractions.
- Graph a line from slope-intercept or standard form, and know that parallel lines have equal slopes.
- Solve a system of two linear equations by graphing, substitution or elimination (Algebra I).
- Graph a linear inequality in two variables with a solid or dashed boundary (Algebra I).
- Evaluate functions and expressions such as $y = ax^2 + bx + c$ at given values (Lessons 1.1–1.2).

MASTERY CHECKLIST

- ☐ I can solve a system of two linear equations by graphing, substitution or elimination, and choose an efficient method.
- ☐ I can explain why elimination works and tell from the algebra whether a system has one, no or infinitely many solutions.
- ☐ I can write and solve a two-variable system for a ticket, mixture, rate or cost problem.
- ☐ I can graph the solution region of a system of linear inequalities using solid or dashed boundaries and test points.
- ☐ I can find the vertices of a solution region and tell whether the region is bounded or unbounded.
- ☐ I can write constraints and an objective function from a word problem.
- ☐ I can explain why a linear objective function reaches its maximum or minimum at a vertex, and use the corner-point method.
- ☐ I can recognize when an unbounded region has no maximum and when two corners tie.
- ☐ I can solve a system of three linear equations in three variables by elimination and check the solution in all three equations.
- ☐ I can find the parabola

$y =$ through three points and solve
ax^2 three-variable word problems.
$bx +$
c

LESSONS IN THIS UNIT

2.1	**Systems of Two Linear Equations: A Review**	45 minutes
2.2	**Systems of Linear Inequalities**	2 sessions of 45 minutes
2.3	**Linear Programming**	2 sessions of 45 minutes
2.4	**Systems of Three Equations in Three Variables**	2 sessions of 45 minutes

Unit 2 test

You have 60 minutes. A scientific calculator is allowed, but no graphing calculator. Show your work on every problem: write your variables, your equations or inequalities, and each step on its own line. Check solutions of systems in the original equations. Give exact answers unless a problem says how to round, and include units in answers to word problems.

1. **5 pts**
   Solve by substitution: $x = 2y + 1$ and $3x - 4y = 9$.

2. **6 pts**
   Solve by elimination: $5x + 2y = 4$ and $3x - 5y = 21$.

3. **6 pts**
   Without graphing, decide how many solutions each system has. (a) $4x - 6y = 8$ and $-2x + 3y = -4$. (b) $3x + y = 2$ and $6x + 2y = 5$.

4. **7 pts**
   A hardware store mixes grass seed that sells for $6 per pound with seed that sells for $10 per pound to make 40 pounds of a blend that sells for $7 per pound. How many pounds of each kind go into the blend?

5. **5 pts**

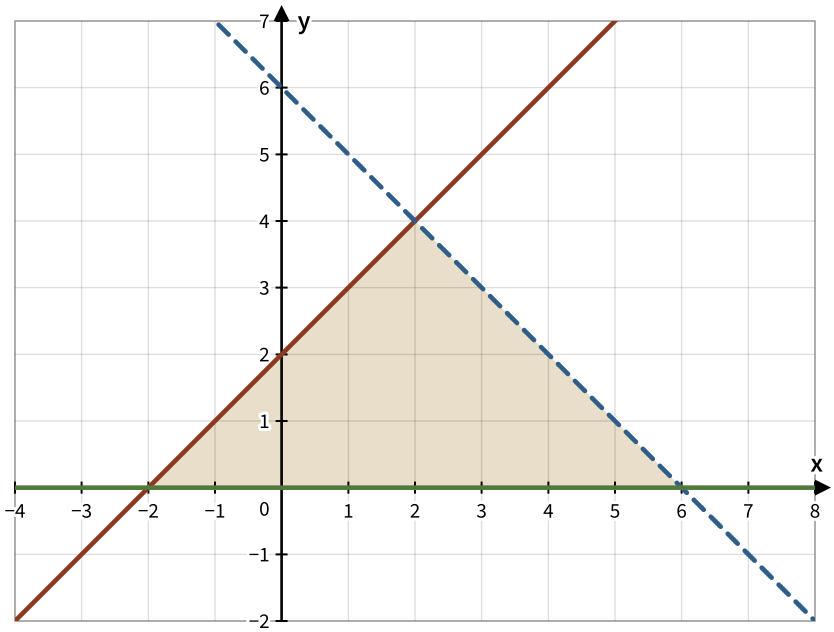
Is $(1, 2)$ a solution of the system $y \geq 3x - 2$ and $2x + y < 4$? Explain.

6. **8 pts**

Graph the system $x \geq 0$, $y \geq 2$, $x + 2y \leq 12$, $2x + y \leq 12$, and find all the vertices of the region.

7. **7 pts**

Write the system of three inequalities whose solution region is shaded. Read the boundary lines from the grid and note which one is dashed.



8. **10 pts**

Henry's woodshop builds chairs and benches. A chair needs 1 hour of cutting and 2 hours of finishing; a bench needs 2 hours of cutting and 1 hour of finishing. Each week there are at most 16 hours for cutting and 14 hours for finishing. The profit is \$30 per chair and \$50 per bench. How many chairs and benches should be built each week to make the most profit, and what is that profit?

9. **8 pts**

   Find the minimum value of $C = 3x + 2y$ subject to $x \geq 0$, $y \geq 0$, $x + y \geq 5$, $2x + y \geq 8$, and the point where it occurs. Does C have a maximum on this region?

10. **6 pts**

    A classmate says the maximum of $P = 2x + 3y$ over a feasible region might be at a point in the middle of the region, away from every edge. Explain why this cannot happen.

11. **8 pts**

    Solve: $x + 2y + z = 0$, $2x - y + 3z = 11$, $x + y - 2z = -1$.

12. **5 pts**

    How many solutions does the system $x + 2y - z = 3$, $2x + 4y - 2z = 6$, $x - y + z = 0$ have? Explain briefly.

13. **9 pts**

    Find the equation of the parabola $y = ax^2 + bx + c$ that passes through $(-1, 2)$, $(1, 6)$ and $(3, 2)$.

14. **10 pts**

    The Miller family buys 20 young trees for a new orchard: apple trees at \$30 each, pear trees at \$35 each and cherry trees at \$40 each. They spend \$690 and buy twice as many apple trees as cherry trees. How many of each kind do they buy?

ANSWER KEY

1. $3(2y+1)-4y=9$, so $2y=6$, $y=3$ and $x=7$: $(7,3)$.
2. Multiply by 5 and by 2: $25x+10y=20$ and $6x-10y=42$; adding gives $31x=62$, so $x=2$ and $y=-3$: $(2,-3)$.
3. (a) Multiplying the second equation by -2 gives the first, so adding gives $0=0$: infinitely many solutions. (b) Doubling the first gives $6x+2y=4$, and subtracting gives $0=1$: no solution.
4. $x+y=40$ and $6x+10y=280$; $6x+10(40-x)=280$ gives $4x=120$. 30 pounds of the \$6 seed and 10 pounds of the \$10 seed.
5. No: $2\geq 1$ is true, but $2(1)+2=4<4$ is false (the point is on the dashed boundary).
6. $(0,2)$, $(5,2)$, $(4,4)$, $(0,6)$. For $(4,4)$: subtracting $x+2y=12$ from $2(2x+y)=24$ gives $3x=12$.
7. $y\leq x+2$, $x+y<6$, $y\geq 0$.
8. Constraints $x+2y\leq 16$, $2x+y\leq 14$, $x,y\geq 0$; vertices $(0,0)$, $(7,0)$, $(4,6)$, $(0,8)$ give \$0, \$210, \$420, \$400. Build 4 chairs and 6 benches for \$420.
9. Vertices $(0,8)$, $(3,2)$, $(5,0)$ give $C=16,13,15$; the minimum is 13 at $(3,2)$. There is no maximum, because the region is unbounded and C grows without limit.
10. At a point in the middle of the region there is room to move a little in every direction and stay in the region. Moving a little to the right (or up) increases $2x+3y$, so that point cannot give the largest value. The maximum must be on the boundary, and by sliding the parallel level lines $2x+3y=c$ outward, the last point they touch is a corner.
11. (2) − 2 × (1): $-5y+z=11$; (1) − (3): $y+3z=1$. Then $y=1-3z$ gives $16z=16$, so $z=1$, $y=-2$, $x=3$: $(3,-2,1)$.
12. The second equation is twice the first, so it is the same plane; subtracting gives $0=0$. That plane meets the third plane in a line, so there are infinitely many solutions.
13. $a-b+c=2$, $a+b+c=6$, $9a+3b+c=2$. Subtracting the first two gives $b=2$; then $a+c=4$ and $9a+c=-4$, so $a=-1$, $c=5$: $y=-x^2+2x+5$.
14. $a+p+c=20$, $30a+35p+40c=690$, $a=2c$. Then $p=20-3c$ and $60c+700-105c+40c=690$, so $c=2$: 4 apple, 14 pear and 2 cherry trees. Check: $120+490+80=690$ ✓.

UNIT 3

Quadratic Functions and Complex Numbers

Quadratic functions describe thrown balls, arches, the area of a garden and the revenue of a small business. In Algebra I you learned to graph parabolas and to solve quadratic equations. This unit pulls that work together, moving between the three forms of a quadratic and choosing the best method to solve, and then explains why the methods work, including a derivation of the quadratic formula and of the facts about the sum and product of the roots.

The unit also enlarges the number system. The equation $x^2 = -1$ has no real solution, so we bring in the imaginary unit i with $i^2 = -1$ and learn to compute with complex numbers. With them, every quadratic equation has solutions, and the discriminant tells whether they are real or non-real. The last two lessons use quadratics to solve inequalities and to build models from data, heights and prices.

This unit leads straight into Unit 4, where the same ideas (zeros, factors, conjugate pairs and sign charts) are used for polynomials of any degree. Families can expect steady practice in exact algebra and in checking answers in the original equation.

PREREQUISITES

- Factoring trinomials, the difference of squares and perfect square trinomials (Algebra I)
- Simplifying square roots, such as $\sqrt{72} = 6\sqrt{2}$ (Algebra I)
- Graphing parabolas and using vertex form (Algebra I)
- Interval notation and solving linear inequalities (Algebra I)
- Transformations of parent functions (Lesson 1.2)
- Solving systems of three linear equations by elimination (Lesson 2.4)

MASTERY CHECKLIST

- ☐ I can read the vertex, axis of symmetry, intercepts and direction of opening from standard, vertex and factored form, and convert among the forms.
- ☐ I can explain why the axis of symmetry lies halfway between the zeros.
- ☐ I can solve a quadratic equation by factoring, square roots, completing the square or the quadratic formula, and choose an efficient method.
- ☐ I can derive the quadratic formula by completing the square.
- ☐ I can add, subtract, multiply and divide complex numbers and simplify powers of

i.

- ☐ I can plot complex numbers and find their absolute values.
- ☐ I can solve quadratics with non-real solutions and explain why they come in conjugate pairs.
- ☐ I can use the discriminant to describe the solutions and the x-intercepts of the graph.
- ☐ I can use the sum

$-\frac{b}{a}$ and product

$\frac{c}{a}$ of the roots to check solutions and to write an equation from its roots.

- ☐ I can solve quadratic inequalities with a sign chart or a graph and

write the answer in interval notation.

- ☐ I can write a quadratic model from a vertex, from zeros or from three points, and use projectile and revenue models.

LESSONS IN THIS UNIT

3.1	**Quadratic Functions and Their Three Forms**	45 minutes
3.2	**Solving Quadratic Equations: A Review**	45 minutes
3.3	**Complex Numbers**	2 sessions of 45 minutes
3.4	**Complex Solutions and the Discriminant**	2 sessions of 45 minutes
3.5	**Quadratic Inequalities**	2 sessions of 45 minutes
3.6	**Quadratic Models**	2 sessions of 45 minutes

Unit 3 Test

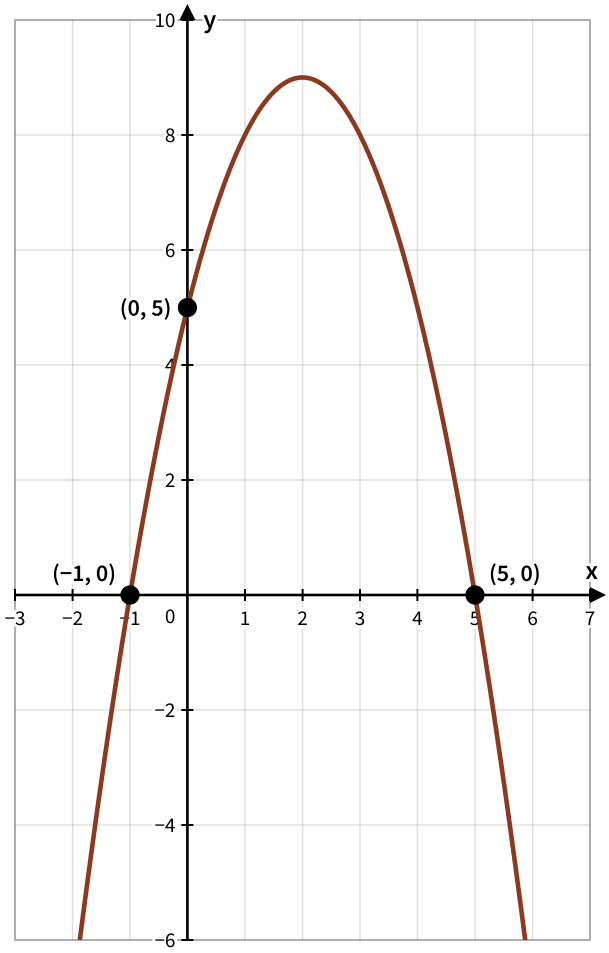
You have 60 minutes. A scientific calculator is allowed, but show all your work: answers without supporting work earn little credit. Give exact answers (integers, fractions, simplest radical form, or $a + bi$) unless a problem asks you to round. Write inequality solutions in interval notation unless told otherwise, and check solutions in the original equation.

1. **5 pts**
   For $y = -4(x + 3)^2 + 7$, give the vertex, the axis of symmetry and the maximum or minimum value.

2. **5 pts**
   Write $y = x^2 + 8x + 12$ in vertex form and in factored form.

3. **5 pts**

The graph shows a parabola with its x-intercepts and y-intercept marked. (a) Write its equation in factored form. (b) Find its vertex.



4. **6 pts**

Solve by factoring: $5x^2 - 3x - 2 = 0$.

5. **6 pts**

Solve by completing the square: $x^2 + 10x + 18 = 0$. Give exact answers.

6. **7 pts**

The Miller family is building a rectangular wooden deck that is 6 feet longer than it is wide, with an area of 112 square feet. Find its dimensions.

7. **5 pts**

Simplify: (a) $(3 - 2i)(4 + 5i)$ (b) i^{42}

8. **7 pts**

   (a) Write $\dfrac{7 - i}{1 + i}$ in the form $a + bi$. (b) Find the absolute value of your answer.

9. **7 pts**

   Solve $x^2 - 8x + 20 = 0$. What does the discriminant tell you about the graph of $y = x^2 - 8x + 20$?

10. **5 pts**

    One root of a quadratic equation with real coefficients is $3 - i\sqrt{2}$. Find the other root and write the equation with leading coefficient 1.

11. **7 pts**

    Solve $2x^2 - x - 15 < 0$. Write the answer in interval notation.

12. **5 pts**

    Solve $x^2 + 3 \geq 2x$.

13. **8 pts**

    Samuel builds birdhouses in his shop. When he charges x dollars each, his monthly profit is $P = -2x^2 + 80x - 600$ dollars. What prices give a monthly profit of at least \$150?

14. **8 pts**

    The table gives the height of a kicked ball. (a) Find the model $h = at^2 + bt + c$. (b) Find the greatest height, to the nearest tenth of a meter.

    Height of a kicked ball

t (seconds)	h (meters)
0	1
1	16.1
3	16.9

15. **7 pts**

A church bakery sells 300 cinnamon rolls a week at \$2.00 each. Each \$0.10 increase in price loses 10 sales a week. What price gives the greatest weekly revenue, and what is that revenue?

16. **7 pts**

Explain why, when a quadratic equation $ax^2 + bx + c = 0$ with real coefficients has a negative discriminant, its two solutions are complex conjugates.

ANSWER KEY

1. Vertex $(-3, 7)$, axis $x = -3$, maximum value 7 (it opens down)
2. $y = (x+4)^2 - 16 + 12 = (x+4)^2 - 4$; $y = (x+2)(x+6)$
3. (a) $y = a(x+1)(x-5)$ with $5 = a(1)(-5)$, so $a = -1$: $y = -(x+1)(x-5)$ (b) $x = \dfrac{-1+5}{2} = 2$, $y = -(3)(-3) = 9$: $(2, 9)$
4. $(5x+2)(x-1) = 0$, so $x = -\dfrac{2}{5}$ or $x = 1$
5. $(x+5)^2 = -18 + 25 = 7$, so $x = -5 \pm \sqrt{7}$
6. $w(w+6) = 112$, $w^2 + 6w - 112 = 0$, $(w+14)(w-8) = 0$, $w = 8$: 8 feet by 14 feet (check: $8 \times 14 = 112$ ✓)
7. (a) $12 + 15i - 8i - 10i^2 = 22 + 7i$ (b) $42 = 40 + 2$, so $i^{42} = i^2 = -1$
8. (a) $\dfrac{(7-i)(1-i)}{(1+i)(1-i)} = \dfrac{7 - 8i + i^2}{2} = \dfrac{6 - 8i}{2} = 3 - 4i$ (b) $\sqrt{9 + 16} = 5$
9. $D = 64 - 80 = -16$; $x = \dfrac{8 \pm 4i}{2} = 4 \pm 2i$. Since $D < 0$, the parabola has no x-intercepts (it lies entirely above the x-axis).
10. Other root $3 + i\sqrt{2}$; sum 6, product $9 + 2 = 11$: $x^2 - 6x + 11 = 0$
11. $(2x+5)(x-3) < 0$; zeros $-\dfrac{5}{2}$ and 3; opens up, so negative between: $\left(-\dfrac{5}{2}, 3\right)$
12. $x^2 - 2x + 3 \geq 0$; $D = 4 - 12 = -8 < 0$ and the parabola opens up, so it is always positive: all real numbers, $(-\infty, \infty)$
13. $-2x^2 + 80x - 750 \geq 0$; divide by -2: $x^2 - 40x + 375 \leq 0$, $(x-15)(x-25) \leq 0$: prices from \$15 to \$25, that is, $[15, 25]$
14. (a) $c = 1$; $a + b = 15.1$ and $9a + 3b = 15.9$ give $a = -4.9$, $b = 20$: $h = -4.9t^2 + 20t + 1$ (b) $t = \dfrac{20}{9.8} \approx 2.04$ s, $h \approx 21.4$ meters
15. $R = (2 + 0.1n)(300 - 10n) = -n^2 + 10n + 600$; $n = 5$: price \$2.50, selling 250 rolls, revenue \$625
16. The formula gives $x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$. The first part is a real number p. When $b^2 - 4ac < 0$, its square root is $i\sqrt{|b^2 - 4ac|}$, so the second part is a pure imaginary number qi. The $\pm$ gives $p + qi$ and $p - qi$, which are conjugates.

3.1 Quadratic Functions and Their Three Forms

OBJECTIVES

- Identify the vertex, axis of symmetry, intercepts and direction of opening from standard, vertex and factored form.
- Convert a quadratic among standard, vertex and factored form by expanding, completing the square and factoring.
- Explain why the vertex form shows the vertex and why the axis of symmetry lies halfway between the zeros.
- Write a quadratic function from a graph or a description.

WATCH

Vertex Form of a Parabola (1 of 2: Why it matters)

Eddie Woo · 7:42

https://www.youtube.com/watch?v=wOTfkuK_GQU

> **WHEN TO WATCH**
> Before Example 1: why vertex form shows the turning point.

Note: Eddie Woo teaches in Australia: he says 'turning point' for vertex.

BACKUP VIDEO

Convert from Standard Form to Vertex Form of a Quadratic (Parabola) 2 Methods

Mario's Math Tutoring · 7:15

https://www.youtube.com/watch?v=d31AA71rJqI

After Example 1: converting standard form to vertex form by completing the square, two ways.

KEY TERMS

quadratic function

A function that can be written $f(x) = ax^2 + bx + c$ with $a \neq 0$. Its graph is a parabola.

vertex

The turning point of a parabola: its lowest point if it opens up, its highest point if it opens down.

axis of symmetry

The vertical line through the vertex. The parabola is its own mirror image across this line.

zero

An input that makes the function equal 0. The real zeros are the x-coordinates of the x-intercepts.

vertex form

$y = a(x - h)^2 + k$, which shows the vertex (h, k).

factored (intercept) form

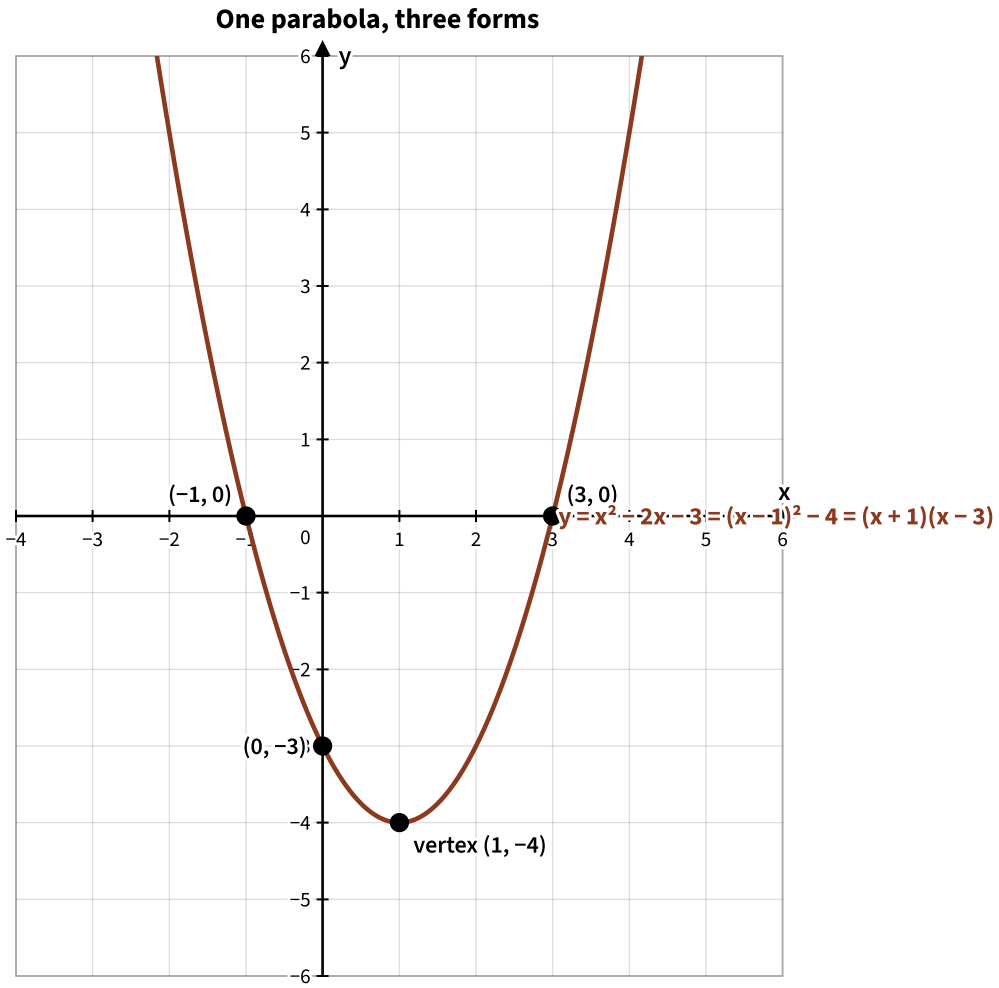
$y = a(x - p)(x - q)$, which shows the zeros p and q.

CORE IDEAS

A **quadratic function** can be written $f(x) = ax^2 + bx + c$ with $a \neq 0$, and its graph is a **parabola.** In Algebra I you met three ways to write the same quadratic, and each makes different features easy to read. **Standard form** $y = ax^2 + bx + c$ shows the y-intercept c (put in $x = 0$). **Vertex form** $y = a(x - h)^2 + k$ shows the vertex (h, k). **Factored form** $y = a(x - p)(x - q)$ shows the zeros p and q. The number a is the same in all three: if $a > 0$ the parabola opens up, if $a < 0$ it opens down, and the larger $|a|$ is, the narrower the parabola.

Why does vertex form show the vertex? The square $(x - h)^2$ is never negative, and it equals 0 only when $x = h$. So if $a > 0$, the value $a(x - h)^2 + k$ is never less than k, and it equals k exactly at $x = h$: the point (h, k) is the lowest point. If $a < 0$, the same reasoning makes (h, k) the highest point. Notice also that $x = h + d$ and $x = h - d$ both give $(x - h)^2 = d^2$, so points the same distance on either side of $x = h$ have the same height. That is why the graph is symmetric about the vertical line $x = h$, the **axis of symmetry**.

For example, $y = x^2 - 2x - 3$, $y = (x - 1)^2 - 4$ and $y = (x + 1)(x - 3)$ are the same function (multiply each out to check). Standard form gives the y-intercept -3, vertex form gives the vertex $(1, -4)$, and factored form gives the zeros -1 and 3.



Standard to vertex form: complete the square. Factor a out of the x-terms, add the square of half the x-coefficient inside the parentheses, and subtract the same amount (times a) outside so the value does not change. Doing this to the general quadratic shows where the vertex formula comes from: $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$. So $h = -\frac{b}{2a}$, and the easiest way to get k is to substitute: $k = f(h)$.

Standard to factored form: factor (first take out a or any common factor). **Vertex or factored form to standard form: multiply out. Factored to vertex form: use symmetry.** The two zeros are the same height (0), so they are mirror images across the axis, and the axis must lie exactly halfway between them: $h = \frac{p+q}{2}$. The algebra agrees: $a(x-p)(x-q) = ax^2 - a(p+q)x + apq$, so $-\frac{b}{2a} = \frac{a(p+q)}{2a} = \frac{p+q}{2}$. Then find k by substituting h.

Not every quadratic can be factored over the integers, and some have no real zeros at all. The parabola $y = x^2 + 4$ never reaches the x-axis because $x^2 + 4 \geq 4$. Such a quadratic has no real factored form; Lessons 3.3 and 3.4 show what its zeros are.

KEY FORMULAS

$$y = ax^2 + bx + c \qquad y = a(x - h)^2 + k \qquad y = a(x - p)(x - q)$$

$$h = -\frac{b}{2a}, \qquad k = f(h)$$

$$\text{axis of symmetry: } x = \frac{p + q}{2} \text{ (halfway between the zeros)}$$

Practice habit. After every conversion, multiply your new form back out and compare it with standard form. It takes thirty seconds and catches nearly every sign error.

WORKED EXAMPLES

Example 1. Write $y = 2x^2 - 12x + 10$ in vertex form and in factored form. Give the vertex, the axis of symmetry and the zeros.

Show solution

1. Vertex form. Factor 2 out of the x-terms: $y = 2(x^2 - 6x) + 10$.
2. Half of -6 is -3, and $(-3)^2 = 9$. Add 9 inside; that adds $2 \times 9 = 18$, so subtract 18 outside: $y = 2(x^2 - 6x + 9) + 10 - 18$.
3. $y = 2(x - 3)^2 - 8$, so the vertex is $(3, -8)$ and the axis is $x = 3$.
4. Factored form. Take out 2: $y = 2(x^2 - 6x + 5)$. Two numbers with product 5 and sum -6: -1 and -5. So $y = 2(x - 1)(x - 5)$, with zeros 1 and 5.
5. Check: the axis should be halfway between the zeros: $\frac{1 + 5}{2} = 3$ ✓. Expand vertex form: $2(x^2 - 6x + 9) - 8 = 2x^2 - 12x + 10$ ✓.

Answer: $y = 2(x - 3)^2 - 8$ and $y = 2(x - 1)(x - 5)$; vertex $(3, -8)$, axis $x = 3$, zeros 1 and 5

Example 2. A stone arch over a garden path is shaped like a parabola. Its feet stand on the ground 8 feet apart, and its highest point is 6 feet above the ground. Put the left foot at the origin with x in feet along the ground. Write the arch's equation in factored, vertex and standard form, and find its height 2 feet in from the left foot.

Show solution

1. The zeros are $x = 0$ and $x = 8$, so the factored form is $y = a(x - 0)(x - 8) = ax(x - 8)$.
2. The top is halfway between the zeros, at $x = 4$, with height 6. So $6 = a(4)(4 - 8) = -16a$, and $a = -\dfrac{6}{16} = -\dfrac{3}{8}$.
3. Factored form: $y = -\dfrac{3}{8}x(x - 8)$. Vertex form: $y = -\dfrac{3}{8}(x - 4)^2 + 6$.
4. Standard form: $-\dfrac{3}{8}(x^2 - 8x) = -\dfrac{3}{8}x^2 + 3x$.
5. Check vertex form: $-\dfrac{3}{8}(x^2 - 8x + 16) + 6 = -\dfrac{3}{8}x^2 + 3x - 6 + 6$ ✓.
6. Height at $x = 2$: $-\dfrac{3}{8}(2)(2 - 8) = -\dfrac{3}{8}(-12) = 4.5$ feet.

Answer: $y = -\dfrac{3}{8}x(x - 8) = -\dfrac{3}{8}(x - 4)^2 + 6 = -\dfrac{3}{8}x^2 + 3x$; the height 2 feet in is 4.5 feet

PRACTICE

1. Easy For $y = 3(x - 2)^2 + 5$, give the vertex, the axis of symmetry, the direction of opening and the minimum or maximum value.

   Show answer

   Vertex $(2, 5)$, axis $x = 2$, opens up, minimum value 5

2. Easy For $y = -2(x + 1)(x - 5)$, give the zeros, the axis of symmetry and the vertex.

   Show answer

   Zeros -1 and 5; axis $x = \dfrac{-1 + 5}{2} = 2$; $y(2) = -2(3)(-3) = 18$, so the vertex is $(2, 18)$

3. **Easy** Write $y = (x - 4)^2 - 9$ in standard form.

   Show answer

   $y = x^2 - 8x + 16 - 9 = x^2 - 8x + 7$

4. **Easy** Henry tosses a ball up from a porch. Its height in feet after t seconds is $h = -16t^2 + 32t + 6$. Use $t = -\dfrac{b}{2a}$ to find when the ball is highest and how high it gets.

   Show answer

   $t = -\dfrac{32}{2(-16)} = 1$ second; $h(1) = -16 + 32 + 6 = 22$ feet

5. **Medium** Write $y = x^2 - 10x + 21$ in vertex form and in factored form.

   Show answer

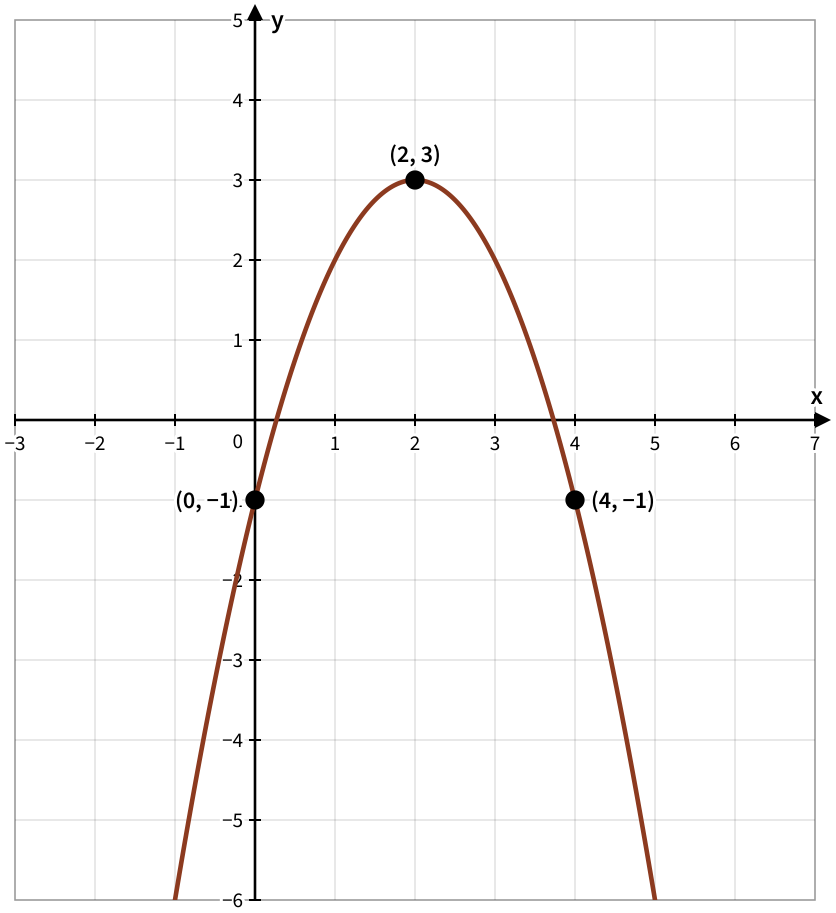
   $y = (x - 5)^2 - 25 + 21 = (x - 5)^2 - 4$; $y = (x - 3)(x - 7)$ (check: $\dfrac{3 + 7}{2} = 5$ ✓)

6. **Medium** Write $y = -x^2 + 4x + 5$ in vertex form and in factored form.

   Show answer

   $y = -(x^2 - 4x + 4) + 4 + 5 = -(x - 2)^2 + 9$; $y = -(x^2 - 4x - 5) = -(x - 5)(x + 1)$

7. **Medium** The graph shows a parabola with its vertex and two other points marked. Write its equation in vertex form.



Show answer

Vertex $(2, 3)$, so $y = a(x-2)^2 + 3$; through $(0, -1)$: $-1 = 4a + 3$, $a = -1$; $y = -(x-2)^2 + 3$

8. **Medium** Write $y = 3x^2 + 12x + 7$ in vertex form and give the vertex.

Show answer

$y = 3(x^2 + 4x + 4) + 7 - 12 = 3(x+2)^2 - 5$; vertex $(-2, -5)$

9. **Medium** A lawn sprinkler at ground level shoots a stream of water whose height is $y = -0.5(x-4)^2 + 8$ feet when it is x feet from the sprinkler. (a) What is the greatest height of the stream? (b) How far from the sprinkler does the water land?

Show answer

(a) 8 feet (the vertex is $(4, 8)$). (b) $0.5(x-4)^2 = 8$, $(x-4)^2 = 16$, $x = 0$ or $x = 8$; it lands 8 feet away

10. Hard A parabola has zeros -3 and 5 and passes through $(1, -8)$. Write its equation in factored, vertex and standard form.

Show answer

$y = a(x+3)(x-5)$; $-8 = a(4)(-4)$, $a = \frac{1}{2}$. Factored $y = \frac{1}{2}(x+3)(x-5)$; axis $x = 1$, so vertex form $y = \frac{1}{2}(x-1)^2 - 8$; standard form $y = \frac{1}{2}x^2 - x - \frac{15}{2}$

11. Hard Ruth sells jars of honey at a farmers' market. If she charges x dollars per jar, her weekly profit is $P = -5(x-4)(x-20)$ dollars. (a) At what prices does she just break even? (b) What price gives the greatest profit, and what is that profit?

Show answer

(a) \$4 and \$20 (the zeros). (b) Halfway between: \$12; $P(12) = -5(8)(-8) = 320$, so \$320

12. Hard Explain why, if a parabola $y = a(x-p)(x-q)$ has zeros p and q, its axis of symmetry is $x = \frac{p+q}{2}$. Give an algebraic reason.

Show answer

Multiply out: $a(x-p)(x-q) = ax^2 - a(p+q)x + apq$, so $b = -a(p+q)$. The axis is $x = -\frac{b}{2a} = \frac{a(p+q)}{2a} = \frac{p+q}{2}$, the midpoint of the zeros. (This matches symmetry: the zeros have equal heights, so they are mirror images across the axis.)

13. Challenge The parabola $y = x^2 + bx + c$ has its vertex at $(3, -2)$. (a) Find b and c. (b) Find the zeros in exact form.

Show answer

(a) $y = (x-3)^2 - 2 = x^2 - 6x + 7$, so $b = -6$, $c = 7$. (b) $(x-3)^2 = 2$, $x = 3 \pm \sqrt{2}$

COMMON MISTAKES

- Reading the sign of h wrong: $y = (x+2)^2 - 5$ has vertex $(-2, -5)$, not $(2, -5)$, because $x + 2 = x - (-2)$. Rewrite with a minus sign before you read h.
- Completing the square with $a \neq 1$ and forgetting that the number added inside the parentheses is multiplied by a. In $2(x^2 - 6x + 9)$ you added 18, not 9, so subtract 18 outside.

- Dropping a when factoring: $2x^2 - 12x + 10 = 2(x - 1)(x - 5)$, not $(x - 1)(x - 5)$. The zeros are the same, but the function is not. Multiply back out to check.

> **Make it a game · Three Forms Trading Cards.** Make nine cards from three quadratics, each written in standard, vertex and factored form, such as $x^2 - 2x - 3$, $(x - 1)^2 - 4$ and $(x - 3)(x + 1)$. Shuffle them face down and race to find each trio. For every complete set, state the vertex, axis of symmetry, zeros and direction of opening.

3.2 Solving Quadratic Equations: A Review

OBJECTIVES

- Solve quadratic equations by factoring, by square roots, by completing the square and with the quadratic formula.
- Derive the quadratic formula by completing the square.
- Use the discriminant to tell how many real solutions a quadratic equation has.
- Choose an efficient method and give answers in exact form or rounded as asked.

WATCH

Deriving the Quadratic Formula

Eddie Woo · 11:38

https://www.youtube.com/watch?v=3tvIxLGe8K4

> **WHEN TO WATCH**
> After the Core ideas: deriving the quadratic formula by completing the square.

BACKUP VIDEO

Solving Quadratic Equations (5 Methods)

Mario's Math Tutoring · 8:43

https://www.youtube.com/watch?v=puWdJ-s9y_Y

Before the practice set: five methods side by side, to help choose a method.

KEY TERMS

quadratic equation

An equation that can be written $ax^2 + bx + c = 0$ with $a \neq 0$.

completing the square

Adding the right constant to $x^2 + bx$ to make a perfect square trinomial: $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$.

Zero Product Property

If a product of numbers is 0, then at least one of the factors is 0.

discriminant

The number $b^2 - 4ac$ under the square root in the quadratic formula. Its sign tells how many real solutions there are.

exact form

An answer written with integers, fractions and radicals, such as $3 \pm 2\sqrt{5}$, with no rounding.

CORE IDEAS

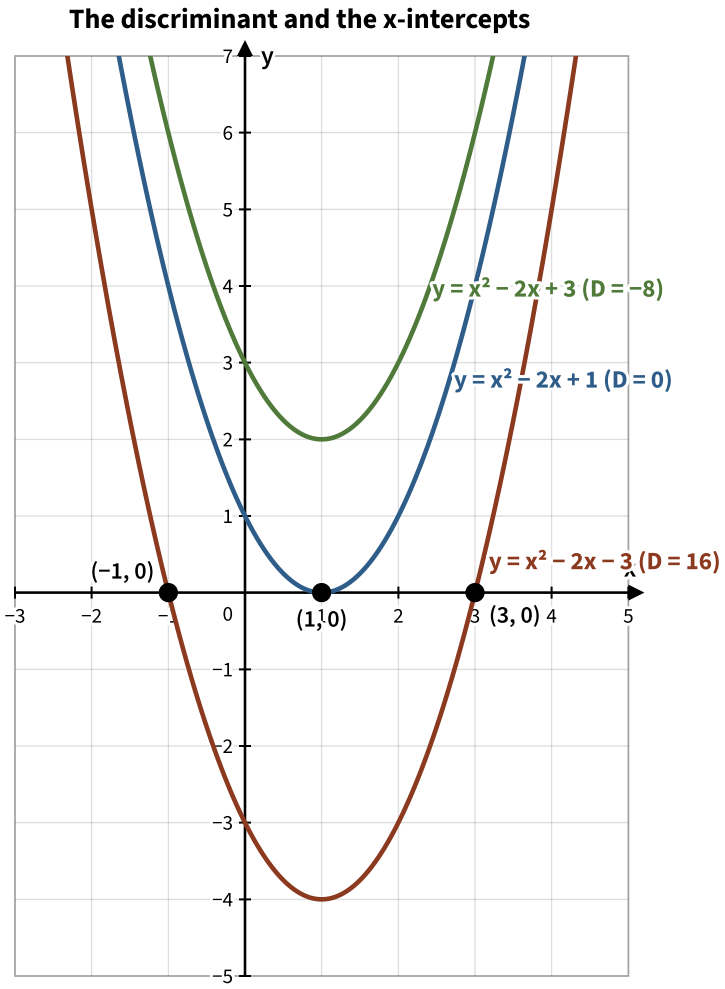
Solving $ax^2 + bx + c = 0$ means finding the inputs where the parabola $y = ax^2 + bx + c$ meets the x-axis, so the solutions of the equation are the zeros of the function from Lesson 3.1. You already know four methods from Algebra I; this lesson reviews them quickly and shows how they fit together.

Factoring. Get 0 on one side, factor, and use the Zero Product Property: if $(x - 3)(x + 5) = 0$, then $x - 3 = 0$ or $x + 5 = 0$. This only works because the other side is 0; a product equal to 6 tells you nothing about each factor. Never divide both sides by a variable: dividing $3x^2 = 12x$ by x loses the solution $x = 0$. Instead write $3x^2 - 12x = 0$ and factor: $3x(x - 4) = 0$.

Square roots. If $(x - h)^2 = d$ with $d \geq 0$, then $x - h = \pm\sqrt{d}$, because both $\sqrt{d}$ and $-\sqrt{d}$ square to d. Forgetting the $\pm$ loses a solution. **Completing the square** turns any quadratic into this form: move the constant, divide by a, then add $\left(\frac{b}{2a}\right)^2$ to both sides.

The quadratic formula is completing the square done once for every equation. Start with $ax^2 + bx + c = 0$ and divide by a: $x^2 + \frac{b}{a}x = -\frac{c}{a}$. Add $\frac{b^2}{4a^2}$ to both sides: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$. Take square roots: $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$. Subtract $\frac{b}{2a}$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Look at the step where we took square roots. It needs $b^2 - 4ac \geq 0$ to give real numbers. That is why the **discriminant** $D = b^2 - 4ac$ counts the real solutions: if $D > 0$ there are two ($\pm$ gives two different numbers), if $D = 0$ there is one (the $\pm$ adds and subtracts 0), and if $D < 0$ there are no real solutions. On the graph: two x-intercepts, one (the vertex touches the axis), or none.



Choosing a method. If there is no x-term, use square roots. If the quadratic factors easily, factor. If $a = 1$ and b is even, completing the square is quick. Otherwise use the formula, which always works. Give exact answers unless the problem asks you to round, and keep the exact form until the very last step so rounding errors do not pile up.

KEY FORMULAS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = b^2 - 4ac: \quad D > 0 \text{ two real solutions}, \quad D = 0 \text{ one}, \quad D < 0 \text{ none}$$

$$(x - h)^2 = d,\ d \geq 0 \implies x = h \pm \sqrt{d}$$

Practice habit. Keep exact answers until the last step, then check each solution in the ORIGINAL equation (a decimal check that comes out close to 0 is fine for irrational answers).

WORKED EXAMPLES

Example 1. Solve $2x^2 + 4x - 7 = 0$. Give exact answers and answers rounded to the nearest hundredth.

Show solution

1. It does not factor over the integers, so use the formula with $a = 2$, $b = 4$, $c = -7$.
2. $D = 4^2 - 4(2)(-7) = 16 + 56 = 72$, and $\sqrt{72} = 6\sqrt{2}$.
3. $x = \dfrac{-4 \pm 6\sqrt{2}}{4} = \dfrac{-2 \pm 3\sqrt{2}}{2}$ (divide every term by 2).
4. $3\sqrt{2} \approx 4.24264$, so $x \approx \dfrac{2.24264}{2} \approx 1.12$ or $x \approx \dfrac{-6.24264}{2} \approx -3.12$.
5. Check $x \approx 1.12132$: $2(1.25736) + 4.48528 - 7 = 2.51472 + 4.48528 - 7 = 0$ ✓.

Answer: $x = \dfrac{-2 \pm 3\sqrt{2}}{2}$, so $x \approx -3.12$ or $x \approx 1.12$

Example 2. A church garden is a rectangle 20 feet by 30 feet. The congregation will lay a gravel walkway of the same width all the way around it, so that the garden and walkway together cover 816 square feet. How wide is the walkway?

Show solution

1. Let x be the width in feet. The outer rectangle is $20 + 2x$ by $30 + 2x$ (the walkway adds x on each side).
2. $(20 + 2x)(30 + 2x) = 816$, so $600 + 100x + 4x^2 = 816$.
3. $4x^2 + 100x - 216 = 0$; divide by 4: $x^2 + 25x - 54 = 0$.
4. Factor: $(x + 27)(x - 2) = 0$, so $x = -27$ or $x = 2$.
5. A width cannot be negative, so $x = 2$.
6. Check: $(24)(34) = 816$ ✓.

Answer: The walkway is 2 feet wide.

PRACTICE

1. **Easy** Solve by factoring: $x^2 - 7x + 12 = 0$.

   Show answer

   $(x - 3)(x - 4) = 0$, so $x = 3$ or $x = 4$

2. **Easy** Solve by square roots: $4x^2 = 49$.

   Show answer

   $x^2 = \frac{49}{4}$, so $x = \pm\frac{7}{2}$

3. **Easy** Solve $(x - 3)^2 = 20$. Give exact answers in simplest radical form.

   Show answer

   $x - 3 = \pm\sqrt{20} = \pm 2\sqrt{5}$, so $x = 3 \pm 2\sqrt{5}$

4. **Easy** Find the discriminant of $2x^2 - 3x + 5 = 0$ and state how many real solutions the equation has.

   Show answer

   $D = (-3)^2 - 4(2)(5) = 9 - 40 = -31$; no real solutions

5. **Medium** Solve by completing the square: $x^2 + 8x - 5 = 0$. Give exact answers.

   Show answer

   $x^2 + 8x + 16 = 5 + 16$, $(x + 4)^2 = 21$, $x = -4 \pm \sqrt{21}$

6. **Medium** Solve by factoring: $6x^2 - x - 2 = 0$.

   Show answer

   $(3x - 2)(2x + 1) = 0$, so $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

7. **Medium** Solve $x^2 - 4x - 1 = 0$. Give exact answers and answers rounded to the nearest hundredth.

   Show answer

   $x = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$, so $x \approx -0.24$ or $x \approx 4.24$

8. **Medium** Solve $3x^2 = 12x$. (Be careful not to lose a solution.)

   Show answer

   $3x^2 - 12x = 0$, $3x(x - 4) = 0$, so $x = 0$ or $x = 4$

9. Medium Samuel drops a stone from a footbridge 144 feet above a river. Its height in feet after t seconds is $h = -16t^2 + 144$. After how many seconds does it hit the water?

Show answer

$16t^2 = 144$, $t^2 = 9$, $t = 3$ (reject $t = -3$); 3 seconds

10. Hard A ball is thrown upward from a height of 5 feet at 40 feet per second, so its height after t seconds is $h = -16t^2 + 40t + 5$ feet. When does it hit the ground? Round to the nearest tenth of a second.

Show answer

$16t^2 - 40t - 5 = 0$; $t = \dfrac{40 \pm \sqrt{1600 + 320}}{32} = \dfrac{40 \pm \sqrt{1920}}{32}$; the positive root is $\dfrac{40 + 43.818}{32} \approx 2.6$ seconds

11. Hard For what values of k does $x^2 + kx + 9 = 0$ have exactly one real solution?

Show answer

Set $D = k^2 - 36 = 0$: $k = -6$ or $k = 6$

12. Hard The Miller family wants a rectangular vegetable bed with an area of 60 square feet and a perimeter of 34 feet. What should its dimensions be?

Show answer

$L + W = 17$ and $LW = 60$, so $W(17 - W) = 60$, $W^2 - 17W + 60 = 0$, $(W - 5)(W - 12) = 0$; 5 feet by 12 feet

13. Hard Solve $2x^2 - 6x + 1 = 0$ by completing the square, and show that your answer agrees with the quadratic formula.

Show answer

$x^2 - 3x = -\dfrac{1}{2}$; $\left(x - \dfrac{3}{2}\right)^2 = \dfrac{9}{4} - \dfrac{1}{2} = \dfrac{7}{4}$; $x = \dfrac{3 \pm \sqrt{7}}{2}$. Formula: $\dfrac{6 \pm \sqrt{36 - 8}}{4} = \dfrac{6 \pm 2\sqrt{7}}{4} = \dfrac{3 \pm \sqrt{7}}{2}$ ✓

14. **Challenge** Starting from $ax^2 + bx + c = 0$ with $a \neq 0$, derive the quadratic formula by completing the square. Point out the step that requires $b^2 - 4ac \geq 0$ for real solutions.

Show answer

Divide by a: $x^2 + \frac{b}{a}x = -\frac{c}{a}$. Add $\frac{b^2}{4a^2}$: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$. Take square roots (this step needs $b^2 - 4ac \geq 0$ to give real numbers): $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$. So $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

15. **Challenge** The product of two consecutive odd integers is 323. Find both pairs of integers that work.

Show answer

$n(n+2) = 323$, $n^2 + 2n - 323 = 0$, $D = 4 + 1292 = 1296 = 36^2$, $n = \frac{-2 \pm 36}{2} = 17$ or -19; the pairs are 17 and 19, and -19 and -17

COMMON MISTAKES

- Using the Zero Product Property when the other side is not 0. From $x(x-2) = 8$ you cannot conclude $x = 8$; first write $x^2 - 2x - 8 = 0$ and factor: $(x-4)(x+2) = 0$.
- Forgetting the $\pm$ when taking a square root, which loses half the solutions. $x^2 = 25$ means $x = 5$ or $x = -5$.
- Dividing only part of the numerator: $\frac{-4 \pm 6\sqrt{2}}{4}$ is $\frac{-2 \pm 3\sqrt{2}}{2}$, not $-4 \pm \frac{3\sqrt{2}}{2}$. Divide every term on top by the same number, or none of them.
- Mixing up signs in the formula when b or c is negative. Write a, b and c with their signs first, and put negative values in parentheses: $(-3)^2 - 4(2)(5)$.

> **Make it a game · Two Solvers, One Answer.** Race the parent on four quadratic equations. One of you factors or uses square roots, while the other uses the quadratic formula. Compare answers at the end of each problem; they must match exactly. For the final boss round, recite the steps of deriving the quadratic formula by completing the square, with the parent checking each line.

3.3 Complex Numbers

OBJECTIVES

- Explain why the imaginary unit i is needed and write square roots of negative numbers using i.
- Add, subtract, multiply and divide complex numbers, writing answers in the form $a + bi$.
- Simplify powers of i and explain why they repeat every four.
- Plot complex numbers in the complex plane and find their absolute values.

WATCH

Introduction to Complex Numbers (2 of 2: Why Algebra Requires Complex Numbers)

Eddie Woo · 11:12

https://www.youtube.com/watch?v=0OHiSZUvn0I

> **WHEN TO WATCH**
> Before the Core ideas: why algebra needs the number i.

Note: Eddie Woo teaches in Australia: he writes $\sqrt{-1}$ and talks about 'surds' for radicals.

BACKUP VIDEO

Complex and Imaginary Numbers (Add, Subtract, Multiply, Divide, Standard a+bi form)

Mario's Math Tutoring · 10:44

https://www.youtube.com/watch?v=8xWE2ns8T6Q

Before Example 1: adding, subtracting, multiplying and dividing complex numbers.

KEY TERMS

imaginary unit

The number i defined by $i^2 = -1$.

complex number

A number $a + bi$ where a and b are real. The real number a is the real part and the real number b is the imaginary part.

pure imaginary number

A complex number bi with real part 0 and $b \neq 0$, such as $5i$.

complex conjugate

The conjugate of $a + bi$ is $a - bi$: the same real part, the opposite imaginary part.

complex plane

A coordinate plane where $a + bi$ is plotted as the point (a, b): the horizontal axis is the real axis and the vertical axis is the imaginary axis.

absolute value (modulus)

The distance from $a + bi$ to 0 in the complex plane: $|a + bi| = \sqrt{a^2 + b^2}$.

CORE IDEAS

Every real number squared is 0 or positive, so the equation $x^2 = -1$ has no real solution. Mathematicians have met this kind of problem before: $x + 5 = 2$ has no solution among the counting numbers, so negative numbers were accepted, and $2x = 3$ needs fractions. Each time, the number system was enlarged so the old rules keep working. Here we add one new number, the **imaginary unit** i, whose only special property is $i^2 = -1$. Everything else follows from the usual rules of algebra.

With i we can take the square root of any negative number. For $k > 0$ we define $\sqrt{-k} = i\sqrt{k}$, which works because $(i\sqrt{k})^2 = i^2 \cdot k = -k$. So $\sqrt{-25} = 5i$ and $\sqrt{-12} = i\sqrt{12} = 2i\sqrt{3}$. A **complex number** has the form $a + bi$ with a and b real. Real numbers are the complex numbers with $b = 0$, so the complex numbers include every number you already know. Two complex numbers are equal only when their real parts are equal and their imaginary parts are equal.

Adding, subtracting and multiplying work just like polynomials in i: combine like terms, multiply with the distributive property (FOIL), and then replace i^2 by -1. For example, $(2 + 3i)(4 - i) = 8 - 2i + 12i - 3i^2 = 8 + 10i + 3 = 11 + 10i$. Always finish in the form $a + bi$.

Powers of i repeat every four. $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, and $i^4 = (i^2)^2 = (-1)^2 = 1$. Since $i^4 = 1$, multiplying by i^4 changes nothing, so the pattern starts over: $i^5 = i$, $i^6 = -1$, and so on. To simplify i^n, divide n by 4 and use the remainder: $i^{27} = i^{24} \cdot i^3 = 1 \cdot (-i) = -i$.

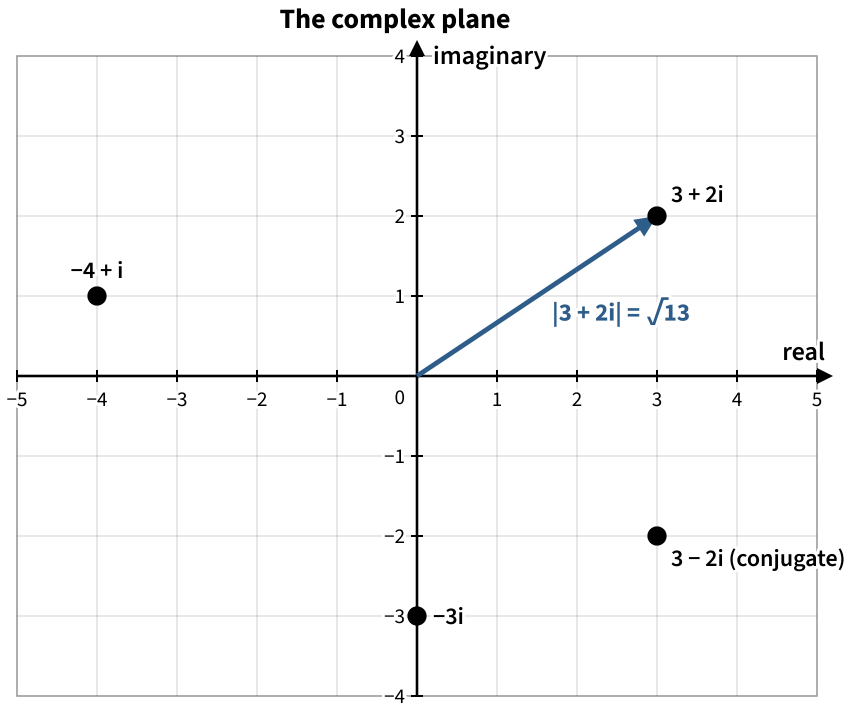
Powers of i

power	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
value	i	-1	$-i$	1	i	-1	$-i$	1

Conjugates and division. The product of $a + bi$ and its conjugate $a - bi$ is $a^2 - abi + abi - b^2 i^2 = a^2 + b^2$. The imaginary terms cancel, and $-b^2 i^2 = +b^2$, so the product is always a real number (and never negative). That is the key to dividing: multiply the top and bottom of the fraction by the conjugate of the denominator, which makes the denominator real, just as you rationalize a denominator with radicals. For example, $\dfrac{3+i}{1-2i} \cdot \dfrac{1+2i}{1+2i} = \dfrac{3+7i+2i^2}{1+4} = \dfrac{1+7i}{5} = \dfrac{1}{5} + \dfrac{7}{5}i$.

A warning. The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ was proved only for $a, b \geq 0$, and it fails when both are negative: $\sqrt{-4} \cdot \sqrt{-9} = (2i)(3i) = 6i^2 = -6$, while $\sqrt{(-4)(-9)} = \sqrt{36} = 6$. To stay safe, always rewrite square roots of negative numbers with i first, then compute.

The complex plane. Plot $a + bi$ as the point (a, b), with real parts across and imaginary parts up. A conjugate is the reflection of a number in the real axis. The distance from $a + bi$ to the origin is, by the Pythagorean theorem, $|a + bi| = \sqrt{a^2 + b^2}$; for a real number this is the ordinary absolute value. Notice that $(a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2$.



KEY FORMULAS

$$i^2 = -1, \qquad \sqrt{-k} = i\sqrt{k} \ \ (k > 0)$$

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$$

$$(a + bi)(a - bi) = a^2 + b^2$$

$$|a + bi| = \sqrt{a^2 + b^2}$$

From history. Gerolamo Cardano met square roots of negative numbers in his book Ars Magna (1545) while solving equations, and Rafael Bombelli gave rules for computing with them in 1572. René Descartes called such numbers "imaginary" in 1637, a name that stuck even though they are as useful as any other numbers. Leonhard Euler introduced the letter i in 1777.

Practice habit. Know the powers of i cold: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and then the cycle repeats. Whenever you see i^2 in your work, replace it with -1 right away.

WORKED EXAMPLES

Example 1. Simplify each and write in the form $a+bi$: (a) $(5-2i)-(3-6i)$ (b) $(4+3i)(2-5i)$ (c) i^{27}

Show solution

1. (a) Distribute the minus sign: $5-2i-3+6i$.
2. Combine real parts and imaginary parts: $2+4i$.
3. (b) FOIL: $8-20i+6i-15i^2$.
4. Replace i^2 with -1: $8-14i+15=23-14i$.
5. (c) $27=4\times 6+3$, so $i^{27}=(i^4)^6\cdot i^3=1\cdot(-i)=-i$.

Answer: (a) $2+4i$ (b) $23-14i$ (c) $-i$

Example 2. Electrical engineers describe alternating-current circuits with complex numbers. Voltage V (volts), current I (amperes) and impedance Z (ohms) are related by $V=IZ$. If $V=8+i$ and $Z=2-i$, find the current $I=\dfrac{V}{Z}$, then find its size $|I|$ to the nearest hundredth of an ampere.

Show solution

1. $I=\dfrac{8+i}{2-i}$. Multiply top and bottom by the conjugate $2+i$.
2. Top: $(8+i)(2+i)=16+8i+2i+i^2=16+10i-1=15+10i$.
3. Bottom: $(2-i)(2+i)=4+1=5$.
4. $I=\dfrac{15+10i}{5}=3+2i$.
5. Check: $(3+2i)(2-i)=6-3i+4i-2i^2=6+i+2=8+i$ ✓.
6. $|I|=\sqrt{3^2+2^2}=\sqrt{13}\approx 3.61$ amperes.

Answer: $I=3+2i$ amperes; $|I|=\sqrt{13}\approx 3.61$ amperes

PRACTICE

1. Easy Write each using i, in simplest form: (a) $\sqrt{-49}$ (b) $\sqrt{-18}$

   Show answer

   (a) $7i$ (b) $i\sqrt{18}=3i\sqrt{2}$

2. **Easy** Add: $(6 + 2i) + (-4 + 5i)$.

   Show answer

   $2 + 7i$

3. **Easy** Multiply and write in the form $a + bi$: $3i(4 - 2i)$.

   Show answer

   $12i - 6i^2 = 6 + 12i$

4. **Easy** Simplify: (a) i^{14} (b) i^{33}

   Show answer

   (a) $14 = 12 + 2$, so $i^{14} = i^2 = -1$ (b) $33 = 32 + 1$, so $i^{33} = i$

5. **Medium** Multiply: $(2 - 3i)(4 + i)$.

   Show answer

   $8 + 2i - 12i - 3i^2 = 8 - 10i + 3 = 11 - 10i$

6. **Medium** Square: $(5 + 2i)^2$.

   Show answer

   $25 + 20i + 4i^2 = 21 + 20i$

7. **Medium** In an alternating-current circuit, $V = IZ$. If the current is $I = 2 + i$ amperes and the impedance is $Z = 4 - 3i$ ohms, find the voltage V.

   Show answer

   $(2 + i)(4 - 3i) = 8 - 6i + 4i - 3i^2 = 8 - 2i + 3 = 11 - 2i$ volts

8. **Medium** A surveyor draws a farm map on the complex plane with 1 unit = 100 feet and the farmhouse at 0. The well is at $6 + 8i$. How far is the well from the farmhouse, in feet?

   Show answer

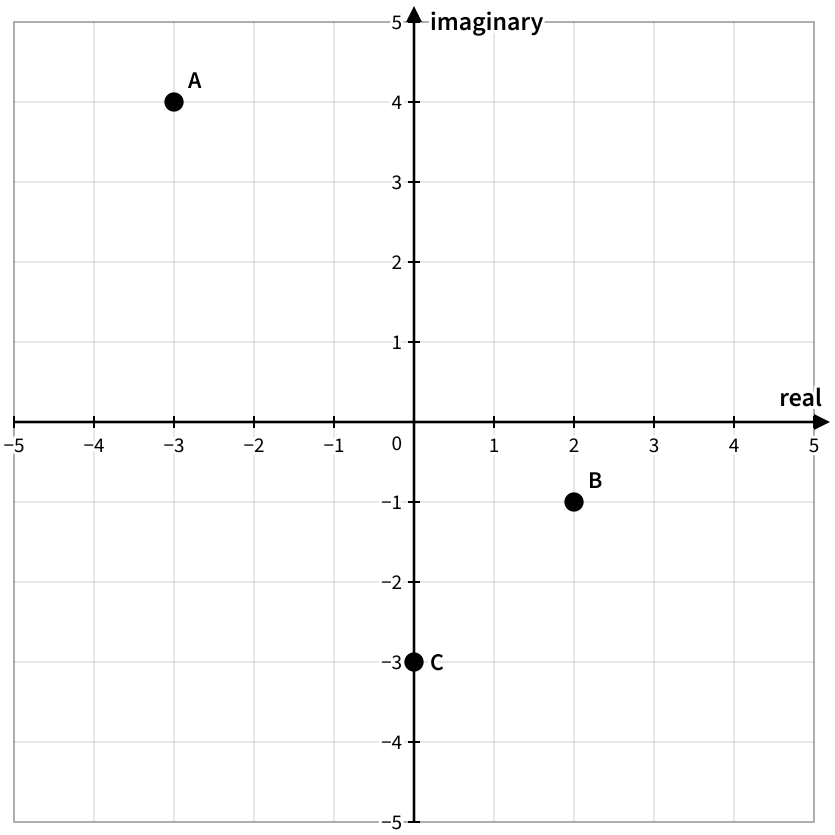
   $|6 + 8i| = \sqrt{36 + 64} = 10$ units, which is 1000 feet

9. **Medium** (a) Multiply $(3 - i)(3 + i)$. (b) Use your result to write $\dfrac{10}{3 - i}$ in the form $a + bi$.

   Show answer

   (a) $9 + 1 = 10$ (b) $\dfrac{10(3 + i)}{10} = 3 + i$

10. **Medium** The graph shows the complex plane. (a) Write the complex numbers at A, B and C. (b) Find $|A|$.



Show answer

(a) A $= -3 + 4i$, B $= 2 - i$, C $= -3i$ (b) $|A| = \sqrt{9 + 16} = 5$

11. **Hard** Divide and write in the form $a + bi$: $\dfrac{1 + 5i}{1 - i}$.

Show answer

$\dfrac{(1 + 5i)(1 + i)}{(1 - i)(1 + i)} = \dfrac{1 + 6i + 5i^2}{2} = \dfrac{-4 + 6i}{2} = -2 + 3i$; check: $(-2 + 3i)(1 - i) = -2 + 5i + 3 = 1 + 5i$ ✓

12. **Hard** When two parts of a circuit are connected in series, their impedances add. Find the total impedance of $Z_1 = 3 + 4i$ ohms and $Z_2 = 5 - 2i$ ohms, and its size $|Z|$ in simplest radical form.

Show answer

$Z = 8 + 2i$ ohms; $|Z| = \sqrt{64 + 4} = \sqrt{68} = 2\sqrt{17} \approx 8.25$ ohms

13. Hard (a) Compute $\sqrt{-4} \cdot \sqrt{-9}$ correctly. (b) Explain why the answer is not $\sqrt{36} = 6$.

Show answer

(a) $(2i)(3i) = 6i^2 = -6$. (b) The rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds only when $a, b \geq 0$ (then $\sqrt{a}\sqrt{b}$ is a nonnegative number whose square is ab). For negative numbers you must write each root with i first, and $i \cdot i = -1$ brings in a minus sign.

14. Hard Is $1 + 2i$ a solution of $x^2 - 2x + 5 = 0$? Show your check.

Show answer

Yes: $(1 + 2i)^2 = 1 + 4i - 4 = -3 + 4i$, and $-3 + 4i - 2(1 + 2i) + 5 = -3 + 4i - 2 - 4i + 5 = 0$ ✓

15. Challenge Find the sum $i + i^2 + i^3 + i^4 + \cdots + i^{100}$ and explain your reasoning.

Show answer

Each group of four consecutive powers adds to $i - 1 - i + 1 = 0$. The 100 terms form 25 such groups, so the sum is 0.

COMMON MISTAKES

- Leaving i^2 in an answer or treating it as $+1$. Every i^2 becomes -1, so $-3i^2 = +3$.
- Multiplying square roots of negative numbers before writing them with i: $\sqrt{-4}\sqrt{-9}$ is -6, not 6. Write $\sqrt{-4} = 2i$ and $\sqrt{-9} = 3i$ first.
- Dividing by multiplying only the denominator by the conjugate. Multiply the top and the bottom by the same conjugate, or you change the value of the fraction.
- Computing $|a + bi|$ as $a + b$ or as $\sqrt{a^2 - b^2}$. It is a distance, so use the Pythagorean theorem: $|3 - 4i| = \sqrt{9 + 16} = 5$.

> **Make it a game · The Cycle of i.** The parent calls powers such as i^5, i^{10}, i^{27} and i^{40}, and you answer in three seconds by spotting the cycle i, -1, -i, 1. Then multiply (3 + 2i)(1 - i), divide by a conjugate, and plot the answer on a complex plane drawn on paper. Try to beat your own streak of correct powers.

3.4 Complex Solutions and the Discriminant

OBJECTIVES

- Solve quadratic equations with negative discriminants, giving non-real solutions in the form $a + bi$.
- Explain why non-real solutions of a quadratic with real coefficients come in conjugate pairs.
- Use the discriminant to classify solutions as two real, one repeated real or two non-real, and connect each case to the graph.
- Derive and use the sum $-\frac{b}{a}$ and product $\frac{c}{a}$ of the roots to check solutions and to write a quadratic equation from its roots.

WATCH

Discriminant of a Quadratic Equation - Determine Number and Type of Solutions

Mario's Math Tutoring · 9:44

https://www.youtube.com/watch?v=vu--ssdvN8g

> **WHEN TO WATCH**
> Before Example 1: using the discriminant to predict the number and type of solutions.

Note: Eddie Woo teaches in Australia: the backup calls the roots α and β and calls these 'Viète's results'.

BACKUP VIDEO

Sum & Product of Roots (1 of 5: Introduction to Viete's Results)

Eddie Woo · 11:39

https://www.youtube.com/watch?v=yOVpa0QDgfg

Before Example 1, which checks its answers this way: the sum and product of the roots.

KEY TERMS

non-real solution

A solution $p + qi$ with $q \neq 0$. It is a complex number that is not a real number.

conjugate pair

Two numbers $p + qi$ and $p - qi$ that are complex conjugates of each other.

repeated (double) root

A solution that occurs twice, as $x = 3$ does in $(x - 3)^2 = 0$. It happens when the discriminant is 0.

roots

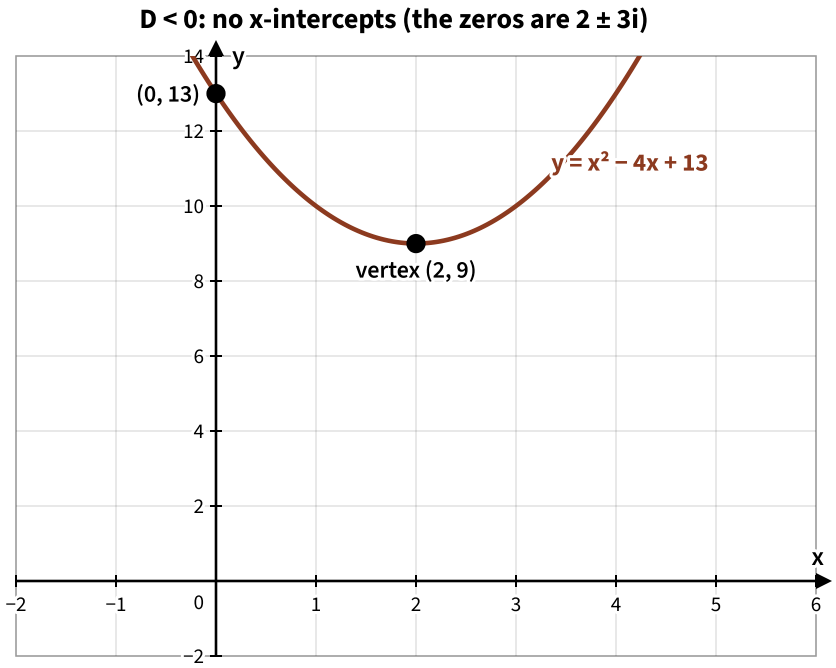
Another word for the solutions of an equation such as $ax^2 + bx + c = 0$.

CORE IDEAS

In Lesson 3.2, a negative discriminant meant "no real solutions." Now that we have complex numbers, every quadratic equation has solutions. Use the quadratic formula exactly as before, and write the square root of the negative discriminant with i. For $x^2 - 4x + 13 = 0$: $D = 16 - 52 = -36$, so $x = \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$. Completing the square gives the same answer: $(x - 2)^2 = -13 + 4 = -9$, so $x - 2 = \pm 3i$.

Why non-real solutions come in conjugate pairs. Write the formula as $x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{D}}{2a}$. When a, b and c are real and $D < 0$, the first part $-\dfrac{b}{2a}$ is a real number p, and $\sqrt{D} = i\sqrt{|D|}$ makes the second part a pure imaginary number qi. The $\pm$ then gives $p + qi$ and $p - qi$, a conjugate pair. So a quadratic with real coefficients can never have just one non-real solution.

The discriminant tells the whole story. If $D > 0$, there are two different real solutions, and the parabola $y = ax^2 + bx + c$ crosses the x-axis twice. If $D = 0$, there is one repeated real solution, and the vertex sits on the x-axis. If $D < 0$, there are two non-real conjugate solutions, and the parabola never touches the x-axis: it lies entirely above it (if $a > 0$) or entirely below it (if $a < 0$). Non-real solutions are not points on the graph, but there is a link: their real part $p = -\dfrac{b}{2a}$ is the x-coordinate of the vertex.



The sum and product of the roots. If r_1 and r_2 are the roots, the parabola can be written in factored form $a(x - r_1)(x - r_2)$. Multiplying out gives $ax^2 - a(r_1 + r_2)x + ar_1r_2$. Comparing with $ax^2 + bx + c$: $b = -a(r_1 + r_2)$ and $c = ar_1r_2$, so $r_1 + r_2 = -\dfrac{b}{a}$ and $r_1r_2 = \dfrac{c}{a}$. You can confirm this straight from the formula: adding the two roots cancels the $\pm$ parts, $\dfrac{-b + \sqrt{D}}{2a} + \dfrac{-b - \sqrt{D}}{2a} = \dfrac{-2b}{2a} = -\dfrac{b}{a}$, and multiplying them is a difference of squares, $\dfrac{b^2 - D}{4a^2} = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{c}{a}$. These facts hold for real and non-real roots alike.

The sum and product make a fast check. For $x^2 - 4x + 13 = 0$ the roots $2 \pm 3i$ have sum $4 = -\dfrac{-4}{1}$ ✓ and product $(2 + 3i)(2 - 3i) = 4 + 9 = 13 = \dfrac{13}{1}$ ✓.

Writing an equation from its roots. Dividing $ax^2 + bx + c = 0$ by a gives $x^2 - (r_1 + r_2)x + r_1r_2 = 0$: the coefficient of x is minus the sum and the constant is the product. For roots $\dfrac{1}{2}$ and -3: the sum is $-\dfrac{5}{2}$ and the product is $-\dfrac{3}{2}$, so $x^2 + \dfrac{5}{2}x - \dfrac{3}{2} = 0$, or $2x^2 + 5x - 3 = 0$ with integer coefficients. For roots $1 \pm 4i$: the sum is 2 and the product is $1 + 16 = 17$, so $x^2 - 2x + 17 = 0$.

KEY FORMULAS

$D = b^2 - 4ac$: $D > 0$ two real, $D = 0$ one repeated real, $D < 0$ two non-real conjugates

$$r_1 + r_2 = -\frac{b}{a}, \qquad r_1 r_2 = \frac{c}{a}$$

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

Practice habit. Check every pair of roots with the sum and product: they must add to $-\frac{b}{a}$ and multiply to $\frac{c}{a}$. This catches most errors in seconds, even with non-real roots.

WORKED EXAMPLES

Example 1. Solve $2x^2 + 4x + 5 = 0$. Check with the sum and product of the roots.

Show solution

1. $a = 2$, $b = 4$, $c = 5$. $D = 16 - 40 = -24$, so there are two non-real solutions.
2. $\sqrt{-24} = i\sqrt{24} = 2i\sqrt{6}$.
3. $x = \dfrac{-4 \pm 2i\sqrt{6}}{4} = -1 \pm \dfrac{\sqrt{6}}{2}i$.
4. Sum: $(-1) + (-1) = -2$, and $-\dfrac{b}{a} = -\dfrac{4}{2} = -2$ ✓.
5. Product: $(-1)^2 + \left(\dfrac{\sqrt{6}}{2}\right)^2 = 1 + \dfrac{6}{4} = \dfrac{5}{2}$, and $\dfrac{c}{a} = \dfrac{5}{2}$ ✓.

Answer: $x = -1 \pm \dfrac{\sqrt{6}}{2}i$

Example 2. Grace kicks a ball straight up from a height of 4 feet at 48 feet per second, so its height after t seconds is $h = -16t^2 + 48t + 4$ feet. Use the discriminant to decide whether the ball ever reaches 50 feet.

Show solution

1. Set $h = 50$: $-16t^2 + 48t + 4 = 50$.
2. Rearrange: $-16t^2 + 48t - 46 = 0$; multiply by $-\frac{1}{2}$: $8t^2 - 24t + 23 = 0$.
3. $D = (-24)^2 - 4(8)(23) = 576 - 736 = -160$.
4. $D < 0$, so there is no real time t when $h = 50$: the ball never reaches 50 feet.
5. Check with the vertex: $t = -\dfrac{48}{2(-16)} = 1.5$ and $h(1.5) = -36 + 72 + 4 = 40$ feet, the greatest height, which is less than 50 ✓. (The non-real solutions $t = \dfrac{3}{2} \pm \dfrac{\sqrt{10}}{4}i$ have real part 1.5, the time of the peak.)

Answer: No. The discriminant is $-160 < 0$; the ball rises only to 40 feet.

PRACTICE

1. Easy Find the discriminant and describe the solutions (two real, one repeated real or two non-real): (a) $x^2 + 6x + 9 = 0$ (b) $3x^2 - 2x + 4 = 0$

   Show answer

   (a) $D = 36 - 36 = 0$: one repeated real solution (b) $D = 4 - 48 = -44$: two non-real solutions

2. Easy Solve $x^2 = -16$.

   Show answer

   $x = \pm\sqrt{-16} = \pm 4i$

3. Easy Without solving, find the sum and the product of the roots of $2x^2 - 6x + 7 = 0$.

   Show answer

   Sum $= -\dfrac{-6}{2} = 3$; product $= \dfrac{7}{2}$

4. **Easy** A rectangular church banner has an area of 24 square feet and a perimeter of 20 feet. Its length and width are the roots of a quadratic with sum 10 and product 24. Write that equation and find the dimensions.

   Show answer

   $x^2 - 10x + 24 = 0$, $(x-4)(x-6)=0$; 4 feet by 6 feet

5. **Medium** Solve $x^2 - 6x + 13 = 0$.

   Show answer

   $D = 36 - 52 = -16$; $x = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$

6. **Medium** Solve by completing the square: $x^2 + 4x + 7 = 0$.

   Show answer

   $(x+2)^2 = -7 + 4 = -3$, so $x = -2 \pm i\sqrt{3}$

7. **Medium** One root of a quadratic equation with real coefficients is $5 - 2i$. (a) What is the other root? (b) Write the equation with leading coefficient 1.

   Show answer

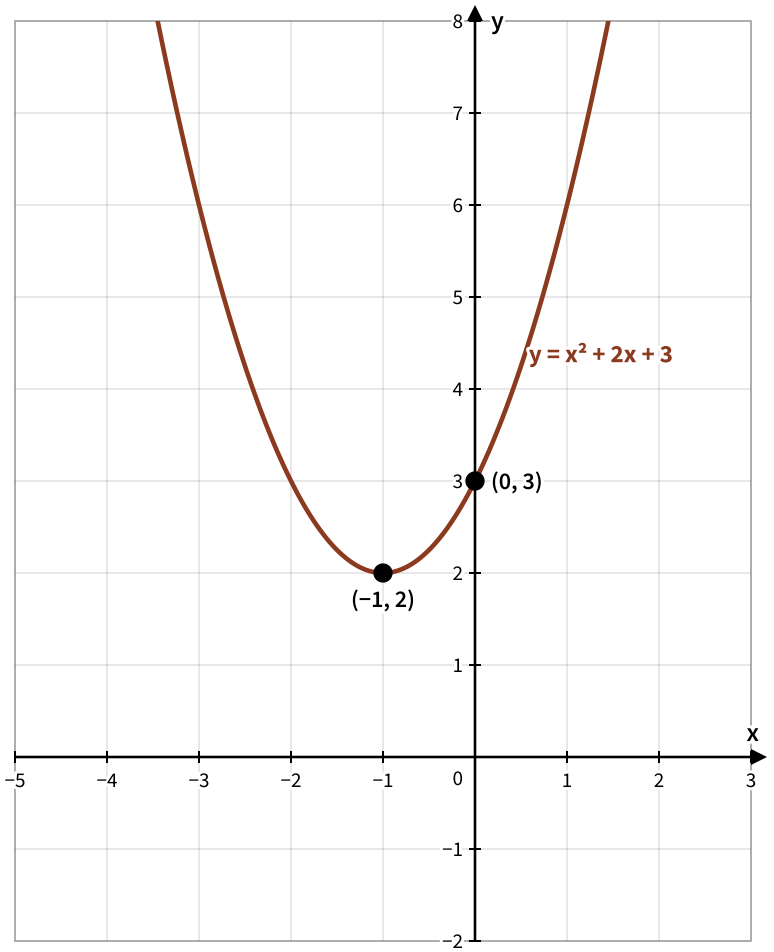
   (a) $5 + 2i$ (b) sum 10, product $25 + 4 = 29$: $x^2 - 10x + 29 = 0$

8. **Medium** Write a quadratic equation with integer coefficients whose roots are -4 and $\dfrac{3}{2}$.

   Show answer

   Sum $-\dfrac{5}{2}$, product -6: $x^2 + \dfrac{5}{2}x - 6 = 0$, so $2x^2 + 5x - 12 = 0$ (check: $(x+4)(2x-3) = 2x^2 + 5x - 12$ ✓)

9. **Medium** The graph of $y = x^2 + 2x + 3$ is shown. (a) How can you tell from the graph that the equation $x^2 + 2x + 3 = 0$ has no real solutions? (b) Find its solutions.



Show answer

(a) The parabola opens up and its lowest point $(-1, 2)$ is above the x-axis, so it never crosses the axis. (b) $D = 4 - 12 = -8$; $x = \dfrac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$

10. **Medium** A model rocket is launched from the ground at 64 feet per second, so its height is $h = -16t^2 + 64t$ feet after t seconds. Use the discriminant to decide whether it reaches 80 feet.

Show answer

$-16t^2 + 64t = 80$ gives $t^2 - 4t + 5 = 0$; $D = 16 - 20 = -4 < 0$, so no. (Its greatest height is $h(2) = 64$ feet.)

11. Hard Solve $3x^2 - 2x + 1 = 0$ and check with the sum and product of the roots.

Show answer

$D = 4 - 12 = -8$; $x = \dfrac{2 \pm 2i\sqrt{2}}{6} = \dfrac{1}{3} \pm \dfrac{\sqrt{2}}{3}i$. Sum $\dfrac{2}{3} = -\dfrac{b}{a}$ ✓; product $\dfrac{1}{9} + \dfrac{2}{9} = \dfrac{1}{3} = \dfrac{c}{a}$ ✓

12. Hard Henry has 40 feet of fencing and wants a rectangular garden with an area of 120 square feet. Is this possible? Use a quadratic and its discriminant to decide.

Show answer

$L + W = 20$ and $LW = 120$, so $W^2 - 20W + 120 = 0$; $D = 400 - 480 = -80 < 0$, so no real dimensions work. (A 40-foot perimeter encloses at most 100 square feet, with a 10-by-10 square.)

13. Hard For what values of k does $x^2 - 6x + k = 0$ have two non-real solutions?

Show answer

$D = 36 - 4k < 0$, so $k > 9$

14. Hard Use the quadratic formula to prove that the roots r_1 and r_2 of $ax^2 + bx + c = 0$ satisfy $r_1 + r_2 = -\dfrac{b}{a}$ and $r_1 r_2 = \dfrac{c}{a}$.

Show answer

Let $D = b^2 - 4ac$, $r_1 = \dfrac{-b + \sqrt{D}}{2a}$, $r_2 = \dfrac{-b - \sqrt{D}}{2a}$. Sum: the $\sqrt{D}$ terms cancel, leaving $\dfrac{-2b}{2a} = -\dfrac{b}{a}$. Product: $\dfrac{(-b)^2 - (\sqrt{D})^2}{4a^2} = \dfrac{b^2 - b^2 + 4ac}{4a^2} = \dfrac{c}{a}$. (This works whether D is positive, zero or negative.)

15. Challenge Can a quadratic equation with real coefficients have the solutions $2 + i$ and 3? Explain.

Show answer

No. If $2 + i$ is a solution, the discriminant must be negative, and then the $\pm$ in the formula makes the other solution its conjugate $2 - i$. So the solutions would be $2 \pm i$, and 3 could not be one of them.

COMMON MISTAKES

- Writing "no solution" when $D < 0$. There are no REAL solutions, but there are two non-real ones. Read whether the problem asks for real solutions or all solutions.

- Dividing only part of the numerator: $\frac{6 \pm 4i}{2} = 3 \pm 2i$, not $3 \pm 4i$.
- Getting the sign of the sum wrong when writing an equation from its roots. The equation is $x^2 - (\text{sum})x + \text{product} = 0$; roots 2 and 5 give $x^2 - 7x + 10 = 0$, not $x^2 + 7x + 10 = 0$.
- Squaring i wrongly in the product of conjugates: $(p + qi)(p - qi) = p^2 + q^2$, so $(5 - 2i)(5 + 2i) = 29$, not 21.

> **Make it a game · Reverse Engineer.** Choose a conjugate pair of roots, such as 2 + 3i and 2 - 3i, and build the quadratic that has them: x^2 - 4x + 13. Hand the parent only the equation. The parent uses the discriminant to solve it and gets the roots back. Explain aloud why non-real roots of a real-coefficient quadratic come in pairs. Trade three times.

3.5 Quadratic Inequalities

OBJECTIVES

- Solve quadratic inequalities in one variable using the zeros and a sign chart, and write the solution in interval notation.
- Explain why a quadratic can change sign only at its zeros, and read solutions from a graph.
- Handle the special cases of no real zeros or one repeated zero.
- Graph quadratic inequalities in two variables and solve word problems about heights, prices and areas.

WATCH

Solving Quadratic Inequalities

Eddie Woo · 4:08

https://www.youtube.com/watch?v=hpFnHMPCYn4

> **WHEN TO WATCH**
> Before Example 1: solving a quadratic inequality from the graph of the parabola.

BACKUP VIDEO

Solving Quadratic Inequalities Using Sign Analysis

Mario's Math Tutoring · 2:07

https://www.youtube.com/watch?v=NvfAei2S320

After Example 1: the sign-analysis (test-point) method, a second way to solve.

KEY TERMS

quadratic inequality

An inequality that can be written $ax^2 + bx + c > 0$ (or with $<$, $\leq$, $\geq$), with $a \neq 0$.

sign chart

A number line divided at the zeros of an expression, marked + or − on each interval to show where the expression is positive or negative.

test point

A single value chosen inside an interval (or a point chosen in a region) to find out whether the whole interval or region satisfies the inequality.

boundary

The curve or points where the two sides are equal. It is included (solid or closed) for $\leq$ and $\geq$, and left out (dashed or open) for $<$ and $>$.

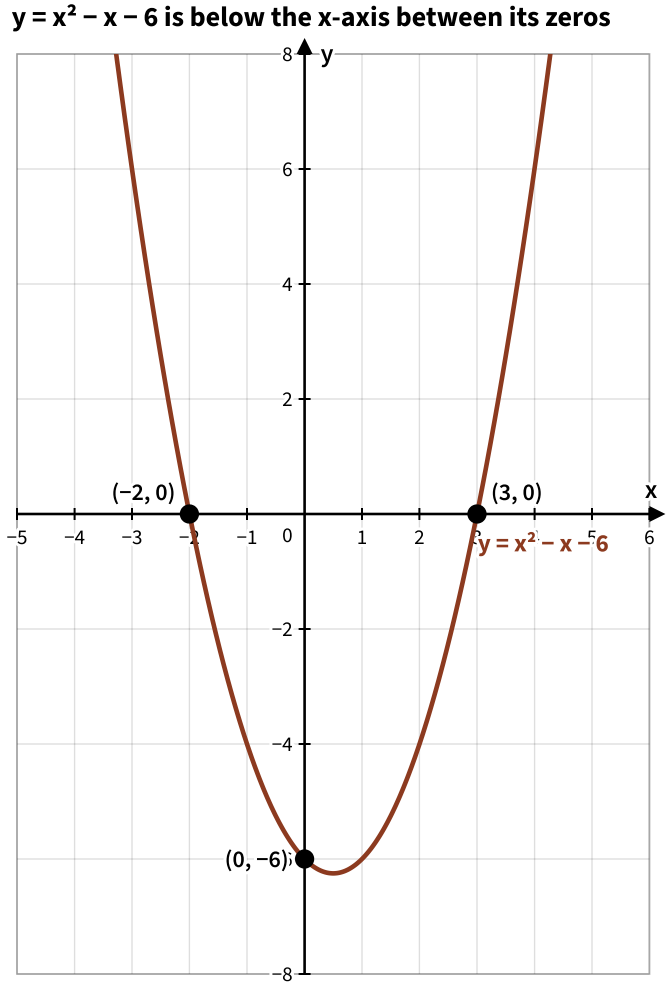
CORE IDEAS

A **quadratic inequality** such as $x^2 - x - 6 < 0$ asks: for which x is the parabola $y = x^2 - x - 6$ below the x-axis? Its answer is usually an interval or a pair of intervals, not one or two numbers. The zeros you learned to find in Lessons 3.2 and 3.4 are the key.

Why the zeros split the line. A parabola is an unbroken curve, so its height can go from positive to negative only by passing through 0. Between two neighboring zeros, and beyond the outer zeros, the sign of $ax^2 + bx + c$ cannot change. You can also see it algebraically: in $a(x - p)(x - q)$, the factor $x - p$ is negative to the left of p and positive to the right, and changes sign nowhere else; the same is true of $x - q$. So the product keeps one sign on each interval, and one test value tells you the sign of the whole interval.

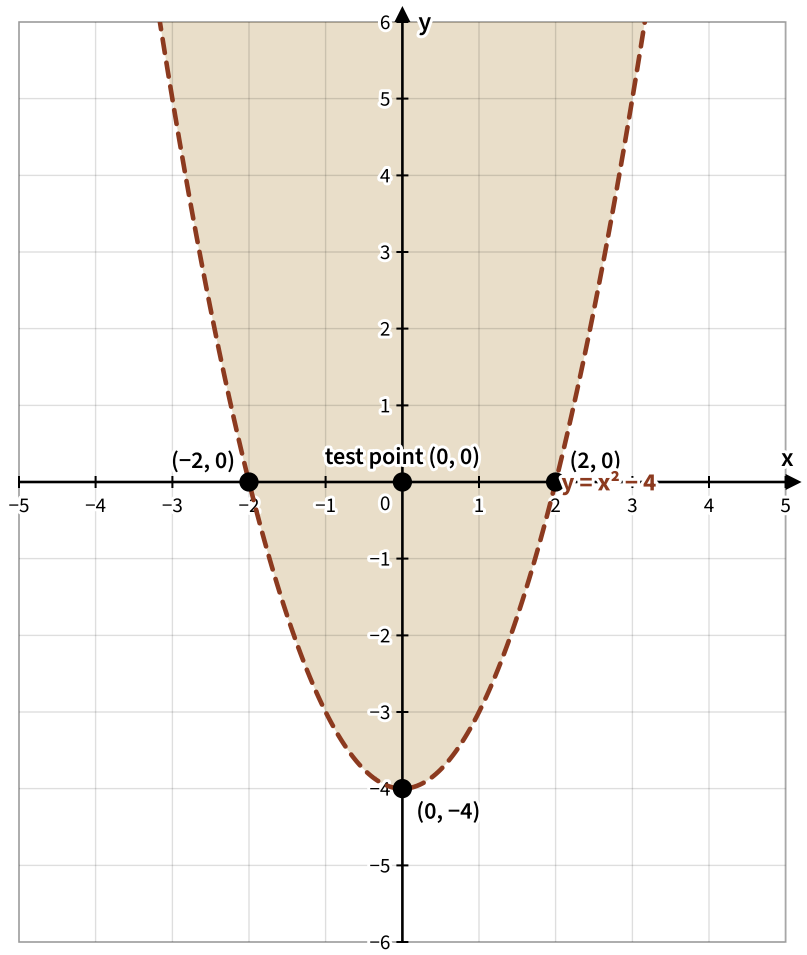
The method. (1) Move everything to one side so the other side is 0. (2) Find the real zeros. (3) Mark them on a number line, which splits it into intervals. (4) Test one value in each interval, or use the factors, and mark + or −. (5) Choose the intervals that fit the inequality. Include the zeros (closed endpoints, square brackets) for $\leq$ or $\geq$; leave them out (open endpoints, parentheses) for $<$ or $>$.

For $x^2 - x - 6 = (x + 2)(x - 3)$ the zeros are -2 and 3. The graph opens up, so it is below the x-axis between the zeros and above it outside them. So $x^2 - x - 6 < 0$ on $(-2, 3)$, and $x^2 - x - 6 \geq 0$ on $(-\infty, -2] \cup [3, \infty)$. In general, when $a > 0$ the quadratic is negative between its zeros and positive outside; when $a < 0$ it is the other way around.



Special cases. If $D < 0$ there are no real zeros, so the parabola never crosses the axis and the quadratic always has the sign of a. Then $x^2 + 2x + 5 > 0$ is true for all real numbers, and $x^2 + 2x + 5 < 0$ has no solution. If $D = 0$ the parabola just touches the axis: $(x - 3)^2 \geq 0$ for all real numbers, $(x - 3)^2 > 0$ for every x except 3, $(x - 3)^2 \leq 0$ only at $x = 3$, and $(x - 3)^2 < 0$ never.

Inequalities in two variables. The graph of $y > x^2 - 4$ is every point above the parabola $y = x^2 - 4$. Draw the parabola dashed for $<$ or $>$ (its points are not included) and solid for $\leq$ or $\geq$, then shade above for $y >$ or $y \geq$ and below for $y <$ or $y \leq$. Confirm with a test point not on the curve: $(0, 0)$ gives $0 > 0 - 4$, which is true, so the region containing the origin (inside the cup) is shaded.



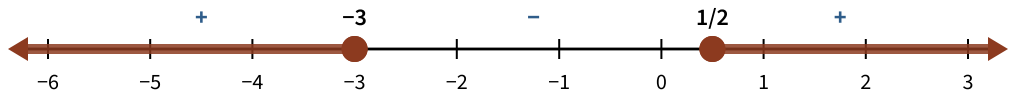
KEY FORMULAS

$$a > 0: \; ax^2 + bx + c < 0 \text{ between the zeros}, \quad > 0 \text{ outside them}$$

$$a < 0: \; ax^2 + bx + c > 0 \text{ between the zeros}, \quad < 0 \text{ outside them}$$

Practice habit. Sketch before you compute: a ten-second sketch of the parabola, with its direction of opening and its zeros, shows at a glance which intervals you want. Then confirm with one test value.

Example 1. Solve $2x^2 + 5x - 3 \geq 0$. Write the answer in interval notation.

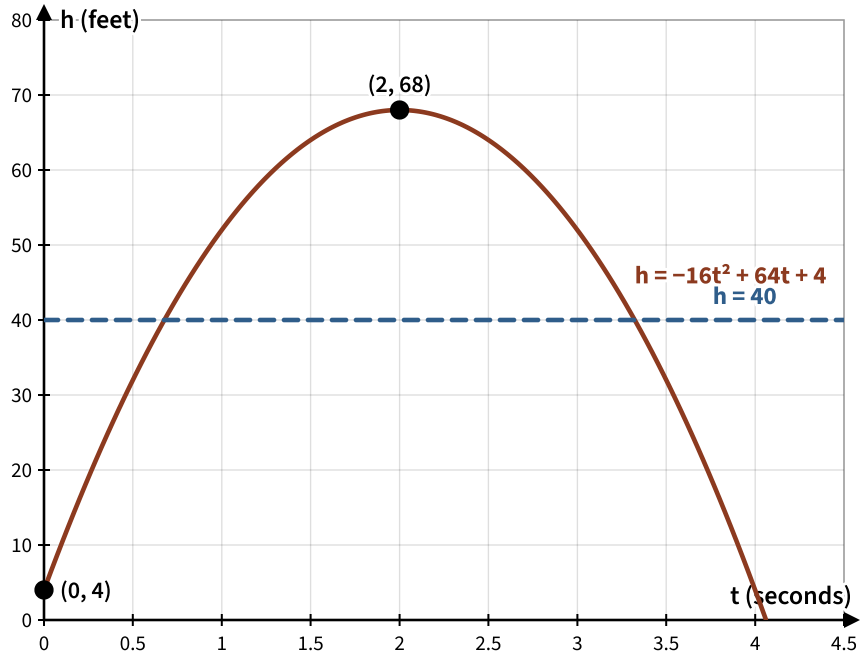


Show solution

1. Factor: $(2x - 1)(x + 3) \geq 0$.
2. Zeros: $x = \dfrac{1}{2}$ and $x = -3$. They split the line into three intervals.
3. Test $x = -4$: $2(16) - 20 - 3 = 9 > 0$, so +.
4. Test $x = 0$: $-3 < 0$, so −.
5. Test $x = 1$: $2 + 5 - 3 = 4 > 0$, so +.
6. We want ≥ 0: the + intervals, with the zeros included (closed endpoints).
7. This matches the picture: $a = 2 > 0$, so the parabola is above the axis outside its zeros.

Answer: $(-\infty, -3] \cup \left[\dfrac{1}{2}, \infty\right)$

Example 2. Samuel throws a ball straight up from a height of 4 feet at 64 feet per second, so its height after t seconds is $h = -16t^2 + 64t + 4$ feet. During what time interval is the ball higher than 40 feet? Round to the nearest hundredth of a second.



Show solution

1. Set up: $-16t^2 + 64t + 4 > 40$, so $-16t^2 + 64t - 36 > 0$.
2. Divide by -4 and reverse the inequality: $4t^2 - 16t + 9 < 0$.
3. Zeros: $t = \dfrac{16 \pm \sqrt{256 - 144}}{8} = \dfrac{16 \pm \sqrt{112}}{8} = \dfrac{16 \pm 4\sqrt{7}}{8} = 2 \pm \dfrac{\sqrt{7}}{2}$.
4. $\dfrac{\sqrt{7}}{2} \approx 1.32288$, so $t \approx 0.68$ and $t \approx 3.32$.
5. $4t^2 - 16t + 9$ opens up, so it is negative between its zeros: $0.68 < t < 3.32$.
6. Check $t = 2$ (inside): $h = -64 + 128 + 4 = 68 > 40$ ✓. Check $t = 0$ (outside): $h = 4$, not above 40 ✓.

Answer: From about 0.68 seconds to about 3.32 seconds after the throw

PRACTICE

1. **Easy** Solve $(x-1)(x-5)<0$. Write the answer in interval notation.

Show answer

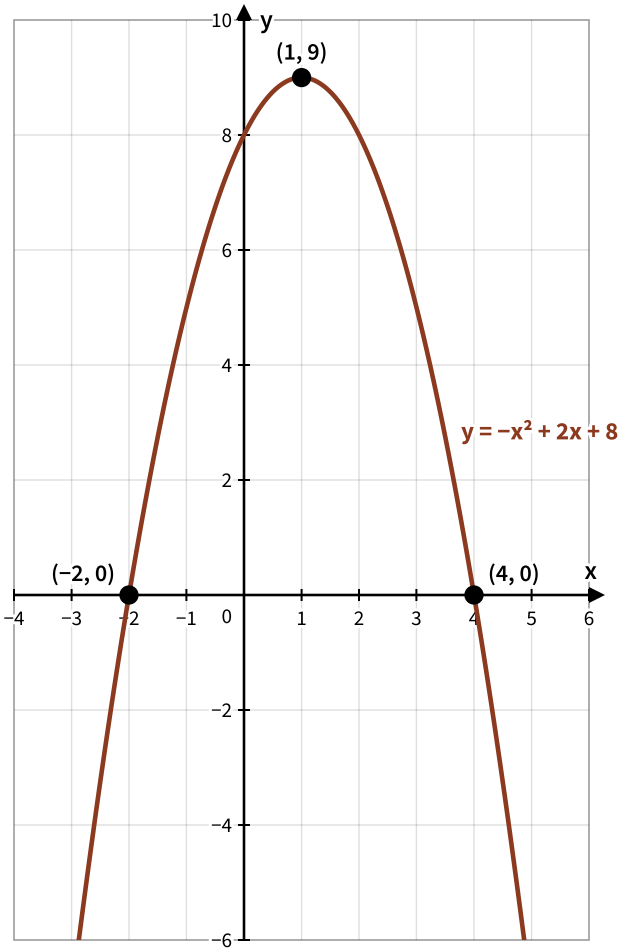
Zeros 1 and 5; the parabola opens up, so it is negative between them: $(1, 5)$

2. **Easy** Solve $x^2 - 9 \geq 0$. Write the answer in interval notation.

Show answer

$(x+3)(x-3) \geq 0$: $(-\infty, -3] \cup [3, \infty)$

3. **Easy** Use the graph of $y = -x^2 + 2x + 8$ to solve (a) $-x^2 + 2x + 8 > 0$ (b) $-x^2 + 2x + 8 \leq 0$. Use interval notation.



Show answer

(a) Above the axis between the zeros: $(-2, 4)$ (b) On or below the axis: $(-\infty, -2] \cup [4, \infty)$

4. **Easy** Solve $x^2 + 4x > 0$. Write the answer in interval notation.

   Show answer

   $x(x+4) > 0$; zeros -4 and 0: $(-\infty, -4) \cup (0, \infty)$

5. **Medium** Solve $x^2 - 3x - 10 \le 0$. Write the answer in interval notation.

   Show answer

   $(x-5)(x+2) \le 0$: $[-2, 5]$

6. **Medium** Solve $-x^2 + x + 12 > 0$. Write the answer in interval notation.

   Show answer

   Multiply by -1 and reverse: $x^2 - x - 12 < 0$, $(x-4)(x+3) < 0$: $(-3, 4)$

7. **Medium** Solve $3x^2 + 5x - 2 \ge 0$. Write the answer in interval notation.

   Show answer

   $(3x-1)(x+2) \ge 0$; zeros -2 and $\frac{1}{3}$: $(-\infty, -2] \cup \left[\frac{1}{3}, \infty\right)$

8. **Medium** Solve (a) $x^2 + 2x + 5 > 0$ (b) $x^2 + 2x + 5 < 0$.

   Show answer

   $D = 4 - 20 = -16 < 0$ and $a > 0$, so the parabola is always above the axis. (a) All real numbers (b) No solution

9. **Medium** Solve (a) $x^2 - 6x + 9 \le 0$ (b) $x^2 - 6x + 9 > 0$.

   Show answer

   $x^2 - 6x + 9 = (x-3)^2$, which is never negative and is 0 only at 3. (a) $x = 3$ only (b) all real numbers except 3: $(-\infty, 3) \cup (3, \infty)$

10. **Medium** Ruth is planning a rectangular flower bed whose length is 4 feet more than its width. What widths give an area of at least 60 square feet?

    Show answer

    $w(w+4) \ge 60$, $w^2 + 4w - 60 \ge 0$, $(w+10)(w-6) \ge 0$; since $w > 0$, the width must be at least 6 feet: $w \ge 6$

11. Hard A ball is tossed up from a height of 8 feet at 48 feet per second, so its height is $h = -16t^2 + 48t + 8$ feet after t seconds. For how long is the ball at least 40 feet high?

Show answer

$-16t^2 + 48t + 8 \geq 40$ gives $t^2 - 3t + 2 \leq 0$, $(t-1)(t-2) \leq 0$, so $1 \leq t \leq 2$: for 1 second

12. Hard A family bakery finds that when it charges x dollars for a pie, its weekly profit from pies is $P = -20x^2 + 260x - 600$ dollars. What prices give a weekly profit of at least \$200?

Show answer

$-20x^2 + 260x - 600 \geq 200$, so $-20x^2 + 260x - 800 \geq 0$; divide by -20: $x^2 - 13x + 40 \leq 0$, $(x-5)(x-8) \leq 0$. Prices from \$5 to \$8: $5 \leq x \leq 8$

13. Hard Consider the graph of $y \leq -x^2 + 2x + 3$. (a) Is the boundary solid or dashed, and is the shading above or below it? (b) Is $(1, 2)$ a solution? (c) Is $(3, 1)$ a solution?

Show answer

(a) Solid (the symbol is $\leq$), shaded below the parabola. (b) $-1 + 2 + 3 = 4$ and $2 \leq 4$: yes. (c) $-9 + 6 + 3 = 0$ and $1 \leq 0$ is false: no

14. Hard Explain why, for a quadratic $a(x-p)(x-q)$ with $p < q$, one test value in each of the intervals $(-\infty, p)$, (p, q) and (q, ∞) tells you the sign on the whole interval.

Show answer

The factor $x - p$ is negative for $x < p$ and positive for $x > p$; it changes sign only at p. Likewise $x - q$ changes sign only at q. Inside any one of the three intervals neither factor changes sign, so the product $a(x-p)(x-q)$ keeps a single sign there, and any one value shows what it is.

15. Challenge For what values of k is $x^2 + kx + 4 > 0$ true for every real number x?

Show answer

The parabola opens up, so it is always positive exactly when it has no real zeros: $D = k^2 - 16 < 0$, so $-4 < k < 4$

COMMON MISTAKES

- Solving $x^2 < 9$ as $x < \pm 3$ or just $x < 3$. Write $x^2 - 9 < 0$, find the zeros ± 3, and test: the answer is $(-3, 3)$.
- Forgetting to reverse the inequality after multiplying or dividing by a negative number, as when changing $-x^2 + x + 12 > 0$ to $x^2 - x - 12 < 0$.
- Using brackets for $<$ or $>$, or parentheses for $\leq$ or $\geq$. The zeros make the expression equal 0, so they belong to the answer only when the inequality allows equality. Infinity always gets a parenthesis.
- Ignoring the situation in a word problem. A width or a time cannot be negative, so throw away the intervals that make no sense, as in $w \geq 6$ rather than $w \leq -10$ or $w \geq 6$.

> **Make it a game · Sign Chart Stepping.** Solve $x^2 - x - 6 > 0$. Lay paper slips on the floor labeled with the zeros -2 and 3. Stand in each region, pick a test number, and call 'positive' or 'negative'. The parent records the signs on a chart. Write the answer in interval notation, then check with a number from the graph. Do two more inequalities.

3.6 Quadratic Models

OBJECTIVES

- Recognize quadratic data by constant second differences and explain why the second differences are constant.
- Write a quadratic model from its vertex and one point, from its zeros and one point, or through any three points.
- Use projectile models in feet and in meters to find heights, times and maximum heights.
- Build revenue models when raising the price loses customers, and find and interpret the maximum.

WATCH

Write a Quadratic Function Given a Table

Mario's Math Tutoring · 9:39

https://www.youtube.com/watch?v=wnZpMRgGBmg

> **WHEN TO WATCH**
> Before Example 2: writing a quadratic function from a table of values.

BACKUP VIDEO

Solving Projectile Motion Word Problems Using Quadratics

Mario's Math Tutoring · 11:55

https://www.youtube.com/watch?v=BIUphudCHF4

Before Example 1: projectile-motion word problems.

KEY TERMS

first differences

The changes between consecutive outputs in a table whose inputs go up by equal steps.

second differences

The changes between consecutive first differences. They are constant (and not zero) exactly for quadratic data.

projectile

An object that is thrown, kicked, hit or launched and then moves only under gravity (ignoring air resistance).

revenue

The total money taken in from sales: price per item times the number of items sold.

quadratic model

A quadratic function used to describe a real situation, with its inputs and outputs in stated units.

CORE IDEAS

Recognizing quadratic data. In Algebra I you saw that linear data have constant first differences. When the inputs go up by equal steps, quadratic data have constant second differences. Here is why. For $f(x) = ax^2 + bx + c$, the first difference is $f(x+1) - f(x) = a(x+1)^2 + b(x+1) + c - (ax^2 + bx + c) = 2ax + a + b$. That is a linear expression in x, so its own differences are constant: each one is $2a$. So the second differences are $2a$, which also tells you a immediately.

Differences for $y = 2x^2 - 3x + 1$

x	y	first difference	second difference
0	1	—	—
1	0	−1	—
2	3	3	4
3	10	7	4
4	21	11	4
5	36	15	4

Three ways to write a model. (1) If you know the vertex (h, k) and one other point, use vertex form $y = a(x-h)^2 + k$, put in the point and solve for a. (2) If you know the zeros p and q and one other point, use factored form $y = a(x-p)(x-q)$ the same way. (3) If you know any three points, use standard form $y = ax^2 + bx + c$.

Each point gives one linear equation in a, b and c, so three points give a system of three equations in three variables, which you solve by elimination as in Lesson 2.4. Three unknowns need three facts; that is why two points are not enough for a parabola.

Projectiles. An object moving straight up and down under gravity has height $h = -16t^2 + v_0 t + h_0$ in feet after t seconds, where v_0 is the starting upward speed (feet per second) and h_0 is the starting height (feet). In meters the model is $h = -4.9t^2 + v_0 t + h_0$. The numbers 16 and 4.9 are half of the acceleration due to gravity, 32 feet per second per second or 9.8 meters per second per second. The highest point comes at the vertex, $t = -\dfrac{b}{2a}$, which in feet is $t = \dfrac{v_0}{32}$. The same models describe the height of a thrown or kicked ball whose path also moves sideways.

Revenue when a higher price loses customers. A farm stand sells 200 dozen eggs a week at \$3.00 a dozen. Each \$0.25 increase in price loses 10 sales a week. After n increases, the price is $3 + 0.25n$ and the number sold is $200 - 10n$, so the revenue is $R = (3 + 0.25n)(200 - 10n) = -2.5n^2 + 20n + 600$. Raising the price earns more on each dozen but sells fewer, and the product of an increasing factor and a decreasing factor is a parabola that opens down, so there is a best price. Its zeros are $n = -12$ and $n = 20$, so the vertex is halfway, at $n = 4$: charge \$4.00, sell 160 dozen and take in \$640 a week.

Interpreting a model. Always ask what the variables mean and which inputs make sense. Time and lengths cannot be negative, a ball's model stops when it hits the ground, and n in a pricing model must keep the number sold from going below 0. Give answers with units and say what they mean: "the ball is highest, 106 feet, after 2.5 seconds."

KEY FORMULAS

$$\text{second differences (step 1)} = 2a$$

$$h = -16t^2 + v_0 t + h_0 \text{ (feet)}, \qquad h = -4.9t^2 + v_0 t + h_0 \text{ (meters)}$$

$$\text{revenue} = (\text{price per item}) \times (\text{number sold})$$

> **Practice habit.** Write one step per line when you solve a system for a, b and c, and then check your model in every point you were given. A model that misses one of its own data points is wrong.

Example 1. A garden fountain's nozzle is 1 foot above the ground. The jet of water reaches its greatest height, 5 feet, at a point 2 feet out from the nozzle. Put the base of the nozzle at the origin, with x the horizontal distance in feet. Write a model for the jet and find how far from the nozzle the water hits the ground, to the nearest hundredth of a foot.

Show solution

1. The vertex is $(2, 5)$, so $y = a(x - 2)^2 + 5$.
2. The jet starts at $(0, 1)$: $1 = a(0 - 2)^2 + 5 = 4a + 5$, so $a = -1$.
3. Model: $y = -(x - 2)^2 + 5$.
4. Ground: $-(x - 2)^2 + 5 = 0$, so $(x - 2)^2 = 5$ and $x = 2 \pm \sqrt{5}$.
5. Only the positive distance in front of the nozzle makes sense: $x = 2 + \sqrt{5} \approx 4.24$ feet.
6. Check: $y(0) = -4 + 5 = 1$ ✓.

Answer: $y = -(x - 2)^2 + 5$; the water lands $2 + \sqrt{5} \approx 4.24$ feet from the nozzle

Example 2. Henry hits a pop fly straight up. The table gives its height at three times. Find a quadratic model $h = at^2 + bt + c$, state the height and upward speed at which the ball left the bat, and find the greatest height.

Height of a pop fly

t (seconds)	h (feet)
1	70
2	102
3	102

Show solution

1. Each point gives an equation: $a + b + c = 70$, $4a + 2b + c = 102$, $9a + 3b + c = 102$.
2. Subtract the first from the second: $3a + b = 32$.
3. Subtract the second from the third: $5a + b = 0$.
4. Subtract these two: $2a = -32$, so $a = -16$. Then $b = 32 - 3(-16) = 80$.
5. From the first equation: $c = 70 - (-16) - 80 = 6$. So $h = -16t^2 + 80t + 6$.
6. Check $t = 3$: $-144 + 240 + 6 = 102$ ✓. The ball left the bat at 6 feet high, moving up at 80 feet per second.
7. Vertex: $t = -\dfrac{80}{2(-16)} = 2.5$ seconds; $h(2.5) = -100 + 200 + 6 = 106$ feet.

Answer: $h = -16t^2 + 80t + 6$; it started 6 feet high at 80 feet per second; its greatest height is 106 feet, at 2.5 seconds

PRACTICE

1. Easy Is the data in the table quadratic? If so, what is a?

x	0	1	2	3	4
y	3	5	11	21	35

Show answer

First differences 2, 6, 10, 14; second differences all 4, so yes, and $2a = 4$ gives $a = 2$

2. Easy Write the equation of the parabola with vertex $(1, -3)$ that passes through $(3, 5)$.

Show answer

$5 = a(2)^2 - 3$, $a = 2$: $y = 2(x - 1)^2 - 3$

3. Easy Write the equation of the parabola with zeros 2 and 6 that passes through $(4, -8)$.

Show answer

$-8 = a(2)(-2)$, $a = 2$: $y = 2(x - 2)(x - 6)$

4. Easy Grace throws a stone upward at 15 meters per second from the top of a cliff 20 meters above the sea. (a) Write a model for its height h in meters after t seconds. (b) How high is it after 2 seconds?

Show answer

(a) $h = -4.9t^2 + 15t + 20$ (b) $-19.6 + 30 + 20 = 30.4$ meters

5. Medium Use second differences to find a, then find the quadratic $y = ax^2 + bx + c$ that fits the table.

x	1	2	3	4	5
y	2	7	14	23	34

Show answer

Differences 5, 7, 9, 11; second differences 2, so $a = 1$. Then $1 + b + c = 2$ and $4 + 2b + c = 7$ give $b = 2$, $c = -1$: $y = x^2 + 2x - 1$

6. **Medium** Find the quadratic function whose graph passes through $(0, 5)$, $(1, 2)$ and $(2, 3)$.

Show answer

$c = 5$; $a + b = -3$ and $4a + 2b = -2$, so $2a + b = -1$, $a = 2$, $b = -5$: $y = 2x^2 - 5x + 5$

7. **Medium** A baseball is hit from a height of 3 feet with an upward speed of 80 feet per second. Write its height model and find when it is highest and how high it gets.

Show answer

$h = -16t^2 + 80t + 3$; $t = \dfrac{80}{32} = 2.5$ seconds; $h = -100 + 200 + 3 = 103$ feet

8. **Medium** The main cable of a footbridge hangs in a parabola. Its lowest point is 4 feet above the deck at the center, and it is attached 29 feet above the deck at towers 100 feet apart. Put the center of the deck at the origin. (a) Write the equation of the cable. (b) How high is the cable 20 feet from the center?

Show answer

(a) $y = ax^2 + 4$ through $(50, 29)$: $2500a = 25$, $a = 0.01$, so $y = 0.01x^2 + 4$ (b) $0.01(400) + 4 = 8$ feet

9. **Medium** A parabola has x-intercepts -1 and 5 and y-intercept 10. Write its equation in factored form and find its maximum value.

Show answer

$10 = a(1)(-5)$, $a = -2$: $y = -2(x + 1)(x - 5)$; at $x = 2$, $y = -2(3)(-3) = 18$

10. **Hard** A coach films a volleyball serve and finds that the ball is 5 feet high at $t = 0$, 17 feet high at $t = 0.5$ second and 21 feet high at $t = 1$ second. (a) Find the model $h = at^2 + bt + c$. (b) When does the ball reach the floor, to the nearest tenth of a second, if nothing stops it?

Show answer

(a) $c = 5$; $0.25a + 0.5b = 12$ and $a + b = 16$ give $a = -16$, $b = 32$: $h = -16t^2 + 32t + 5$ (b) $t = \dfrac{32 + \sqrt{1024 + 320}}{32} \approx 2.1$ seconds

11. Hard An orchard sells apple cider at \$6 per gallon and sells 120 gallons a week. Each \$0.50 increase in price loses 6 gallons of sales a week. What price gives the greatest weekly revenue, and what is that revenue?

Show answer

$R = (6 + 0.5n)(120 - 6n) = -3n^2 + 24n + 720$; $n = 4$, price \$8, selling 96 gallons: \$768

12. Hard Find the quadratic function whose graph passes through $(-1, 6)$, $(1, 2)$ and $(2, 3)$.

Show answer

$a - b + c = 6$, $a + b + c = 2$, $4a + 2b + c = 3$. Subtracting the first two: $b = -2$; then $a + c = 4$ and $4a + c = 7$, so $a = 1$, $c = 3$: $y = x^2 - 2x + 3$

13. Hard Prove that for $f(x) = ax^2 + bx + c$, the second differences of the values $f(0), f(1), f(2), \ldots$ are all equal to $2a$.

Show answer

First difference: $f(x+1) - f(x) = a(x^2 + 2x + 1) + bx + b + c - ax^2 - bx - c = 2ax + a + b$. Second difference: $[2a(x+1) + a + b] - [2ax + a + b] = 2a$, the same for every x.

14. Challenge A farmer has 600 meters of fence to make a rectangular field along a straight river. No fence is needed along the river, and one extra fence perpendicular to the river divides the field into two pens. What dimensions give the greatest total area, and what is that area?

Show answer

With x the length of each of the three fences perpendicular to the river, the side parallel to it is $600 - 3x$, so $A = x(600 - 3x) = -3x^2 + 600x$. Vertex: $x = 100$; the field is 100 m by 300 m, with area 30,000 square meters

COMMON MISTAKES

- Using second differences when the inputs are not equally spaced. Check the x-values first; the test works only for equal steps.
- Mixing units in projectile problems: $-16t^2$ goes with feet and $-4.9t^2$ with meters. Read the problem's units before writing the model.
- Answering a revenue problem with the number of increases n instead of the price. If $n = 4$ increases of \$0.50, the price is $6 + 0.5(4) = 8$ dollars.

- Making an arithmetic slip in a 3-by-3 system and never noticing. Check the final model in all three given points.

> **Make it a game · Bean Triangles.** Build triangles of beans with 1, then 3, then 6, then 10, then 15 beans. Record the counts and calculate first and second differences; note that the second difference stays constant at 1. Using the first three points, find the quadratic model. Predict the 20th triangle, then check by pattern. The closest guess before calculating earns a point.

UNIT 3

Quadratic Functions and Complex Numbers

Quadratic functions describe thrown balls, arches, the area of a garden and the revenue of a small business. In Algebra I you learned to graph parabolas and to solve quadratic equations. This unit pulls that work together, moving between the three forms of a quadratic and choosing the best method to solve, and then explains why the methods work, including a derivation of the quadratic formula and of the facts about the sum and product of the roots.

The unit also enlarges the number system. The equation $x^2 = -1$ has no real solution, so we bring in the imaginary unit i with $i^2 = -1$ and learn to compute with complex numbers. With them, every quadratic equation has solutions, and the discriminant tells whether they are real or non-real. The last two lessons use quadratics to solve inequalities and to build models from data, heights and prices.

This unit leads straight into Unit 4, where the same ideas (zeros, factors, conjugate pairs and sign charts) are used for polynomials of any degree. Families can expect steady practice in exact algebra and in checking answers in the original equation.

PREREQUISITES

- Factoring trinomials, the difference of squares and perfect square trinomials (Algebra I)
- Simplifying square roots, such as $\sqrt{72} = 6\sqrt{2}$ (Algebra I)
- Graphing parabolas and using vertex form (Algebra I)
- Interval notation and solving linear inequalities (Algebra I)
- Transformations of parent functions (Lesson 1.2)
- Solving systems of three linear equations by elimination (Lesson 2.4)

MASTERY CHECKLIST

- ☐ I can read the vertex, axis of symmetry, intercepts and direction of opening from standard, vertex and factored form, and convert among the forms.
- ☐ I can explain why the axis of symmetry lies halfway between the zeros.
- ☐ I can solve a quadratic equation by factoring, square roots, completing the square or the quadratic formula, and choose an efficient method.
- ☐ I can derive the quadratic formula by completing the square.
- ☐ I can add, subtract, multiply and divide complex numbers and simplify powers of

i.

- ☐ I can plot complex numbers and find their absolute values.
- ☐ I can solve quadratics with non-real solutions and explain why they come in conjugate pairs.
- ☐ I can use the discriminant to describe the solutions and the x-intercepts of the graph.
- ☐ I can use the sum

$-\frac{b}{a}$ and product

$\frac{c}{a}$ of the roots to check solutions and to write an equation from its roots.

- ☐ I can solve quadratic inequalities with a sign chart or a graph and

write the answer in interval notation.

- ☐ I can write a quadratic model from a vertex, from zeros or from three points, and use projectile and revenue models.

LESSONS IN THIS UNIT

3.1	**Quadratic Functions and Their Three Forms**	45 minutes
3.2	**Solving Quadratic Equations: A Review**	45 minutes
3.3	**Complex Numbers**	2 sessions of 45 minutes
3.4	**Complex Solutions and the Discriminant**	2 sessions of 45 minutes
3.5	**Quadratic Inequalities**	2 sessions of 45 minutes
3.6	**Quadratic Models**	2 sessions of 45 minutes

Unit 3 Test

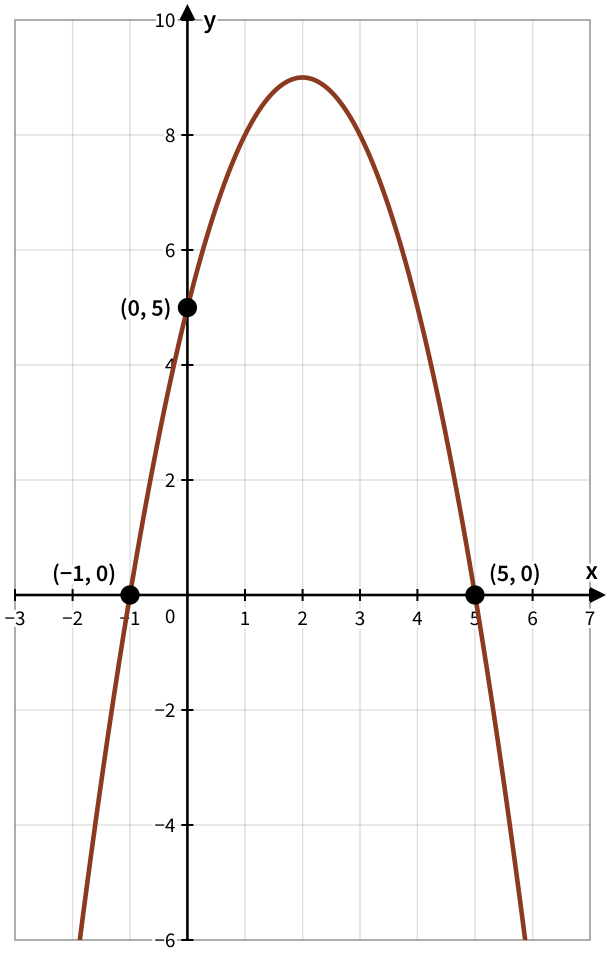
You have 60 minutes. A scientific calculator is allowed, but show all your work: answers without supporting work earn little credit. Give exact answers (integers, fractions, simplest radical form, or $a + bi$) unless a problem asks you to round. Write inequality solutions in interval notation unless told otherwise, and check solutions in the original equation.

1. **5 pts**
   For $y = -4(x + 3)^2 + 7$, give the vertex, the axis of symmetry and the maximum or minimum value.

2. **5 pts**
   Write $y = x^2 + 8x + 12$ in vertex form and in factored form.

3. **5 pts**

The graph shows a parabola with its x-intercepts and y-intercept marked. (a) Write its equation in factored form. (b) Find its vertex.



4. **6 pts**

Solve by factoring: $5x^2 - 3x - 2 = 0$.

5. **6 pts**

Solve by completing the square: $x^2 + 10x + 18 = 0$. Give exact answers.

6. **7 pts**

The Miller family is building a rectangular wooden deck that is 6 feet longer than it is wide, with an area of 112 square feet. Find its dimensions.

7. **5 pts**

Simplify: (a) $(3 - 2i)(4 + 5i)$ (b) i^{42}

8. **7 pts**

   (a) Write $\dfrac{7 - i}{1 + i}$ in the form $a + bi$. (b) Find the absolute value of your answer.

9. **7 pts**

   Solve $x^2 - 8x + 20 = 0$. What does the discriminant tell you about the graph of $y = x^2 - 8x + 20$?

10. **5 pts**

    One root of a quadratic equation with real coefficients is $3 - i\sqrt{2}$. Find the other root and write the equation with leading coefficient 1.

11. **7 pts**

    Solve $2x^2 - x - 15 < 0$. Write the answer in interval notation.

12. **5 pts**

    Solve $x^2 + 3 \geq 2x$.

13. **8 pts**

    Samuel builds birdhouses in his shop. When he charges x dollars each, his monthly profit is $P = -2x^2 + 80x - 600$ dollars. What prices give a monthly profit of at least \$150?

14. **8 pts**

    The table gives the height of a kicked ball. (a) Find the model $h = at^2 + bt + c$. (b) Find the greatest height, to the nearest tenth of a meter.

    Height of a kicked ball

t (seconds)	h (meters)
0	1
1	16.1
3	16.9

15. **7 pts**

A church bakery sells 300 cinnamon rolls a week at \$2.00 each. Each \$0.10 increase in price loses 10 sales a week. What price gives the greatest weekly revenue, and what is that revenue?

16. **7 pts**

Explain why, when a quadratic equation $ax^2 + bx + c = 0$ with real coefficients has a negative discriminant, its two solutions are complex conjugates.

ANSWER KEY

1. Vertex $(-3, 7)$, axis $x = -3$, maximum value 7 (it opens down)
2. $y = (x+4)^2 - 16 + 12 = (x+4)^2 - 4$; $y = (x+2)(x+6)$
3. (a) $y = a(x+1)(x-5)$ with $5 = a(1)(-5)$, so $a = -1$: $y = -(x+1)(x-5)$ (b) $x = \dfrac{-1+5}{2} = 2$, $y = -(3)(-3) = 9$: $(2, 9)$
4. $(5x+2)(x-1) = 0$, so $x = -\dfrac{2}{5}$ or $x = 1$
5. $(x+5)^2 = -18 + 25 = 7$, so $x = -5 \pm \sqrt{7}$
6. $w(w+6) = 112$, $w^2 + 6w - 112 = 0$, $(w+14)(w-8) = 0$, $w = 8$: 8 feet by 14 feet (check: $8 \times 14 = 112$ ✓)
7. (a) $12 + 15i - 8i - 10i^2 = 22 + 7i$ (b) $42 = 40 + 2$, so $i^{42} = i^2 = -1$
8. (a) $\dfrac{(7-i)(1-i)}{(1+i)(1-i)} = \dfrac{7 - 8i + i^2}{2} = \dfrac{6 - 8i}{2} = 3 - 4i$ (b) $\sqrt{9 + 16} = 5$
9. $D = 64 - 80 = -16$; $x = \dfrac{8 \pm 4i}{2} = 4 \pm 2i$. Since $D < 0$, the parabola has no x-intercepts (it lies entirely above the x-axis).
10. Other root $3 + i\sqrt{2}$; sum 6, product $9 + 2 = 11$: $x^2 - 6x + 11 = 0$
11. $(2x+5)(x-3) < 0$; zeros $-\dfrac{5}{2}$ and 3; opens up, so negative between: $\left(-\dfrac{5}{2}, 3\right)$
12. $x^2 - 2x + 3 \geq 0$; $D = 4 - 12 = -8 < 0$ and the parabola opens up, so it is always positive: all real numbers, $(-\infty, \infty)$
13. $-2x^2 + 80x - 750 \geq 0$; divide by -2: $x^2 - 40x + 375 \leq 0$, $(x-15)(x-25) \leq 0$: prices from \$15 to \$25, that is, $[15, 25]$
14. (a) $c = 1$; $a + b = 15.1$ and $9a + 3b = 15.9$ give $a = -4.9$, $b = 20$: $h = -4.9t^2 + 20t + 1$ (b) $t = \dfrac{20}{9.8} \approx 2.04$ s, $h \approx 21.4$ meters
15. $R = (2 + 0.1n)(300 - 10n) = -n^2 + 10n + 600$; $n = 5$: price \$2.50, selling 250 rolls, revenue \$625
16. The formula gives $x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$. The first part is a real number p. When $b^2 - 4ac < 0$, its square root is $i\sqrt{|b^2 - 4ac|}$, so the second part is a pure imaginary number qi. The $\pm$ gives $p + qi$ and $p - qi$, which are conjugates.

UNIT 4

Polynomial Functions

In this unit you move from quadratics to polynomials of any degree. You will learn how a polynomial's leading term controls the ends of its graph, how to multiply and divide polynomials (including long division and its shortcut, synthetic division), and how the Remainder and Factor Theorems link the value $f(c)$ to the factor $x - c$. These ideas lead to the Rational Root Theorem and the Fundamental Theorem of Algebra, which together tell you where to look for zeros and how many to expect.

The unit builds on factoring and the quadratic formula from Algebra I, on complex numbers and sign charts from Unit 3, and on function behavior from Unit 1. It prepares you directly for Unit 5, where rational functions are quotients of polynomials and long division gives slant asymptotes. Along the way you will model real objects: the volume of an open box cut from a sheet, the dimensions of a chest or a garden bed, and the growth of savings.

PREREQUISITES

- Factoring trinomials, differences of squares and perfect-square trinomials (Algebra I)
- Solving quadratic equations by factoring and by the quadratic formula (Lesson 3.2)
- Complex numbers and complex solutions of quadratics (Lessons 3.3 and 3.4)
- Sign charts and interval notation for quadratic inequalities (Lesson 3.5)
- Function notation, end behavior in words and relative maxima and minima (Lessons 1.1 and 1.6)

MASTERY CHECKLIST

- ☐ I can give the degree, leading coefficient and constant term of a polynomial and describe its end behavior from the leading term, and explain why the leading term controls the ends.
- ☐ I can add, subtract and multiply polynomials, expand $(a \pm b)^3$ and, write the volume of an open box as a polynomial with its domain.
- ☐ I can divide polynomials by long division, using placeholders, and write the result as $q(x) + \frac{r(x)}{d(x)}$.
- ☐ I can divide by $x - c$ with synthetic division and explain why it matches long division.
- ☐ I can prove the Remainder Theorem and use synthetic substitution to evaluate a polynomial.
- ☐ I can use the Factor Theorem and one known zero to factor a polynomial completely.
- ☐ I can factor sums and differences of cubes, four-term polynomials by grouping, and polynomials in quadratic form.
- ☐ I can list possible rational zeros, explain why the Rational Root

Theorem works, and find all real and non-real zeros of a polynomial.

- ☐ I can write a polynomial with given zeros, including conjugate pairs.
- ☐ I can sketch a polynomial from its zeros, multiplicities, end behavior and y-intercept, and explain why it crosses or touches at each zero.
- ☐ I can solve polynomial inequalities with a sign chart and use a graphing tool to find maxima and minima in a model.

LESSONS IN THIS UNIT

4.1	**Polynomial Functions and End Behavior**	2 sessions of 45 minutes
4.2	**Operations with Polynomials**	2 sessions of 45 minutes
4.3	**Polynomial Long Division**	2 sessions of 45 minutes
4.4	**Synthetic Division and the Remainder Theorem**	2 sessions of 45 minutes
4.5	**The Factor Theorem and Factoring Polynomials**	2 sessions of 45 minutes
4.6	**Finding Zeros of Polynomials**	2 sessions of 45 minutes
4.7	**Graphing Polynomial Functions**	2 sessions of 45 minutes

Unit 4 test

Time: 60 minutes. A scientific calculator is allowed for arithmetic; no graphing calculator or graphing app. Show your work for every problem: a correct answer without work earns partial credit only. Give exact answers (fractions, radicals, i) unless a problem asks for a rounded answer. Write inequality answers in interval notation. The test is worth 100 points.

1. **5 pts**

   For $f(x) = 3x^2 - x^5 + 4x - 8$, give the degree, the leading coefficient and the constant term, and describe the end behavior.

2. **5 pts**

   Without multiplying out, find the degree, the leading coefficient and the y-intercept of $g(x) = (4 - x)^2(2x + 1)(x + 5)$. Then describe its end behavior.

3. **6 pts**

   Write each product in standard form. (a) $(3x - 2)(x^2 - 4x + 5)$ (b) $(x - 5)^3$

4. **6 pts**

   A sheet-metal shop makes an open tray from a 14-inch by 10-inch sheet by cutting a square of side x inches from each corner and folding up the sides. (a) Write the volume $V(x)$ in standard form. (b) Find the volume when $x = 1.5$.

5. **7 pts**

   Use long division to divide $(2x^3 + x^2 - 13x + 9) \div (x^2 + 2x - 3)$. Write the result in the form $q(x) + \dfrac{r(x)}{d(x)}$.

6. **6 pts**

   Use long division to divide $(4x^3 - 7x + 5) \div (2x + 1)$. Give the quotient and remainder.

7. **6 pts**

   Use synthetic division to divide $(x^4 - 3x^3 + 5x - 6) \div (x + 2)$. Then use the Remainder Theorem to check your remainder.

8. **7 pts**

   (a) Prove that $x - 1$ is a factor of $x^n - 1$ for every positive whole number n. (b) Use synthetic division to find the quotient when $x^5 - 1$ is divided by $x - 1$.

9. **6 pts**

   Factor completely over the integers. (a) $4x^4 - 37x^2 + 9$ (b) $27x^3 + 8$

10. **6 pts**

    Given that -3 is a zero of $f(x) = 2x^3 + 5x^2 - 4x - 3$, solve $f(x) = 0$.

11. **7 pts**

Find all zeros, real and non-real, of $f(x) = x^4 - 2x^3 - x^2 - 4x - 6$.

12. **7 pts**

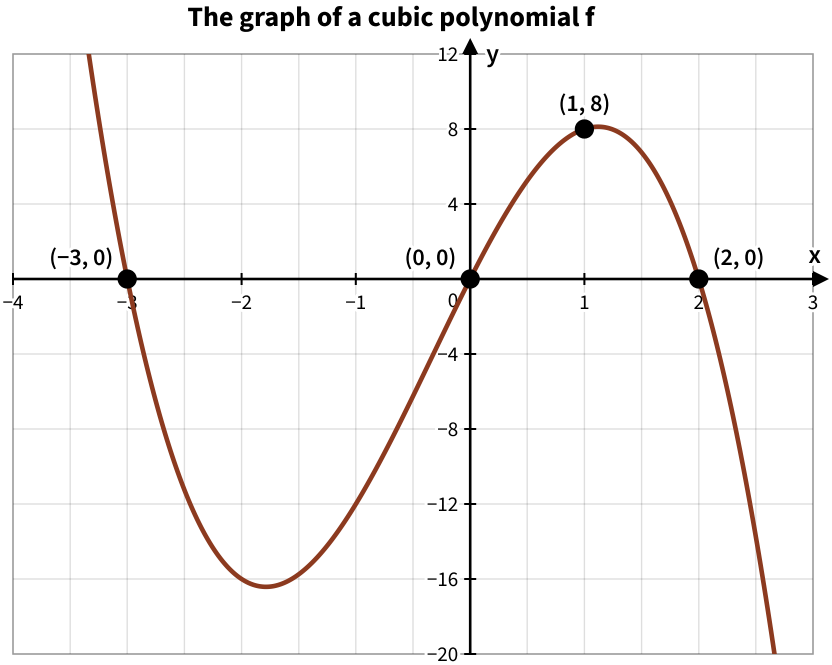
A farmer builds a wooden hay bin shaped like a box. Its width is 1 foot less than twice its height, and its length is 3 feet more than twice its height. The bin must hold 80 cubic feet. Find the height of the bin, and show that no other height works.

13. **6 pts**

Write a polynomial of least degree in standard form, with real coefficients and leading coefficient 1, that has zeros -2 and $1 - 3i$.

14. **6 pts**

The graph of a cubic polynomial f is shown. Write an equation for f in factored form.



15. **6 pts**

Solve $(x + 3)(x - 1)^2(x - 2) \leq 0$. Write the answer in interval notation.

16. **8 pts**

For the tray in Problem 4, $V(x) = 4x^3 - 48x^2 + 140x$ with $0 < x < 5$. (a) Show that $x = 1$ gives a volume of 96 cubic inches. (b) Find every cut size x that gives a volume of 96 cubic inches. (c) For which cut sizes is the volume at least 96 cubic inches? Use a sign chart.

ANSWER KEY

1. Standard form $-x^5 + 3x^2 + 4x - 8$: degree 5, leading coefficient -1, constant term -8; rises on the left and falls on the right (as $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to -\infty$)
2. Leading term $(x^2)(2x)(x) = 2x^4$: degree 4, leading coefficient 2; $g(0) = 16 \times 1 \times 5 = 80$; both ends rise
3. (a) $3x^3 - 12x^2 + 15x - 2x^2 + 8x - 10 = 3x^3 - 14x^2 + 23x - 10$; (b) $x^3 - 15x^2 + 75x - 125$
4. (a) $V(x) = x(14 - 2x)(10 - 2x) = 4x^3 - 48x^2 + 140x$; (b) $V(1.5) = 1.5 \times 11 \times 7 = 115.5$ cubic inches
5. Quotient $2x - 3$, remainder $-x$: $2x - 3 - \dfrac{x}{x^2 + 2x - 3}$ (check: $(x^2 + 2x - 3)(2x - 3) - x = 2x^3 + x^2 - 13x + 9$ ✓)
6. With the placeholder $0x^2$: quotient $2x^2 - x - 3$, remainder 8 (check: $(2x + 1)(2x^2 - x - 3) + 8 = 4x^3 - 7x + 5$ ✓)
7. Coefficients 1, -3, 0, 5, -6 with $c = -2$: bottom row 1, -5, 10, -15 | 24. Quotient $x^3 - 5x^2 + 10x - 15$, remainder 24; check $f(-2) = 16 + 24 - 10 - 6 = 24$ ✓
8. (a) Let $f(x) = x^n - 1$. Then $f(1) = 1^n - 1 = 0$, so by the Factor Theorem $x - 1$ is a factor of $f(x)$. (b) Coefficients 1, 0, 0, 0, 0, -1 with $c = 1$: bottom row 1, 1, 1, 1, 1 | 0, so the quotient is $x^4 + x^3 + x^2 + x + 1$
9. (a) With $u = x^2$: $(4u - 1)(u - 9)$, so $(2x - 1)(2x + 1)(x - 3)(x + 3)$; (b) $(3x + 2)(9x^2 - 6x + 4)$
10. Synthetic division with $c = -3$: bottom row 2, -1, -1 | 0, so $f(x) = (x + 3)(2x^2 - x - 1) = (x + 3)(2x + 1)(x - 1)$. Solutions $x = -3, -\dfrac{1}{2}, 1$
11. Candidates $\pm 1, \pm 2, \pm 3, \pm 6$. $f(-1) = 0$: quotient $x^3 - 3x^2 + 2x - 6$. Then $x = 3$ works: quotient $x^2 + 2$, so $x = \pm i\sqrt{2}$. Zeros $-1, 3, -i\sqrt{2}, i\sqrt{2}$
12. $x(2x - 1)(2x + 3) = 80$ gives $4x^3 + 4x^2 - 3x - 80 = 0$. Testing candidates, $x = \dfrac{5}{2}$ works: synthetic division gives 4, 14, 32 | 0, and $4x^2 + 14x + 32$ has discriminant $196 - 512 < 0$, so there are no other real solutions. Height 2.5 ft (width 4 ft, length 8 ft; check $2.5 \times 4 \times 8 = 80$ ✓)
13. $1 + 3i$ is also a zero. $(x - 1)^2 + 9 = x^2 - 2x + 10$, and $(x + 2)(x^2 - 2x + 10) = x^3 + 6x + 20$
14. Zeros -3, 0 and 2, each crossing: $f(x) = a\,x(x + 3)(x - 2)$. Using $(1, 8)$: $a(1)(4)(-1) = -4a = 8$, so $a = -2$: $f(x) = -2x(x + 3)(x - 2)$

15. Zeros -3, 1 (even multiplicity) and 2. Signs: $+$ on $(-\infty, -3)$, $-$ on $(-3, 1)$, $-$ on $(1, 2)$, $+$ on $(2, \infty)$; the zeros are included. Solution $[-3, 2]$

16. (a) $V(1) = 1 \times 12 \times 8 = 96$ ✓; (b) $V(x) - 96 = 4(x^3 - 12x^2 + 35x - 24) = 4(x - 1)(x - 3)(x - 8)$, so $x = 1$ or $x = 3$ ($x = 8$ is outside the domain); (c) on $0 < x < 5$, $(x - 1)(x - 3)(x - 8)$ is $-$ on $(0, 1)$, $+$ on $(1, 3)$ and $-$ on $(3, 5)$, so $1 \le x \le 3$, that is $[1, 3]$

4.1 Polynomial Functions and End Behavior

OBJECTIVES

- Identify the degree, leading coefficient and constant term of a polynomial written in standard form or as a product of factors.
- Describe the graphs of $y = x^n$ for even and odd whole numbers n.
- Explain why the leading term controls end behavior, and describe end behavior in words and with arrow notation.
- State the greatest possible number of real zeros and turning points of a polynomial of degree n.

WATCH

Graphing Polynomials (1 of 4: Fundamental graphs)

Eddie Woo · 8:49

https://www.youtube.com/watch?v=p2FRcEZezGM

> **WHEN TO WATCH**
> Before Example 1: the basic shapes of polynomial graphs.

BACKUP VIDEO

End Behavior of Polynomials Using Leading Coefficient Test

Mario's Math Tutoring · 10:23

https://www.youtube.com/watch?v=gD8QxjrVMS4

Before Example 2: end behavior with the leading coefficient test.

KEY TERMS

polynomial function

A function that is a sum of terms of the form ax^k, where each coefficient a is a real number and each exponent k is a whole number (0, 1, 2, ...).

standard form

A polynomial written with its terms in order of decreasing exponents, such as $4x^3 - 2x^2 + 7$.

degree

The greatest exponent of the variable among the terms with nonzero coefficients.

leading coefficient

The coefficient of the term of highest degree. That term is called the leading term.

end behavior

What happens to the values $f(x)$ as x moves far to the right ($x \to \infty$) and far to the left ($x \to -\infty$).

turning point

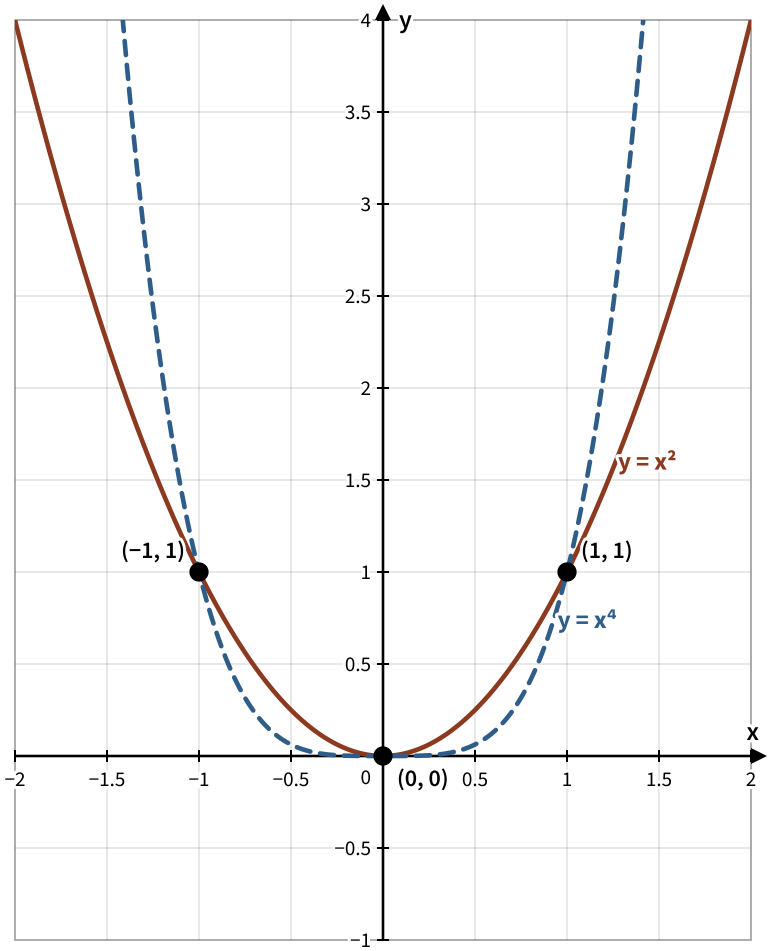
A point where a graph changes from rising to falling or from falling to rising; it is a relative maximum or relative minimum.

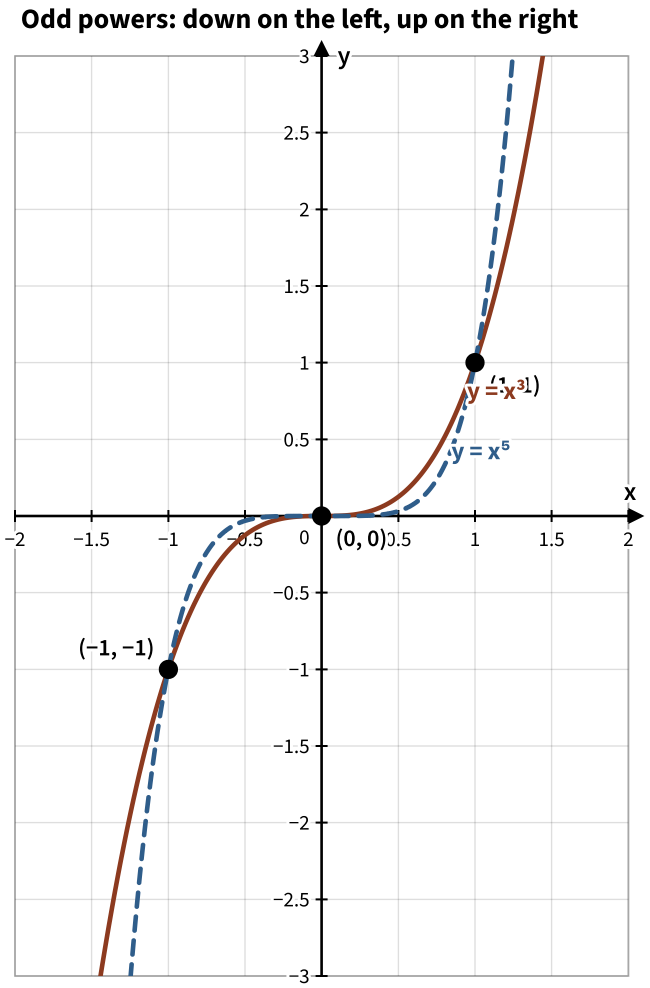
CORE IDEAS

A **polynomial function** is built from whole-number powers of x using multiplication by constants and addition. In standard form, $f(x) = 4x^3 - 2x^2 + 7$ has degree 3, leading coefficient 4 and constant term 7. The constant term is always $f(0)$, the y-intercept. Functions such as $\dfrac{1}{x}$ (a negative power), $\sqrt{x}$ (not a whole-number power) and 2^x (the variable is in the exponent) are not polynomials.

The simplest polynomials are the powers $y = x^n$. When n is even, $(-x)^n = x^n$, so the graph is symmetric in the y-axis and both ends point up; every even power passes through $(-1, 1)$, $(0, 0)$ and $(1, 1)$. When n is odd, $(-x)^n = -x^n$, so the graph is symmetric about the origin, falling on the left and rising on the right, through $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Higher powers are flatter between -1 and 1 (for example $0.5^4 = 0.0625$ is less than $0.5^2 = 0.25$) and steeper outside.

Even powers: both ends up





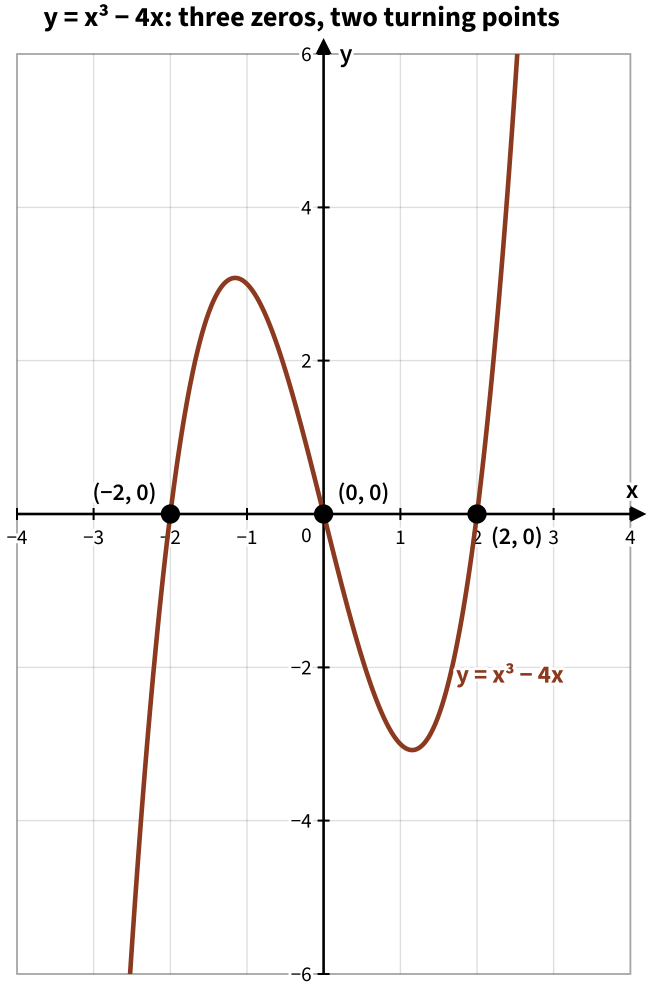
Why the leading term controls the ends. Take $f(x) = 2x^3 - 30x^2$. At $x = 100$ the leading term is 2,000,000 and $f(100) = 2{,}000{,}000 - 300{,}000 = 1{,}700{,}000$, which is 85% of the leading term. At $x = 1000$, $f(1000) = 2{,}000{,}000{,}000 - 30{,}000{,}000 = 1{,}970{,}000{,}000$, which is 98.5% of it. In general, factor out the leading term: $f(x) = a_n x^n \left(1 + \dfrac{a_{n-1}}{a_n x} + \cdots + \dfrac{a_0}{a_n x^n}\right)$. When $|x|$ is large, every fraction in the parentheses is close to 0, so the parentheses are close to 1 and $f(x)$ behaves like $a_n x^n$. The other terms matter in the middle of the graph, but far out the leading term is much bigger than all the others together.

So end behavior depends only on two facts about the leading term: whether the degree is even or odd, and whether the leading coefficient is positive or negative. We write it with arrows: for $f(x) = -2x^3 + x$, "as $x \to \infty$, $f(x) \to -\infty$, and as $x \to -\infty$, $f(x) \to \infty$," or in words, "rises on the left and falls on the right."

End behavior from the leading term

Leading coefficient	Even degree	Odd degree
positive	up on the left, up on the right	down on the left, up on the right
negative	down on the left, down on the right	up on the left, down on the right

A polynomial of degree n has **at most n real zeros** (in Lesson 4.5 you will see that each zero gives a factor $x - c$, and more than n such factors would make the degree too big) and **at most $n - 1$ turning points** (the reason needs calculus; take it as a fact for now). The cubic $y = x^3 - 4x = x(x - 2)(x + 2)$ below uses all of its allowance: three zeros and two turning points. Notice also that an odd-degree polynomial must cross the x-axis at least once, because its two ends are on opposite sides of the axis and the graph has no breaks. An even-degree polynomial such as $x^2 + 1$ may have no real zeros at all.



KEY FORMULAS

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}\right)$$

Degree n: at most n real zeros and at most $n - 1$ turning points.

Practice habit. Know the parent powers cold: $y = x^2$ and $y = x^4$ pass through $(-1, 1)$, $(0, 0)$ and $(1, 1)$; $y = x^3$ and $y = x^5$ pass through $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Before describing any polynomial, find its leading term first.

WORKED EXAMPLES

Example 1. Henry is building a wooden toolbox. Its height is x inches, its width is $x + 1$ inches and its length is $x + 4$ inches. (a) Write the volume $V(x)$ in standard form. (b) Give the degree, leading coefficient and constant term. (c) Find the volume when the height is 8 inches.

Show solution

1. Volume is length times width times height: $V(x) = x(x + 1)(x + 4)$.
2. Multiply the binomials first: $(x + 1)(x + 4) = x^2 + 5x + 4$.
3. Multiply by x: $V(x) = x^3 + 5x^2 + 4x$.
4. The leading term is x^3, so the degree is 3 and the leading coefficient is 1. There is no constant term, so the constant term is 0 (and $V(0) = 0$).
5. Evaluate: $V(8) = 8^3 + 5(8^2) + 4(8) = 512 + 320 + 32 = 864$.
6. Check with the dimensions: $8 \times 9 \times 12 = 864$ ✓

Answer: (a) $V(x) = x^3 + 5x^2 + 4x$; (b) degree 3, leading coefficient 1, constant term 0; (c) 864 cubic inches

Example 2. Let $f(x) = (2 - x)(x + 1)^2(x - 3)$. Without multiplying everything out, find (a) the degree and leading coefficient, (b) the end behavior, (c) the y-intercept and (d) the greatest possible number of turning points.

Show solution

1. The leading term of a product is the product of the leading terms of the factors: $(-x)(x^2)(x) = -x^4$.
2. So the degree is 4 and the leading coefficient is -1.
3. Even degree with a negative leading coefficient: both ends point down. As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$.
4. The y-intercept is $f(0) = (2)(1)^2(-3) = -6$.
5. A degree-4 polynomial has at most $4 - 1 = 3$ turning points.

Answer: (a) degree 4, leading coefficient -1; (b) falls on both ends; (c) y-intercept -6; (d) at most 3 turning points

PRACTICE

1. **Easy** Is each function a polynomial? (a) $f(x) = 3x^4 - x + 7$ (b) $g(x) = \dfrac{5}{x} + 2$ (c) $h(x) = \dfrac{x^3}{2} - \sqrt{3}\,x$

   Show answer

   (a) yes; (b) no, $\dfrac{5}{x}$ is a negative power of x; (c) yes, the coefficients $\dfrac{1}{2}$ and $-\sqrt{3}$ are real numbers and the exponents are whole numbers

2. **Easy** Write $7 - 3x^2 + x^5 - 4x$ in standard form. Then give its degree, leading coefficient and constant term.

   Show answer

   $x^5 - 3x^2 - 4x + 7$; degree 5, leading coefficient 1, constant term 7

3. **Easy** Let $f(x) = 2x^3 - 5x^2 + x - 4$. Find (a) $f(3)$ and (b) $f(-2)$.

   Show answer

   (a) $54 - 45 + 3 - 4 = 8$; (b) $-16 - 20 - 2 - 4 = -42$

4. **Easy** Describe the end behavior of $f(x) = -4x^3 + x^2 + 9$ in words and with arrow notation.

   Show answer

   Leading term $-4x^3$: odd degree, negative coefficient. Rises on the left, falls on the right: as $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to -\infty$.

5. **Medium** A polynomial has degree 6. (a) What is the greatest number of real zeros it can have? (b) What is the greatest number of turning points it can have?

   Show answer

   (a) 6; (b) 5

6. **Medium** Describe the end behavior of $g(x) = 5x^2 - 2x^6 + x$. (Careful: it is not written in standard form.)

   Show answer

   The leading term is $-2x^6$: even degree, negative coefficient, so both ends fall. As $x \to \pm\infty$, $g(x) \to -\infty$.

7. **Medium** A shipping crate has height x feet, length $2x + 1$ feet and width $x + 3$ feet. (a) Write its volume $V(x)$ in standard form. (b) Find the volume when $x = 2$.

Show answer

(a) $V(x) = x(2x + 1)(x + 3) = x(2x^2 + 7x + 3) = 2x^3 + 7x^2 + 3x$; (b) $V(2) = 16 + 28 + 6 = 50$ cubic feet (check: $2 \times 5 \times 5 = 50$ ✓)

8. **Medium** Without multiplying out, find the degree and leading coefficient of $f(x) = (3x - 1)(x + 2)^2(1 - x)$, and describe its end behavior.

Show answer

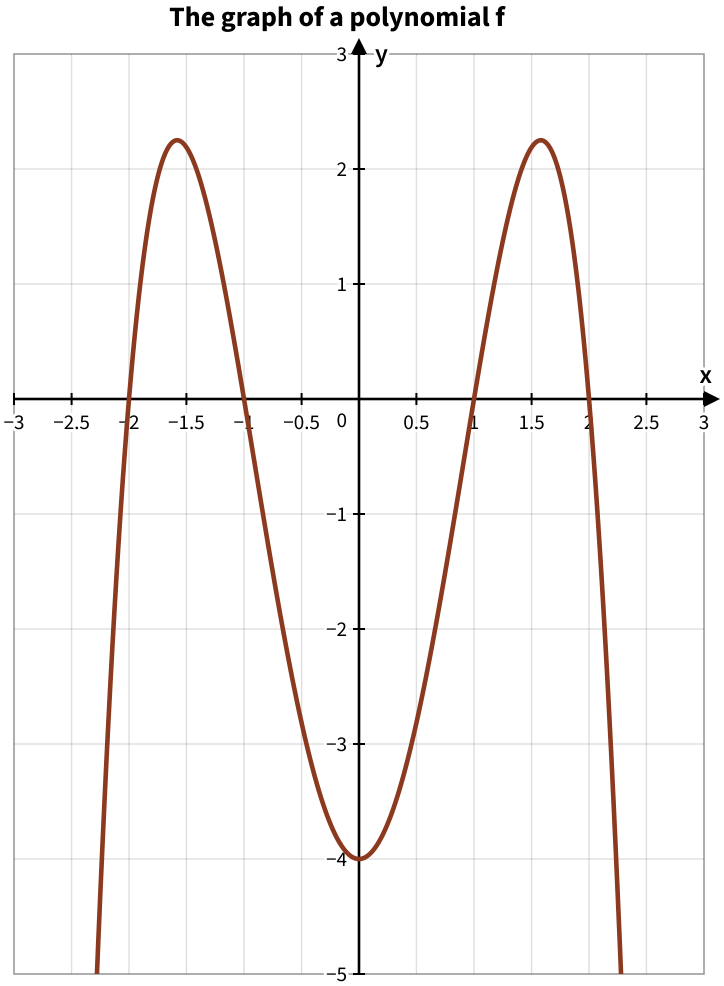
Leading term $(3x)(x^2)(-x) = -3x^4$: degree 4, leading coefficient -3; both ends fall.

9. **Medium** A soccer ball is kicked upward. Its height in feet after t seconds is $h(t) = -16t^2 + 48t + 6$. (a) What is the degree of h? (b) Describe the end behavior of the formula and explain why the model only makes sense for a limited time. (c) Find $h(2)$.

Show answer

(a) 2; (b) both ends fall (leading term $-16t^2$); the formula would give negative heights after the ball lands, so it only applies from the kick until the ball hits the ground; (c) $h(2) = -64 + 96 + 6 = 38$ feet

10. **Medium** The graph of a polynomial f is shown. (a) Is the degree of f even or odd? (b) Is the leading coefficient positive? (c) Counting turning points, what is the least possible degree of f?



Show answer

(a) even, since both ends point the same way (down); (b) no, the ends point down, so it is negative; (c) the graph has 3 turning points, so the degree is at least $3 + 1 = 4$

11. Hard The Miller family models the number of tomatoes ripening per day in their garden by $T(d) = -0.01d^3 + 0.6d^2$, where d is the number of days since the first tomato ripened. (a) Find $T(30)$. (b) Find $T(60)$. (c) Use the end behavior of T to explain why the model should only be used for $0 \le d \le 60$.

Show answer

(a) $-0.01(27{,}000) + 0.6(900) = -270 + 540 = 270$ tomatoes; (b) $-2160 + 2160 = 0$ tomatoes; (c) the leading term $-0.01d^3$ makes $T(d) \to -\infty$ as d grows, so after day 60 the model gives negative numbers of tomatoes, which is impossible

12. Hard Let $f(x) = x^3 - 50x^2$. (a) Find $f(100)$. (b) Find $f(1000)$. (c) Compare each value with x^3 and explain why f has the same end behavior as $y = x^3$.

Show answer

(a) $1{,}000{,}000 - 500{,}000 = 500{,}000$; (b) $1{,}000{,}000{,}000 - 50{,}000{,}000 = 950{,}000{,}000$; (c) $f(100)$ is 50% of 100^3 and $f(1000)$ is 95% of 1000^3. Since $f(x) = x^3\left(1 - \dfrac{50}{x}\right)$ and $\dfrac{50}{x} \to 0$ as $|x|$ grows, $f(x)$ gets closer and closer to a full share of x^3, so the ends behave like x^3.

13. Hard Can a polynomial of degree 3 have no real zeros? Explain.

Show answer

No. A cubic's ends point in opposite directions (one up, one down), so its graph is below the x-axis at one end and above it at the other. A polynomial graph has no breaks, so it must cross the x-axis somewhere.

14. Challenge The graph of a polynomial rises on the left, falls on the right, has exactly 3 x-intercepts and has 4 turning points. (a) What is the least possible degree? (b) Is the leading coefficient positive?

Show answer

(a) 4 turning points require degree at least 5; rising left and falling right means odd degree, and 5 is odd, so the least degree is 5; (b) no, odd degree that falls on the right has a negative leading coefficient

COMMON MISTAKES

- Taking the first term written as the leading term. In $5x^2 - 2x^6 + x$ the leading term is $-2x^6$; rewrite in standard form or look for the highest power first.

- Confusing the degree with the number of terms. $x^5 + 1$ has two terms but degree 5.
- Thinking the constant term or middle terms change the end behavior. They move the graph in the middle; only the leading term decides the ends.
- Reading "at most n zeros" as "exactly n real zeros." The graph of $x^2 + 1$ has degree 2 but never touches the x-axis.

> **Make it a game · End Behavior Arms.** The parent calls polynomials such as $-2x^5 + 3x^2 - 1$ or $x^4 - 5x^2$. Name the degree, leading coefficient and constant term, then show the end behavior with your arms: left arm for x going far left, right arm for far right. Up means large positive, down means large negative. Add a sketch of $y = x^n$ for even and odd n.

4.2 Operations with Polynomials

OBJECTIVES

- Add, subtract and multiply polynomials of any size and write the result in standard form.
- Derive the cube of a binomial, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, by multiplying out.
- Verify the sum and difference of cubes formulas by multiplying.
- Write and evaluate the volume polynomial $V = x(L-2x)(W-2x)$ for an open box and state its domain.

WATCH

Polynomial Operations (1 of 2: Addition, subtraction, multiplication)

Eddie Woo · 6:14

https://www.youtube.com/watch?v=4lYy7GUTdJ4

> **WHEN TO WATCH**
> Before Example 1: adding, subtracting and multiplying polynomials.

BACKUP VIDEO

Cubic Identities (2 of 3: Cubes of sum & difference)

Eddie Woo · 9:21

https://www.youtube.com/watch?v=HvvFfTmK2pE

Before the practice set: expanding $(a+b)^3$ and $(a-b)^3$.

KEY TERMS

like terms

Terms with the same variable raised to the same power, such as $3x^2$ and $-7x^2$. Only like terms can be combined.

distributive property

The rule $a(b+c) = ab + ac$. Multiplying polynomials is the distributive property used over and over: every term of one factor multiplies every term of the other.

cube of a binomial

A product of the form $(a+b)^3$ or $(a-b)^3$, a binomial used as a factor three times.

sum and difference of cubes

Expressions of the form a^3+b^3 and a^3-b^3, which have the special factorizations $(a+b)(a^2-ab+b^2)$ and $(a-b)(a^2+ab+b^2)$.

CORE IDEAS

Adding and subtracting (review). Combine like terms. When subtracting, the minus sign applies to every term in the parentheses: $(5x^3-2x+1)-(x^3+4x^2-6) = 5x^3-2x+1-x^3-4x^2+6 = 4x^3-4x^2-2x+7$.

Multiplying. Every term of the first polynomial multiplies every term of the second, then like terms are combined. A product of a 2-term and a 3-term polynomial has $2\times 3=6$ small products before combining. A grid keeps them organized. For $(x+2)(x^2-3x+4)$, the six products are shown below; combining gives $x^3-3x^2+4x+2x^2-6x+8 = x^3-x^2-2x+8$. Since the leading terms multiply, the degree of a product is the sum of the degrees.

Grid for (x + 2)(x² − 3x + 4)

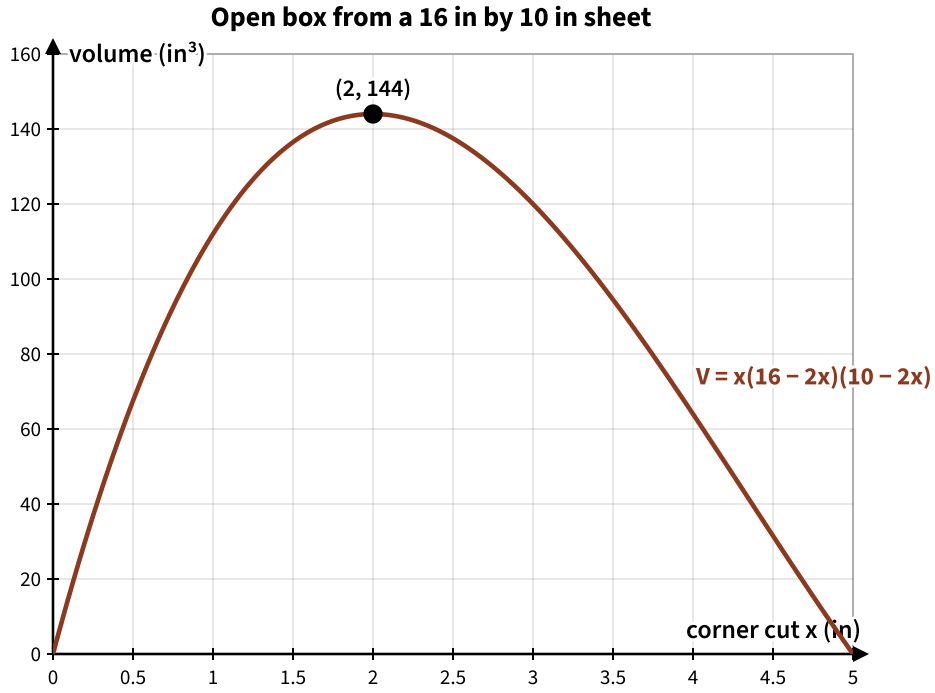
×	x^2	$-3x$	4
x	x^3	$-3x^2$	4x
2	$2x^2$	−6x	8

A quick check. Two polynomials that are equal give the same value for every x. Substitute an easy number such as $x=1$ into both the original product and your answer. For the product above: $(1+2)(1-3+4)=3\times 2=6$ and $1-1-2+8=6$ ✓. (A match does not prove you are right, but a mismatch proves you are wrong.)

The cube of a binomial. Write $(a+b)^3=(a+b)(a+b)^2=(a+b)(a^2+2ab+b^2)$. Distributing gives $a^3+2a^2b+ab^2+a^2b+2ab^2+b^3=a^3+3a^2b+3ab^2+b^3$. Replacing b by $-b$ gives $(a-b)^3=a^3-3a^2b+3ab^2-b^3$: the signs alternate because odd powers of $-b$ are negative. Notice that $(a+b)^3$ is NOT a^3+b^3; the middle terms $3a^2b+3ab^2$ are real and usually large. For example $(10+1)^3=1000+300+30+1=1331$.

Sum and difference of cubes. Multiply: $(a+b)(a^2-ab+b^2) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3$. The middle terms cancel in pairs. In the same way $(a-b)(a^2+ab+b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3$. You will use these to factor in Lesson 4.5.

An open box. Cut a square of side x from each corner of an L-by-W sheet and fold up the sides. The box has height x, length $L - 2x$ and width $W - 2x$, so $V(x) = x(L-2x)(W-2x)$, a cubic. Every dimension must be positive, so the domain is $0 < x < \dfrac{W}{2}$ when W is the shorter side. For a 16-by-10-inch sheet, $V(x) = x(16-2x)(10-2x) = 4x^3 - 52x^2 + 160x$ on $0 < x < 5$; the graph shows the volume rising and then falling, with $V(2) = 2 \times 12 \times 6 = 144$ cubic inches.



KEY FORMULAS

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$V(x) = x(L-2x)(W-2x), \quad 0 < x < \frac{W}{2} \ (W \le L)$$

Practice habit. Know the special products cold: $(a \pm b)^2$, $(a \pm b)^3$, $a^3 + b^3$ and $a^3 - b^3$. After every multiplication, check by substituting $x = 1$ (or $x = 2$) into the problem and your answer.

WORKED EXAMPLES

Example 1. Simplify each expression and write it in standard form. (a) $(3x^2 - 5x + 2) - (x^2 - 4x - 7)$ (b) $(2x - 3)(x^2 + 4x - 5)$

Show solution

1. (a) Distribute the minus sign: $3x^2 - 5x + 2 - x^2 + 4x + 7$.
2. Combine like terms: $2x^2 - x + 9$.
3. (b) Multiply $2x$ by each term: $2x^3 + 8x^2 - 10x$.
4. Multiply -3 by each term: $-3x^2 - 12x + 15$.
5. Combine: $2x^3 + (8 - 3)x^2 + (-10 - 12)x + 15 = 2x^3 + 5x^2 - 22x + 15$.
6. Check (b) with $x = 2$: $(1)(4 + 8 - 5) = 7$ and $16 + 20 - 44 + 15 = 7$ ✓

Answer: (a) $2x^2 - x + 9$; (b) $2x^3 + 5x^2 - 22x + 15$

Example 2. Grace is making an open tray from a 20-inch by 12-inch sheet of tin by cutting a square of side x inches from each corner and folding up the sides. (a) Write the volume $V(x)$ in standard form. (b) State the domain. (c) Find the volume when $x = 3$.

Show solution

1. Length $20 - 2x$, width $12 - 2x$, height x: $V(x) = x(20 - 2x)(12 - 2x)$.
2. Multiply the binomials: $(20 - 2x)(12 - 2x) = 240 - 40x - 24x + 4x^2 = 4x^2 - 64x + 240$.
3. Multiply by x: $V(x) = 4x^3 - 64x^2 + 240x$.
4. Every dimension must be positive: $x > 0$ and $12 - 2x > 0$, so $x < 6$ (then $20 - 2x > 0$ too). Domain: $0 < x < 6$.
5. $V(3) = 4(27) - 64(9) + 240(3) = 108 - 576 + 720 = 252$.
6. Check with dimensions: $3 \times 14 \times 6 = 252$ ✓

Answer: (a) $V(x) = 4x^3 - 64x^2 + 240x$; (b) $0 < x < 6$; (c) 252 cubic inches

PRACTICE

1. Easy Add: $(4x^3 - 2x^2 + 7) + (-x^3 + 5x^2 - 3x - 9)$.

   Show answer

   $3x^3 + 3x^2 - 3x - 2$

2. Easy Subtract: $(5x^2 - 3x + 1) - (2x^2 + 4x - 6)$.

   Show answer

   $5x^2 - 3x + 1 - 2x^2 - 4x + 6 = 3x^2 - 7x + 7$

3. Easy Multiply: $-3x^2(2x^3 - x + 4)$.

   Show answer

   $-6x^5 + 3x^3 - 12x^2$

4. Easy Multiply: $(x + 4)(x^2 - 2x + 3)$.

   Show answer

   $x^3 - 2x^2 + 3x + 4x^2 - 8x + 12 = x^3 + 2x^2 - 5x + 12$

5. Medium Multiply: $(x^2 + 3x - 1)(2x^2 - x + 4)$.

   Show answer

   $2x^4 - x^3 + 4x^2 + 6x^3 - 3x^2 + 12x - 2x^2 + x - 4 = 2x^4 + 5x^3 - x^2 + 13x - 4$

6. Medium Expand $(x + 2)^3$.

   Show answer

   $x^3 + 3(x^2)(2) + 3(x)(4) + 8 = x^3 + 6x^2 + 12x + 8$

7. Medium Expand $(2x - 3)^3$.

   Show answer

   $(2x)^3 - 3(2x)^2(3) + 3(2x)(3)^2 - 3^3 = 8x^3 - 36x^2 + 54x - 27$

8. Medium Multiply: $(x - 1)(x + 2)(x + 3)$.

   Show answer

   $(x - 1)(x^2 + 5x + 6) = x^3 + 4x^2 + x - 6$

9. Medium Multiply $(x-4)(x^2+4x+16)$ and explain why the answer has only two terms.

Show answer

$x^3+4x^2+16x-4x^2-16x-64 = x^3-64$. The middle terms cancel in pairs, which is the difference of cubes a^3-b^3 with $a=x$, $b=4$.

10. Hard Samuel cuts squares of side x inches from the corners of an 18-inch by 12-inch piece of cardboard to make an open box. (a) Write $V(x)$ in standard form. (b) State the domain. (c) Find $V(2)$.

Show answer

(a) $(18-2x)(12-2x) = 4x^2-60x+216$, so $V(x) = 4x^3-60x^2+216x$; (b) $0<x<6$; (c) $V(2) = 32-240+432 = 224$ cubic inches (check: $2\times 14\times 8 = 224$ ✓)

11. Hard A carpenter has a cube-shaped wooden block with edge x inches. A second block has every edge 2 inches longer. (a) Write a polynomial for how much more volume the second block has. (b) Evaluate it for $x=10$.

Show answer

(a) $(x+2)^3 - x^3 = 6x^2+12x+8$ cubic inches; (b) $600+120+8 = 728$ cubic inches (check: $1728-1000=728$ ✓)

12. Hard Ruth deposits \$500 at the start of each of three years into a savings account. Each year the balance is multiplied by the growth factor x. (a) Explain why the balance at the end of the third year is $500x^3+500x^2+500x$. (b) Find the balance when the account pays 4% per year ($x=1.04$), to the nearest cent.

Show answer

(a) The first deposit grows for 3 years, the second for 2 and the third for 1; (b) $500(1.124864+1.0816+1.04) = 500(3.246464) \approx \$1623.23$

13. Hard Prove that $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$ by multiplying.

Show answer

$(a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2-2ab+b^2) = a^3-2a^2b+ab^2-a^2b+2ab^2-b^3 = a^3-3a^2b+3ab^2-b^3$.

14. **Challenge** Find the numbers a and b so that $(x^2 + ax + 3)(x + b) = x^3 + 5x^2 + 9x + 6$.

Show answer

The product is $x^3 + (a + b)x^2 + (ab + 3)x + 3b$. Matching: $3b = 6$, so $b = 2$; $a + b = 5$, so $a = 3$; check $ab + 3 = 9$ ✓

COMMON MISTAKES

- Subtracting only the first term: $(5x^2 - 3x) - (2x^2 + 4x)$ is $3x^2 - 7x$, not $3x^2 + x$. Rewrite with the minus sign distributed before combining.
- Writing $(a + b)^3 = a^3 + b^3$. The middle terms $3a^2b + 3ab^2$ are missing; test with $a = b = 1$: $8 \neq 2$.
- Missing a product. Count: an m-term times an n-term polynomial gives mn products before combining. A grid helps.
- Combining unlike terms such as x^3 and x^2. Only terms with the same power combine.

> **Make it a game · Plug-In Lie Detector.** The parent multiplies $(x^2 + 2x - 1)(x - 3)$ and secretly makes a mistake half the time. You do not redo the product. Instead, test the claimed answer by plugging in $x = 1$ and $x = 2$ on both sides, and report whether the parent is lying. Then expand $(x + 2)^3$ yourself and check it the same way.

4.3 Polynomial Long Division

OBJECTIVES

- Divide a polynomial by a polynomial of degree one or two using long division.
- Use zero placeholders for missing terms in the dividend.
- Write the result as dividend = divisor × quotient + remainder and as $q(x) + \frac{r(x)}{d(x)}$.
- Check a division by multiplying.

WATCH

Division of Polynomials (2 of 2: Using Long Division to divide Polynomials)

Eddie Woo · 10:23

https://www.youtube.com/watch?v=JnE23UkRSqo

> **WHEN TO WATCH**
> Before Example 1: polynomial long division.

BACKUP VIDEO

How to do Polynomial Long Division Step by Step

Mario's Math Tutoring · 8:17

https://www.youtube.com/watch?v=5ujv2OlMnjo

Before Example 2: a second long division, step by step, including a missing term.

KEY TERMS

dividend

The polynomial being divided, $f(x)$.

divisor

The polynomial you divide by, $d(x)$.

quotient

The result of the division, $q(x)$, built one term at a time.

remainder

What is left, $r(x)$, when its degree is less than the degree of the divisor. If the remainder is 0, the divisor is a factor of the dividend.

placeholder

A term with coefficient 0, such as $0x^2$, written in for a missing power so that the columns line up.

CORE IDEAS

Polynomial long division works exactly like the long division of whole numbers. To divide 745 by 6: 7 divided by 6 is 1; multiply $1 \times 6 = 6$; subtract to get 1; bring down the 4 to make 14. Repeat: 14 divided by 6 is 2, $2 \times 6 = 12$, subtract to get 2, bring down the 5 to make 25. Then 25 divided by 6 is 4, $4 \times 6 = 24$, and 1 is left. So $745 = 6 \times 124 + 1$. The cycle is **divide, multiply, subtract, bring down**.

With polynomials, "divide" means: divide the leading term of what is left by the leading term of the divisor. Here is $(x^3 + 2x^2 - 5x - 6) \div (x - 2)$, one cycle per row. The quotient is the sum of the second column's answers, $x^2 + 4x + 3$, and the remainder is 0. So $x - 2$ is a factor: $x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + 4x + 3) = (x - 2)(x + 1)(x + 3)$.

$(x^3 + 2x^2 - 5x - 6) \div (x - 2)$, one cycle per row

Cycle	Divide leading terms	Multiply by $x - 2$	Subtract and bring down
1	$x^3 \div x = x^2$	$x^3 - 2x^2$	$4x^2 - 5x$
2	$4x^2 \div x = 4x$	$4x^2 - 8x$	$3x - 6$
3	$3x \div x = 3$	$3x - 6$	0 (remainder)

Why it works and when to stop. Each cycle subtracts a multiple of the divisor chosen to cancel the leading term, so the degree of what is left goes down every time. What you subtract altogether is (divisor) × (quotient), and what is left is the remainder. That gives the **division statement** $f(x) = d(x) \cdot q(x) + r(x)$. You stop as soon as the remainder's degree is less than the divisor's degree, because then its leading term can no longer be divided by the divisor's leading term to give a polynomial term. (With whole numbers you stop when the remainder is less than the divisor.)

Placeholders. If a power is missing, write it with a zero coefficient, just as 405 has a 0 in the tens place. To divide $x^3 - 1$, write the dividend as $x^3 + 0x^2 + 0x - 1$. Without

placeholders the columns slip and the subtraction combines unlike terms.

Divisors of degree two. The cycle is the same. Dividing by $x^2 + 1$ or $x^2 + x - 1$, you keep going until what is left has degree 1 or 0, so the remainder may be a linear expression such as $2x + 4$.

Two ways to write the answer. Just as $\frac{745}{6} = 124 + \frac{1}{6}$, the division statement divided by $d(x)$ gives $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. Always check by multiplying: $d(x) \cdot q(x) + r(x)$ must give back the dividend.

KEY FORMULAS

$$f(x) = d(x) \cdot q(x) + r(x), \quad \text{degree of } r < \text{degree of } d$$

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

> **Practice habit.** Write one step per line and keep like terms in the same column, with zero placeholders for missing powers. Finish every division by checking that divisor × quotient + remainder gives back the dividend.

WORKED EXAMPLES

Example 1. A wooden crate has volume $x^3 + 7x^2 + 14x + 8$ cubic feet, and its base has area $x^2 + 5x + 4$ square feet. Find a polynomial for its height.

Show solution

1. Height = volume ÷ base area, so divide $(x^3 + 7x^2 + 14x + 8) \div (x^2 + 5x + 4)$.
2. Divide leading terms: $x^3 \div x^2 = x$. Multiply: $x(x^2 + 5x + 4) = x^3 + 5x^2 + 4x$.
3. Subtract: $(x^3 + 7x^2 + 14x) - (x^3 + 5x^2 + 4x) = 2x^2 + 10x$; bring down $+8$: $2x^2 + 10x + 8$.
4. Divide: $2x^2 \div x^2 = 2$. Multiply: $2(x^2 + 5x + 4) = 2x^2 + 10x + 8$.
5. Subtract: remainder 0. The quotient is $x + 2$.
6. Check: $(x^2 + 5x + 4)(x + 2) = x^3 + 2x^2 + 5x^2 + 10x + 4x + 8 = x^3 + 7x^2 + 14x + 8$ ✓

Answer: The height is $x + 2$ feet.

Example 2. Divide $(x^4 - 3x^2 + 2x + 5) \div (x^2 + x - 1)$. Write the result in the form $q(x) + \dfrac{r(x)}{d(x)}$.

Show solution

1. Insert the placeholder: the dividend is $x^4 + 0x^3 - 3x^2 + 2x + 5$.
2. Divide: $x^4 \div x^2 = x^2$. Multiply: $x^2(x^2 + x - 1) = x^4 + x^3 - x^2$. Subtract: $-x^3 - 2x^2$; bring down $+2x$: $-x^3 - 2x^2 + 2x$.
3. Divide: $-x^3 \div x^2 = -x$. Multiply: $-x(x^2 + x - 1) = -x^3 - x^2 + x$. Subtract: $-x^2 + x$; bring down $+5$: $-x^2 + x + 5$.
4. Divide: $-x^2 \div x^2 = -1$. Multiply: $-1(x^2 + x - 1) = -x^2 - x + 1$. Subtract: $2x + 4$.
5. The remainder $2x + 4$ has degree 1, less than 2, so stop. Quotient $x^2 - x - 1$, remainder $2x + 4$.
6. Check: $(x^2 + x - 1)(x^2 - x - 1) = x^4 - 3x^2 + 1$, and $x^4 - 3x^2 + 1 + 2x + 4 = x^4 - 3x^2 + 2x + 5$ ✓

Answer: $x^2 - x - 1 + \dfrac{2x + 4}{x^2 + x - 1}$

PRACTICE

1. Easy Divide: $(x^2 + 7x + 10) \div (x + 2)$.

   Show answer

   $x + 5$, remainder 0

2. Easy Divide: $(2x^2 - 5x - 3) \div (x - 3)$.

   Show answer

   $2x + 1$, remainder 0 (check: $(x - 3)(2x + 1) = 2x^2 - 5x - 3$ ✓)

3. Easy Divide: $(x^2 + 3x - 7) \div (x + 5)$. Give the quotient and remainder.

   Show answer

   Quotient $x - 2$, remainder 3 (check: $(x + 5)(x - 2) + 3 = x^2 + 3x - 7$ ✓)

4. Easy Divide: $(x^3 + 4x^2 + x - 6) \div (x + 2)$.

   Show answer

   $x^2 + 2x - 3$, remainder 0

5. **Medium** Divide: $(2x^3 - 3x^2 + 4x + 1) \div (x - 1)$.

Show answer

Quotient $2x^2 - x + 3$, remainder 4 (check: $(x-1)(2x^2 - x + 3) + 4 = 2x^3 - 3x^2 + 4x + 1$ ✓)

6. **Medium** Divide: $(x^3 - 8) \div (x - 2)$. Use placeholders.

Show answer

Write $x^3 + 0x^2 + 0x - 8$; quotient $x^2 + 2x + 4$, remainder 0

7. **Medium** Divide $(x^3 + 2x - 4) \div (x + 1)$ and write the answer in the form $q(x) + \dfrac{r}{x+1}$.

Show answer

Quotient $x^2 - x + 3$, remainder -7: $x^2 - x + 3 - \dfrac{7}{x+1}$

8. **Medium** Divide: $(6x^3 + 5x^2 - 8x + 3) \div (2x + 3)$.

Show answer

Quotient $3x^2 - 2x - 1$, remainder 6 (check: $(2x+3)(3x^2 - 2x - 1) + 6 = 6x^3 + 5x^2 - 8x + 3$ ✓)

9. **Medium** Divide: $(x^4 + 2x^3 - x^2 + 4x - 1) \div (x^2 + 1)$.

Show answer

Quotient $x^2 + 2x - 2$, remainder $2x + 1$ (check: $(x^2+1)(x^2 + 2x - 2) + 2x + 1 = x^4 + 2x^3 - x^2 + 4x - 1$ ✓)

10. **Hard** The floor of a church's new kitchen is a rectangle with area $6x^2 + 7x - 20$ square feet. One side is $2x + 5$ feet long. (a) Find a polynomial for the other side. (b) Find the length of that side when $x = 8$.

Show answer

(a) $(6x^2 + 7x - 20) \div (2x + 5) = 3x - 4$ feet; (b) $3(8) - 4 = 20$ feet (the room is 21 ft by 20 ft)

11. Hard A box has volume $x^3 + 9x^2 + 26x + 24$ cubic inches and height $x + 2$ inches. (a) Find the area of its base as a polynomial. (b) Factor the base area to find possible expressions for the length and width.

Show answer

(a) $x^2 + 7x + 12$ square inches, remainder 0; (b) $(x + 3)(x + 4)$, so the base is $x + 3$ by $x + 4$ inches

12. Hard The Millers pick $6x^2 + 5x + 4$ apples and share them equally among $2x + 1$ crates, where x is a whole number greater than 1. (a) Divide to find the quotient and remainder. (b) When $x = 20$, how many apples go in each crate? (c) How many apples are left over?

Show answer

(a) Quotient $3x + 1$, remainder 3; (b) $3(20) + 1 = 61$ apples per crate; (c) 3 left over (check: 2504 apples $= 41 \times 61 + 3$ ✓)

13. Hard Ruth says $(x^3 + 1) \div (x + 1) = x^2 + 1$ because "$x^3 \div x = x^2$ and $1 \div 1 = 1$." Show that she is wrong and find the correct quotient.

Show answer

Check her answer: $(x + 1)(x^2 + 1) = x^3 + x^2 + x + 1 \neq x^3 + 1$. Polynomials cannot be divided term by term by a binomial. Long division with placeholders $(x^3 + 0x^2 + 0x + 1)$ gives $x^2 - x + 1$, remainder 0; check: $(x + 1)(x^2 - x + 1) = x^3 + 1$ ✓

14. Challenge (a) Divide $(x^3 + 3x^2 + 5x + 7) \div (x + 2)$. (b) Substitute $x = 10$ into the dividend, divisor, quotient and remainder, and explain what whole-number division you get.

Show answer

(a) Quotient $x^2 + x + 3$, remainder 1; (b) at $x = 10$ the dividend is 1357, the divisor is 12, the quotient is 113 and the remainder is 1, so $1357 = 12 \times 113 + 1$: polynomial division is whole-number division with the digits as coefficients.

15. **Challenge** Use long division to find the value of k for which $x + 2$ is a factor of $x^3 + kx + 6$.

Show answer

Dividing $x^3 + 0x^2 + kx + 6$ by $x + 2$ gives quotient $x^2 - 2x + (k + 4)$ and remainder $6 - 2(k + 4) = -2k - 2$. Set $-2k - 2 = 0$: $k = -1$. Check: $(x + 2)(x^2 - 2x + 3) = x^3 - x + 6$ ✓

COMMON MISTAKES

- Leaving out placeholders. In $(x^3 - 8) \div (x - 2)$, write $x^3 + 0x^2 + 0x - 8$ or the columns will not line up.
- Sign errors when subtracting. Subtract the WHOLE product: change every sign of the product line, then add.
- Stopping too early or too late. Stop exactly when the remainder's degree is less than the divisor's degree.
- Writing the remainder as a plain term, as in $x^2 - x + 3 - 7$. The remainder must be divided by the divisor: $x^2 - x + 3 - \dfrac{7}{x + 1}$.

> **Make it a game · Placeholder Patrol.** The parent writes dividends with missing terms, such as $x^3 - 8$ divided by $x - 2$. Rewrite each with zero placeholders, as $x^3 + 0x^2 + 0x - 8$, then do long division. Prove the answer by multiplying quotient by divisor and adding the remainder. Do four problems in ten minutes and keep a tally of caught missing terms.

4.4 Synthetic Division and the Remainder Theorem

OBJECTIVES

- Divide a polynomial by $x - c$ using synthetic division, and explain why it gives the same result as long division.
- Prove the Remainder Theorem from the division statement $f(x) = (x - c)q(x) + r$.
- Evaluate a polynomial quickly by synthetic substitution.

WATCH

Synthetic Division

Eddie Woo · 6:59

https://www.youtube.com/watch?v=vyZ6EmIOPGA

> **WHEN TO WATCH**
> Before Example 1: synthetic division.

BACKUP VIDEO

Remainder & Factor Theorem (1 of 2: How do we calculate remainders without dividing?)

Eddie Woo · 8:39

https://www.youtube.com/watch?v=xBD2sbfmtB8

After Example 1: the remainder theorem, finding a remainder without dividing.

KEY TERMS

synthetic division

A shortcut for dividing a polynomial by $x - c$ that works only with the coefficients: bring down, multiply by c, add, and repeat.

synthetic substitution

Using synthetic division to find the value $f(c)$; the last number in the bottom row is $f(c)$.

Remainder Theorem

When a polynomial $f(x)$ is divided by $x - c$, the remainder is $f(c)$.

CORE IDEAS

Look back at the long division of $x^3 + 2x^2 - 5x - 6$ by $x - 2$ in Lesson 4.3. Every line had the same powers of x in the same columns, and the leading term of each product just copied the term above it. All the real information is in the coefficients. Synthetic division keeps only those numbers.

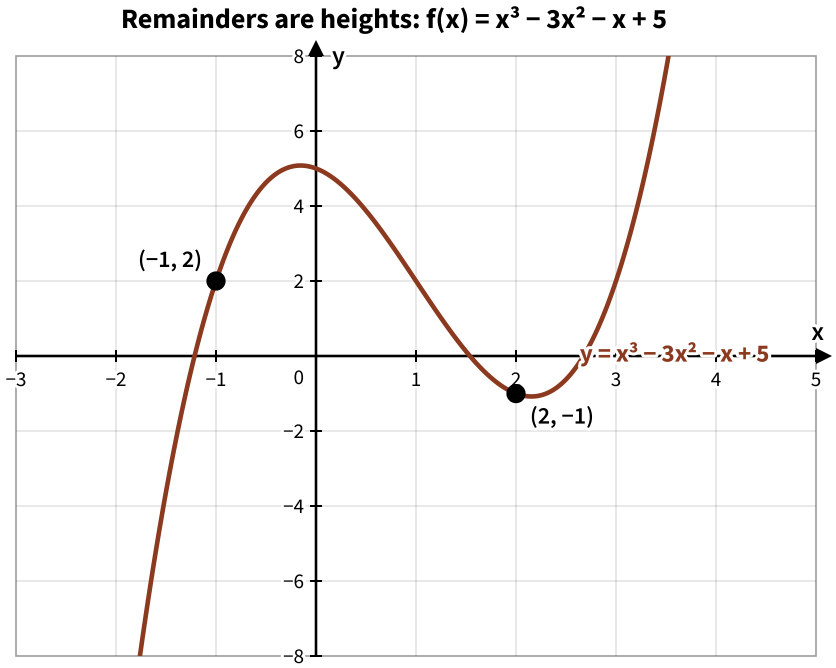
How to do it. For a divisor $x - c$, write c on the left and the dividend's coefficients (with zeros for missing powers) in a row. Bring down the first coefficient. Then repeat: multiply the number you just wrote by c, write the product under the next coefficient, and add. For $(x^3 + 2x^2 - 5x - 6) \div (x - 2)$, use $c = 2$: bring down 1; $1 \times 2 = 2$, $2 + 2 = 4$; $4 \times 2 = 8$, $-5 + 8 = 3$; $3 \times 2 = 6$, $-6 + 6 = 0$. The bottom row 1, 4, 3 gives the quotient $x^2 + 4x + 3$ (one degree less than the dividend), and the last number, 0, is the remainder. This matches Lesson 4.3.

Synthetic division: $(x^3 + 2x^2 - 5x - 6) \div (x - 2)$

c = 2	x^3	x^2	x	constant
coefficients	1	2	−5	−6
multiply by 2		2	8	6
add	1	4	3	0 = remainder

Why it is the same as long division. In long division by $x - c$, each cycle multiplies the divisor by a term ax^k and subtracts, which subtracts $-c \cdot a$ from the next column. Subtracting $-c \cdot a$ is the same as adding $c \cdot a$. Synthetic division simply writes c instead of $-c$ so that every subtraction becomes an addition. That is why you use $c = 2$ for $x - 2$ and $c = -3$ for $x + 3$.

The Remainder Theorem. Dividing $f(x)$ by $x - c$ gives $f(x) = (x - c)q(x) + r$, where r is a number because its degree must be less than 1. This is true for every x, so substitute $x = c$: $f(c) = (c - c)q(c) + r = 0 \cdot q(c) + r = r$. So the remainder is $f(c)$. On a graph, the remainder from dividing by $x - c$ is the height of the graph at $x = c$: for $f(x) = x^3 - 3x^2 - x + 5$, dividing by $x - 2$ leaves remainder $f(2) = -1$ and dividing by $x + 1$ leaves remainder $f(-1) = 2$.



Synthetic substitution. To find $f(c)$, run synthetic division and read the last number. It is often faster and safer than substituting, because it uses only multiplications by c and additions; there are no large powers to compute. It is the same as evaluating the nested form: $x^3 + 2x^2 - 5x - 6 = ((x + 2)x - 5)x - 6$.

Divisors like $2x - 1$. Synthetic division needs a divisor of the form $x - c$. Since $2x - 1 = 2\left(x - \frac{1}{2}\right)$, divide by $x - \frac{1}{2}$ and then divide the quotient (not the remainder) by 2.

KEY FORMULAS

$$f(x) = (x - c)\,q(x) + r \quad \Longrightarrow \quad r = f(c)$$

Divisor $x - c$: use c. Divisor $x + c$: use $-c$.

From history. The Italian mathematician and physician Paolo Ruffini published this method in 1809, and the English schoolmaster William Horner published an equivalent method in 1819. It is still called Ruffini's rule or Horner's method, and computers use the same nested form to evaluate polynomials quickly.

Practice habit. Before you divide, write the full row of coefficients with a 0 for every missing power, and write c with the correct sign: for $x + 3$, write -3.

WORKED EXAMPLES

Example 1. Divide $(3x^3 - 4x^2 - 5x + 8) \div (x - 2)$ using synthetic division. Check the remainder with the Remainder Theorem.

Show solution

1. Divisor $x - 2$, so $c = 2$. Coefficients: 3, -4, -5, 8.
2. Bring down 3.
3. $3 \times 2 = 6$; $-4 + 6 = 2$.
4. $2 \times 2 = 4$; $-5 + 4 = -1$.
5. $-1 \times 2 = -2$; $8 + (-2) = 6$.
6. Bottom row 3, 2, -1 | 6: quotient $3x^2 + 2x - 1$, remainder 6.
7. Check: $f(2) = 3(8) - 4(4) - 5(2) + 8 = 24 - 16 - 10 + 8 = 6$ ✓

Answer: Quotient $3x^2 + 2x - 1$, remainder 6

Example 2. Divide $(2x^4 - 9x^2 + x + 5) \div (x + 2)$ using synthetic division, and write the division statement.

Show solution

1. Divisor $x + 2 = x - (-2)$, so $c = -2$.
2. There is no x^3 term, so the coefficients are 2, 0, -9, 1, 5.
3. Bring down 2. $2 \times (-2) = -4$; $0 + (-4) = -4$.
4. $-4 \times (-2) = 8$; $-9 + 8 = -1$.
5. $-1 \times (-2) = 2$; $1 + 2 = 3$.
6. $3 \times (-2) = -6$; $5 + (-6) = -1$.
7. Quotient $2x^3 - 4x^2 - x + 3$, remainder -1.
8. Check: $f(-2) = 2(16) - 9(4) + (-2) + 5 = 32 - 36 - 2 + 5 = -1$ ✓
9. Division statement: $2x^4 - 9x^2 + x + 5 = (x + 2)(2x^3 - 4x^2 - x + 3) - 1$.

Answer: $2x^4 - 9x^2 + x + 5 = (x + 2)(2x^3 - 4x^2 - x + 3) - 1$

PRACTICE

1. **Easy** Use synthetic division: $(x^2 + 5x + 7) \div (x + 2)$.

   Show answer

   $c = -2$: bottom row $1,\ 3 \mid 1$. Quotient $x + 3$, remainder 1

2. **Easy** Use synthetic division: $(x^3 - 6x^2 + 11x - 6) \div (x - 1)$.

   Show answer

   Bottom row $1,\ -5,\ 6 \mid 0$. Quotient $x^2 - 5x + 6$, remainder 0

3. **Easy** Use synthetic division: $(2x^3 + x^2 - 8x + 5) \div (x - 3)$.

   Show answer

   Bottom row $2,\ 7,\ 13 \mid 44$. Quotient $2x^2 + 7x + 13$, remainder 44

4. **Easy** Use the Remainder Theorem to find the remainder when $f(x) = x^3 - 4x^2 + 2x - 1$ is divided by $x - 3$.

   Show answer

   $f(3) = 27 - 36 + 6 - 1 = -4$

5. **Medium** Use synthetic division: $(x^4 - 16) \div (x - 2)$.

   Show answer

   Coefficients $1,\ 0,\ 0,\ 0,\ -16$; bottom row $1,\ 2,\ 4,\ 8 \mid 0$. Quotient $x^3 + 2x^2 + 4x + 8$, remainder 0

6. **Medium** Use synthetic division: $(3x^4 + 2x^3 - x + 6) \div (x + 1)$.

   Show answer

   Coefficients $3,\ 2,\ 0,\ -1,\ 6$ with $c = -1$; bottom row $3,\ -1,\ 1,\ -2 \mid 8$. Quotient $3x^3 - x^2 + x - 2$, remainder 8

7. **Medium** Use synthetic substitution to find $f(4)$ for $f(x) = 2x^4 - 3x^3 + x^2 - 5x + 2$.

   Show answer

   Bottom row $2,\ 5,\ 21,\ 79 \mid 318$, so $f(4) = 318$

8. **Medium** Use synthetic substitution to find $f(-3)$ for $f(x) = x^5 + 2x^4 - 3x^2 + 7$.

Show answer

Coefficients 1, 2, 0, -3, 0, 7; bottom row 1, -1, 3, -12, 36 | -101, so $f(-3) = -101$

9. **Medium** A punter kicks a football. Its height in feet after t seconds is $h(t) = -16t^2 + 64t + 5$. Use synthetic substitution to find the height after 1.5 seconds.

Show answer

Coefficients -16, 64, 5 with $c = 1.5$: bottom row -16, 40 | 65. The ball is 65 feet high.

10. **Hard** A garden shed has volume $V(x) = x^3 + 6x^2 + 8x$ cubic feet, where x is its height in feet. Use synthetic substitution to find the volume when the height is 7.5 feet.

Show answer

Coefficients 1, 6, 8, 0 with $c = 7.5$: bottom row 1, 13.5, 109.25 | 819.375, so 819.375 cubic feet (check: $7.5 \times 9.5 \times 11.5 = 819.375$ ✓)

11. **Hard** Grace deposits \$300 at the start of each of three years in an account paying 5% interest compounded yearly. Her balance at the end of the third year is $B(x) = 300x^3 + 300x^2 + 300x$, where $x = 1.05$. Use synthetic substitution to find the balance to the nearest cent.

Show answer

Coefficients 300, 300, 300, 0 with $c = 1.05$: bottom row 300, 615, 945.75 | 993.0375, so about \$993.04

12. **Hard** When $x^3 + kx^2 - 4x + 3$ is divided by $x - 2$, the remainder is 7. Find k.

Show answer

By the Remainder Theorem, $f(2) = 8 + 4k - 8 + 3 = 4k + 3 = 7$, so $k = 1$

13. Hard Prove the Remainder Theorem: if a polynomial $f(x)$ is divided by $x - c$, the remainder is $f(c)$.

Show answer

Long division gives $f(x) = (x - c)q(x) + r$, where the remainder r has degree less than 1, so it is a constant. The equation holds for every x, so let $x = c$: $f(c) = (c - c)q(c) + r = 0 + r = r$. Therefore the remainder equals $f(c)$.

14. Challenge Divide $(4x^3 - 2x^2 + 6x - 1) \div (2x - 1)$ by first using synthetic division with $c = \frac{1}{2}$.

Show answer

Synthetic division with $\frac{1}{2}$: bottom row 4, 0, 6 | 2, so $f(x) = \left(x - \frac{1}{2}\right)(4x^2 + 6) + 2 = (2x - 1)(2x^2 + 3) + 2$. Quotient $2x^2 + 3$, remainder 2

15. Challenge Without dividing, find the remainder when $f(x) = x^{50} - 3x^{25} + 2$ is divided by $x + 1$.

Show answer

$f(-1) = (-1)^{50} - 3(-1)^{25} + 2 = 1 + 3 + 2 = 6$

COMMON MISTAKES

- Using the wrong sign for c. For $x + 4$, use $c = -4$; for $x - 4$, use $c = 4$.
- Forgetting zeros for missing powers. $x^4 - 16$ has coefficients 1, 0, 0, 0, -16.
- Subtracting in the columns instead of adding. Synthetic division always adds; the sign change is already built into c.
- Misreading the bottom row. The last number is the remainder, and the quotient has degree one less than the dividend.

> **Make it a game · Two Roads, One Remainder.** Race the parent on P(x) = x³ - 4x² + x + 6. You use synthetic division by x - 2. The parent evaluates P(2) directly. Whoever finishes first announces the remainder; the answers must match. Try three more values of c. Then explain why the Remainder Theorem follows from f(x) = (x - c)q(x) + r.

4.5 The Factor Theorem and Factoring Polynomials

OBJECTIVES

- State and prove the Factor Theorem from the Remainder Theorem.
- Use a known zero and synthetic division to factor a polynomial completely.
- Factor sums and differences of cubes, four-term polynomials by grouping, and polynomials in quadratic form.
- Solve polynomial equations by factoring.

WATCH

Remainder & Factor Theorem (2 of 2: How can we identify factors without dividing?)

Eddie Woo · 13:06

https://www.youtube.com/watch?v=jcKeCq24-ps

> **WHEN TO WATCH**
> Before Example 1: the factor theorem.

Note: Eddie Woo teaches in Australia: he says 'factorise' for factor.

BACKUP VIDEO

Factoring Sum and Difference of Cubes

Mario's Math Tutoring · 5:35

https://www.youtube.com/watch?v=hixEq9KKcFE

Before the practice set: factoring sums and differences of cubes.

KEY TERMS

Factor Theorem

A polynomial $f(x)$ has $x - c$ as a factor exactly when $f(c) = 0$.

zero of a polynomial

A number c with $f(c) = 0$. Real zeros are the x-intercepts of the graph and the solutions of $f(x) = 0$.

factor completely

Write a polynomial as a product of factors that cannot be factored further over the integers.

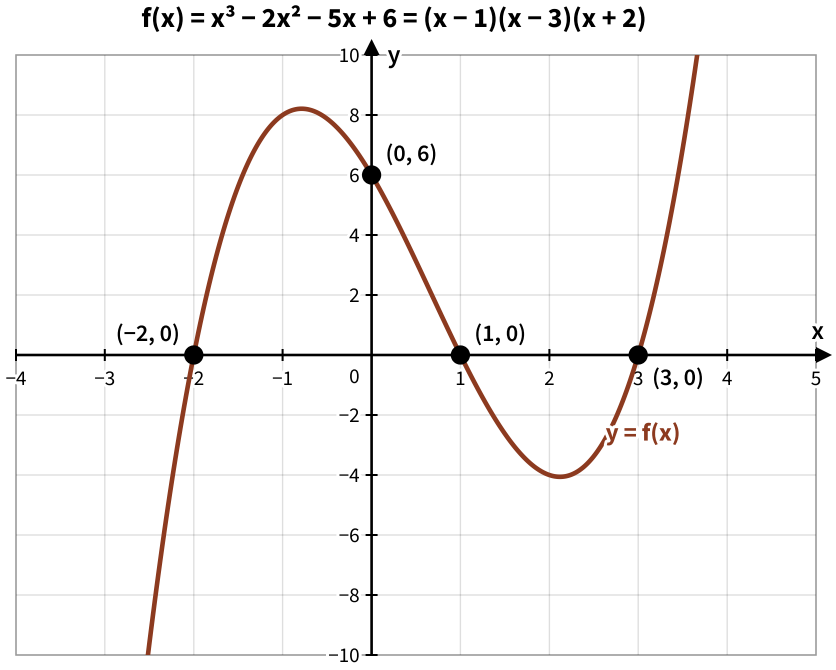
quadratic form

A polynomial that looks like $au^2 + bu + c$ when some expression such as $u = x^2$ is used as the variable, for example $x^4 - 13x^2 + 36$.

CORE IDEAS

The Factor Theorem. By the Remainder Theorem, dividing $f(x)$ by $x - c$ gives $f(x) = (x - c)q(x) + f(c)$. If $f(c) = 0$, this says $f(x) = (x - c)q(x)$, so $x - c$ is a factor. In the other direction, if $f(x) = (x - c)q(x)$, then $f(c) = 0 \cdot q(c) = 0$. So $x - c$ **is a factor of** $f(x)$ **exactly when** $f(c) = 0$. Zeros and factors are two views of the same fact.

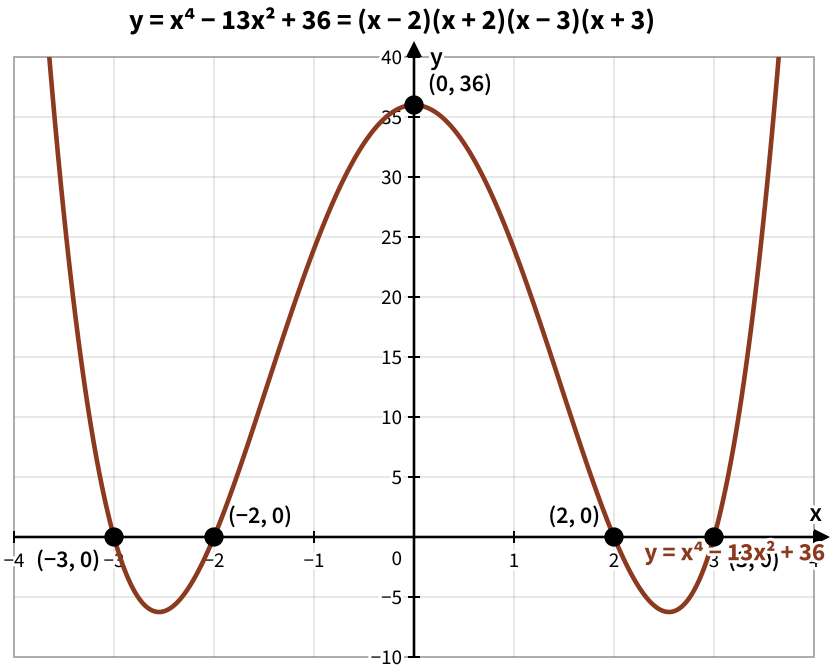
One zero opens the door. Take $f(x) = x^3 - 2x^2 - 5x + 6$. Since $f(1) = 1 - 2 - 5 + 6 = 0$, $x - 1$ is a factor. Synthetic division by $x - 1$ gives the bottom row $1, -1, -6 \mid 0$, so $f(x) = (x - 1)(x^2 - x - 6)$. The quadratic factors as before: $f(x) = (x - 1)(x - 3)(x + 2)$, with zeros -2, 1 and 3, the x-intercepts on the graph. This also explains Lesson 4.1: each zero c gives a factor $x - c$, and a polynomial of degree n cannot have more than n such factors, so it has at most n zeros.



Sums and differences of cubes. In Lesson 4.2 you multiplied to verify $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. The Factor Theorem predicts the linear factor: $x^3 - 8$ is 0 at $x = 2$, so $x - 2$ is a factor. Memory aid for the signs: the binomial has the **same** sign as the original, the middle term of the trinomial has the **opposite** sign, and the last term is **always positive**. The trinomial factor does not factor further over the real numbers: as a quadratic in a, $a^2 \pm ab + b^2$ has discriminant $b^2 - 4b^2 = -3b^2$, which is negative whenever $b \neq 0$. Example: $27x^3 - 64 = (3x)^3 - 4^3 = (3x - 4)(9x^2 + 12x + 16)$.

Grouping. A four-term polynomial may factor in pairs: $x^3 + 3x^2 - 4x - 12 = x^2(x + 3) - 4(x + 3) = (x + 3)(x^2 - 4) = (x + 3)(x - 2)(x + 2)$. Always look at what is left; here $x^2 - 4$ is a difference of squares.

Quadratic form. In $x^4 - 13x^2 + 36$ the exponents are 4, 2 and 0, so let $u = x^2$: $u^2 - 13u + 36 = (u - 4)(u - 9)$. Put x^2 back: $(x^2 - 4)(x^2 - 9) = (x - 2)(x + 2)(x - 3)(x + 3)$. The graph shows the four zeros ± 2 and ± 3.



Solving polynomial equations. Move everything to one side so the other side is 0, factor completely, and set each factor equal to 0 (the zero-product property). Never divide both sides by x or by any factor that contains x: that factor could be 0, and you would lose a solution. For $x^3 = 9x$, write $x^3 - 9x = 0$, so $x(x - 3)(x + 3) = 0$ and $x = -3, 0, 3$; dividing by x would have lost $x = 0$.

KEY FORMULAS

$$x - c \text{ is a factor of } f(x) \iff f(c) = 0$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Perfect cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000

Practice habit. Factor before anything else, in this order: take out the greatest common factor; then count terms (two terms: squares or cubes; three terms: a trinomial or quadratic form; four terms: grouping); then check whether any factor still factors. Know the perfect cubes to $10^3 = 1000$ cold.

WORKED EXAMPLES

Example 1. Show that $x + 2$ is a factor of $f(x) = x^3 + 4x^2 + x - 6$, and then factor $f(x)$ completely.

Show solution

1. $x + 2 = x - (-2)$, so evaluate $f(-2) = -8 + 16 - 2 - 6 = 0$. By the Factor Theorem, $x + 2$ is a factor.
2. Synthetic division with $c = -2$ on 1, 4, 1, -6: bring down 1; -2, $4 - 2 = 2$; -4, $1 - 4 = -3$; 6, $-6 + 6 = 0$.
3. So $f(x) = (x + 2)(x^2 + 2x - 3)$.
4. Factor the quadratic: $x^2 + 2x - 3 = (x + 3)(x - 1)$.
5. $f(x) = (x + 2)(x + 3)(x - 1)$.
6. Check by multiplying: $(x + 2)(x + 3) = x^2 + 5x + 6$, and $(x^2 + 5x + 6)(x - 1) = x^3 + 4x^2 + x - 6$ ✓

Answer: $f(x) = (x + 2)(x + 3)(x - 1)$

Example 2. Henry is making a wooden keepsake box. Its width is 2 inches more than its height, and its length is 5 inches more than its height. The box must hold 120 cubic inches. Find its dimensions, and show that no other height works.

Show solution

1. Let x be the height. Then $x(x+2)(x+5) = 120$.
2. Expand and set to 0: $x^3 + 7x^2 + 10x - 120 = 0$.
3. Try small whole numbers: $x = 1$ gives $1 \times 3 \times 6 = 18$; $x = 2$ gives $2 \times 4 \times 7 = 56$; $x = 3$ gives $3 \times 5 \times 8 = 120$ ✓
4. So $x - 3$ is a factor. Synthetic division with $c = 3$ on 1, 7, 10, -120: bottom row 1, 10, 40 | 0.
5. The equation is $(x-3)(x^2 + 10x + 40) = 0$.
6. For $x^2 + 10x + 40 = 0$ the discriminant is $100 - 160 = -60 < 0$, so there are no other real solutions.
7. Height 3 in, width 5 in, length 8 in. Check: $3 \times 5 \times 8 = 120$ ✓

Answer: 3 inches high, 5 inches wide and 8 inches long; $x = 3$ is the only real solution.

PRACTICE

1. Easy Use the Factor Theorem to decide whether $x - 2$ is a factor of $f(x) = x^3 - 3x^2 + 4$.

   Show answer

   $f(2) = 8 - 12 + 4 = 0$, so yes

2. Easy Use the Factor Theorem to decide whether $x + 1$ is a factor of $f(x) = 2x^3 + x^2 - 5x + 3$.

   Show answer

   $f(-1) = -2 + 1 + 5 + 3 = 7 \neq 0$, so no

3. Easy Factor: $x^3 - 27$.

   Show answer

   $(x-3)(x^2 + 3x + 9)$

4. Easy Factor: $8x^3 + 125$.

Show answer

$(2x)^3 + 5^3 = (2x + 5)(4x^2 - 10x + 25)$

5. Medium Factor by grouping: $x^3 - 5x^2 - 4x + 20$.

Show answer

$x^2(x - 5) - 4(x - 5) = (x - 5)(x^2 - 4) = (x - 5)(x - 2)(x + 2)$

6. Medium Given that 2 is a zero of $f(x) = x^3 - 7x + 6$, factor $f(x)$ completely.

Show answer

Synthetic division on $1, 0, -7, 6$ with $c = 2$: bottom row $1, 2, -3 \mid 0$. $f(x) = (x - 2)(x^2 + 2x - 3) = (x - 2)(x - 1)(x + 3)$

7. Medium Factor completely: $x^4 - 5x^2 + 4$.

Show answer

Let $u = x^2$: $u^2 - 5u + 4 = (u - 1)(u - 4)$, so $(x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$

8. Medium Solve: $2x^3 + 3x^2 - 8x - 12 = 0$.

Show answer

$x^2(2x + 3) - 4(2x + 3) = (2x + 3)(x - 2)(x + 2) = 0$, so $x = -2, -\dfrac{3}{2}, 2$

9. Medium Factor completely over the integers: $x^4 + 3x^2 - 4$.

Show answer

$(x^2 + 4)(x^2 - 1) = (x^2 + 4)(x - 1)(x + 1)$; the factor $x^2 + 4$ does not factor over the integers

10. Hard Factor completely: $2x^4 - 16x$.

Show answer

GCF first: $2x(x^3 - 8) = 2x(x - 2)(x^2 + 2x + 4)$

11. Hard The height, width and length of a raised garden bed are three consecutive whole numbers of feet, with the height the smallest. The bed holds 60 cubic feet of soil. Write and solve a polynomial equation to find the dimensions.

Show answer

$x(x+1)(x+2) = 60$, so $x^3 + 3x^2 + 2x - 60 = 0$. $x = 3$ works; synthetic division gives $(x-3)(x^2+6x+20) = 0$, and $x^2 + 6x + 20$ has discriminant $36 - 80 < 0$. The bed is 3 ft high, 4 ft wide and 5 ft long.

12. Hard Two cube-shaped feed bins have edges x feet and $x + 2$ feet. The larger bin holds 152 cubic feet more than the smaller. Find both edge lengths.

Show answer

$(x+2)^3 - x^3 = 152$; by the difference of cubes, $(x+2)^3 - x^3 = 2[(x+2)^2 + (x+2)x + x^2] = 6x^2 + 12x + 8$. So $6x^2 + 12x - 144 = 0$, $x^2 + 2x - 24 = 0$, $(x+6)(x-4) = 0$, and $x = 4$ (length is positive). Edges 4 ft and 6 ft; check $216 - 64 = 152$ ✓

13. Hard An open box is made from a 16-inch by 10-inch sheet by cutting squares of side x inches from the corners, so $V(x) = 4x^3 - 52x^2 + 160x$ for $0 < x < 5$. Cutting 3-inch squares gives a volume of 120 cubic inches. Use this fact to find the other cut size that gives 120 cubic inches, exactly and to the nearest hundredth.

Show answer

$4x^3 - 52x^2 + 160x = 120$ gives $x^3 - 13x^2 + 40x - 30 = 0$. Since $x = 3$ is a solution, synthetic division gives $(x-3)(x^2 - 10x + 10) = 0$, so $x = 5 \pm \sqrt{15}$. Only $5 - \sqrt{15} \approx 1.13$ is between 0 and 5. The cuts are $x = 5 - \sqrt{15} \approx 1.13$ in or $x = 3$ in.

14. Hard Prove the Factor Theorem: $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

Show answer

By the Remainder Theorem, $f(x) = (x-c)q(x) + f(c)$. If $f(c) = 0$, then $f(x) = (x-c)q(x)$, so $x - c$ is a factor. Conversely, if $f(x) = (x-c)q(x)$ for some polynomial q, then $f(c) = (c-c)q(c) = 0$.

15. **Challenge** Factor completely over the integers: $x^6 - 64$. (Hint: start with a difference of squares.)

Show answer

$$(x^3 - 8)(x^3 + 8) = (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4)$$

COMMON MISTAKES

- Getting the signs of the cubes formulas wrong. Same sign in the binomial, opposite sign for the middle term of the trinomial, last term always positive: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.
- Trying to factor $a^2 + ab + b^2$ or $a^2 - ab + b^2$ further. Its discriminant is negative, so it stays as it is.
- Dividing both sides of an equation by x. Factor out x instead, so the solution $x = 0$ is not lost.
- Stopping too soon. After grouping or quadratic form, check every factor again, for example $x^2 - 4 = (x - 2)(x + 2)$.

> **Make it a game · Guess My Factor.** The parent writes $x^3 - 6x^2 + 11x - 6$ and secretly knows its three zeros. You may test candidates by synthetic division, and each test costs a turn. Find a zero, reduce the polynomial to a quadratic, then factor completely. Fewest tests wins. Explain how the Factor Theorem lets you turn a zero into a factor.

4.6 Finding Zeros of Polynomials

OBJECTIVES

- Explain why the Rational Root Theorem is true, and use it to list the possible rational zeros of a polynomial with integer coefficients.
- Find all zeros of a polynomial, real and non-real, by testing candidates with synthetic division and finishing with the quadratic formula.
- State the Fundamental Theorem of Algebra and explain why non-real zeros of real polynomials come in conjugate pairs.
- Write a polynomial with given zeros.

WATCH

Rational Root Theorem (Rational Zero Theorem p/q)

Mario's Math Tutoring · 9:20

https://www.youtube.com/watch?v=ZtaGPHBf7Y4

> **WHEN TO WATCH**
> Before Example 1: listing possible rational zeros with the p/q test.

BACKUP VIDEO

Find a Polynomial with Real Coefficients that has the Given Zeros

Mario's Math Tutoring · 3:06

https://www.youtube.com/watch?v=4P6MfqEHB90

Before Example 2: building a polynomial from its zeros, including complex conjugate pairs.

KEY TERMS

Rational Root Theorem

If a polynomial with integer coefficients has a rational zero $\frac{p}{q}$ in lowest terms, then p divides the constant term and q divides the leading coefficient.

candidate

A number on the list of possible rational zeros $\pm\frac{p}{q}$. Each one must be tested; most will not be zeros.

multiplicity

The number of times a factor $x - c$ appears in the factored form. In $(x-2)^3(x+1)$, the zero 2 has multiplicity 3.

Fundamental Theorem of Algebra

Every polynomial of degree $n \geq 1$ has exactly n complex zeros, counting each zero as many times as its multiplicity.

conjugate pair

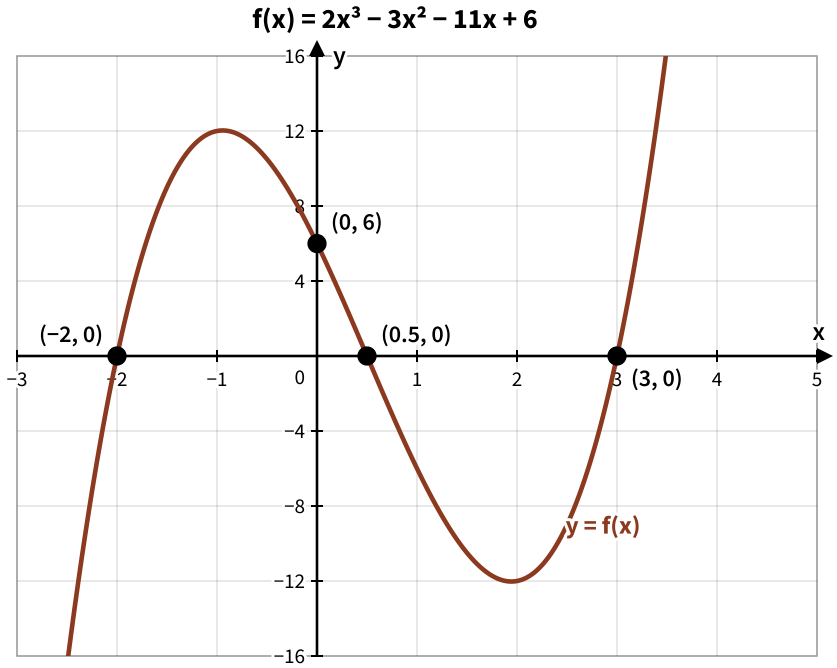
Two numbers $a + bi$ and $a - bi$ (complex conjugates), or $a + \sqrt{b}$ and $a - \sqrt{b}$ (radical conjugates).

CORE IDEAS

The Factor Theorem lets you factor once you know one zero, but where do you start? Guessing at random could take forever. The **Rational Root Theorem** gives a short list of the only rational numbers that could possibly be zeros.

Why it is true. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ have integer coefficients, and suppose $\frac{p}{q}$ is a zero in lowest terms (so p and q share no factor). Then $a_n \frac{p^n}{q^n} + \cdots + a_1 \frac{p}{q} + a_0 = 0$. Multiply by q^n: $a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$. Every term except the last contains p, so $a_0 q^n = -p(a_n p^{n-1} + \cdots + a_1 q^{n-1})$ is a multiple of p. Since p shares no factor with q, p must divide a_0. In the same way, every term except the first contains q, so $a_n p^n$ is a multiple of q, and q must divide a_n.

Using it. For $f(x) = 2x^3 - 3x^2 - 11x + 6$: p divides 6, so p is 1, 2, 3 or 6; q divides 2, so q is 1 or 2. The candidates are $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$. A quick graph shows which are worth testing first. Test 3 by synthetic division: the bottom row is 2, 3, $-2 \mid 0$, so $f(x) = (x-3)(2x^2 + 3x - 2) = (x-3)(2x-1)(x+2)$, and the zeros are -2, $\frac{1}{2}$ and 3.



The Fundamental Theorem of Algebra. Every polynomial of degree $n \geq 1$ has exactly n complex zeros, counting multiplicity. (The proof is beyond this course.) So a polynomial of degree n can always be written as $a(x - r_1)(x - r_2) \cdots (x - r_n)$, where some of the r's may be equal and some may be non-real. Each time you find a zero, synthetic division lowers the degree by one; once you reach a quadratic, the quadratic formula finds the last two zeros.

Conjugate pairs. If a polynomial has real coefficients and $a + bi$ is a zero, then $a - bi$ is also a zero. Why: taking the conjugate of a sum or a product gives the sum or product of the conjugates, and a real coefficient is its own conjugate. So if z is a zero, the conjugate of $f(z)$ is $f(\overline{z})$, and the conjugate of 0 is 0, so $f(\overline{z}) = 0$. You saw the same thing in the quadratic formula, where the $\pm$ produces both. Similarly, if a polynomial has rational coefficients and $a + \sqrt{b}$ is a zero, where a and b are rational and $\sqrt{b}$ is irrational, then $a - \sqrt{b}$ is a zero too.

Writing a polynomial from its zeros. Make a factor $x - r$ for each zero, include each conjugate partner, and multiply. For zeros 2 and $1 + i$ with real coefficients, $1 - i$ must also be a zero. Pair the conjugates first: $(x - 1 - i)(x - 1 + i) = (x - 1)^2 - i^2 = x^2 - 2x + 2$. Then $(x - 2)(x^2 - 2x + 2) = x^3 - 4x^2 + 6x - 4$.

Strategy for finding all zeros. (1) List the candidates $\pm\frac{p}{q}$. (2) Use a graph to choose likely ones. (3) Test with synthetic division; a remainder of 0 means you have found a

zero and a smaller quotient. (4) Repeat on the quotient until it is quadratic, then use factoring or the quadratic formula.

KEY FORMULAS

$$\text{Possible rational zeros} = \pm\frac{\text{factors of the constant term}}{\text{factors of the leading coefficient}}$$

$$f(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$$

$$(x - (a + bi))(x - (a - bi)) = (x - a)^2 + b^2$$

From history. Scipione del Ferro, Niccolò Tartaglia and Gerolamo Cardano found a formula for solving cubic equations, which Cardano published in his Ars Magna in 1545. Carl Friedrich Gauss gave a proof of the Fundamental Theorem of Algebra in his doctoral dissertation in 1799. In 1824 Niels Henrik Abel proved that no formula of that kind exists for every equation of degree five.

Practice habit. Sketch before you compute: a quick graph of the polynomial (by hand or with a graphing tool) shows roughly where the real zeros are, so you test the right candidates first instead of all of them.

WORKED EXAMPLES

Example 1. Find all zeros of $f(x) = x^3 + x^2 - 7x - 3$. Give exact answers.

Show solution

1. Leading coefficient 1, constant -3: the candidates are ± 1 and ± 3.
2. Test: $f(1) = 1 + 1 - 7 - 3 = -8$; $f(-1) = -1 + 1 + 7 - 3 = 4$; $f(3) = 27 + 9 - 21 - 3 = 12$; $f(-3) = -27 + 9 + 21 - 3 = 0$ ✓
3. Synthetic division with $c = -3$ on $1, 1, -7, -3$: bring down 1; $-3,\ 1 - 3 = -2$; $6,\ -7 + 6 = -1$; $3,\ -3 + 3 = 0$.
4. So $f(x) = (x + 3)(x^2 - 2x - 1)$.
5. Quadratic formula: $x = \dfrac{2 \pm \sqrt{4 + 4}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$.
6. The irrational zeros come as a pair $1 \pm \sqrt{2}$, as expected. Check: $(1 + \sqrt{2})^2 - 2(1 + \sqrt{2}) - 1 = 3 + 2\sqrt{2} - 2 - 2\sqrt{2} - 1 = 0$ ✓

Answer: $x = -3$, $x = 1 - \sqrt{2}$, $x = 1 + \sqrt{2}$

Example 2. Ruth is building a cedar chest. Its length is 1 foot more than twice its height, and its width is 1 foot less than twice its height. It must hold 12 cubic feet. Find the height.

Show solution

1. Let x be the height. Then $x(2x+1)(2x-1) = 12$, so $x(4x^2-1) = 12$ and $4x^3 - x - 12 = 0$.
2. Candidates: p divides 12 and q divides 4. A height is positive, so test positive candidates: $1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}$.
3. $f(1) = 4 - 1 - 12 = -9$ and $f(2) = 32 - 2 - 12 = 18$, so a zero lies between 1 and 2. Try $\frac{3}{2}$.
4. Synthetic division with $c = \frac{3}{2}$ on 4, 0, -1, -12: bring down 4; 6, $0 + 6 = 6$; 9, $-1 + 9 = 8$; 12, $-12 + 12 = 0$ ✓
5. So $(x - \frac{3}{2})(4x^2 + 6x + 8) = 0$. The quadratic has discriminant $36 - 128 = -92 < 0$, so its zeros are not real.
6. Height $\frac{3}{2}$ ft = 1.5 ft; length $2(1.5) + 1 = 4$ ft; width $2(1.5) - 1 = 2$ ft. Check: $1.5 \times 4 \times 2 = 12$ ✓

Answer: The chest is 1.5 feet high (4 feet long and 2 feet wide).

PRACTICE

1. Easy List all possible rational zeros of $f(x) = x^3 - 4x^2 + x + 6$.

   Show answer

   $\pm 1, \pm 2, \pm 3, \pm 6$

2. Easy List all possible rational zeros of $f(x) = 2x^3 + x^2 - 7x - 6$.

   Show answer

   $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

3. Easy How many complex zeros, counting multiplicity, does $f(x) = x^5 - 3x^2 + 1$ have?

   Show answer

   5, by the Fundamental Theorem of Algebra

4. **Easy** A polynomial with real coefficients has $3 - 2i$ as a zero. Name another zero.

   Show answer

   $3 + 2i$

5. **Medium** Find all zeros of $f(x) = x^3 - 4x^2 + x + 6$.

   Show answer

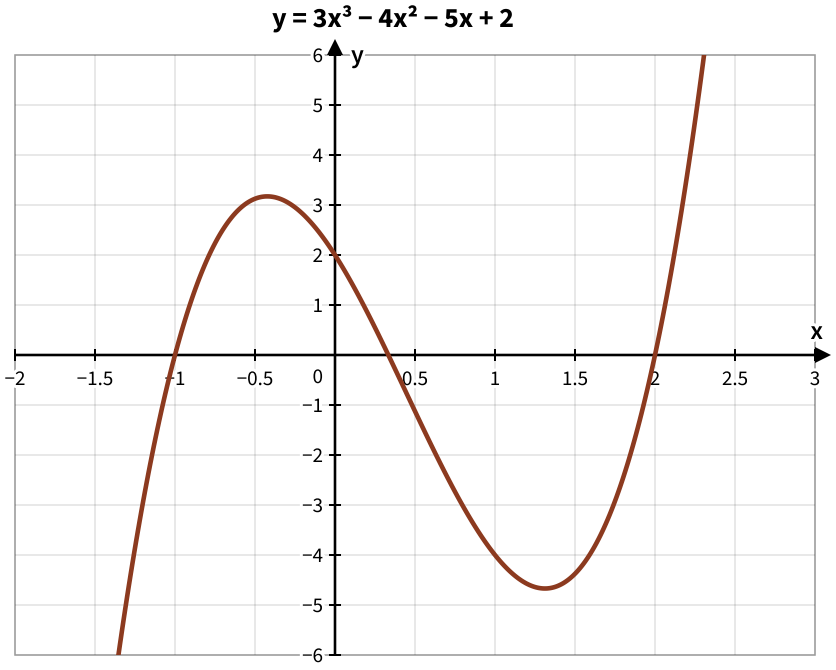
   $f(-1) = 0$; synthetic division gives $(x+1)(x^2 - 5x + 6) = (x+1)(x-2)(x-3)$; zeros $-1, 2, 3$

6. **Medium** Find all zeros of $f(x) = 2x^3 + x^2 - 7x - 6$.

   Show answer

   $f(2) = 16 + 4 - 14 - 6 = 0$; synthetic division gives $(x-2)(2x^2 + 5x + 3) = (x-2)(2x+3)(x+1)$; zeros $-\dfrac{3}{2}, -1, 2$

7. **Medium** The graph of $f(x) = 3x^3 - 4x^2 - 5x + 2$ is shown. Use the graph and the Rational Root Theorem to find all zeros exactly.



Show answer

Candidates $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$. The graph suggests 2: synthetic division gives 3, 2, -1 | 0, so $f(x) = (x-2)(3x^2+2x-1) = (x-2)(3x-1)(x+1)$. Zeros $-1, \frac{1}{3}, 2$

8. **Medium** Find all zeros, real and non-real, of $f(x) = x^3 - 3x^2 + 4x - 12$.

Show answer

$f(3) = 27 - 27 + 12 - 12 = 0$; $f(x) = (x-3)(x^2+4)$; $x^2 = -4$ gives $x = \pm 2i$. Zeros $3, 2i, -2i$

9. **Medium** Write a polynomial in standard form with leading coefficient 1 and zeros 1, -2 and 4.

Show answer

$(x-1)(x+2)(x-4) = (x^2+x-2)(x-4) = x^3 - 3x^2 - 6x + 8$

10. **Hard** Write a polynomial of least degree in standard form, with real coefficients and leading coefficient 1, that has zeros 3 and $2 + i$.

    Show answer

    $2 - i$ is also a zero. $(x - 2 - i)(x - 2 + i) = (x - 2)^2 + 1 = x^2 - 4x + 5$, and $(x - 3)(x^2 - 4x + 5) = x^3 - 7x^2 + 17x - 15$

11. **Hard** A planter box has a square base, and its height is 2 feet less than the side of the base. It holds 32 cubic feet of soil. (a) Find the side of the base. (b) Find the height.

    Show answer

    $x^2(x - 2) = 32$, so $x^3 - 2x^2 - 32 = 0$. $x = 4$ works; synthetic division gives $(x - 4)(x^2 + 2x + 8) = 0$, and $x^2 + 2x + 8 = 0$ has the non-real zeros $-1 \pm i\sqrt{7}$. (a) 4 ft; (b) 2 ft

12. **Hard** A wooden feed trough is a box whose width is twice its height and whose length is 3 feet more than its width. It holds 27 cubic feet. Find its height.

    Show answer

    $x(2x)(2x + 3) = 27$, so $4x^3 + 6x^2 - 27 = 0$. Testing candidates, $\frac{3}{2}$ works: synthetic division gives 4, 12, 18 | 0, and $4x^2 + 12x + 18$ has discriminant $144 - 288 < 0$. Height 1.5 ft (width 3 ft, length 6 ft; check $1.5 \times 3 \times 6 = 27$ ✓)

13. **Hard** An open box is made from a 9-inch by 12-inch sheet by cutting squares of side x inches from the corners. For what values of x is the volume 44 cubic inches? Give exact answers and decimals to the nearest hundredth.

    Show answer

    $x(9 - 2x)(12 - 2x) = 44$ gives $2x^3 - 21x^2 + 54x - 22 = 0$. $x = \frac{1}{2}$ works (synthetic division: 2, -20, 44 | 0), leaving $x^2 - 10x + 22 = 0$, so $x = 5 \pm \sqrt{3}$. The domain is $0 < x < 4.5$, so $x = \frac{1}{2}$ in or $x = 5 - \sqrt{3} \approx 3.27$ in

14. Hard Explain why every polynomial of degree 3 with real coefficients has at least one real zero.

Show answer

By the Fundamental Theorem of Algebra it has exactly 3 complex zeros, counting multiplicity. Because the coefficients are real, non-real zeros come in conjugate pairs, so the number of non-real zeros is even: 0 or 2. That leaves at least $3 - 2 = 1$ real zero.

15. Challenge Use the Rational Root Theorem to prove that $\sqrt{2}$ is irrational. (Hint: $\sqrt{2}$ is a zero of $x^2 - 2$.)

Show answer

If $\sqrt{2}$ were rational, it would be a rational zero of $x^2 - 2$, so it would be one of the candidates $\pm 1, \pm 2$. But $1^2 - 2 = -1$ and $2^2 - 2 = 2$ (and the negatives give the same values), so none of them is a zero. Therefore $x^2 - 2$ has no rational zeros, and $\sqrt{2}$ is irrational.

COMMON MISTAKES

- Putting the factors upside down. The numerator p comes from the constant term and the denominator q from the leading coefficient.
- Forgetting the negative candidates or the fractions. List $\pm$ every $\frac{p}{q}$.
- Writing a polynomial from the zero $2 + i$ without its partner $2 - i$. With real coefficients the conjugate must be included, or the coefficients will not be real.
- Testing later candidates in the original polynomial after a zero is found. Keep working with the smaller quotient; it is easier and finds repeated zeros.

> **Make it a game · Possible Zeros Lineup.** For $2x^3 - 3x^2 - 11x + 6$, list every possible rational zero using p over q within 90 seconds. Write the factors of the constant and of the leading coefficient first. Count your candidates, then test them with synthetic division until you find all three zeros. Fewer tests means a higher score. Explain why the Rational Root Theorem is true.

4.7 Graphing Polynomial Functions

OBJECTIVES

- Sketch the graph of a polynomial from its zeros, their multiplicities, its end behavior and its y-intercept.
- Explain why a graph crosses the x-axis at a zero of odd multiplicity and touches and turns at a zero of even multiplicity.
- Solve polynomial inequalities with a sign chart and write the answer in interval notation.
- Use a graphing tool to find relative maxima and minima in models such as the volume of a box.

WATCH

Graphing Polynomial Functions Using End Behavior, Zeros, and Multiplicities

Mario's Math Tutoring · 12:08

https://www.youtube.com/watch?v=189INSXn_gk

> **WHEN TO WATCH**
> Before Example 1: sketching from end behavior, zeros and multiplicities.

BACKUP VIDEO

Graphing Cubic Functions (3 of 4: Factored form)

Eddie Woo · 9:55

https://www.youtube.com/watch?v=x4NkROT90-k

Before the practice set: sketching a cubic from its factored form.

KEY TERMS

multiplicity

The number of times a factor $x - c$ appears in the factored form of a polynomial.

crosses

The graph passes from one side of the x-axis to the other at a zero. This happens at zeros of odd multiplicity.

touches and turns

The graph meets the x-axis at a zero but stays on the same side, like a parabola at its vertex. This happens at zeros of even multiplicity.

sign chart

A number line marked with the zeros of a polynomial and the sign (+ or −) of the polynomial on each interval between them.

polynomial inequality

An inequality such as $f(x) > 0$ or $f(x) \le 0$ where f is a polynomial; its solution is the set of x where the graph is above (or below) the x-axis.

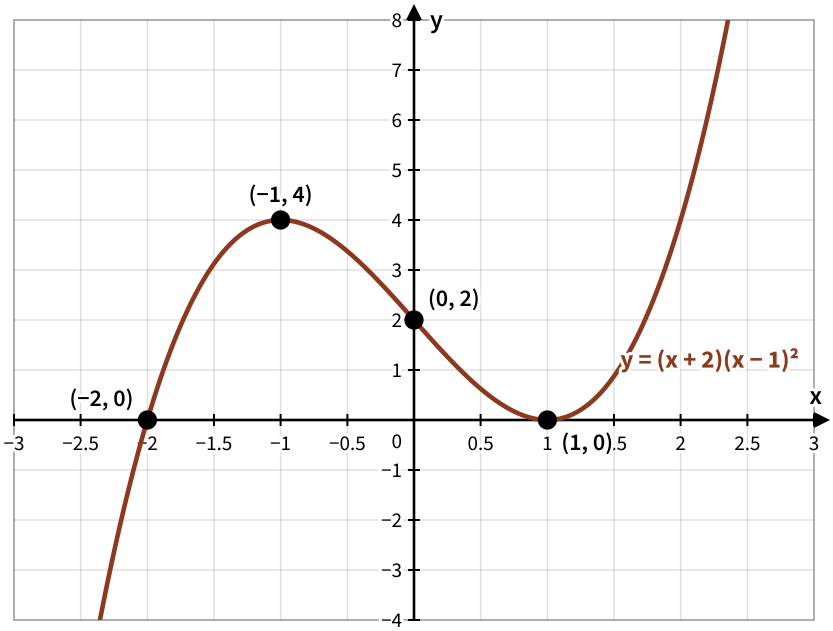
CORE IDEAS

You can sketch a good graph of a factored polynomial from four facts, without plotting many points: (1) the **zeros**, which are the x-intercepts; (2) the **multiplicity** of each zero, which tells whether the graph crosses or touches there; (3) the **end behavior**, from the leading term; and (4) the **y-intercept** $f(0)$.

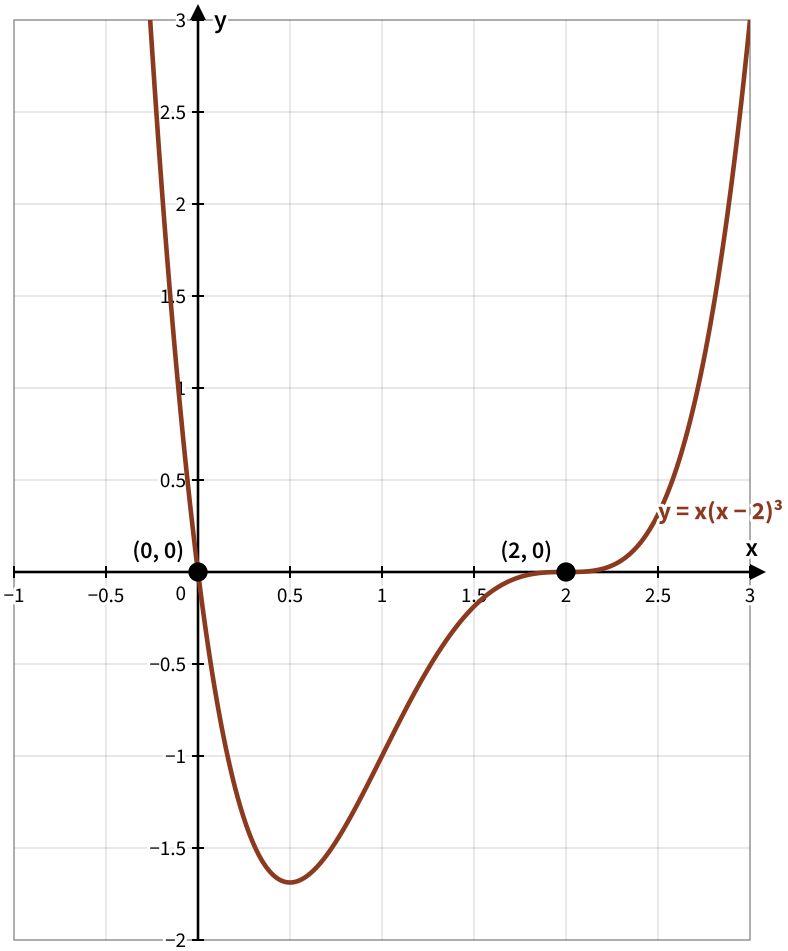
Why multiplicity decides cross or touch. Write $f(x) = (x - c)^m g(x)$, where $g(c) \neq 0$. Close to $x = c$, the factor $g(x)$ stays near $g(c)$, so it keeps one sign. The sign of f near c is therefore controlled by $(x - c)^m$. Just left of c, $x - c$ is a small negative number; just right of c, it is positive. A negative number to an odd power is negative, but to an even power it is positive. So $(x - c)^m$ changes sign at c only when m is odd: **odd multiplicity crosses, even multiplicity touches and turns.** Near the zero the graph looks like $y = (x - c)^m$: a straight crossing for $m = 1$, a parabola-like touch for $m = 2$, and a flattened, x^3-like crossing for $m = 3$.

Example. For $f(x) = (x + 2)(x - 1)^2$: the zeros are -2 (multiplicity 1, crosses) and 1 (multiplicity 2, touches). The leading term is $x \cdot x^2 = x^3$, so the graph falls on the left and rises on the right. The y-intercept is $f(0) = (2)(1) = 2$. Coming up from the lower left, the graph crosses at -2, rises to a turning point, comes down to touch the axis at 1, and rises again. A graphing tool shows the relative maximum at $(-1, 4)$; check: $f(-1) = (1)(4) = 4$.

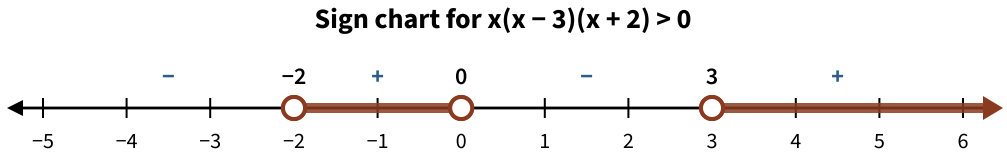
f(x) = (x + 2)(x − 1)²: crosses at −2, touches at 1



y = x(x − 2)³: crosses at 0, flattens and crosses at 2



Sign charts and inequalities. To solve $x^3 - x^2 - 6x > 0$, factor: $x(x-3)(x+2) > 0$. The zeros -2, 0 and 3 split the number line into four intervals, and the sign can change only at a zero. Test one number in each interval: $x = -3$ gives $(-3)(-6)(-1) = -18$, negative; $x = -1$ gives $(-1)(-4)(1) = 4$, positive; $x = 1$ gives $(1)(-2)(3) = -6$, negative; $x = 4$ gives $(4)(1)(6) = 24$, positive. The solution is where the sign is +: $(-2, 0) \cup (3, \infty)$. The zeros are not included because the inequality is strict; for $\geq$ they would be. (Shortcut: the sign flips at a zero of odd multiplicity and stays the same at a zero of even multiplicity.)



Turning points and models. Finding the exact relative maxima and minima of a cubic needs calculus, so in this course use a graphing tool for them, after you have done the algebra for the zeros, the domain and the overall shape. In a model, keep only the part of the graph that makes sense: the volume of a box from a sheet is a cubic, but only x-values that give positive dimensions count. Data whose third differences are constant (for equally spaced inputs) fit a cubic, just as constant second differences fit a quadratic.

KEY FORMULAS

Zero c of odd multiplicity: the graph crosses the x-axis. Zero of even multiplicity: it touches and turns.

$$f(x) = (x-c)^m g(x),\ g(c) \neq 0 :\ \text{the sign of } f \text{ changes at } c \text{ exactly when } m \text{ is odd}$$

Practice habit. Sketch before you compute: mark the zeros, write "cross" or "touch" above each, draw the two ends from the leading term, plot the y-intercept, and only then connect the pieces or reach for a graphing tool.

Example 1. Describe the graph of $f(x) = -(x-2)^2(x+1)$: give the zeros and whether the graph crosses or touches at each, the y-intercept and the end behavior.

Show solution

1. Zeros: $x = 2$ (multiplicity 2) and $x = -1$ (multiplicity 1).
2. At -1 (odd) the graph crosses; at 2 (even) it touches and turns.
3. y-intercept: $f(0) = -(0-2)^2(0+1) = -(4)(1) = -4$.
4. Leading term: $-(x^2)(x) = -x^3$. Odd degree, negative coefficient: rises on the left, falls on the right.
5. Consistency check with test points: $f(-2) = -(16)(-1) = 16 > 0$ (above, as the left end requires); $f(3) = -(1)(4) = -4 < 0$ (below on both sides of 2, so it touches from below) ✓
6. Shape: from the upper left, down through $(-1, 0)$, down to a low point at $(0, -4)$ (a graphing tool confirms this is the relative minimum), back up to touch at $(2, 0)$, then down to the lower right.

Answer: Crosses at $(-1, 0)$, touches at $(2, 0)$, y-intercept -4, rises on the left and falls on the right.

Example 2. Samuel makes an open box from a 12-inch by 12-inch sheet of tin by cutting squares of side x inches from the corners, so $V(x) = x(12 - 2x)^2$. (a) Find the zeros of V and their multiplicities. (b) State the domain for the model. (c) Use a graphing tool to find the cut that gives the greatest volume, and that volume.

Show solution

1. Write $(12 - 2x)^2 = 4(x - 6)^2$, so $V(x) = 4x(x - 6)^2$.
2. (a) Zeros: 0 (multiplicity 1, crosses) and 6 (multiplicity 2, touches).
3. (b) Dimensions must be positive: $x > 0$ and $12 - 2x > 0$, so $0 < x < 6$, the part of the graph between the zeros.
4. Leading term $4x^3$: the full graph falls on the left and rises on the right, crossing at 0 and touching at 6, so between them it is above the axis with one turning point.
5. (c) A graphing tool shows the relative maximum at $x = 2$.
6. $V(2) = 2(8)^2 = 128$. Check nearby: $V(1.9) = 1.9(8.2)^2 \approx 127.76$ and $V(2.1) = 2.1(7.8)^2 \approx 127.76$, both smaller ✓

Answer: (a) 0 (multiplicity 1) and 6 (multiplicity 2); (b) $0 < x < 6$; (c) cut 2-inch squares for a volume of 128 cubic inches.

PRACTICE

1. Easy List the zeros of $f(x) = (x - 4)(x + 1)^3(x - 2)^2$ with their multiplicities, and say whether the graph crosses or touches at each.

   Show answer

   -1 (multiplicity 3, crosses), 2 (multiplicity 2, touches), 4 (multiplicity 1, crosses)

2. Easy For $f(x) = -2(x + 3)(x - 1)^2$, find the y-intercept and describe the end behavior.

   Show answer

   $f(0) = -2(3)(1) = -6$; leading term $-2x^3$, so it rises on the left and falls on the right

3. **Easy** Does the graph of $f(x) = x^2(x-5)$ cross or touch the x-axis at $x = 0$? At $x = 5$?

Show answer

Touches at 0 (multiplicity 2); crosses at 5 (multiplicity 1)

4. **Easy** Write a polynomial in factored form with leading coefficient 1 whose graph touches the x-axis at -1 and crosses it at 3, and has degree 3.

Show answer

$f(x) = (x+1)^2(x-3)$

5. **Medium** Factor $f(x) = x^3 - 4x^2 - 4x + 16$ and use the factors to give the zeros, the y-intercept and the end behavior.

Show answer

$x^2(x-4) - 4(x-4) = (x-4)(x-2)(x+2)$; zeros $-2, 2, 4$, all crossing; y-intercept 16; falls on the left, rises on the right

6. **Medium** Solve $(x+1)(x-2)(x-5) < 0$. Write the answer in interval notation.

Show answer

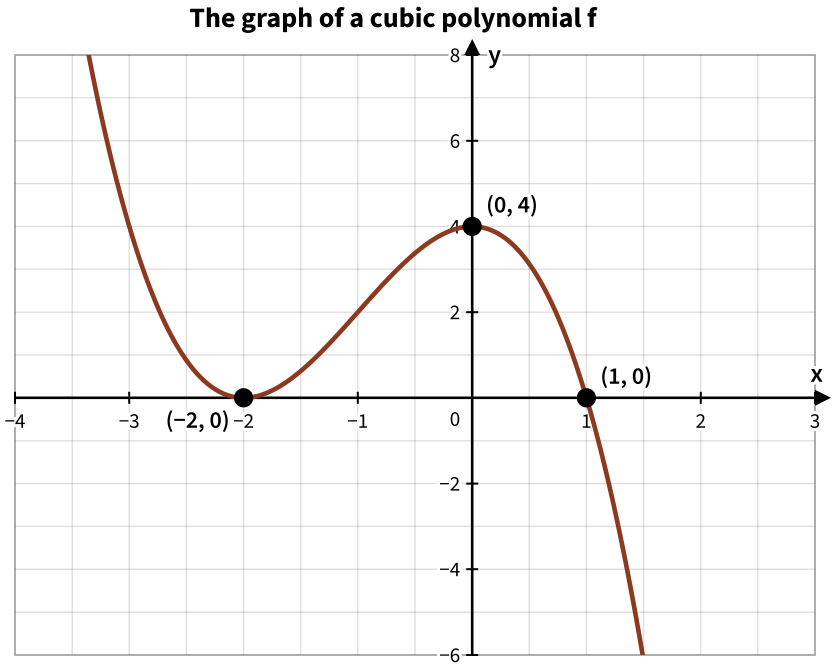
Signs: $-$ on $(-\infty, -1)$, $+$ on $(-1, 2)$, $-$ on $(2, 5)$, $+$ on $(5, \infty)$. Solution $(-\infty, -1) \cup (2, 5)$

7. **Medium** Solve $x^3 - 9x \geq 0$. Write the answer in interval notation.

Show answer

$x(x-3)(x+3) \geq 0$. Signs: $-$, $+$, $-$, $+$ on the intervals split by $-3, 0, 3$. Solution $[-3, 0] \cup [3, \infty)$

8. **Medium** The graph of a cubic polynomial f is shown. Write an equation for f in factored form. (Use the y-intercept to find the leading coefficient.)



Show answer

Touches at -2 and crosses at 1, so $f(x) = a(x+2)^2(x-1)$. $f(0) = a(4)(-1) = -4a = 4$, so $a = -1$: $f(x) = -(x+2)^2(x-1)$

9. **Medium** Solve $(x-1)^2(x+4) > 0$. Write the answer in interval notation.

Show answer

$(x-1)^2 > 0$ except at $x = 1$, where it is 0. So we need $x + 4 > 0$ and $x \neq 1$: $(-4, 1) \cup (1, \infty)$

10. **Hard** An open box is made from an 18-inch by 12-inch sheet of cardboard by cutting squares of side x inches from the corners, so $V(x) = x(18-2x)(12-2x)$. (a) State the domain. (b) Use a graphing tool to find, to the nearest hundredth of an inch, the cut that gives the greatest volume. (c) Find that volume to the nearest tenth of a cubic inch.

Show answer

(a) $0 < x < 6$; (b) $x \approx 2.35$ in; (c) $V \approx 228.2$ cubic inches

11. Hard A family woodshop's weekly profit, in hundreds of dollars, from making x dozen birdhouses is modeled by $P(x) = -x^3 + 9x^2 - 18x$ for $0 \le x \le 8$. For which values of x is the profit positive? Use a sign chart.

Show answer

$P(x) = -x(x-3)(x-6)$. Test: $P(1) = -10$, $P(4) = 8$, $P(7) = -28$. Profit is positive for $3 < x < 6$, between 3 and 6 dozen birdhouses

12. Hard A grocer stacks oranges in a square pyramid; the table shows the total number of oranges for each number of layers. (a) Show that the third differences are constant, so a cubic model fits. (b) The model is $f(n) = \dfrac{n(n+1)(2n+1)}{6}$. How many oranges are in a 10-layer pyramid?

Oranges in a square pyramid

Layers n	1	2	3	4	5	6
Total oranges	1	5	14	30	55	91

Show answer

(a) First differences 4, 9, 16, 25, 36; second differences 5, 7, 9, 11; third differences 2, 2, 2: constant; (b) $f(10) = \dfrac{10 \times 11 \times 21}{6} = 385$ oranges

13. Hard Explain why the graph of $f(x) = (x-3)^2(x+1)$ touches the x-axis at $x = 3$ but crosses it at $x = -1$.

Show answer

Near $x = 3$, the factor $x + 1$ is close to 4, so it is positive. The factor $(x-3)^2$ is positive on both sides of 3 because it is a square. So $f(x) > 0$ on both sides of 3: the graph touches and turns. Near $x = -1$, $(x-3)^2$ is close to 16 (positive), but $x + 1$ is negative just left of -1 and positive just right of it, so f changes sign: the graph crosses.

14. Challenge Write an equation for a degree-4 polynomial whose graph touches the x-axis at -1, crosses it at 0 and at 3, and passes through $(1, -8)$.

Show answer

$f(x) = a\,x(x+1)^2(x-3)$. $f(1) = a(1)(4)(-2) = -8a = -8$, so $a = 1$: $f(x) = x(x+1)^2(x-3)$

COMMON MISTAKES

- Making the graph cross at every zero. Check the multiplicity: at a squared factor the graph touches and turns.
- Including the zeros in the answer to a strict inequality such as $f(x) > 0$. Use parentheses there, and brackets only for $\geq$ or $\leq$.
- Assuming the signs simply alternate on a sign chart. They alternate only across zeros of odd multiplicity; test a point in each interval to be sure.
- Reading the turning point from a graphing tool without the algebra. Find the zeros, domain and end behavior first, then use the tool to locate the maximum or minimum.

> **Make it a game · Roller Coaster Blueprint.** Design a roller coaster by choosing zeros with multiplicities, such as -2 (once), 1 (twice) and 3 (once), and a positive leading coefficient. Give them to the parent, who sketches the track: crossing at odd multiplicity, touching and turning at even. Check the y-intercept and end behavior. Then swap and sketch the parent's design.

UNIT 4

Polynomial Functions

In this unit you move from quadratics to polynomials of any degree. You will learn how a polynomial's leading term controls the ends of its graph, how to multiply and divide polynomials (including long division and its shortcut, synthetic division), and how the Remainder and Factor Theorems link the value $f(c)$ to the factor $x - c$. These ideas lead to the Rational Root Theorem and the Fundamental Theorem of Algebra, which together tell you where to look for zeros and how many to expect.

The unit builds on factoring and the quadratic formula from Algebra I, on complex numbers and sign charts from Unit 3, and on function behavior from Unit 1. It prepares you directly for Unit 5, where rational functions are quotients of polynomials and long division gives slant asymptotes. Along the way you will model real objects: the volume of an open box cut from a sheet, the dimensions of a chest or a garden bed, and the growth of savings.

PREREQUISITES

- Factoring trinomials, differences of squares and perfect-square trinomials (Algebra I)
- Solving quadratic equations by factoring and by the quadratic formula (Lesson 3.2)
- Complex numbers and complex solutions of quadratics (Lessons 3.3 and 3.4)
- Sign charts and interval notation for quadratic inequalities (Lesson 3.5)
- Function notation, end behavior in words and relative maxima and minima (Lessons 1.1 and 1.6)

MASTERY CHECKLIST

- ☐ I can give the degree, leading coefficient and constant term of a polynomial and describe its end behavior from the leading term, and explain why the leading term controls the ends.
- ☐ I can add, subtract and multiply polynomials, expand $(a \pm b)^3$ and, write the volume of an open box as a polynomial with its domain.
- ☐ I can divide polynomials by long division, using placeholders, and write the result as $q(x) + \frac{r(x)}{d(x)}$.
- ☐ I can divide by $x - c$ with synthetic division and explain why it matches long division.
- ☐ I can prove the Remainder Theorem and use synthetic substitution to evaluate a polynomial.
- ☐ I can use the Factor Theorem and one known zero to factor a polynomial completely.
- ☐ I can factor sums and differences of cubes, four-term polynomials by grouping, and polynomials in quadratic form.
- ☐ I can list possible rational zeros, explain why the Rational Root

Theorem works, and find all real and non-real zeros of a polynomial.

- ☐ I can write a polynomial with given zeros, including conjugate pairs.
- ☐ I can sketch a polynomial from its zeros, multiplicities, end behavior and y-intercept, and explain why it crosses or touches at each zero.
- ☐ I can solve polynomial inequalities with a sign chart and use a graphing tool to find maxima and minima in a model.

LESSONS IN THIS UNIT

4.1	**Polynomial Functions and End Behavior**	2 sessions of 45 minutes
4.2	**Operations with Polynomials**	2 sessions of 45 minutes
4.3	**Polynomial Long Division**	2 sessions of 45 minutes
4.4	**Synthetic Division and the Remainder Theorem**	2 sessions of 45 minutes
4.5	**The Factor Theorem and Factoring Polynomials**	2 sessions of 45 minutes
4.6	**Finding Zeros of Polynomials**	2 sessions of 45 minutes
4.7	**Graphing Polynomial Functions**	2 sessions of 45 minutes

Unit 4 test

Time: 60 minutes. A scientific calculator is allowed for arithmetic; no graphing calculator or graphing app. Show your work for every problem: a correct answer without work earns partial credit only. Give exact answers (fractions, radicals, i) unless a problem asks for a rounded answer. Write inequality answers in interval notation. The test is worth 100 points.

1. **5 pts**

   For $f(x) = 3x^2 - x^5 + 4x - 8$, give the degree, the leading coefficient and the constant term, and describe the end behavior.

2. **5 pts**

   Without multiplying out, find the degree, the leading coefficient and the y-intercept of $g(x) = (4 - x)^2(2x + 1)(x + 5)$. Then describe its end behavior.

3. **6 pts**

   Write each product in standard form. (a) $(3x - 2)(x^2 - 4x + 5)$ (b) $(x - 5)^3$

4. **6 pts**

   A sheet-metal shop makes an open tray from a 14-inch by 10-inch sheet by cutting a square of side x inches from each corner and folding up the sides. (a) Write the volume $V(x)$ in standard form. (b) Find the volume when $x = 1.5$.

5. **7 pts**

   Use long division to divide $(2x^3 + x^2 - 13x + 9) \div (x^2 + 2x - 3)$. Write the result in the form $q(x) + \dfrac{r(x)}{d(x)}$.

6. **6 pts**

   Use long division to divide $(4x^3 - 7x + 5) \div (2x + 1)$. Give the quotient and remainder.

7. **6 pts**

   Use synthetic division to divide $(x^4 - 3x^3 + 5x - 6) \div (x + 2)$. Then use the Remainder Theorem to check your remainder.

8. **7 pts**

   (a) Prove that $x - 1$ is a factor of $x^n - 1$ for every positive whole number n. (b) Use synthetic division to find the quotient when $x^5 - 1$ is divided by $x - 1$.

9. **6 pts**

   Factor completely over the integers. (a) $4x^4 - 37x^2 + 9$ (b) $27x^3 + 8$

10. **6 pts**

    Given that -3 is a zero of $f(x) = 2x^3 + 5x^2 - 4x - 3$, solve $f(x) = 0$.

11. **7 pts**

Find all zeros, real and non-real, of $f(x) = x^4 - 2x^3 - x^2 - 4x - 6$.

12. **7 pts**

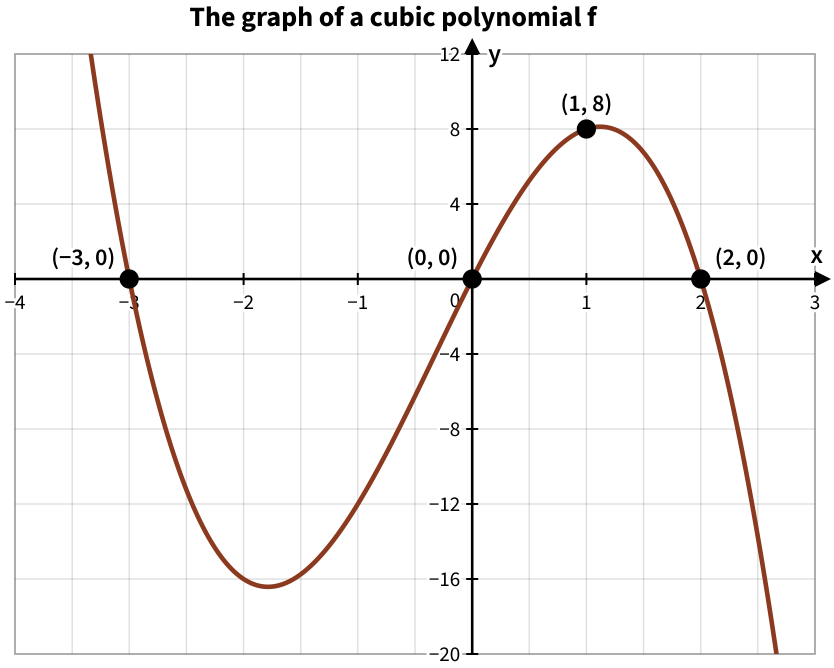
A farmer builds a wooden hay bin shaped like a box. Its width is 1 foot less than twice its height, and its length is 3 feet more than twice its height. The bin must hold 80 cubic feet. Find the height of the bin, and show that no other height works.

13. **6 pts**

Write a polynomial of least degree in standard form, with real coefficients and leading coefficient 1, that has zeros -2 and $1 - 3i$.

14. **6 pts**

The graph of a cubic polynomial f is shown. Write an equation for f in factored form.



15. **6 pts**

Solve $(x + 3)(x - 1)^2(x - 2) \leq 0$. Write the answer in interval notation.

16. **8 pts**

For the tray in Problem 4, $V(x) = 4x^3 - 48x^2 + 140x$ with $0 < x < 5$. (a) Show that $x = 1$ gives a volume of 96 cubic inches. (b) Find every cut size x that gives a volume of 96 cubic inches. (c) For which cut sizes is the volume at least 96 cubic inches? Use a sign chart.

ANSWER KEY

1. Standard form $-x^5 + 3x^2 + 4x - 8$: degree 5, leading coefficient -1, constant term -8; rises on the left and falls on the right (as $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to -\infty$)
2. Leading term $(x^2)(2x)(x) = 2x^4$: degree 4, leading coefficient 2; $g(0) = 16 \times 1 \times 5 = 80$; both ends rise
3. (a) $3x^3 - 12x^2 + 15x - 2x^2 + 8x - 10 = 3x^3 - 14x^2 + 23x - 10$; (b) $x^3 - 15x^2 + 75x - 125$
4. (a) $V(x) = x(14 - 2x)(10 - 2x) = 4x^3 - 48x^2 + 140x$; (b) $V(1.5) = 1.5 \times 11 \times 7 = 115.5$ cubic inches
5. Quotient $2x - 3$, remainder $-x$: $2x - 3 - \dfrac{x}{x^2 + 2x - 3}$ (check: $(x^2 + 2x - 3)(2x - 3) - x = 2x^3 + x^2 - 13x + 9$ ✓)
6. With the placeholder $0x^2$: quotient $2x^2 - x - 3$, remainder 8 (check: $(2x + 1)(2x^2 - x - 3) + 8 = 4x^3 - 7x + 5$ ✓)
7. Coefficients 1, -3, 0, 5, -6 with $c = -2$: bottom row 1, -5, 10, -15 | 24. Quotient $x^3 - 5x^2 + 10x - 15$, remainder 24; check $f(-2) = 16 + 24 - 10 - 6 = 24$ ✓
8. (a) Let $f(x) = x^n - 1$. Then $f(1) = 1^n - 1 = 0$, so by the Factor Theorem $x - 1$ is a factor of $f(x)$. (b) Coefficients 1, 0, 0, 0, 0, -1 with $c = 1$: bottom row 1, 1, 1, 1, 1 | 0, so the quotient is $x^4 + x^3 + x^2 + x + 1$
9. (a) With $u = x^2$: $(4u - 1)(u - 9)$, so $(2x - 1)(2x + 1)(x - 3)(x + 3)$; (b) $(3x + 2)(9x^2 - 6x + 4)$
10. Synthetic division with $c = -3$: bottom row 2, -1, -1 | 0, so $f(x) = (x + 3)(2x^2 - x - 1) = (x + 3)(2x + 1)(x - 1)$. Solutions $x = -3, -\dfrac{1}{2}, 1$
11. Candidates $\pm 1, \pm 2, \pm 3, \pm 6$. $f(-1) = 0$: quotient $x^3 - 3x^2 + 2x - 6$. Then $x = 3$ works: quotient $x^2 + 2$, so $x = \pm i\sqrt{2}$. Zeros $-1, 3, -i\sqrt{2}, i\sqrt{2}$
12. $x(2x - 1)(2x + 3) = 80$ gives $4x^3 + 4x^2 - 3x - 80 = 0$. Testing candidates, $x = \dfrac{5}{2}$ works: synthetic division gives 4, 14, 32 | 0, and $4x^2 + 14x + 32$ has discriminant $196 - 512 < 0$, so there are no other real solutions. Height 2.5 ft (width 4 ft, length 8 ft; check $2.5 \times 4 \times 8 = 80$ ✓)
13. $1 + 3i$ is also a zero. $(x - 1)^2 + 9 = x^2 - 2x + 10$, and $(x + 2)(x^2 - 2x + 10) = x^3 + 6x + 20$
14. Zeros -3, 0 and 2, each crossing: $f(x) = a\,x(x + 3)(x - 2)$. Using $(1, 8)$: $a(1)(4)(-1) = -4a = 8$, so $a = -2$: $f(x) = -2x(x + 3)(x - 2)$

15. Zeros -3, 1 (even multiplicity) and 2. Signs: $+$ on $(-\infty, -3)$, $-$ on $(-3, 1)$, $-$ on $(1, 2)$, $+$ on $(2, \infty)$; the zeros are included. Solution $[-3, 2]$

16. (a) $V(1) = 1 \times 12 \times 8 = 96$ ✓; (b) $V(x) - 96 = 4(x^3 - 12x^2 + 35x - 24) = 4(x - 1)(x - 3)(x - 8)$, so $x = 1$ or $x = 3$ ($x = 8$ is outside the domain); (c) on $0 < x < 5$, $(x - 1)(x - 3)(x - 8)$ is $-$ on $(0, 1)$, $+$ on $(1, 3)$ and $-$ on $(3, 5)$, so $1 \le x \le 3$, that is $[1, 3]$

UNIT 5

Rational Expressions and Functions

A rational expression is a fraction made of polynomials, and a rational function is a function with such a rule. In this unit you will simplify, multiply, divide, add and subtract rational expressions, using the factoring of Unit 4 at every step. Then you will study the graphs of rational functions: their vertical, horizontal and slant asymptotes, their holes, and their intercepts, and you will learn why each of these features appears.

Rational functions describe many everyday situations: the average cost per item as a business grows, the time for a round trip on a river, two pumps filling a tank, the brightness of a lamp as you move away from it, and the strength of a punch as juice is added. Solving rational equations brings a new caution: the solving steps can produce extraneous solutions, so every answer must be checked against the excluded values.

This unit builds on polynomial factoring and long division (Unit 4) and on transformations of parent functions (Unit 1). The habit of checking domains and excluded values carries straight into Unit 6, where radical equations can also have extraneous solutions, and into Unit 8, where logarithms have restricted domains.

PREREQUISITES

- Factoring trinomials, differences of squares, sums and differences of cubes, grouping and quadratic form (Lessons 4.2 and 4.5)
- Polynomial long division (Lesson 4.3)
- Transformations $y = a \cdot f(x - h) + k$ of the parent function $y = \frac{1}{x}$ (Lesson 1.2)
- Solving linear and quadratic equations, including by the quadratic formula (Lesson 3.2)
- Adding, subtracting, multiplying and dividing numerical fractions

MASTERY CHECKLIST

- ☐ I can state the excluded values of a rational expression and simplify it by dividing out common factors, never terms.
- ☐ I can multiply and divide rational expressions, including the excluded values from the divisor's numerator.
- ☐ I can add and subtract rational expressions using the LCD, with parentheses around subtracted numerators.
- ☐ I can simplify complex fractions by two methods.
- ☐ I can graph $f(x) = \frac{a}{x - h} + k$ and state its domain, range and asymptotes.
- ☐ I can find vertical and horizontal asymptotes and explain why they occur.
- ☐ I can find holes, intercepts and slant asymptotes and sketch a rational function with a sign chart.
- ☐ I can solve rational equations and identify extraneous solutions, and explain why they appear.
- ☐ I can solve work, travel and proportion problems with rational equations.
- ☐ I can write and use direct, inverse, joint and combined variation equations, including inverse-square laws.

- ☐ I can use the shared-work formula and explain why the average speed of a round trip is $\dfrac{2ab}{a+b}$.

LESSONS IN THIS UNIT

5.1	**Rational Expressions: Simplifying, Multiplying and Dividing**	45 minutes
5.2	**Adding and Subtracting Rational Expressions**	2 sessions of 45 minutes
5.3	**Rational Functions and Asymptotes**	2 sessions of 45 minutes
5.4	**Graphing Rational Functions**	2 sessions of 45 minutes
5.5	**Solving Rational Equations**	2 sessions of 45 minutes
5.6	**Variation and Rational Models**	2 sessions of 45 minutes

Unit 5 Test

Time: 60 minutes. A scientific calculator is allowed; a graphing calculator is not. Show all work. State excluded values wherever you simplify or solve, and check every solution of an equation in the original equation. Give exact answers unless a problem asks you to round.

1. **5 pts**

   Simplify $\dfrac{x^2 - 49}{x^2 - 4x - 21}$ and state the excluded values.

2. **6 pts**

   Simplify $\dfrac{8x^3 - 1}{4x^2 - 1}$ and state the excluded values.

3. **7 pts**

   Divide and simplify, stating all excluded values: $\dfrac{x^2 + 2x - 15}{x^2 - 4} \div \dfrac{x^2 - 9}{x^2 + 2x}$.

4. **6 pts**

   Add and simplify: $\dfrac{3}{x+4} + \dfrac{24}{x^2-16}$.

5. **6 pts**

   Subtract: $\dfrac{2x}{x^2+5x+6} - \dfrac{x-1}{x^2+4x+3}$.

6. **6 pts**

   Simplify the complex fraction $\dfrac{\frac{1}{x} - \frac{1}{3}}{x-3}$.

7. **6 pts**

   For $f(x) = -\dfrac{4}{x+1} + 2$, find the vertical and horizontal asymptotes and the x- and y-intercepts.

8. **7 pts**

   A family orchard buys a cider press for \$1800. Apples and jugs cost \$2.50 for each gallon of cider. (a) Write the average cost per gallon $\overline{C}(x)$ for x gallons. (b) Give the horizontal asymptote. (c) How many gallons bring the average cost down to \$4 per gallon?

9. **8 pts**

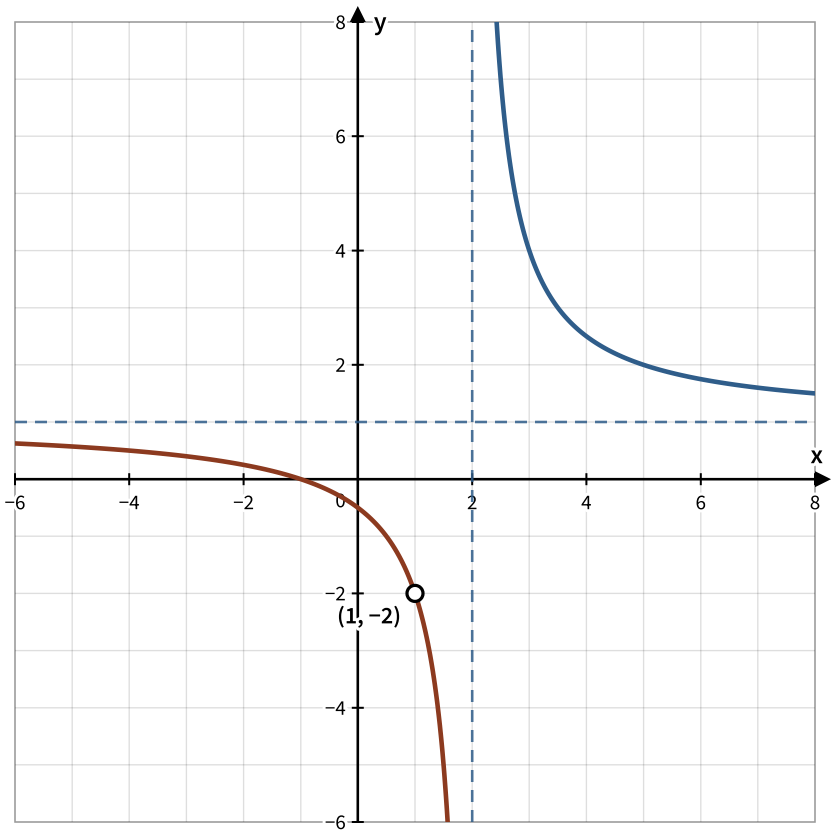
   For $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$, find any hole, the vertical and horizontal asymptotes, and the intercepts.

10. **7 pts**

    For $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, find the vertical asymptote, the slant asymptote and the intercepts.

11. **6 pts**

The graph of a rational function f is shown; its asymptotes are dashed and it has one hole. (a) State the domain of f. (b) Use the graph to solve $f(x) > 0$. Answer both in interval notation.



12. **7 pts**

Solve $\dfrac{3}{x-2} + \dfrac{4}{x+2} = \dfrac{12}{x^2-4} + 1$.

13. **7 pts**

A small plane flies 600 miles with a 30 mph tailwind in the same time it flies 480 miles against the same wind. Find the plane's speed in still air.

14. **6 pts**

The time needed to shingle a roof varies directly as the roof's area and inversely as the number of workers. Six volunteers shingle 2400 square feet of the church roof in 8 hours. How long would 8 volunteers take to shingle 3200 square feet?

15. **6 pts**

A train travels from one city to another at 90 mph and returns on the same track at 60 mph. What is its average speed for the round trip?

16. **4 pts**

In Problem 12, $x = 2$ solves the equation $x^2 - 7x + 10 = 0$ that you obtained. Explain why it is not a solution of the original equation, and why the solving steps produced it.

ANSWER KEY

1. $\dfrac{(x-7)(x+7)}{(x-7)(x+3)} = \dfrac{x+7}{x+3}$, with $x \neq -3, 7$
2. $\dfrac{(2x-1)(4x^2+2x+1)}{(2x-1)(2x+1)} = \dfrac{4x^2+2x+1}{2x+1}$, with $x \neq -\dfrac{1}{2}, \dfrac{1}{2}$
3. $\dfrac{(x+5)(x-3)}{(x-2)(x+2)} \cdot \dfrac{x(x+2)}{(x-3)(x+3)} = \dfrac{x(x+5)}{(x-2)(x+3)}$, with $x \neq -3, -2, 0, 2, 3$
4. $\dfrac{3(x-4)+24}{(x-4)(x+4)} = \dfrac{3(x+4)}{(x-4)(x+4)} = \dfrac{3}{x-4}$, with $x \neq -4, 4$
5. LCD $(x+1)(x+2)(x+3)$; numerator $2x(x+1) - (x-1)(x+2) = x^2 + x + 2$; answer $\dfrac{x^2+x+2}{(x+1)(x+2)(x+3)}$
6. Multiply top and bottom by $3x$: $\dfrac{3-x}{3x(x-3)} = -\dfrac{1}{3x}$, with $x \neq 0, 3$
7. Vertical $x = -1$; horizontal $y = 2$; x-intercept: $\dfrac{4}{x+1} = 2$ gives $(1, 0)$; y-intercept $(0, -2)$
8. (a) $\overline{C}(x) = \dfrac{1800 + 2.5x}{x}$; (b) $y = 2.5$: the average approaches \$2.50, the cost of materials; (c) $2.5 + \dfrac{1800}{x} = 4$ gives $x = 1200$ gallons
9. $\dfrac{(x+3)(x-2)}{(x-2)(x+2)} = \dfrac{x+3}{x+2}$: hole $\left(2, \dfrac{5}{4}\right)$; vertical $x = -2$; horizontal $y = 1$; x-intercept $(-3, 0)$; y-intercept $\left(0, \dfrac{3}{2}\right)$
10. $x^2 - 2x - 3 = (x+2)(x-4) + 5$; vertical $x = -2$; slant $y = x - 4$; x-intercepts $(-1, 0)$ and $(3, 0)$; y-intercept $\left(0, -\dfrac{3}{2}\right)$
11. (a) $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$; (b) $(-\infty, -1) \cup (2, \infty)$
12. Excluded $x \neq \pm 2$. Multiply by $(x-2)(x+2)$: $3(x+2) + 4(x-2) = 12 + x^2 - 4$, so $x^2 - 7x + 10 = 0$ and $x = 2$ or $x = 5$. Reject $x = 2$ (excluded). Check $x = 5$: $1 + \dfrac{4}{7} = \dfrac{12}{21} + 1$ ✓. Solution: $x = 5$
13. $\dfrac{600}{r+30} = \dfrac{480}{r-30}$; $600r - 18000 = 480r + 14400$, so $120r = 32400$ and $r = 270$ mph (check: 2 hours each way ✓)
14. $T = \dfrac{kA}{n}$: $8 = \dfrac{2400k}{6}$ gives $k = 0.02$; $T = \dfrac{0.02 \times 3200}{8} = 8$ hours
15. $\dfrac{2(90)(60)}{90+60} = \dfrac{10800}{150} = 72$ mph

16. At $x = 2$ the denominators $x - 2$ and $x^2 - 4$ are 0, so the original equation is undefined there. The steps multiplied both sides by $(x - 2)(x + 2)$, which is 0 at $x = 2$; multiplying by an expression that can be 0 can add solutions, so the polynomial equation has a root the original equation does not.

5.1 Rational Expressions: Simplifying, Multiplying and Dividing

OBJECTIVES

- State the excluded values of a rational expression by factoring its denominator.
- Simplify rational expressions by factoring completely (including cubes, grouping and quadratic form) and dividing out common factors.
- Explain why only common factors, never terms, may be divided out.
- Multiply and divide rational expressions, listing every excluded value, including those from the divisor's numerator.

WATCH

Algebraic Fractions (2 of 2: Multiplication & Division)

Eddie Woo · 5:31

https://www.youtube.com/watch?v=EBiXoFI5k_Y

> **WHEN TO WATCH**
> Before Example 1: multiplying and dividing rational expressions by factoring first.

Note: Eddie Woo teaches in Australia: he says 'algebraic fractions' for rational expressions and 'factorise' for factor.

BACKUP VIDEO

Dividing Rational Expressions and Identifying Excluded Values

Mario's Math Tutoring · 7:19

https://www.youtube.com/watch?v=l64MJOZH5ko

Before Example 2: listing every excluded value when you divide.

KEY TERMS

rational expression

A fraction whose numerator and denominator are polynomials, such as $\frac{x+1}{x^2-4}$. The denominator may not be the zero polynomial.

excluded value

A value of the variable that makes a denominator equal 0. The expression has no value there, so that number is left out of its domain.

simplest form

A rational expression is in simplest form when its numerator and denominator have no common factor other than 1 or -1.

factor

Something that is multiplied. In $3(x+2)$ the factors are 3 and $x+2$; a term, by contrast, is something that is added.

opposites

Two binomials such as $a-b$ and $b-a$ whose sum is 0; each is -1 times the other.

reciprocal

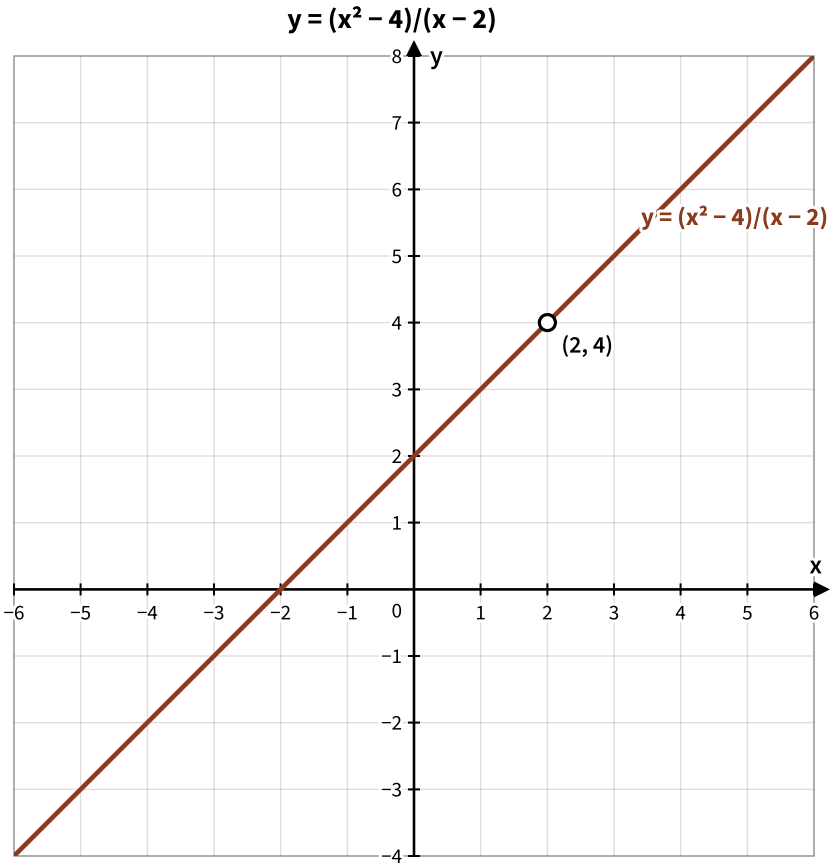
The reciprocal of $\frac{C}{D}$ is $\frac{D}{C}$; their product is 1.

CORE IDEAS

A **rational expression** is to polynomials what a fraction is to whole numbers: $\frac{P}{Q}$, where P and Q are polynomials. Because we can never divide by 0, every rational expression comes with **excluded values**, the numbers that make the denominator 0. Find them by factoring the denominator and setting each factor equal to 0. For $\frac{x+3}{x^2-9} = \frac{x+3}{(x-3)(x+3)}$ the excluded values are $x=3$ and $x=-3$. State them first, from the original expression, before you change anything.

Why only factors divide out. For numbers, $\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}$ as long as $c \neq 0$. This works only because c multiplies the whole numerator and the whole denominator. A term that is added does not behave this way: $\frac{x+6}{x+3}$ is not $\frac{6}{3} = 2$, as $x=1$ shows ($\frac{7}{4} \neq 2$). So the method is always: factor the numerator and the denominator completely, then divide out the common factors.

The simplified expression equals the original one for every allowed value of x, but the excluded values stay excluded. For example, $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$ for $x \neq 2$. The graph of $y = \frac{x^2-4}{x-2}$ is the line $y = x+2$ with one point missing, an open circle at $(2, 4)$. (Lesson 5.4 calls such a gap a hole.)



Opposites. Since $b - a = -(a - b)$, we have $\dfrac{a-b}{b-a} = \dfrac{a-b}{-(a-b)} = -1$ whenever $a \neq b$. For example, $\dfrac{x^2-9}{3-x} = \dfrac{(x-3)(x+3)}{-(x-3)} = -(x+3)$ for $x \neq 3$. Factor -1 out of the binomial that is written backward, and the common factor appears.

Your factoring from Unit 4 does the heavy lifting: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, $a^3 + b^3 = (a+b)(a^2-ab+b^2)$, grouping, and quadratic form. For instance, $\dfrac{x^3-8}{x^2-4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$ for $x \neq \pm 2$. The trinomial x^2+2x+4 has no real zeros (its discriminant is $4 - 16 < 0$), so it does not factor further.

Multiplying. As with fractions, $\dfrac{A}{B} \cdot \dfrac{C}{D} = \dfrac{AC}{BD}$. Factor every numerator and denominator first; then any factor in either numerator may divide out with the same factor in either denominator. Multiply what is left. The excluded values are the zeros of B and of D.

Dividing. Dividing by a number is the same as multiplying by its reciprocal, because $\frac{C}{D} \cdot \frac{D}{C} = 1$. So $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$. Now there are three sources of excluded values: the zeros of B, the zeros of D, and the zeros of C. The last ones are easy to forget: if $C = 0$ the divisor $\frac{C}{D}$ is 0, and we cannot divide by 0. After flipping, C sits in a denominator, which is a good reminder.

KEY FORMULAS

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \quad (b \neq 0,\ c \neq 0)$$

$$\frac{a - b}{b - a} = -1 \quad (a \neq b)$$

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD} \qquad \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} \quad (B, C, D \neq 0)$$

> **Practice habit.** Factor before anything else, and write the excluded values first, from the original expression (for division, include the divisor's numerator). Then, and only then, divide out common factors.

Example 1. A farm's cylindrical water tank has radius r feet and height h feet. Its total surface area (top, bottom and side) is $S = 2\pi r^2 + 2\pi r h$ and its volume is $V = \pi r^2 h$. The ratio $\dfrac{S}{V}$ tells how much surface there is for each cubic foot of water (more surface means faster warming and cooling). (a) Write $\dfrac{S}{V}$ in simplest form. (b) Evaluate it for $r = 5$ and $h = 10$.

Show solution

1. Write the ratio: $\dfrac{S}{V} = \dfrac{2\pi r^2 + 2\pi r h}{\pi r^2 h}$.
2. Factor the numerator: $2\pi r^2 + 2\pi r h = 2\pi r(r + h)$.
3. Write the denominator as a product that shows the common factor: $\pi r^2 h = \pi r \cdot r h$.
4. Divide out the common factor πr: $\dfrac{2\pi r(r + h)}{\pi r \cdot r h} = \dfrac{2(r + h)}{r h}$.
5. Stop here: the r inside $r + h$ is a term, not a factor, so it cannot divide out with the r in the denominator.
6. (b) Substitute: $\dfrac{2(5 + 10)}{5 \times 10} = \dfrac{30}{50} = 0.6$.
7. Check with the original formulas: $S = 2\pi(25) + 2\pi(50) = 150\pi$ and $V = \pi(25)(10) = 250\pi$, so $\dfrac{S}{V} = \dfrac{150}{250} = 0.6$ ✓.

Answer: (a) $\dfrac{S}{V} = \dfrac{2(r + h)}{r h}$; (b) 0.6 square foot of surface per cubic foot of water

Example 2. Divide and simplify. State all excluded values.

$$\frac{x^3 - 2x^2 + 4x - 8}{x^2 - 9} \div \frac{x^2 - 4}{x^2 + 3x}$$

Show solution

1. Factor the first numerator by grouping: $x^2(x-2) + 4(x-2) = (x-2)(x^2+4)$.
2. Factor the rest: $x^2 - 9 = (x-3)(x+3)$, $x^2 - 4 = (x-2)(x+2)$, $x^2 + 3x = x(x+3)$.
3. Excluded values: $x^2 - 9 = 0$ gives ± 3; $x^2 + 3x = 0$ gives 0 and -3; the divisor's numerator $x^2 - 4 = 0$ gives ± 2. So $x \neq -3, -2, 0, 2, 3$.
4. Multiply by the reciprocal: $\dfrac{(x-2)(x^2+4)}{(x-3)(x+3)} \cdot \dfrac{x(x+3)}{(x-2)(x+2)}$.
5. Divide out $x - 2$ and $x + 3$: $\dfrac{x(x^2+4)}{(x-3)(x+2)}$.
6. Check with $x = 1$ in the original: $\dfrac{-5}{-8} \div \dfrac{-3}{4} = \dfrac{5}{8} \cdot \left(-\dfrac{4}{3}\right) = -\dfrac{5}{6}$. Simplified form: $\dfrac{1 \cdot 5}{(-2)(3)} = -\dfrac{5}{6}$ ✓.

Answer: $\dfrac{x(x^2+4)}{(x-3)(x+2)}$, with $x \neq -3, -2, 0, 2, 3$

PRACTICE

1. Easy State the excluded values of $\dfrac{x+5}{x^2 - 7x + 10}$.

   Show answer

   $x^2 - 7x + 10 = (x-2)(x-5)$, so $x \neq 2$ and $x \neq 5$

2. Easy Simplify $\dfrac{6x^3y}{15xy^4}$. Assume no variable equals 0.

   Show answer

   $$\frac{2x^2}{5y^3}$$

3. Easy Simplify $\dfrac{x^2 - 25}{x^2 + 3x - 10}$ and state the excluded values.

Show answer

$\dfrac{(x-5)(x+5)}{(x+5)(x-2)} = \dfrac{x-5}{x-2}$, with $x \neq -5, 2$

4. Easy Simplify $\dfrac{12 - 3x}{x^2 - 16}$ and state the excluded values.

Show answer

$\dfrac{-3(x-4)}{(x-4)(x+4)} = -\dfrac{3}{x+4}$, with $x \neq -4, 4$

5. Easy For the church bake sale, Grace bakes $x^2 + 8x + 15$ cookies and packs them in bags of $x + 3$ cookies each. (a) Write the number of bags as a simplified expression. (b) How many bags does she fill when $x = 9$?

Show answer

(a) $\dfrac{(x+3)(x+5)}{x+3} = x + 5$; (b) 14 bags (168 cookies in bags of 12)

6. Medium Simplify $\dfrac{x^3 + 125}{x^2 - 25}$ and state the excluded values.

Show answer

$\dfrac{(x+5)(x^2 - 5x + 25)}{(x+5)(x-5)} = \dfrac{x^2 - 5x + 25}{x-5}$, with $x \neq -5, 5$

7. Medium Simplify $\dfrac{x^4 - 13x^2 + 36}{x^2 + x - 6}$ and state the excluded values.

Show answer

$\dfrac{(x-2)(x+2)(x-3)(x+3)}{(x+3)(x-2)} = (x+2)(x-3)$, with $x \neq -3, 2$

8. Medium Simplify $\dfrac{x^3 + 3x^2 - 4x - 12}{x^2 + x - 6}$ and state the excluded values.

Show answer

Grouping: $x^2(x+3) - 4(x+3) = (x+3)(x-2)(x+2)$; the denominator is $(x+3)(x-2)$; result $x + 2$, with $x \neq -3, 2$

9. **Medium** Multiply and simplify, stating the excluded values: $\dfrac{x^2-9}{2x+8}\cdot\dfrac{x^2+4x}{x^2-3x}$.

Show answer

$\dfrac{(x-3)(x+3)}{2(x+4)}\cdot\dfrac{x(x+4)}{x(x-3)}=\dfrac{x+3}{2}$, with $x\neq -4, 0, 3$

10. **Medium** Divide and simplify, stating the excluded values: $\dfrac{x^2-x-12}{x^2-16}\div\dfrac{x+3}{x^2+4x}$.

Show answer

$\dfrac{(x-4)(x+3)}{(x-4)(x+4)}\cdot\dfrac{x(x+4)}{x+3}=x$, with $x\neq -4, -3, 0, 4$ (-3 comes from the divisor's numerator)

11. **Hard** A ball of bread dough is a sphere of radius r inches, with surface area $S=4\pi r^2$ and volume $V=\dfrac{4}{3}\pi r^3$. (a) Simplify $\dfrac{S}{V}$. (b) Evaluate it for a dinner roll with $r=1$. (c) Evaluate it for a round loaf with $r=3$. (Heat enters through the surface, which is one reason small rolls bake faster.)

Show answer

(a) $\dfrac{4\pi r^2}{\frac{4}{3}\pi r^3}=\dfrac{3}{r}$; (b) 3 square inches per cubic inch; (c) 1 square inch per cubic inch

12. **Hard** Divide and simplify, stating the excluded values: $\dfrac{25-x^2}{x^2+2x-15}\div\dfrac{x^2+10x+25}{x^2-9}$.

Show answer

$\dfrac{-(x-5)(x+5)}{(x+5)(x-3)}\cdot\dfrac{(x-3)(x+3)}{(x+5)^2}=-\dfrac{(x-5)(x+3)}{(x+5)^2}$, with $x\neq -5, -3, 3$

13. Hard At a grain elevator, one conveyor moves $x^2 + 5x$ bushels of wheat in $x + 2$ minutes, and a second conveyor moves $x^2 - 25$ bushels in $x^2 - 3x - 10$ minutes (assume $x > 5$). (a) Write the first rate divided by the second rate in simplest form. (b) Check your answer with $x = 10$ by computing both rates in bushels per minute.

Show answer

(a) $\dfrac{x(x+5)}{x+2} \div \dfrac{(x-5)(x+5)}{(x-5)(x+2)} = \dfrac{x(x+5)}{x+2} \cdot \dfrac{x+2}{x+5} = x$; (b) first: 150 bushels in 12 minutes, 12.5 per minute; second: 75 bushels in 60 minutes, 1.25 per minute; $12.5 \div 1.25 = 10$ ✓

14. Challenge Henry simplifies $\dfrac{x+6}{x+3}$ to 2 by "canceling the x's and dividing 6 by 3." (a) Show with a number that he is wrong. (b) Explain his error. (c) Explain why $\dfrac{x^2+6x}{x^2+3x}$ can be simplified, and simplify it.

Show answer

(a) At $x = 1$: $\dfrac{7}{4} \neq 2$. (b) x, 6 and 3 are terms (added), not factors of the whole numerator and denominator; only common factors divide out, because $\dfrac{ac}{bc} = \dfrac{a}{b} \cdot \dfrac{c}{c}$. (c) Here x is a factor of both: $\dfrac{x(x+6)}{x(x+3)} = \dfrac{x+6}{x+3}$, with $x \neq 0, -3$.

15. Challenge Simplify $\dfrac{x^6 - 64}{(x^3+8)(x^2-4)}$ and state the excluded values.

Show answer

$x^6 - 64 = (x^3 - 8)(x^3 + 8)$, so the expression is $\dfrac{x^3 - 8}{x^2 - 4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$, with $x \neq -2, 2$

COMMON MISTAKES

- Canceling terms instead of factors, as in $\dfrac{x+6}{x+3} = 2$. Factor completely first; divide out only factors that multiply the entire numerator and the entire denominator.
- Finding excluded values from the simplified answer. A factor that divided out still makes the original undefined; list excluded values from the original expression before simplifying.

- In division, forgetting that the divisor's numerator cannot be 0. After you flip the divisor, that numerator is a denominator, so its zeros are excluded too.
- Treating $a - b$ and $b - a$ as equal. They are opposites: $\frac{a-b}{b-a} = -1$. Factor out -1 and keep the negative sign in the answer.

> **Make it a game · Excluded First, Then Cancel.** The parent writes six rational expressions, such as $(x^3 - 8)/(x^2 - 4)$ and $(x^2 - 5x + 6)/(x^2 - 9)$. Score three points for each: one for stating the excluded values from the denominator, one for factoring completely, and one for the correctly simplified answer. Make the order a rule: excluded values come before you cancel anything.

5.2 Adding and Subtracting Rational Expressions

OBJECTIVES

- Find the least common denominator of polynomial denominators by factoring them first.
- Add and subtract rational expressions, using parentheses when subtracting a numerator with several terms, and simplify the result.
- Simplify complex fractions by two methods: multiplying top and bottom by the LCD, and rewriting as a division.
- Write the total time of a round trip as a single rational expression.

WATCH

Adding & Subtracting Rational Expressions Easily

Mario's Math Tutoring · 6:49

https://www.youtube.com/watch?v=QlNuU3Ytjd8

> **WHEN TO WATCH**
> Before Example 1: adding and subtracting with a common denominator.

BACKUP VIDEO

Simplifying Complex Fractions (2 Methods)

Mario's Math Tutoring · 4:02

https://www.youtube.com/watch?v=IAAzKsFtZ6Y

Before the practice set: simplifying complex fractions, two ways.

KEY TERMS

least common denominator (LCD)

The simplest expression that every denominator divides into: the product of every different factor, each raised to the highest power it has in any one denominator.

equivalent expressions

Expressions that have the same value for every allowed value of the variable, such as $\frac{3}{x}$ and $\frac{3(x+1)}{x(x+1)}$ for $x \neq 0, -1$.

complex fraction

A fraction that has a fraction in its numerator, its denominator, or both, such as $\frac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}-\frac{1}{2}}$.

CORE IDEAS

Same denominator. Fractions with the same denominator add by adding numerators: $\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C}$. This is just the distributive property, since $A\cdot\frac{1}{C}+B\cdot\frac{1}{C}=(A+B)\cdot\frac{1}{C}$. Notice what does not happen: the denominators are not added. $\frac{1}{x}+\frac{1}{y}$ is never $\frac{2}{x+y}$.

Different denominators. Recall how you add $\frac{1}{12}+\frac{1}{18}$: factor $12=2^2\cdot 3$ and $18=2\cdot 3^2$; the LCD uses each prime to its highest power, $2^2\cdot 3^2=36$. Polynomials work the same way, with factors in place of primes. For $\frac{1}{x^2-4}$ and $\frac{1}{x^2+4x+4}$: the denominators are $(x-2)(x+2)$ and $(x+2)^2$, so the LCD is $(x-2)(x+2)^2$. Multiplying the denominators together also gives a common denominator, but usually a bigger one, which leaves more to simplify later.

To rewrite a fraction over the LCD, multiply it by a form of 1 made from the missing factors. For example, $\frac{3}{x-2}=\frac{3}{x-2}\cdot\frac{x+2}{x+2}=\frac{3(x+2)}{(x-2)(x+2)}$. The value does not change, because we multiplied by 1.

Subtracting needs parentheses. The fraction bar groups the whole numerator, so subtracting a fraction subtracts every term of its numerator: $\frac{2x+1}{x-3}-\frac{x+4}{x-3}=\frac{2x+1-(x+4)}{x-3}=\frac{x-3}{x-3}=1$ for $x\neq 3$. Without the parentheses you would get $2x+1-x+4$, which is wrong.

Simplify the result. After combining, factor the new numerator. It may share a factor with the LCD, and that factor divides out. The excluded values are still those of the original denominators.

Complex fractions, two ways. Simplify $\frac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}-\frac{1}{2}}$. *Method 1:* multiply the top and the bottom by $2x$, the LCD of all the small fractions. This is allowed because multiplying

numerator and denominator by the same nonzero quantity multiplies by 1: $\dfrac{2+x}{2-x}$.

Method 2: combine the top into $\dfrac{2+x}{2x}$ and the bottom into $\dfrac{2-x}{2x}$, then divide: $\dfrac{2+x}{2x} \cdot \dfrac{2x}{2-x} = \dfrac{2+x}{2-x}$. Both give the same answer, with $x \neq 0$ (the small fractions) and $x \neq 2$ (the main denominator).

Round trips. Since time = distance ÷ speed, a trip in two parts takes $\dfrac{d_1}{r_1} + \dfrac{d_2}{r_2}$ hours, a sum of rational expressions. A boat that goes r mph in still water travels $r - c$ mph against a current of c mph and $r + c$ mph with it.

KEY FORMULAS

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C} \qquad \frac{A}{C} - \frac{B}{C} = \frac{A-(B)}{C}$$

$$\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Practice habit. Factor every denominator before you look for the LCD, and put parentheses around any numerator you subtract. Write one step per line.

Example 1. Subtract and simplify. State the excluded values.

$$\frac{5x+2}{x^2-4} - \frac{3}{x-2}$$

Show solution

1. Factor: $x^2-4=(x-2)(x+2)$. Excluded values: $x \neq 2, -2$.
2. The LCD is $(x-2)(x+2)$. Rewrite the second fraction: $\frac{3}{x-2} = \frac{3(x+2)}{(x-2)(x+2)}$.
3. Subtract the whole numerator, in parentheses: $\frac{5x+2-3(x+2)}{(x-2)(x+2)}$.
4. Simplify the numerator: $5x+2-3x-6=2x-4=2(x-2)$.
5. Divide out $x-2$: $\frac{2(x-2)}{(x-2)(x+2)} = \frac{2}{x+2}$.
6. Check with $x=0$: original $\frac{2}{-4} - \frac{3}{-2} = -0.5+1.5=1$; answer $\frac{2}{2}=1$ ✓.

Answer: $\frac{2}{x+2}$, with $x \neq -2, 2$

Example 2. The Miller family takes their boat 12 miles up a river and then 12 miles back. The current flows at 2 mph, and the boat goes r mph in still water ($r > 2$). (a) Write the total time as a single rational expression in r. (b) Find the total time when $r = 6$.

Round trip of 12 miles each way, current 2 mph

r (mph in still water)	Upstream: 12/(r − 2) h	Downstream: 12/(r + 2) h	Total time (h)
4	6	2	8
6	3	1.5	4.5
8	2	1.2	3.2
10	1.5	1	2.5

Show solution

1. Upstream the speed is $r - 2$, so that leg takes $\dfrac{12}{r-2}$ hours. Downstream it is $r + 2$, so $\dfrac{12}{r+2}$ hours.
2. Total: $T = \dfrac{12}{r-2} + \dfrac{12}{r+2}$. The LCD is $(r-2)(r+2)$.
3. $T = \dfrac{12(r+2) + 12(r-2)}{(r-2)(r+2)} = \dfrac{12r + 24 + 12r - 24}{r^2 - 4} = \dfrac{24r}{r^2 - 4}$.
4. (b) $T = \dfrac{24 \times 6}{36 - 4} = \dfrac{144}{32} = 4.5$ hours.
5. Check leg by leg: $\dfrac{12}{4} + \dfrac{12}{8} = 3 + 1.5 = 4.5$ ✓. The table shows the same pattern for other speeds.

Answer: (a) $T = \dfrac{24r}{r^2 - 4}$ hours; (b) 4.5 hours

PRACTICE

1. Easy Subtract and simplify: $\dfrac{2x}{x-4} - \dfrac{8}{x-4}$.

   Show answer

   $\dfrac{2x - 8}{x - 4} = \dfrac{2(x-4)}{x-4} = 2$, with $x \neq 4$

2. Easy Find the LCD of $\dfrac{1}{x^2-9}$ and $\dfrac{1}{x^2+6x+9}$.

Show answer

$(x-3)(x+3)$ and $(x+3)^2$ give LCD $(x-3)(x+3)^2$

3. Easy Henry hikes 6 miles up a trail at x mph and 6 miles back down at $2x$ mph. Write his total hiking time as a single simplified expression.

Show answer

$\dfrac{6}{x}+\dfrac{6}{2x}=\dfrac{6}{x}+\dfrac{3}{x}=\dfrac{9}{x}$ hours

4. Easy Subtract: $\dfrac{5}{6x^2}-\dfrac{1}{4x}$.

Show answer

LCD $12x^2$: $\dfrac{10}{12x^2}-\dfrac{3x}{12x^2}=\dfrac{10-3x}{12x^2}$

5. Medium Subtract and simplify, stating the excluded values: $\dfrac{x}{x-3}-\dfrac{18}{x^2-9}$.

Show answer

$\dfrac{x(x+3)-18}{(x-3)(x+3)}=\dfrac{x^2+3x-18}{(x-3)(x+3)}=\dfrac{(x+6)(x-3)}{(x-3)(x+3)}=\dfrac{x+6}{x+3}$, with $x\neq -3, 3$

6. Medium Add: $\dfrac{x+1}{x^2-5x+6}+\dfrac{2}{x^2-4}$.

Show answer

LCD $(x-2)(x-3)(x+2)$; numerator $(x+1)(x+2)+2(x-3)=x^2+5x-4$; answer $\dfrac{x^2+5x-4}{(x-2)(x-3)(x+2)}$

7. Medium Add and simplify: $\dfrac{4}{x-5}+\dfrac{3}{5-x}$.

Show answer

$\dfrac{3}{5-x}=-\dfrac{3}{x-5}$, so the sum is $\dfrac{4-3}{x-5}=\dfrac{1}{x-5}$

8. Medium Subtract and simplify, stating the excluded values: $\dfrac{4x-1}{x^2-1}-\dfrac{x+2}{x^2-1}$.

Show answer

$\dfrac{4x-1-(x+2)}{x^2-1}=\dfrac{3x-3}{(x-1)(x+1)}=\dfrac{3}{x+1}$, with $x\neq -1, 1$

9. **Medium** Simplify the complex fraction and state the excluded values: $\dfrac{1-\frac{4}{x^2}}{1+\frac{2}{x}}$.

Show answer

Multiply top and bottom by x^2: $\dfrac{x^2-4}{x^2+2x} = \dfrac{(x-2)(x+2)}{x(x+2)} = \dfrac{x-2}{x}$, with $x \neq 0, -2$

10. **Medium** Ruth drives 60 miles to her grandparents' farm at r mph and drives home 10 mph faster. (a) Write her total driving time as a single rational expression. (b) Find the total time when $r = 50$.

Show answer

(a) $\dfrac{60}{r} + \dfrac{60}{r+10} = \dfrac{60(r+10)+60r}{r(r+10)} = \dfrac{120(r+5)}{r(r+10)}$ hours; (b) $\dfrac{120 \times 55}{50 \times 60} = 2.2$ hours (1.2 h + 1 h)

11. **Hard** Simplify, stating the excluded values: $\dfrac{2}{x+1} + \dfrac{3}{x-1} - \dfrac{6}{x^2-1}$.

Show answer

LCD $(x-1)(x+1)$: $\dfrac{2(x-1)+3(x+1)-6}{(x-1)(x+1)} = \dfrac{5x-5}{(x-1)(x+1)} = \dfrac{5}{x+1}$, with $x \neq -1, 1$

12. **Hard** For $f(x) = \dfrac{1}{x}$, the difference quotient from Lesson 1.6 is $\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$. Simplify it (assume $h \neq 0$).

Show answer

Multiply top and bottom by $x(x+h)$: $\dfrac{x-(x+h)}{hx(x+h)} = \dfrac{-h}{hx(x+h)} = -\dfrac{1}{x(x+h)}$

13. **Hard** Samuel can paint the farm fence alone in x hours, and his sister Ruth takes $x+2$ hours. Working together for one hour, they paint $\dfrac{1}{x} + \dfrac{1}{x+2}$ of the fence. (a) Write this as a single expression. (b) Evaluate it for $x = 4$, as a fraction and as a decimal to six significant digits.

Show answer

(a) $\dfrac{(x+2)+x}{x(x+2)} = \dfrac{2(x+1)}{x(x+2)}$; (b) $\dfrac{10}{24} = \dfrac{5}{12} \approx 0.416667$ of the fence

14. **Challenge** Grace wrote $\frac{x+1}{x-2} - \frac{x-3}{x-2} = \frac{x+1-x-3}{x-2} = \frac{-2}{x-2}$. (a) Explain her mistake. (b) Give the correct answer.

Show answer

(a) Subtracting the whole numerator means $-(x-3) = -x+3$. She wrote $-x-3$, changing the sign of the x but not of the -3. (b) $\frac{x+1-x+3}{x-2} = \frac{4}{x-2}$

15. **Challenge** Simplify and state the excluded values: $\frac{\frac{x}{x-1} - \frac{1}{x+1}}{\frac{x^2+1}{x^2-1}}$.

Show answer

Top: $\frac{x(x+1)-(x-1)}{(x-1)(x+1)} = \frac{x^2+1}{(x-1)(x+1)}$; dividing by $\frac{x^2+1}{(x-1)(x+1)}$ gives 1, with $x \neq -1, 1$ (x^2+1 is never 0)

COMMON MISTAKES

- Adding denominators: $\frac{1}{x} + \frac{1}{y} \neq \frac{2}{x+y}$. Rewrite both fractions over a common denominator, then add only the numerators.
- Forgetting parentheses when subtracting a numerator with several terms. Write $-(x+4)$, then distribute the minus sign to every term.
- Choosing an LCD without factoring, which misses shared factors and gives huge expressions. Factor each denominator and use each factor to its highest power.
- Stopping before simplifying. Factor the combined numerator; it often shares a factor with the denominator.

> **Make it a game · Factor Card LCD.** Write each factor of the denominators on separate cards, such as x - 2, x + 3 and x. For $1/(x^2 - 4) + 3/(x^2 + x - 6)$, factor the denominators and build the LCD by laying down each factor at its highest power. Then rewrite and add the fractions. Use parentheses when subtracting a numerator with several terms.

5.3 Rational Functions and Asymptotes

OBJECTIVES

- Graph $f(x) = \dfrac{a}{x-h} + k$ as a transformation of $y = \dfrac{1}{x}$ and state its domain, range and asymptotes.
- Find vertical asymptotes where the denominator is 0 and the numerator is not, and explain why the function's values grow without bound there.
- Find horizontal asymptotes by comparing degrees, and explain the rule by dividing every term by the highest power of x.
- Model average cost per item with a rational function and interpret its horizontal asymptote.

WATCH

Graphing Rational Functions (1 of 4: Understanding asymptotes)

Eddie Woo · 12:57

https://www.youtube.com/watch?v=cUoQy4UDMcs

> **WHEN TO WATCH**
> Before Example 1: what vertical and horizontal asymptotes are and why they happen.

BACKUP VIDEO

Graphing Rational Functions of the Form y=a/(x-h)+k

Mario's Math Tutoring · 11:56

https://www.youtube.com/watch?v=zZuNmMJR2kE

After Example 1: graphing y = a/(x − h) + k.

KEY TERMS

rational function

A function whose rule is a rational expression, $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials and Q is not the zero polynomial.

vertical asymptote

A vertical line $x = h$ that the graph approaches as its values grow without bound (toward ∞ or $-\infty$) when x gets close to h.

horizontal asymptote

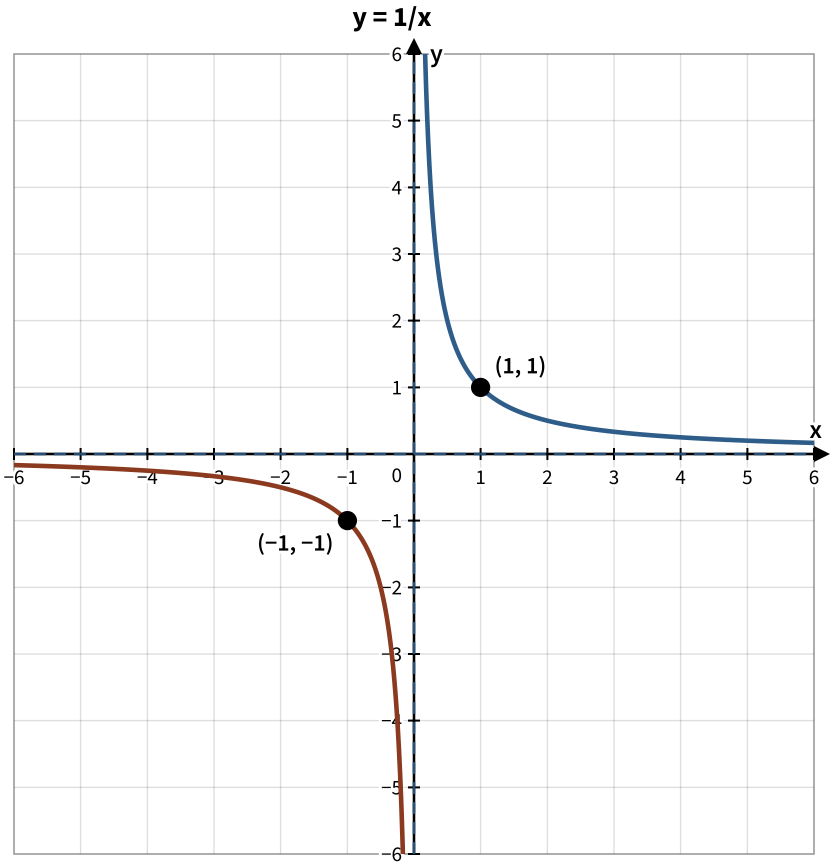
A horizontal line $y = k$ that the graph approaches as x goes far to the right or far to the left. It describes end behavior; the graph may cross it at some middle point.

average cost

Total cost divided by the number of items made: $\overline{C}(x) = \dfrac{\text{fixed cost} + (\text{cost per item})x}{x}$.

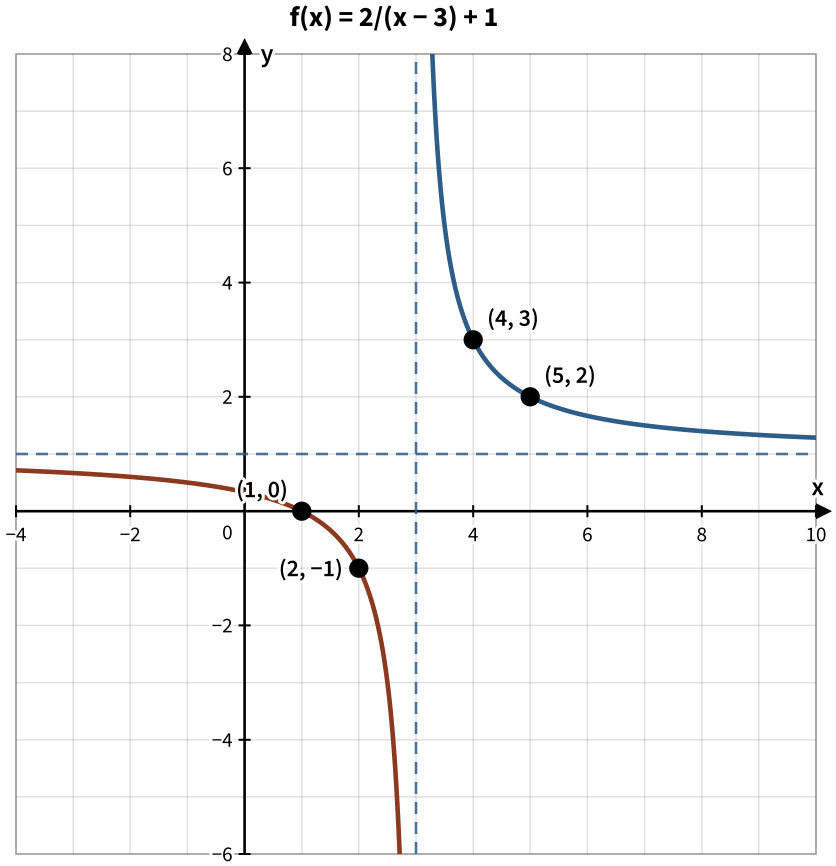
CORE IDEAS

The parent function $y = \dfrac{1}{x}$ (from Lesson 1.2) has two branches. Its key points are $(1, 1)$ and $(-1, -1)$. It is never 0, because 1 divided by any number is not 0, and it is undefined at $x = 0$. So the graph hugs the x-axis far out (the horizontal asymptote $y = 0$) and shoots up or down near the y-axis (the vertical asymptote $x = 0$).



Using the transformations of Lesson 1.2, $f(x) = \dfrac{a}{x - h} + k$ is $y = \dfrac{1}{x}$ stretched or compressed vertically by a factor of $|a|$ (a stretch if $|a| > 1$, a compression if $|a| < 1$; reflected in the x-axis if $a < 0$), shifted h units right and k units up. The asymptotes move with it: $x = h$ and $y = k$. The domain is every real number except h. The range

is every real number except k, because $\dfrac{a}{x-h}$ takes every value except 0, so $f(x)$ takes every value except k. The graph of $f(x) = \dfrac{2}{x-3} + 1$ below has asymptotes $x = 3$ and $y = 1$.



Why values blow up at a vertical asymptote. Near $x = 2$, the denominator of $\dfrac{1}{x-2}$ is tiny, and dividing 1 by a tiny number gives a huge number: $\dfrac{1}{0.001} = 1000$. On the left of 2 the denominator is a tiny negative number, so the values are hugely negative. In general, a rational function in simplest form has a vertical asymptote at each zero of the denominator, because there the numerator stays near a nonzero number while the denominator shrinks to 0. (If the numerator is also 0 there, a common factor is present; Lesson 5.4 shows that this gives a hole instead.)

Values of 1/(x − 2) near x = 2

x	1.9	1.99	1.999	2.001	2.01	2.1
1/(x − 2)	−10	−100	−1000	1000	100	10

Why horizontal asymptotes come from comparing degrees. Divide every term by the highest power of x in the denominator. For $f(x) = \dfrac{3x^2 + 1}{x^2 - 4}$, dividing by x^2 gives $f(x) = \dfrac{3 + \frac{1}{x^2}}{1 - \frac{4}{x^2}}$. As $|x|$ grows, $\frac{1}{x^2}$ and $\frac{4}{x^2}$ shrink toward 0, so $f(x)$ approaches $\dfrac{3}{1} = 3$. Indeed $f(10) \approx 3.135$, $f(100) \approx 3.0013$ and $f(1000) \approx 3.000013$. If the top has lower degree, every term on top still has a power of x in its denominator after dividing, so the top goes to 0 and the asymptote is $y = 0$. If the top has higher degree, the top grows without bound while the bottom approaches a constant, so there is no horizontal asymptote.

Inverse variation and average cost. Inverse variation $y = \dfrac{k}{x}$ is the simplest rational function: halve the speed and the travel time doubles. Average cost is another: if making birdhouses costs \$600 once for tools plus \$8 per birdhouse, the average cost per birdhouse is $\overline{C}(x) = \dfrac{600 + 8x}{x} = 8 + \dfrac{600}{x}$. As more are made, the \$600 is shared among more birdhouses, so the average cost falls toward the horizontal asymptote $y = 8$ but never reaches it.

KEY FORMULAS

$$f(x) = \frac{a}{x - h} + k: \quad \text{asymptotes } x = h \text{ and } y = k$$

$$\text{degree of top} < \text{degree of bottom}: \ y = 0$$

$$\text{equal degrees}: \ y = \frac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}}$$

$$\text{degree of top} > \text{degree of bottom}: \ \text{no horizontal asymptote}$$

$$\overline{C}(x) = \frac{F + cx}{x} = c + \frac{F}{x}$$

Practice habit. Sketch before you compute: draw the asymptotes as dashed lines first, then plot one point on each side of every vertical asymptote. Know the key points $(1, 1)$ and $(-1, -1)$ of $y = \dfrac{1}{x}$ cold.

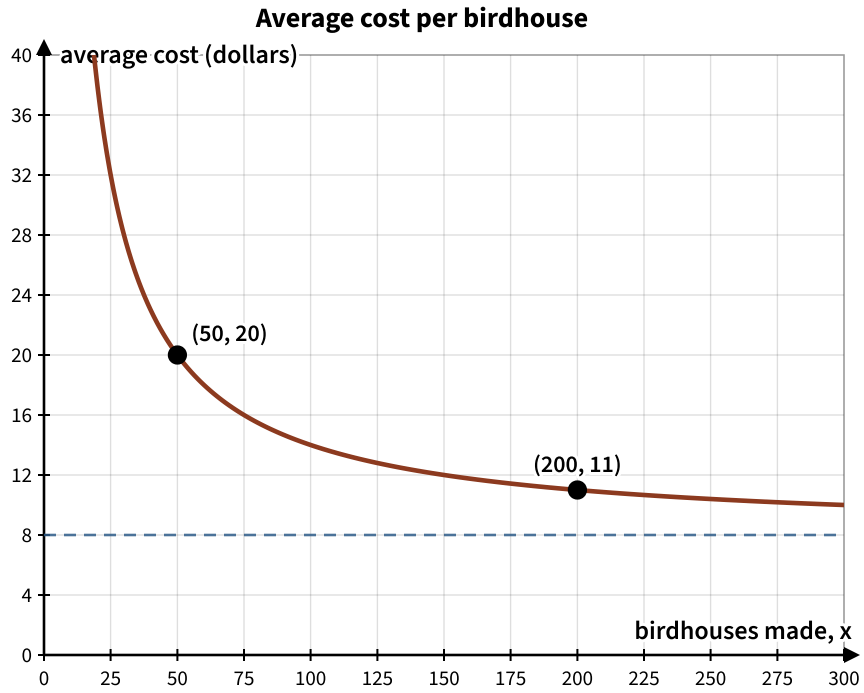
Example 1. Find the vertical and horizontal asymptotes of $f(x) = \dfrac{4x^2 + 1}{2x^2 - 5x - 3}$, and state its domain.

Show solution

1. Factor the denominator: $2x^2 - 5x - 3 = (2x + 1)(x - 3)$. Check: $2x^2 - 6x + x - 3$ ✓.
2. The denominator is 0 at $x = -\dfrac{1}{2}$ and $x = 3$, so the domain is all real numbers except these.
3. Check the numerator there: $4\left(\frac{1}{4}\right) + 1 = 2 \neq 0$ and $4(9) + 1 = 37 \neq 0$. So both are vertical asymptotes: $x = -\dfrac{1}{2}$ and $x = 3$.
4. Divide every term by x^2: $f(x) = \dfrac{4 + \frac{1}{x^2}}{2 - \frac{5}{x} - \frac{3}{x^2}}$.
5. As $|x|$ grows, the small fractions go to 0, so $f(x)$ approaches $\dfrac{4}{2} = 2$. Horizontal asymptote: $y = 2$.
6. Spot check: $f(100) = \dfrac{40001}{19497} \approx 2.05$, close to 2 ✓.

Answer: Vertical asymptotes $x = -\dfrac{1}{2}$ and $x = 3$; horizontal asymptote $y = 2$; domain: all real numbers except $-\dfrac{1}{2}$ and 3

Example 2. Grace starts a small business making wooden birdhouses. Tools and a workbench cost \$600 once, and each birdhouse uses \$8 of wood and hardware. (a) Write the average cost per birdhouse $\overline{C}(x)$ for x birdhouses. (b) Find $\overline{C}(50)$ and $\overline{C}(200)$. (c) Find the horizontal asymptote and explain what it means. (d) How many birdhouses must she make to bring the average cost down to \$10?



Show solution

1. (a) Total cost is $600 + 8x$, so $\overline{C}(x) = \dfrac{600 + 8x}{x} = 8 + \dfrac{600}{x}$.
2. (b) $\overline{C}(50) = 8 + 12 = 20$ and $\overline{C}(200) = 8 + 3 = 11$ (dollars).
3. (c) The degrees of top and bottom are equal (both 1), and the leading coefficients give $y = \dfrac{8}{1} = 8$.
4. Meaning: the average cost approaches \$8, the cost of materials, but never reaches it, because $\dfrac{600}{x}$ is always positive.
5. (d) Solve $8 + \dfrac{600}{x} = 10$: $\dfrac{600}{x} = 2$, so $x = 300$.
6. Check: $\dfrac{600 + 8(300)}{300} = \dfrac{3000}{300} = 10$ ✓.

Answer: (a) $\overline{C}(x) = \dfrac{600 + 8x}{x}$; (b) \$20 and \$11; (c) $y = 8$: the average cost approaches the \$8 materials cost; (d) 300 birdhouses

PRACTICE

1. Easy Find the vertical and horizontal asymptotes of $f(x) = \dfrac{5}{x - 4} + 1$.

   Show answer

   Vertical $x = 4$; horizontal $y = 1$

2. Easy State the domain and range of $f(x) = \dfrac{2}{x + 3} - 5$.

   Show answer

   Domain: all real numbers except -3, $(-\infty, -3) \cup (-3, \infty)$; range: all real numbers except -5, $(-\infty, -5) \cup (-5, \infty)$

3. Easy Find the vertical and horizontal asymptotes of $f(x) = \dfrac{3x - 2}{x^2 + 1}$.

   Show answer

   No vertical asymptote ($x^2 + 1$ is never 0); horizontal $y = 0$ (top degree 1 < bottom degree 2)

4. Easy Describe how to obtain the graph of $y = -\dfrac{2}{x - 1} + 3$ from the graph of $y = \dfrac{1}{x}$.

   Show answer

   Stretch vertically by a factor of 2, reflect in the x-axis, shift 1 unit right and 3 units up. (Asymptotes $x = 1$ and $y = 3$.)

5. Medium Find the vertical and horizontal asymptotes of $f(x) = \dfrac{6x + 1}{2x - 5}$.

   Show answer

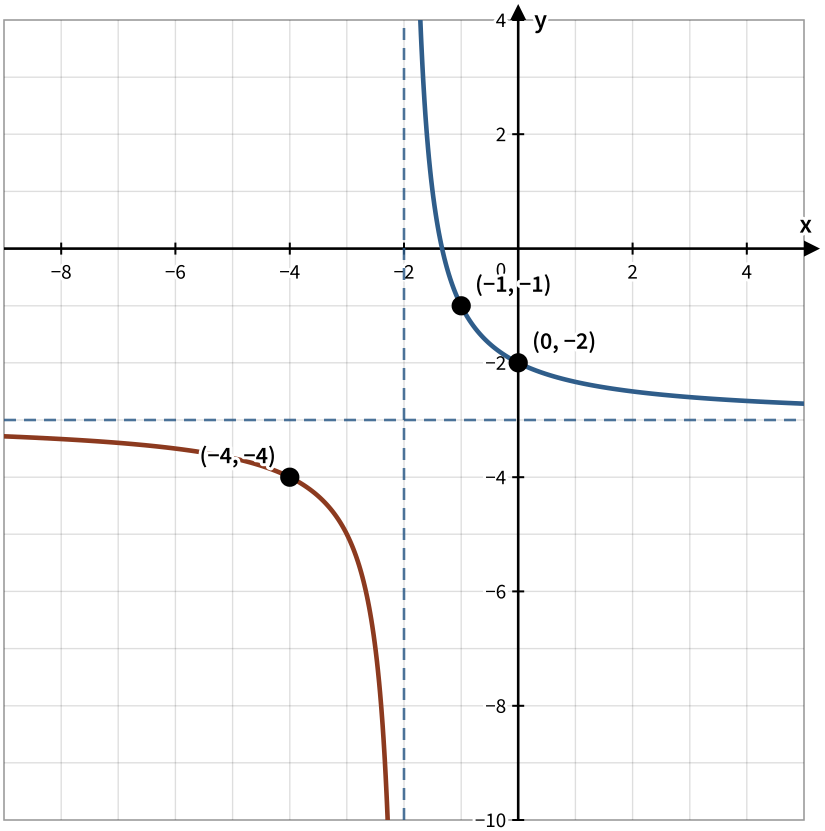
   Vertical $x = \dfrac{5}{2}$ (the numerator there is $16 \neq 0$); horizontal $y = \dfrac{6}{2} = 3$

6. Medium Find the vertical and horizontal asymptotes of $f(x) = \dfrac{x^2 + 2}{x^2 - 9}$.

   Show answer

   Vertical $x = -3$ and $x = 3$; horizontal $y = 1$

7. **Medium** The graph shows a function of the form $y = \dfrac{a}{x-h} + k$; its asymptotes are the dashed lines. Write its equation.



Show answer

Asymptotes $x = -2$ and $y = -3$ give $y = \dfrac{a}{x+2} - 3$; the point $(0, -2)$ gives $-2 = \dfrac{a}{2} - 3$, so $a = 2$: $y = \dfrac{2}{x+2} - 3$

8. **Medium** The time in minutes to fill a 1200-gallon stock tank is $t = \dfrac{1200}{r}$, where r is the flow rate in gallons per minute. (a) What do the asymptotes $r = 0$ and $t = 0$ mean in this setting? (b) How long does it take at 15 gallons per minute?

Show answer

(a) As the flow gets close to 0, the time grows without bound; as the flow gets very large, the time gets close to 0 but never reaches it. (b) 80 minutes

9. **Medium** A church youth group prints a cookbook. Setup costs \$450, and each copy costs \$3 to print. (a) Write the average cost per copy $\overline{C}(x)$. (b) Find $\overline{C}(150)$. (c) Find the horizontal asymptote. (d) How many copies make the average cost \$4.50?

Show answer

(a) $\overline{C}(x) = \dfrac{450 + 3x}{x}$; (b) \$6; (c) $y = 3$; (d) $3 + \dfrac{450}{x} = 4.5$ gives $x = 300$ copies

10. **Hard** Let $f(x) = \dfrac{5x^2 - 3x}{2x^2 + 7}$. (a) Find the horizontal asymptote. (b) Explain why, by dividing every term by x^2.

Show answer

(a) $y = \dfrac{5}{2}$. (b) $f(x) = \dfrac{5 - \frac{3}{x}}{2 + \frac{7}{x^2}}$; as $|x|$ grows, $\frac{3}{x}$ and $\frac{7}{x^2}$ approach 0, so $f(x)$ approaches $\dfrac{5}{2}$.

11. **Hard** Write a function of the form $f(x) = \dfrac{ax^2}{(x - p)(x - q)}$ with vertical asymptotes $x = -1$ and $x = 4$ and horizontal asymptote $y = 2$.

Show answer

$f(x) = \dfrac{2x^2}{(x + 1)(x - 4)}$ (the leading coefficients give $\dfrac{a}{1} = 2$)

12. **Hard** Find the vertical and horizontal asymptotes of $f(x) = \dfrac{x^3 - 1}{x^2 - 4}$.

Show answer

Vertical $x = -2$ and $x = 2$ (the numerator is -9 and 7 there); no horizontal asymptote, since the top has higher degree

13. **Hard** Owning Henry's farm truck costs \$3000 a year for insurance and registration, plus \$0.20 per mile for fuel and upkeep. (a) Find the average cost per mile if he drives 12,000 miles in a year. (b) What is the horizontal asymptote of $\overline{C}(m) = \dfrac{3000 + 0.2m}{m}$? (c) Can the average cost ever equal \$0.20 per mile?

Show answer

(a) $\dfrac{3000}{12000} + 0.20 = \$0.45$ per mile; (b) $y = 0.2$; (c) no, because $\dfrac{3000}{m} > 0$ for every $m > 0$

14. **Challenge** Rewrite $f(x) = \dfrac{2x+1}{x-3}$ in the form $\dfrac{a}{x-h} + k$ (divide $2x+1$ by $x-3$), then state its range.

Show answer

$2x+1 = 2(x-3)+7$, so $f(x) = \dfrac{7}{x-3} + 2$; range: all real numbers except 2

15. **Challenge** Does the graph of $f(x) = \dfrac{2x^2+3x}{x^2+1}$ cross its horizontal asymptote? If so, give the point. Then explain why this does not contradict the meaning of a horizontal asymptote.

Show answer

The asymptote is $y = 2$. Setting $\dfrac{2x^2+3x}{x^2+1} = 2$ gives $2x^2+3x = 2x^2+2$, so $x = \dfrac{2}{3}$: it crosses at $\left(\dfrac{2}{3}, 2\right)$. A horizontal asymptote only describes what happens as $|x|$ grows large; in the middle of the graph it may be crossed.

COMMON MISTAKES

- Setting the numerator equal to 0 to find vertical asymptotes. Vertical asymptotes come from zeros of the denominator (zeros of the numerator are x-intercepts).
- Mixing up the horizontal asymptote rules. Divide by the highest power of x in the denominator if you forget: lower degree on top gives $y = 0$; equal degrees give the ratio of leading coefficients; higher degree on top gives none.
- Reading $f(x) = \dfrac{a}{x+2} - 3$ as having asymptote $x = 2$. Write it as $\dfrac{a}{x-(-2)} - 3$: the asymptote is $x = -2$.
- Thinking a graph can never cross a horizontal asymptote. It can cross in the middle; the asymptote describes only the far left and far right.

> **Make it a game · Fence the Graph.** Draw f(x) = 1/(x - 2) + 1. The parent calls inputs near the fence: 2.1, 2.01, 2.001. Compute each output mentally and say what happens to the values. Draw the vertical asymptote as a dashed fence, then the horizontal asymptote. State the domain and range. Ten correct outputs under a minute earns a gold star.

5.4 Graphing Rational Functions

OBJECTIVES

- Simplify a rational function first and explain why a common factor gives a hole rather than a vertical asymptote.
- Find the intercepts, holes, vertical asymptotes and horizontal or slant asymptotes of a rational function.
- Find a slant asymptote by long division and explain why the graph approaches it.
- Use a sign chart and a few plotted points to sketch a rational function and to tell where it is above or below the x-axis.

WATCH

Graphing Rational Functions (2 of 4: Overall strategy)

Eddie Woo · 7:19

https://www.youtube.com/watch?v=X6QDgb6yMZ4

> **WHEN TO WATCH**
> Before Example 1: an overall strategy for sketching a rational function.

BACKUP VIDEO

Holes, Removable Discontinuities, Graphing Rational Functions

Mario's Math Tutoring · 3:57

https://www.youtube.com/watch?v=irSg7rstDf0

After Example 1: holes, where a factor cancels.

KEY TERMS

hole (removable discontinuity)

A single missing point on a graph, at an x-value where a common factor of the numerator and denominator is 0. It is drawn as an open circle.

slant (oblique) asymptote

A slanted line $y = mx + b$ that the graph approaches as $|x|$ grows. It occurs when the numerator's degree is exactly one more than the denominator's.

sign chart

A number line marked with the zeros of the numerator and denominator, showing whether the function is positive (+) or negative (−) on each interval between them.

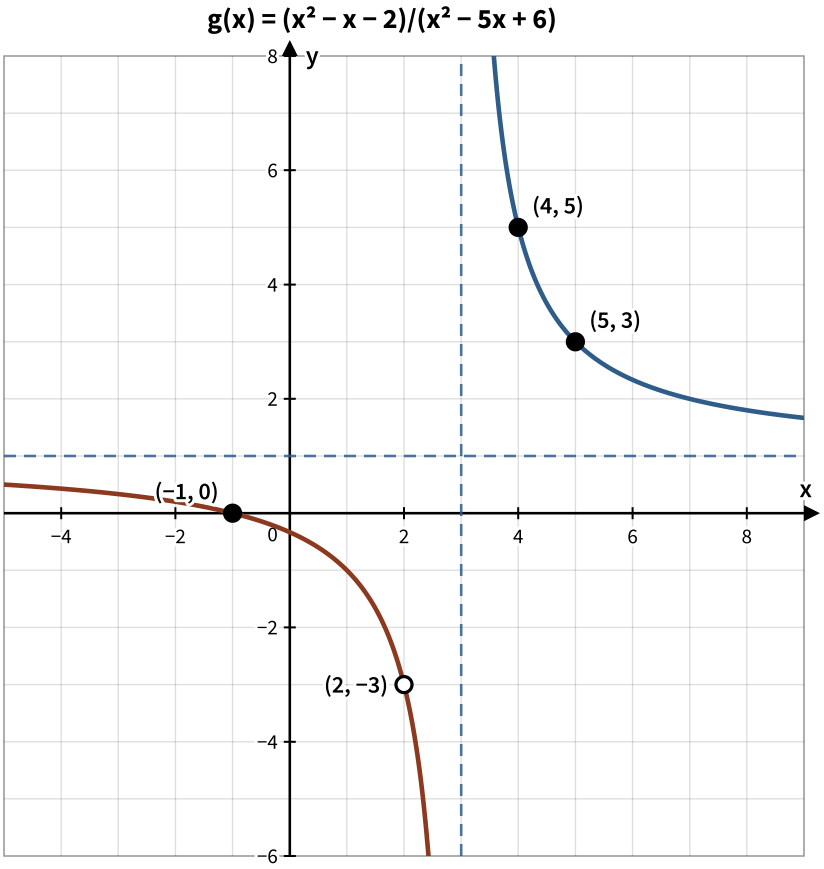
zero of a rational function

An x-value where the function equals 0; after simplifying, these are the zeros of the numerator. Each gives an x-intercept.

CORE IDEAS

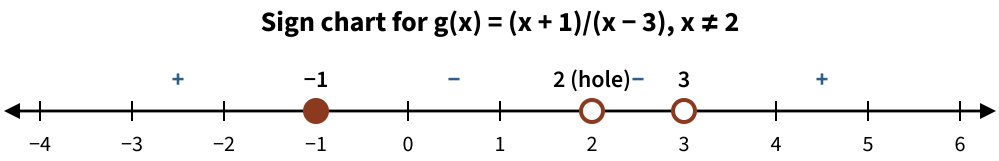
Graphing a rational function is a checklist, and step one is always: **factor and simplify**. Factor the numerator and denominator, note every excluded value, and divide out common factors. Then find, in order: holes, vertical asymptotes, intercepts, the horizontal or slant asymptote, the sign on each interval, and a few points.

Why a common factor gives a hole. Take $g(x) = \dfrac{x^2 - x - 2}{x^2 - 5x + 6} = \dfrac{(x-2)(x+1)}{(x-2)(x-3)}$. For every $x \neq 2$, $g(x) = \dfrac{x+1}{x-3}$, so the two graphs are identical except at $x = 2$, where g is undefined. Near $x = 2$ the values do not blow up: they approach $\dfrac{2+1}{2-3} = -3$. So the graph has a single missing point, a hole at $(2, -3)$. A vertical asymptote happens only where the simplified denominator is 0, here at $x = 3$, because only there does a nonzero numerator get divided by a number shrinking to 0.

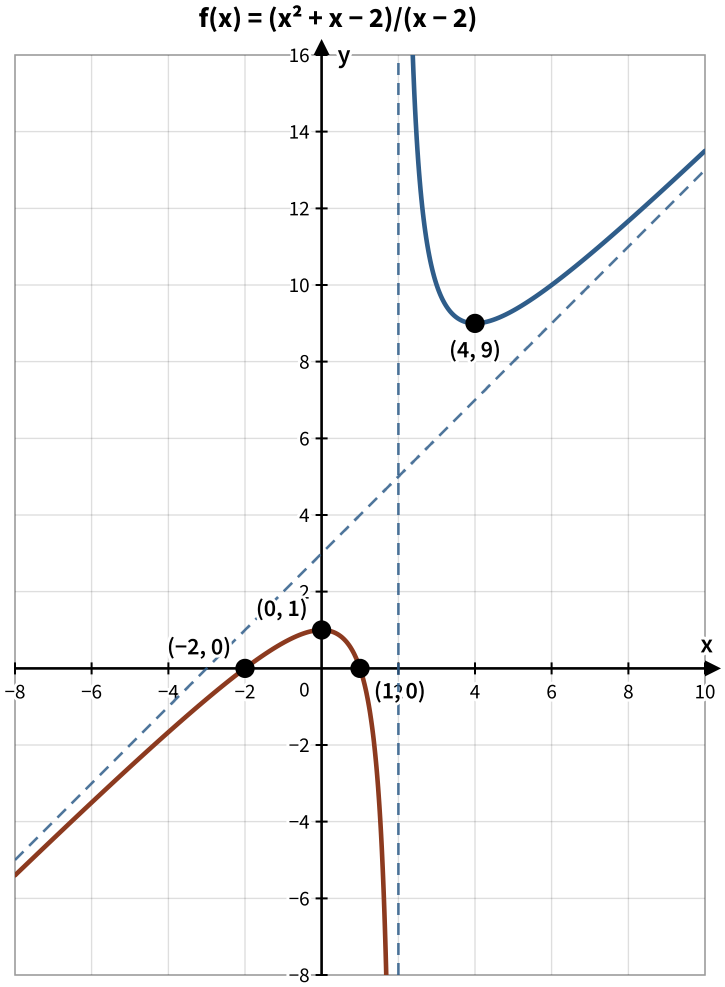


Intercepts. The x-intercepts are the zeros of the simplified numerator (not at a hole): for g, $x + 1 = 0$ gives $(-1, 0)$. The y-intercept is $g(0)$, if 0 is in the domain: $g(0) = \dfrac{1}{-3}$, so $\left(0, -\dfrac{1}{3}\right)$. Since the degrees are equal, the horizontal asymptote of g is $y = \dfrac{1}{1} = 1$.

Sign chart. A rational function can change sign only where its numerator or denominator is 0, because the factors keep their signs in between. Mark those numbers on a number line and test one value in each interval. For g: at $x = -2$, $\dfrac{-1}{-5} > 0$; at $x = 0$, $\dfrac{1}{-3} < 0$; at $x = 4$, $\dfrac{5}{1} > 0$. So the graph is above the x-axis on $(-\infty, -1)$ and $(3, \infty)$ and below it on $(-1, 3)$ (with the hole at 2 missing).



Slant asymptotes. When the numerator's degree is one more than the denominator's, long division (Lesson 4.3) gives $f(x) = q(x) + \dfrac{r(x)}{d(x)}$ with $q(x)$ linear and $r(x)$ of lower degree than $d(x)$. By the rule of Lesson 5.3, $\dfrac{r(x)}{d(x)}$ approaches 0 as $|x|$ grows, so the graph approaches the line $y = q(x)$. For example, $x^2 + x - 2 = (x - 2)(x + 3) + 4$, so $\dfrac{x^2 + x - 2}{x - 2} = x + 3 + \dfrac{4}{x - 2}$: the slant asymptote is $y = x + 3$, and the vertical asymptote is $x = 2$. Because $\dfrac{4}{x - 2}$ is positive for $x > 2$, the right branch lies above the line; the left branch lies below it.



Sketching. Draw the asymptotes as dashed lines, plot the intercepts and any hole (open circle), then use the sign chart and one or two plotted points in each region to see the shape. A graphing tool is a good way to check your sketch, but the features come from the algebra.

KEY FORMULAS

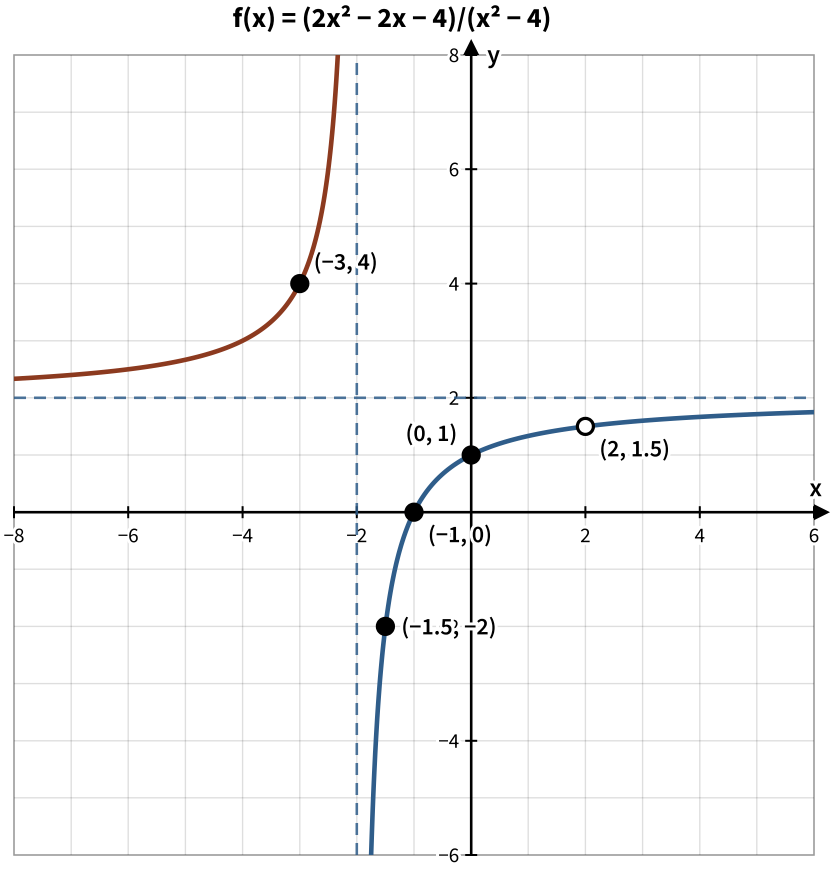
$$\text{common factor } (x - c) \text{ in top and bottom} \Rightarrow \text{hole at } x = c$$

$$\text{zero of the simplified denominator} \Rightarrow \text{vertical asymptote}$$

$$f(x) = q(x) + \frac{r(x)}{d(x)},\ \deg r < \deg d \Rightarrow \text{slant asymptote } y = q(x) \text{ when } q \text{ is linear}$$

Practice habit. Factor before anything else, then work the checklist in order: domain, holes, vertical asymptotes, intercepts, horizontal or slant asymptote, sign chart, a few points. Sketch before you trust a graphing tool.

Example 1. Graph $f(x) = \dfrac{2x^2 - 2x - 4}{x^2 - 4}$. Find any hole, the vertical and horizontal asymptotes, and the intercepts.



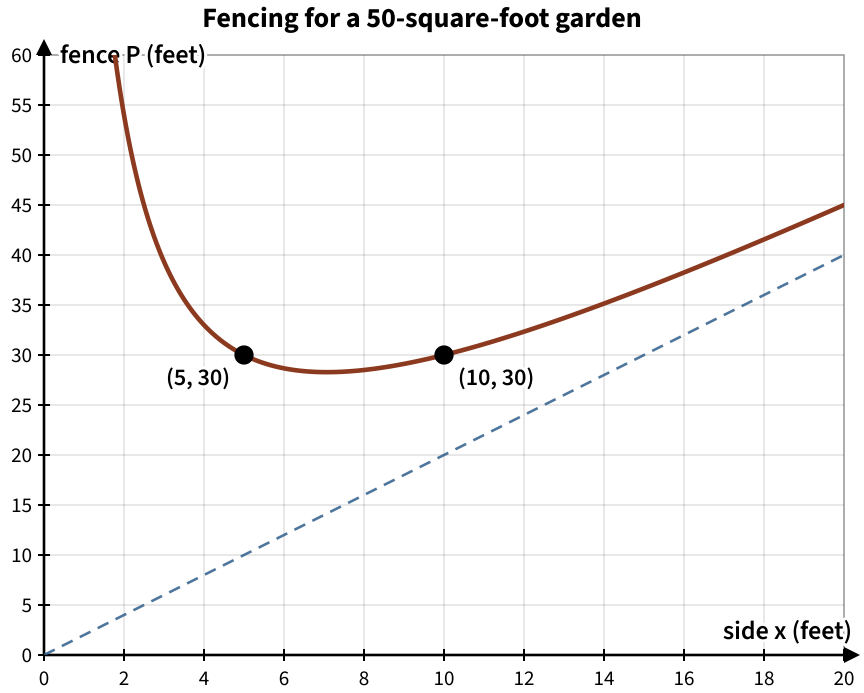
Show solution

1. Factor: $f(x) = \dfrac{2(x^2 - x - 2)}{x^2 - 4} = \dfrac{2(x-2)(x+1)}{(x-2)(x+2)}$. Domain: $x \neq -2, 2$.
2. Simplify: $f(x) = \dfrac{2(x+1)}{x+2}$ for $x \neq 2$.
3. Hole at $x = 2$: $y = \dfrac{2(3)}{4} = 1.5$, so an open circle at $(2, 1.5)$.
4. Vertical asymptote where the simplified denominator is 0: $x = -2$ (the numerator there is $-2 \neq 0$).
5. Equal degrees: horizontal asymptote $y = \dfrac{2}{1} = 2$.

6. x-intercept: $2(x+1)=0$ gives $(-1,0)$. y-intercept: $f(0)=\dfrac{2}{2}=1$, so $(0,1)$. (Check in the original: $\dfrac{-4}{-4}=1$ ✓.)
7. Sign chart with -2 and -1: $f(-3)=\dfrac{-4}{-1}=4>0$; $f(-1.5)=\dfrac{-1}{0.5}=-2<0$; $f(0)=1>0$.
8. Plot the points, draw the dashed asymptotes and the open circle, and connect each branch.

Answer: Hole $(2,1.5)$; vertical asymptote $x=-2$; horizontal asymptote $y=2$; x-intercept $(-1,0)$; y-intercept $(0,1)$

Example 2. Henry wants a rectangular garden with an area of 50 square feet, fenced on all four sides. One side is x feet. (a) Write the perimeter $P(x)$ as a single rational function. (b) Find the vertical and slant asymptotes and explain what the slant asymptote means. (c) Find $P(5)$ and $P(10)$, and explain why they are equal.



Show solution

1. The other side is $\frac{50}{x}$ feet, so $P(x) = 2x + 2 \cdot \frac{50}{x} = 2x + \frac{100}{x}$.
2. (a) Over a common denominator: $P(x) = \frac{2x^2 + 100}{x}$.
3. (b) The denominator is 0 at $x = 0$ and the numerator is 100 there: vertical asymptote $x = 0$.
4. Dividing, $P(x) = 2x + \frac{100}{x}$, and $\frac{100}{x}$ approaches 0 as x grows: slant asymptote $y = 2x$.
5. Meaning: a very long, thin garden needs about twice its long side in fence, because the two short sides add almost nothing.
6. (c) $P(5) = 10 + 20 = 30$ and $P(10) = 20 + 10 = 30$ feet.
7. They are equal because a 5-by-10 garden and a 10-by-5 garden are the same rectangle turned around. (The graph's low point lies between them, near

$x \approx 7$, close to a square.)

Answer: (a) $P(x) = \dfrac{2x^2 + 100}{x}$; (b) vertical $x = 0$, slant $y = 2x$; (c) 30 feet each

PRACTICE

1. Easy Find the hole in the graph of $f(x) = \dfrac{x^2 - 9}{x - 3}$.

   Show answer

   $f(x) = x + 3$ for $x \neq 3$; hole at $(3, 6)$

2. Easy Find the x- and y-intercepts of $f(x) = \dfrac{x - 4}{x + 2}$.

   Show answer

   x-intercept $(4, 0)$; y-intercept $\left(0, \dfrac{-4}{2}\right) = (0, -2)$

3. Easy For $f(x) = \dfrac{x + 5}{(x + 5)(x - 1)}$, decide whether $x = -5$ and $x = 1$ give holes or vertical asymptotes. Give the coordinates of any hole.

   Show answer

   $f(x) = \dfrac{1}{x - 1}$ for $x \neq -5$: hole at $\left(-5, -\dfrac{1}{6}\right)$; vertical asymptote $x = 1$

4. Easy Find the slant asymptote of $f(x) = \dfrac{x^2 + 3x + 5}{x + 1}$.

   Show answer

   $x^2 + 3x + 5 = (x + 1)(x + 2) + 3$, so $f(x) = x + 2 + \dfrac{3}{x + 1}$: slant asymptote $y = x + 2$

5. Medium For $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - x - 12}$, find any hole, the asymptotes and the intercepts.

   Show answer

   $\dfrac{(x + 3)(x - 1)}{(x + 3)(x - 4)} = \dfrac{x - 1}{x - 4}$: hole $\left(-3, \dfrac{4}{7}\right)$; vertical $x = 4$; horizontal $y = 1$; x-intercept $(1, 0)$; y-intercept $\left(0, \dfrac{1}{4}\right)$

6. **Medium** Use a sign chart to find where the graph of $f(x) = \dfrac{(x-1)(x+2)}{x-3}$ is above the x-axis. Answer in interval notation.

Show answer

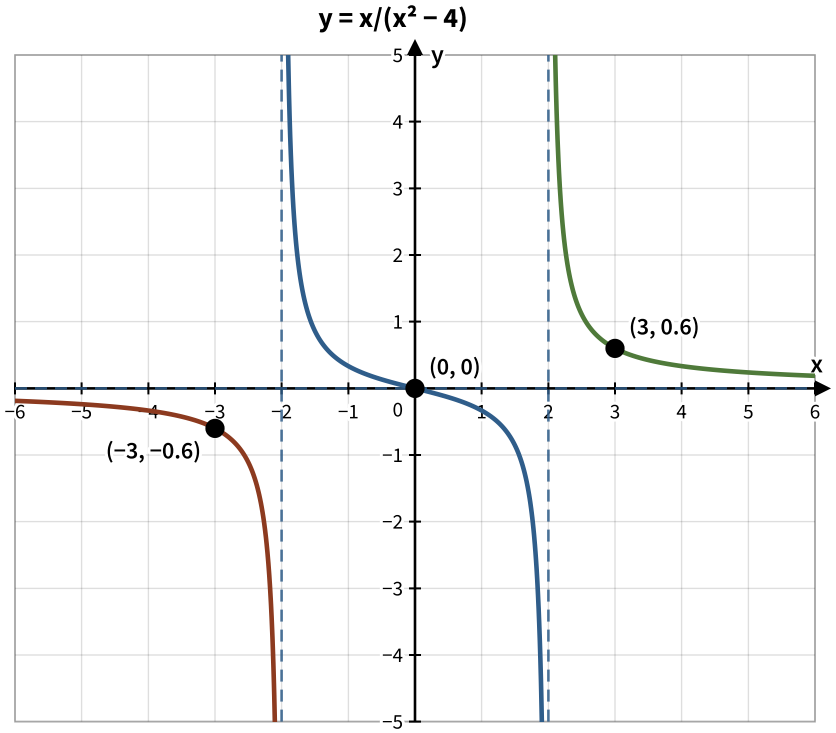
Critical numbers $-2, 1, 3$; test values give $f(-3) < 0$, $f(0) = \dfrac{2}{3} > 0$, $f(2) = -4 < 0$, $f(4) = 18 > 0$; above on $(-2, 1) \cup (3, \infty)$

7. **Medium** For $f(x) = \dfrac{2x^2 - 3x - 5}{x - 3}$, find the vertical asymptote, the slant asymptote and the intercepts.

Show answer

$2x^2 - 3x - 5 = (x-3)(2x+3) + 4$; vertical $x = 3$; slant $y = 2x + 3$; x-intercepts from $(2x-5)(x+1) = 0$: $(-1, 0)$ and $\left(\dfrac{5}{2}, 0\right)$; y-intercept $\left(0, \dfrac{5}{3}\right)$

8. **Medium** The graph of $f(x) = \dfrac{x}{x^2 - 4}$ is shown, with its asymptotes dashed. Use it to solve $f(x) > 0$. Answer in interval notation.



Show answer

The graph is above the x-axis on $(-2, 0) \cup (2, \infty)$

9. Medium A boat goes 24 miles upstream and 24 miles back on a river whose current is 3 mph. Its speed in still water is r mph. (a) Write the total time $T(r)$ as a single rational function. (b) Which vertical asymptote matters here, and what does it mean? (c) Find $T(9)$.

Show answer

(a) $T(r) = \dfrac{24}{r-3} + \dfrac{24}{r+3} = \dfrac{48r}{r^2-9}$; (b) $r = 3$: as the boat's speed falls toward the current's speed, it barely moves upstream, so the time grows without bound; (c) $\dfrac{432}{72} = 6$ hours (4 h up, 2 h back)

10. Hard Ruth fences a rectangular pen of 72 square feet against the long wall of a barn, so fence is needed on only three sides. The two sides perpendicular to the barn are each x feet. (a) Write the fence length $F(x)$ as a single rational function. (b) Find its vertical and slant asymptotes. (c) Find $F(6)$.

Show answer

(a) $F(x) = 2x + \dfrac{72}{x} = \dfrac{2x^2+72}{x}$; (b) vertical $x = 0$, slant $y = 2x$; (c) $12 + 12 = 24$ feet

11. Hard For $f(x) = \dfrac{x^3 - x}{x^3 + 2x^2 - 3x}$, find all holes, the asymptotes and the intercepts.

Show answer

$\dfrac{x(x-1)(x+1)}{x(x+3)(x-1)} = \dfrac{x+1}{x+3}$: holes $\left(0, \dfrac{1}{3}\right)$ and $\left(1, \dfrac{1}{2}\right)$; vertical $x = -3$; horizontal $y = 1$; x-intercept $(-1, 0)$; no y-intercept (0 is not in the domain)

12. Hard A wooden storage bin with an open top has a square base of side x feet and holds 32 cubic feet, so its height is $\dfrac{32}{x^2}$ feet. The wood needed is $S(x) = x^2 + 4x \cdot \dfrac{32}{x^2}$ square feet. (a) Write $S(x)$ as a single rational function. (b) Find the vertical asymptote. (c) Find $S(2)$, $S(4)$ and $S(8)$.

Show answer

(a) $S(x) = x^2 + \dfrac{128}{x} = \dfrac{x^3+128}{x}$; (b) $x = 0$ (there is no horizontal or slant asymptote, since the top's degree is two more than the bottom's); (c) 68, 48 and 80 square feet, so of these the 4-foot base uses the least wood

13. **Challenge** Explain why $f(x) = \dfrac{x^2 - 9}{x - 3}$ has a hole at $x = 3$, while $g(x) = \dfrac{x^2 + 9}{x - 3}$ has a vertical asymptote at $x = 3$.

Show answer

$f(x) = \dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so near 3 the values approach 6 and only the single point $(3, 6)$ is missing. For g, the numerator near $x = 3$ is close to 18, not 0, while the denominator shrinks to 0, so the quotient grows without bound: a vertical asymptote.

14. **Challenge** Find the value of k for which $f(x) = \dfrac{x^2 + kx - 6}{x - 2}$ has a hole (not a vertical asymptote) at $x = 2$. Then give the coordinates of the hole.

Show answer

The numerator must be 0 at $x = 2$: $4 + 2k - 6 = 0$, so $k = 1$. Then $\dfrac{(x+3)(x-2)}{x-2} = x + 3$, and the hole is $(2, 5)$.

COMMON MISTAKES

- Calling every zero of the original denominator a vertical asymptote. Simplify first: a factor that divides out gives a hole.
- Finding the hole's y-value from the original expression (which gives $\frac{0}{0}$). Substitute into the simplified expression instead.
- Listing an x-intercept at a hole. Zeros of the numerator count only if they are not also zeros of the denominator.
- Looking for a horizontal asymptote when the numerator's degree is one more. Divide instead: the quotient gives the slant asymptote.

> **Make it a game · Hole or Wall?.** The parent writes rational functions, such as (x - 1)(x + 2)/((x - 1)(x - 3)). Shout 'hole' or 'wall' for each factor of the denominator, then give the coordinates, using a coin for each hole. Add the intercepts and decide whether there is a horizontal or slant asymptote by comparing degrees. Beat the clock on five functions.

5.5 Solving Rational Equations

OBJECTIVES

- Solve rational equations by multiplying both sides by the LCD, and check every solution against the excluded values and in the original equation.
- Explain why extraneous solutions can appear when both sides are multiplied by an expression that can be 0.
- Solve a rational equation graphically as the intersection of two graphs.
- Solve proportion, work and travel problems with rational equations.

WATCH

Solving Rational Equations (Easy Method)

Mario's Math Tutoring · 11:14

https://www.youtube.com/watch?v=q7EBOc8uFXI

> **WHEN TO WATCH**
> Before Example 1: solving rational equations by clearing denominators.

Note: Eddie Woo teaches in Australia: the backup's first examples are radical equations; the same checking idea applies here.

BACKUP VIDEO

Extraneous Solutions

Eddie Woo · 6:28

https://www.youtube.com/watch?v=6Q9uyDiKblU

After Example 1: why some answers are extraneous and must be checked.

KEY TERMS

rational equation

An equation that contains at least one rational expression with a variable in a denominator, such as $\frac{3}{x} + \frac{1}{2} = \frac{5}{x}$.

extraneous solution

A number produced by the solving steps that does not satisfy the original equation, usually because it is an excluded value.

proportion

An equation stating that two ratios are equal, $\frac{a}{b} = \frac{c}{d}$; it can be solved by cross multiplying, $ad = bc$, with $b, d \neq 0$.

work rate

The fraction of a job done per unit of time. Someone who finishes a job in a hours works at $\frac{1}{a}$ of the job per hour.

CORE IDEAS

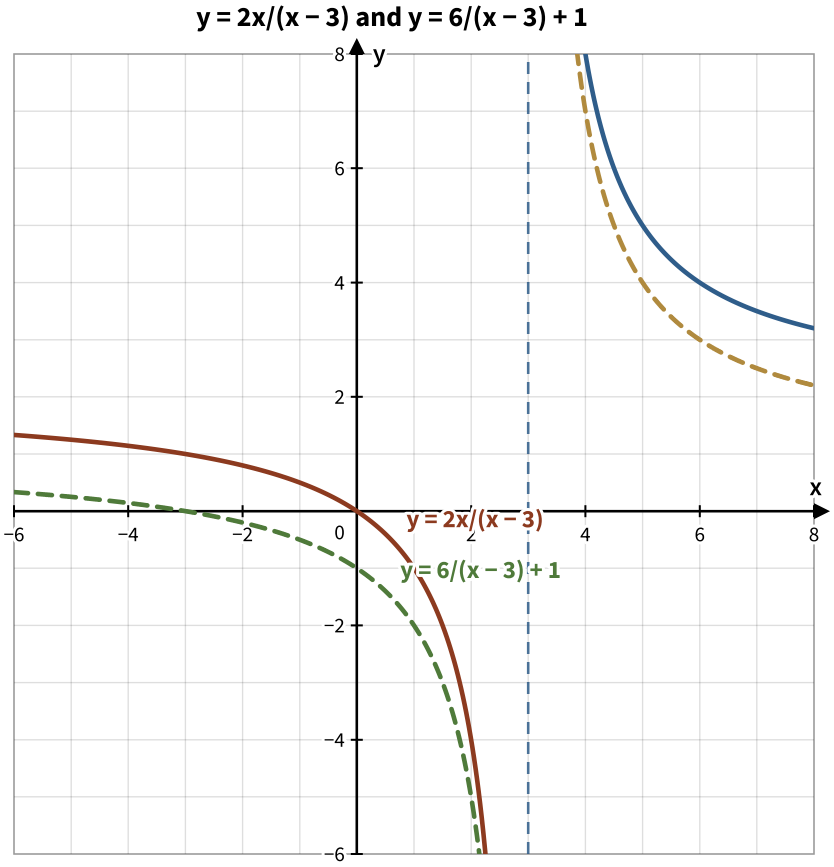
To solve an equation with fractions, clear the fractions: multiply every term on both sides by the LCD of all the denominators. What is left is a polynomial equation (linear or quadratic) that you already know how to solve. Before you multiply, write down the excluded values.

For example, solve $\frac{3}{x} + \frac{1}{2} = \frac{5}{x}$. Excluded: $x \neq 0$. Multiply every term by $2x$: $6 + x = 10$, so $x = 4$. Check: $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ ✓.

Why extraneous solutions appear. Multiplying both sides of an equation by the same nonzero number gives an equivalent equation: no solutions are gained or lost. But the LCD contains x, and for some values of x it equals 0. Multiplying both sides by 0 turns any equation into $0 = 0$, which is true. So the new polynomial equation can be true at a number where the original equation was not even defined. Example: $\frac{x}{x-2} = \frac{2}{x-2} + 1$. Multiplying by $x - 2$ gives $x = 2 + (x - 2)$, that is $x = x$, true for every x, yet $x = 2$ makes the original undefined, so the solution set is all real numbers except 2. In other problems the polynomial equation has a few solutions and one of them is an excluded value; that one is extraneous and must be thrown out.

That is why checking is part of the solution, not an extra. Any answer that equals an excluded value is rejected; every other answer should be substituted into the ORIGINAL equation.

Solving graphically. The solutions of $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ meet. A graph can show why an algebraic answer is extraneous: at that x one graph has a hole or an asymptote, so there is no intersection point there.



Work problems. If Samuel fills a stock tank in a hours, he does $\dfrac{1}{a}$ of the job each hour. Rates add when people (or pumps) work together, so if together they need t hours, $\dfrac{t}{a} + \dfrac{t}{b} = 1$: each one's rate times the time is the part of the job done, and the parts make one whole job.

Travel problems. Use time = distance ÷ speed. When two trips take the same time, or their times add to a known total, set up an equation with those rational expressions. Against a current of c mph a boat that goes r mph in still water travels $r - c$ mph; with the current it travels $r + c$ mph.

KEY FORMULAS

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc \quad (b, d \neq 0)$$

$$\frac{t}{a} + \frac{t}{b} = 1 \quad \text{(two workers who alone take } a \text{ and } b \text{ hours, together } t \text{ hours)}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Practice habit. State the excluded values first, and check every answer in the ORIGINAL equation. An answer that equals an excluded value is crossed out, no matter how nicely it came out.

WORKED EXAMPLES

Example 1. Solve $\dfrac{x}{x-3} - \dfrac{1}{x+3} = \dfrac{18}{x^2-9}$. Check for extraneous solutions.

Show solution

1. Factor: $x^2 - 9 = (x-3)(x+3)$. Excluded values: $x \neq 3, -3$. The LCD is $(x-3)(x+3)$.
2. Multiply every term by the LCD: $x(x+3) - (x-3) = 18$.
3. Expand: $x^2 + 3x - x + 3 = 18$, so $x^2 + 2x - 15 = 0$.
4. Factor: $(x+5)(x-3) = 0$, so $x = -5$ or $x = 3$.
5. $x = 3$ is an excluded value, so it is extraneous: reject it.
6. Check $x = -5$ in the original: $\dfrac{-5}{-8} - \dfrac{1}{-2} = \dfrac{5}{8} + \dfrac{4}{8} = \dfrac{9}{8}$, and $\dfrac{18}{25-9} = \dfrac{18}{16} = \dfrac{9}{8}$ ✓.

Answer: $x = -5$ ($x = 3$ is extraneous)

Example 2. On the Miller farm, a large pump can fill the cattle's stock tank in 3 hours. With a second, smaller pump running too, the tank fills in 2 hours. How long would the smaller pump take alone?

Show solution

1. Let s be the hours the small pump needs alone ($s > 0$). Its rate is $\dfrac{1}{s}$ tank per hour; the large pump's rate is $\dfrac{1}{3}$.
2. In 2 hours together they fill one tank: $\dfrac{2}{3} + \dfrac{2}{s} = 1$.
3. Multiply every term by the LCD $3s$: $2s + 6 = 3s$.
4. Solve: $s = 6$. It is not excluded ($s \neq 0$).
5. Check: in 2 hours the large pump fills $\dfrac{2}{3}$ and the small pump $\dfrac{2}{6} = \dfrac{1}{3}$; together $\dfrac{2}{3} + \dfrac{1}{3} = 1$ tank ✓.

Answer: 6 hours

PRACTICE

1. **Easy** Solve $\frac{4}{x} + \frac{1}{3} = \frac{2}{x}$.

   Show answer

   Multiply by $3x$: $12 + x = 6$, so $x = -6$; check: $-\frac{2}{3} + \frac{1}{3} = -\frac{1}{3}$ and $\frac{2}{-6} = -\frac{1}{3}$ ✓

2. **Easy** Solve the proportion $\frac{x+2}{5} = \frac{x-1}{2}$.

   Show answer

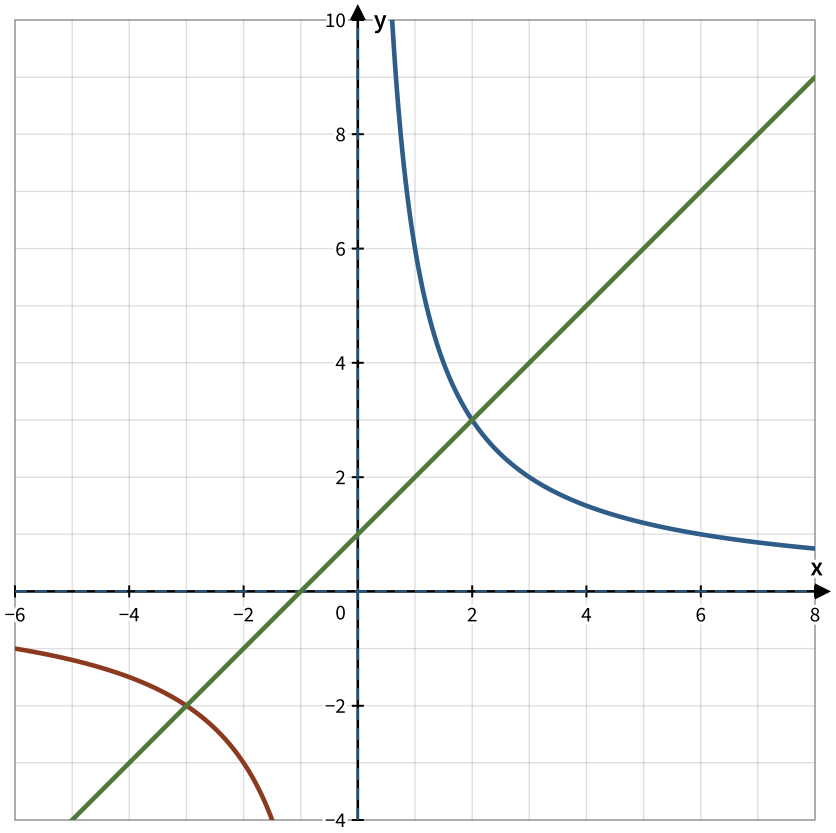
   $2(x+2) = 5(x-1)$, so $2x + 4 = 5x - 5$ and $x = 3$; check: $\frac{5}{5} = \frac{2}{2}$ ✓

3. **Easy** A recipe for Grace's bakery uses 3 cups of flour for every 2 loaves. Solve the proportion $\frac{3}{2} = \frac{x}{14}$ to find how much flour 14 loaves need.

   Show answer

   $2x = 42$, so $x = 21$ cups

4. Easy Solve $\dfrac{6}{x} = x + 1$ algebraically. Then check with the graph, which shows $y = \dfrac{6}{x}$ and $y = x + 1$.



Show answer

Multiply by x: $6 = x^2 + x$, so $x^2 + x - 6 = (x + 3)(x - 2) = 0$: $x = -3$ or $x = 2$; the graphs cross at the grid points $(-3, -2)$ and $(2, 3)$ ✓

5. Medium Solve $\dfrac{x}{x - 4} = \dfrac{4}{x - 4} + 3$.

Show answer

Excluded $x \neq 4$. Multiply by $x - 4$: $x = 4 + 3(x - 4)$, so $x = 3x - 8$ and $x = 4$. But 4 is excluded, so there is no solution.

6. Medium Solve $\dfrac{2}{x - 1} + \dfrac{3}{x + 1} = \dfrac{4}{x^2 - 1}$.

Show answer

Excluded $x \neq \pm 1$. Multiply by $(x - 1)(x + 1)$: $2(x + 1) + 3(x - 1) = 4$, so $5x - 1 = 4$ and $x = 1$, which is excluded. No solution.

7. **Medium** Solve $x - \dfrac{8}{x} = 2$.

Show answer

Multiply by x: $x^2 - 8 = 2x$, so $(x-4)(x+2) = 0$: $x = -2$ or $x = 4$; check: $-2 + 4 = 2$ ✓ and $4 - 2 = 2$ ✓

8. **Medium** Samuel can paint the barn in 12 hours, and his father can paint it in 6 hours. How long will it take them working together?

Show answer

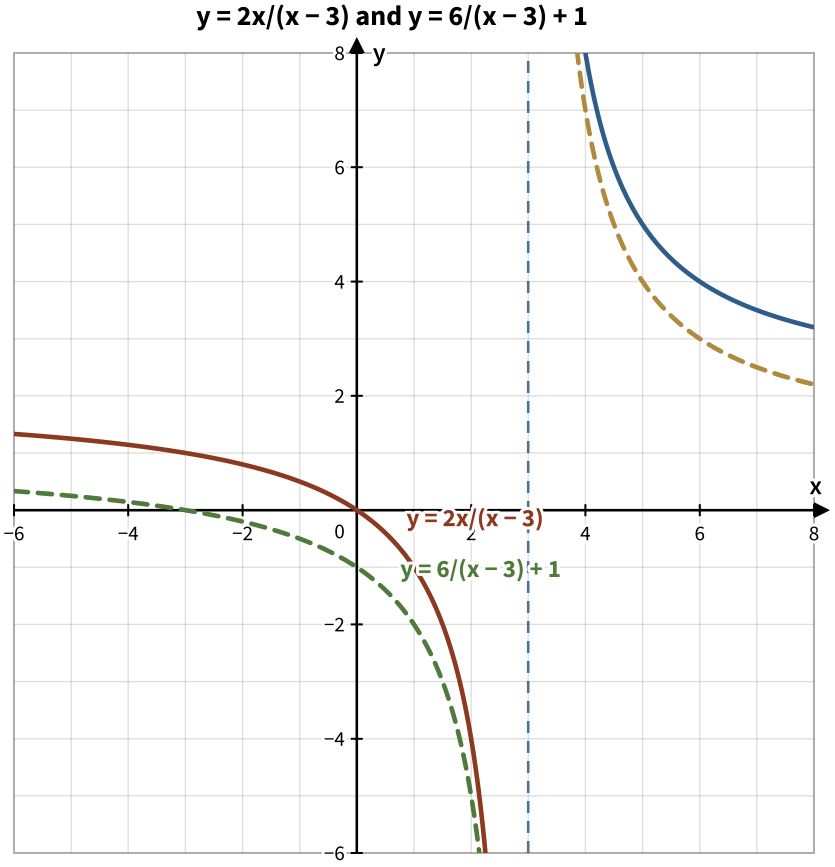
$\dfrac{t}{12} + \dfrac{t}{6} = 1$; multiply by 12: $t + 2t = 12$, so $t = 4$ hours

9. **Medium** A boat goes 15 miles downstream in the same time it takes to go 9 miles upstream. The current is 2 mph. Find the boat's speed in still water.

Show answer

$\dfrac{15}{r+2} = \dfrac{9}{r-2}$; $15(r-2) = 9(r+2)$, so $6r = 48$ and $r = 8$ mph; check: $\dfrac{15}{10} = 1.5$ h and $\dfrac{9}{6} = 1.5$ h ✓

10. Hard Solve $\dfrac{2x}{x-3} = \dfrac{6}{x-3} + 1$. Then use the graph to explain your answer.



Show answer

Excluded $x \neq 3$. Multiply by $x-3$: $2x = 6 + x - 3$, so $x = 3$, which is excluded: no solution. On the graph, $\dfrac{2x}{x-3} = 2 + \dfrac{6}{x-3}$ and $\dfrac{6}{x-3} + 1$ differ by exactly 1 everywhere, so the curves never meet.

11. Hard Ruth drives 150 miles to a church conference on the highway, then 40 miles on country roads where she averages 20 mph less. The whole trip takes 3 hours. Find her highway speed to the nearest tenth of a mile per hour.

Show answer

$\dfrac{150}{r} + \dfrac{40}{r-20} = 3$; multiply by $r(r-20)$: $150(r-20) + 40r = 3r(r-20)$, so $3r^2 - 250r + 3000 = 0$; $r = \dfrac{250 \pm \sqrt{62500 - 36000}}{6} = \dfrac{250 \pm 162.79}{6}$, giving $r \approx 68.8$ or $r \approx 14.5$. Since $r - 20$ must be positive, her highway speed is about 68.8 mph (check: $2.18 + 0.82 \approx 3$ h ✓)

12. Hard Working together, Henry and Grace can weed the church garden in 4 hours. Alone, Henry takes 6 hours longer than Grace. How long does Grace take alone?

Show answer

$\frac{4}{g}+\frac{4}{g+6}=1$; multiply by $g(g+6)$: $4(g+6)+4g=g^2+6g$, so $g^2-2g-24=0$ and $(g-6)(g+4)=0$. Reject $g=-4$ (a time cannot be negative): Grace takes 6 hours (Henry 12; check: $\frac{4}{6}+\frac{4}{12}=1$ ✓)

13. Hard Solve $\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5}$.

Show answer

Multiply by $5x(x+2)$: $15x-5(x+2)=x(x+2)$, so $x^2-8x+10=0$ and $x=4\pm\sqrt{6}$ (both allowed; $x\approx 6.449$ and $x\approx 1.551$)

14. Challenge Explain why multiplying both sides of an equation by the LCD can produce an extraneous solution, but multiplying both sides by 5 never can.

Show answer

Multiplying by a nonzero number can be undone by dividing by it, so the new equation has exactly the same solutions. The LCD contains the variable and equals 0 at the excluded values; at such a value both sides were multiplied by 0, so the new equation can be true there even though the original equation is undefined. So a solution of the new equation may be an excluded value, which is extraneous. The number 5 is never 0, so nothing is gained.

15. Challenge A small pump and a large pump fill a stock tank together in $\frac{12}{5}$ hours. Alone, the small pump takes twice as long as the large pump. How long does each pump take alone?

Show answer

$\frac{12/5}{b}+\frac{12/5}{2b}=1$; multiply by $2b$: $\frac{24}{5}+\frac{12}{5}=2b$, so $2b=\frac{36}{5}$ and $b=3.6$ hours for the large pump, 7.2 hours for the small one (check: $\frac{1}{3.6}+\frac{1}{7.2}=\frac{3}{7.2}=\frac{5}{12}$ tank per hour ✓)

COMMON MISTAKES

- Multiplying only some terms by the LCD. Every term on both sides, including whole-number terms like $+1$, must be multiplied.

- Keeping an answer that is an excluded value. Write the excluded values first and cross out any answer that matches one.
- In work problems, adding times instead of rates. Two people who take 6 and 12 hours do not take 18 (or 9) hours together; add the rates $\frac{1}{6} + \frac{1}{12}$.
- Accepting a solution that makes no sense in the situation, such as a negative time or a speed slower than the current when going upstream.

> **Make it a game · Stamp the Impostor.** The parent writes three rational equations, including x/(x - 3) = 3/(x - 3) + 2. Solve each by multiplying by the LCD, then check every answer against the excluded values and in the original. Mark any extraneous answer with a stamp. Explain why multiplying by an expression that could be zero lets impostors in.

5.6 Variation and Rational Models

OBJECTIVES

- Write and use equations for direct, inverse, joint and combined variation, finding the constant of variation from one set of values.
- Apply inverse-square laws for light and gravity.
- Derive and use the shared-work formula $\frac{1}{t} = \frac{1}{a} + \frac{1}{b}$ and the round-trip average speed $\frac{2ab}{a+b}$.
- Model the concentration of a mixture as a rational function of the amount added.

WATCH

Combined Variation

Mario's Math Tutoring · 4:41

https://www.youtube.com/watch?v=3AVVxJkBJTU

> **WHEN TO WATCH**
> Before Example 1: direct, inverse, joint and combined variation.

BACKUP VIDEO

Combined Rate Example (2 People Working Together)

Mario's Math Tutoring · 3:15

https://www.youtube.com/watch?v=YfpkGXSrdYI

Before the practice set: a work-rate problem with two people working together.

KEY TERMS

direct variation

$y = kx$ with $k \neq 0$: the ratio $\frac{y}{x}$ is always the same, so doubling x doubles y.

inverse variation

$y = \frac{k}{x}$ with $k \neq 0$: the product xy is always the same, so doubling x halves y.

joint variation

$y = kxz$: y varies directly as the product of two or more variables.

combined variation

A mix of direct and inverse variation, such as $y = \dfrac{kx}{z^2}$ (y varies directly as x and inversely as the square of z).

constant of variation

The fixed number k in a variation equation. Find it by substituting one known set of values.

harmonic mean

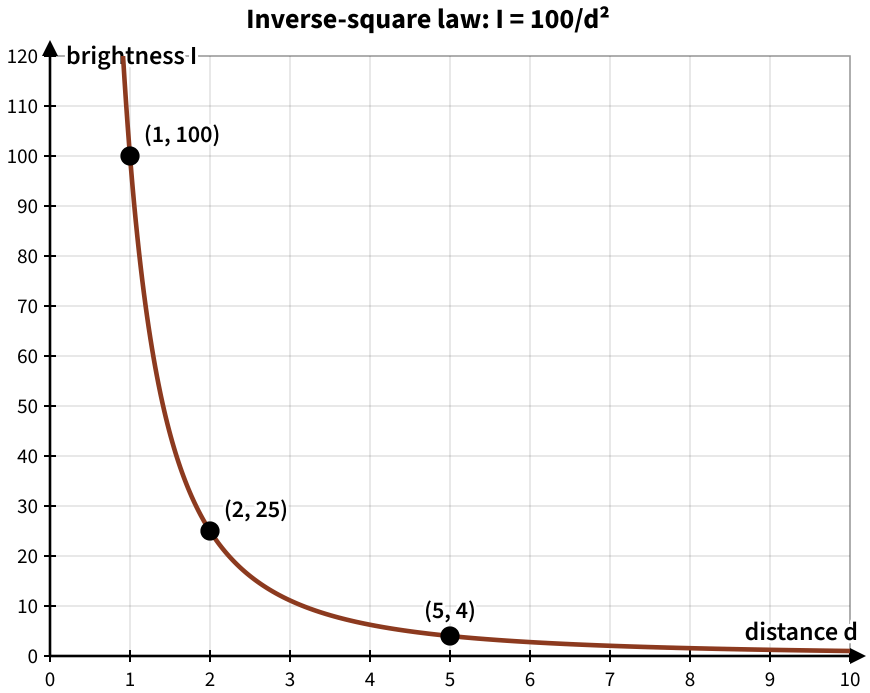
For two positive numbers a and b, the number $\dfrac{2ab}{a+b}$. It is the true average speed of a round trip driven at a and b.

CORE IDEAS

The language of variation. "y varies directly as x" means $y = kx$. "y varies inversely as x" means $y = \dfrac{k}{x}$. "y varies jointly as x and z" means $y = kxz$. Phrases combine: "y varies directly as x and inversely as the square of z" means $y = \dfrac{kx}{z^2}$. The method is always the same: write the equation with k, substitute one known set of values to find k, then use the equation. Inverse variation is the simplest rational model: its graph for $x > 0$ is a branch of $y = \dfrac{k}{x}$ (Lesson 5.3).

Variation equations also answer "what if" questions without finding k. If $y = \dfrac{kx}{z^2}$ and z is doubled, the new value is $\dfrac{kx}{(2z)^2} = \dfrac{1}{4} \cdot \dfrac{kx}{z^2}$: y is divided by 4.

Inverse-square laws. A small lamp sends its light out evenly in every direction. At distance d that light is spread over a sphere of area $4\pi d^2$, so the brightness on each square foot is $\dfrac{\text{lamp's output}}{4\pi d^2}$, which has the form $I = \dfrac{k}{d^2}$. Double the distance and the same light covers 4 times the area, so the page is $\dfrac{1}{4}$ as bright. Gravity follows the same kind of law, as Newton described: the pull between two bodies varies jointly as their masses and inversely as the square of the distance between their centers, $F = \dfrac{km_1m_2}{d^2}$.

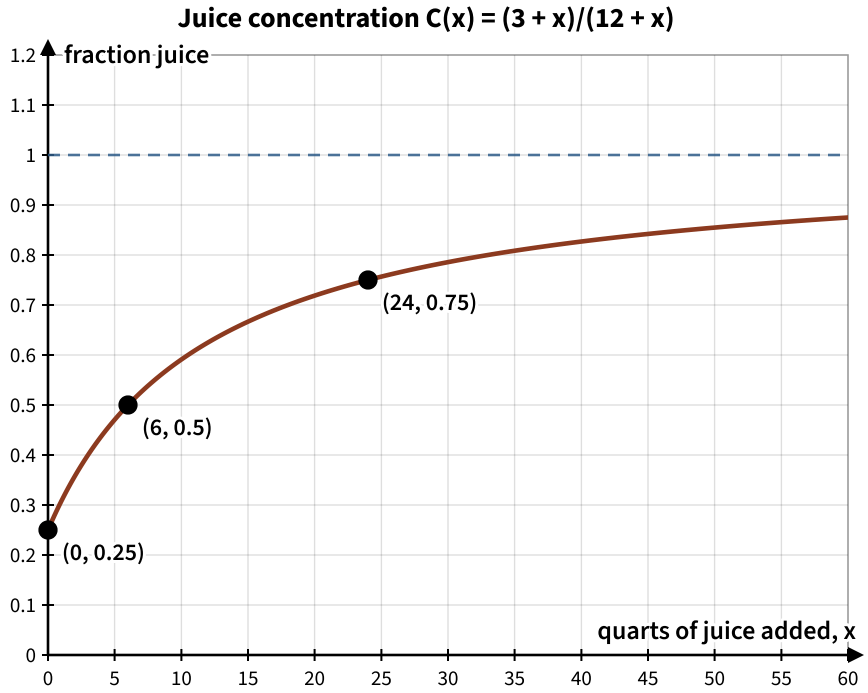


Shared work. If one pump fills a tank in a hours, it fills $\frac{1}{a}$ of the tank per hour; a second pump fills $\frac{1}{b}$ per hour. Working together, their rates add. If together they need t hours, their combined rate is $\frac{1}{t}$ tank per hour, so $\frac{1}{t} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, and taking reciprocals, $t = \frac{ab}{a+b}$. For pumps that take 3 and 6 hours, $t = \frac{18}{9} = 2$ hours. Notice that t is always less than either time alone, as it should be.

The average speed of a round trip is not the average of the two speeds. Drive d miles out at a mph and d miles back at b mph. The total time is $\frac{d}{a} + \frac{d}{b} = \frac{d(a+b)}{ab}$, and the total distance is $2d$. So the average speed is $\frac{2d}{\frac{d(a+b)}{ab}} = \frac{2ab}{a+b}$, the harmonic mean. The distance d cancels, so the answer does not depend on how far you go. For 60 mph out and 40 mph back, the average is $\frac{2(60)(40)}{100} = 48$ mph, not 50. Why lower? You spend more time at the slower speed: for 60 miles each way, 1 hour at 60 mph but 1.5 hours at 40 mph, and $\frac{120}{2.5} = 48$.

Concentration. Ruth has 12 quarts of punch containing 3 quarts of juice, so it is $\frac{3}{12} = 25\%$ juice. If she stirs in x quarts of pure juice, both the juice and the total grow: $C(x) = \frac{3+x}{12+x}$. Since the degrees are equal, the horizontal asymptote is $y = 1$:

the punch gets closer and closer to 100% juice but never reaches it, and each added quart raises the concentration a little less than the one before.



KEY FORMULAS

$$y = kx \qquad y = \frac{k}{x} \qquad y = kxz \qquad y = \frac{kx}{z^2}$$

$$I = \frac{k}{d^2} \qquad F = \frac{km_1m_2}{d^2}$$

$$\frac{1}{t} = \frac{1}{a} + \frac{1}{b} \quad \Longrightarrow \quad t = \frac{ab}{a+b}$$

$$\text{average speed of a round trip} = \frac{2ab}{a+b}$$

Practice habit. Translate the words into an equation with k before touching the numbers: write $y = \dfrac{kx}{z^2}$ first, find k second, answer the question third.

WORKED EXAMPLES

Example 1. The illumination on Henry's workbench varies inversely as the square of its distance from the lamp. At 4 feet the illumination is 30 foot-candles. (a) Find the constant of variation. (b) Find the illumination at 3 feet, to the nearest tenth. (c) At what distance is the illumination 120 foot-candles?

Show solution

1. Write the equation: $I = \dfrac{k}{d^2}$.
2. (a) Substitute: $30 = \dfrac{k}{16}$, so $k = 480$.
3. (b) $I = \dfrac{480}{9} \approx 53.3$ foot-candles.
4. (c) $120 = \dfrac{480}{d^2}$, so $d^2 = 4$ and $d = 2$ feet (distance is positive).
5. Check (c) by reasoning: going from 4 feet to 2 feet halves the distance, which should multiply the brightness by 4: $30 \times 4 = 120$ ✓.

Answer: (a) $k = 480$; (b) about 53.3 foot-candles; (c) 2 feet

Example 2. The load a rectangular wooden beam can safely carry varies jointly as its width and the square of its depth, and inversely as its length. A beam 2 inches wide, 6 inches deep and 10 feet long can carry 1440 pounds. How much can a beam 2 inches wide, 8 inches deep and 16 feet long carry?

Show solution

1. Write the equation: $S = \dfrac{kwh^2}{L}$, with w the width, h the depth and L the length.
2. Substitute the known beam: $1440 = \dfrac{k(2)(6^2)}{10} = \dfrac{72k}{10} = 7.2k$.
3. Solve: $k = \dfrac{1440}{7.2} = 200$.
4. New beam: $S = \dfrac{200(2)(8^2)}{16} = \dfrac{200 \times 128}{16} = 1600$ pounds.
5. Check by ratios: depth $\times \frac{8}{6}$ multiplies the load by $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$; length $\times 1.6$ divides it by 1.6. $1440 \times \frac{16}{9} \div 1.6 = 2560 \div 1.6 = 1600$ ✓.

Answer: 1600 pounds

PRACTICE

1. **Easy** y varies directly as x, and $y = 12$ when $x = 4$. Find k, and find y when $x = 10$.

   Show answer

   $k = 3$; $y = 30$

2. **Easy** y varies inversely as x, and $y = 6$ when $x = 8$. Find k, and find y when $x = 12$.

   Show answer

   $k = 48$; $y = \dfrac{48}{12} = 4$

3. **Easy** z varies jointly as x and y, and $z = 60$ when $x = 3$ and $y = 5$. Find z when $x = 4$ and $y = 7$.

   Show answer

   $60 = 15k$, so $k = 4$; $z = 4(4)(7) = 112$

4. **Easy** The time for the Miller family's trip to their cousins' farm varies inversely as their average speed. At 60 mph the trip takes 3 hours. How long does it take at 45 mph?

   Show answer

   $k = 180$, so $t = \dfrac{180}{45} = 4$ hours

5. **Medium** A reading lamp 3 feet from a page gives an illumination of 40 foot-candles. Using the inverse-square law, find the illumination when the lamp is moved to 2 feet from the page.

   Show answer

   $k = 40 \times 9 = 360$, so $I = \dfrac{360}{4} = 90$ foot-candles

6. **Medium** y varies directly as x and inversely as the square of z, and $y = 10$ when $x = 5$ and $z = 2$. Find k, and find y when $x = 12$ and $z = 4$.

   Show answer

   $10 = \dfrac{5k}{4}$, so $k = 8$; $y = \dfrac{8 \times 12}{16} = 6$

7. **Medium** One hose fills a livestock pool in 3 hours and another in 6 hours. How long do the two hoses take together?

   Show answer

   $$t = \frac{3 \times 6}{3 + 6} = 2 \text{ hours}$$

8. **Medium** Samuel drives to town at 30 mph and back home along the same road at 60 mph. What is his average speed for the round trip?

   Show answer

   $$\frac{2(30)(60)}{30 + 60} = \frac{3600}{90} = 40 \text{ mph (not 45)}$$

9. **Medium** Grace has 8 cups of punch that contains 2 cups of juice. She adds x cups of pure juice. (a) Write the concentration of juice $C(x)$ as a rational function. (b) How many cups must she add to make the punch 50% juice?

   Show answer

   (a) $C(x) = \dfrac{2 + x}{8 + x}$; (b) $\dfrac{2 + x}{8 + x} = \dfrac{1}{2}$ gives $4 + 2x = 8 + x$, so $x = 4$ cups (check: $\dfrac{6}{12} = 0.5$ ✓)

10. **Hard** An object's weight varies inversely as the square of its distance from the center of the earth, whose radius is about 4000 miles. An astronaut with gear weighs 180 pounds on the surface. What would the astronaut weigh 1000 miles above the surface?

    Show answer

    $$W = 180 \times \left(\frac{4000}{5000}\right)^2 = 180 \times 0.64 = 115.2 \text{ pounds}$$

11. **Hard** The force of the wind on a barn wall varies jointly as the wall's area and the square of the wind speed. A 100-square-foot wall in a 20 mph wind feels a force of 100 pounds. (a) Find the force on a 300-square-foot wall in a 40 mph wind. (b) How many times the original 100-pound force is this?

    Show answer

    (a) $k = \dfrac{100}{100 \times 400} = 0.0025$, so $F = 0.0025 \times 300 \times 1600 = 1200$ pounds; (b) 12 times (3×2^2)

12. Hard Three pumps can each drain a flooded cellar alone in 4 hours, 6 hours and 12 hours. How long do they take running together?

Show answer

$\frac{1}{t} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{1}{2}$, so $t = 2$ hours

13. Challenge (a) Show that a round trip of d miles each way at a mph out and b mph back has average speed $\frac{2ab}{a+b}$. (b) Prove that this is less than $\frac{a+b}{2}$ whenever $a \neq b$.

Show answer

(a) Time $= \frac{d}{a} + \frac{d}{b} = \frac{d(a+b)}{ab}$; average speed $= \frac{2d}{d(a+b)/(ab)} = \frac{2ab}{a+b}$. (b) $\frac{a+b}{2} - \frac{2ab}{a+b} = \frac{(a+b)^2 - 4ab}{2(a+b)} = \frac{(a-b)^2}{2(a+b)}$, which is positive because $(a-b)^2 > 0$ when $a \neq b$ and $a+b > 0$.

14. Challenge Ruth drives to a choir festival at 40 mph. (a) How fast must she drive home on the same road to average 48 mph for the round trip? (b) Could she average 80 mph for the round trip by driving fast enough on the way home? Explain.

Show answer

(a) $\frac{2(40)b}{40+b} = 48$ gives $80b = 1920 + 48b$, so $b = 60$ mph. (b) No: $\frac{80b}{40+b} = 80$ gives $80b = 3200 + 80b$, which is impossible. She has already used up all the time an 80 mph average would allow on the way out.

COMMON MISTAKES

- Averaging the two speeds of a round trip. Use total distance ÷ total time, which gives $\frac{2ab}{a+b}$.
- Adding work times instead of work rates. Add $\frac{1}{a} + \frac{1}{b}$, then take the reciprocal to get the time.
- Forgetting the square in an inverse-square law. Doubling the distance divides the brightness by 4, not 2.
- Using a new set of values before finding k. Find k from the one complete set of values, then substitute the new values.

Make it a game · Say It, Write It. The parent says a sentence, such as 'y varies jointly with x and z and inversely with the square of w'. Write the equation, find k from one set of values, and predict a new value. Time limit 30 seconds. Then answer: if you double your distance from a lamp, what fraction of the light remains?

UNIT 5

Rational Expressions and Functions

A rational expression is a fraction made of polynomials, and a rational function is a function with such a rule. In this unit you will simplify, multiply, divide, add and subtract rational expressions, using the factoring of Unit 4 at every step. Then you will study the graphs of rational functions: their vertical, horizontal and slant asymptotes, their holes, and their intercepts, and you will learn why each of these features appears.

Rational functions describe many everyday situations: the average cost per item as a business grows, the time for a round trip on a river, two pumps filling a tank, the brightness of a lamp as you move away from it, and the strength of a punch as juice is added. Solving rational equations brings a new caution: the solving steps can produce extraneous solutions, so every answer must be checked against the excluded values.

This unit builds on polynomial factoring and long division (Unit 4) and on transformations of parent functions (Unit 1). The habit of checking domains and excluded values carries straight into Unit 6, where radical equations can also have extraneous solutions, and into Unit 8, where logarithms have restricted domains.

PREREQUISITES

- Factoring trinomials, differences of squares, sums and differences of cubes, grouping and quadratic form (Lessons 4.2 and 4.5)
- Polynomial long division (Lesson 4.3)
- Transformations $y = a \cdot f(x - h) + k$ of the parent function $y = \frac{1}{x}$ (Lesson 1.2)
- Solving linear and quadratic equations, including by the quadratic formula (Lesson 3.2)
- Adding, subtracting, multiplying and dividing numerical fractions

MASTERY CHECKLIST

- ☐ I can state the excluded values of a rational expression and simplify it by dividing out common factors, never terms.
- ☐ I can multiply and divide rational expressions, including the excluded values from the divisor's numerator.
- ☐ I can add and subtract rational expressions using the LCD, with parentheses around subtracted numerators.
- ☐ I can simplify complex fractions by two methods.
- ☐ I can graph $f(x) = \frac{a}{x - h} + k$ and state its domain, range and asymptotes.
- ☐ I can find vertical and horizontal asymptotes and explain why they occur.
- ☐ I can find holes, intercepts and slant asymptotes and sketch a rational function with a sign chart.
- ☐ I can solve rational equations and identify extraneous solutions, and explain why they appear.
- ☐ I can solve work, travel and proportion problems with rational equations.
- ☐ I can write and use direct, inverse, joint and combined variation equations, including inverse-square laws.

- ☐ I can use the shared-work formula and explain why the average speed of a round trip is $\dfrac{2ab}{a+b}$.

LESSONS IN THIS UNIT

5.1	**Rational Expressions: Simplifying, Multiplying and Dividing**	45 minutes
5.2	**Adding and Subtracting Rational Expressions**	2 sessions of 45 minutes
5.3	**Rational Functions and Asymptotes**	2 sessions of 45 minutes
5.4	**Graphing Rational Functions**	2 sessions of 45 minutes
5.5	**Solving Rational Equations**	2 sessions of 45 minutes
5.6	**Variation and Rational Models**	2 sessions of 45 minutes

Unit 5 Test

Time: 60 minutes. A scientific calculator is allowed; a graphing calculator is not. Show all work. State excluded values wherever you simplify or solve, and check every solution of an equation in the original equation. Give exact answers unless a problem asks you to round.

1. **5 pts**

   Simplify $\dfrac{x^2 - 49}{x^2 - 4x - 21}$ and state the excluded values.

2. **6 pts**

   Simplify $\dfrac{8x^3 - 1}{4x^2 - 1}$ and state the excluded values.

3. **7 pts**

   Divide and simplify, stating all excluded values: $\dfrac{x^2 + 2x - 15}{x^2 - 4} \div \dfrac{x^2 - 9}{x^2 + 2x}$.

4. **6 pts**

   Add and simplify: $\dfrac{3}{x+4} + \dfrac{24}{x^2-16}$.

5. **6 pts**

   Subtract: $\dfrac{2x}{x^2+5x+6} - \dfrac{x-1}{x^2+4x+3}$.

6. **6 pts**

   Simplify the complex fraction $\dfrac{\frac{1}{x} - \frac{1}{3}}{x-3}$.

7. **6 pts**

   For $f(x) = -\dfrac{4}{x+1} + 2$, find the vertical and horizontal asymptotes and the x- and y-intercepts.

8. **7 pts**

   A family orchard buys a cider press for \$1800. Apples and jugs cost \$2.50 for each gallon of cider. (a) Write the average cost per gallon $\overline{C}(x)$ for x gallons. (b) Give the horizontal asymptote. (c) How many gallons bring the average cost down to \$4 per gallon?

9. **8 pts**

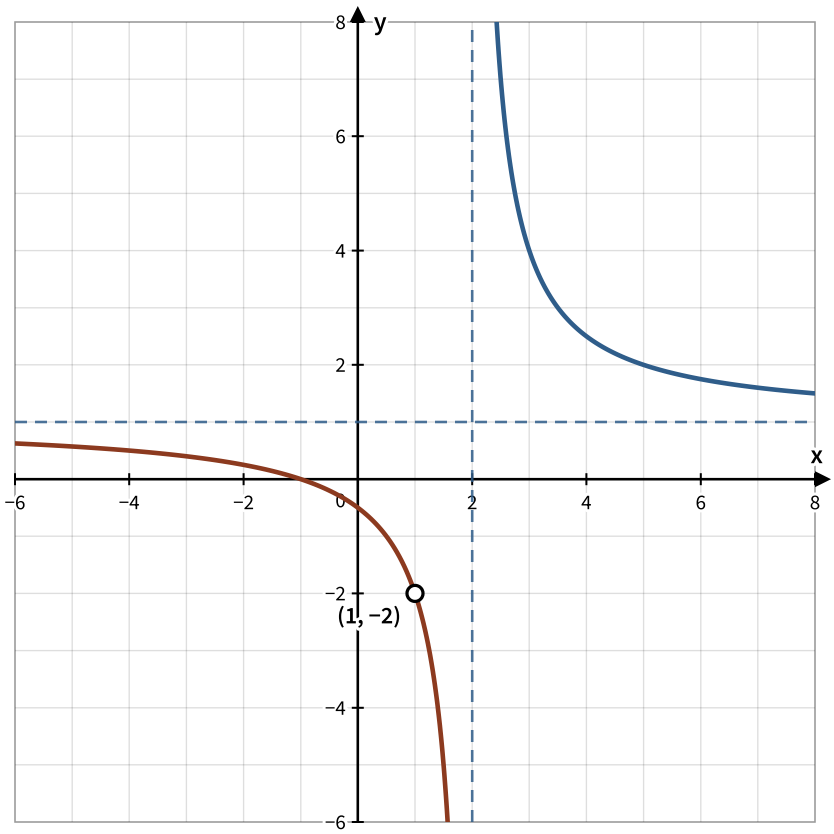
   For $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$, find any hole, the vertical and horizontal asymptotes, and the intercepts.

10. **7 pts**

    For $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, find the vertical asymptote, the slant asymptote and the intercepts.

11. **6 pts**

The graph of a rational function f is shown; its asymptotes are dashed and it has one hole. (a) State the domain of f. (b) Use the graph to solve $f(x) > 0$. Answer both in interval notation.



12. **7 pts**

Solve $\dfrac{3}{x-2} + \dfrac{4}{x+2} = \dfrac{12}{x^2-4} + 1$.

13. **7 pts**

A small plane flies 600 miles with a 30 mph tailwind in the same time it flies 480 miles against the same wind. Find the plane's speed in still air.

14. **6 pts**

The time needed to shingle a roof varies directly as the roof's area and inversely as the number of workers. Six volunteers shingle 2400 square feet of the church roof in 8 hours. How long would 8 volunteers take to shingle 3200 square feet?

15. **6 pts**

A train travels from one city to another at 90 mph and returns on the same track at 60 mph. What is its average speed for the round trip?

16. **4 pts**

In Problem 12, $x = 2$ solves the equation $x^2 - 7x + 10 = 0$ that you obtained. Explain why it is not a solution of the original equation, and why the solving steps produced it.

ANSWER KEY

1. $\dfrac{(x-7)(x+7)}{(x-7)(x+3)} = \dfrac{x+7}{x+3}$, with $x \neq -3, 7$
2. $\dfrac{(2x-1)(4x^2+2x+1)}{(2x-1)(2x+1)} = \dfrac{4x^2+2x+1}{2x+1}$, with $x \neq -\dfrac{1}{2}, \dfrac{1}{2}$
3. $\dfrac{(x+5)(x-3)}{(x-2)(x+2)} \cdot \dfrac{x(x+2)}{(x-3)(x+3)} = \dfrac{x(x+5)}{(x-2)(x+3)}$, with $x \neq -3, -2, 0, 2, 3$
4. $\dfrac{3(x-4)+24}{(x-4)(x+4)} = \dfrac{3(x+4)}{(x-4)(x+4)} = \dfrac{3}{x-4}$, with $x \neq -4, 4$
5. LCD $(x+1)(x+2)(x+3)$; numerator $2x(x+1) - (x-1)(x+2) = x^2 + x + 2$; answer $\dfrac{x^2+x+2}{(x+1)(x+2)(x+3)}$
6. Multiply top and bottom by $3x$: $\dfrac{3-x}{3x(x-3)} = -\dfrac{1}{3x}$, with $x \neq 0, 3$
7. Vertical $x = -1$; horizontal $y = 2$; x-intercept: $\dfrac{4}{x+1} = 2$ gives $(1, 0)$; y-intercept $(0, -2)$
8. (a) $\overline{C}(x) = \dfrac{1800 + 2.5x}{x}$; (b) $y = 2.5$: the average approaches \$2.50, the cost of materials; (c) $2.5 + \dfrac{1800}{x} = 4$ gives $x = 1200$ gallons
9. $\dfrac{(x+3)(x-2)}{(x-2)(x+2)} = \dfrac{x+3}{x+2}$: hole $\left(2, \dfrac{5}{4}\right)$; vertical $x = -2$; horizontal $y = 1$; x-intercept $(-3, 0)$; y-intercept $\left(0, \dfrac{3}{2}\right)$
10. $x^2 - 2x - 3 = (x+2)(x-4) + 5$; vertical $x = -2$; slant $y = x - 4$; x-intercepts $(-1, 0)$ and $(3, 0)$; y-intercept $\left(0, -\dfrac{3}{2}\right)$
11. (a) $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$; (b) $(-\infty, -1) \cup (2, \infty)$
12. Excluded $x \neq \pm 2$. Multiply by $(x-2)(x+2)$: $3(x+2) + 4(x-2) = 12 + x^2 - 4$, so $x^2 - 7x + 10 = 0$ and $x = 2$ or $x = 5$. Reject $x = 2$ (excluded). Check $x = 5$: $1 + \dfrac{4}{7} = \dfrac{12}{21} + 1$ ✓. Solution: $x = 5$
13. $\dfrac{600}{r+30} = \dfrac{480}{r-30}$; $600r - 18000 = 480r + 14400$, so $120r = 32400$ and $r = 270$ mph (check: 2 hours each way ✓)
14. $T = \dfrac{kA}{n}$: $8 = \dfrac{2400k}{6}$ gives $k = 0.02$; $T = \dfrac{0.02 \times 3200}{8} = 8$ hours
15. $\dfrac{2(90)(60)}{90+60} = \dfrac{10800}{150} = 72$ mph

16. At $x = 2$ the denominators $x - 2$ and $x^2 - 4$ are 0, so the original equation is undefined there. The steps multiplied both sides by $(x - 2)(x + 2)$, which is 0 at $x = 2$; multiplying by an expression that can be 0 can add solutions, so the polynomial equation has a root the original equation does not.

UNIT 6

Radicals and Rational Exponents

In Algebra I you worked with square roots and with whole-number exponents. This unit joins those two ideas. You will see why an exponent like $\frac{1}{3}$ has to mean a cube root if the exponent rules are to keep working, and then use that idea to evaluate, simplify and combine expressions with roots of any index. You will graph square root and cube root functions as transformations of their parent graphs and see that $y = \sqrt{x}$ is the inverse of $y = x^2$ on a restricted domain.

The unit ends with radical equations, where squaring both sides can quietly add wrong answers, so you will learn why extraneous solutions appear and how a careful check catches them. Along the way you will use radicals to find the period and length of a pendulum, how far you can see from a height, and how fast water leaves a tank.

Rational exponents prepare the way for Unit 7, where exponents can be any real number and the number e appears, and for Unit 8, where logarithms undo exponents. The habit of checking answers in the original equation returns with logarithmic equations. The midterm covering Units 1 to 6 follows this unit.

PREREQUISITES

- Exponent rules for integer exponents, including zero and negative exponents (Algebra I)
- Simplifying square roots and rationalizing one-term denominators (Algebra I)
- Multiplying binomials and the special products, including the difference of squares (Lesson 4.2)
- Solving quadratic equations by factoring (Lesson 3.2)
- Transformations of parent functions, $y = a \cdot f(x - h) + k$ (Lesson 1.2)
- Finding inverse functions and restricting a domain (Lesson 1.5)

MASTERY CHECKLIST

- ☐ I can find real nth roots and tell when an even root of a number is not real.
- ☐ I can explain why $a^{1/n}$ means $\sqrt[n]{a}$ and evaluate expressions like $27^{2/3}$ and $32^{-3/5}$ by hand.
- ☐ I can convert between radical form and rational exponent form.
- ☐ I can simplify expressions with rational exponents using the exponent rules.
- ☐ I can simplify nth roots, including roots with variables, and explain why $\sqrt{x^2} = |x|$.
- ☐ I can rationalize denominators with one term, such as $\frac{5}{\sqrt[3]{4}}$, and with two terms using conjugates.
- ☐ I can add, subtract and multiply radical expressions, and combine radicals with different indexes using rational exponents.
- ☐ I can graph $y = a\sqrt{x - h} + k$ and $y = a\sqrt[3]{x - h} + k$ and state their domains and ranges.

- ☐ I can find inverses of simple quadratic and cubic functions, restricting the domain when needed.
- ☐ I can solve radical equations and equations with rational exponents, and explain why extraneous solutions can appear.
- ☐ I can use radical models for pendulums, the distance to the horizon and the speed of water from a tank.

LESSONS IN THIS UNIT

6.1	**nth Roots and Rational Exponents**	2 sessions of 45 minutes
6.2	**Properties of Rational Exponents and Radicals**	2 sessions of 45 minutes
6.3	**Operations with Radical Expressions**	2 sessions of 45 minutes
6.4	**Radical Functions and Their Graphs**	2 sessions of 45 minutes
6.5	**Solving Radical Equations**	2 sessions of 45 minutes

Unit 6 test

Time: 60 minutes. Do not use a calculator except on problems that ask for a rounded decimal answer; then a scientific calculator is allowed. Show your work on every problem. Give exact answers in simplest form (simplified radicals, rational denominators) unless a problem asks you to round. When simplifying expressions, assume all variables are positive unless a problem says otherwise; when solving equations, give every real solution. Check every solution of an equation in the original equation and state any extraneous solutions.

1. **6 pts**

   Evaluate by hand: (a) $125^{2/3}$ (b) $16^{-5/4}$ (c) $(-32)^{3/5}$

2. **5 pts**

   (a) Write $\sqrt[7]{x^4}$ with a rational exponent. (b) Write $y^{-3/5}$ in radical form with no negative exponent.

3. **6 pts**

   Simplify: $\dfrac{\left(a^{3/4}b^{-1/2}\right)^4}{a^2 b^{-3}}$

4. **6 pts**

   Simplify: $\sqrt[3]{250a^8b^4}$

5. **6 pts**

   Rationalize the denominator and simplify: $\dfrac{12}{\sqrt[3]{6}}$

6. **6 pts**

   A rectangular herb garden beside a church hall measures $\sqrt{45}$ meters by $\sqrt{20}$ meters. Find (a) its exact perimeter in simplest radical form and (b) its area.

7. **6 pts**

   Multiply and simplify: $(4-\sqrt{6})(2+3\sqrt{6})$

8. **7 pts**

   Rationalize the denominator and simplify: $\dfrac{10}{\sqrt{7}-\sqrt{2}}$

9. **5 pts**

   Write $\sqrt{x}\cdot\sqrt[5]{x^2}$ as a single radical.

10. **5 pts**

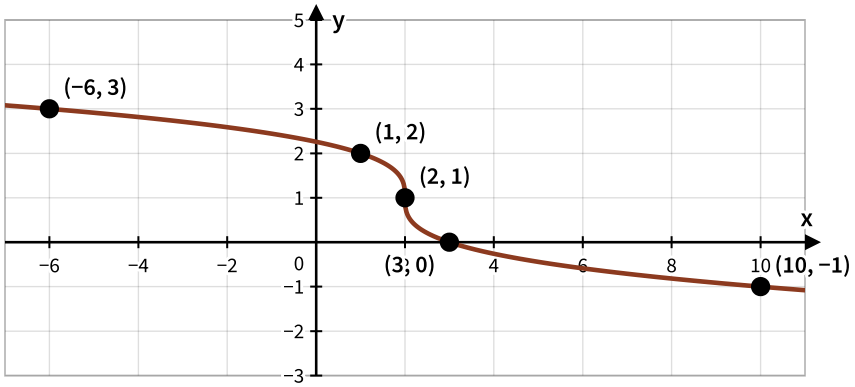
    From a height of h feet you can see about $d = 1.22\sqrt{h}$ miles to the horizon. How far can riders see from a hot-air balloon 900 feet above a flat county fairground? Give the answer to the nearest tenth of a mile.

11. **6 pts**

    For $f(x) = 3\sqrt{6-2x} - 5$, give (a) the domain and (b) the range in interval notation.

12. **7 pts**

The graph has the form $y = a\sqrt[3]{x-h}+k$, and $(2,1)$ is the point where it changes from bending one way to the other (the image of $(0,0)$). Use the labeled points to write its equation.



13. **7 pts**

Let $f(x) = (x+3)^2 - 4$ with $x \geq -3$. (a) Find $f^{-1}(x)$. (b) Give the domain of f^{-1}.

14. **6 pts**

Explain why, if the rule $(a^m)^n = a^{mn}$ is to hold for fractions, $5^{1/3}$ must equal $\sqrt[3]{5}$.

15. **10 pts**

Solve and check. Identify any extraneous solutions. (a) $\sqrt{3x+10} = x+2$ (b) $(2x-3)^{2/3} = 9$

16. **6 pts**

Water leaves the spout of a rain barrel at about $v = 8\sqrt{h}$ feet per second, where h is the depth of water above the spout in feet. How many inches of water must be above the spout for the water to leave at 6 feet per second?

ANSWER KEY

1. (a) $5^2 = 25$ (b) $\dfrac{1}{2^5} = \dfrac{1}{32}$ (c) $(-2)^3 = -8$
2. (a) $x^{4/7}$ (b) $\dfrac{1}{\sqrt[5]{y^3}}$
3. $\dfrac{a^3 b^{-2}}{a^2 b^{-3}} = ab$
4. $\sqrt[3]{125a^6b^3} \cdot \sqrt[3]{2a^2b} = 5a^2b\sqrt[3]{2a^2b}$
5. Multiply by $\dfrac{\sqrt[3]{36}}{\sqrt[3]{36}}$: $\dfrac{12\sqrt[3]{36}}{\sqrt[3]{216}} = \dfrac{12\sqrt[3]{36}}{6} = 2\sqrt[3]{36}$
6. (a) $\sqrt{45} = 3\sqrt{5}$ and $\sqrt{20} = 2\sqrt{5}$, so $P = 2(3\sqrt{5} + 2\sqrt{5}) = 10\sqrt{5}$ m (b) $\sqrt{45 \cdot 20} = \sqrt{900} = 30$ m^2
7. $8 + 12\sqrt{6} - 2\sqrt{6} - 18 = -10 + 10\sqrt{6}$
8. $\dfrac{10(\sqrt{7} + \sqrt{2})}{7 - 2} = 2\sqrt{7} + 2\sqrt{2}$
9. $x^{1/2 + 2/5} = x^{9/10} = \sqrt[10]{x^9}$
10. $1.22\sqrt{900} = 1.22 \times 30 = 36.6$ miles
11. (a) $6 - 2x \geq 0$, so $(-\infty, 3]$ (b) $3\sqrt{6 - 2x} \geq 0$, so $[-5, \infty)$
12. $h = 2$, $k = 1$. From $(10, -1)$: $-1 = a\sqrt[3]{8} + 1$, so $a = -1$. $y = -\sqrt[3]{x - 2} + 1$
13. (a) $x = (y + 3)^2 - 4$, $y + 3 = \sqrt{x + 4}$ (positive since $y \geq -3$), so $f^{-1}(x) = -3 + \sqrt{x + 4}$ (b) $x \geq -4$
14. By the rule, $\left(5^{1/3}\right)^3 = 5^{\frac{1}{3} \cdot 3} = 5^1 = 5$. So $5^{1/3}$ is a number whose cube is 5. The only real number whose cube is 5 is $\sqrt[3]{5}$, so $5^{1/3} = \sqrt[3]{5}$.
15. (a) $3x + 10 = x^2 + 4x + 4$, $x^2 + x - 6 = 0$, $x = 2$ or $x = -3$. $x = 2$ checks ($\sqrt{16} = 4$); $x = -3$ is extraneous ($\sqrt{1} = 1 \neq -1$). So $x = 2$. (b) $\sqrt[3]{2x - 3} = \pm 3$, $2x - 3 = \pm 27$, $x = 15$ or $x = -12$; both check: $27^{2/3} = 9$ and $(-27)^{2/3} = 9$.
16. $\sqrt{h} = \dfrac{6}{8} = 0.75$, so $h = 0.5625$ ft $= 0.5625 \times 12 = 6.75$ in; check: $8\sqrt{0.5625} = 8(0.75) = 6$ ✓

6.1 nth Roots and Rational Exponents

OBJECTIVES

- Find real nth roots of numbers and decide when an even root is not a real number.
- Explain why $a^{1/n}$ must mean $\sqrt[n]{a}$ if the exponent rules are to keep working.
- Evaluate expressions such as $27^{2/3}$, $32^{-3/5}$ and $\left(\frac{8}{27}\right)^{-2/3}$ by hand.
- Convert expressions between radical form and rational exponent form.

WATCH

Fractional Indices (1 of 2: Deriving definitions from index laws)

Eddie Woo · 6:01

https://www.youtube.com/watch?v=5V8V-Q47tPs

> **WHEN TO WATCH**
> Before Example 1: where the meaning of a fractional exponent comes from.

Note: Eddie Woo teaches in Australia: he says 'fractional indices' for rational exponents and 'index laws' for exponent rules.

BACKUP VIDEO

How to Convert Rational Exponents to Radical Form & Evaluate (Fractional Exponents)

Mario's Math Tutoring · 13:25

https://www.youtube.com/watch?v=B5DTsB1dOoE

Before the practice set: converting between rational exponents and radicals, and evaluating.

KEY TERMS

nth root

A number b is an nth root of a if $b^n = a$. For example, -2 is a cube root of -8 because $(-2)^3 = -8$.

index

The small number n in $\sqrt[n]{a}$ that tells which root to take. A square root has index 2, which is usually not written.

radicand

The number or expression under the radical sign. In $\sqrt[4]{81}$ the radicand is 81.

principal root

The root that the radical sign stands for: for an even index, the nonnegative root (so $\sqrt[4]{16} = 2$, not -2); for an odd index, the one real root.

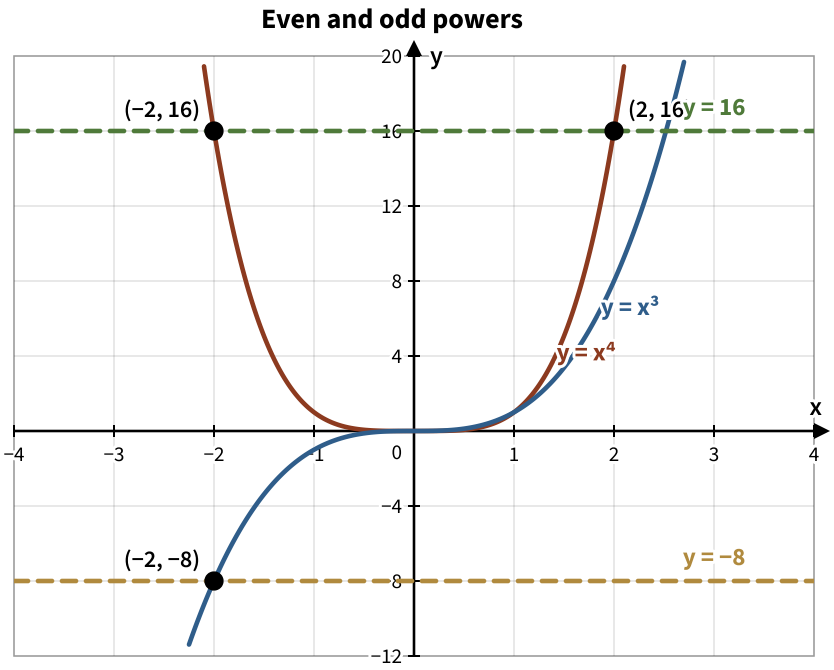
rational exponent

An exponent that is a fraction, such as $\frac{2}{3}$ in $27^{2/3}$. The denominator is the index of a root and the numerator is a power.

CORE IDEAS

In Algebra I you met square roots: 9 has two square roots, 3 and -3, because both square to 9, and the symbol $\sqrt{9}$ means the nonnegative one, 3. The same idea works for any whole number $n \geq 2$: b is an **nth root** of a when $b^n = a$. So 2 is a fourth root of 16, and -2 is a cube root of -8. We write the principal nth root as $\sqrt[n]{a}$.

Whether a root exists depends on whether the index is even or odd. An even power is never negative, since $b^4 = (b^2)^2 \geq 0$; so a negative number has **no real even root**, and a positive number has two real even roots, $\pm\sqrt[n]{a}$, of which $\sqrt[n]{a}$ means the positive one. An odd power keeps the sign of its base, so **every real number has exactly one real odd root**: $\sqrt[3]{-8} = -2$. The graph shows this: the horizontal line $y = 16$ meets $y = x^4$ twice, at $x = \pm 2$, while $y = x^4$ never reaches $y = -8$ at all; the line $y = -8$ meets $y = x^3$ exactly once, at $x = -2$.



Now, what should a fractional exponent such as $a^{1/3}$ mean? We are free to define it, but we want the exponent rules to keep working. The power-of-a-power rule $(a^m)^n = a^{mn}$ would give $\left(a^{1/3}\right)^3 = a^{\frac{1}{3}\cdot 3} = a^1 = a$. So $a^{1/3}$ must be a number whose cube is a: a cube root of a. In the same way $\left(a^{1/n}\right)^n = a$, so the only sensible choice is $a^{1/n} = \sqrt[n]{a}$. For example, $9^{1/2} = 3$, $8^{1/3} = 2$ and $81^{1/4} = 3$. For an even n we need $a \geq 0$; for an odd n any real a works, so $(-8)^{1/3} = -2$.

Other fractions follow from the same rule. Since $\dfrac{m}{n} = \dfrac{1}{n} \cdot m = m \cdot \dfrac{1}{n}$, we get $a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m$ and also $a^{m/n} = \left(a^m\right)^{1/n} = \sqrt[n]{a^m}$. Both give the same number, but taking the root first keeps the numbers small: $27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$, while $\sqrt[3]{27^2} = \sqrt[3]{729}$ asks you to recognize 729 as 9^3. A way to remember the form: the denominator is the root, the numerator is the power.

Negative exponents mean reciprocals, exactly as in Algebra I: $a^{-m/n} = \dfrac{1}{a^{m/n}}$. A negative exponent never makes the answer negative: $32^{-3/5} = \dfrac{1}{\left(\sqrt[5]{32}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

For a fraction, a negative exponent flips it: $\left(\dfrac{8}{27}\right)^{-2/3} = \left(\dfrac{27}{8}\right)^{2/3} = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$.

To work these by hand you need to recognize perfect powers quickly. The table lists the ones that come up most often.

Perfect powers to know

n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125

Watch the parentheses. In $-16^{1/2}$ the exponent applies only to 16, so $-16^{1/2} = -4$; but $(-16)^{1/2}$ is not a real number. Also, in this course a negative base is used with a rational exponent only when the denominator (in lowest terms) is odd, as in $(-27)^{2/3} = \left(\sqrt[3]{-27}\right)^2 = 9$.

KEY FORMULAS

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a \quad (b \geq 0 \text{ when } n \text{ is even})$$

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

$$a^{-m/n} = \frac{1}{a^{m/n}}, \quad a \neq 0$$

Practice habit. Know these perfect powers cold: the squares to 15^2, the cubes to 10^3, the powers of 2 to $2^{10} = 1024$ and the powers of 3 to $3^5 = 243$. When you see $243^{3/5}$ you should think "$243 = 3^5$" at once.

WORKED EXAMPLES

Example 1. Evaluate by hand: (a) $27^{2/3}$ (b) $32^{-3/5}$ (c) $\left(\frac{8}{27}\right)^{-2/3}$

Show solution

1. (a) The denominator 3 is the root and the numerator 2 is the power: $27^{2/3} = \left(\sqrt[3]{27}\right)^2$.
2. $\sqrt[3]{27} = 3$, so $27^{2/3} = 3^2 = 9$.
3. Check the other order: $\sqrt[3]{27^2} = \sqrt[3]{729} = 9$, since $9^3 = 729$ ✓.
4. (b) A negative exponent means a reciprocal: $32^{-3/5} = \frac{1}{32^{3/5}}$.
5. $32^{3/5} = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8$, so $32^{-3/5} = \frac{1}{8}$.
6. (c) A negative exponent flips the fraction: $\left(\frac{8}{27}\right)^{-2/3} = \left(\frac{27}{8}\right)^{2/3}$.
7. $\sqrt[3]{\frac{27}{8}} = \frac{3}{2}$, so the value is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Answer: (a) 9 (b) $\frac{1}{8}$ (c) $\frac{9}{4}$

Example 2. Kepler's third law says that an object orbiting the sun at an average distance of a astronomical units (AU) takes $T = a^{3/2}$ years to go around once. Find by hand the period of an asteroid whose average distance is (a) 4 AU (b) 2.25 AU.

Show solution

1. Write the exponent as a root and a power: $T = a^{3/2} = \left(\sqrt{a}\right)^3$.
2. (a) $\sqrt{4} = 2$, so $T = 2^3 = 8$ years.
3. (b) $\sqrt{2.25} = 1.5$, since $1.5^2 = 2.25$.
4. $T = 1.5^3 = 1.5 \times 2.25 = 3.375$ years.
5. Check with $T^2 = a^3$: (a) $8^2 = 64 = 4^3$ ✓; (b) $3.375^2 = 11.390625$ and $2.25^3 = 11.390625$ ✓.

Answer: (a) 8 years (b) 3.375 years

PRACTICE

1. **Easy** Find each real root: (a) $\sqrt[3]{-125}$ (b) $\sqrt[4]{81}$ (c) $\sqrt[5]{-32}$

   Show answer

   (a) -5 (b) 3 (c) -2

2. **Easy** Is each a real number? Answer yes or no. (a) $\sqrt[4]{-16}$ (b) $\sqrt[3]{-64}$ (c) $\sqrt[6]{-1}$

   Show answer

   (a) no (an even root of a negative number) (b) yes, it is -4 (c) no

3. **Easy** Evaluate by hand: (a) $49^{1/2}$ (b) $125^{1/3}$ (c) $81^{1/4}$

   Show answer

   (a) 7 (b) 5 (c) 3

4. **Easy** (a) Write $x^{3/5}$ in radical form. (b) Write $\sqrt[3]{y^2}$ with a rational exponent. (c) Write $\dfrac{1}{\sqrt{x}}$ with a rational exponent.

   Show answer

   (a) $\sqrt[5]{x^3}$ or $\left(\sqrt[5]{x}\right)^3$ (b) $y^{2/3}$ (c) $x^{-1/2}$

5. **Medium** Evaluate by hand: (a) $8^{2/3}$ (b) $16^{3/4}$ (c) $9^{3/2}$

   Show answer

   (a) $2^2 = 4$ (b) $2^3 = 8$ (c) $3^3 = 27$

6. **Medium** Evaluate by hand. Give exact fractions. (a) $25^{-1/2}$ (b) $81^{-3/4}$

   Show answer

   (a) $\dfrac{1}{5}$ (b) $\dfrac{1}{3^3} = \dfrac{1}{27}$

7. **Medium** Evaluate by hand: (a) $(-27)^{1/3}$ (b) $(-8)^{2/3}$ (c) $-16^{1/2}$

   Show answer

   (a) -3 (b) $(-2)^2 = 4$ (c) $-(16^{1/2}) = -4$

8. **Medium** Evaluate by hand. Give exact answers. (a) $\left(\dfrac{4}{9}\right)^{3/2}$ (b) $\left(\dfrac{1}{16}\right)^{-3/4}$

   Show answer

   (a) $\left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$ (b) $16^{3/4} = 2^3 = 8$

9. **Medium** A cube-shaped water tank on a farm holds 343 cubic feet. Its edge is $s = V^{1/3}$ and the area of its bottom is $V^{2/3}$, where V is the volume. Find by hand (a) the edge length and (b) the area of the bottom.

Show answer

(a) $343^{1/3} = 7$ ft (b) $343^{2/3} = 7^2 = 49$ ft²

10. **Hard** Use Kepler's law $T = a^{3/2}$ (T in years, a in AU) to find the period of Jupiter, whose average distance from the sun is about 5.2 AU. Round to the nearest tenth of a year.

Show answer

$T = 5.2^{1.5} \approx 11.858$, so about 11.9 years

11. **Hard** Evaluate by hand. Give exact answers. (a) $\left(\frac{16}{81}\right)^{-3/4}$ (b) $0.04^{-3/2}$

Show answer

(a) $\left(\frac{81}{16}\right)^{3/4} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$ (b) $0.04 = \frac{1}{25}$, so $25^{3/2} = 5^3 = 125$

12. **Hard** Explain why $(-16)^{1/4}$ is not a real number while $(-8)^{1/3}$ is.

Show answer

$(-16)^{1/4}$ would have to be a number whose fourth power is -16. But any real number to the fourth power is $(b^2)^2 \geq 0$, so no real number works. A cube keeps the sign of its base, and $(-2)^3 = -8$, so $(-8)^{1/3} = -2$.

13. **Hard** A carpenter builds a cube-shaped wooden chest with a volume of 3.375 cubic feet. The surface area of a cube of volume V is $S = 6V^{2/3}$. Find by hand how many square feet of wood cover all six faces.

Show answer

$3.375 = \frac{27}{8}$, so $V^{2/3} = \left(\frac{3}{2}\right)^{2} = 2.25$ and $S = 6(2.25) = 13.5$ ft²

14. **Challenge** For which counting numbers n is $64^{1/n}$ a whole number? Explain.

Show answer

$64 = 2^6$, so $64^{1/n} = 2^{6/n}$, a whole number exactly when n divides 6: $n = 1, 2, 3, 6$ (giving 64, 8, 4, 2).

COMMON MISTAKES

- Reading $a^{1/n}$ as $\frac{a}{n}$: $8^{1/3}$ is 2, not $\frac{8}{3}$. The exponent $\frac{1}{n}$ means the nth root.
- Thinking a negative exponent makes the answer negative: $32^{-3/5} = \frac{1}{8}$, not -8. A negative exponent means a reciprocal.
- Ignoring parentheses: $-16^{1/2} = -4$, but $(-16)^{1/2}$ is not real. Ask what the exponent is attached to.
- Raising to the power first and getting stuck with large numbers. Take the root first: $16^{3/4} = 2^3$, not $\sqrt[4]{4096}$.

> **Make it a game · Root Riddles.** The parent asks riddles. 'I am the real fourth root of 81. Who am I?' 'I am the cube root of -27.' 'I am an even root of -16.' Answer each, including 'no real number'. Then explain why 8^(1/3) must equal 2: raise both sides to the third power and watch the exponent rules work. Swap and write riddles.

6.2 Properties of Rational Exponents and Radicals

OBJECTIVES

- Simplify expressions with rational exponents using the exponent rules.
- Use the product and quotient properties of radicals to simplify nth roots by taking out perfect nth powers.
- Simplify radicals that contain variables and explain why $\sqrt{x^2} = |x|$.
- Rationalize denominators such as $\frac{5}{\sqrt[3]{4}}$.

WATCH

Radicals How to Simplify Square Roots (& Cube Roots)

Mario's Math Tutoring · 5:08

https://www.youtube.com/watch?v=6f9j3NA25Bo

> **WHEN TO WATCH**
> Before Example 1: simplifying square roots and cube roots.

BACKUP VIDEO

Rationalize Cube Roots in the Denominator

Mario's Math Tutoring · 4:38

https://www.youtube.com/watch?v=AWZ6JkKofTE

Before Example 2: rationalizing a cube root in the denominator.

KEY TERMS

perfect nth power

A number or expression that equals something raised to the nth power, such as $27 = 3^3$ (a perfect cube) or $x^8 = (x^2)^4$ (a perfect fourth power).

product property of radicals

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ when both roots are real: the root of a product is the product of the roots.

quotient property of radicals

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ when both roots are real and $b \neq 0$.

rationalizing the denominator

Rewriting a fraction so that no radical is left in the denominator, by multiplying top and bottom by a well-chosen factor.

simplest radical form

A radical expression with no perfect nth power factors left under an nth root, no fractions under a radical, and no radicals in a denominator.

CORE IDEAS

In Lesson 6.1 we defined $a^{m/n}$ so that the power-of-a-power rule keeps working. The payoff is that **all** the exponent rules from Algebra I hold for rational exponents when the base is positive: $a^r a^s = a^{r+s}$, $\frac{a^r}{a^s} = a^{r-s}$, $(a^r)^s = a^{rs}$, $(ab)^r = a^r b^r$ and $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$. So $x^{1/2} \cdot x^{1/3} = x^{3/6+2/6} = x^{5/6}$ and $\left(x^{2/3}\right)^{3/4} = x^{1/2}$. Why insist on positive bases? With a negative base the rules can fail: $\left((-8)^2\right)^{1/6} = 64^{1/6} = 2$, but $(-8)^{2/6} = (-8)^{1/3} = -2$.

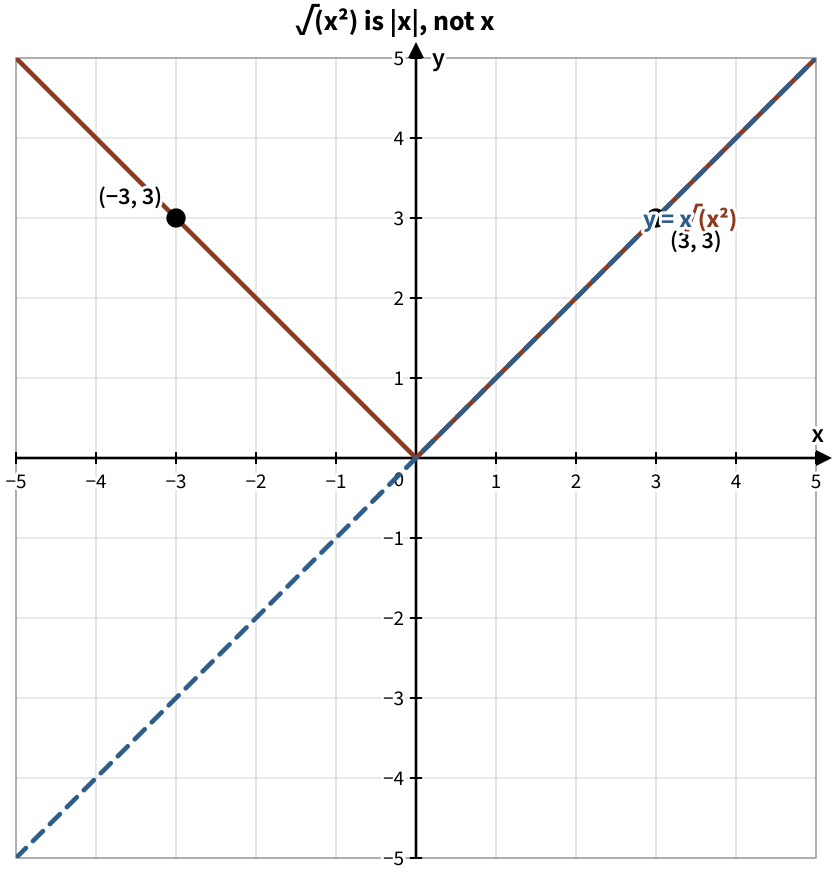
The rules for radicals come straight from these. Writing a root as an exponent, $\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a} \cdot \sqrt[n]{b}$. That is the **product property**. In the same way $\left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}}$ gives the **quotient property** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

To simplify an nth root, split the radicand into a perfect nth power times what is left, then take the root of the perfect power. For a cube root look for factors 8, 27, 64, 125, ...; for a fourth root, 16, 81, 256, ... So $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$ and $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = 2\sqrt[4]{3}$. Be careful to use the right kind of power: 9 is a perfect square but not a perfect cube, so it cannot come out of a cube root.

Variables work the same way: divide each exponent by the index. The quotient comes out and the remainder stays in. For example, $\sqrt[3]{x^7} = \sqrt[3]{x^6 \cdot x} = x^2\sqrt[3]{x}$, because $7 \div 3$ is 2 with remainder 1.

There is one trap with even roots. Try $x = -3$ in $\sqrt{x^2}$: you get $\sqrt{9} = 3$, not -3. The principal square root is never negative, so in general $\sqrt{x^2} = |x|$, as the graph shows: $y = \sqrt{x^2}$ is the V of $y = |x|$, not the line $y = x$. Odd roots have no such problem:

$\sqrt[3]{x^3} = x$ for every real x. To keep the work clean, **when simplifying expressions in this unit, assume every variable is positive unless a problem says otherwise**; then $\sqrt{x^2} = x$. (This does not apply when solving equations or finding domains: there every real value counts.)



A radical in a denominator is usually removed (this is called rationalizing). The tool is multiplying by a form of 1 that makes the radicand a perfect nth power. For $\frac{5}{\sqrt[3]{4}}$, notice $4 = 2^2$; one more factor of 2 makes the perfect cube 8. So $\frac{5}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{5\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{5\sqrt[3]{2}}{2}$. Multiplying by $\frac{\sqrt[3]{4}}{\sqrt[3]{4}}$ would not work, because $\sqrt[3]{16}$ is still not a whole number. The value has not changed (we multiplied by 1); only the form has.

KEY FORMULAS

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}, \quad (a^r)^s = a^{rs} \quad (a > 0)$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt{x^2} = |x|, \qquad \sqrt[3]{x^3} = x$$

Practice habit. Factor before anything else: before simplifying a root, write the radicand as a perfect nth power times what is left (for example $54 = 27 \cdot 2$), and write each variable's exponent as a multiple of the index plus a remainder.

WORKED EXAMPLES

Example 1. Simplify. Assume all variables are positive. (a) $\left(8x^{3/4}\right)^{2/3}$ (b) $\sqrt[3]{54x^7y^3}$

Show solution

1. (a) Raise each factor to the $\frac{2}{3}$ power: $8^{2/3} \cdot \left(x^{3/4}\right)^{2/3}$.
2. $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.
3. Multiply the exponents: $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$, so the result is $4x^{1/2}$ (or $4\sqrt{x}$).
4. (b) Split off perfect cubes: $54 = 27 \cdot 2$, $x^7 = x^6 \cdot x$, and y^3 is already a cube.
5. $\sqrt[3]{54x^7y^3} = \sqrt[3]{27x^6y^3} \cdot \sqrt[3]{2x}$.
6. $\sqrt[3]{27x^6y^3} = 3x^2y$, so the result is $3x^2y\sqrt[3]{2x}$.
7. Check (b): $(3x^2y)^3 \cdot 2x = 27x^6y^3 \cdot 2x = 54x^7y^3$ ✓.

Answer: (a) $4x^{1/2}$ (b) $3x^2y\sqrt[3]{2x}$

Example 2. Grace is building a cube-shaped wooden box that holds $\frac{1}{4}$ cubic foot. The edge of a cube with volume V is $\sqrt[3]{V}$. (a) Write the edge length in feet in simplest radical form. (b) Find the edge length to the nearest tenth of an inch.

Show solution

1. (a) $s = \sqrt[3]{\frac{1}{4}} = \frac{\sqrt[3]{1}}{\sqrt[3]{4}} = \frac{1}{\sqrt[3]{4}}$ by the quotient property.
2. $4 = 2^2$, so one more factor of 2 makes a perfect cube. Multiply by $\frac{\sqrt[3]{2}}{\sqrt[3]{2}}$.
3. $s = \frac{\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{\sqrt[3]{2}}{2}$ ft.
4. (b) $\sqrt[3]{2} \approx 1.259921$, so $s \approx 0.629961$ ft.
5. Convert: $0.629961 \times 12 \approx 7.56$ in, about 7.6 in.
6. Check: $7.56^3 \approx 432$ cubic inches, and $\frac{1}{4}$ of $12^3 = 1728$ is 432 ✓.

Answer: (a) $\frac{\sqrt[3]{2}}{2}$ ft (b) about 7.6 in

PRACTICE

1. Easy Simplify. Assume all variables are positive. (a) $x^{1/2} \cdot x^{1/4}$ (b) $\left(y^{2/3}\right)^3$ (c) $\frac{z^{5/6}}{z^{1/3}}$

   Show answer

   (a) $x^{3/4}$ (b) y^2 (c) $z^{5/6-2/6} = z^{1/2}$

2. Easy Simplify: (a) $\sqrt{50}$ (b) $\sqrt[3]{24}$ (c) $\sqrt[4]{32}$

   Show answer

   (a) $5\sqrt{2}$ (b) $\sqrt[3]{8 \cdot 3} = 2\sqrt[3]{3}$ (c) $\sqrt[4]{16 \cdot 2} = 2\sqrt[4]{2}$

3. Easy Simplify: (a) $\sqrt[3]{-250}$ (b) $\sqrt[5]{64}$

   Show answer

   (a) $\sqrt[3]{-125 \cdot 2} = -5\sqrt[3]{2}$ (b) $\sqrt[5]{32 \cdot 2} = 2\sqrt[5]{2}$

4. Easy Use the quotient property to simplify: (a) $\sqrt[3]{\frac{5}{8}}$ (b) $\sqrt[4]{\frac{3}{16}}$

   Show answer

   (a) $\frac{\sqrt[3]{5}}{2}$ (b) $\frac{\sqrt[4]{3}}{2}$

5. **Medium** Simplify, assuming all variables are positive: $\sqrt[3]{16x^5y^9}$

Show answer

$\sqrt[3]{8x^3y^9}\cdot\sqrt[3]{2x^2} = 2xy^3\sqrt[3]{2x^2}$

6. **Medium** Simplify, assuming all variables are positive. Write the answer with positive exponents: $\left(27a^6b^{-3}\right)^{1/3}$

Show answer

$3a^2b^{-1} = \dfrac{3a^2}{b}$

7. **Medium** Simplify, assuming all variables are positive: $\dfrac{\left(x^{1/3}y^{1/2}\right)^6}{xy^2}$

Show answer

$\dfrac{x^2y^3}{xy^2} = xy$

8. **Medium** Rationalize the denominator and simplify: (a) $\dfrac{6}{\sqrt{3}}$ (b) $\dfrac{3}{\sqrt[3]{9}}$

Show answer

(a) $\dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$ (b) multiply by $\dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$: $\dfrac{3\sqrt[3]{3}}{\sqrt[3]{27}} = \sqrt[3]{3}$

9. **Medium** (a) Find $\sqrt{(-7)^2}$. (b) For which real numbers x is $\sqrt{x^2} = x$ true? Write an inequality.

Show answer

(a) $\sqrt{49} = 7$ (b) $x \geq 0$ (for negative x, $\sqrt{x^2} = |x| = -x$)

10. **Hard** Rationalize the denominator and simplify: $\dfrac{10}{\sqrt[4]{8}}$

Show answer

$8 = 2^3$, so multiply by $\dfrac{\sqrt[4]{2}}{\sqrt[4]{2}}$: $\dfrac{10\sqrt[4]{2}}{\sqrt[4]{16}} = \dfrac{10\sqrt[4]{2}}{2} = 5\sqrt[4]{2}$

11. **Hard** A cube-shaped concrete base for a windmill tower holds 54 cubic feet of concrete. (a) Write its edge length in simplest radical form. (b) Find the edge length to the nearest hundredth of a foot.

Show answer

(a) $\sqrt[3]{54} = \sqrt[3]{27\cdot 2} = 3\sqrt[3]{2}$ ft (b) about 3.78 ft

12. Hard A family bakery's yearly sales grew from \$20,000 to \$26,620 in 3 years. The average yearly growth factor is $\left(\frac{26{,}620}{20{,}000}\right)^{1/3}$. Find (a) the growth factor by hand and (b) the average yearly percent growth.

Show answer

(a) $\frac{26{,}620}{20{,}000} = 1.331 = 1.1^3$, so the factor is $1.331^{1/3} = 1.1$ (b) 10% per year

13. Hard Explain why $\sqrt[3]{x^3} = x$ for every real number x, but $\sqrt{x^2} = x$ is false for some real numbers.

Show answer

A cube keeps the sign of its base, so the one real cube root of x^3 is x itself. But x^2 is the same for x and $-x$, and the square root sign always gives the nonnegative root. For $x = -3$, $\sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$. In general $\sqrt{x^2} = |x|$.

14. Challenge Assume $x > 0$. (a) Write $\sqrt{\sqrt[3]{x}}$ as a single radical. (b) Rewrite $\sqrt[6]{x^4}$ as a radical with a smaller index.

Show answer

(a) $\left(x^{1/3}\right)^{1/2} = x^{1/6} = \sqrt[6]{x}$ (b) $x^{4/6} = x^{2/3} = \sqrt[3]{x^2}$

COMMON MISTAKES

- Multiplying exponents when you should add them: $x^{1/2} \cdot x^{1/3} = x^{5/6}$, not $x^{1/6}$. Multiply exponents only for a power of a power.
- Taking out a factor that is not the right kind of power: $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$; writing $\sqrt[3]{9 \cdot 6} = 3\sqrt[3]{6}$ is wrong because 9 is a square, not a cube.
- Rationalizing a cube root by multiplying by the same cube root: $\frac{5}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$ leaves $\sqrt[3]{16}$ in the denominator. Multiply by what makes a perfect cube: $\sqrt[3]{2}$.
- Writing $\sqrt{x^2} = x$ when x might be negative. In general it is $|x|$; drop the absolute value only when you know $x \geq 0$.

> **Make it a game · Exponent Translator.** Write ten cards that mix forms: x^(2/3), the cube root of x squared, 16^(3/4) and the fourth root of 81. Translate each into the other form in two minutes, then simplify the numbers where you can. Next, simplify the cube root of 54 into 3 times the cube root of 2. Record your score and try again tomorrow.

6.3 Operations with Radical Expressions

OBJECTIVES

- Add and subtract radical expressions by simplifying first and combining like radicals.
- Multiply radical expressions with the same index, including binomials that contain radicals.
- Use conjugates to rationalize binomial denominators such as $\frac{6}{3-\sqrt{5}}$.
- Combine radicals with different indexes by writing them with rational exponents.

WATCH

Binomial Expansions with Surds

Eddie Woo · 8:17

https://www.youtube.com/watch?v=_yqLjaYjCQs

> **WHEN TO WATCH**
> Before Example 1: multiplying binomials that contain radicals.

Note: Eddie Woo teaches in Australia: he says 'surds' for radicals.

BACKUP VIDEO

Rationalize the Denominator with Radicals (Monomials & Binomials)

Mario's Math Tutoring · 5:12

https://www.youtube.com/watch?v=WI10UlLgHwg

Before the practice set: rationalizing a denominator with a conjugate.

KEY TERMS

like radicals

Radicals with the same index and the same radicand, such as $3\sqrt[3]{2}$ and $-5\sqrt[3]{2}$. Only like radicals can be combined by adding or subtracting.

conjugates

Two binomials of the form $a+\sqrt{b}$ and $a-\sqrt{b}$ (the same terms with the middle sign changed). Their product, a^2-b, has no radical.

binomial denominator

A denominator with two terms, such as $3 - \sqrt{5}$.

CORE IDEAS

Adding radicals works exactly like combining like terms. Since $\sqrt{2}$ is just a number, the distributive property gives $3\sqrt{2} + 5\sqrt{2} = (3+5)\sqrt{2} = 8\sqrt{2}$, the same way $3x + 5x = 8x$. But **unlike radicals do not combine**: $\sqrt{2} + \sqrt{3}$ is not $\sqrt{5}$. The table checks this with decimals.

Like and unlike radicals

Expression	Decimal value
√2 + √3	1.414 + 1.732 = 3.146
√5	2.236
√2 + √2	1.414 + 1.414 = 2.828
2√2	2.828

Radicals that look unlike are sometimes like radicals in disguise, so **simplify first**: $\sqrt{50} + \sqrt{18} = 5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$, and $\sqrt[3]{16} - \sqrt[3]{54} = 2\sqrt[3]{2} - 3\sqrt[3]{2} = -\sqrt[3]{2}$.

To multiply radicals with the same index, use the product property in reverse and then simplify: $\sqrt{6} \cdot \sqrt{15} = \sqrt{90} = \sqrt{9 \cdot 10} = 3\sqrt{10}$. For sums, use the distributive property (FOIL) just as with polynomials: $(2 + \sqrt{3})(4 - \sqrt{3}) = 8 - 2\sqrt{3} + 4\sqrt{3} - 3 = 5 + 2\sqrt{3}$, since $\sqrt{3} \cdot \sqrt{3} = 3$.

Conjugates are special because their product is a difference of squares: $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - a\sqrt{b} + a\sqrt{b} - (\sqrt{b})^2 = a^2 - b$. The middle terms cancel and the radical disappears. For example $(3 - \sqrt{5})(3 + \sqrt{5}) = 9 - 5 = 4$. (This is the same idea as multiplying a complex number by its conjugate in Lesson 3.3.)

That gives a way to rationalize a binomial denominator: multiply top and bottom by the conjugate of the denominator.

$$\frac{6}{3-\sqrt{5}}\cdot\frac{3+\sqrt{5}}{3+\sqrt{5}}=\frac{6(3+\sqrt{5})}{9-5}=\frac{18+6\sqrt{5}}{4}=\frac{9+3\sqrt{5}}{2}.$$

Check with decimals: $6\div 0.763932\approx 7.854102$ and $(9+6.708204)\div 2=7.854102$ ✓.

Radicals with different indexes cannot be multiplied with the product property directly, but rational exponents handle them. Write each as a power of the same base and add exponents: $\sqrt{x}\cdot\sqrt[3]{x}=x^{1/2}\cdot x^{1/3}=x^{3/6+2/6}=x^{5/6}=\sqrt[6]{x^5}$. The common denominator of the exponents becomes the index of the answer.

Here is a geometry use. Radicals appear whenever you use the distance formula, and a perimeter is a sum of distances, so you simplify each side and then combine like radicals.

KEY FORMULAS

$$a\sqrt[n]{x}+b\sqrt[n]{x}=(a+b)\sqrt[n]{x}$$

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$$

$$(a+\sqrt{b})(a-\sqrt{b})=a^2-b$$

Practice habit. Simplify every radical before you add, subtract or multiply. Then write one step per line, especially when you FOIL, so that each product of radicals is visible.

WORKED EXAMPLES

Example 1. Simplify: (a) $\sqrt{75}-\sqrt{12}+\sqrt{27}$ (b) $(2+\sqrt{3})(4-\sqrt{3})$

Show solution

1. (a) Simplify each radical: $\sqrt{75}=\sqrt{25\cdot 3}=5\sqrt{3}$, $\sqrt{12}=2\sqrt{3}$, $\sqrt{27}=3\sqrt{3}$.
2. All three are like radicals: $5\sqrt{3}-2\sqrt{3}+3\sqrt{3}=(5-2+3)\sqrt{3}=6\sqrt{3}$.
3. (b) FOIL: $2\cdot 4-2\sqrt{3}+4\sqrt{3}-\sqrt{3}\cdot\sqrt{3}$.
4. $=8-2\sqrt{3}+4\sqrt{3}-3$.
5. Combine: $5+2\sqrt{3}$.
6. Check (b) with decimals: $3.732\times 2.268\approx 8.464$ and $5+3.464=8.464$ ✓.

Answer: (a) $6\sqrt{3}$ (b) $5+2\sqrt{3}$

Example 2. A carpenter builds a rectangular window frame 2 feet wide and $1+\sqrt{5}$ feet long (a golden rectangle, a shape builders have long admired). (a) Write the ratio $\dfrac{\text{width}}{\text{length}}$ in simplest radical form with a rational denominator. (b) Show that $\dfrac{\text{length}}{\text{width}} - \dfrac{\text{width}}{\text{length}}$ is a whole number, and find it.

Show solution

1. (a) The ratio is $\dfrac{2}{1+\sqrt{5}}$. Multiply top and bottom by the conjugate $\sqrt{5}-1$.
2. Denominator: $(\sqrt{5}+1)(\sqrt{5}-1) = 5-1 = 4$.
3. $\dfrac{2(\sqrt{5}-1)}{4} = \dfrac{\sqrt{5}-1}{2} \approx 0.618$.
4. (b) $\dfrac{\text{length}}{\text{width}} = \dfrac{1+\sqrt{5}}{2}$.
5. Subtract: $\dfrac{1+\sqrt{5}}{2} - \dfrac{\sqrt{5}-1}{2} = \dfrac{1+\sqrt{5}-\sqrt{5}+1}{2} = \dfrac{2}{2} = 1$.
6. Check with decimals: $1.618 - 0.618 = 1$ ✓.

Answer: (a) $\dfrac{\sqrt{5}-1}{2}$ (b) the difference is 1

PRACTICE

1. Easy Simplify: (a) $4\sqrt{7}+9\sqrt{7}$ (b) $5\sqrt[3]{2}-8\sqrt[3]{2}$

   Show answer

   (a) $13\sqrt{7}$ (b) $-3\sqrt[3]{2}$

2. Easy Simplify: $\sqrt{18}+\sqrt{32}$

   Show answer

   $3\sqrt{2}+4\sqrt{2} = 7\sqrt{2}$

3. Easy Multiply and simplify: (a) $\sqrt{6}\cdot\sqrt{15}$ (b) $\sqrt[3]{4}\cdot\sqrt[3]{10}$

   Show answer

   (a) $\sqrt{90} = 3\sqrt{10}$ (b) $\sqrt[3]{40} = 2\sqrt[3]{5}$

4. Easy Multiply and simplify: $\sqrt{3}(\sqrt{6}+2)$

   Show answer

   $\sqrt{18}+2\sqrt{3} = 3\sqrt{2}+2\sqrt{3}$

5. **Medium** Simplify: $\sqrt[3]{54} + \sqrt[3]{16} - \sqrt[3]{250}$

Show answer

$3\sqrt[3]{2} + 2\sqrt[3]{2} - 5\sqrt[3]{2} = 0$

6. **Medium** Multiply and simplify: $(3 + \sqrt{2})(5 - 2\sqrt{2})$

Show answer

$15 - 6\sqrt{2} + 5\sqrt{2} - 4 = 11 - \sqrt{2}$

7. **Medium** Multiply and simplify: (a) $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$ (b) $(2\sqrt{5} - 1)^2$

Show answer

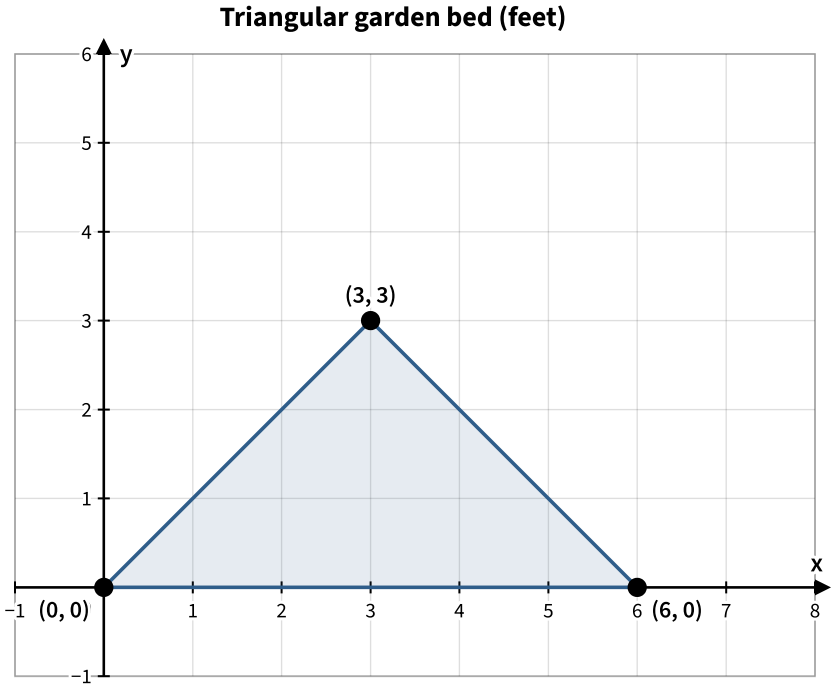
(a) $7 - 3 = 4$ (b) $20 - 4\sqrt{5} + 1 = 21 - 4\sqrt{5}$

8. **Medium** Rationalize the denominator and simplify: $\dfrac{8}{3 + \sqrt{5}}$

Show answer

$\dfrac{8(3 - \sqrt{5})}{9 - 5} = 2(3 - \sqrt{5}) = 6 - 2\sqrt{5}$

9. **Medium** A triangular garden bed has corners at the points shown (units are feet). (a) Find its exact perimeter in simplest radical form. (b) Round the perimeter to the nearest hundredth of a foot.



Show answer

(a) Sides: 6, $\sqrt{3^2+3^2} = 3\sqrt{2}$ and $3\sqrt{2}$, so the perimeter is $6 + 6\sqrt{2}$ ft (b) about 14.49 ft

10. **Hard** Rationalize the denominator and simplify: $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

Show answer

$$\frac{(\sqrt{3}+1)^2}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

11. **Hard** Write each as a single power or radical, assuming $x > 0$: (a) $\sqrt[3]{x}\cdot\sqrt[4]{x}$ (b) $\sqrt{2}\cdot\sqrt[3]{4}$ (c) $\dfrac{\sqrt[3]{x^2}}{\sqrt{x}}$

Show answer

(a) $x^{1/3+1/4} = x^{7/12}$ (b) $2^{1/2}\cdot 2^{2/3} = 2^{7/6} = 2\sqrt[6]{2}$ (c) $x^{2/3-1/2} = x^{1/6} = \sqrt[6]{x}$

12. Hard Henry writes $\sqrt{9}+\sqrt{16}=\sqrt{25}=5$. Explain his error and give the correct value.

Show answer

Square roots do not add that way: $\sqrt{a}+\sqrt{b}$ is not $\sqrt{a+b}$. Only like radicals combine. Correctly, $\sqrt{9}+\sqrt{16}=3+4=7$.

13. Hard An object dropped from a height of h feet hits the ground after $t=\dfrac{\sqrt{h}}{4}$ seconds (from $0=-16t^2+h$). A stone is dropped from a 48-foot silo roof and another from a 12-foot ladder. (a) Find the exact difference in their fall times in simplest radical form. (b) Round it to the nearest hundredth of a second.

Show answer

(a) $\dfrac{\sqrt{48}}{4}-\dfrac{\sqrt{12}}{4}=\dfrac{4\sqrt{3}-2\sqrt{3}}{4}=\dfrac{\sqrt{3}}{2}$ s (b) about 0.87 s

14. Challenge Show that $2+\sqrt{3}$ and $2-\sqrt{3}$ are both solutions of $x^2-4x+1=0$. Then find (a) their sum and (b) their product.

Show answer

$(2\pm\sqrt{3})^2-4(2\pm\sqrt{3})+1=(7\pm4\sqrt{3})-8\mp4\sqrt{3}+1=0$ ✓. (a) sum $=4$, matching $-\dfrac{b}{a}=4$ (b) product $=4-3=1$, matching $\dfrac{c}{a}=1$ (Lesson 3.4)

15. Challenge Rationalize the denominator of $\dfrac{1}{\sqrt[3]{2}-1}$. (Hint: use $(a-b)(a^2+ab+b^2)=a^3-b^3$ with $a=\sqrt[3]{2}$ and $b=1$.)

Show answer

Multiply top and bottom by $\sqrt[3]{4}+\sqrt[3]{2}+1$. The denominator becomes $(\sqrt[3]{2})^3-1^3=1$, so the result is $\sqrt[3]{4}+\sqrt[3]{2}+1$.

COMMON MISTAKES

- Adding radicands: $\sqrt{2}+\sqrt{3}\neq\sqrt{5}$. Only like radicals combine, and you combine the coefficients, not the radicands.
- Forgetting to simplify before deciding that radicals are unlike: $\sqrt{50}$ and $\sqrt{18}$ are both multiples of $\sqrt{2}$.
- Rationalizing $\dfrac{6}{3-\sqrt{5}}$ by multiplying by $\dfrac{\sqrt{5}}{\sqrt{5}}$. That leaves a radical in the denominator; multiply by the conjugate $3+\sqrt{5}$ instead.

- Squaring a binomial term by term: $(2\sqrt{5}-1)^2$ is not $20+1$. Write it as $(2\sqrt{5}-1)(2\sqrt{5}-1)$ and FOIL to get the middle term $-4\sqrt{5}$.

> **Make it a game · Vanishing Radicals.** Perform a math magic trick. The parent names a binomial with a radical, such as $3+\sqrt{2}$. Write its conjugate and multiply the pair so the radical vanishes, giving 7. Say 'ta-da'. Repeat with three more binomials. Then add $\sqrt{48}+\sqrt{27}$ by simplifying first, and explain why the denominator trick works for $2/(1+\sqrt{3})$.

6.4 Radical Functions and Their Graphs

OBJECTIVES

- Graph $y = a\sqrt{x-h}+k$ and $y = a\sqrt[3]{x-h}+k$ as transformations of the parent functions and state their domains and ranges.
- Explain why $y = \sqrt{x}$ is the inverse of $y = x^2$ for $x \geq 0$.
- Find inverses of simple quadratic and cubic functions, restricting the domain when needed.
- Use radical models for pendulums, the distance to the horizon and the speed of water leaving a tank.

WATCH

Graph Square Root and Cube Root Functions

Mario's Math Tutoring · 6:28

https://www.youtube.com/watch?v=PPVW33Qnv4U

> **WHEN TO WATCH**
> Before Example 1: graphing square root and cube root functions with transformations.

BACKUP VIDEO

Investigating Inverse Functions (2 of 3: The Problem with Quadratics)

Eddie Woo · 6:02

https://www.youtube.com/watch?v=pW3ocb4w4nI

After Example 2: why the square root function is the inverse of half a parabola.

KEY TERMS

radical function

A function whose rule contains the variable under a radical, such as $f(x) = 2\sqrt{x-1}$ or $g(x) = \sqrt[3]{x+4}$.

square root function

The parent function $y = \sqrt{x}$, with domain $[0, \infty)$ and range $[0, \infty)$.

cube root function

The parent function $y = \sqrt[3]{x}$, with domain and range all real numbers.

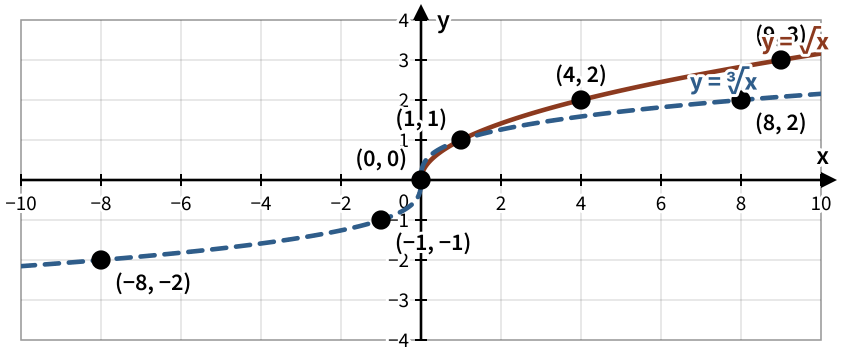
restricted domain

A smaller domain chosen on purpose, often to make a function one-to-one so that it has an inverse function; for example $y = x^2$ with $x \geq 0$.

CORE IDEAS

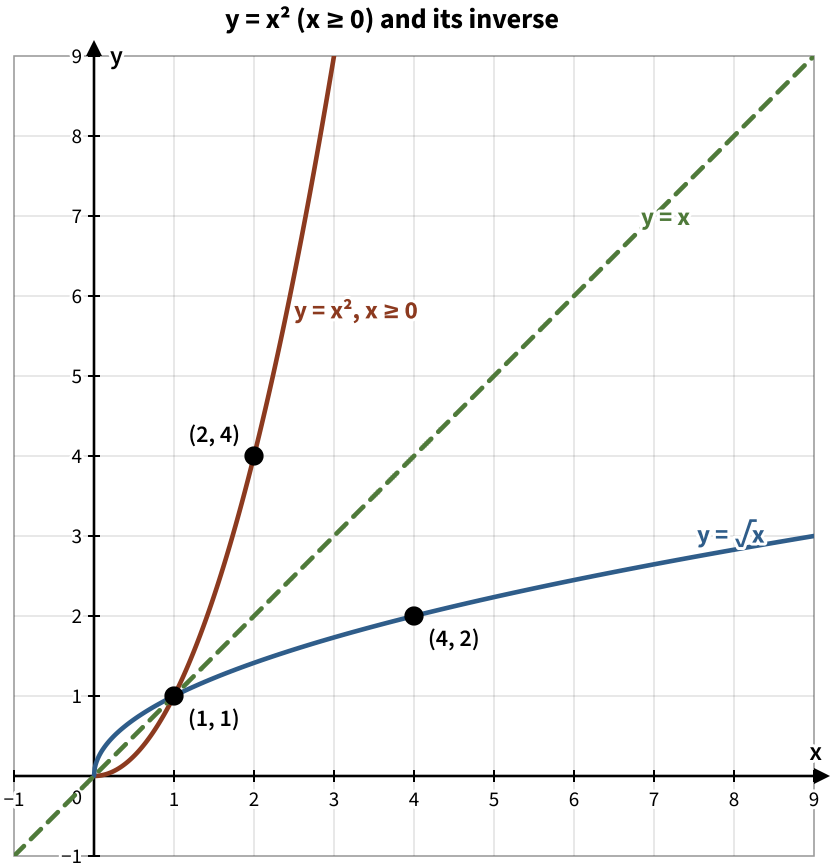
Start with the two parent functions. The **square root function** $y = \sqrt{x}$ is defined only for $x \geq 0$; its key points are $(0, 0)$, $(1, 1)$, $(4, 2)$ and $(9, 3)$, and it rises more and more slowly. The **cube root function** $y = \sqrt[3]{x}$ is defined for every real number, since odd roots of negatives are real; its key points are $(-8, -2)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$ and $(8, 2)$, and it is symmetric about the origin.

The parent radical functions



Everything from Lesson 1.2 applies. In $y = a\sqrt{x - h} + k$, the graph shifts h units right and k units up, and a stretches it vertically (and reflects it in the x-axis if $a < 0$). The starting point $(0, 0)$ moves to (h, k). The **domain** comes from the radicand: we need $x - h \geq 0$, so $x \geq h$. The **range** comes from the fact that $\sqrt{x - h} \geq 0$: if $a > 0$ then $y \geq k$, and if $a < 0$ then $y \leq k$. For $y = a\sqrt[3]{x - h} + k$, the point $(0, 0)$ moves to (h, k), but the domain and range are both all real numbers.

Why is $\sqrt{x}$ the inverse of x^2? The function $y = x^2$ is not one-to-one (2^2 and $(-2)^2$ are both 4), so it has no inverse function on all real numbers. Restrict it to $x \geq 0$. Swapping x and y gives $x = y^2$ with $y \geq 0$, so $y = \sqrt{x}$. As in Lesson 1.5, the graphs are reflections in the line $y = x$: the point $(2, 4)$ on $y = x^2$ becomes $(4, 2)$ on $y = \sqrt{x}$.



The same steps find other inverses. For $f(x) = (x-1)^2 + 2$ with $x \geq 1$, swap to get $x = (y-1)^2 + 2$, so $(y-1)^2 = x - 2$. Because $y \geq 1$, take the positive root: $y - 1 = \sqrt{x-2}$, and $f^{-1}(x) = 1 + \sqrt{x-2}$ with domain $x \geq 2$ (the range of f). A cubic like $g(x) = x^3 - 4$ is already one-to-one, so no restriction is needed: $x = y^3 - 4$ gives $g^{-1}(x) = \sqrt[3]{x+4}$.

Radical functions describe many real things. A pendulum L feet long swings back and forth in $T = 2\pi\sqrt{\dfrac{L}{32}}$ seconds. From a height of h feet you can see about $d \approx 1.22\sqrt{h}$ miles to the horizon. Water leaves a hole in a tank at about $v = 8\sqrt{h}$ feet per second, where h is the depth of water above the hole in feet. In each, multiplying the input by 4 only doubles the output, because $\sqrt{4h} = 2\sqrt{h}$.

KEY FORMULAS

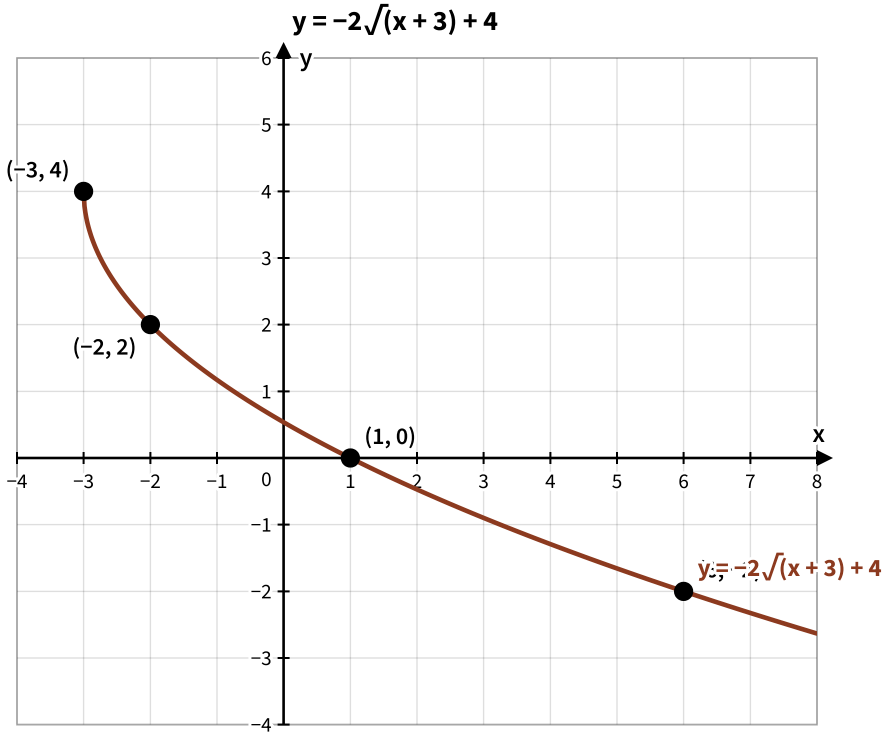
$$y = a\sqrt{x-h}+k: \quad \text{domain } x \geq h;\ \text{range } y \geq k\ (a>0) \text{ or } y \leq k\ (a<0)$$

$$y = a\sqrt[3]{x-h}+k: \quad \text{domain and range all real numbers}$$

$$T = 2\pi\sqrt{\frac{L}{32}}, \qquad d \approx 1.22\sqrt{h}, \qquad v = 8\sqrt{h}$$

Practice habit. Sketch before you compute: plot the starting point (h, k) first, then use the key points of the parent function, $(0,0)$, $(1,1)$, $(4,2)$, $(9,3)$ for $\sqrt{x}$ and $(-8,-2)$, $(-1,-1)$, $(0,0)$, $(1,1)$, $(8,2)$ for $\sqrt[3]{x}$, stretched by a.

Example 1. Graph $y = -2\sqrt{x+3} + 4$. Describe it as a transformation of $y = \sqrt{x}$ and state its domain and range in interval notation.



Show solution

1. Write it as $y = -2\sqrt{x-(-3)} + 4$: $h = -3$, $k = 4$, $a = -2$.
2. The graph shifts 3 left and 4 up, is stretched vertically by 2, and is reflected in the x-axis (it goes down from its starting point).
3. Starting point: $(-3, 4)$.
4. More points: $x = -2$ gives $-2\sqrt{1} + 4 = 2$; $x = 1$ gives $-2\sqrt{4} + 4 = 0$; $x = 6$ gives $-2\sqrt{9} + 4 = -2$.
5. Domain: $x + 3 \geq 0$, so $x \geq -3$, that is $[-3, \infty)$.
6. Range: $\sqrt{x+3} \geq 0$, so $-2\sqrt{x+3} \leq 0$ and $y \leq 4$, that is $(-\infty, 4]$.

Answer: Domain $[-3, \infty)$; range $(-\infty, 4]$

Example 2. A pendulum L feet long has period $T = 2\pi\sqrt{\dfrac{L}{32}}$ seconds (one full swing out and back). (a) Find the period of a 2-foot pendulum in a family's wall clock, exactly and to the nearest hundredth of a second. (b) If the pendulum is made 4 times as long, by what factor does the period change? Explain.

Show solution

1. (a) $\dfrac{L}{32} = \dfrac{2}{32} = \dfrac{1}{16}$.
2. $\sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$, so $T = 2\pi \cdot \dfrac{1}{4} = \dfrac{\pi}{2}$ seconds.
3. $\dfrac{\pi}{2} \approx 1.57$ seconds.
4. (b) Replace L by $4L$: $2\pi\sqrt{\dfrac{4L}{32}} = 2\pi \cdot \sqrt{4} \cdot \sqrt{\dfrac{L}{32}}$.
5. $= 2 \cdot 2\pi\sqrt{\dfrac{L}{32}} = 2T$, so the period doubles.
6. Check: an 8-foot pendulum has $T = 2\pi\sqrt{\dfrac{8}{32}} = 2\pi \cdot \dfrac{1}{2} = \pi$, which is twice $\dfrac{\pi}{2}$ ✓.

Answer: (a) $\dfrac{\pi}{2} \approx 1.57$ s (b) it doubles (factor 2), because $\sqrt{4L} = 2\sqrt{L}$

PRACTICE

1. Easy For $y = \sqrt{x - 5} + 1$, give (a) the domain and (b) the range in interval notation.

   Show answer

   (a) $[5, \infty)$ (b) $[1, \infty)$

2. Easy (a) Describe how the graph of $y = \sqrt[3]{x + 2} - 3$ is obtained from $y = \sqrt[3]{x}$. (b) Where does the point $(0, 0)$ of the parent graph move?

   Show answer

   (a) Shift 2 units left and 3 units down. (b) $(-2, -3)$

3. Easy For $f(x) = 3\sqrt{x - 1}$, find $f(1)$, $f(5)$ and $f(10)$.

   Show answer

   0, $3\sqrt{4} = 6$, $3\sqrt{9} = 9$

4. **Easy** From a height of h feet you can see about $d = 1.22\sqrt{h}$ miles to the horizon. How far can Samuel see from the top of a 64-foot grain elevator on flat prairie? Round to the nearest tenth of a mile.

Show answer

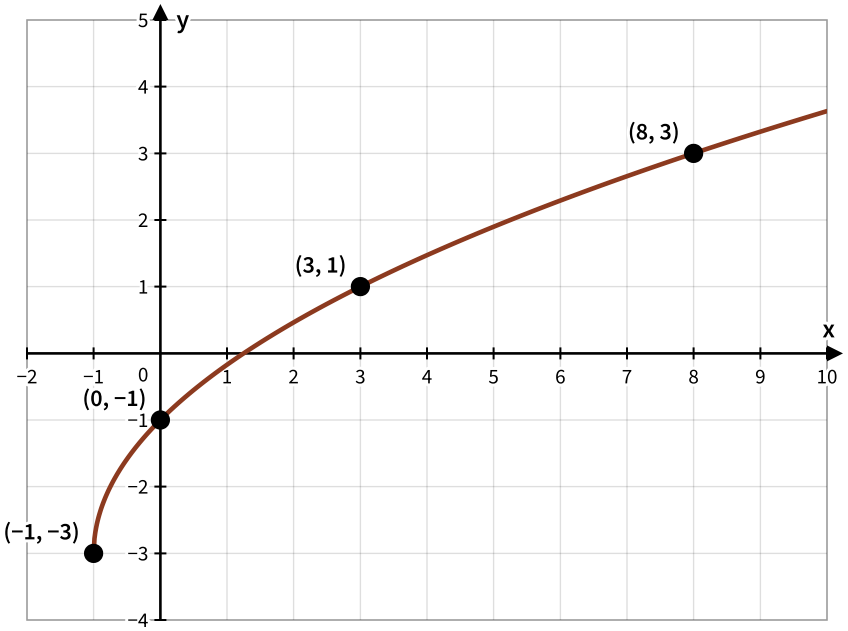
$1.22\sqrt{64} = 1.22 \times 8 = 9.76$, about 9.8 miles

5. **Medium** For $y = -\sqrt{4 - x}$, give (a) the domain and (b) the range in interval notation.

Show answer

(a) $4 - x \geq 0$, so $(-\infty, 4]$ (b) $\sqrt{4 - x} \geq 0$, so $y \leq 0$: $(-\infty, 0]$

6. **Medium** The graph has the form $y = a\sqrt{x - h} + k$. Use the labeled points to write its equation.



Show answer

Starting point $(-1, -3)$ gives $h = -1$, $k = -3$. From $(3, 1)$: $1 = a\sqrt{4} - 3$, so $a = 2$. $y = 2\sqrt{x + 1} - 3$

7. **Medium** Let $f(x) = x^2 + 3$ with $x \geq 0$. (a) Find $f^{-1}(x)$. (b) Give the domain of f^{-1}.

Show answer

(a) $x = y^2 + 3$, $y = \sqrt{x - 3}$ (positive root since $y \geq 0$), so $f^{-1}(x) = \sqrt{x - 3}$ (b) $x \geq 3$

8. **Medium** Find the inverse of $g(x) = 2x^3 - 1$.

Show answer

$x = 2y^3 - 1$, so $y^3 = \dfrac{x + 1}{2}$ and $g^{-1}(x) = \sqrt[3]{\dfrac{x + 1}{2}}$

9. **Medium** Water flows out of a hole near the bottom of a farm's stock tank at about $v = 8\sqrt{h}$ feet per second, where h is the depth of water above the hole in feet. Find the speed when the depth is (a) 9 feet and (b) 2.25 feet.

Show answer

(a) $8 \times 3 = 24$ ft/s (b) $8 \times 1.5 = 12$ ft/s

10. **Hard** Use $T = 2\pi\sqrt{\dfrac{L}{32}}$ to find the period of a 3.25-foot pendulum in a grandfather clock. Round to the nearest hundredth of a second.

Show answer

$\dfrac{3.25}{32} = 0.1015625$, $\sqrt{0.1015625} \approx 0.318689$, $T \approx 2\pi(0.318689) \approx 2.00$ s

11. **Hard** Let $f(x) = (x - 2)^2 + 1$ with $x \geq 2$. (a) Find $f^{-1}(x)$. (b) Give the domain of f^{-1}.

Show answer

(a) $x = (y - 2)^2 + 1$, $y - 2 = \sqrt{x - 1}$ (positive since $y \geq 2$), so $f^{-1}(x) = 2 + \sqrt{x - 1}$ (b) $x \geq 1$, the range of f

12. **Hard** Explain why $y = \sqrt{x}$ is the inverse of $y = x^2$ only when the domain of $y = x^2$ is restricted to $x \geq 0$.

Show answer

$y = x^2$ is not one-to-one: for example $(-3)^2 = 9 = 3^2$, so its graph fails the horizontal line test and an inverse could not decide whether 9 goes back to 3 or -3. On $x \geq 0$ each output comes from only one input, and $\sqrt{x}$ always gives the nonnegative one, so $\sqrt{x^2} = x$ there.

13. Hard A lighthouse lamp is 81 feet above the sea, and a sailor's lookout on a ship is 25 feet above the sea. The lamp first comes into view when the ship is the sum of the two horizon distances away, $1.22\sqrt{81} + 1.22\sqrt{25}$ miles. How far away is that? Round to the nearest tenth of a mile.

Show answer

$1.22(9 + 5) = 1.22 \times 14 = 17.08$, about 17.1 miles

14. Challenge Let $f(x) = -\sqrt{x + 4} + 1$. (a) Find $f^{-1}(x)$. (b) Give the domain of f^{-1}.

Show answer

(a) $x = -\sqrt{y + 4} + 1$ gives $\sqrt{y + 4} = 1 - x$, so $y = (1 - x)^2 - 4$ (b) the range of f is $y \leq 1$, so the domain of f^{-1} is $x \leq 1$

COMMON MISTAKES

- Shifting the wrong way: $y = \sqrt{x + 3}$ moves 3 units left, not right, because the input must be 3 smaller to give the same output.
- Giving the domain of $y = \sqrt{x - h} + k$ as $x \geq k$. The domain comes from the radicand ($x \geq h$); k affects the range.
- Writing $f^{-1}(x) = \pm\sqrt{x - 3}$. An inverse function gives one output; use the original restricted domain to choose the sign.
- Forgetting that a cube root function has all real numbers as its domain. Only even roots need a nonnegative radicand.

> **Make it a game · Swap the Coordinates.** Plot y = x² for x ≥ 0 using the points (0, 0), (1, 1), (2, 4) and (3, 9). Now swap each point's coordinates and plot the new ones; the curve is y = √x, the reflection in y = x. Then transform y = 2√(x - 1) + 3 by moving the key points, and state its domain and range.

6.5 Solving Radical Equations

OBJECTIVES

- Solve radical equations by isolating a radical and raising both sides to the power of the index.
- Explain why raising both sides to an even power can introduce extraneous solutions, and check every answer in the original equation.
- Solve equations with two square roots and equations with rational exponents such as $x^{3/2} = 27$ and $(x+1)^{2/3} = 4$.
- Use radical equations to find the length of a pendulum for a given period and the height needed to see a given distance.

WATCH

Solving Equations with Radical Functions (1 of 2: Initial steps & discussion)

Eddie Woo · 10:58

https://www.youtube.com/watch?v=T5Be98VX24M

> **WHEN TO WATCH**
> Before Example 1: isolating the radical, squaring and checking.

Note: Eddie Woo teaches in Australia: he says 'surds' for radicals.

BACKUP VIDEO

Solve Radical Equations with 2 Square Roots

Mario's Math Tutoring · 6:29

https://www.youtube.com/watch?v=c0jONsjJ_eU

Before the practice set: equations with two square roots.

KEY TERMS

radical equation

An equation with the variable inside a radical, such as $\sqrt{x+2} = x$, or raised to a rational exponent, such as $x^{3/2} = 27$.

isolate a radical

Use inverse operations so that a radical stands alone on one side of the equation before you raise both sides to a power.

extraneous solution

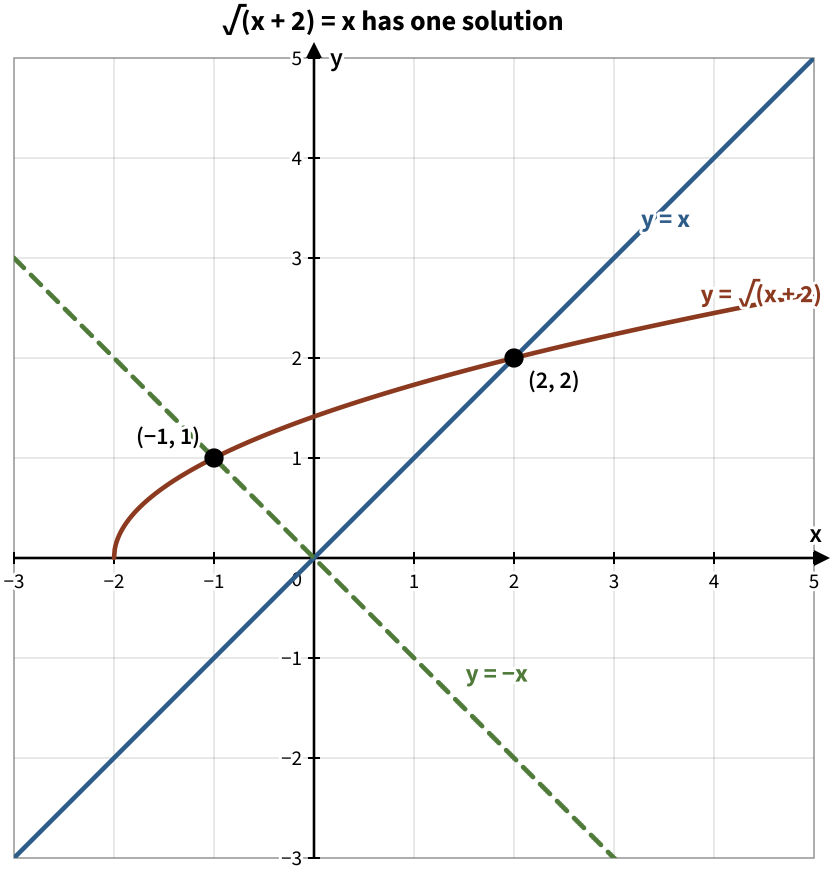
A number produced by the solving steps that does not satisfy the original equation. It must be rejected.

CORE IDEAS

To undo a square root, square; to undo a cube root, cube. The plan for a radical equation is: (1) isolate the radical, (2) raise both sides to the power of the index, (3) solve the resulting equation, (4) check every answer in the original equation. Step 2 is allowed because equal numbers have equal powers: if $a = b$, then $a^n = b^n$.

But the reverse is not true for even powers. If $a^2 = b^2$, then $a = b$ **or** $a = -b$; squaring erases the difference between a number and its opposite, since $3^2 = (-3)^2$. So the squared equation can have solutions that belong to a different equation, one with the sign changed. Those are **extraneous solutions**.

Watch it happen. Solve $\sqrt{x+2} = x$. Squaring gives $x + 2 = x^2$, so $x^2 - x - 2 = 0$ and $(x-2)(x+1) = 0$: $x = 2$ or $x = -1$. Check: $\sqrt{2+2} = 2$ ✓, but $\sqrt{-1+2} = 1 \neq -1$ ✗. The number -1 actually solves $\sqrt{x+2} = -x$; squaring could not tell the two equations apart. The graph shows it: $y = \sqrt{x+2}$ meets $y = x$ only at $(2, 2)$, and meets $y = -x$ at $(-1, 1)$. So the only solution is $x = 2$.



Odd powers do not cause this trouble, because $a^3 = b^3$ forces $a = b$ (a cube keeps its sign). Still, check anyway: a check also catches arithmetic slips.

When an equation has **two square roots**, isolate one, square, then isolate the one that remains and square again. For $\sqrt{x+5} - \sqrt{x} = 1$, write $\sqrt{x+5} = 1 + \sqrt{x}$. Squaring the right side needs FOIL: $x + 5 = 1 + 2\sqrt{x} + x$. Then $4 = 2\sqrt{x}$, $\sqrt{x} = 2$ and $x = 4$. Check: $\sqrt{9} - \sqrt{4} = 3 - 2 = 1$ ✓.

For a **rational exponent**, raise both sides to the reciprocal power, because $\left(x^{m/n}\right)^{n/m} = x$. So $x^{3/2} = 27$ gives $x = 27^{2/3} = 9$; check $9^{3/2} = 3^3 = 27$ ✓. Be careful when the numerator of the exponent is even: $(x+1)^{2/3} = 4$ means $\left(\sqrt[3]{x+1}\right)^2 = 4$, so $\sqrt[3]{x+1} = \pm 2$. Then $x + 1 = \pm 8$, giving $x = 7$ or $x = -9$, and both check: $8^{2/3} = 4$ and $(-8)^{2/3} = (-2)^2 = 4$. Simply raising both sides to the $\dfrac{3}{2}$ power would have lost $x = -9$.

KEY FORMULAS

$$\text{If } a = b, \text{ then } a^n = b^n \quad (\text{but } a^2 = b^2 \text{ only gives } a = \pm b)$$

$$x^{m/n} = c \;\Rightarrow\; x = c^{n/m} \quad (\text{use } \pm \text{ when } m \text{ is even})$$

$$T = 2\pi\sqrt{\frac{L}{32}} \iff L = \frac{8T^2}{\pi^2}$$

Practice habit. Check every answer in the ORIGINAL equation, not in a squared or rearranged one. An answer that fails the check is extraneous: cross it out and say so.

WORKED EXAMPLES

Example 1. Solve each equation and check: (a) $\sqrt{2x+7} - x = 2$ (b) $(x+1)^{2/3} = 4$

Show solution

1. (a) Isolate the radical: $\sqrt{2x+7} = x + 2$.
2. Square both sides: $2x + 7 = x^2 + 4x + 4$.
3. Rearrange: $x^2 + 2x - 3 = 0$, so $(x+3)(x-1) = 0$ and $x = -3$ or $x = 1$.
4. Check $x = 1$: $\sqrt{9} - 1 = 3 - 1 = 2$ ✓.
5. Check $x = -3$: $\sqrt{1} - (-3) = 1 + 3 = 4 \neq 2$ ✗, so -3 is extraneous.
6. (b) Write it as $\left(\sqrt[3]{x+1}\right)^2 = 4$, so $\sqrt[3]{x+1} = 2$ or $\sqrt[3]{x+1} = -2$.
7. Cube: $x + 1 = 8$ or $x + 1 = -8$, so $x = 7$ or $x = -9$.
8. Check: $8^{2/3} = 2^2 = 4$ ✓ and $(-8)^{2/3} = (-2)^2 = 4$ ✓.

Answer: (a) $x = 1$ ($x = -3$ is extraneous) (b) $x = -9$ or $x = 7$

Example 2. (a) A clockmaker wants a pendulum with a period of exactly 2 seconds. Use $T = 2\pi\sqrt{\dfrac{L}{32}}$ (L in feet) to find the length, to the nearest hundredth of a foot. (b) From a height of h feet you can see about $d = 1.22\sqrt{h}$ miles. How high must a fire lookout tower on a hilltop reach above the surrounding plain to see 15 miles? Round to the nearest tenth of a foot.

Show solution

1. (a) Divide by 2π: $\sqrt{\dfrac{L}{32}} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$.
2. Square: $\dfrac{L}{32} = \dfrac{1}{\pi^2}$, so $L = \dfrac{32}{\pi^2} \approx 3.2423$ ft, about 3.24 ft.
3. Check: $2\pi\sqrt{\dfrac{3.2423}{32}} \approx 2\pi(0.31831) \approx 2.000$ s ✓.
4. (b) Divide by 1.22: $\sqrt{h} = \dfrac{15}{1.22} \approx 12.29508$.
5. Square: $h \approx 151.169$, about 151.2 ft.
6. Check: $1.22\sqrt{151.169} \approx 1.22(12.29508) \approx 15.00$ miles ✓.

Answer: (a) $L = \dfrac{32}{\pi^2} \approx 3.24$ ft (b) about 151.2 ft

PRACTICE

1. Easy Solve and check: $\sqrt{x - 3} = 5$

   Show answer

   $x - 3 = 25$, so $x = 28$; check: $\sqrt{25} = 5$ ✓

2. Easy Solve and check: $\sqrt[3]{2x + 1} = 3$

   Show answer

   $2x + 1 = 27$, so $x = 13$; check: $\sqrt[3]{27} = 3$ ✓

3. Easy Solve and check: $\sqrt{3x} + 4 = 10$

   Show answer

   $\sqrt{3x} = 6$, $3x = 36$, $x = 12$; check: $\sqrt{36} + 4 = 10$ ✓

4. **Easy** Solve: $\sqrt{x+1}+6=2$

Show answer

No solution: $\sqrt{x+1}=-4$, but a principal square root is never negative. (Squaring would give $x=15$, which fails: $\sqrt{16}+6=10\neq 2$.)

5. **Medium** Solve and check: $\sqrt{x+6}=x$

Show answer

$x^2-x-6=0$, $(x-3)(x+2)=0$. $x=3$ checks ($\sqrt{9}=3$); $x=-2$ fails ($\sqrt{4}=2\neq -2$). So $x=3$.

6. **Medium** Solve: (a) $x^{3/2}=8$ (b) $x^{2/3}=9$

Show answer

(a) $x=8^{2/3}=4$; check $4^{3/2}=8$ ✓ (b) $\sqrt[3]{x}=\pm 3$, so $x=-27$ or $x=27$; both check

7. **Medium** Solve and check: $(x-2)^{3/4}=8$

Show answer

$x-2=8^{4/3}=2^4=16$, so $x=18$; check: $16^{3/4}=2^3=8$ ✓

8. **Medium** Solve and check: $\sqrt{5x-4}=\sqrt{x+8}$

Show answer

$5x-4=x+8$, $x=3$; check: $\sqrt{11}=\sqrt{11}$ ✓

9. **Medium** Using $d=1.22\sqrt{h}$, how high (in feet) must Ruth climb on a hillside overlooking a flat valley to see 12.2 miles across it?

Show answer

$\sqrt{h}=\dfrac{12.2}{1.22}=10$, so $h=100$ ft

10. **Hard** Solve and check: $\sqrt{x+7}-\sqrt{x}=1$

Show answer

$\sqrt{x+7}=1+\sqrt{x}$, $x+7=1+2\sqrt{x}+x$, $\sqrt{x}=3$, $x=9$; check: $\sqrt{16}-\sqrt{9}=1$ ✓

11. Hard Solve and check: $\sqrt{2x-1} = x - 2$

Show answer

$2x - 1 = x^2 - 4x + 4$, $x^2 - 6x + 5 = 0$, $x = 1$ or $x = 5$. $x = 5$ checks ($\sqrt{9} = 3$); $x = 1$ fails ($\sqrt{1} = 1 \neq -1$). So $x = 5$.

12. Hard Henry is building a pendulum for a science fair that should have a period of 1.5 seconds. Use $T = 2\pi\sqrt{\dfrac{L}{32}}$ to find the length in feet, to the nearest hundredth.

Show answer

$$L = \frac{8T^2}{\pi^2} = \frac{8(2.25)}{\pi^2} = \frac{18}{\pi^2} \approx 1.82 \text{ ft}$$

13. Hard Squaring $\sqrt{x} = -3$ gives $x = 9$. Explain why 9 is not a solution and why squaring produced it.

Show answer

$\sqrt{9} = 3$, not -3; a principal square root is never negative, so the equation has no solution. Squaring gives $x = 9$ because 3^2 and $(-3)^2$ are both 9: squaring cannot tell $\sqrt{x} = -3$ from $\sqrt{x} = 3$, and 9 solves the second equation.

14. Challenge Solve and check: $\sqrt{x+5} + \sqrt{x-3} = 4$

Show answer

$\sqrt{x+5} = 4 - \sqrt{x-3}$, $x + 5 = 16 - 8\sqrt{x-3} + x - 3$, $8\sqrt{x-3} = 8$, $\sqrt{x-3} = 1$, $x = 4$; check: $\sqrt{9} + \sqrt{1} = 4$ ✓

COMMON MISTAKES

- Squaring before isolating the radical. In $\sqrt{x} + 2 = 5$, squaring the left side gives $x + 4\sqrt{x} + 4$, not $x + 4$. Subtract 2 first.
- Squaring a binomial term by term: $(x + 2)^2 = x^2 + 4x + 4$, not $x^2 + 4$. Always FOIL.
- Skipping the check, or checking in the squared equation. Extraneous solutions satisfy the squared equation, so only the original equation can catch them.
- Losing a solution with an even exponent numerator: $x^{2/3} = 9$ has two solutions, ± 27, because $\sqrt[3]{x} = \pm 3$.

Make it a game · Scrambled Solution. The parent writes the steps for solving $\sqrt{(x + 7)} = x - 5$ on separate paper strips and shuffles them. Put the steps in the right order, then check both $x = 9$ and $x = 2$ in the original equation. Explain why squaring brought in the extraneous one. Then scramble a cube-root solution for the parent.

UNIT 6

Radicals and Rational Exponents

In Algebra I you worked with square roots and with whole-number exponents. This unit joins those two ideas. You will see why an exponent like $\frac{1}{3}$ has to mean a cube root if the exponent rules are to keep working, and then use that idea to evaluate, simplify and combine expressions with roots of any index. You will graph square root and cube root functions as transformations of their parent graphs and see that $y = \sqrt{x}$ is the inverse of $y = x^2$ on a restricted domain.

The unit ends with radical equations, where squaring both sides can quietly add wrong answers, so you will learn why extraneous solutions appear and how a careful check catches them. Along the way you will use radicals to find the period and length of a pendulum, how far you can see from a height, and how fast water leaves a tank.

Rational exponents prepare the way for Unit 7, where exponents can be any real number and the number e appears, and for Unit 8, where logarithms undo exponents. The habit of checking answers in the original equation returns with logarithmic equations. The midterm covering Units 1 to 6 follows this unit.

PREREQUISITES

- Exponent rules for integer exponents, including zero and negative exponents (Algebra I)
- Simplifying square roots and rationalizing one-term denominators (Algebra I)
- Multiplying binomials and the special products, including the difference of squares (Lesson 4.2)
- Solving quadratic equations by factoring (Lesson 3.2)
- Transformations of parent functions, $y = a \cdot f(x - h) + k$ (Lesson 1.2)
- Finding inverse functions and restricting a domain (Lesson 1.5)

MASTERY CHECKLIST

- ☐ I can find real nth roots and tell when an even root of a number is not real.
- ☐ I can explain why $a^{1/n}$ means $\sqrt[n]{a}$ and evaluate expressions like $27^{2/3}$ and $32^{-3/5}$ by hand.
- ☐ I can convert between radical form and rational exponent form.
- ☐ I can simplify expressions with rational exponents using the exponent rules.
- ☐ I can simplify nth roots, including roots with variables, and explain why $\sqrt{x^2} = |x|$.
- ☐ I can rationalize denominators with one term, such as $\frac{5}{\sqrt[3]{4}}$, and with two terms using conjugates.
- ☐ I can add, subtract and multiply radical expressions, and combine radicals with different indexes using rational exponents.
- ☐ I can graph $y = a\sqrt{x - h} + k$ and $y = a\sqrt[3]{x - h} + k$ and state their domains and ranges.

- ☐ I can find inverses of simple quadratic and cubic functions, restricting the domain when needed.
- ☐ I can solve radical equations and equations with rational exponents, and explain why extraneous solutions can appear.
- ☐ I can use radical models for pendulums, the distance to the horizon and the speed of water from a tank.

LESSONS IN THIS UNIT

6.1	**nth Roots and Rational Exponents**	2 sessions of 45 minutes
6.2	**Properties of Rational Exponents and Radicals**	2 sessions of 45 minutes
6.3	**Operations with Radical Expressions**	2 sessions of 45 minutes
6.4	**Radical Functions and Their Graphs**	2 sessions of 45 minutes
6.5	**Solving Radical Equations**	2 sessions of 45 minutes

Unit 6 test

Time: 60 minutes. Do not use a calculator except on problems that ask for a rounded decimal answer; then a scientific calculator is allowed. Show your work on every problem. Give exact answers in simplest form (simplified radicals, rational denominators) unless a problem asks you to round. When simplifying expressions, assume all variables are positive unless a problem says otherwise; when solving equations, give every real solution. Check every solution of an equation in the original equation and state any extraneous solutions.

1. **6 pts**

   Evaluate by hand: (a) $125^{2/3}$ (b) $16^{-5/4}$ (c) $(-32)^{3/5}$

2. **5 pts**

   (a) Write $\sqrt[7]{x^4}$ with a rational exponent. (b) Write $y^{-3/5}$ in radical form with no negative exponent.

3. **6 pts**

   Simplify: $\dfrac{\left(a^{3/4}b^{-1/2}\right)^4}{a^2 b^{-3}}$

4. **6 pts**

   Simplify: $\sqrt[3]{250a^8b^4}$

5. **6 pts**

   Rationalize the denominator and simplify: $\dfrac{12}{\sqrt[3]{6}}$

6. **6 pts**

   A rectangular herb garden beside a church hall measures $\sqrt{45}$ meters by $\sqrt{20}$ meters. Find (a) its exact perimeter in simplest radical form and (b) its area.

7. **6 pts**

   Multiply and simplify: $(4-\sqrt{6})(2+3\sqrt{6})$

8. **7 pts**

   Rationalize the denominator and simplify: $\dfrac{10}{\sqrt{7}-\sqrt{2}}$

9. **5 pts**

   Write $\sqrt{x}\cdot\sqrt[5]{x^2}$ as a single radical.

10. **5 pts**

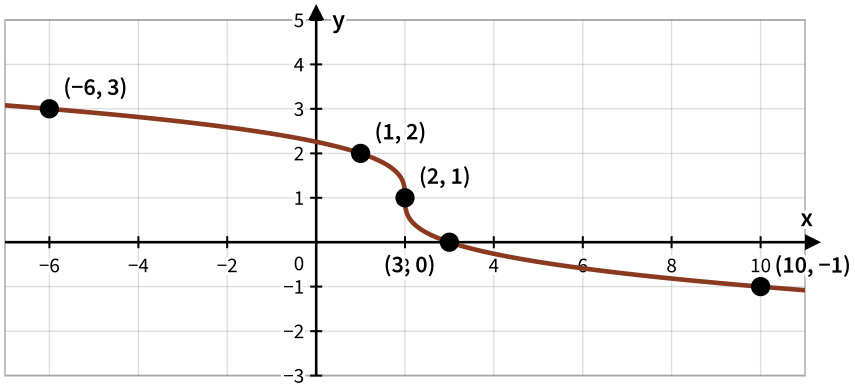
    From a height of h feet you can see about $d = 1.22\sqrt{h}$ miles to the horizon. How far can riders see from a hot-air balloon 900 feet above a flat county fairground? Give the answer to the nearest tenth of a mile.

11. **6 pts**

    For $f(x) = 3\sqrt{6-2x} - 5$, give (a) the domain and (b) the range in interval notation.

12. **7 pts**

The graph has the form $y = a\sqrt[3]{x-h}+k$, and $(2,1)$ is the point where it changes from bending one way to the other (the image of $(0,0)$). Use the labeled points to write its equation.



13. **7 pts**

Let $f(x) = (x+3)^2 - 4$ with $x \geq -3$. (a) Find $f^{-1}(x)$. (b) Give the domain of f^{-1}.

14. **6 pts**

Explain why, if the rule $(a^m)^n = a^{mn}$ is to hold for fractions, $5^{1/3}$ must equal $\sqrt[3]{5}$.

15. **10 pts**

Solve and check. Identify any extraneous solutions. (a) $\sqrt{3x+10} = x+2$ (b) $(2x-3)^{2/3} = 9$

16. **6 pts**

Water leaves the spout of a rain barrel at about $v = 8\sqrt{h}$ feet per second, where h is the depth of water above the spout in feet. How many inches of water must be above the spout for the water to leave at 6 feet per second?

ANSWER KEY

1. (a) $5^2 = 25$ (b) $\dfrac{1}{2^5} = \dfrac{1}{32}$ (c) $(-2)^3 = -8$
2. (a) $x^{4/7}$ (b) $\dfrac{1}{\sqrt[5]{y^3}}$
3. $\dfrac{a^3 b^{-2}}{a^2 b^{-3}} = ab$
4. $\sqrt[3]{125a^6b^3} \cdot \sqrt[3]{2a^2b} = 5a^2b\sqrt[3]{2a^2b}$
5. Multiply by $\dfrac{\sqrt[3]{36}}{\sqrt[3]{36}}$: $\dfrac{12\sqrt[3]{36}}{\sqrt[3]{216}} = \dfrac{12\sqrt[3]{36}}{6} = 2\sqrt[3]{36}$
6. (a) $\sqrt{45} = 3\sqrt{5}$ and $\sqrt{20} = 2\sqrt{5}$, so $P = 2(3\sqrt{5} + 2\sqrt{5}) = 10\sqrt{5}$ m (b) $\sqrt{45 \cdot 20} = \sqrt{900} = 30$ m^2
7. $8 + 12\sqrt{6} - 2\sqrt{6} - 18 = -10 + 10\sqrt{6}$
8. $\dfrac{10(\sqrt{7} + \sqrt{2})}{7 - 2} = 2\sqrt{7} + 2\sqrt{2}$
9. $x^{1/2 + 2/5} = x^{9/10} = \sqrt[10]{x^9}$
10. $1.22\sqrt{900} = 1.22 \times 30 = 36.6$ miles
11. (a) $6 - 2x \geq 0$, so $(-\infty, 3]$ (b) $3\sqrt{6 - 2x} \geq 0$, so $[-5, \infty)$
12. $h = 2$, $k = 1$. From $(10, -1)$: $-1 = a\sqrt[3]{8} + 1$, so $a = -1$. $y = -\sqrt[3]{x - 2} + 1$
13. (a) $x = (y + 3)^2 - 4$, $y + 3 = \sqrt{x + 4}$ (positive since $y \geq -3$), so $f^{-1}(x) = -3 + \sqrt{x + 4}$ (b) $x \geq -4$
14. By the rule, $\left(5^{1/3}\right)^3 = 5^{\frac{1}{3} \cdot 3} = 5^1 = 5$. So $5^{1/3}$ is a number whose cube is 5. The only real number whose cube is 5 is $\sqrt[3]{5}$, so $5^{1/3} = \sqrt[3]{5}$.
15. (a) $3x + 10 = x^2 + 4x + 4$, $x^2 + x - 6 = 0$, $x = 2$ or $x = -3$. $x = 2$ checks ($\sqrt{16} = 4$); $x = -3$ is extraneous ($\sqrt{1} = 1 \neq -1$). So $x = 2$. (b) $\sqrt[3]{2x - 3} = \pm 3$, $2x - 3 = \pm 27$, $x = 15$ or $x = -12$; both check: $27^{2/3} = 9$ and $(-27)^{2/3} = 9$.
16. $\sqrt{h} = \dfrac{6}{8} = 0.75$, so $h = 0.5625$ ft $= 0.5625 \times 12 = 6.75$ in; check: $8\sqrt{0.5625} = 8(0.75) = 6$ ✓

UNIT 7

Exponential Functions

This unit is about quantities that change by multiplying: savings that earn interest on interest, equipment that loses a fixed percent of its value each year, radioactive samples that lose half their mass in each half-life, and a cup of cocoa cooling toward room temperature. You will graph exponential functions and their transformations, see why exponential growth eventually beats any polynomial, and meet the number $e \approx 2.71828$, which comes from compounding interest more and more often.

The unit builds on the exponential functions and compound interest of Algebra I, the transformations of Lesson 1.2 and the rational exponents of Unit 6. It ends with logistic functions, which describe growth that levels off, such as fish in a stocked pond or news spreading through a town. Unit 8 then introduces logarithms, the tool for solving exponential equations for the time or the rate; Unit 9 uses these same growth factors to understand savings plans and loans.

PREREQUISITES

- Exponent rules, including zero, negative and rational exponents (Lessons 6.1 and 6.2)
- Exponential growth and decay $y = a \cdot b^x$ and compound interest from Algebra I
- Transformations of graphs $y = a \cdot f(x - h) + k$ (Lesson 1.2)
- Domain and range in interval notation (Lesson 1.1)
- Solving equations with nth roots, such as $b^3 = \frac{27}{8}$ (Lesson 6.5)

MASTERY CHECKLIST

- ☐ I can explain why the base of an exponential function must be positive and not 1.
- ☐ I can graph $y = a \cdot b^{x-h} + k$ and give its asymptote, domain, range and y-intercept.
- ☐ I can convert between a growth or decay rate and a growth factor.
- ☐ I can explain why 2^x eventually overtakes x^3.
- ☐ I can derive and use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to find balances and deposits.
- ☐ I can explain where the number e comes from and why continuous compounding gives $A = Pe^{rt}$.
- ☐ I can write an exponential model from two data points.
- ☐ I can use half-life, doubling-time and continuous models, and Newton's law of cooling.
- ☐ I can tell linear data from exponential data using differences and ratios.
- ☐ I can find the carrying capacity, the starting value and A for a logistic model and explain its S-shape.

- ☐ I can evaluate $\sigma(x)$ and prove that $\sigma(-x) = 1 - \sigma(x)$

LESSONS IN THIS UNIT

7.1	**Exponential Functions and Their Graphs**	2 sessions of 45 minutes
7.2	**Compound Interest and the Number e**	2 sessions of 45 minutes
7.3	**Exponential Models**	2 sessions of 45 minutes
7.4	**Logistic Functions**	2 sessions of 45 minutes

Unit 7 test

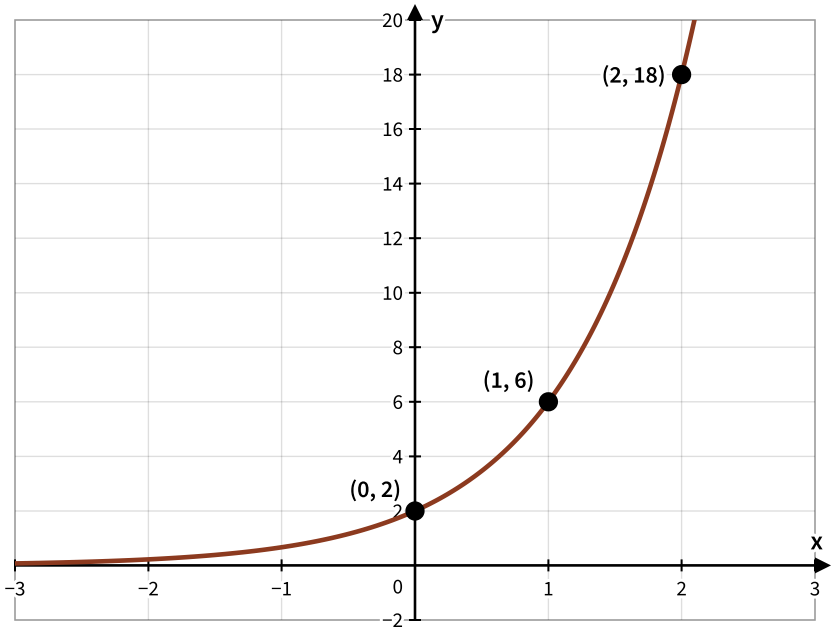
You have 60 minutes. A scientific calculator is allowed except where a problem says "by hand"; use the e^x key rather than a rounded value of e. Show your work for every problem. Give exact answers unless the problem asks you to round, and round only at the final step, as the problem directs. Give units in word problems.

1. **5 pts**
   By hand, evaluate $f(x) = 6\left(\frac{1}{3}\right)^x$ at $x = -2$, $x = 0$ and $x = 2$.

2. **6 pts**
   For $g(x) = -3 \cdot 2^{x-1} + 4$, give the horizontal asymptote, the domain, the range and the y-intercept.

3. **6 pts**

The graph shows a function $y = a \cdot b^x$ through the labeled points. Write its equation.



4. **6 pts**

Ruth's goat herd has 40 goats and grows 9% per year. (a) What is the yearly growth factor? (b) How many goats will she have after 4 years, to the nearest goat?

5. **7 pts**

The Miller family deposits \$3,500 at 4.2% compounded monthly. Find the balance after 5 years, to the nearest cent.

6. **6 pts**

Find the balance after 12 years when \$8,000 is invested at 3.5% compounded continuously, to the nearest cent.

7. **8 pts**

Starting from $A = P\left(1 + \frac{r}{n}\right)^{nt}$, explain why compounding more and more often leads to the formula $A = Pe^{rt}$.

8. **7 pts**

   Find an exponential function $y = a \cdot b^x$ whose graph passes through $(1, 24)$ and $(3, 54)$.

9. **7 pts**

   A hospital uses a 120-milligram sample of iodine-131, which has a half-life of 8 days. (a) How much remains after 24 days? (b) How much remains after 30 days, to the nearest tenth of a milligram?

10. **7 pts**

    A loaf of bread comes out of the oven at 190°F into a 68°F kitchen and cools by Newton's law of cooling with $k = 0.05$ per minute. (a) Find its temperature after 20 minutes, to the nearest degree. (b) What temperature does it approach?

11. **7 pts**

    One of the columns P and Q in the table is linear and the other is exponential. Write a rule for each.

x	P	Q
0	50	50
1	60	60
2	70	72
3	80	86.4
4	90	103.68

12. **7 pts**

    A hay baler is worth \$40,500 one year after it was bought and \$32,805 three years after it was bought. Assuming exponential depreciation, find the yearly rate of depreciation and the price of the baler when new.

13. **7 pts**

The number of deer in a nature preserve after t years is modeled by $f(t) = \dfrac{1500}{1 + 14e^{-0.35t}}$. Find (a) the carrying capacity, (b) the starting number of deer, (c) the number after 6 years, to the nearest deer.

14. **6 pts**

Given that $\sigma(1.5) \approx 0.8176$, find $\sigma(-1.5)$ without evaluating $e^{1.5}$, and name the property you used.

15. **8 pts**

A new hay-drying method spreads among the 1,200 farms of a county. At $t = 0$, 24 farms use it, and the number of farms using it after t years is $f(t) = \dfrac{1200}{1 + Ae^{-0.7t}}$. (a) Find A. (b) How many farms use the method after 5 years, to the nearest farm?

ANSWER KEY

1. $f(-2) = 6 \cdot 9 = 54$, $f(0) = 6$, $f(2) = \frac{6}{9} = \frac{2}{3}$
2. Asymptote $y = 4$; domain $(-\infty, \infty)$; range $(-\infty, 4)$; y-intercept $-3 \cdot \frac{1}{2} + 4 = 2.5$.
3. $a = 2$ (the y-intercept) and each step right multiplies y by 3: $y = 2 \cdot 3^x$.
4. (a) 1.09 (b) $40(1.09)^4 = 40 \times 1.41158161 \approx 56.46$, about 56 goats.
5. $3500\left(1 + \frac{0.042}{12}\right)^{60} = 3500(1.0035)^{60} \approx 3500 \times 1.2332258 \approx \$4{,}316.29$
6. $8000e^{0.035 \times 12} = 8000e^{0.42} \approx 8000 \times 1.5219616 \approx \$12{,}175.69$
7. Let $m = \frac{n}{r}$, so $\frac{r}{n} = \frac{1}{m}$ and $nt = m \cdot rt$. Then $\left(1 + \frac{r}{n}\right)^{nt} = \left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}$. As n grows, m grows, and $\left(1 + \frac{1}{m}\right)^m$ approaches e. So the balance approaches Pe^{rt}.
8. $\frac{54}{24} = b^2 = 2.25$, so $b = 1.5$; $24 = 1.5a$, so $a = 16$: $y = 16(1.5)^x$. Check: $16 \times 3.375 = 54$ ✓
9. (a) 3 half-lives: $120\left(\frac{1}{2}\right)^3 = 15$ mg. (b) $120\left(\frac{1}{2}\right)^{30/8} = 120\left(\frac{1}{2}\right)^{3.75} \approx 8.9$ mg.
10. (a) $T = 68 + 122e^{-0.05t}$; $T(20) = 68 + 122e^{-1} \approx 68 + 44.88 \approx 113°$F. (b) 68°F.
11. P adds 10 each step: $y = 50 + 10x$. Q multiplies by 1.2 each step: $y = 50(1.2)^x$.
12. $b^2 = \frac{32805}{40500} = 0.81$, so $b = 0.9$: it loses 10% per year. New price: $\frac{40500}{0.9} = \$45{,}000$.
13. (a) 1,500 (b) $\frac{1500}{15} = 100$ (c) $\frac{1500}{1+14e^{-2.1}} \approx \frac{1500}{2.71439} \approx 553$ deer.
14. $\sigma(-x) = 1 - \sigma(x)$, so $\sigma(-1.5) \approx 1 - 0.8176 = 0.1824$.
15. (a) $A = \frac{1200}{24} - 1 = 49$. (b) $f(5) = \frac{1200}{1+49e^{-3.5}} \approx \frac{1200}{2.47967} \approx 484$ farms.

7.1 Exponential Functions and Their Graphs

OBJECTIVES

- Explain why the base of an exponential function must be positive and not 1, and why b^x has a value for every real x.
- Graph $y = a \cdot b^{x-h} + k$ and state its y-intercept, horizontal asymptote, domain and range.
- Convert between a growth or decay rate and a growth factor, and use them to write and evaluate models.
- Explain why an exponential function with base greater than 1 eventually overtakes any polynomial.

WATCH

How to Graph Exponential Functions with Transformations (3 Examples)

Mario's Math Tutoring · 8:50

https://www.youtube.com/watch?v=gVkONK_G8bg

> **WHEN TO WATCH**
> Before Example 1: graphing y = a·b^x with shifts and reflections, and its asymptote.

Note: Neither video compares exponential growth with polynomial growth. Teach that from the Core ideas. Eddie Woo teaches in Australia: he says 'dilation' for a stretch. The backup is from a series where the last video also covers logarithm graphs (Lesson 8.2).

BACKUP VIDEO

Graphing Exponentials & Logs (2 of 5: Overview of transformations)

Eddie Woo · 9:55

https://www.youtube.com/watch?v=XG4tqFMLzgU

Before the practice set: a review of translations and dilations applied to exponential graphs.

KEY TERMS

exponential function

A function of the form $f(x) = a \cdot b^x$ with $a \neq 0$, $b > 0$ and $b \neq 1$; the variable is in the exponent.

growth factor

The base b: the number the output is multiplied by each time x increases by 1. If $b > 1$ the function shows growth; if $0 < b < 1$ it shows decay.

growth rate

The percent added each step, r, where $b = 1 + r$. A decay rate is the percent lost each step, where $b = 1 - r$.

horizontal asymptote

A horizontal line $y = k$ that a graph gets closer and closer to as x moves far to the right or far to the left, without ever reaching it in this case.

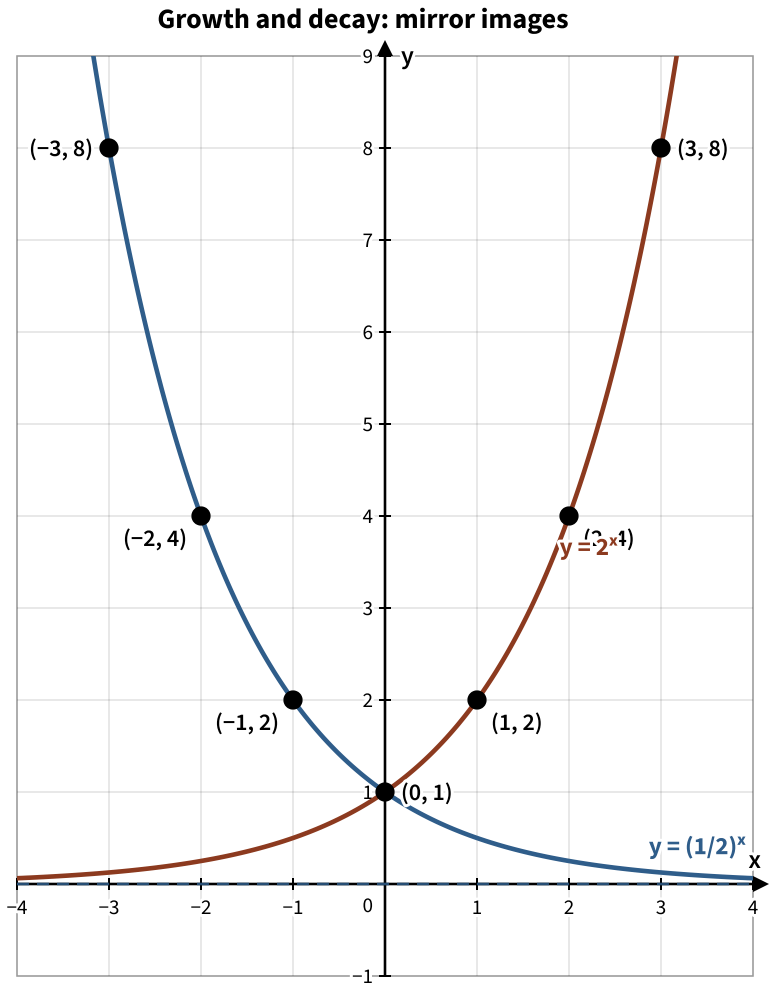
exponential decay

A pattern in which a quantity is multiplied by the same number between 0 and 1 in each equal step, so it shrinks by the same percent each time.

CORE IDEAS

In Algebra I you met exponential functions $f(x) = a \cdot b^x$, where the variable sits in the exponent. The **base** b must be positive and not equal to 1. If $b = 1$, then $1^x = 1$ for every x, and the function is just the constant a. If b were negative, b^x would not even be real for many inputs: $(-4)^{1/2} = \sqrt{-4}$ is not a real number, so the graph would be full of gaps. With $b > 0$, Lesson 6.1 gives a meaning to b^x for every rational x, such as $2^{3/2} = \sqrt{8}$. For an irrational exponent, $2^{\sqrt{2}}$ is the number that $2^{1.4}, 2^{1.41}, 2^{1.414}, \ldots$ approach (about 2.665), so b^x has a value for every real x and the graph is one unbroken curve.

Look at $y = b^x$ itself. It passes through $(0, 1)$ because $b^0 = 1$, and through $(1, b)$. Its outputs are always positive, since a positive number raised to any power is positive. For $b > 1$, moving left makes the outputs shrink: $2^{-10} = \dfrac{1}{2^{10}} = \dfrac{1}{1024}$. The outputs get as close to 0 as you like but never reach it, so the x-axis, $y = 0$, is a **horizontal asymptote**. The domain is all real numbers and the range is $(0, \infty)$.

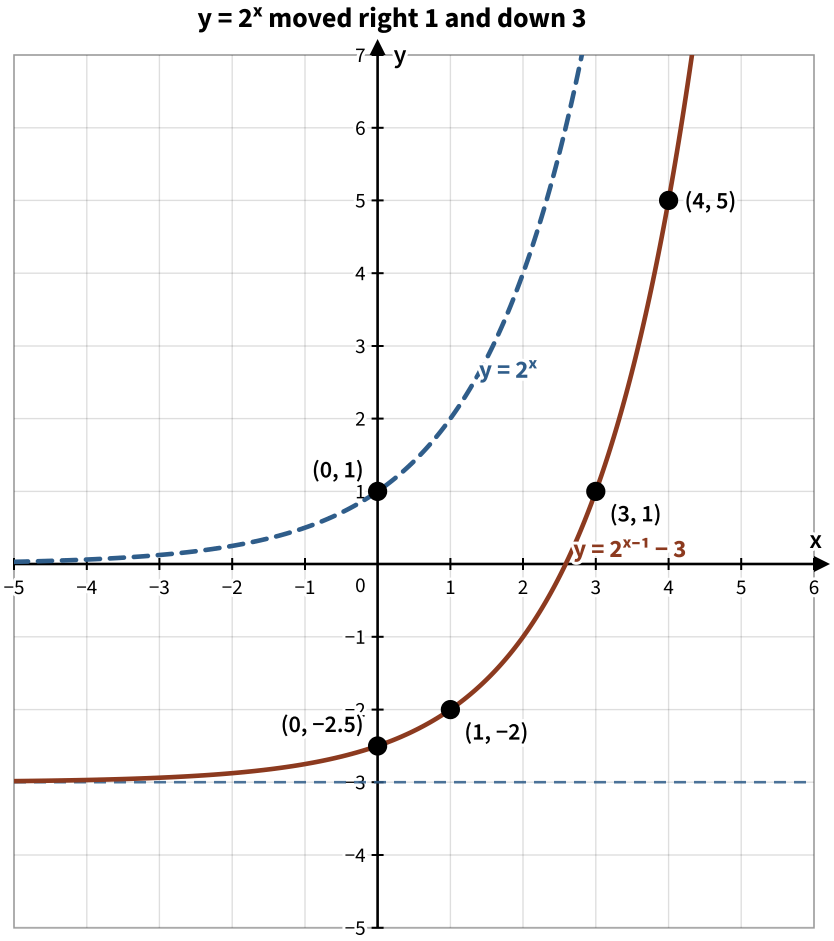


Since $\left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$, the graph of $y = \left(\frac{1}{2}\right)^x$ is the graph of $y = 2^x$ with x replaced by $-x$: a reflection in the y-axis (Lesson 1.2). So every decay curve is a growth curve seen in a mirror. Each step to the right multiplies the output by b: by 2 on the rising curve, by $\frac{1}{2}$ on the falling one.

A **growth rate** and a **growth factor** are different numbers for the same idea. If a herd grows 6% a year, each year you keep the whole herd and add 6% of it: $A + 0.06A = 1.06A$. So the growth factor is $b = 1 + r = 1.06$, not 0.06 and not 6. If a truck loses 15% of its value each year, it keeps 85%: $A - 0.15A = 0.85A$, so $b = 1 - r = 0.85$. A factor above 1 means growth; a factor between 0 and 1 means decay.

The transformations of Lesson 1.2 work here exactly as before. In $y = a \cdot b^{x-h} + k$, the graph of $y = b^x$ is stretched by a (and reflected in the x-axis if $a < 0$), moved right h and moved up k. The key point $(0, 1)$ moves to $(h, a + k)$, and the asymptote $y = 0$

moves to $y = k$. The domain stays all real numbers. Because $a \cdot b^{x-h}$ is never zero and has the same sign as a, the range is (k, ∞) when $a > 0$ and $(-\infty, k)$ when $a < 0$.



Exponential growth with $b > 1$ always wins in the long run, even against a powerful polynomial like x^3. The table shows x^3 ahead for a while, but 2^x passes it at $x = 10$ and then leaves it far behind. Here is why. Each step from x to $x + 1$ doubles 2^x, but multiplies x^3 by $\left(\frac{x+1}{x}\right)^3$. For $x \geq 4$ that factor is at most $\left(\frac{5}{4}\right)^3 = \frac{125}{64} \approx 1.95$, which is less than 2, and it keeps shrinking toward 1. So once $x \geq 4$, 2^x gains on x^3 at every step, and after it catches up it can never fall behind again. The same argument works for any base $b > 1$ against any power x^n.

x	2^x	x^3	Larger
1	2	1	2^x

x	2^x	x^3	Larger
2	4	8	x^3
5	32	125	x^3
8	256	512	x^3
9	512	729	x^3
10	1024	1000	2^x
12	4096	1728	2^x
15	32768	3375	2^x
20	1048576	8000	2^x

KEY FORMULAS

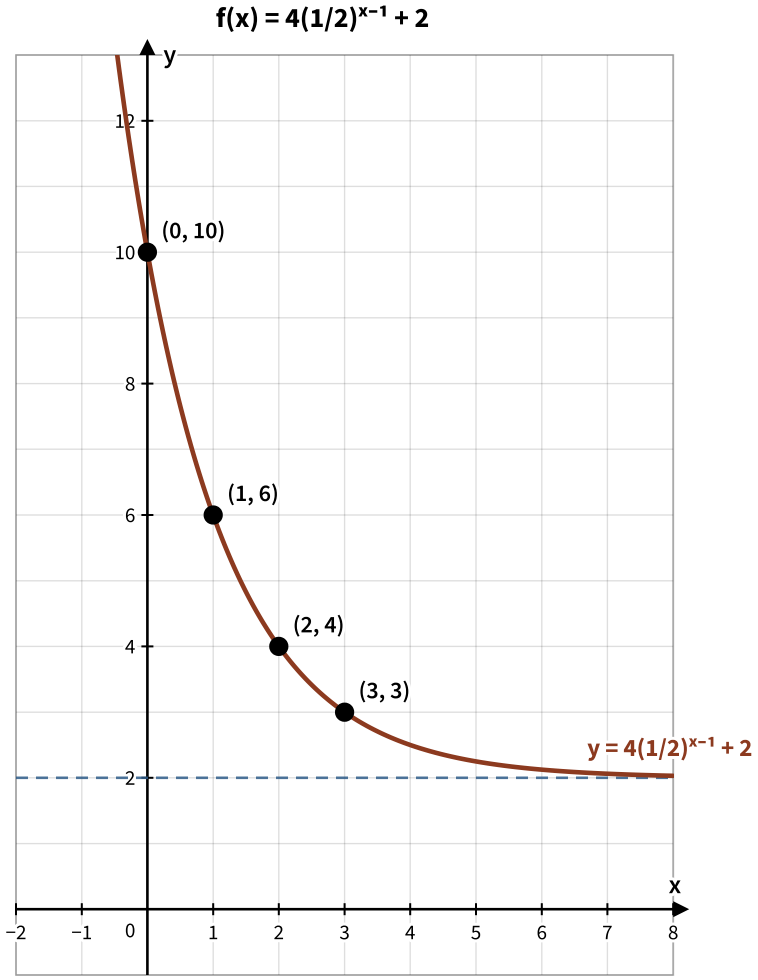
$$f(x) = a \cdot b^x, \qquad b > 0,\ b \neq 1$$

$$\text{growth: } b = 1 + r \qquad \text{decay: } b = 1 - r$$

$$y = a \cdot b^{x-h} + k: \quad \text{asymptote } y = k,\ \text{domain } (-\infty, \infty),\ \text{range } (k, \infty) \text{ if } a > 0,\ (-\infty, k) \text{ if } a < 0$$

Practice habit. Know the powers of 2 up to $2^{10} = 1024$ and the powers of 3 up to $3^5 = 243$ cold. They let you evaluate, graph and check exponential functions by hand in seconds.

Example 1. For $f(x) = 4\left(\frac{1}{2}\right)^{x-1} + 2$, decide whether the graph shows growth or decay, and give the horizontal asymptote, domain, range and y-intercept. Then sketch the graph.



Show solution

1. Compare with $y = a \cdot b^{x-h} + k$: $a = 4$, $b = \frac{1}{2}$, $h = 1$, $k = 2$.
2. $0 < b < 1$ and $a > 0$, so the graph falls from left to right: decay.
3. The asymptote $y = 0$ moves up 2: the asymptote is $y = 2$.
4. Domain: all real numbers, $(-\infty, \infty)$.
5. $4\left(\frac{1}{2}\right)^{x-1}$ is always positive, so $f(x) > 2$: range $(2, \infty)$.
6. y-intercept: $f(0) = 4\left(\frac{1}{2}\right)^{-1} + 2 = 4(2) + 2 = 10$.

7. More points: $f(1) = 4(1) + 2 = 6$, $f(2) = 4\left(\frac{1}{2}\right) + 2 = 4$, $f(3) = 4\left(\frac{1}{4}\right) + 2 = 3$. Each step right halves the distance above the asymptote: 8, 4, 2, 1.
8. Plot the points, draw the dashed asymptote $y = 2$, and draw a smooth curve that flattens toward it on the right.

Answer: Decay; asymptote $y = 2$; domain $(-\infty, \infty)$; range $(2, \infty)$; y-intercept 10.

Example 2. Henry's family cider press sold 2,500 gallons of cider this year, and sales are growing 8% per year. The family also bought a used tractor for $60,000 that loses 15% of its value each year. (a) Write a model for cider sales t years from now and find the sales 5 years from now, to the nearest gallon. (b) Write a model for the tractor's value and find its value after 4 years, to the nearest cent.

Show solution

1. (a) Growth rate 8%, so the growth factor is $1 + 0.08 = 1.08$.
2. $S(t) = 2500(1.08)^t$.
3. $1.08^5 = 1.4693280768$, so $S(5) = 2500 \times 1.4693280768 \approx 3673.32$.
4. Sales are about 3,673 gallons.
5. (b) Decay rate 15%, so the tractor keeps 85% each year: factor $1 - 0.15 = 0.85$.
6. $V(t) = 60000(0.85)^t$.
7. $0.85^2 = 0.7225$ and $0.85^4 = 0.7225^2 = 0.52200625$.
8. $V(4) = 60000 \times 0.52200625 = 31320.375$, about $31,320.38.
9. Check the sense: sales grew by less than half in 5 years, and the tractor kept a bit more than half its value after 4 years. Both are reasonable for rates of 8% and 15%.

Answer: (a) $S(t) = 2500(1.08)^t$; about 3,673 gallons. (b) $V(t) = 60000(0.85)^t$; about $31,320.38.

PRACTICE

1. Easy Grace's savings club grows according to $y = 300(1.04)^t$. State the growth factor and the growth rate (as a decimal).

   Show answer

   Growth factor 1.04; growth rate 0.04, which is 4% per period.

2. **Easy** By hand, evaluate $f(x) = 5 \cdot 2^x$ at $x = 3$, $x = 0$ and $x = -2$.

Show answer

$f(3) = 40$, $f(0) = 5$, $f(-2) = 5 \cdot \frac{1}{4} = \frac{5}{4}$

3. **Easy** Write the growth factor b for a quantity that (a) grows 12% per year, (b) decays 7% per year, (c) decays 40% per year.

Show answer

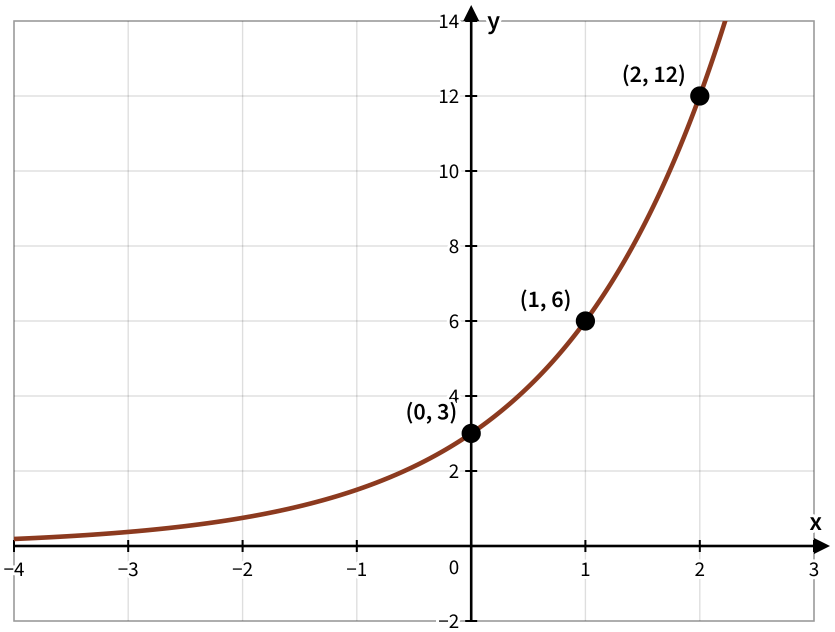
(a) 1.12 (b) 0.93 (c) 0.6

4. **Easy** For $y = 2 \cdot 3^x + 1$, give the horizontal asymptote, the domain, the range and the y-intercept.

Show answer

Asymptote $y = 1$; domain $(-\infty, \infty)$; range $(1, \infty)$; y-intercept $2(1) + 1 = 3$.

5. **Medium** The graph shows a function of the form $y = a \cdot b^x$ passing through the labeled points. Write its equation.



Show answer

At $x = 0$, $y = a = 3$. Each step right doubles y (3, 6, 12), so $b = 2$: $y = 3 \cdot 2^x$.

6. **Medium** Describe how to obtain the graph of $y = -2^{x+3} + 5$ from the graph of $y = 2^x$ (the negative sign is outside the power). Then give the asymptote, the range and the y-intercept.

Show answer

Move left 3, reflect in the x-axis, move up 5. Asymptote $y = 5$; range $(-\infty, 5)$; y-intercept $-2^3 + 5 = -3$.

7. **Medium** Explain why the graph of $y = \left(\frac{1}{3}\right)^x$ is the reflection of the graph of $y = 3^x$ in the y-axis.

Show answer

$\left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x}$. So $y = \left(\frac{1}{3}\right)^x$ is $y = 3^x$ with x replaced by $-x$, and replacing x by $-x$ reflects a graph in the y-axis: the point $(a, 3^a)$ on one graph matches the point $(-a, 3^a)$ on the other.

8. **Medium** The Miller family's blueberry patch gave 40 pints its first summer, and the yield is expected to grow 25% per year for the next few years. Write a model for the yield y after t years and find the yield 3 years later, to the nearest pint.

Show answer

$y = 40(1.25)^t$; $y(3) = 40 \times 1.953125 = 78.125$, about 78 pints.

9. **Medium** A used pickup truck is bought for \$24,000 and loses 12% of its value each year. Find its value after 6 years, to the nearest dollar.

Show answer

$V = 24000(0.88)^6 \approx 24000 \times 0.464404 \approx 11145.70$, about \$11,146.

10. **Hard** An exponential function $y = a \cdot b^x + k$ has horizontal asymptote $y = -2$ and passes through $(0, 1)$ and $(1, 4)$. Find its equation.

Show answer

$k = -2$. At $(0, 1)$: $a - 2 = 1$, so $a = 3$. At $(1, 4)$: $3b - 2 = 4$, so $b = 2$.
$y = 3 \cdot 2^x - 2$.

11. **Hard** Make a table of 2^x and x^4 for whole numbers near 15. What is the smallest whole number $x > 1$ for which $2^x > x^4$?

Show answer

$2^{15} = 32768 < 15^4 = 50625$; $2^{16} = 65536 = 16^4$; $2^{17} = 131072 > 17^4 = 83521$. So $x = 17$.

12. Hard Samuel saves 1 cent on day 1, 2 cents on day 2, 4 cents on day 3, doubling his deposit every day. His sister Ruth saves $10 every day. On which day is Samuel's deposit for that single day first larger than Ruth's $10?

Show answer

Samuel's deposit on day n is $0.01 \cdot 2^{n-1}$ dollars. We need $2^{n-1} > 1000$; $2^9 = 512$ and $2^{10} = 1024$, so $n - 1 = 10$: day 11 (a deposit of $10.24).

13. Challenge (a) Write 2^{x+3} in the form $a \cdot 2^x$. (b) Explain why moving the graph of $y = 2^x$ to the left 3 units gives the same graph as stretching it vertically by a factor of 8.

Show answer

(a) $2^{x+3} = 2^3 \cdot 2^x = 8 \cdot 2^x$. (b) Moving left 3 gives $y = 2^{x+3}$, and by the product rule for exponents this equals $8 \cdot 2^x$, which is the vertical stretch by 8. For exponential functions, a horizontal shift is always the same as a vertical stretch or shrink, because $b^{x-h} = b^{-h} \cdot b^x$.

COMMON MISTAKES

- Using the rate as the base: 6% growth is 1.06^t, not 0.06^t or 6^t. Write $b = 1 + r$ or $b = 1 - r$ before you write the model.
- Multiplying before raising to the power: $3 \cdot 2^4 = 3 \cdot 16 = 48$, not 6^4. The exponent applies only to the base right under it.
- Thinking a negative exponent makes a negative number: $2^{-3} = \frac{1}{8}$, which is positive. That is why $y = b^x$ never goes below its asymptote.
- Forgetting to move the asymptote: in $y = a \cdot b^{x-h} + k$ the asymptote is $y = k$, and the range starts at k, not at 0.

> **Make it a game · Bad Base Detective.** Make tables for y = b^x with x = -1, 0, 1, 2 for b = 2, b = 1/2, b = 1 and b = -2. Spot why base 1 is boring and why a negative base fails for x = 1/2. Then graph y = 2^(x - 1) + 3 and name its y-intercept, horizontal asymptote, domain and range. Two points per correct feature.

7.2 Compound Interest and the Number e

OBJECTIVES

- Derive the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ from repeated growth and use it to find balances and deposits.
- Explain why compounding more often helps less and less, and describe the number e as the value that $\left(1 + \frac{1}{n}\right)^{n}$ approaches.
- Derive and use the continuous compounding formula $A = Pe^{rt}$.
- Compare savings offers with different rates and compounding periods.

WATCH

Developing the Compound Interest Formula

Eddie Woo · 11:08

https://www.youtube.com/watch?v=4Y6T7R_DCVw

> **WHEN TO WATCH**
> Before Example 1: building the compound interest formula from repeated growth.

Note: Eddie Woo teaches in Australia: amounts are in dollars, and he says 'per annum' for per year.

BACKUP VIDEO

What is the number "e" and where does it come from?

Eddie Woo · 7:58

https://www.youtube.com/watch?v=pg827uDPFqA

After Example 1: where the number e comes from.

KEY TERMS

principal

The amount of money first deposited or borrowed, P.

compounding

Adding the interest earned to the balance so that later interest is earned on the interest too. Compounding n times a year means the interest is added n times each year.

annual rate

The yearly interest rate r, written as a decimal in formulas (5% is $r = 0.05$). Each compounding period uses the rate $\frac{r}{n}$.

the number e

The number, about 2.71828, that $\left(1 + \frac{1}{n}\right)^n$ gets closer and closer to as n grows. Like π, it is irrational.

continuous compounding

The limit of compounding more and more often; the balance is $A = Pe^{rt}$.

natural exponential function

The function $y = e^x$, the exponential function whose base is e.

CORE IDEAS

Suppose you deposit P dollars at an annual rate r compounded n times a year. Each period the bank pays the rate $\frac{r}{n}$ on the current balance, so the balance B becomes $B + \frac{r}{n}B = B\left(1 + \frac{r}{n}\right)$. Every period multiplies the balance by the same factor $1 + \frac{r}{n}$, and in t years there are nt periods. So $A = P\left(1 + \frac{r}{n}\right)^{nt}$. This is simply an exponential function of t with growth factor $\left(1 + \frac{r}{n}\right)^n$ per year.

Does compounding more often help? The table shows \$1,000 at 6% for one year. Going from once a year to twice a year adds 90 cents; going from monthly to daily adds only 15 cents. Compounding more often always helps, because interest starts earning interest sooner, but the gains shrink quickly and the amounts seem to settle toward a ceiling.

Compounding	n	Balance after 1 year (dollars)
annually	1	1060.00
semiannually	2	1060.90
quarterly	4	1061.36
monthly	12	1061.68
daily	365	1061.83
continuously	—	1061.84

To find that ceiling, the mathematician Jacob Bernoulli asked the simplest version: \$1 at 100% interest for one year, compounded n times. The balance is $\left(1+\frac{1}{n}\right)^n$. The values climb, but more and more slowly, and they approach a definite number, 2.71828… This number is called e. It is irrational, so its decimals never end or repeat; $e \approx 2.718281828$.

n	$(1+1/n)^n$
1	2
2	2.25
4	2.441406
12	2.613035
100	2.704814
365	2.714567
1000	2.716924
10,000	2.718146
1,000,000	2.718280

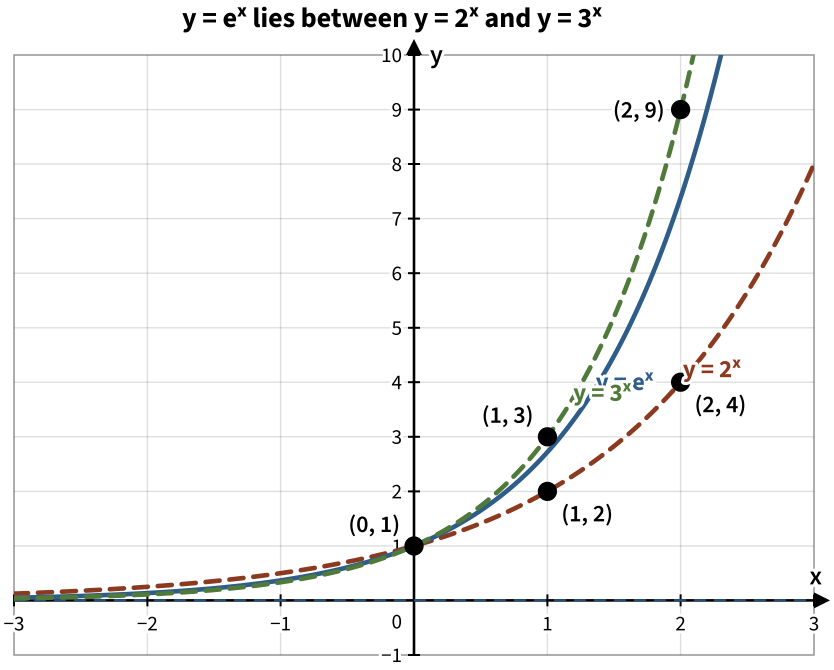
Now take any rate r. Let $m = \frac{n}{r}$, so that $\frac{r}{n} = \frac{1}{m}$ and $nt = m \cdot rt$. Then by the power-of-a-power rule,

$$\left(1+\frac{r}{n}\right)^{nt} = \left(1+\frac{1}{m}\right)^{m \cdot rt} = \left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

As the compounding gets more frequent, n grows, so m grows, and the bracket approaches e. So the balance approaches $A = Pe^{rt}$: this is **continuous compounding**. In the table, $1000e^{0.06} \approx 1061.84$, just one cent above daily compounding.

The function $y = e^x$ is called the **natural exponential function**. Since $2 < e < 3$, its graph lies between the graphs of $y = 2^x$ and $y = 3^x$, passing through $(0, 1)$ and $(1, e)$,

with the x-axis as its asymptote. Your calculator has an e^x key; use it rather than typing 2.718, so you do not lose accuracy.



KEY FORMULAS

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$e = \text{the value approached by } \left(1 + \frac{1}{n}\right)^{n} \approx 2.718281828$$

$$A = Pe^{rt} \quad \text{(continuous compounding)}$$

From history. Jacob Bernoulli studied this compound-interest question in 1683 and saw that the balance $\left(1 + \frac{1}{n}\right)^{n}$ approaches a number between 2 and 3. Leonhard Euler began using the letter e for this number in the 1730s, and in his Introductio of 1748 he computed it to 23 decimal places.

Practice habit. Keep exact answers until the last step. Type the whole expression, such as $6000 \div (1 + 0.048/12)^{48}$, into your calculator at once and round only the final answer to the nearest cent; rounding the growth factor halfway through can cost several dollars.

Example 1. Ruth deposits \$4,000 at 5% annual interest for 6 years. Find the balance, to the nearest cent, if the interest is (a) compounded quarterly, (b) compounded continuously. How much more does continuous compounding earn?

Show solution

1. (a) $P = 4000$, $r = 0.05$, $n = 4$, $t = 6$. Each quarter the factor is $1 + \frac{0.05}{4} = 1.0125$, and there are $4 \times 6 = 24$ quarters.
2. $A = 4000(1.0125)^{24}$.
3. $(1.0125)^{24} \approx 1.3473510$, so $A \approx 5389.40$.
4. (b) $A = 4000e^{0.05 \times 6} = 4000e^{0.3}$.
5. $e^{0.3} \approx 1.3498588$, so $A \approx 5399.44$.
6. Difference: $5399.44 - 5389.40 = 10.04$.
7. Check the sense: continuous compounding should earn a little more than quarterly, and it does.

Answer: (a) \$5,389.40 (b) \$5,399.44; continuous compounding earns \$10.04 more.

Example 2. Grace wants \$6,000 in 4 years to buy a used car. Bank A pays 4.8% compounded monthly; Bank B pays 4.75% compounded continuously. How much must she deposit today at each bank to reach \$6,000, to the nearest cent? Which offer is better?

Show solution

1. Bank A: $6000 = P\left(1 + \frac{0.048}{12}\right)^{12\times 4} = P(1.004)^{48}$.
2. Divide: $P = \dfrac{6000}{(1.004)^{48}}$.
3. $(1.004)^{48} \approx 1.2112066$, so $P \approx 4953.74$.
4. Bank B: $6000 = Pe^{0.0475\times 4} = Pe^{0.19}$.
5. $P = \dfrac{6000}{e^{0.19}} = 6000e^{-0.19} \approx 6000 \times 0.8269591 \approx 4961.75$.
6. Bank A needs the smaller deposit, so Bank A is the better offer here: its slightly higher rate matters more than the more frequent compounding at Bank B.
7. Check Bank A: $4953.74 \times 1.2112066 \approx 6000.00$ ✓

Answer: Bank A: \$4,953.74; Bank B: \$4,961.75. Bank A is better (about \$8 less to deposit).

PRACTICE

1. Easy Samuel deposits \$1,000 at 4% compounded annually. Find the balance after 3 years, to the nearest cent.

   Show answer

   $$A = 1000(1.04)^3 = 1000 \times 1.124864 \approx \$1{,}124.86$$

2. Easy A family puts \$2,500 in a savings account at 6% compounded semiannually. Find the balance after 2 years, to the nearest cent.

   Show answer

   $$A = 2500(1.03)^4 = 2500 \times 1.12550881 \approx \$2{,}813.77$$

3. Easy Use a calculator to evaluate e^{2}, e^{-1} and $e^{0.5}$ to four decimal places.

   Show answer

   $e^{2} \approx 7.3891$, $e^{-1} \approx 0.3679$, $e^{0.5} \approx 1.6487$

4. **Easy** Find the balance after 10 years when \$3,000 is invested at 5% compounded continuously, to the nearest cent.

   Show answer

   $A = 3000e^{0.5} \approx 3000 \times 1.6487213 \approx \$4{,}946.16$

5. **Medium** By hand, compute $\left(1 + \frac{1}{n}\right)^n$ exactly as a fraction for $n = 2$ and $n = 3$. Are the values moving toward e?

   Show answer

   $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$ and $\left(\frac{4}{3}\right)^3 = \frac{64}{27} \approx 2.370$. Yes, they increase toward $e \approx 2.718$.

6. **Medium** Henry has \$5,000 to save for 8 years. Offer 1 pays 6% compounded monthly; Offer 2 pays 5.9% compounded continuously. Find each balance to the nearest cent and say which is better.

   Show answer

   Offer 1: $5000(1.005)^{96} \approx \$8{,}070.71$. Offer 2: $5000e^{0.472} \approx \$8{,}015.99$. Offer 1 is better.

7. **Medium** Samuel wants \$10,000 in 5 years for a down payment on a tractor. His account pays 4% compounded continuously. How much must he deposit today, to the nearest cent?

   Show answer

   $10000 = Pe^{0.2}$, so $P = 10000e^{-0.2} \approx \$8{,}187.31$.

8. **Medium** Use the table to explain why switching from annual to monthly compounding adds more than a dollar per \$1,000 in a year, while switching from daily to continuous compounding adds only about a cent.

Compounding	n	Balance after 1 year (dollars)
annually	1	1060.00
semiannually	2	1060.90
quarterly	4	1061.36
monthly	12	1061.68
daily	365	1061.83
continuously	—	1061.84

Show answer

Monthly compounding lets interest start earning interest after the first month instead of waiting a year, which matters noticeably. But as the periods get shorter, each change gives the interest only a little more time to earn, and the balances approach the ceiling $1000e^{0.06} \approx 1061.84$. Daily compounding is already within about a cent of that ceiling.

9. **Medium** For $f(x) = e^x - 2$, give the horizontal asymptote, the range and the y-intercept.

Show answer

Asymptote $y = -2$; range $(-2, \infty)$; y-intercept $e^0 - 2 = -1$.

10. **Hard** For one year, which is better: 5% compounded monthly, or 4.95% compounded continuously? Find the one-year growth factor of each to six decimal places.

Show answer

Monthly: $\left(1 + \frac{0.05}{12}\right)^{12} \approx 1.051162$. Continuous: $e^{0.0495} \approx 1.050746$. 5% compounded monthly is better.

11. Hard Grace invests \$1,000 at 7% compounded continuously. Will her money double in 10 years? Find the balance after 10 years to the nearest cent.

Show answer

$1000e^{0.7} \approx 1000 \times 2.0137527 \approx \$2{,}013.75$, so yes, it just more than doubles.

12. Hard A church building fund has \$15,000 and needs \$20,000 in 6 years. What annual interest rate, compounded annually, would get it there? Give the rate to the nearest hundredth of a percent.

Show answer

$15000(1+r)^6 = 20000$, so $(1+r)^6 = \frac{4}{3}$ and $1+r = \left(\frac{4}{3}\right)^{1/6} \approx 1.049115$. The rate is about 4.91%.

13. Challenge (a) Explain why money at annual rate r compounded continuously grows by the factor e^r each year, so its effective yearly growth rate is $e^r - 1$. (b) Find that effective rate for $r = 10\%$, as a percent to the nearest hundredth.

Show answer

(a) $A = Pe^{rt} = P(e^r)^t$, so each year multiplies the balance by e^r; the percent added each year is $e^r - 1$. (b) $e^{0.1} - 1 \approx 0.105171$, about 10.52%.

COMMON MISTAKES

- Using the annual rate for each period: at 6% compounded monthly the factor is $1 + \frac{0.06}{12} = 1.005$, not 1.06, and the exponent is $12t$, not t.
- Writing the rate as a whole number: 4.8% is $r = 0.048$. Writing $e^{4.8 \cdot 4}$ gives an absurd balance; a quick sense check catches it.
- Rounding e or the growth factor early. Use the e^x key and round only the final dollar amount.
- Thinking continuous compounding makes money grow without limit in a year. It only reaches the ceiling e^r, which for normal rates is barely above daily compounding.

> **Make it a game · Compounding Showdown.** Start with 1 pretend dollar at 100 percent interest per year. Guess the final value if it compounds once, twice, four times or 12 times. Calculate once, twice and four times by hand: 2, 2.25 and about 2.44. Guess where the numbers are heading, use a calculator if you have one, and meet the number e, about 2.718.

7.3 Exponential Models

OBJECTIVES

- Write an exponential model $y = a \cdot b^t$ from two data points by dividing one equation by the other.
- Use half-life, doubling-time and continuous models $y = a\left(\frac{1}{2}\right)^{t/h}$, $y = a \cdot 2^{t/d}$ and $y = ae^{kt}$.
- Apply Newton's law of cooling $T = T_s + (T_0 - T_s)e^{-kt}$ and explain its horizontal asymptote.
- Decide whether data are linear or exponential by checking for constant differences or constant ratios.

WATCH

Exponential Functions Word Problems - Half Life, Doubling, Growth and Decay

Mario's Math Tutoring · 13:37

https://www.youtube.com/watch?v=BQlcDFwaJVU

> **WHEN TO WATCH**
> Before Example 1: half-life, doubling and growth and decay word problems.

Note: Eddie Woo teaches in Australia: temperatures are in degrees Celsius.

BACKUP VIDEO

Exponential Decay Example 1: The Fridge

Eddie Woo · 7:48

https://www.youtube.com/watch?v=eIoPjK_b_Wo

Before Example 2: a cooling model with a horizontal asymptote, like Newton's law of cooling.

KEY TERMS

half-life

The time it takes a decaying quantity to fall to half of its amount. It is the same no matter where you start.

doubling time

The time it takes a growing quantity to double. For exponential growth it is always the same.

continuous growth rate

The constant k in $y = ae^{kt}$; $k > 0$ means growth and $k < 0$ means decay.

Newton's law of cooling

A model saying that the difference between an object's temperature and the room's temperature shrinks exponentially: $T = T_s + (T_0 - T_s)e^{-kt}$.

depreciation

The loss in value of equipment over time; often modeled as losing the same percent each year.

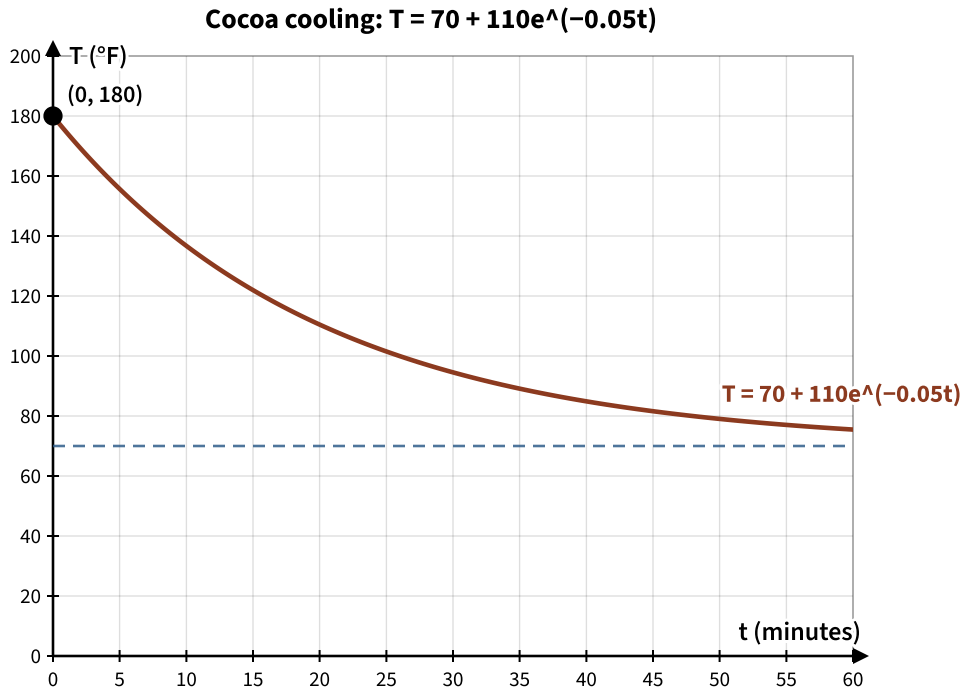
CORE IDEAS

An exponential model $y = a \cdot b^t$ has two unknowns, so two data points determine it. Substituting (t_1, y_1) and (t_2, y_2) gives $y_1 = ab^{t_1}$ and $y_2 = ab^{t_2}$. Dividing the second equation by the first cancels a: $\dfrac{y_2}{y_1} = \dfrac{ab^{t_2}}{ab^{t_1}} = b^{t_2 - t_1}$. Now take a root to find b, and substitute back to find a. Dividing works for exponential models just as subtracting works for linear ones, because exponential models change by multiplying.

A **half-life** model is $y = a\left(\frac{1}{2}\right)^{t/h}$, where a is the starting amount and h is the half-life. Why does this fit? When $t = h$ the exponent is 1, so $y = \frac{a}{2}$; when $t = 2h$ the exponent is 2, so $y = \frac{a}{4}$; every h units of time multiply the amount by $\frac{1}{2}$. In the same way, $y = a \cdot 2^{t/d}$ doubles every d units of time. Both are ordinary exponential functions: $a\left(\frac{1}{2}\right)^{t/h} = a\left[\left(\frac{1}{2}\right)^{1/h}\right]^t$, with base $\left(\frac{1}{2}\right)^{1/h}$.

Scientists often write models with base e: $y = ae^{kt}$. Here a is the amount at $t = 0$ and k is the **continuous rate**, just as r was in Pe^{rt}. If $k > 0$ the quantity grows; if $k < 0$ it decays. In this lesson k is always given; in Unit 8 you will learn to find it from data with logarithms.

Newton's law of cooling. Hot cocoa set on a table does not cool toward 0°F; it cools toward the room temperature T_s. What decays exponentially is the *gap* between the cocoa and the room. If the gap starts at $T_0 - T_s$ and shrinks by the same factor every minute, then the gap is $(T_0 - T_s)e^{-kt}$, and the temperature is the room temperature plus the gap: $T = T_s + (T_0 - T_s)e^{-kt}$. As t grows, e^{-kt} approaches 0, so the graph has horizontal asymptote $T = T_s$. The cocoa cools quickly at first, when the gap is large, and more and more slowly as it nears room temperature.



Linear or exponential? For inputs that are equally spaced, a linear function adds the same amount each step (constant differences), while an exponential function multiplies by the same factor each step (constant ratios). In the table below, both start at 100 and both reach 120 after one step. The linear one keeps adding 20; the exponential one keeps multiplying by 1.2, and the gap widens every step. To choose a model for data, compute the differences and the ratios and see which is (nearly) constant.

x	Linear: y = 100 + 20x	Exponential: $y = 100(1.2)^x$
0	100	100
1	120	120
2	140	144
3	160	172.8
4	180	207.36

Depreciation. A tractor that loses 15% of its value each year keeps 85%, so its value is $V = V_0(0.85)^t$. This fits real equipment better than a straight-line loss, because a

linear model would eventually give a negative value, while the exponential model only approaches 0.

KEY FORMULAS

$$\frac{y_2}{y_1} = b^{t_2 - t_1}$$

$$y = a\left(\frac{1}{2}\right)^{t/h} \qquad y = a \cdot 2^{t/d}$$

$$y = ae^{kt}$$

$$T = T_s + (T_0 - T_s)e^{-kt}$$

Practice habit. Write one step per line. When you find a model from two points, write both equations, divide, solve for b, then solve for a, each on its own line, and finish by checking that the model gives both data points.

WORKED EXAMPLES

Example 1. Find an exponential function $y = a \cdot b^x$ whose graph passes through $(2, 72)$ and $(5, 243)$.

Show solution

1. Write both equations: $72 = ab^2$ and $243 = ab^5$.
2. Divide the second by the first: $\dfrac{243}{72} = \dfrac{ab^5}{ab^2} = b^3$.
3. Simplify: $\dfrac{243}{72} = \dfrac{27}{8}$, so $b^3 = \dfrac{27}{8}$.
4. Take the cube root: $b = \dfrac{3}{2} = 1.5$.
5. Substitute: $72 = a(1.5)^2 = 2.25a$, so $a = 32$.
6. Model: $y = 32(1.5)^x$.
7. Check: $32(1.5)^2 = 32 \times 2.25 = 72$ ✓ and $32(1.5)^5 = 32 \times 7.59375 = 243$ ✓

Answer: $y = 32(1.5)^x$

Example 2. An apple pie comes out of the oven at 200°F and is set on the counter in a 70°F kitchen. Its temperature follows Newton's law of cooling with $k = 0.04$ per minute. (a) Write the model. (b) Find the temperature after 30 minutes, to the nearest degree. (c) Find the temperature after 45 minutes, to the nearest degree.

Show solution

1. (a) $T_s = 70$ and $T_0 = 200$, so the starting gap is $200 - 70 = 130$.
2. $T = 70 + 130e^{-0.04t}$.
3. (b) $T(30) = 70 + 130e^{-1.2}$.
4. $e^{-1.2} \approx 0.3011942$, so $130e^{-1.2} \approx 39.155$.
5. $T(30) \approx 70 + 39.155 = 109.155$, about 109°F.
6. (c) $T(45) = 70 + 130e^{-1.8} \approx 70 + 130 \times 0.1652989 \approx 70 + 21.489 = 91.489$, about 91°F.
7. Check the sense: the pie lost 91 degrees in the first 30 minutes but only 18 more in the next 15, because it cools more slowly as it nears room temperature.

Answer: (a) $T = 70 + 130e^{-0.04t}$ (b) about 109°F (c) about 91°F

PRACTICE

1. Easy Show that the data in the table are exponential, and write an equation $y = a \cdot b^x$ for them.

x	y
0	5
1	15
2	45
3	135
4	405

Show answer

Each ratio is $\frac{15}{5} = \frac{45}{15} = \cdots = 3$, so the data are exponential with $a = 5$, $b = 3$: $y = 5 \cdot 3^x$.

2. Easy A sample of a radioactive substance has a mass of 80 grams and a half-life of 6 hours. How much remains after 18 hours?

Show answer

18 hours is 3 half-lives: $80\left(\frac{1}{2}\right)^3 = 10$ grams.

3. Easy A jar of sourdough starter in a warm kitchen holds 2 million yeast cells, and the number doubles every 4 hours. Write a model for the number y (in millions) after t hours, and find the number after 12 hours.

Show answer

$y = 2 \cdot 2^{t/4}$; $y(12) = 2 \cdot 2^3 = 16$ million cells.

4. Easy A new tractor costs \$80,000 and loses 10% of its value each year. Write a model for its value V after t years and find its value after 3 years.

Show answer

$V = 80000(0.9)^t$; $V(3) = 80000 \times 0.729 = \$58{,}320$.

5. **Medium** Find an exponential function $y = a \cdot b^x$ through $(0, 400)$ and $(3, 50)$.

Show answer

$a = 400$; $b^3 = \frac{50}{400} = \frac{1}{8}$, so $b = \frac{1}{2}$: $y = 400\left(\frac{1}{2}\right)^x$.

6. **Medium** Find an exponential function $y = a \cdot b^x$ through $(1, 12)$ and $(4, 324)$.

Show answer

$b^3 = \frac{324}{12} = 27$, so $b = 3$; $12 = 3a$, so $a = 4$: $y = 4 \cdot 3^x$. Check: $4 \cdot 81 = 324$ ✓

7. **Medium** A sample of radon gas decays according to $A = 500e^{-0.181t}$, where A is in milligrams and t in days. How much remains after 7 days, to the nearest milligram?

Show answer

$A(7) = 500e^{-1.267} \approx 500 \times 0.28168 \approx 140.8$, about 141 mg.

8. **Medium** One table is linear and one is exponential. Decide which is which and write a rule for each.

x	Table A	Table B
0	200	200
1	230	230
2	260	264.5
3	290	304.175

Show answer

Table A adds 30 each step: $y = 200 + 30x$. Table B multiplies by 1.15 each step ($\frac{230}{200} = \frac{264.5}{230} = 1.15$): $y = 200(1.15)^x$.

9. **Medium** In the half-life model $y = a\left(\frac{1}{2}\right)^{t/h}$, explain why the amount after $2h$ is $\frac{a}{4}$, and why the model never reaches 0.

Show answer

At $t = 2h$ the exponent is $\frac{2h}{h} = 2$, so $y = a\left(\frac{1}{2}\right)^2 = \frac{a}{4}$. Each half-life multiplies the amount by $\frac{1}{2}$, and half of a positive number is still positive, so y gets as close to 0 as you like but never equals 0.

10. Hard A cup of cocoa cools according to $T = 72 + 108e^{-0.06t}$, with t in minutes and T in °F. (a) Find its temperature after 10 minutes, to the nearest degree. (b) What temperature does the model approach as time goes on?

    Show answer

    (a) $72 + 108e^{-0.6} \approx 72 + 108 \times 0.548812 \approx 131.3$, about 131°F. (b) 72°F, the room temperature.

11. Hard Grace sold 150 jars of honey in 2022 and 216 jars in 2024, and her sales are growing exponentially. Find the yearly growth factor and predict her sales in 2027, to the nearest jar.

    Show answer

    With $t = 0$ in 2022: $b^2 = \frac{216}{150} = 1.44$, so $b = 1.2$ (20% per year). In 2027, $t = 5$: $150(1.2)^5 = 150 \times 2.48832 \approx 373$ jars.

12. Hard A tractor worth \$90,000 new is worth \$57,600 after 2 years. Assuming exponential depreciation, find the yearly rate of depreciation and the value after 5 years.

    Show answer

    $b^2 = \frac{57600}{90000} = 0.64$, so $b = 0.8$: it loses 20% per year. $V(5) = 90000(0.8)^5 = 90000 \times 0.32768 = \$29{,}491.20$.

13. Hard A sample of a radioactive substance has a mass of 96 grams. Fifteen days later it has 12 grams. Find its half-life.

    Show answer

    $\frac{12}{96} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$, so 15 days is 3 half-lives: the half-life is 5 days.

14. Challenge (a) Show that $y = a \cdot 2^{t/d}$ can be written as $y = a \cdot b^t$, and give b in terms of d. (b) A flock of chickens on a farm doubles every 5 years. Find its yearly growth rate as a percent, to the nearest tenth.

    Show answer

    (a) $2^{t/d} = \left(2^{1/d}\right)^t$ by the power-of-a-power rule, so $b = 2^{1/d}$. (b) $b = 2^{1/5} \approx 1.148698$, a growth rate of about 14.9% per year.

COMMON MISTAKES

- Subtracting instead of dividing when finding b from two points. The ratio $\frac{y_2}{y_1}$ equals $b^{t_2 - t_1}$; remember to take the root, not just divide by the number of steps.

- Writing the half-life model as $a\left(\frac{1}{2}\right)^{th}$ or $a\left(\frac{1}{2}\right)^{h/t}$. The exponent counts half-lives, so it is $\frac{t}{h}$; check that $t = h$ gives $\frac{a}{2}$.
- In Newton's law of cooling, using T_0 instead of the gap $T_0 - T_s$ in front of e^{-kt}. Check that $t = 0$ gives T_0 and that the graph levels off at T_s.
- Calling data exponential because they increase. Linear data increase too; compute the ratios and the differences to decide.

> **Make it a game · Pond Doubling Tale.** Tell a story chain about a pond where lily pads double every day and cover the pond on day 30. Take turns adding one day. When did it become half covered? Next, the parent gives two data points, such as 12 at t = 2 and 96 at t = 5. Divide the equations to find b and a and write the model.

7.4 Logistic Functions

OBJECTIVES

- Identify the carrying capacity, the starting value and the constant A in a logistic function $f(t) = \dfrac{L}{1 + Ae^{-kt}}$, and build a model from a description.
- Explain why a logistic graph is S-shaped, with horizontal asymptotes $y = 0$ and $y = L$ and fastest growth at $y = \frac{L}{2}$.
- Evaluate the standard logistic function $\sigma(x) = \dfrac{1}{1 + e^{-x}}$ and derive $\sigma(-x) = 1 - \sigma(x)$.
- Use logistic models for stocked ponds, spreading crops and news in a small town.

KEY TERMS

logistic function

A function of the form $f(t) = \dfrac{L}{1 + Ae^{-kt}}$ with $L > 0$, $A > 0$ and $k > 0$; it grows almost exponentially at first and then levels off.

carrying capacity

The number L that a logistic model approaches but never reaches: the largest population the pond, valley or town can support.

midpoint (inflection point)

The point on a logistic graph where $f = \frac{L}{2}$; there the curve is rising fastest and changes from bending upward to bending downward.

standard logistic function

The function $\sigma(x) = \dfrac{1}{1 + e^{-x}}$, called the sigmoid. It turns any real number into a number between 0 and 1.

CORE IDEAS

Exponential growth cannot go on forever. A pond holds only so many fish; a valley has only so many acres for a new apple variety; a town has only so many people to hear a piece of news. Growth that starts out exponential and then levels off is described by a **logistic function**

$$f(t) = \frac{L}{1 + Ae^{-kt}},$$

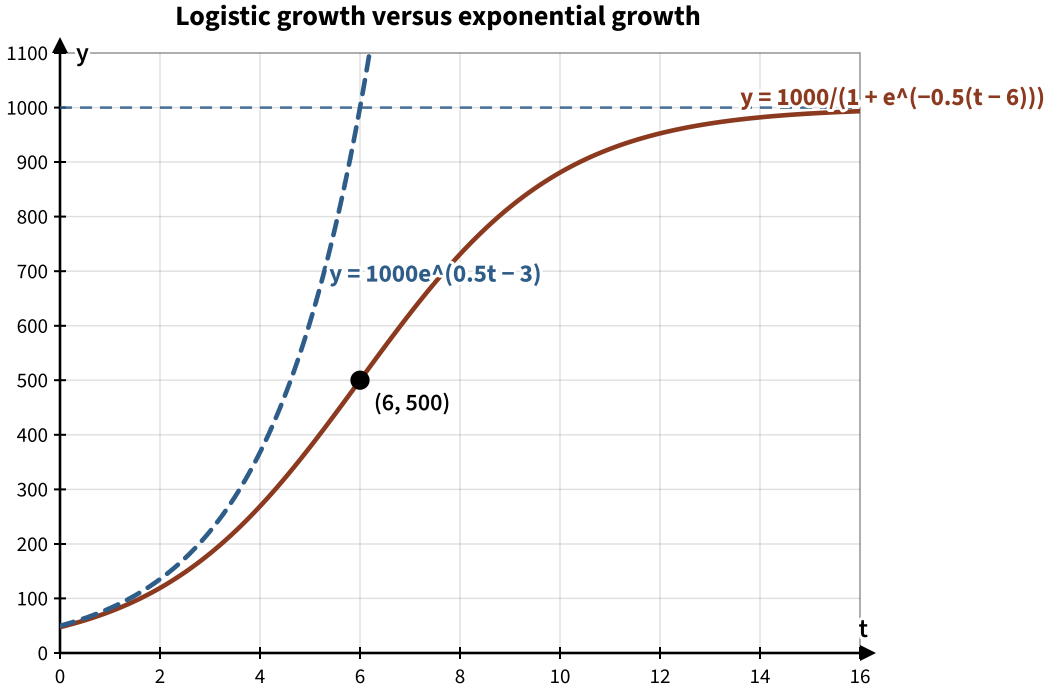
with positive constants L, A and k.

Read the constants from the formula. At $t = 0$, $e^{0} = 1$, so the starting value is $f(0) = \dfrac{L}{1 + A}$; solving for A gives $A = \dfrac{L}{f(0)} - 1$. As t grows, e^{-kt} shrinks toward 0, so

the denominator shrinks toward 1 and $f(t)$ approaches L, the **carrying capacity**. Since the denominator is always greater than 1, $f(t)$ is always less than L; going back in time, the denominator grows huge and $f(t)$ approaches 0. So the graph has two horizontal asymptotes, $y = 0$ and $y = L$, and the range is $(0, L)$.

Why the S-shape? Early on, when Ae^{-kt} is large, the 1 in the denominator hardly matters, so $f(t) \approx \dfrac{L}{Ae^{-kt}} = \dfrac{L}{A}e^{kt}$: nearly exponential growth. Later, Ae^{-kt} becomes small, the denominator is close to 1, and $f(t)$ creeps up toward L. In between, when $Ae^{-kt} = 1$, the value is $f = \dfrac{L}{1+1} = \dfrac{L}{2}$. This **midpoint** is where the population is growing fastest (a fact proved in calculus); after it, growth slows down.

A handy second form puts the midpoint in plain sight. Since $e^{-k(t-m)} = e^{km} \cdot e^{-kt}$, the function $f(t) = \dfrac{L}{1 + e^{-k(t-m)}}$ is a logistic function with $A = e^{km}$, and at $t = m$ the exponent is 0, so $f(m) = \dfrac{L}{2}$. The graph below shows $f(t) = \dfrac{1000}{1 + e^{-0.5(t-6)}}$ with its midpoint $(6, 500)$ and, dashed, the exponential $y = \dfrac{L}{A}e^{kt} = 1000e^{0.5t-3}$ that it follows at first and then leaves behind.

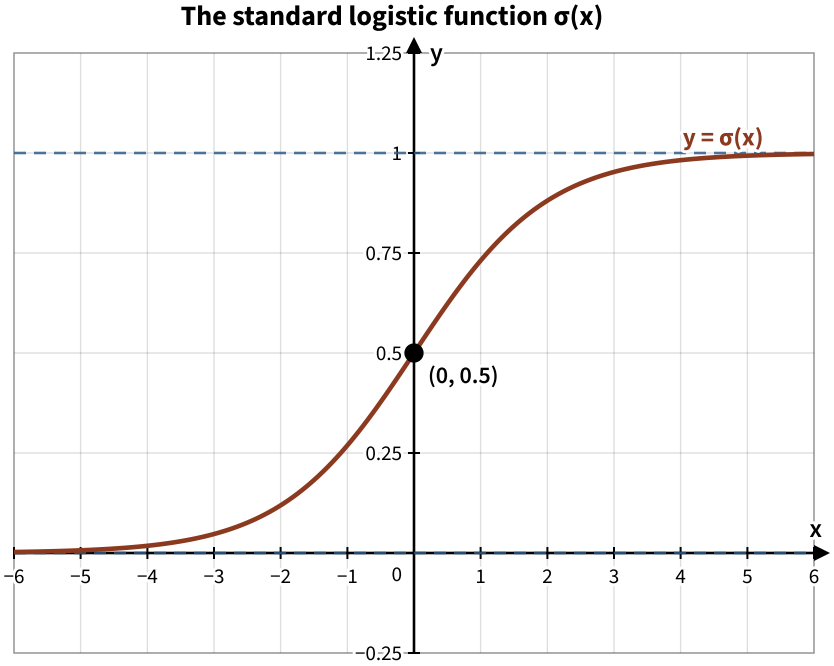


The simplest logistic function is the **standard logistic function** (sigmoid) $\sigma(x) = \dfrac{1}{1 + e^{-x}}$, with $L = 1$, $A = 1$, $k = 1$. Since $e^{-x} > 0$, the denominator is greater than 1,

so $0 < \sigma(x) < 1$ for every real x; and $\sigma(0) = \dfrac{1}{1+1} = \dfrac{1}{2}$. Its graph is symmetric about the point $(0, \frac{1}{2})$:

$$\sigma(-x) = 1 - \sigma(x).$$

Proof: $\sigma(-x) = \dfrac{1}{1+e^{x}}$; multiply top and bottom by e^{-x} to get $\dfrac{e^{-x}}{e^{-x}+1}$. And $1 - \sigma(x) = \dfrac{1+e^{-x}-1}{1+e^{-x}} = \dfrac{e^{-x}}{1+e^{-x}}$. The two are equal.



Because σ turns any real number, however large or small, into a number strictly between 0 and 1, it is a natural way to turn a score into something like a probability or a measure of confidence. That is why statisticians and computer scientists use it constantly: it appears again in the Statistics course and in the later course on the mathematics of neural networks.

KEY FORMULAS

$$f(t) = \frac{L}{1 + Ae^{-kt}}, \qquad f(0) = \frac{L}{1+A}, \qquad A = \frac{L}{f(0)} - 1$$

$$f(t) = \frac{L}{1 + e^{-k(t-m)}} \quad \text{has midpoint } \left(m, \tfrac{L}{2}\right)$$

$$\sigma(x) = \frac{1}{1+e^{-x}}, \qquad \sigma(0) = \frac{1}{2}, \qquad \sigma(-x) = 1 - \sigma(x)$$

From history. The Belgian mathematician Pierre-François Verhulst proposed the logistic model in 1838 as a better description of population growth than unlimited exponential growth. He argued that growth must slow as a population nears the limit its surroundings can support.

Practice habit. Sketch before you compute. Draw the asymptotes $y = 0$ and $y = L$, mark the starting value $\frac{L}{1+A}$ and the level $\frac{L}{2}$, and then check every calculated value against the sketch: it must lie strictly between 0 and L.

WORKED EXAMPLES

Example 1. A farm pond is stocked with bass, and the number of fish after t years is modeled by $f(t) = \dfrac{600}{1 + 5e^{-0.3t}}$. (a) What is the carrying capacity? (b) How many fish were stocked? (c) How many fish are there after 4 years, to the nearest fish?

Show solution

1. (a) As t grows, $5e^{-0.3t}$ approaches 0, so $f(t)$ approaches $\frac{600}{1} = 600$. The carrying capacity is 600 fish.
2. (b) $f(0) = \dfrac{600}{1 + 5e^{0}} = \dfrac{600}{6} = 100$ fish.
3. (c) $f(4) = \dfrac{600}{1 + 5e^{-1.2}}$.
4. $e^{-1.2} \approx 0.3011942$, so $5e^{-1.2} \approx 1.5059711$.
5. $f(4) \approx \dfrac{600}{2.5059711} \approx 239.4$, about 239 fish.
6. Check against a sketch: 239 lies between the start, 100, and the midpoint level, 300, so the pond is still in its fast-growing early stage.

Answer: (a) 600 fish (b) 100 fish (c) about 239 fish

Example 2. News of a barn-raising spreads through a small town of 2,400 people. At 8 a.m. ($t = 0$), 30 people have heard. Suppose the number who have heard after t hours is $f(t) = \dfrac{2400}{1 + Ae^{-0.8t}}$. (a) Find A. (b) How many people have heard after 5 hours, to the nearest person? (c) Use a table of values to find the first whole hour at which more than half the town has heard.

Show solution

1. (a) $A = \dfrac{L}{f(0)} - 1 = \dfrac{2400}{30} - 1 = 80 - 1 = 79$.
2. Check: $f(0) = \dfrac{2400}{1 + 79} = 30$ ✓
3. (b) $f(5) = \dfrac{2400}{1 + 79e^{-4}}$; $e^{-4} \approx 0.0183156$, so $79e^{-4} \approx 1.4469355$.
4. $f(5) \approx \dfrac{2400}{2.4469355} \approx 980.8$, about 981 people.
5. (c) Half the town is 1,200 people. At $t = 5$, only 981 have heard.
6. $f(6) = \dfrac{2400}{1 + 79e^{-4.8}} \approx \dfrac{2400}{1 + 79 \times 0.0082297} \approx \dfrac{2400}{1.6501500} \approx 1454.4$.
7. So hour 6 (2 p.m.) is the first whole hour when more than half the town has heard. The news spread fastest between hours 5 and 6, as it passed the midpoint of 1,200.

Answer: (a) $A = 79$ (b) about 981 people (c) hour 6

PRACTICE

1. Easy For $f(t) = \dfrac{500}{1 + 4e^{-0.2t}}$, find the carrying capacity and the starting value $f(0)$.

   Show answer

   Carrying capacity 500; $f(0) = \frac{500}{5} = 100$.

2. Easy Find $\sigma(0)$ exactly, where $\sigma(x) = \dfrac{1}{1 + e^{-x}}$.

   Show answer

   $\sigma(0) = \frac{1}{1+1} = \frac{1}{2}$

3. **Easy** For $f(t) = \dfrac{800}{1 + 3e^{-0.5t}}$, find $f(0)$ and the value of f at the midpoint of the curve.

Show answer

$f(0) = \frac{800}{4} = 200$; the midpoint value is $\frac{800}{2} = 400$.

4. **Easy** A logistic model $f(t) = \dfrac{L}{1 + Ae^{-kt}}$ has carrying capacity 1,200 and starting value 40. Find A.

Show answer

$A = \frac{1200}{40} - 1 = 30 - 1 = 29$

5. **Medium** Evaluate $\sigma(2)$ and $\sigma(-2)$ to four decimal places, and check that they add to 1.

Show answer

$\sigma(2) = \frac{1}{1+e^{-2}} \approx 0.8808$; $\sigma(-2) = \frac{1}{1+e^{2}} \approx 0.1192$; $0.8808 + 0.1192 = 1$ ✓

6. **Medium** The number of trout in a stocked pond after t years is $f(t) = \dfrac{900}{1 + 8e^{-0.6t}}$. How many trout were stocked, and how many are there after 3 years, to the nearest fish?

Show answer

$f(0) = \frac{900}{9} = 100$. $f(3) = \frac{900}{1+8e^{-1.8}} \approx \frac{900}{2.32239} \approx 387.5$, about 388 trout.

7. **Medium** A new apple variety spreads across a valley, and the number of acres planted after t years is $f(t) = \dfrac{5000}{1 + e^{-0.5(t-8)}}$. At what time t is the planting growing fastest, and how many acres are planted then?

Show answer

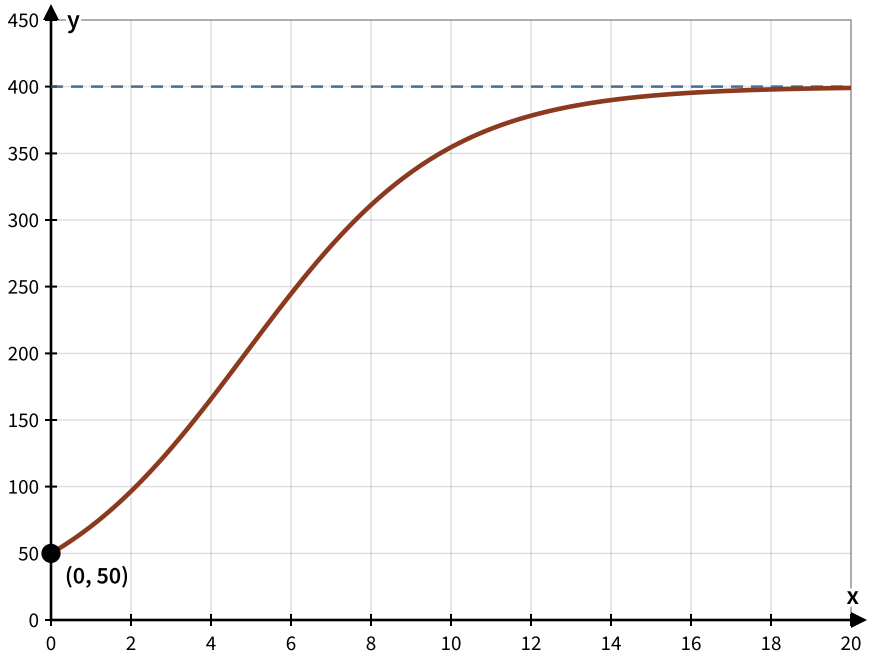
The midpoint is at $t = 8$, where $f(8) = \frac{5000}{1+e^{0}} = 2500$ acres.

8. **Medium** Explain why $0 < \sigma(x) < 1$ for every real number x, and why $\sigma(x)$ never equals 0 or 1.

Show answer

e^{-x} is positive for every x, so the denominator $1 + e^{-x}$ is greater than 1. A fraction with numerator 1 and a denominator greater than 1 is positive and less than 1. It cannot equal 1, because that would need $e^{-x} = 0$, and it cannot equal 0, because its numerator is 1.

9. **Medium** The graph shows a logistic model $y = \dfrac{L}{1 + Ae^{-0.4x}}$ with its starting point and its upper asymptote. Find L and A, and write the equation.



Show answer

The upper asymptote is $y = 400$, so $L = 400$. The starting value is 50, so $A = \frac{400}{50} - 1 = 7$: $y = \dfrac{400}{1 + 7e^{-0.4x}}$.

10. **Hard** A town has 3,000 people. At $t = 0$, 60 of them have heard about a new library opening, and news spreads with $k = 0.9$ per day. Write the logistic model $f(t) = \dfrac{L}{1 + Ae^{-kt}}$ and find how many have heard after 4 days, to the nearest person.

Show answer

$A = \frac{3000}{60} - 1 = 49$: $f(t) = \dfrac{3000}{1 + 49e^{-0.9t}}$. $f(4) = \frac{3000}{1+49e^{-3.6}} \approx \frac{3000}{2.33886} \approx 1282.7$, about 1,283 people.

11. Hard Compare $f(t) = \dfrac{1000}{1 + 99e^{-t}}$ with the exponential $g(t) = 10e^{t}$ (both start at 10). Find $f(1)$, $g(1)$, $f(5)$ and $g(5)$ to the nearest whole number, and explain what you see.

Show answer

$f(1) \approx 27$, $g(1) \approx 27$, $f(5) \approx 600$, $g(5) \approx 1484$. At first the logistic model is nearly exponential, but later it falls far behind because it levels off below 1,000.

12. Hard Acres planted in a new strawberry variety are modeled by $f(t) = \dfrac{2000}{1 + e^{-0.5(t-6)}}$. (a) Write the model in the form $\dfrac{L}{1 + Ae^{-kt}}$, giving A exactly. (b) How many acres were planted at $t = 0$, to the nearest acre?

Show answer

(a) $e^{-0.5(t-6)} = e^{3}e^{-0.5t}$, so $A = e^{3}$ and $f(t) = \dfrac{2000}{1 + e^{3}e^{-0.5t}}$. (b) $f(0) = \frac{2000}{1+e^{3}} \approx \frac{2000}{21.0855} \approx 94.9$, about 95 acres.

13. Challenge Prove that $\sigma(x) = \dfrac{e^{x}}{e^{x} + 1}$ and that $\dfrac{\sigma(x)}{1 - \sigma(x)} = e^{x}$.

Show answer

Multiply the top and bottom of $\frac{1}{1+e^{-x}}$ by e^{x}: $\sigma(x) = \frac{e^{x}}{e^{x}+1}$. Then $1 - \sigma(x) = \frac{e^{x}+1-e^{x}}{e^{x}+1} = \frac{1}{e^{x}+1}$, so $\frac{\sigma(x)}{1-\sigma(x)} = \frac{e^{x}}{e^{x}+1} \cdot \frac{e^{x}+1}{1} = e^{x}$.

COMMON MISTAKES

- Taking A to be the starting value. The starting value is $\frac{L}{1+A}$; find A from $A = \frac{L}{f(0)} - 1$ and check by substituting $t = 0$.
- Typing the formula without parentheses around the denominator: $L \div (1 + Ae^{-kt})$, not $L \div 1 + Ae^{-kt}$. A value above L or below 0 means a keying error.
- Dropping the negative sign in e^{-kt}. With e^{kt} the function would decrease toward 0 instead of growing toward L.
- Saying the population reaches L. The model approaches L but always stays below it.

Make it a game · Sketch the S. The parent describes a neighborhood of 500 people who hear some news. At the start, 5 know it. Sketch the S-shaped logistic curve, marking the carrying capacity, the two horizontal asymptotes and the midpoint at 250. Find A from f(0) = 5 in f(t) = 500/(1 + Ae^(-kt)). Then swap and describe a different spread situation.

UNIT 7

Exponential Functions

This unit is about quantities that change by multiplying: savings that earn interest on interest, equipment that loses a fixed percent of its value each year, radioactive samples that lose half their mass in each half-life, and a cup of cocoa cooling toward room temperature. You will graph exponential functions and their transformations, see why exponential growth eventually beats any polynomial, and meet the number $e \approx 2.71828$, which comes from compounding interest more and more often.

The unit builds on the exponential functions and compound interest of Algebra I, the transformations of Lesson 1.2 and the rational exponents of Unit 6. It ends with logistic functions, which describe growth that levels off, such as fish in a stocked pond or news spreading through a town. Unit 8 then introduces logarithms, the tool for solving exponential equations for the time or the rate; Unit 9 uses these same growth factors to understand savings plans and loans.

PREREQUISITES

- Exponent rules, including zero, negative and rational exponents (Lessons 6.1 and 6.2)
- Exponential growth and decay $y = a \cdot b^x$ and compound interest from Algebra I
- Transformations of graphs $y = a \cdot f(x - h) + k$ (Lesson 1.2)
- Domain and range in interval notation (Lesson 1.1)
- Solving equations with nth roots, such as $b^3 = \frac{27}{8}$ (Lesson 6.5)

MASTERY CHECKLIST

- ☐ I can explain why the base of an exponential function must be positive and not 1.
- ☐ I can graph $y = a \cdot b^{x-h} + k$ and give its asymptote, domain, range and y-intercept.
- ☐ I can convert between a growth or decay rate and a growth factor.
- ☐ I can explain why 2^x eventually overtakes x^3.
- ☐ I can derive and use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to find balances and deposits.
- ☐ I can explain where the number e comes from and why continuous compounding gives $A = Pe^{rt}$.
- ☐ I can write an exponential model from two data points.
- ☐ I can use half-life, doubling-time and continuous models, and Newton's law of cooling.
- ☐ I can tell linear data from exponential data using differences and ratios.
- ☐ I can find the carrying capacity, the starting value and A for a logistic model and explain its S-shape.

- ☐ I can evaluate $\sigma(x)$ and prove that $\sigma(-x) = 1 - \sigma(x)$

LESSONS IN THIS UNIT

7.1	**Exponential Functions and Their Graphs**	2 sessions of 45 minutes
7.2	**Compound Interest and the Number e**	2 sessions of 45 minutes
7.3	**Exponential Models**	2 sessions of 45 minutes
7.4	**Logistic Functions**	2 sessions of 45 minutes

Unit 7 test

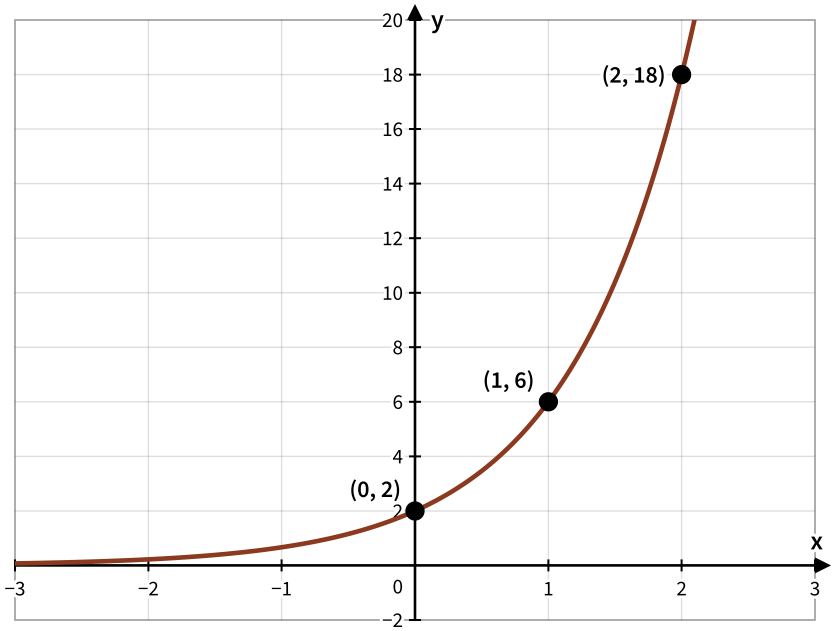
You have 60 minutes. A scientific calculator is allowed except where a problem says "by hand"; use the e^x key rather than a rounded value of e. Show your work for every problem. Give exact answers unless the problem asks you to round, and round only at the final step, as the problem directs. Give units in word problems.

1. **5 pts**
   By hand, evaluate $f(x) = 6\left(\frac{1}{3}\right)^x$ at $x = -2$, $x = 0$ and $x = 2$.

2. **6 pts**
   For $g(x) = -3 \cdot 2^{x-1} + 4$, give the horizontal asymptote, the domain, the range and the y-intercept.

3. **6 pts**

The graph shows a function $y = a \cdot b^x$ through the labeled points. Write its equation.



4. **6 pts**

Ruth's goat herd has 40 goats and grows 9% per year. (a) What is the yearly growth factor? (b) How many goats will she have after 4 years, to the nearest goat?

5. **7 pts**

The Miller family deposits \$3,500 at 4.2% compounded monthly. Find the balance after 5 years, to the nearest cent.

6. **6 pts**

Find the balance after 12 years when \$8,000 is invested at 3.5% compounded continuously, to the nearest cent.

7. **8 pts**

Starting from $A = P\left(1 + \frac{r}{n}\right)^{nt}$, explain why compounding more and more often leads to the formula $A = Pe^{rt}$.

8. **7 pts**

   Find an exponential function $y = a \cdot b^x$ whose graph passes through $(1, 24)$ and $(3, 54)$.

9. **7 pts**

   A hospital uses a 120-milligram sample of iodine-131, which has a half-life of 8 days. (a) How much remains after 24 days? (b) How much remains after 30 days, to the nearest tenth of a milligram?

10. **7 pts**

    A loaf of bread comes out of the oven at 190°F into a 68°F kitchen and cools by Newton's law of cooling with $k = 0.05$ per minute. (a) Find its temperature after 20 minutes, to the nearest degree. (b) What temperature does it approach?

11. **7 pts**

    One of the columns P and Q in the table is linear and the other is exponential. Write a rule for each.

x	P	Q
0	50	50
1	60	60
2	70	72
3	80	86.4
4	90	103.68

12. **7 pts**

    A hay baler is worth \$40,500 one year after it was bought and \$32,805 three years after it was bought. Assuming exponential depreciation, find the yearly rate of depreciation and the price of the baler when new.

13. **7 pts**

The number of deer in a nature preserve after t years is modeled by $f(t) = \dfrac{1500}{1 + 14e^{-0.35t}}$. Find (a) the carrying capacity, (b) the starting number of deer, (c) the number after 6 years, to the nearest deer.

14. **6 pts**

Given that $\sigma(1.5) \approx 0.8176$, find $\sigma(-1.5)$ without evaluating $e^{1.5}$, and name the property you used.

15. **8 pts**

A new hay-drying method spreads among the 1,200 farms of a county. At $t = 0$, 24 farms use it, and the number of farms using it after t years is $f(t) = \dfrac{1200}{1 + Ae^{-0.7t}}$. (a) Find A. (b) How many farms use the method after 5 years, to the nearest farm?

ANSWER KEY

1. $f(-2) = 6 \cdot 9 = 54$, $f(0) = 6$, $f(2) = \frac{6}{9} = \frac{2}{3}$
2. Asymptote $y = 4$; domain $(-\infty, \infty)$; range $(-\infty, 4)$; y-intercept $-3 \cdot \frac{1}{2} + 4 = 2.5$.
3. $a = 2$ (the y-intercept) and each step right multiplies y by 3: $y = 2 \cdot 3^x$.
4. (a) 1.09 (b) $40(1.09)^4 = 40 \times 1.41158161 \approx 56.46$, about 56 goats.
5. $3500\left(1 + \frac{0.042}{12}\right)^{60} = 3500(1.0035)^{60} \approx 3500 \times 1.2332258 \approx \$4{,}316.29$
6. $8000e^{0.035 \times 12} = 8000e^{0.42} \approx 8000 \times 1.5219616 \approx \$12{,}175.69$
7. Let $m = \frac{n}{r}$, so $\frac{r}{n} = \frac{1}{m}$ and $nt = m \cdot rt$. Then $\left(1 + \frac{r}{n}\right)^{nt} = \left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}$. As n grows, m grows, and $\left(1 + \frac{1}{m}\right)^m$ approaches e. So the balance approaches Pe^{rt}.
8. $\frac{54}{24} = b^2 = 2.25$, so $b = 1.5$; $24 = 1.5a$, so $a = 16$: $y = 16(1.5)^x$. Check: $16 \times 3.375 = 54$ ✓
9. (a) 3 half-lives: $120\left(\frac{1}{2}\right)^3 = 15$ mg. (b) $120\left(\frac{1}{2}\right)^{30/8} = 120\left(\frac{1}{2}\right)^{3.75} \approx 8.9$ mg.
10. (a) $T = 68 + 122e^{-0.05t}$; $T(20) = 68 + 122e^{-1} \approx 68 + 44.88 \approx 113°$F. (b) 68°F.
11. P adds 10 each step: $y = 50 + 10x$. Q multiplies by 1.2 each step: $y = 50(1.2)^x$.
12. $b^2 = \frac{32805}{40500} = 0.81$, so $b = 0.9$: it loses 10% per year. New price: $\frac{40500}{0.9} = \$45{,}000$.
13. (a) 1,500 (b) $\frac{1500}{15} = 100$ (c) $\frac{1500}{1+14e^{-2.1}} \approx \frac{1500}{2.71439} \approx 553$ deer.
14. $\sigma(-x) = 1 - \sigma(x)$, so $\sigma(-1.5) \approx 1 - 0.8176 = 0.1824$.
15. (a) $A = \frac{1200}{24} - 1 = 49$. (b) $f(5) = \frac{1200}{1+49e^{-3.5}} \approx \frac{1200}{2.47967} \approx 484$ farms.

UNIT 8

Logarithmic Functions

In Unit 7 you used exponential functions to find how much: how much money after 10 years, how much of a substance after 3 half-lives. This unit answers the reverse question, how long: how many years until savings double, how many minutes until a pie cools. That question asks for an exponent, and the tool that finds exponents is the logarithm. You will learn that a logarithm is an exponent, that $y = \log_b x$ is the inverse of $y = b^x$ (so its graph is a reflection in $y = x$), and that the properties of logarithms are the rules of exponents in a new form.

With those properties you will solve exponential and logarithmic equations exactly and with a calculator, and learn why some candidate answers are extraneous and must be checked against the domain. The unit ends with logarithmic scales that scientists use every day (pH, decibels, earthquake and star magnitudes), the rule of 72 for doubling time, and fitting exponential models to data by taking logs.

Logarithms come back often in later work: in sequences and loan formulas in Unit 9, in the Statistics course, and in calculus, where $\ln x$ and e^x are the most important functions of all. They also appear in the mathematics of modern computing, where the logistic function from Lesson 7.4 and logarithms work side by side.

PREREQUISITES

- Use the exponent rules, including zero, negative and rational exponents (Lesson 6.1).
- Find inverse functions and understand that the graph of an inverse is a reflection in $y = x$ (Lesson 1.5).
- Transform graphs with $y = a\,f(x - h) + k$ (Lesson 1.2).
- Work with exponential functions, compound interest and the number e (Lessons 7.1–7.3).
- Solve quadratic equations by factoring and solve quadratic and rational inequalities with a sign chart (Lessons 3.2, 3.5 and 5.4).

MASTERY CHECKLIST

- ☐ I can explain what a logarithm is and switch between logarithmic and exponential form.
- ☐ I can evaluate logarithms such as $\log_2 32$, $\log_3 \frac{1}{9}$ and $\log_5 \sqrt{5}$ by hand, and estimate a logarithm between two whole numbers.

- ☐ I can graph $y = a\log_b(x-h)+k$ and give its domain, range, asymptote and intercepts, and find the domain of the logarithm of an expression.
- ☐ I can derive the product, quotient and power properties from the exponent rules and use them to expand and condense logarithms.
- ☐ I can derive the change-of-base formula and use it to evaluate any logarithm on a calculator.
- ☐ I can solve exponential equations by using a common base or by taking logarithms, including equations with e and equations in quadratic form.
- ☐ I can solve logarithmic equations and explain and reject extraneous solutions.
- ☐ I can solve growth, decay, cooling and savings models for time.
- ☐ I can compute and compare values on logarithmic scales: pH, decibels, earthquake magnitudes and star magnitudes.
- ☐ I can derive and use the rule of 72, and fit an exponential model to data by taking logarithms.

LESSONS IN THIS UNIT

8.1	**Logarithms**	2 sessions of 45 minutes
8.2	**Graphs of Logarithmic Functions**	2 sessions of 45 minutes

8.3	**Properties of Logarithms**	2 sessions of 45 minutes
8.4	**Solving Exponential Equations**	2 sessions of 45 minutes
8.5	**Solving Logarithmic Equations**	2 sessions of 45 minutes
8.6	**Logarithmic Scales and Models**	2 sessions of 45 minutes

Unit 8 test

You have 60 minutes. A scientific calculator is allowed except on problems marked "by hand." Show your work on every problem, and check each solution of an equation in the original equation. Give exact answers unless a problem asks you to round; when it does, round only at the last step.

1. **5 pts**

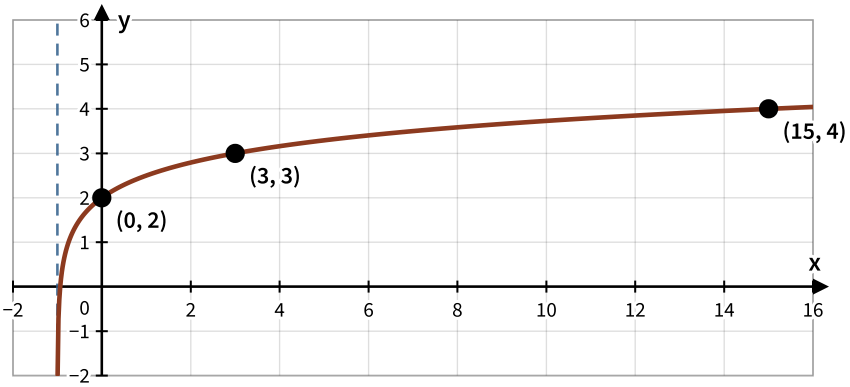
   By hand, evaluate: (a) $\log_4 64$ (b) $\log_2 \dfrac{1}{32}$ (c) $\log_9 3$

2. **5 pts**

   By hand, solve for x: (a) $\log_x 125 = 3$ (b) $\log_8 x = \dfrac{2}{3}$

3. **6 pts**

   The graph shows a function of the form $y = \log_b(x - h) + k$, with its asymptote dashed. Find b, h and k, and explain how you found each.



4. **6 pts**

   For $f(x) = \log_3(x + 6) - 2$, give the domain, the vertical asymptote and the x-intercept.

5. **6 pts**

   Expand completely (all variables positive): $\log_2 \dfrac{x^2\sqrt{y}}{16}$

6. **6 pts**

   Condense into a single logarithm: $4\ln x - \ln(x+1) + 2\ln y$

7. **5 pts**

   Use the change-of-base formula to evaluate $\log_6 45$ to the nearest thousandth.

8. **7 pts**

   Solve: (a) $16^{x} = 8^{x+1}$ (b) $5 \cdot 4^{x} + 2 = 47$. Give an exact answer for (b) and round it to the nearest hundredth.

9. **7 pts**

   Solve exactly: $e^{2x} - e^{x} - 12 = 0$

10. **7 pts**

    Solve $\log_2 x + \log_2(x+6) = 4$ and check for extraneous solutions.

11. **7 pts**

    Solve exactly: $\log_3(x+2) = 2 - \log_3 x$

12. **7 pts**

    Ruth deposits \$1,500 in an account paying 3.6% interest compounded quarterly. How long will it take for the balance to reach \$2,400? Round to the nearest tenth of a year.

13. **7 pts**

    A mug of cocoa at 180°F is set on a table in a 68°F room. Its temperature after t minutes is $T = 68 + 112e^{-0.05t}$. When will it cool to 140°F? Round to the nearest tenth of a minute.

14. **7 pts**

(a) A bakery's dough mixer runs at 85 dB, and the kitchen refrigerator hums at 45 dB. How many times as intense is the mixer's sound? (b) A jar of pickle brine has $[\mathrm{H}^+] = 3.2 \times 10^{-4}$ moles per liter. Find its pH to the nearest tenth.

15. **5 pts**

A family puts savings for a new tractor in a fund growing 3% per year, compounded yearly. (a) Estimate the doubling time with the rule of 72. (b) Find the exact doubling time to the nearest tenth of a year.

16. **7 pts**

(a) Prove the product property $\log_b(MN) = \log_b M + \log_b N$ for $M, N > 0$ from the rules of exponents. (b) Solve $\log_2 x + \log_2(x - 3) = 2$, and explain why one candidate must be rejected even though it solves the condensed equation.

ANSWER KEY

1. (a) 3 (b) -5 (c) $\dfrac{1}{2}$, since $9^{1/2} = 3$
2. (a) $x^3 = 125$, so $x = 5$ (b) $x = 8^{2/3} = (\sqrt[3]{8})^2 = 4$
3. The asymptote is $x = -1$, so $h = -1$. At $(0, 2)$ the argument $x + 1$ is 1, so the log is 0 and $k = 2$. At $(3, 3)$ the argument is 4 and $\log_b 4 + 2 = 3$, so $\log_b 4 = 1$ and $b = 4$: $y = \log_4(x + 1) + 2$
4. Domain $(-6, \infty)$; asymptote $x = -6$; $\log_3(x + 6) = 2$ gives $x + 6 = 9$, so the x-intercept is 3
5. $2\log_2 x + \dfrac{1}{2}\log_2 y - 4$
6. $\ln \dfrac{x^4 y^2}{x + 1}$
7. $\dfrac{\ln 45}{\ln 6} \approx \dfrac{3.80666}{1.79176} \approx 2.125$
8. (a) $2^{4x} = 2^{3x+3}$, so $x = 3$; check: $16^3 = 4096 = 8^4$ ✓ (b) $4^x = 9$, so $x = \dfrac{\ln 9}{\ln 4} = \log_2 3 \approx 1.58$
9. With $u = e^x$: $(u - 4)(u + 3) = 0$. Since $e^x > 0$, $e^x = -3$ is impossible, so $e^x = 4$ and $x = \ln 4$
10. $x(x + 6) = 16$, $x^2 + 6x - 16 = 0$, $(x + 8)(x - 2) = 0$. $x = 2$ works: $\log_2 2 + \log_2 8 = 1 + 3 = 4$ ✓; $x = -8$ is extraneous ($\log_2(-8)$ is undefined)
11. $\log_3 (x(x + 2)) = 2$, so $x^2 + 2x - 9 = 0$ and $x = -1 \pm \sqrt{10}$. Only $x = -1 + \sqrt{10} \approx 2.16$ makes both arguments positive; check: $x(x + 2) = (\sqrt{10} - 1)(\sqrt{10} + 1) = 9$ ✓
12. $1500(1.009)^{4t} = 2400$, so $(1.009)^{4t} = 1.6$ and $t = \dfrac{\ln 1.6}{4 \ln 1.009} \approx 13.1$ years
13. $112e^{-0.05t} = 72$, so $e^{-0.05t} = \dfrac{9}{14}$ and $t = \dfrac{\ln(9/14)}{-0.05} \approx 8.8$ minutes
14. (a) $10^{(85-45)/10} = 10^4 = 10{,}000$ times (b) $\text{pH} = -\log(3.2 \times 10^{-4}) = 4 - \log 3.2 \approx 3.5$
15. (a) $72 \div 3 = 24$ years (b) $\dfrac{\ln 2}{\ln 1.03} \approx \dfrac{0.693147}{0.029559} \approx 23.4$ years
16. (a) Let $m = \log_b M$ and $n = \log_b N$, so $M = b^m$ and $N = b^n$. Then $MN = b^m b^n = b^{m+n}$, so $\log_b(MN) = m + n = \log_b M + \log_b N$. (b) $x(x - 3) = 4$, $(x - 4)(x + 1) = 0$. $x = 4$ works: $\log_2 4 + \log_2 1 = 2$ ✓. $x = -1$ satisfies $\log_2 (x(x - 3)) = 2$ because $(-1)(-4) = 4$, but in the original $\log_2(-1)$ is undefined; condensing enlarged the domain, so $x = -1$ is extraneous.

8.1 Logarithms

OBJECTIVES

- Explain what a logarithm is and translate between logarithmic form and exponential form.
- Evaluate logarithms by hand by writing the number as a power of the base.
- Explain why $y = \log_b x$ is the inverse of $y = b^x$ and use the facts $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$ and $b^{\log_b x} = x$.
- Estimate a logarithm between two consecutive whole numbers.

WATCH

Introduction to Logarithms (1 of 2: Definition)

Eddie Woo · 7:52

https://www.youtube.com/watch?v=ntBWrcbAhaY

> **WHEN TO WATCH**
> Before Example 1: what a logarithm is, as the inverse of an exponential.

BACKUP VIDEO

Evaluating Logarithms Explained (Step-by-Step Examples)

Mario's Math Tutoring · 10:52

https://www.youtube.com/watch?v=KcFswfR37K0

After Example 1: more logarithms evaluated step by step.

KEY TERMS

logarithm

For $b > 0$, $b \neq 1$ and $x > 0$, $\log_b x$ is the exponent you put on b to get x: $\log_b x = y$ means $b^y = x$.

base

The number b in $\log_b x$; it is the same as the base of the power b^y.

argument

The number x whose logarithm is taken in $\log_b x$; it must be positive.

common logarithm

The logarithm with base 10, written $\log x$ with no base shown, so $\log x = \log_{10} x$.

natural logarithm

The logarithm with base $e \approx 2.71828$, written $\ln x$, so $\ln x = \log_e x$.

CORE IDEAS

In Unit 7 you answered questions like "how much will the account hold after 10 years?" by computing a power. Now turn the question around: "how many years until it doubles?" That question asks for an **exponent**. Some such exponents are easy: $2^y = 8$ gives $y = 3$, because you know $2^3 = 8$. But what exponent makes $2^y = 10$? It must be between 3 and 4, and we need a name for it. That name is the logarithm.

Definition. For a base $b > 0$ with $b \neq 1$ and a number $x > 0$, $\log_b x = y$ means $b^y = x$. Read $\log_b x$ as "the exponent you put on b to get x." So $\log_2 8 = 3$ because $2^3 = 8$, and the exponent we wanted above is $\log_2 10$. A logarithm is an exponent. Every logarithm statement is an exponent statement in disguise, and you can always translate: $\log_3 81 = 4$ says exactly the same thing as $3^4 = 81$.

The table lists powers of 2. Read from right to left it is the exponential function $y = 2^x$'s table; read from left to right it answers the logarithm question "what exponent on 2 gives this number?"

Powers of 2 read backward

$x = 2^y$	$y = \log_2 x$
1/8	−3
1/4	−2
1/2	−1
1	0
2	1
4	2
8	3

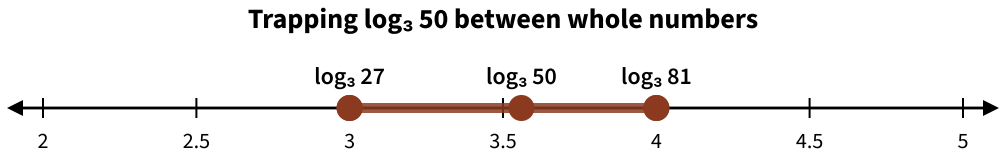
$x = 2^y$	$y = \log_2 x$
16	4
32	5

To evaluate a logarithm by hand, write the number as a power of the base. For $\log_3 \frac{1}{9}$: since $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$, the exponent is -2, so $\log_3 \frac{1}{9} = -2$. For $\log_5 \sqrt{5}$: since $\sqrt{5} = 5^{1/2}$, $\log_5 \sqrt{5} = \frac{1}{2}$. Two bases are used so often that they have their own names. The **common logarithm** $\log x$ (no base written) means $\log_{10} x$, so $\log 1000 = 3$ because $10^3 = 1000$. The **natural logarithm** $\ln x$ means $\log_e x$, using the number $e \approx 2.71828$ from Lesson 7.2. Your calculator has a LOG key and an LN key for these two bases.

Why is a logarithm always there to find when $x > 0$? The function $f(x) = b^x$ is always increasing (when $b > 1$) or always decreasing (when $0 < b < 1$), so it passes the horizontal line test, and by Lesson 1.5 it has an inverse. To find the inverse, swap x and y in $y = b^x$ to get $x = b^y$. Solving that for y means asking "what exponent on b gives x?", which is exactly $y = \log_b x$. So **the logarithm is the inverse of the exponential function**: if $f(x) = b^x$, then $f^{-1}(x) = \log_b x$. The outputs of b^x are all the positive numbers, so the inputs of $\log_b x$ are exactly the positive numbers: $\log_2 0$ and $\log_2(-4)$ are undefined, because no power of 2 is 0 or negative. The base cannot be 1, because $1^y = 1$ for every y, so no power of 1 could ever equal 5.

Four facts follow straight from the definition. Since $b^0 = 1$, $\log_b 1 = 0$. Since $b^1 = b$, $\log_b b = 1$. The exponent you put on b to get b^x is x, so $\log_b b^x = x$. And $\log_b x$ is by definition the exponent that turns b into x, so $b^{\log_b x} = x$. The last two are the inverse checks $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ from Lesson 1.5: raising b to a power and taking $\log_b$ undo each other. For example, $\log_7 7^{12} = 12$, $10^{\log 6} = 6$ and $\ln e^3 = 3$.

Most logarithms are not whole numbers, but you can always trap one between two whole numbers. To estimate $\log_3 50$, find the powers of 3 on either side of 50: $3^3 = 27$ and $3^4 = 81$. Larger exponents give larger powers of 3, so the exponent that gives 50 is between 3 and 4. (In Lesson 8.3 you will see how a calculator finds it: $\log_3 50 \approx 3.56$.)



KEY FORMULAS

$$\log_b x = y \quad \text{means} \quad b^y = x \qquad (b > 0,\ b \neq 1,\ x > 0)$$

$$\log_b 1 = 0 \qquad \log_b b = 1$$

$$\log_b b^x = x \qquad b^{\log_b x} = x$$

$$\log x = \log_{10} x \qquad \ln x = \log_e x$$

From history. The Scottish mathematician John Napier published the first table of logarithms in 1614, and the English mathematician Henry Briggs made tables of base-10 logarithms in 1617 and 1624. With a table of logarithms, a long multiplication could be turned into an addition (you will see why in Lesson 8.3). Astronomers and navigators, who multiplied large numbers all day, welcomed them as an enormous saving of time and of errors.

Practice habit. Know the powers of 2 up to $2^{10} = 1024$ and the powers of 3 up to $3^5 = 243$ cold, along with the powers of 10. Then most hand-evaluated logarithms are just a matter of recognizing a power.

WORKED EXAMPLES

Example 1. Evaluate each logarithm by hand: (a) $\log_2 32$ (b) $\log_3 \frac{1}{9}$ (c) $\log_5 \sqrt{5}$ (d) $\log 1000$

Show solution

1. (a) Ask: 2 to what power is 32? Since $32 = 2^5$, $\log_2 32 = 5$.
2. (b) Write $\frac{1}{9}$ as a power of 3: $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$.
3. So $\log_3 \frac{1}{9} = -2$. Check: $3^{-2} = \frac{1}{9}$ ✓
4. (c) Write $\sqrt{5}$ as a power of 5: $\sqrt{5} = 5^{1/2}$, so $\log_5 \sqrt{5} = \frac{1}{2}$.
5. (d) log with no base means base 10, and $1000 = 10^3$, so $\log 1000 = 3$.

Answer: (a) 5 (b) -2 (c) $\frac{1}{2}$ (d) 3

Example 2. A county's phone tree for school snow closings works in rounds. In round 1 the school office calls 3 families; in round 2 each of those families calls 3 new families; and so on, so round n reaches 3^n new families. (a) In which round are 729 new families called? Write the answer as a logarithm and evaluate it. (b) Use $\log_3 2000$ to find the first round that reaches at least 2000 new families.

Show solution

1. (a) We need $3^n = 729$, so $n = \log_3 729$.
2. Count up the powers of 3: $3^4 = 81$, $3^5 = 243$, $3^6 = 729$.
3. So $n = \log_3 729 = 6$: round 6 reaches 729 families.
4. (b) We need $3^n \geq 2000$. The exact exponent with $3^n = 2000$ is $\log_3 2000$.
5. Trap it: $3^6 = 729 < 2000 < 2187 = 3^7$, so $6 < \log_3 2000 < 7$.
6. Rounds are whole numbers, so round 6 is not enough and round 7 is the first that reaches at least 2000 families. Check: $3^7 = 2187 \geq 2000$ ✓

Answer: (a) $\log_3 729 = 6$, round 6 (b) $6 < \log_3 2000 < 7$, so round 7

PRACTICE

1. Easy Rewrite in exponential form: (a) $\log_5 125 = 3$ (b) $\log 0.01 = -2$

   Show answer

   (a) $5^3 = 125$ (b) $10^{-2} = 0.01$

2. Easy Rewrite in logarithmic form: (a) $3^4 = 81$ (b) $2^{-3} = \dfrac{1}{8}$

   Show answer

   (a) $\log_3 81 = 4$ (b) $\log_2 \dfrac{1}{8} = -3$

3. Easy Evaluate by hand: (a) $\log_2 64$ (b) $\log_7 49$ (c) $\log 10{,}000$

   Show answer

   (a) 6 (b) 2 (c) 4

4. Easy Evaluate by hand: (a) $\log_4 1$ (b) $\log_9 9$ (c) $\log_6 6^{11}$ (d) $5^{\log_5 13}$

   Show answer

   (a) 0 (b) 1 (c) 11 (d) 13

5. **Medium** Evaluate by hand: (a) $\log_2 \dfrac{1}{16}$ (b) $\log_{1/3} 27$ (c) $\log 0.001$

Show answer

(a) -4, since $2^{-4} = \dfrac{1}{16}$ (b) -3, since $\left(\dfrac{1}{3}\right)^{-3} = 3^3 = 27$ (c) -3, since $10^{-3} = 0.001$

6. **Medium** Evaluate by hand by writing both numbers as powers of the same base: (a) $\log_8 2$ (b) $\log_4 8$ (c) $\log_{27} 9$

Show answer

(a) $8^y = 2$ means $2^{3y} = 2^1$, so $y = \dfrac{1}{3}$ (b) $2^{2y} = 2^3$, so $y = \dfrac{3}{2}$ (c) $3^{3y} = 3^2$, so $y = \dfrac{2}{3}$

7. **Medium** Evaluate by hand: (a) $\log_2(4\sqrt{2})$ (b) $\log \sqrt{1000}$

Show answer

(a) $4\sqrt{2} = 2^2 \cdot 2^{1/2} = 2^{5/2}$, so $\dfrac{5}{2}$ (b) $\sqrt{1000} = 10^{3/2}$, so $\dfrac{3}{2}$

8. **Medium** Evaluate: (a) $\ln 1$ (b) $\ln e$ (c) $\ln e^{-4}$ (d) $e^{\ln 7}$

Show answer

(a) 0 (b) 1 (c) -4 (d) 7

9. **Medium** Solve each equation for x by rewriting it in exponential form: (a) $\log_3 x = 4$ (b) $\log_x 64 = 3$ (c) $\log_{25} x = \dfrac{1}{2}$

Show answer

(a) $x = 3^4 = 81$ (b) $x^3 = 64$, so $x = 4$ (c) $x = 25^{1/2} = 5$

10. **Hard** (a) Between which two consecutive integers is $\log_2 50$? (b) A farm stand sold 3500 ears of sweet corn this summer. Between which two consecutive integers is $\log 3500$?

Show answer

(a) $2^5 = 32 < 50 < 64 = 2^6$, so between 5 and 6 (b) $10^3 = 1000 < 3500 < 10{,}000 = 10^4$, so between 3 and 4

11. Hard Henry's sourdough starter doubles in size every 4 hours, so after n doubling periods it is 2^n times its starting size. (a) Write the number of doubling periods needed to reach 128 times the starting size as a logarithm and evaluate it. (b) How many hours is that?

Show answer

(a) $n = \log_2 128 = 7$ (b) $7 \times 4 = 28$ hours

12. Hard Imagine folding a sheet of paper 0.1 mm thick in half again and again; each fold doubles the thickness, so after n folds it is 2^n times as thick. (a) After how many folds is it 1024 times as thick? (b) What is the fewest number of folds that would make it at least 1 meter (1000 mm) thick? Use powers of 2.

Show answer

(a) $\log_2 1024 = 10$ folds (b) It must be at least $\dfrac{1000}{0.1} = 10{,}000$ times as thick; $2^{13} = 8192 < 10{,}000 < 16{,}384 = 2^{14}$, so 14 folds

13. Hard Let $f(x) = 5^x$. (a) Write $f^{-1}(x)$. (b) Find $f^{-1}(125)$ and explain what it means in terms of f.

Show answer

(a) $f^{-1}(x) = \log_5 x$ (b) $f^{-1}(125) = \log_5 125 = 3$, because $f(3) = 5^3 = 125$

14. Challenge Explain why $\log_2 0$ and $\log_2(-4)$ are undefined, and why 1 is not allowed as the base of a logarithm.

Show answer

$\log_2 x$ is the exponent y with $2^y = x$. Every power of 2 is positive, so no exponent gives 0 or -4; those logarithms do not exist. With base 1, $1^y = 1$ for every y, so $\log_1 5$ would need $1^y = 5$, which never happens, while $\log_1 1$ could be any number at all. Neither situation gives a function, so the base 1 is not allowed.

15. Challenge (a) Solve $\log_2(\log_3 x) = 2$. (b) Evaluate $\log_3(\log_2 512)$ by hand.

Show answer

(a) $\log_3 x = 2^2 = 4$, so $x = 3^4 = 81$; check: $\log_3 81 = 4$ and $\log_2 4 = 2$ ✓ (b) $\log_2 512 = 9$, and $\log_3 9 = 2$

COMMON MISTAKES

- Treating $\log_2 8$ as a division, a product or a power ($8 \div 2 = 4$, $2 \cdot 8 = 16$ or $2^8 = 256$). Always ask the question "2 to what power gives 8?" The answer is 3.
- Mixing up which number is the exponent when translating. In $\log_b x = y$, the base stays the base and the logarithm y is the exponent: $b^y = x$. Say it aloud: "the log is the exponent."
- Getting the sign wrong for fractions, such as writing $\log_3 \frac{1}{9} = \frac{1}{2}$ or $-\frac{1}{9}$. A fraction less than 1 with base greater than 1 needs a negative exponent: $\frac{1}{9} = 3^{-2}$.
- Trying to take the logarithm of 0 or a negative number. The argument must be positive, because every power of a positive base is positive.

> **Make it a game · Exponent Hunt.** Write powers of 2, 3, 5 and 10 from memory in 60 seconds, as far as you can. Then the parent calls logs: $\log_2 32$, $\log_3 81$, $\log_{10} 0.01$ and $\log_4 (1/16)$. Remember that a log is an exponent and say it aloud: 'to what power do I raise the base to get this?' Convert three to exponential form.

8.2 Graphs of Logarithmic Functions

OBJECTIVES

- Explain why the graph of $y = \log_b x$ is the reflection of $y = b^x$ in the line $y = x$, and state its domain, range, asymptote and intercept.
- Graph $y = a\log_b(x - h) + k$ by transforming key points, and give its domain and vertical asymptote.
- Find the domain of the logarithm of an expression by requiring the argument to be positive.
- Describe how slowly logarithmic functions grow and use a logarithmic model.

WATCH

Graphing Logarithmic Functions (1 of 2: Overview)

Eddie Woo · 11:11

https://www.youtube.com/watch?v=rcuqWUCa-FQ

> **WHEN TO WATCH**
> Before Example 1: the shape of a logarithm graph and how it mirrors the exponential graph in the line y = x.

BACKUP VIDEO

Graphing Logarithmic Functions with Transformations

Mario's Math Tutoring · 9:39

https://www.youtube.com/watch?v=geGH75wCh64

After Example 1: graphing log functions with transformations.

KEY TERMS

logarithmic function

A function $f(x) = \log_b x$ with $b > 0$ and $b \neq 1$; it is the inverse of $g(x) = b^x$.

vertical asymptote

A vertical line $x = h$ that the graph approaches more and more closely, plunging (or climbing) without bound, but never touches.

key points

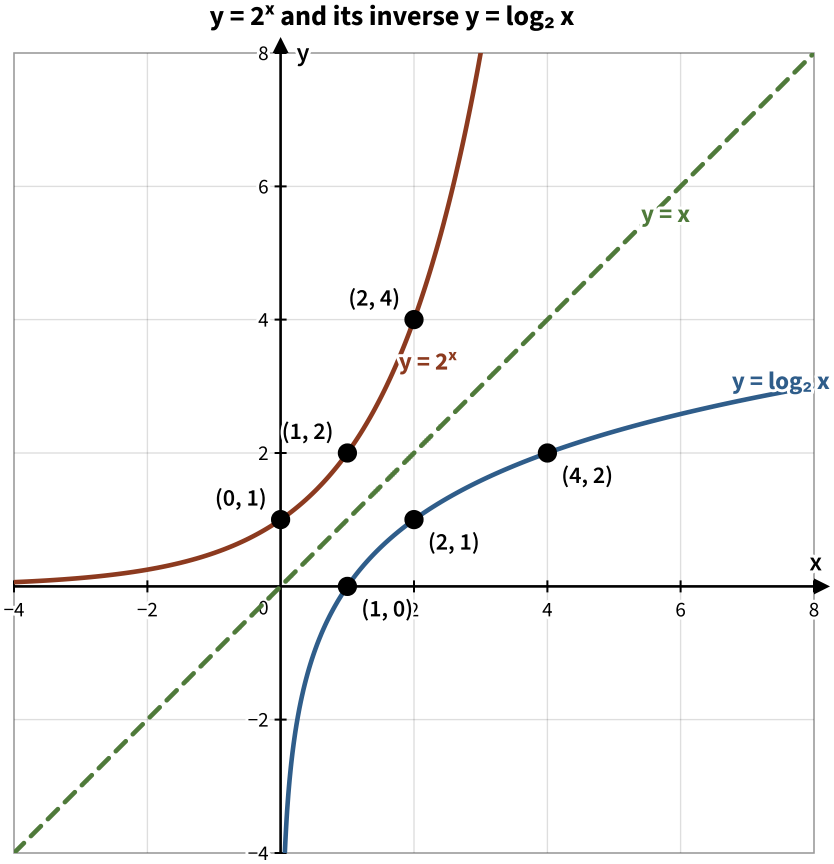
For $y = \log_b x$, the points $\left(\frac{1}{b}, -1\right)$, $(1, 0)$ and $(b, 1)$, where the argument is a simple power of the base.

argument

The expression inside a logarithm, such as $x - 3$ in $\log_2(x - 3)$; it must be positive.

CORE IDEAS

In Lesson 8.1 you saw that $\log_b x$ is the inverse of b^x. In Lesson 1.5 you learned that the graph of an inverse is the reflection of the original graph in the line $y = x$, because a point (a, c) on one graph becomes the point (c, a) on the other. So the graph of $y = \log_2 x$ comes for free: the points $(0, 1)$, $(1, 2)$ and $(2, 4)$ on $y = 2^x$ become $(1, 0)$, $(2, 1)$ and $(4, 2)$ on $y = \log_2 x$.



Every feature of $y = 2^x$ reflects into a feature of $y = \log_2 x$. The domain of 2^x (all real numbers) becomes the range of $\log_2 x$; the range of 2^x (all $y > 0$) becomes the domain $x > 0$. The horizontal asymptote $y = 0$ becomes the **vertical asymptote** $x = 0$, and the y-intercept $(0, 1)$ becomes the x-intercept $(1, 0)$. You can also see the asymptote directly: a small positive input needs a very negative exponent, for

example $\log_2 \dfrac{1}{1024} = -10$, so the graph plunges as x approaches 0 from the right. The same is true for every base $b > 1$. (When $0 < b < 1$ the graph falls instead of rises; for example $\left(\dfrac{1}{2}\right)^{y} = x$ means $2^{-y} = x$, so $\log_{1/2} x = -\log_2 x$, the mirror image in the x-axis.)

Transformations work exactly as in Lesson 1.2. The graph of $y = a\log_b(x - h) + k$ is the graph of $y = \log_b x$ shifted right h and up k, stretched vertically by a (and reflected in the x-axis if $a < 0$). The asymptote moves with the graph to $x = h$, and the domain becomes $x > h$. Why: the argument $x - h$ must be positive, so $x > h$, and as x approaches h the argument approaches 0, where the logarithm plunges. To sketch quickly, plot the two points where the argument equals 1 and equals b: there the logarithm is 0 and 1, giving the points $(h + 1, k)$ and $(h + b, a + k)$.

For the logarithm of any expression, the domain is wherever the argument is positive. For $y = \log(5 - x)$, solve $5 - x > 0$ to get $x < 5$: this graph lives to the left of its asymptote $x = 5$ (it is $y = \log x$ reflected in the y-axis and shifted). For $y = \log_3(x^2 - 9)$, solve $x^2 - 9 > 0$ as in Lesson 3.5: $x < -3$ or $x > 3$.

For a base $b > 1$, a logarithmic graph rises forever, since any height y is reached at $x = b^y$, so there is no horizontal asymptote. But they rise very slowly. The table shows how far out you must go for $y = \log_2 x$ to climb. That slow growth makes logarithms good models for things that improve quickly at first and then more and more slowly, such as the height of a young tree or a student's skill with practice.

How slowly y = log₂ x grows

x	y = $\log_2 x$
2	1
16	4
1024	10
1,048,576	20
1,073,741,824	30

KEY FORMULAS

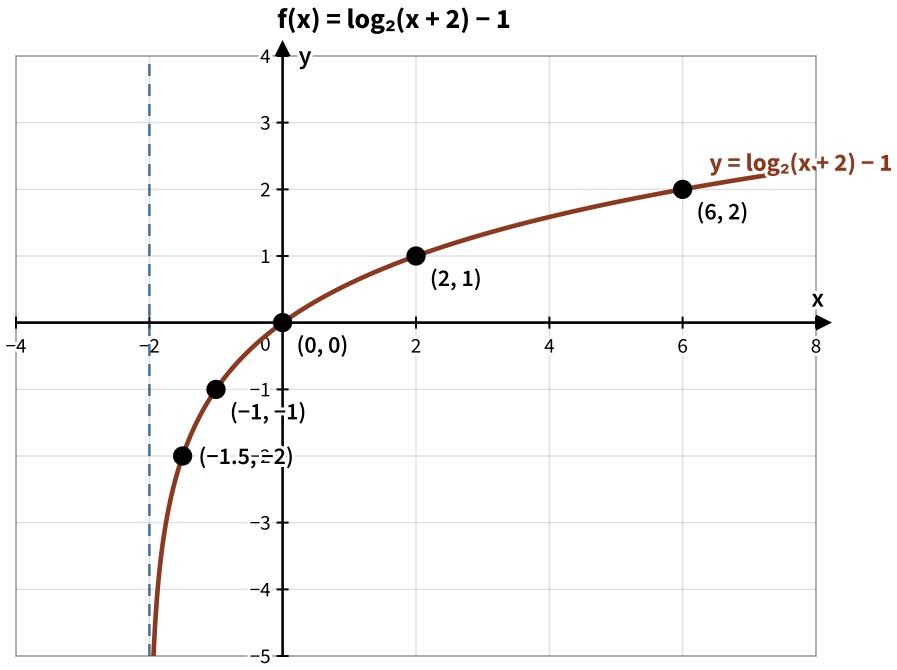
$$y = \log_b x: \quad \text{domain } x > 0, \quad \text{range all real numbers}, \quad \text{asymptote } x = 0, \quad x\text{-intercept } 1$$

$$y = a\log_b(x - h) + k: \quad \text{domain } x > h, \quad \text{asymptote } x = h$$

$$\text{key points: } (h + 1,\ k) \quad \text{and} \quad (h + b,\ a + k)$$

Practice habit. Sketch before you compute: for any logarithmic graph, first draw the dashed asymptote $x = h$, then plot the two points where the argument equals 1 and equals the base, then draw the curve rising slowly away from the asymptote.

Example 1. Graph $f(x) = \log_2(x + 2) - 1$. State its domain, range and vertical asymptote, and find its x-intercept.



Show solution

1. Compare with $y = a\log_b(x - h) + k$: here $b = 2$, $a = 1$, $h = -2$, $k = -1$. The graph is $y = \log_2 x$ shifted left 2 and down 1.
2. Key points of $y = \log_2 x$: $(0.5, -1)$, $(1, 0)$, $(2, 1)$, $(4, 2)$, $(8, 3)$.
3. Shift each one left 2 and down 1: $(-1.5, -2)$, $(-1, -1)$, $(0, 0)$, $(2, 1)$, $(6, 2)$.
4. The asymptote $x = 0$ moves left 2 to $x = -2$.
5. Domain: the argument must be positive, $x + 2 > 0$, so $x > -2$. Range: all real numbers (shifting down does not change that).
6. x-intercept: set $f(x) = 0$: $\log_2(x + 2) = 1$, so $x + 2 = 2^1$ and $x = 0$. Check: $\log_2 2 - 1 = 0$ ✓

Answer: Domain $(-2, \infty)$; range all real numbers; asymptote $x = -2$; x-intercept 0 (the graph passes through the origin)

Example 2. A tree farmer models the height of a young spruce by $h(t) = 2 + 3\log_2(t+1)$ feet, where t is the number of years since planting. (a) How tall was the tree when planted? (b) How tall is it after 7 years? (c) When will it reach 14 feet?

Show solution

1. (a) $h(0) = 2 + 3\log_2 1 = 2 + 3(0) = 2$ feet.
2. (b) $h(7) = 2 + 3\log_2 8 = 2 + 3(3) = 11$ feet.
3. (c) Set $2 + 3\log_2(t+1) = 14$. Subtract 2: $3\log_2(t+1) = 12$. Divide by 3: $\log_2(t+1) = 4$.
4. Rewrite in exponential form: $t + 1 = 2^4 = 16$, so $t = 15$ years.
5. Check: $h(15) = 2 + 3\log_2 16 = 2 + 12 = 14$ ✓
6. Notice the slow growth: the tree gains 9 feet in its first 7 years but only 3 more feet in the next 8 years.

Answer: (a) 2 feet (b) 11 feet (c) after 15 years

PRACTICE

1. Easy For $y = \log_4 x$, state the domain, the range, the vertical asymptote and the x-intercept.

   Show answer

   Domain $(0, \infty)$; range all real numbers; asymptote $x = 0$; x-intercept 1

2. Easy The points $\left(-1, \frac{1}{3}\right)$, $(0, 1)$, $(1, 3)$ and $(2, 9)$ lie on $y = 3^x$. Give the corresponding four points on $y = \log_3 x$.

   Show answer

   $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, $(3, 1)$, $(9, 2)$ (swap the coordinates)

3. Easy For $f(x) = \log_2 x$, find $f\left(\frac{1}{4}\right)$, $f(1)$ and $f(16)$.

   Show answer

   -2, 0, 4

4. **Easy** (a) Describe how the graph of $y = \log(x - 5) + 3$ is obtained from the graph of $y = \log x$. (b) Give its vertical asymptote.

Show answer

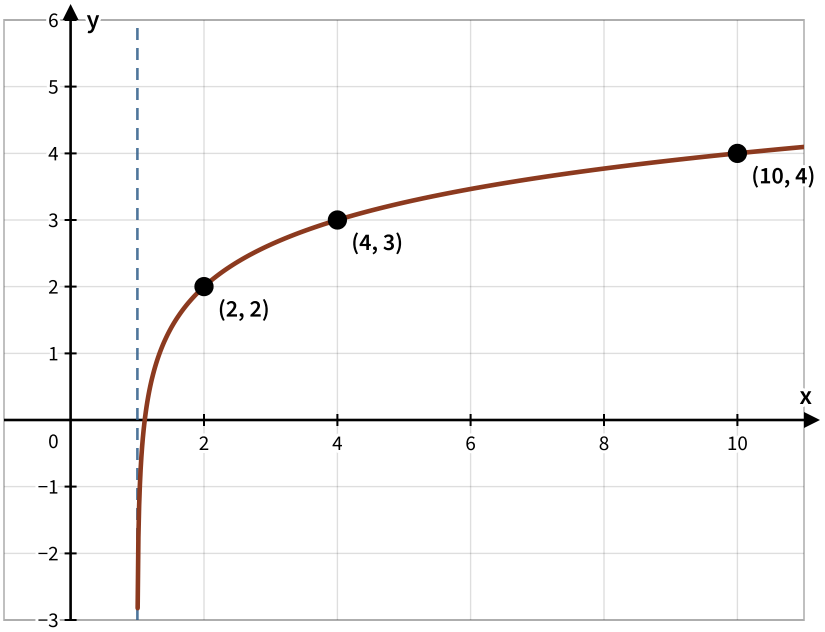
(a) Shift right 5 and up 3 (b) $x = 5$

5. **Medium** Find the domain of each function in interval notation: (a) $y = \log_3(x + 4)$ (b) $y = \log(6 - 2x)$

Show answer

(a) $x + 4 > 0$: $(-4, \infty)$ (b) $6 - 2x > 0$: $(-\infty, 3)$

6. **Medium** The graph shows a function of the form $y = \log_3(x - h) + k$, with its asymptote dashed. Find h and k and write the equation.



Show answer

The asymptote is $x = 1$, so $h = 1$; at $(2, 2)$ the argument is 1 and the log is 0, so $k = 2$: $y = \log_3(x - 1) + 2$. Check $(4, 3)$: $\log_3 3 + 2 = 3$ ✓

7. **Medium** Find the x-intercept of each graph: (a) $y = \log_2(x - 3)$ (b) $y = \log_3 x - 2$

Show answer

(a) $x - 3 = 1$, so $x = 4$ (b) $\log_3 x = 2$, so $x = 9$

8. **Medium** For $f(x) = 2\log_5(x+1) - 4$: (a) give the vertical asymptote and the domain; (b) evaluate $f(4)$ and $f(24)$.

Show answer

(a) $x = -1$; $(-1, \infty)$ (b) $f(4) = 2\log_5 5 - 4 = -2$; $f(24) = 2\log_5 25 - 4 = 0$

9. **Medium** Samuel draws $y = \log_2 x$ on a giant grid where each unit is 1 inch. (a) How far along the x-axis must he go for the graph to be 10 inches high? (b) For the graph to be 20 inches high, how far along the x-axis must he go, in miles, to the nearest tenth? (1 mile = 5280 feet.)

Show answer

(a) $x = 2^{10} = 1024$ inches (b) $2^{20} = 1{,}048{,}576$ inches $= 87{,}381.3$ feet ≈ 16.5 miles

10. **Hard** Find the domain of each function: (a) $y = \log(x^2 - 16)$ (b) $y = \ln\left(\dfrac{x-2}{x+3}\right)$

Show answer

(a) $x^2 - 16 > 0$: $(-\infty, -4) \cup (4, \infty)$ (b) a sign chart for $\dfrac{x-2}{x+3} > 0$ with zeros at -3 and 2: $(-\infty, -3) \cup (2, \infty)$

11. **Hard** The Miller family models the yield of a young apple tree by $B(t) = 2 + 4\log_3 t$ bushels in year t (for $t \geq 1$). (a) Find $B(1)$ and $B(9)$. (b) In what year does the model predict 14 bushels?

Show answer

(a) $B(1) = 2$, $B(9) = 2 + 4(2) = 10$ bushels (b) $4\log_3 t = 12$, $\log_3 t = 3$, $t = 27$: year 27

12. **Hard** The graph of $y = \log_2 x$ is shifted left 3 units and down 2 units. (a) Write its equation. (b) Give its domain. (c) Find its x-intercept.

Show answer

(a) $y = \log_2(x+3) - 2$ (b) $(-3, \infty)$ (c) $\log_2(x+3) = 2$, $x + 3 = 4$, $x = 1$

13. **Challenge** Using the graph of $y = 2^x$, explain why the graph of $y = \log_2 x$ has the vertical asymptote $x = 0$ and no horizontal asymptote.

Show answer

The graph of $y = \log_2 x$ is the reflection of $y = 2^x$ in $y = x$. The graph of $y = 2^x$ gets closer and closer to the horizontal line $y = 0$ as $x \to -\infty$, so its reflection gets closer and closer to the vertical line $x = 0$ as $y \to -\infty$. The graph of $y = 2^x$ is defined for every x and has no vertical asymptote, so its reflection reaches every height y (at $x = 2^y$) and keeps rising: there is no horizontal asymptote.

14. **Challenge** Let $f(x) = \log_3(x - 1) + 2$. (a) Find $f^{-1}(x)$. (b) State the range of f^{-1} and explain how it relates to f.

Show answer

(a) Swap: $x = \log_3(y - 1) + 2$, so $\log_3(y - 1) = x - 2$, $y - 1 = 3^{x-2}$: $f^{-1}(x) = 3^{x-2} + 1$ (b) $y > 1$, which is the domain of f

COMMON MISTAKES

- Moving the asymptote the wrong way. In $y = \log_2(x + 2)$, the argument is 0 when $x = -2$, so the asymptote is $x = -2$, not $x = 2$. Find the asymptote by setting the argument equal to 0.
- Giving the domain as all real numbers. Only positive arguments are allowed: solve "argument > 0" every time.
- Drawing the graph crossing or touching its asymptote, or leveling off to a horizontal asymptote. A logarithmic graph never reaches $x = h$ and keeps rising slowly forever.
- Thinking a shift up or down changes the domain. Only changes inside the argument (horizontal shifts and reflections) change the domain; k changes nothing about the domain.

> **Make it a game · Flip the Table.** Write a table for $y = 2^x$ on index cards, with x from -2 to 3. Flip each card to swap the columns, and you have a table for $y = \log_2 x$. Plot both on one grid and fold along y = x to compare. State the domain, range, asymptote and intercept of the log graph. Then move $y = 2\log_2(x - 3) + 1$ by its key points.

8.3 Properties of Logarithms

OBJECTIVES

- Derive the product, quotient and power properties of logarithms from the rules of exponents.
- Expand a logarithm of a product, quotient or power, and condense a sum or difference of logarithms into a single logarithm.
- Recognize the common errors $\log(a+b) \neq \log a + \log b$ and $\dfrac{\log a}{\log b} \neq \log \dfrac{a}{b}$.
- Derive the change-of-base formula and use it to evaluate any logarithm with a calculator.

WATCH

Logarithm Laws (1 of 3: Adding logarithms)

Eddie Woo · 9:28

https://www.youtube.com/watch?v=2Bu5NG1KIL4

> **WHEN TO WATCH**
> Before Example 1: why log(xy) = log x + log y.

Note: Eddie Woo teaches in Australia: he says 'log laws' for the properties of logarithms.

BACKUP VIDEO

Logarithms / Change of Base (2 of 2: Deriving the Law)

Eddie Woo · 6:40

https://www.youtube.com/watch?v=xJYpNwWPeA4

Before the practice set: deriving the change-of-base formula.

KEY TERMS

expand

To rewrite the logarithm of a product, quotient or power as a sum, difference or multiple of simpler logarithms.

condense

To rewrite a sum, difference or multiple of logarithms with the same base as a single logarithm.

change-of-base formula

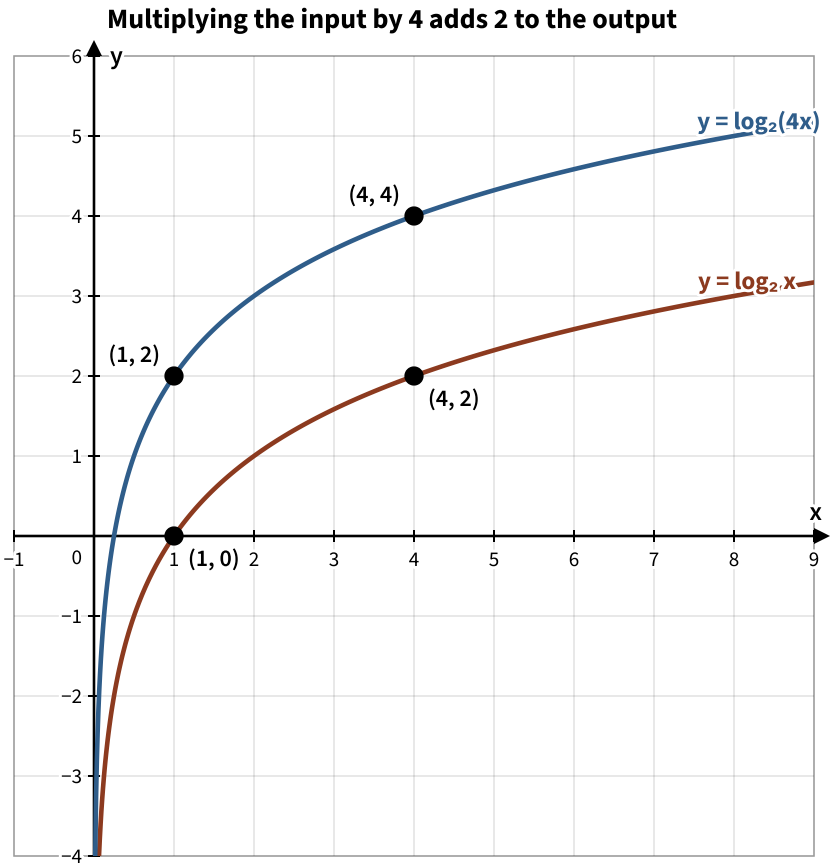
The rule $\log_b x = \dfrac{\log_c x}{\log_c b}$, which turns a logarithm in any base b into logarithms in a base c your calculator has (usually 10 or e).

CORE IDEAS

A logarithm is an exponent, so the rules of logarithms are just the rules of exponents in a new language. Each proof below uses the same trick: give the logarithms names. Let $m = \log_b M$ and $n = \log_b N$, where $M > 0$ and $N > 0$. In exponential form that says $M = b^m$ and $N = b^n$.

Product property. Multiply: $MN = b^m \cdot b^n = b^{m+n}$ (add exponents when multiplying powers). Translating back, the exponent on b that gives MN is $m + n$, so $\log_b(MN) = \log_b M + \log_b N$. **Quotient property.** Divide: $\dfrac{M}{N} = \dfrac{b^m}{b^n} = b^{m-n}$, so $\log_b \dfrac{M}{N} = \log_b M - \log_b N$. **Power property.** Raise to a power p: $M^p = (b^m)^p = b^{mp}$, so $\log_b M^p = p \cdot m = p \log_b M$. In short: the log of a product is the sum of the logs, the log of a quotient is the difference, and an exponent inside comes out front as a multiplier.

You can see the product property in a graph. Since $\log_2(4x) = \log_2 4 + \log_2 x = 2 + \log_2 x$, the graph of $y = \log_2(4x)$ is exactly the graph of $y = \log_2 x$ lifted up 2 units: multiplying the input by 4 adds 2 to the output.

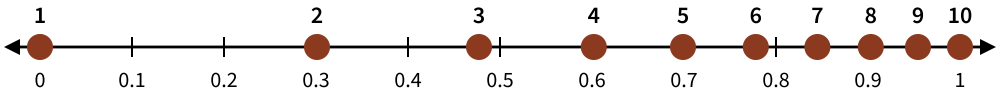


Use the properties in both directions. To **expand** $\log_3 \dfrac{9x^4}{\sqrt{y}}$, split the quotient, then the product, then bring down the powers (remember $\sqrt{y} = y^{1/2}$). To **condense** $2\ln x + \ln(x+1) - 3\ln y$, first move the multipliers back up as exponents, then combine sums into products and differences into quotients. Watch for two tempting errors. There is **no** rule for the log of a sum: $\log(a+b)$ is not $\log a + \log b$ (try $a = b = 10$: $\log 20 \approx 1.301$, but $\log 10 + \log 10 = 2$). And a quotient of logs is not the log of a quotient: $\dfrac{\log 100}{\log 10} = \dfrac{2}{1} = 2$, but $\log \dfrac{100}{10} = \log 10 = 1$.

Change of base. Calculators have keys only for log (base 10) and ln (base e). To find $\log_b x$, let $y = \log_b x$, so $b^y = x$. Take the logarithm in any base c of both sides: $\log_c b^y = \log_c x$. The power property gives $y \log_c b = \log_c x$, so $y = \dfrac{\log_c x}{\log_c b}$. Therefore $\log_b x = \dfrac{\log x}{\log b} = \dfrac{\ln x}{\ln b}$; both give the same answer. For example, $\log_3 50 = \dfrac{\ln 50}{\ln 3} \approx \dfrac{3.912}{1.099} \approx 3.56$, the value promised in Lesson 8.1.

The product property is why logarithms were such a gift before calculators: to multiply two numbers, look up their logarithms, **add** them, and look up which number has that sum as its logarithm. A slide rule does the same with lengths. On the scale below, each number is placed at a distance proportional to its common logarithm. Sliding one copy of the scale along another adds two lengths, and adding the lengths for 2 and 3 lands exactly on 6, because $\log 2 + \log 3 = \log 6$.

A slide-rule scale: each number placed at distance log x



KEY FORMULAS

$$\log_b(MN) = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b M^p = p \log_b M$$

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

From history. The slide rule was invented in England in the 1620s, only a few years after Napier's logarithms. It multiplied numbers by adding lengths on logarithmic scales, exactly the product property in physical form. Engineers, surveyors and scientists carried slide rules for more than three hundred years, until electronic calculators replaced them in the 1970s.

Practice habit. Know three logarithm facts cold: $\log 2 \approx 0.301$, $\log 3 \approx 0.477$ and $\ln 2 \approx 0.693$. With the properties they give many others by hand, such as $\log 6 = \log 2 + \log 3 \approx 0.778$ and $\log 5 = \log 10 - \log 2 \approx 0.699$.

Example 1. (a) Expand $\log_3 \dfrac{9x^4}{\sqrt{y}}$ completely. (b) Condense $2\ln x + \ln(x+1) - 3\ln y$ into a single logarithm. (Assume all variables are positive.)

Show solution

1. (a) Quotient property: $\log_3 \dfrac{9x^4}{\sqrt{y}} = \log_3(9x^4) - \log_3 \sqrt{y}$.
2. Product property on the first term: $= \log_3 9 + \log_3 x^4 - \log_3 y^{1/2}$.
3. Evaluate $\log_3 9 = 2$ and bring down the powers: $= 2 + 4\log_3 x - \dfrac{1}{2}\log_3 y$.
4. (b) Power property in reverse: $2\ln x = \ln x^2$ and $3\ln y = \ln y^3$, so the expression is $\ln x^2 + \ln(x+1) - \ln y^3$.
5. Product property: $\ln x^2 + \ln(x+1) = \ln\big(x^2(x+1)\big)$.
6. Quotient property: $\ln\big(x^2(x+1)\big) - \ln y^3 = \ln \dfrac{x^2(x+1)}{y^3}$.

Answer: (a) $2 + 4\log_3 x - \dfrac{1}{2}\log_3 y$ (b) $\ln \dfrac{x^2(x+1)}{y^3}$

Example 2. Grace's savings account grows by 6% each year, so after t years her balance has been multiplied by 1.06^t. The number of years until her money triples is $t = \log_{1.06} 3$, because $1.06^t = 3$. Use the change-of-base formula to find t to the nearest tenth of a year, and check the answer.

Show solution

1. Change of base: $\log_{1.06} 3 = \dfrac{\ln 3}{\ln 1.06}$.
2. Calculator: $\ln 3 \approx 1.098612$ and $\ln 1.06 \approx 0.058269$.
3. Divide: $t \approx \dfrac{1.098612}{0.058269} \approx 18.854$, so $t \approx 18.9$ years.
4. Using common logs instead gives the same result: $\dfrac{\log 3}{\log 1.06} \approx \dfrac{0.477121}{0.025306} \approx 18.854$.
5. Check by bracketing: $1.06^{18} \approx 2.854$ (not yet triple) and $1.06^{19} \approx 3.026$ (just past triple) ✓

Answer: $t = \dfrac{\ln 3}{\ln 1.06} \approx 18.9$ years

PRACTICE

1. **Easy** Expand: $\log_2(8x)$

   Show answer

   $\log_2 8 + \log_2 x = 3 + \log_2 x$

2. **Easy** Expand: $\log \dfrac{x}{100}$

   Show answer

   $\log x - \log 100 = \log x - 2$

3. **Easy** Expand: $\ln x^5$

   Show answer

   $5 \ln x$

4. **Easy** (a) Condense $\log_3 x + \log_3 7$. (b) Condense and evaluate $\log 50 + \log 2$.

   Show answer

   (a) $\log_3(7x)$ (b) $\log 100 = 2$

5. **Medium** Evaluate by hand by condensing first: (a) $\log_6 4 + \log_6 9$ (b) $\log_2 40 - \log_2 5$ (c) $\log_5 75 - \log_5 3$

   Show answer

   (a) $\log_6 36 = 2$ (b) $\log_2 8 = 3$ (c) $\log_5 25 = 2$

6. **Medium** Use $\log 2 \approx 0.3010$ and $\log 3 \approx 0.4771$ (no calculator) to approximate (a) $\log 6$ (b) $\log 1.5$ (c) $\log 18$.

   Show answer

   (a) $\log 2 + \log 3 \approx 0.7781$ (b) $\log 3 - \log 2 \approx 0.1761$ (c) $\log 2 + 2\log 3 \approx 1.2552$

7. **Medium** Expand completely: $\log_2 \dfrac{x^3 y}{z^2}$ (all variables positive).

   Show answer

   $3\log_2 x + \log_2 y - 2\log_2 z$

8. **Medium** Expand completely and simplify: $\ln \dfrac{\sqrt{x}}{e^2 y}$ (all variables positive).

   Show answer

   $\dfrac{1}{2}\ln x - 2 - \ln y$

9. Medium Condense into a single logarithm: $3\log x - 2\log y + \log z$

Show answer

$$\log \frac{x^3 z}{y^2}$$

10. Hard Condense and simplify for $x > 3$: $\ln(x^2 - 9) - \ln(x - 3)$

Show answer

$$\ln \frac{(x-3)(x+3)}{x-3} = \ln(x+3)$$

11. Hard Use the change-of-base formula to evaluate to the nearest thousandth: (a) $\log_3 20$ (b) $\log_7 0.5$

Show answer

(a) $\dfrac{\ln 20}{\ln 3} \approx \dfrac{2.99573}{1.09861} \approx 2.727$ (b) $\dfrac{\ln 0.5}{\ln 7} \approx \dfrac{-0.69315}{1.94591} \approx -0.356$

12. Hard A family's flock of laying hens grows by about 8% a year. The number of years for the flock to double is $t = \log_{1.08} 2$. Find t to the nearest tenth of a year.

Show answer

$$t = \frac{\ln 2}{\ln 1.08} \approx \frac{0.693147}{0.076961} \approx 9.0 \text{ years}$$

13. Hard On a slide rule whose scale is 25 cm long from the mark 1 to the mark 10, each number x is marked $25\log x$ cm from the 1 mark. (a) How far from the 1 mark is the 2 mark, to the nearest tenth of a centimeter? (b) How far is the 3 mark? (c) The two lengths together reach which mark, and why?

Show answer

(a) $25\log 2 \approx 7.5$ cm (b) $25\log 3 \approx 11.9$ cm (c) $25(\log 2 + \log 3) = 25\log 6$, so they reach the 6 mark (about 19.5 cm)

14. Challenge Prove the quotient property $\log_b \dfrac{M}{N} = \log_b M - \log_b N$ for $M, N > 0$, using only the definition of a logarithm and the rules of exponents.

Show answer

Let $m = \log_b M$ and $n = \log_b N$, so $M = b^m$ and $N = b^n$. Then $\dfrac{M}{N} = \dfrac{b^m}{b^n} = b^{m-n}$ by the quotient rule for exponents. By the definition of a logarithm, $\log_b \dfrac{M}{N} = m - n = \log_b M - \log_b N$.

15. **Challenge** Ruth wrote $\log(x+10) = \log x + 1$ and also $\dfrac{\log 8}{\log 2} = \log 4$. Explain the error in each, and give the correct value of $\dfrac{\log 8}{\log 2}$.

Show answer

There is no property for the log of a sum: $\log x + 1 = \log x + \log 10 = \log(10x)$, not $\log(x+10)$ (for $x = 10$: $\log 20 \approx 1.301$ but $\log 10 + 1 = 2$). The quotient property is about the log of a quotient, not a quotient of logs. By change of base, $\dfrac{\log 8}{\log 2} = \log_2 8 = 3$, while $\log 4 \approx 0.602$.

COMMON MISTAKES

- Splitting the log of a sum: $\log(x+5) \neq \log x + \log 5$. The properties work only for products, quotients and powers inside the logarithm. If the argument is a sum, leave it alone (or factor it first).
- Treating a quotient of logarithms as a log of a quotient: $\dfrac{\log a}{\log b}$ is $\log_b a$ (change of base), not $\log \dfrac{a}{b}$.
- Applying the power property to only part of the argument. $\log\left((3x)^2\right) = 2\log(3x)$, but $\log 3x^2 = \log 3 + 2\log x$: the exponent belongs only to what it is attached to.
- Forgetting the order in change of base: the argument goes on top. $\log_3 20 = \dfrac{\ln 20}{\ln 3}$, not $\dfrac{\ln 3}{\ln 20}$. Check by estimating: $3^2 = 9$ and $3^3 = 27$, so $\log_3 20$ must be between 2 and 3.

> **Make it a game · Add the Exponents.** Check the product rule with real numbers. Write 8 as 2^3 and 4 as 2^2, then show $\log_2 8 + \log_2 4 = 5 = \log_2 32$. Do the same for quotient and power rules. Next, the parent says, 'log(a + b) equals log a plus log b.' Disprove it with numbers. Then expand and condense three expressions of your choice.

8.4 Solving Exponential Equations

OBJECTIVES

- Solve exponential equations by rewriting both sides with a common base.
- Solve exponential equations by taking the logarithm of both sides, giving exact answers in terms of logarithms and decimal approximations.
- Solve equations with e using ln, and equations in quadratic form such as $e^{2x} - 5e^{x} + 6 = 0$.
- Solve for time in growth and decay models: doubling time, half-life, cooling and savings goals.

WATCH

Logarithmic Arithmetic (3 of 3: Using Logs to solve Exponential Equations)

Eddie Woo · 5:18

https://www.youtube.com/watch?v=3DOrApDak-Q

> **WHEN TO WATCH**
> Before Example 1: solving exponential equations by taking logarithms.

BACKUP VIDEO

Solve Exponential Equations by Getting the Bases Equal

Mario's Math Tutoring · 7:18

https://www.youtube.com/watch?v=0KlgmNveUdc

Before the Core ideas, as a warm-up: solving by rewriting both sides with the same base.

KEY TERMS

exponential equation

An equation with the variable in an exponent, such as $3^{x} = 20$ or $e^{0.04t} = 2$.

common base

A single base, such as 2, in which both sides of an equation can be written, as in $8^{x} = (2^{3})^{x} = 2^{3x}$.

exact answer in terms of logarithms

An answer such as $x = \log_3 20$ or $x = \dfrac{\ln 20}{\ln 3}$, kept as an expression with logarithms instead of a rounded decimal.

quadratic form

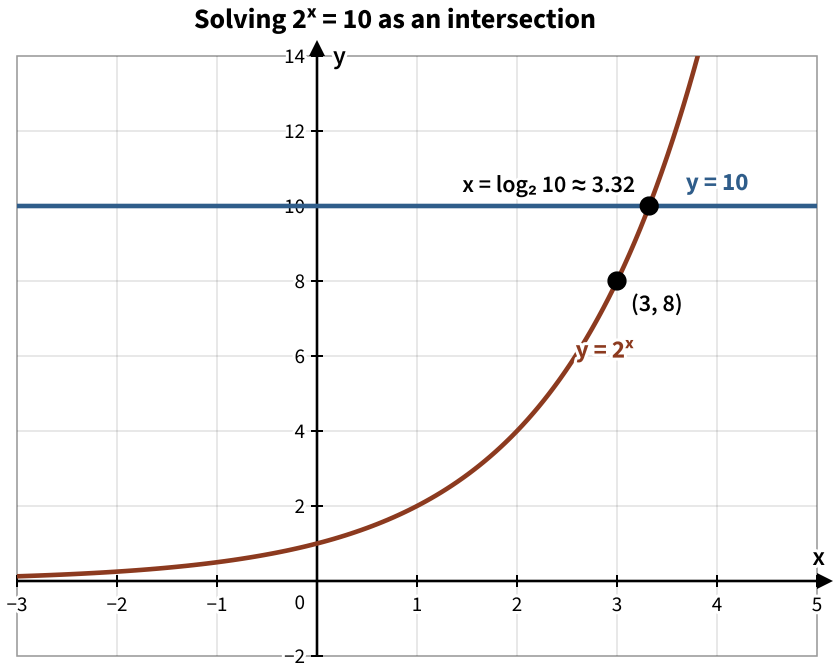
An equation that becomes a quadratic after a substitution, such as $e^{2x} - 5e^x + 6 = 0$, which is $u^2 - 5u + 6 = 0$ with $u = e^x$.

CORE IDEAS

Method 1: a common base. Because $y = b^x$ is one-to-one (it passes the horizontal line test), two powers of the same base are equal only when their exponents are equal: if $b^M = b^N$, then $M = N$. So when both sides can be written as powers of one base, set the exponents equal. For $2^{x+1} = 8^x$, write $8^x = (2^3)^x = 2^{3x}$. Then $2^{x+1} = 2^{3x}$, so $x + 1 = 3x$ and $x = \dfrac{1}{2}$. Check: $2^{1.5} \approx 2.828$ and $8^{0.5} = \sqrt{8} \approx 2.828$ ✓

Method 2: take a logarithm of both sides. Most equations have no common base: in $3^x = 20$, 20 is not a whole power of 3. Both sides are positive, so we may take the logarithm of both sides (equal positive numbers have equal logarithms). The power property then brings the variable down out of the exponent: $\log 3^x = \log 20$ gives $x \log 3 = \log 20$, so $x = \dfrac{\log 20}{\log 3} \approx 2.73$. That is exactly $\log_3 20$, as the definition of a logarithm says it must be. Keep the exact form until the last step and round only at the end. Either log or ln works; use ln when the base is e, because $\ln e^k = k$.

The graph shows the same idea: the solution of $2^x = 10$ is the x-coordinate where the curve $y = 2^x$ meets the line $y = 10$. Algebra gives the exact value $x = \log_2 10 = \dfrac{\ln 10}{\ln 2} \approx 3.32$, and the graph confirms it is a little past 3, since $2^3 = 8$.



Isolate the power first. In $4 \cdot 3^x - 7 = 33$, you cannot take a log right away, because there is no rule for the log of a difference. Add 7 and divide by 4 to get $3^x = 10$, and only then take logarithms. The same order applies to every growth model: isolate the exponential factor, then take a logarithm.

Solving for time. The questions of Unit 7, "how long until…?", are now solvable. For continuous growth at 5%, the doubling time comes from $e^{0.05t} = 2$: take ln of both sides, $0.05t = \ln 2$, so $t = \dfrac{\ln 2}{0.05} \approx 13.9$ years. Half-life and cooling problems work the same way: isolate the exponential factor, then take ln. In doubling and half-life problems the starting amount divides out, which is why doubling time and half-life do not depend on how much you start with.

Quadratic form. Since $e^{2x} = (e^x)^2$, the equation $e^{2x} - 5e^x + 6 = 0$ is a quadratic in disguise. Let $u = e^x$: $u^2 - 5u + 6 = 0$, so $(u - 2)(u - 3) = 0$ and $u = 2$ or $u = 3$. Then $e^x = 2$ gives $x = \ln 2$, and $e^x = 3$ gives $x = \ln 3$. If a value of u is zero or negative, throw it away: e^x is always positive, so $e^x = -4$ has no solution.

KEY FORMULAS

$$b^M = b^N \implies M = N$$

$$b^x = c \implies x = \log_b c = \frac{\log c}{\log b} = \frac{\ln c}{\ln b} \qquad (c > 0)$$

$$e^{kx} = c \implies x = \frac{\ln c}{k}$$

Practice habit. Keep exact answers until the last step: write $x = \dfrac{\ln 20}{\ln 3}$ first, then compute the decimal once. Rounding $\ln 20$ and $\ln 3$ early can change the final digits.

WORKED EXAMPLES

Example 1. Solve each equation. Give exact answers, and round any non-exact answer to the nearest hundredth. (a) $9^x = 27^{x-1}$ (b) $4 \cdot 3^x - 7 = 33$

Show solution

1. (a) Both 9 and 27 are powers of 3: $9^x = (3^2)^x = 3^{2x}$ and $27^{x-1} = (3^3)^{x-1} = 3^{3x-3}$.
2. Set the exponents equal: $2x = 3x - 3$, so $x = 3$.
3. Check in the original: $9^3 = 729$ and $27^2 = 729$ ✓
4. (b) Isolate the power: $4 \cdot 3^x = 40$, so $3^x = 10$.
5. Take ln of both sides: $x \ln 3 = \ln 10$, so $x = \dfrac{\ln 10}{\ln 3} = \log_3 10$.
6. Calculator: $x \approx \dfrac{2.302585}{1.098612} \approx 2.10$.
7. Check: $3^{2.0959} \approx 10.00$, and $4(10) - 7 = 33$ ✓

Answer: (a) $x = 3$ (b) $x = \dfrac{\ln 10}{\ln 3} = \log_3 10 \approx 2.10$

Example 2. Samuel deposits \$2,000 in a savings account that pays 4.5% interest compounded monthly. How long will it take for the balance to reach \$3,000? Give the time to the nearest tenth of a year.

Show solution

1. Compound interest (Lesson 7.2): $2000\left(1+\dfrac{0.045}{12}\right)^{12t}=3000$, that is, $2000(1.00375)^{12t}=3000$.
2. Isolate the power: divide by 2000 to get $(1.00375)^{12t}=1.5$.
3. Take ln of both sides: $12t\ln(1.00375)=\ln 1.5$.
4. Solve: $t=\dfrac{\ln 1.5}{12\ln 1.00375}\approx\dfrac{0.405465}{12(0.0037430)}\approx\dfrac{0.405465}{0.044916}\approx 9.03$.
5. So about 9.0 years. (Interest is added monthly, so the balance first passes \$3,000 at the 109th month, just after 9 years.)
6. Check: $2000(1.00375)^{108.33}\approx 2000(1.5)=3000$ ✓

Answer: $t=\dfrac{\ln 1.5}{12\ln 1.00375}\approx 9.0$ years

PRACTICE

1. Easy Solve: $2^x=64$

   Show answer

   $2^x=2^6$, so $x=6$

2. Easy Solve: $5^{2x-1}=\dfrac{1}{25}$

   Show answer

   $5^{2x-1}=5^{-2}$, so $2x-1=-2$ and $x=-\dfrac{1}{2}$

3. Easy Solve: (a) $4^x=32$ (b) $10^x=0.001$

   Show answer

   (a) $2^{2x}=2^5$, so $x=\dfrac{5}{2}$ (b) $10^x=10^{-3}$, so $x=-3$

4. Easy Solve $e^x=12$. Give the exact answer and the answer to the nearest hundredth.

   Show answer

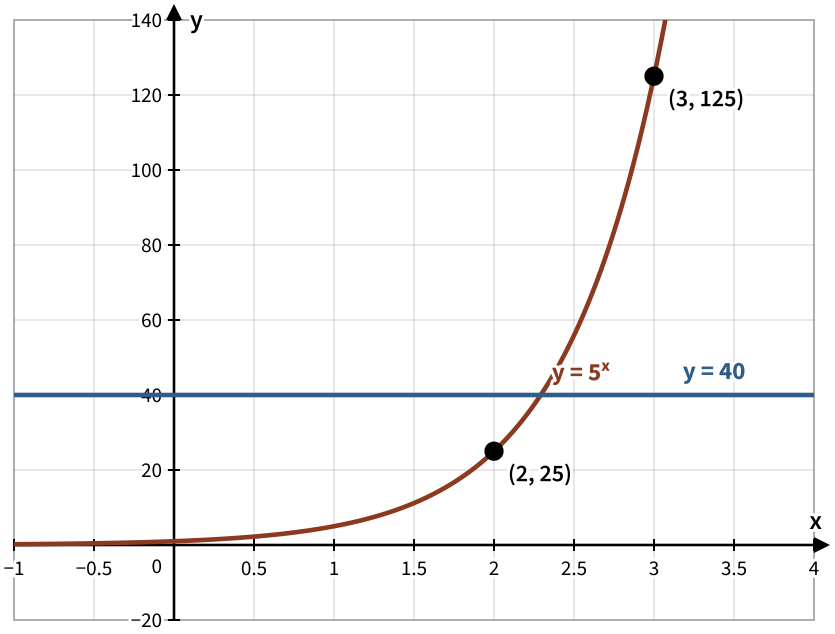
   $x=\ln 12\approx 2.48$

5. **Medium** Solve: $25^{x} = 125^{x+2}$

Show answer

$5^{2x} = 5^{3x+6}$, so $2x = 3x + 6$ and $x = -6$; check: both sides equal 5^{-12} ✓

6. **Medium** The graph shows $y = 5^{x}$ and $y = 40$. (a) Use the labeled points to tell between which two consecutive integers the solution of $5^{x} = 40$ lies. (b) Solve $5^{x} = 40$ exactly, then round to the nearest hundredth.



Show answer

(a) $5^{2} = 25 < 40 < 125 = 5^{3}$, so between 2 and 3 (b) $x = \dfrac{\ln 40}{\ln 5} \approx 2.29$

7. **Medium** Solve $6 + 2 \cdot 3^{x} = 40$ to the nearest hundredth.

Show answer

$3^{x} = 17$, so $x = \dfrac{\ln 17}{\ln 3} \approx 2.58$

8. **Medium** Solve $e^{0.3x} = 5$. Give the exact answer and the answer to the nearest hundredth.

Show answer

$0.3x = \ln 5$, so $x = \dfrac{\ln 5}{0.3} \approx 5.36$

9. **Medium** A school science lab has 100 grams of a radioactive substance with a half-life of 8 days, so the amount left after t days is $A = 100\left(\frac{1}{2}\right)^{t/8}$. How long until only 10 grams remain? Round to the nearest tenth of a day.

Show answer

$\left(\frac{1}{2}\right)^{t/8} = 0.1$, so $\frac{t}{8} = \frac{\ln 0.1}{\ln 0.5} \approx 3.3219$ and $t \approx 26.6$ days

10. **Hard** Solve $3^x = 2^{x+1}$. Give the exact answer and the answer to the nearest hundredth.

Show answer

$x \ln 3 = (x+1)\ln 2$, so $x(\ln 3 - \ln 2) = \ln 2$ and $x = \frac{\ln 2}{\ln 1.5} \approx 1.71$

11. **Hard** Solve exactly: $e^{2x} - 7e^x + 12 = 0$

Show answer

With $u = e^x$: $(u-3)(u-4) = 0$, so $e^x = 3$ or $e^x = 4$: $x = \ln 3$ or $x = \ln 4$

12. **Hard** Ruth takes a cherry pie out of a 350°F oven into a 70°F kitchen. By Newton's law of cooling its temperature after t minutes is $T = 70 + 280e^{-0.04t}$. When will the pie be 120°F? Round to the nearest minute.

Show answer

$280e^{-0.04t} = 50$, so $e^{-0.04t} = \frac{5}{28}$ and $t = \frac{\ln(5/28)}{-0.04} \approx 43$ minutes

13. **Hard** Grace puts \$500 of her summer earnings in an account paying 3% interest compounded continuously, $A = 500e^{0.03t}$. How long until it reaches \$800? Round to the nearest tenth of a year.

Show answer

$e^{0.03t} = 1.6$, so $t = \frac{\ln 1.6}{0.03} \approx 15.7$ years

14. **Challenge** Solve: $4^x - 6 \cdot 2^x + 8 = 0$

Show answer

$4^x = (2^x)^2$; with $u = 2^x$: $u^2 - 6u + 8 = (u-2)(u-4) = 0$, so $2^x = 2$ or $2^x = 4$: $x = 1$ or $x = 2$; check: $4 - 12 + 8 = 0$ ✓ and $16 - 24 + 8 = 0$ ✓

15. **Challenge** Solve $e^{2x} + 3e^{x} - 10 = 0$ exactly, and explain why one of the values you find for e^{x} must be rejected.

Show answer

With $u = e^{x}$: $(u + 5)(u - 2) = 0$, so $u = -5$ or $u = 2$. But e^{x} is positive for every real x, so $e^{x} = -5$ has no solution. Only $e^{x} = 2$ works: $x = \ln 2$.

COMMON MISTAKES

- Taking the logarithm before isolating the power, as in writing $\log(4 \cdot 3^{x} - 7) = \log 4 + x \log 3 - \log 7$. There is no rule for the log of a difference. First get the power alone on one side.
- Dividing the bases instead of the logarithms: from $3^{x} = 20$, writing $x = \dfrac{20}{3}$ or $x = \log \dfrac{20}{3}$. The correct result is $x = \dfrac{\log 20}{\log 3}$.
- Rounding too early. Rounding $\ln 1.00375$ to 0.004 in a savings problem changes the answer by more than half a year. Keep full calculator values until the last step.
- Keeping impossible values in quadratic-form equations. e^{x} and 2^{x} are always positive, so a zero or negative value of u gives no solution.

> **Make it a game · Same Base or Take Logs.** The parent flashes exponential equations such as $4^{x} = 32$ and $3^{x} = 20$. In three seconds, decide whether the numbers share a common base or whether you must take logs. Then solve, giving an exact answer in terms of logarithms and, where needed, a decimal approximation. Five correct decisions in a row earns a bonus point.

8.5 Solving Logarithmic Equations

OBJECTIVES

- Solve logarithmic equations by isolating the logarithm and rewriting in exponential form.
- Condense sums and differences of logarithms before solving, and solve equations with one logarithm on each side using $\log_b M = \log_b N \implies M = N$.
- Explain why extraneous solutions appear and reject them by checking that every argument in the original equation is positive.
- Solve logarithmic models for an unknown quantity.

WATCH

Solving Log Equations w/ Extraneous Solutions (Example 1 of 2)

Eddie Woo · 7:19

https://www.youtube.com/watch?v=BxxXGZ015ZY

> **WHEN TO WATCH**
> Before Example 1: solving a log equation and rejecting extraneous solutions.

BACKUP VIDEO

Solving Logarithmic Equations (5 Examples)

Mario's Math Tutoring · 11:04

https://www.youtube.com/watch?v=kBeHeTIcZ98

Before the practice set: five more log equations.

KEY TERMS

logarithmic equation

An equation in which the variable appears inside a logarithm, such as $\log_2(x+3) = 5$.

extraneous solution

A number produced by the solving steps that does not satisfy the original equation; for logarithmic equations it usually makes some argument zero or negative.

domain check

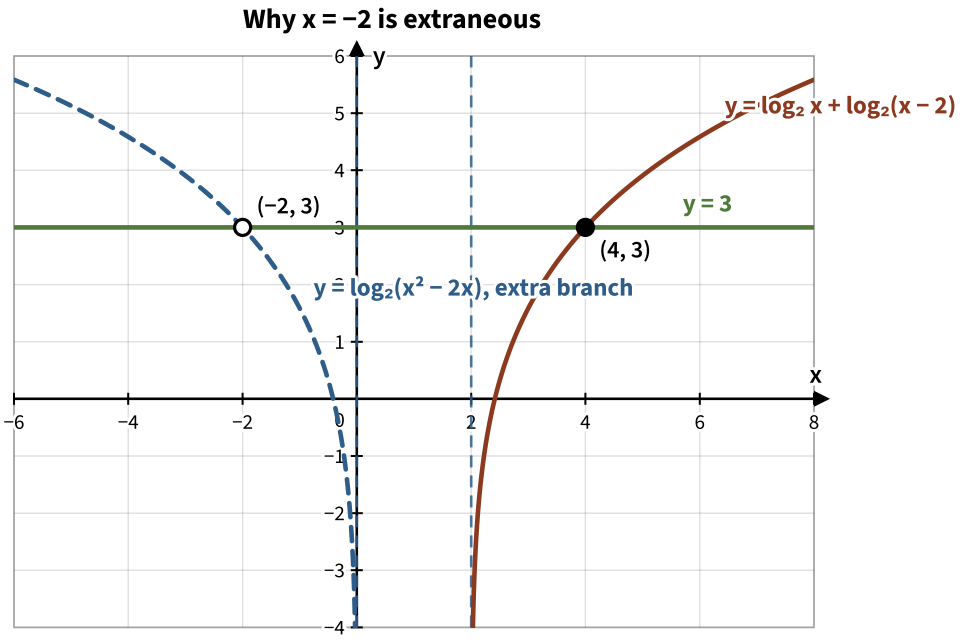
Substituting each candidate into every logarithm of the ORIGINAL equation to confirm each argument is positive.

CORE IDEAS

Rewrite in exponential form. A logarithmic equation with one logarithm is an exponent statement in disguise. To solve $\log_3(2x - 1) = 2$, translate: $2x - 1 = 3^2 = 9$, so $x = 5$. Check: $\log_3(2 \cdot 5 - 1) = \log_3 9 = 2$ ✓. If other terms are present, isolate the logarithm first: $3 \log_5 x - 1 = 5$ becomes $\log_5 x = 2$, so $x = 25$.

Condense first. When an equation has several logarithms with the same base on one side, use the properties of Lesson 8.3 to combine them into one, then rewrite in exponential form. For $\log_2 x + \log_2(x - 2) = 3$: the product property gives $\log_2 (x(x - 2)) = 3$, so $x(x - 2) = 2^3 = 8$. Then $x^2 - 2x - 8 = 0$, $(x - 4)(x + 2) = 0$, and $x = 4$ or $x = -2$.

Why one of those is extraneous. Check $x = -2$ in the ORIGINAL equation: $\log_2(-2)$ is undefined, so $x = -2$ is not a solution. Where did it come from? The original equation needs $x > 0$ **and** $x - 2 > 0$, that is, $x > 2$. The condensed equation $\log_2(x^2 - 2x) = 3$ only needs the product $x(x - 2)$ to be positive, which is also true when both factors are negative. Condensing quietly **enlarged the domain**, and the new part of the domain contained a new solution. The graph shows it: the original left side (solid) exists only for $x > 2$ and meets $y = 3$ once, at $x = 4$; the condensed form (its extra left branch dashed) also meets $y = 3$ at $x = -2$.



One logarithm on each side. Since $y = \log_b x$ is one-to-one, $\log_b M = \log_b N$ means $M = N$ (with M and N positive). For $\log(x+6) = \log x + \log(x-4)$, condense the right side: $\log(x+6) = \log\left(x(x-4)\right)$, so $x + 6 = x^2 - 4x$, $x^2 - 5x - 6 = 0$, $(x-6)(x+1) = 0$. Then $x = 6$ works ($\log 12 = \log 6 + \log 2$ ✓), but $x = -1$ is rejected because $\log(-1)$ is undefined.

Equations with ln. The natural logarithm follows the same rules with base e: $\ln(x+2) = 3$ means $x + 2 = e^3$, so $x = e^3 - 2 \approx 18.09$. Give the exact form with e first, then a decimal if asked.

The steps, in order: (1) note the domain: every argument must be positive; (2) isolate or condense so there is a single logarithm on a side; (3) rewrite in exponential form or set the arguments equal; (4) solve the resulting equation; (5) check every candidate in the ORIGINAL equation and reject any that make an argument zero or negative.

KEY FORMULAS

$$\log_b M = c \implies M = b^c$$

$$\log_b M = \log_b N \implies M = N \qquad (M > 0,\ N > 0)$$

$$\ln M = c \implies M = e^c$$

Practice habit. Check every answer in the ORIGINAL equation, not the condensed one. For each candidate, ask of every logarithm: is its argument positive? If any is zero or negative, the candidate is extraneous.

WORKED EXAMPLES

Example 1. Solve $\log_5(x+3) + \log_5(x-1) = 1$. Check for extraneous solutions.

Show solution

1. Domain: $x + 3 > 0$ and $x - 1 > 0$, so $x > 1$.
2. Condense with the product property: $\log_5\left((x+3)(x-1)\right) = 1$.
3. Rewrite in exponential form: $(x+3)(x-1) = 5^1 = 5$.
4. Expand and solve: $x^2 + 2x - 3 = 5$, so $x^2 + 2x - 8 = 0$ and $(x+4)(x-2) = 0$: $x = -4$ or $x = 2$.
5. Check $x = 2$ in the original: $\log_5 5 + \log_5 1 = 1 + 0 = 1$ ✓
6. Check $x = -4$: $\log_5(-1)$ is undefined, so $x = -4$ is extraneous.

Answer: $x = 2$ (the candidate $x = -4$ is extraneous)

Example 2. Abigail is learning to type. Her speed after t weeks of practice is modeled by $S(t) = 18 + 12\ln(t+1)$ words per minute. (a) What is her speed when she starts? (b) After how many weeks does the model predict 42 words per minute? Give the exact answer and the answer to the nearest tenth of a week.

Show solution

1. (a) $S(0) = 18 + 12\ln 1 = 18 + 0 = 18$ words per minute.
2. (b) Set $18 + 12\ln(t+1) = 42$.
3. Isolate the logarithm: $12\ln(t+1) = 24$, so $\ln(t+1) = 2$.
4. Rewrite in exponential form (base e): $t + 1 = e^2$, so $t = e^2 - 1$.
5. Calculator: $e^2 \approx 7.389$, so $t \approx 6.4$ weeks.
6. Check: $18 + 12\ln(e^2) = 18 + 12(2) = 42$ ✓, and the argument e^2 is positive.

Answer: (a) 18 words per minute (b) $t = e^2 - 1 \approx 6.4$ weeks

PRACTICE

1. Easy Solve: $\log_2 x = 5$

   Show answer

   $x = 2^5 = 32$

2. Easy Solve: $\log_3(x - 4) = 2$

   Show answer

   $x - 4 = 9$, so $x = 13$; check: $\log_3 9 = 2$ ✓

3. Easy Solve: $\log(2x) = 3$

   Show answer

   $2x = 10^3 = 1000$, so $x = 500$

4. Easy Solve $\ln x = 4$. Give the exact answer.

   Show answer

   $x = e^4$

5. Medium Solve: $\log_4(3x + 1) = \log_4(x + 9)$

   Show answer

   $3x + 1 = x + 9$, so $x = 4$; check: both arguments are 13 ✓

6. Medium Solve: $\log_6 x + \log_6(x - 5) = 2$

   Show answer

   $x(x - 5) = 36$, $x^2 - 5x - 36 = 0$, $(x - 9)(x + 4) = 0$; $x = 9$ ($x = -4$ is extraneous); check: $\log_6 9 + \log_6 4 = \log_6 36 = 2$ ✓

7. Medium Solve: $\log_2(x + 6) - \log_2 x = 2$

   Show answer

   $\dfrac{x + 6}{x} = 4$, so $x + 6 = 4x$ and $x = 2$; check: $\log_2 8 - \log_2 2 = 3 - 1 = 2$ ✓

8. Medium Solve: $2\log_3 x = \log_3 49$

   Show answer

   $\log_3 x^2 = \log_3 49$, so $x^2 = 49$ and $x = \pm 7$; but $\log_3(-7)$ is undefined, so $x = 7$

9. Medium A pilot's handbook gives the approximate altitude h (in kilometers) where the air pressure is P atmospheres as $h = -8 \ln P$. At about what pressure is a plane flying at 10 km? Round to the nearest hundredth of an atmosphere.

Show answer

$\ln P = -1.25$, so $P = e^{-1.25} \approx 0.29$ atmosphere

10. Hard Solve: $\log x + \log(x - 21) = 2$

Show answer

$x(x - 21) = 100$, $x^2 - 21x - 100 = 0$, $(x - 25)(x + 4) = 0$; $x = 25$ ($x = -4$ is extraneous); check: $\log 25 + \log 4 = \log 100 = 2$ ✓

11. Hard Solve: $\ln(x + 1) - \ln(x - 2) = \ln 4$

Show answer

$\dfrac{x + 1}{x - 2} = 4$, so $x + 1 = 4x - 8$ and $x = 3$; check: $\ln 4 - \ln 1 = \ln 4$ ✓

12. Hard For money earning 5% interest compounded continuously, the number of years it takes to grow from a deposit P to an amount A is $t = 20 \ln \dfrac{A}{P}$. How much must Henry deposit now so that it grows to \$5,000 in 10 years? Round to the nearest cent.

Show answer

$10 = 20 \ln \dfrac{5000}{P}$, so $\ln \dfrac{5000}{P} = 0.5$, $\dfrac{5000}{P} = e^{0.5}$ and $P = \dfrac{5000}{e^{0.5}} \approx \$3{,}032.65$

13. Hard A common rule of thumb says that a telescope whose main lens or mirror is D millimeters across can show stars as faint as magnitude $m = 2.7 + 5 \log D$. What diameter is needed to see stars of magnitude 12.7?

Show answer

$5 \log D = 10$, so $\log D = 2$ and $D = 10^2 = 100$ mm

14. **Challenge** Solving $\log x + \log(x+3) = 1$, Daniel found $x = 2$ and $x = -5$. Which of these are solutions? Explain why the other number appeared.

Show answer

Condensing gives $x(x+3) = 10$, $(x+5)(x-2) = 0$. Only $x = 2$ is a solution: $\log 2 + \log 5 = \log 10 = 1$ ✓. For $x = -5$, $\log(-5)$ and $\log(-2)$ are undefined. It appeared because the condensed equation $\log\left(x(x+3)\right) = 1$ only needs the product $x(x+3)$ to be positive, which also happens when both factors are negative, so condensing enlarged the domain.

15. **Challenge** Solve: $\log_2 x + \log_4 x = 6$ (Hint: use change of base to write $\log_4 x$ in base 2.)

Show answer

$\log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{1}{2}\log_2 x$, so $\dfrac{3}{2}\log_2 x = 6$, $\log_2 x = 4$ and $x = 16$; check: $4 + 2 = 6$ ✓

COMMON MISTAKES

- Skipping the check. Condensing $\log_b M + \log_b N$ into $\log_b(MN)$ can enlarge the domain, so a candidate can look fine in the condensed equation but fail in the original. Always check every argument of the ORIGINAL equation.
- "Dropping" logs when they are not alone, as in turning $\log x + \log 3 = \log 12$ into $x + 3 = 12$. Condense first: $\log(3x) = \log 12$, so $3x = 12$ and $x = 4$.
- Rejecting a negative candidate without checking, or keeping a positive one without checking. The test is not whether x is negative but whether every argument is positive: in $\log(x+10) = 0$, the solution $x = -9$ is negative but correct, because the argument is $-9 + 10 = 1 > 0$.
- Writing $\log_3(x-4) = 2$ as $x - 4 = 2^3$. The base stays the base: $x - 4 = 3^2$.

> **Make it a game · The Domain Gate.** You are the gatekeeper, and only valid arguments may enter. Solve log2 x + log2 (x - 2) = 3 by condensing, rewriting in exponential form and solving the quadratic. The parent plays each candidate, 4 and -2, and you check whether the argument of every log is positive. Use a dramatic announcer voice when rejecting the impostor.

8.6 Logarithmic Scales and Models

OBJECTIVES

- Explain why logarithmic scales are useful for quantities that range over many powers of ten.
- Compute and compare pH values, sound levels in decibels, earthquake magnitudes and star magnitudes.
- Derive the rule of 72 (and 70) from $\ln 2 \approx 0.693$ and use it to estimate doubling times.
- Fit an exponential model $y = a \cdot b^x$ to data by taking logarithms, using the fact that $\log y$ is linear in x.

KEY TERMS

logarithmic scale

A scale that records the logarithm of a quantity instead of the quantity itself, so that each step of 1 means multiplying by the base (usually 10).

pH

A measure of acidity: $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per liter. Lower pH means more acidic; 7 is neutral.

decibel (dB)

The unit of sound level $L = 10 \log \dfrac{I}{I_0}$, where I is the sound's intensity and $I_0 = 10^{-12}$ watts per square meter is the faintest sound a person can hear.

magnitude (of a star)

A measure of how bright a star looks; smaller numbers are brighter, and a difference of 5 magnitudes is a factor of 100 in brightness.

rule of 72

A mental estimate: money growing at R percent per year doubles in about $\dfrac{72}{R}$ years.

CORE IDEAS

Some quantities in nature range over many powers of ten. The hydrogen-ion concentration of household liquids runs from about 0.01 to 0.000000001 moles per liter; the loudest sound you can stand is about a trillion times as intense as the faintest one you can hear. On an ordinary scale the small values would all crowd together at zero. A **logarithmic scale** records the exponent instead: $\log 0.01 = -2$

and $\log 0.000000001 = -9$. Now the values are spread out evenly, and each step of 1 on the scale means a factor of 10 in the quantity.

pH. Chemists define $\text{pH} = -\log[\text{H}^+]$. The minus sign makes the usual values positive. Pure water has $[\text{H}^+] = 10^{-7}$, so its pH is 7. Because pH is a logarithm, a difference in pH is a ratio of concentrations: lemon juice (pH 2.3) has $10^{6.6-2.3} = 10^{4.3} \approx 20{,}000$ times the hydrogen-ion concentration of milk (pH 6.6).

Some household liquids

Liquid	[H⁺] (moles per liter)	pH
Lemon juice	about 0.005	2.3
Milk	about 0.00000025	6.6
Pure water	0.0000001	7
Baking soda in water	about 0.000000005	8.3

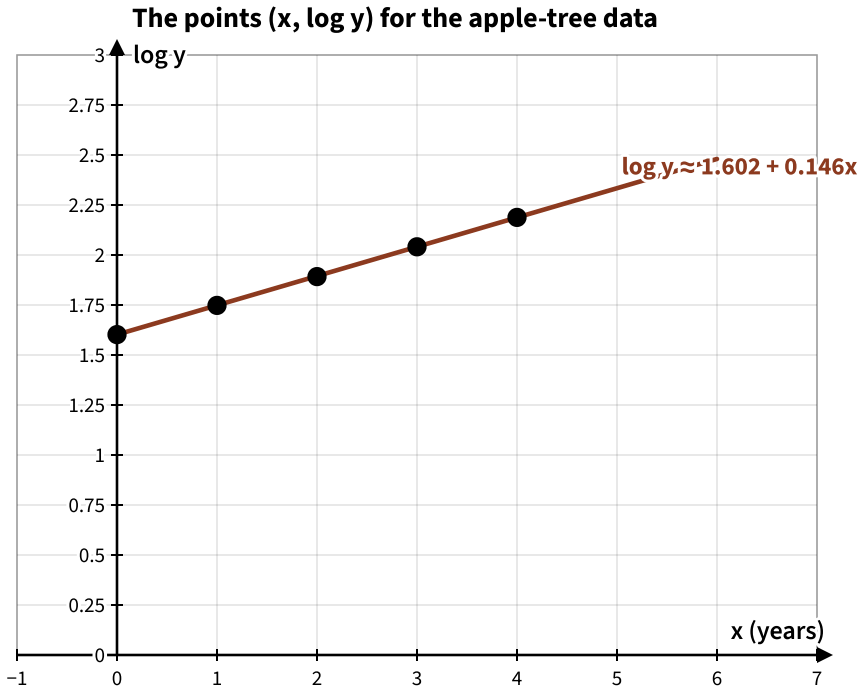
Decibels. Sound level is $L = 10 \log \dfrac{I}{I_0}$. Each increase of 10 dB multiplies the intensity by 10, because $10 \log(10x) = 10 + 10 \log x$. A lawn mower at 90 dB is $10^{(90-60)/10} = 10^3 = 1000$ times as intense as conversation at 60 dB. **Earthquakes** are measured on a magnitude scale built on logarithms as well; each whole step in magnitude means about 32 times as much energy released. **Star magnitudes** run backward (smaller is brighter), and 5 magnitudes is a factor of exactly 100 in brightness. So one magnitude is a factor of $100^{1/5} = 10^{0.4} \approx 2.512$, and two stars whose magnitudes differ by d differ in brightness by a factor of $10^{0.4d}$.

The rule of 72. At an annual growth rate r (as a decimal), the doubling time solves $(1+r)^t = 2$, so $t = \dfrac{\ln 2}{\ln(1+r)}$ (Lesson 8.4). For small r, $\ln(1+r)$ is very close to r itself (for example $\ln 1.05 \approx 0.0488$), so $t \approx \dfrac{0.693}{r} = \dfrac{69.3}{R}$, where R is the rate in percent. For continuous growth $e^{rt} = 2$ gives exactly $t = \dfrac{\ln 2}{r}$, which is where the **rule of 70** comes from. With yearly compounding, $\ln(1+r)$ is a bit smaller than r, so the true time is a bit longer than $\dfrac{69.3}{R}$; using 72 corrects for that at common rates, and 72 divides evenly by 2, 3, 4, 6, 8, 9 and 12.

Doubling times with yearly compounding

Annual rate	Exact time (years)	Rule of 72	Rule of 70
2%	35	36	35
4%	17.7	18	17.5
8%	9	9	8.75
12%	6.1	6	5.8

Fitting exponential data with logs. If $y = a \cdot b^x$, take the common log of both sides and use the product and power properties: $\log y = \log a + x \log b$. That is a **linear** equation in x, with slope $\log b$ and intercept $\log a$. So if you list $\log y$ for each data point and the values go up by nearly the same amount each step, the data are exponential. Find the line through the points $(x, \log y)$, then undo the logs: $a = 10^{\text{intercept}}$ and $b = 10^{\text{slope}}$. The graph shows the logs of the apple-tree data in Worked Example 2 lying almost on a line. Logarithms have turned the repeated multiplication of exponential growth into the repeated addition of a line.



KEY FORMULAS

$$\mathrm{pH} = -\log[\mathrm{H}^+]$$

$$L = 10 \log \frac{I}{I_0}, \qquad I_0 = 10^{-12}\ \mathrm{W/m^2}$$

$$\frac{B_1}{B_2} = 10^{0.4(m_2 - m_1)} \qquad (5 \text{ magnitudes} = \text{a factor of } 100)$$

$$t_{\text{double}} = \frac{\ln 2}{\ln(1 + r)} \approx \frac{72}{R}$$

$$y = a \cdot b^x \iff \log y = \log a + x \log b$$

From history. The star magnitude scale goes back to the star catalogs of the ancient Greek astronomers Hipparchus and Ptolemy, who sorted the visible stars into six classes, from the brightest (first magnitude) to the faintest the eye could see (sixth magnitude). Much later, astronomers made the scale exact by agreeing that 5 magnitudes means a factor of exactly 100 in brightness, which keeps the old classes nearly where they were.

Practice habit. Use the rule of 72 for quick mental estimates: divide 72 by the percent rate to estimate doubling time (6% doubles in about 12 years; 9% in about 8). Then, when the answer matters, compute the exact value $\dfrac{\ln 2}{\ln(1 + r)}$.

WORKED EXAMPLES

Example 1. A quiet library has a sound level of about 40 dB, and a tractor engine heard from the driver's seat is about 100 dB. (a) Find the intensity of the library's sound in watts per square meter. (b) How many times as intense is the tractor's sound as the library's?

Show solution

1. (a) Set $10 \log \frac{I}{10^{-12}} = 40$. Divide by 10: $\log \frac{I}{10^{-12}} = 4$.
2. Rewrite in exponential form: $\frac{I}{10^{-12}} = 10^4$, so $I = 10^4 \times 10^{-12} = 10^{-8}$ W/m².
3. (b) In the same way the tractor has $\frac{I}{10^{-12}} = 10^{10}$, so $I = 10^{-2}$ W/m².
4. Ratio: $\frac{10^{-2}}{10^{-8}} = 10^6$.
5. Shortcut: 100 dB is 60 dB more, and each 10 dB is a factor of 10, so the factor is $10^{60/10} = 10^6 = 1{,}000{,}000$ ✓

Answer: (a) 10^{-8} W/m² (b) $10^6 = 1{,}000{,}000$ times as intense

Example 2. A new apple variety is being planted across a valley. The table gives the number of trees y in year x. (a) Use the $\log y$ column to explain why an exponential model fits. (b) Using the first and last data points, find a model $y = a \cdot b^x$, with b rounded to the nearest hundredth. (c) Use your model to predict the number of trees in year 6, to the nearest tree.

Trees of a new apple variety in a valley

Year x	Trees y	log y
0	40	1.602
1	56	1.748
2	78	1.892
3	110	2.041
4	154	2.188

Show solution

1. (a) The values of $\log y$ go up by 0.146, 0.144, 0.149, 0.147: nearly the same each year. So $\log y$ is nearly linear in x, which means y is nearly exponential.
2. (b) Slope of the line through $(0, 1.602)$ and $(4, 2.188)$: $\dfrac{2.188 - 1.602}{4 - 0} = \dfrac{0.586}{4} \approx 0.1465$. Intercept: 1.602.
3. So $\log y \approx 1.602 + 0.1465x$. Undo the logs: $a = 10^{1.602} \approx 40$ and $b = 10^{0.1465} \approx 1.40$.
4. Model: $y \approx 40(1.40)^x$. Check year 4: $40(1.40)^4 = 40(3.8416) \approx 154$ ✓
5. (c) Year 6: $y \approx 40(1.40)^6 = 40(7.529536) \approx 301$ trees.

Answer: (a) $\log y$ rises by about 0.146 each year (b) $y \approx 40(1.40)^x$ (c) about 301 trees

PRACTICE

1. **Easy** Find the pH of a solution with (a) $[\mathrm{H}^+] = 10^{-4}$ (b) $[\mathrm{H}^+] = 0.001$ moles per liter.

   Show answer

   (a) $-\log 10^{-4} = 4$ (b) $-\log 10^{-3} = 3$

2. **Easy** Find the sound level in decibels of a sound with intensity $I = 10^{-6}$ W/m².

   Show answer

   $$L = 10 \log \frac{10^{-6}}{10^{-12}} = 10 \log 10^{6} = 60 \text{ dB}$$

3. **Easy** Use the rule of 72 to estimate how long it takes money to double at (a) 6% per year (b) 9% per year.

   Show answer

   (a) $72 \div 6 = 12$ years (b) $72 \div 9 = 8$ years

4. **Easy** Star A has magnitude 1 and star B has magnitude 6. Which star looks brighter, and how many times as bright?

   Show answer

   Star A (smaller magnitude is brighter); 5 magnitudes is a factor of 100, so 100 times as bright

5. **Medium** Black coffee has a pH of about 5.0 and tomato juice about 4.3. (a) Find $[\mathrm{H}^+]$ for the coffee. (b) How many times the hydrogen-ion concentration of coffee does tomato juice have? Round to the nearest tenth.

   Show answer

   (a) $10^{-5} = 0.00001$ moles per liter (b) $10^{5.0-4.3} = 10^{0.7} \approx 5.0$ times

6. **Medium** Using the fact that each whole step in earthquake magnitude means about 32 times as much energy, about how many times as much energy does a magnitude 7 earthquake release as a magnitude 5 earthquake?

   Show answer

   $32 \times 32 = 1024$, about 1000 times

7. **Medium** (a) A lawn mower makes 90 dB and a conversation 60 dB. How many times as intense is the mower's sound? (b) By how many decibels does the level rise when the intensity doubles? Round to the nearest tenth.

Show answer

(a) $10^{30/10} = 1000$ times (b) $10\log(2I/I_0) - 10\log(I/I_0) = 10\log 2 \approx 3.0$ dB

8. **Medium** Money in an account earns 3.5% interest compounded continuously. (a) Find the exact doubling time $\dfrac{\ln 2}{0.035}$ to the nearest tenth of a year. (b) What does the rule of 70 estimate?

Show answer

(a) $\dfrac{0.693147}{0.035} \approx 19.8$ years (b) $70 \div 3.5 = 20$ years

9. **Medium** Vega has magnitude about 0.0 and Polaris, the North Star, about 2.0. How many times as bright as Polaris does Vega look? Round to the nearest tenth.

Show answer

$10^{0.4(2.0-0.0)} = 10^{0.8} \approx 6.3$ times

10. **Hard** Henry's savings bond earns 5% interest compounded yearly. (a) Estimate its doubling time with the rule of 72. (b) Find the exact doubling time $\dfrac{\ln 2}{\ln 1.05}$ to the nearest tenth of a year.

Show answer

(a) $72 \div 5 = 14.4$ years (b) $\dfrac{0.693147}{0.048790} \approx 14.2$ years

11. **Hard** For a set of data, the points $(x, \log y)$ lie very close to the line $\log y = 0.477 + 0.301x$. Find an exponential model $y = a \cdot b^x$, rounding a and b to the nearest whole number.

Show answer

$a = 10^{0.477} \approx 3$ and $b = 10^{0.301} \approx 2$, so $y = 3 \cdot 2^x$

12. Hard Sirius, the brightest star in the night sky, has magnitude -1.46. A star of magnitude 4.00 is just visible from a town with some light pollution. How many times as bright as that star does Sirius look? Round to the nearest whole number.

Show answer

The difference is $4.00 - (-1.46) = 5.46$, so the factor is $10^{0.4(5.46)} = 10^{2.184} \approx 153$

13. Hard One singer in the church choir produces a sound level of 60 dB at the back of the sanctuary. Sound intensities add, so n singers singing equally loudly produce n times the intensity. Find the sound level of (a) 10 singers (b) 20 singers, to the nearest tenth of a decibel.

Show answer

The level rises by $10\log n$. (a) $60 + 10\log 10 = 70.0$ dB (b) $60 + 10\log 20 \approx 60 + 13.0 = 73.0$ dB

14. Challenge Explain why, if $y = a \cdot b^{x}$ with $a > 0$ and $b > 0$, the points $(x, \log y)$ all lie on a straight line. What are its slope and its intercept?

Show answer

Take the log of both sides: $\log y = \log(a \cdot b^{x}) = \log a + \log b^{x} = \log a + x \log b$ by the product and power properties. This has the form $Y = mx + c$ with $Y = \log y$, so the points lie on a line with slope $\log b$ and intercept $\log a$.

15. Challenge Seismologists relate the energy E (in joules) released by an earthquake to its magnitude M by $\log E = 4.8 + 1.5M$. (a) Show that increasing M by 1 multiplies E by $10^{1.5}$, and give that factor to the nearest tenth. (b) How many magnitude 6 earthquakes would release as much energy as one magnitude 8 earthquake?

Show answer

(a) $\log E_{M+1} - \log E_{M} = 1.5$, so $\dfrac{E_{M+1}}{E_{M}} = 10^{1.5} \approx 31.6$ (the "about 32" fact) (b) $10^{1.5 \times 2} = 10^{3} = 1000$

COMMON MISTAKES

- Treating a logarithmic scale like an ordinary one: thinking pH 3 is "twice as acidic" as pH 6, or that 80 dB is "twice as loud" as 40 dB. A difference of 1 on a

base-10 log scale is a factor of 10, so compare by subtracting and then raising 10 to that power.

- Getting the direction wrong on scales that run backward. Lower pH means more acidic, and lower star magnitude means brighter.
- Forgetting the factor 10 in decibels or the 0.4 in magnitudes. Each 10 dB (not each 1 dB) is a factor of 10 in intensity, and each 1 magnitude is a factor of $10^{0.4} \approx 2.512$.
- Using the rule of 72 with the rate as a decimal. Divide 72 by the percent: at 6%, $72 \div 6 = 12$ years, not $72 \div 0.06$.

> **Make it a game · How Many Times More?.** The parent names two readings, such as pH 3 and pH 6, earthquake magnitudes 5 and 7, or 60 dB and 80 dB. Write your estimate of how many times stronger one is before computing, then check with powers of ten. Closest guess wins a point. Finish with the rule of 72: how long to double at 6 percent?

UNIT 8

Logarithmic Functions

In Unit 7 you used exponential functions to find how much: how much money after 10 years, how much of a substance after 3 half-lives. This unit answers the reverse question, how long: how many years until savings double, how many minutes until a pie cools. That question asks for an exponent, and the tool that finds exponents is the logarithm. You will learn that a logarithm is an exponent, that $y = \log_b x$ is the inverse of $y = b^x$ (so its graph is a reflection in $y = x$), and that the properties of logarithms are the rules of exponents in a new form.

With those properties you will solve exponential and logarithmic equations exactly and with a calculator, and learn why some candidate answers are extraneous and must be checked against the domain. The unit ends with logarithmic scales that scientists use every day (pH, decibels, earthquake and star magnitudes), the rule of 72 for doubling time, and fitting exponential models to data by taking logs.

Logarithms come back often in later work: in sequences and loan formulas in Unit 9, in the Statistics course, and in calculus, where $\ln x$ and e^x are the most important functions of all. They also appear in the mathematics of modern computing, where the logistic function from Lesson 7.4 and logarithms work side by side.

PREREQUISITES

- Use the exponent rules, including zero, negative and rational exponents (Lesson 6.1).
- Find inverse functions and understand that the graph of an inverse is a reflection in $y = x$ (Lesson 1.5).
- Transform graphs with $y = a\,f(x - h) + k$ (Lesson 1.2).
- Work with exponential functions, compound interest and the number e (Lessons 7.1–7.3).
- Solve quadratic equations by factoring and solve quadratic and rational inequalities with a sign chart (Lessons 3.2, 3.5 and 5.4).

MASTERY CHECKLIST

- ☐ I can explain what a logarithm is and switch between logarithmic and exponential form.
- ☐ I can evaluate logarithms such as $\log_2 32$, $\log_3 \frac{1}{9}$ and $\log_5 \sqrt{5}$ by hand, and estimate a logarithm between two whole numbers.

- ☐ I can graph $y = a \log_b(x - h) + k$ and give its domain, range, asymptote and intercepts, and find the domain of the logarithm of an expression.
- ☐ I can derive the product, quotient and power properties from the exponent rules and use them to expand and condense logarithms.
- ☐ I can derive the change-of-base formula and use it to evaluate any logarithm on a calculator.
- ☐ I can solve exponential equations by using a common base or by taking logarithms, including equations with e and equations in quadratic form.
- ☐ I can solve logarithmic equations and explain and reject extraneous solutions.
- ☐ I can solve growth, decay, cooling and savings models for time.
- ☐ I can compute and compare values on logarithmic scales: pH, decibels, earthquake magnitudes and star magnitudes.
- ☐ I can derive and use the rule of 72, and fit an exponential model to data by taking logarithms.

LESSONS IN THIS UNIT

8.1	**Logarithms**	2 sessions of 45 minutes
8.2	**Graphs of Logarithmic Functions**	2 sessions of 45 minutes

8.3	**Properties of Logarithms**	2 sessions of 45 minutes
8.4	**Solving Exponential Equations**	2 sessions of 45 minutes
8.5	**Solving Logarithmic Equations**	2 sessions of 45 minutes
8.6	**Logarithmic Scales and Models**	2 sessions of 45 minutes

Unit 8 test

You have 60 minutes. A scientific calculator is allowed except on problems marked "by hand." Show your work on every problem, and check each solution of an equation in the original equation. Give exact answers unless a problem asks you to round; when it does, round only at the last step.

1. **5 pts**

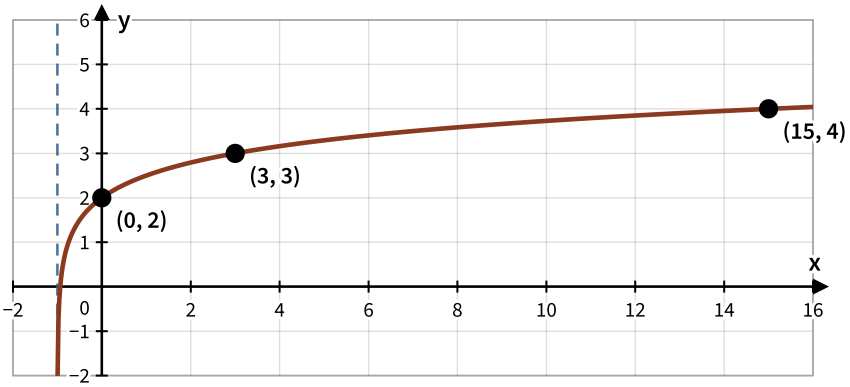
   By hand, evaluate: (a) $\log_4 64$ (b) $\log_2 \dfrac{1}{32}$ (c) $\log_9 3$

2. **5 pts**

   By hand, solve for x: (a) $\log_x 125 = 3$ (b) $\log_8 x = \dfrac{2}{3}$

3. **6 pts**

   The graph shows a function of the form $y = \log_b(x - h) + k$, with its asymptote dashed. Find b, h and k, and explain how you found each.



4. **6 pts**

   For $f(x) = \log_3(x + 6) - 2$, give the domain, the vertical asymptote and the x-intercept.

5. **6 pts**

   Expand completely (all variables positive): $\log_2 \dfrac{x^2\sqrt{y}}{16}$

6. **6 pts**

   Condense into a single logarithm: $4\ln x - \ln(x+1) + 2\ln y$

7. **5 pts**

   Use the change-of-base formula to evaluate $\log_6 45$ to the nearest thousandth.

8. **7 pts**

   Solve: (a) $16^{x} = 8^{x+1}$ (b) $5 \cdot 4^{x} + 2 = 47$. Give an exact answer for (b) and round it to the nearest hundredth.

9. **7 pts**

   Solve exactly: $e^{2x} - e^{x} - 12 = 0$

10. **7 pts**

    Solve $\log_2 x + \log_2(x+6) = 4$ and check for extraneous solutions.

11. **7 pts**

    Solve exactly: $\log_3(x+2) = 2 - \log_3 x$

12. **7 pts**

    Ruth deposits \$1,500 in an account paying 3.6% interest compounded quarterly. How long will it take for the balance to reach \$2,400? Round to the nearest tenth of a year.

13. **7 pts**

    A mug of cocoa at 180°F is set on a table in a 68°F room. Its temperature after t minutes is $T = 68 + 112e^{-0.05t}$. When will it cool to 140°F? Round to the nearest tenth of a minute.

14. **7 pts**

(a) A bakery's dough mixer runs at 85 dB, and the kitchen refrigerator hums at 45 dB. How many times as intense is the mixer's sound? (b) A jar of pickle brine has $[\mathrm{H}^+] = 3.2 \times 10^{-4}$ moles per liter. Find its pH to the nearest tenth.

15. **5 pts**

A family puts savings for a new tractor in a fund growing 3% per year, compounded yearly. (a) Estimate the doubling time with the rule of 72. (b) Find the exact doubling time to the nearest tenth of a year.

16. **7 pts**

(a) Prove the product property $\log_b(MN) = \log_b M + \log_b N$ for $M, N > 0$ from the rules of exponents. (b) Solve $\log_2 x + \log_2(x - 3) = 2$, and explain why one candidate must be rejected even though it solves the condensed equation.

ANSWER KEY

1. (a) 3 (b) -5 (c) $\dfrac{1}{2}$, since $9^{1/2} = 3$
2. (a) $x^3 = 125$, so $x = 5$ (b) $x = 8^{2/3} = (\sqrt[3]{8})^2 = 4$
3. The asymptote is $x = -1$, so $h = -1$. At $(0, 2)$ the argument $x + 1$ is 1, so the log is 0 and $k = 2$. At $(3, 3)$ the argument is 4 and $\log_b 4 + 2 = 3$, so $\log_b 4 = 1$ and $b = 4$: $y = \log_4(x + 1) + 2$
4. Domain $(-6, \infty)$; asymptote $x = -6$; $\log_3(x + 6) = 2$ gives $x + 6 = 9$, so the x-intercept is 3
5. $2\log_2 x + \dfrac{1}{2}\log_2 y - 4$
6. $\ln \dfrac{x^4 y^2}{x + 1}$
7. $\dfrac{\ln 45}{\ln 6} \approx \dfrac{3.80666}{1.79176} \approx 2.125$
8. (a) $2^{4x} = 2^{3x+3}$, so $x = 3$; check: $16^3 = 4096 = 8^4$ ✓ (b) $4^x = 9$, so $x = \dfrac{\ln 9}{\ln 4} = \log_2 3 \approx 1.58$
9. With $u = e^x$: $(u - 4)(u + 3) = 0$. Since $e^x > 0$, $e^x = -3$ is impossible, so $e^x = 4$ and $x = \ln 4$
10. $x(x + 6) = 16$, $x^2 + 6x - 16 = 0$, $(x + 8)(x - 2) = 0$. $x = 2$ works: $\log_2 2 + \log_2 8 = 1 + 3 = 4$ ✓; $x = -8$ is extraneous ($\log_2(-8)$ is undefined)
11. $\log_3\left(x(x + 2)\right) = 2$, so $x^2 + 2x - 9 = 0$ and $x = -1 \pm \sqrt{10}$. Only $x = -1 + \sqrt{10} \approx 2.16$ makes both arguments positive; check: $x(x + 2) = (\sqrt{10} - 1)(\sqrt{10} + 1) = 9$ ✓
12. $1500(1.009)^{4t} = 2400$, so $(1.009)^{4t} = 1.6$ and $t = \dfrac{\ln 1.6}{4 \ln 1.009} \approx 13.1$ years
13. $112e^{-0.05t} = 72$, so $e^{-0.05t} = \dfrac{9}{14}$ and $t = \dfrac{\ln(9/14)}{-0.05} \approx 8.8$ minutes
14. (a) $10^{(85-45)/10} = 10^4 = 10{,}000$ times (b) $\text{pH} = -\log(3.2 \times 10^{-4}) = 4 - \log 3.2 \approx 3.5$
15. (a) $72 \div 3 = 24$ years (b) $\dfrac{\ln 2}{\ln 1.03} \approx \dfrac{0.693147}{0.029559} \approx 23.4$ years
16. (a) Let $m = \log_b M$ and $n = \log_b N$, so $M = b^m$ and $N = b^n$. Then $MN = b^m b^n = b^{m+n}$, so $\log_b(MN) = m + n = \log_b M + \log_b N$. (b) $x(x - 3) = 4$, $(x - 4)(x + 1) = 0$. $x = 4$ works: $\log_2 4 + \log_2 1 = 2$ ✓. $x = -1$ satisfies $\log_2\left(x(x - 3)\right) = 2$ because $(-1)(-4) = 4$, but in the original $\log_2(-1)$ is undefined; condensing enlarged the domain, so $x = -1$ is extraneous.

UNIT 9

Sequences and Series

This unit is about lists of numbers that follow a rule and what happens when you add them up. You will describe sequences with explicit and recursive rules and write sums compactly in sigma notation. Then you will study the two most important kinds of sequences, arithmetic (add the same amount each time) and geometric (multiply by the same amount each time), and derive formulas for their sums. You will see why some infinite sums have a finite value, and why $0.999\ldots$ is exactly 1.

These ideas tie together much of the course. Arithmetic sequences are linear functions on the counting numbers, and geometric sequences are exponential functions, so Units 1, 7 and 8 come back here. The geometric series also explains the formulas behind savings plans, car loans and home mortgages, so you will see how saving, borrowing and giving wisely look in numbers. The unit ends with mathematical induction, a method for proving that a statement is true for every counting number. Induction and sigma notation are used again in Unit 10 (the Binomial Theorem) and in every later mathematics course.

PREREQUISITES

- Function notation, domain and linear functions (Lessons 1.1 and 1.2)
- Exponent rules and exponential growth and decay, including $A = P(1 + r/n)^{nt}$ (Lessons 7.1–7.3)
- Solving exponential equations with logarithms (Lesson 8.4)
- Solving quadratic equations (Lesson 3.2)
- Simplifying algebraic expressions and fractions, and factoring out a common factor
- Reading and writing a short proof (Geometry)

MASTERY CHECKLIST

- ☐ I can find terms of a sequence from an explicit or recursive rule and write a rule from the first several terms.
- ☐ I can read, write and evaluate sums in sigma notation and use the properties of sums.
- ☐ I can use $a_n = a_1 + (n - 1)d$ to find a term, the common difference or the number of terms of an arithmetic sequence, and find arithmetic means.
- ☐ I can derive and use $S_n = \frac{n(a_1 + a_n)}{2}$ including $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- ☐ I can use $a_n = a_1 r^{n-1}$ to find terms, ratios and geometric means of a geometric sequence.
- ☐ I can derive and use $S_n = \frac{a_1(1 - r^n)}{1 - r}$ for raises, doubling amounts and similar problems.
- ☐ I can decide whether an infinite geometric series converges and find its sum $\frac{a_1}{1 - r}$ when $|r| < 1$.

- ☐ I can write a repeating decimal as a fraction and explain why $0.999\ldots = 1$.
- ☐ I can find the future value of regular deposits and the monthly payment and total interest of a loan.
- ☐ I can explain how the loan payment formula comes from a geometric series and build the first rows of an amortization table.
- ☐ I can write a proof by mathematical induction for a sum formula or a divisibility statement.
- ☐ I can explain why checking many cases is not a proof.

LESSONS IN THIS UNIT

9.1	**Sequences and Summation Notation**	2 sessions of 45 minutes
9.2	**Arithmetic Sequences and Series**	2 sessions of 45 minutes
9.3	**Geometric Sequences and Series**	2 sessions of 45 minutes
9.4	**Infinite Geometric Series**	2 sessions of 45 minutes
9.5	**Savings, Loans and Geometric Series**	2 sessions of 45 minutes
9.6	**Mathematical Induction**	2 sessions of 45 minutes

Unit 9 test

Time: 60 minutes. A scientific calculator is allowed, and you will need one for problems 8, 13 and 14. Show your work for every problem: write the formula you use and the values you substitute. Give exact answers unless a problem asks you to

round; round money to the nearest cent unless told otherwise. Write proofs in complete sentences, one step per line.

1. **5 pts**

   Write the first five terms of the sequence $a_1 = 3$, $a_n = 3a_{n-1} - 2$.

2. **5 pts**

   Evaluate $\sum_{k=2}^{5} (k^2 + 1)$.

3. **6 pts**

   Write an explicit rule for the sequence $-2, 4, -8, 16, \ldots$.

4. **6 pts**

   In an arithmetic sequence, $a_3 = 14$ and $a_{10} = 49$. Find the common difference d, the first term a_1, and a_{50}.

5. **7 pts**

   A rancher building a new pasture fence puts up 120 feet of fence on the first day, and each day after that puts up 15 feet more than the day before. (a) How many feet does the rancher put up on day 12? (b) How many feet in all are built in the first 12 days?

6. **6 pts**

   Find the sum of all the even numbers from 50 to 150, including 50 and 150.

7. **6 pts**

   A geometric sequence has $a_1 = 2$ and $r = 3$. Find a_6 and the sum of the first 6 terms.

8. **7 pts**

   A family bakery has sales of \$50,000 in its first year, and its sales grow by 10% each year after that. Find the total sales over the first 8 years, to the nearest dollar.

9. **5 pts**

   Insert two geometric means between 3 and 192.

10. **6 pts**

(a) Find the sum of the infinite geometric series $45 + 15 + 5 + \cdots$. (b) Does the infinite series $0.5 - 1 + 2 - 4 + \cdots$ have a sum? Explain.

11. **5 pts**

Write $0.\overline{27} = 0.272727\ldots$ as a fraction in lowest terms.

12. **6 pts**

A church building-fund drive raises \$800 in its first week, and each week after that the gifts are $\dfrac{3}{4}$ of the previous week's. (a) If this pattern continued without end, what is the most the drive could raise in all? (b) Can the drive reach a goal of \$3500 this way?

13. **7 pts**

Henry deposits \$150 at the end of each month for 4 years in an account paying 3% interest compounded monthly, to buy a used car. (a) How much will he have after the last deposit? (b) How much of it is interest?

14. **8 pts**

Ruth borrows \$12,000 for a pickup truck for her landscaping business at 6% interest compounded monthly, repaid in equal monthly payments over 4 years. (a) Find the monthly payment. (b) Find the total interest she pays.

15. **10 pts**

Prove by mathematical induction that for every counting number n, $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

16. **5 pts**

The expression $n^2 - n + 11$ is prime for $n = 1, 2, \ldots, 10$. (a) Explain why this does not prove that it is prime for every counting number n. (b) Find the first counting number n for which it is not prime.

ANSWER KEY

1. 3, 7, 19, 55, 163
2. $5 + 10 + 17 + 26 = 58$
3. $a_n = (-2)^n$ (check: $a_1 = -2$, $a_4 = 16$ ✓)
4. $7d = 35$, so $d = 5$; $a_1 = 14 - 2(5) = 4$; $a_{50} = 4 + 49(5) = 249$
5. (a) $120 + 11(15) = 285$ feet; (b) $\dfrac{12(120 + 285)}{2} = 2430$ feet
6. $n = \dfrac{150 - 50}{2} + 1 = 51$ terms; sum $= \dfrac{51(50 + 150)}{2} = 5100$
7. $a_6 = 2 \cdot 3^5 = 486$; $S_6 = \dfrac{2(3^6 - 1)}{3 - 1} = 728$
8. $r = 1.1$: $S_8 = \dfrac{50{,}000(1.1^8 - 1)}{0.1} \approx \$571{,}794$
9. $r^3 = \dfrac{192}{3} = 64$, $r = 4$: the means are 12 and 48
10. (a) $r = \dfrac{1}{3}$: $\dfrac{45}{1 - \frac{1}{3}} = 67.5$; (b) no: $r = -2$ and $|r| \geq 1$, so the partial sums never settle
11. $\dfrac{0.27}{1 - 0.01} = \dfrac{27}{99} = \dfrac{3}{11}$
12. (a) $\dfrac{800}{1 - \frac{3}{4}} = \$3200$; (b) no, the total can never exceed \$3200
13. (a) $i = 0.0025$, $n = 48$: $150 \cdot \dfrac{1.0025^{48} - 1}{0.0025} \approx \$7639.68$; (b) $7639.68 - 7200 = \$439.68$
14. (a) $M = \dfrac{12{,}000(0.005)}{1 - 1.005^{-48}} \approx \$281.82$; (b) $48(281.82) - 12{,}000 = \$1527.36$
15. Base case: for $n = 1$ the left side is $1 \cdot 2 = 2$ and the right side is $\dfrac{1 \cdot 2 \cdot 3}{3} = 2$. Inductive hypothesis: assume $1 \cdot 2 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$. Adding the next term $(k+1)(k+2)$: $\dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \dfrac{(k+1)(k+2)(k+3)}{3}$, which is the formula with $k+1$ in place of n. By induction the formula holds for every counting number n.
16. (a) Checking cases, however many, leaves infinitely many counting numbers unchecked; a proof must cover every n (for example, by induction). (b) $n = 11$: $121 - 11 + 11 = 121 = 11 \times 11$

9.1 Sequences and Summation Notation

OBJECTIVES

- Describe a sequence as a function whose domain is the counting numbers, and find its terms from an explicit or a recursive rule.
- Write an explicit or a recursive rule that fits the first several terms of a sequence.
- Read, write and evaluate sums written in sigma notation.
- Use the properties of sums to evaluate a sum quickly and explain why each property is true.

WATCH

Introduction to Series (1 of 2: Interpreting Sigma Notation)

Eddie Woo · 6:03

https://www.youtube.com/watch?v=O8vla0xnAfg

> **WHEN TO WATCH**
> Before the sigma-notation part of the Core ideas.

Note: Eddie Woo teaches in Australia: he says 'series' for the sum of terms, as this course does.

BACKUP VIDEO

3 Ways to Define a Sequence

Eddie Woo · 8:02

https://www.youtube.com/watch?v=HdJ5RjIxdik

Before Example 1: explicit and recursive rules for a sequence.

KEY TERMS

sequence

A list of numbers in a definite order; formally, a function whose inputs are the counting numbers 1, 2, 3, ... The output for input n is the nth term, written a_n.

explicit rule

A formula that gives a_n directly from n, such as $a_n = 3n - 1$.

recursive rule

A rule that gives the first term (or first few terms) and tells how to find each later term from the terms before it, such as $a_1 = 2$, $a_n = a_{n-1} + 3$.

sigma (summation) notation

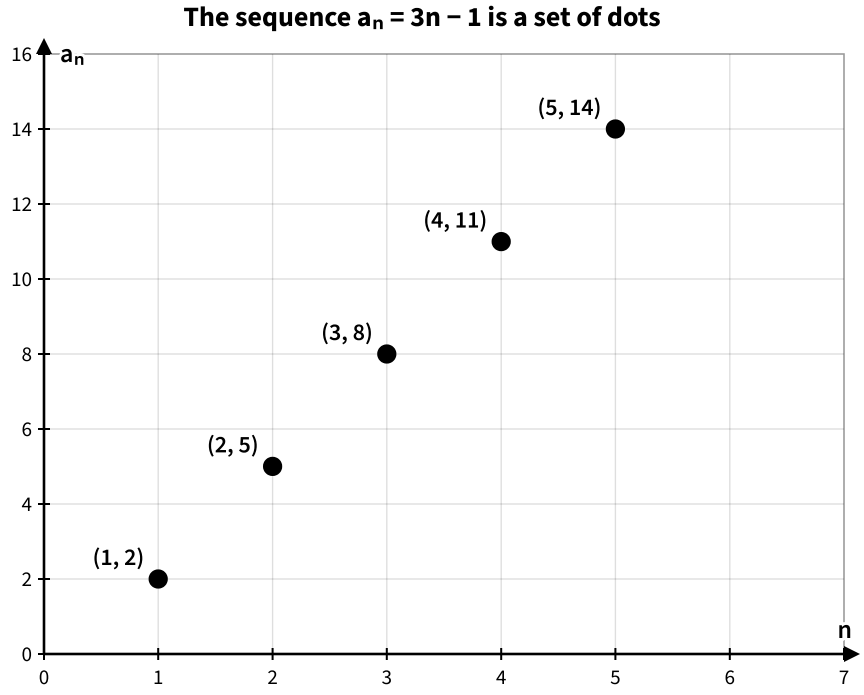
A short way to write a sum using the Greek capital letter sigma, $\sum$. In $\sum_{k=1}^{5} a_k$, k is the index, 1 is the lower limit and 5 is the upper limit.

series

The sum of the terms of a sequence, such as $2 + 5 + 8 + 11$.

CORE IDEAS

A **sequence** is a list of numbers in order: 2, 5, 8, 11, ... You can think of it as a function whose inputs are only the counting numbers 1, 2, 3, ... Instead of writing $f(n)$ we usually write a_n (read "a sub n") for the nth term. For $a_n = 3n - 1$, the terms are $a_1 = 2$, $a_2 = 5$, $a_3 = 8$, and so on. Because the inputs are whole numbers only, the graph of a sequence is a set of separate dots, not a connected line.



There are two common ways to describe a sequence. An **explicit rule** gives any term directly: to find a_{100} for $a_n = 3n - 1$, substitute 100 and get 299. A **recursive rule**

gives a starting term and a step: $a_1 = 2$, $a_n = a_{n-1} + 3$ says "start at 2, and each term is 3 more than the one before." Both rules describe the same sequence. A recursive rule matches a process that repeats (each week the herd grows, each day the tank refills), while an explicit rule is faster for far-off terms. A recursive rule is not complete without its starting value: "add 3 each time" alone could start anywhere.

To write a rule from terms, look at how the terms change. A constant difference (4, 7, 10, 13 goes up by 3) suggests a rule of the form $a_n = 3n + b$; then choose b so a_1 is right: $3(1) + b = 4$ gives $b = 1$. A constant ratio suggests powers, such as 2^n. Compare with familiar patterns (n^2, 2^n, $\frac{1}{n}$), and use $(-1)^n$ for signs that alternate (negative when n is odd). For $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ the numerator is n and the denominator is one more, so $a_n = \frac{n}{n+1}$. Always test your rule on every term you were given.

The **Fibonacci sequence** from Algebra I is the most famous recursive sequence: $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, giving 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... It needs two starting terms because each new term uses the two before it.

Adding the terms of a sequence gives a **series**. **Sigma notation** writes a long sum compactly: $\sum_{k=1}^{5}(2k+1)$ means "let k take the values 1, 2, 3, 4, 5 in the expression $2k + 1$, and add the results": $3 + 5 + 7 + 9 + 11 = 35$. The letter k is the **index**; the numbers below and above the sigma are the **lower and upper limits**. A sum from $k = m$ to $k = n$ has $n - m + 1$ terms. The index is only a placeholder: $\sum_{j=1}^{5}(2j+1)$ is the same sum.

Three properties make sums easier, and each one is just ordinary arithmetic written in sigma form. (1) $\sum_{k=1}^{n} c = nc$, because you are adding the same number c a total of n times. (2) $\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$, because $ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$ by the distributive property. (3) $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$, because addition can be done in any order, so you may add all the a's, then all the b's. For example, $\sum_{k=1}^{5}(2k+1) = 2(1+2+3+4+5) + 5(1) = 2(15) + 5 = 35$, matching the direct sum. There is no such rule for products: $\sum a_k b_k$ is generally not $(\sum a_k)\,(\sum b_k)$.

KEY FORMULAS

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Practice habit. Before evaluating any sum in sigma notation, write out the first three terms and the last term. This shows you the pattern and the number of terms (upper limit − lower limit + 1), and it catches off-by-one errors.

WORKED EXAMPLES

Example 1. Evaluate $\sum_{k=1}^{5}(3k-2)$ in two ways: by writing out the terms, and by using the properties of sums with $\sum_{k=1}^{5} k = 15$.

Show solution

1. Substitute $k = 1, 2, 3, 4, 5$ into $3k - 2$: the terms are 1, 4, 7, 10, 13.
2. Add: $1 + 4 + 7 + 10 + 13 = 35$.
3. Second way: split the sum. $\sum_{k=1}^{5}(3k-2) = 3\sum_{k=1}^{5} k - \sum_{k=1}^{5} 2$.
4. $3(15) - 5(2) = 45 - 10 = 35$.
5. Both ways give 35 ✓.

Answer: 35

Example 2. A stock tank for cattle holds 800 gallons of water at the start of day 1. Each day the cattle drink half of the water in the tank, and then the rancher pumps in 300 gallons. Let a_n be the number of gallons in the tank at the start of day n. (a) Write a recursive rule for a_n. (b) Find a_2, a_3, a_4 and a_5. (c) Write the total amount the cattle drink on days 1 through 4 in sigma notation, and evaluate it.

Show solution

1. (a) Start: $a_1 = 800$. Each day half is drunk, leaving $\frac{1}{2}a_{n-1}$, then 300 gallons are added: $a_n = \frac{1}{2}a_{n-1} + 300$.
2. (b) $a_2 = \frac{1}{2}(800) + 300 = 400 + 300 = 700$.
3. $a_3 = 350 + 300 = 650$, $a_4 = 325 + 300 = 625$, $a_5 = 312.5 + 300 = 612.5$.
4. (c) On day k the cattle drink half of a_k, so the total is $\sum_{k=1}^{4} \frac{1}{2}a_k = \frac{1}{2}\sum_{k=1}^{4} a_k$.
5. $\frac{1}{2}(800 + 700 + 650 + 625) = \frac{1}{2}(2775) = 1387.5$ gallons.
6. Check by adding the daily amounts: $400 + 350 + 325 + 312.5 = 1387.5$ ✓.
7. Notice the levels settle toward 600 gallons: if the level stayed at L, then $L = \frac{1}{2}L + 300$, so $L = 600$.

Answer: (a) $a_1 = 800$, $a_n = \frac{1}{2}a_{n-1} + 300$; (b) 700, 650, 625, 612.5 gallons; (c) $\sum_{k=1}^{4} \frac{1}{2}a_k = 1387.5$ gallons

PRACTICE

1. Easy Write the first four terms of the sequence $a_n = 5n - 2$.

   Show answer

   3, 8, 13, 18

2. Easy Write the first four terms of the sequence $a_n = (-1)^n n^2$.

   Show answer

   $-1, 4, -9, 16$

3. **Easy** Write the first five terms of the sequence $a_1 = 4$, $a_n = 2a_{n-1} + 1$.

Show answer

4, 9, 19, 39, 79 (for example, $a_2 = 2(4) + 1 = 9$)

4. **Easy** Evaluate $\sum_{k=1}^{4} k^2$.

Show answer

$1 + 4 + 9 + 16 = 30$

5. **Medium** The table shows the first five terms of a sequence. Write an explicit rule for a_n.

n	a_n
1	4
2	7
3	10
4	13
5	16

Show answer

The terms go up by 3, so $a_n = 3n + b$ with $3(1) + b = 4$: $a_n = 3n + 1$ (check: $a_5 = 16$ ✓)

6. **Medium** Write an explicit rule for the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Show answer

$$a_n = \frac{n}{n+1}$$

7. **Medium** (a) Write $3 + 6 + 9 + \cdots + 60$ in sigma notation. (b) Evaluate it, using the fact that $\sum_{k=1}^{20} k = 210$.

Show answer

(a) $\sum_{k=1}^{20} 3k$ (since $60 = 3 \times 20$); (b) $3(210) = 630$

8. **Medium** Given that $\sum_{k=1}^{10} k = 55$, use the properties of sums to evaluate $\sum_{k=1}^{10} (2k + 5)$.

Show answer

$2(55) + 10(5) = 110 + 50 = 160$

9. **Medium** A male honeybee has one parent (a female), while a female bee has two parents (a male and a female). Counting back, a male bee has 1 parent, 2 grandparents, 3 great-grandparents, and then 5, 8, … ancestors in each earlier generation: each count is the sum of the two counts before it. (a) How many ancestors does he have in the 10th generation back? (b) How many ancestors does he have in all, in generations 1 through 10?

Show answer

The counts are 1, 2, 3, 5, 8, 13, 21, 34, 55, 89. (a) 89; (b) $1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 = 231$

10. **Hard** Grace's bakery sells 40 loaves of bread on its first day open, and each day after that it sells 6 more loaves than the day before. (a) Write an explicit rule for a_n, the number of loaves sold on day n. (b) Write the total sold in the first 7 days in sigma notation and evaluate it.

Show answer

(a) $a_n = 40 + 6(n - 1) = 6n + 34$; (b) $\sum_{k=1}^{7} (6k + 34) = 6(28) + 7(34) = 168 + 238 = 406$ loaves

11. **Hard** Evaluate $\sum_{k=3}^{6} (k^2 - 2k)$.

Show answer

The terms for $k = 3, 4, 5, 6$ are 3, 8, 15, 24; the sum is 50

12. **Hard** A sequence is defined by $a_1 = 2$ and $a_n = a_{n-1} + 2n$. (a) Find a_2, a_3, a_4 and a_5. (b) Guess an explicit rule and check it on all five terms.

Show answer

(a) 6, 12, 20, 30; (b) $2 = 1 \cdot 2$, $6 = 2 \cdot 3$, $12 = 3 \cdot 4$, $20 = 4 \cdot 5$, $30 = 5 \cdot 6$, so $a_n = n(n + 1)$

13. Hard A grocer stacks oranges in a pyramid with square layers: 1 orange on top, a 2-by-2 layer under it, then 3-by-3, and so on, down to a 6-by-6 layer on the bottom. (a) Write the total number of oranges in sigma notation. (b) How many oranges are in the pyramid?

Show answer

(a) $\sum_{k=1}^{6} k^2$; (b) $1+4+9+16+25+36 = 91$ oranges

14. Challenge (a) Write out the four terms of $\sum_{k=1}^{4} \left(\frac{1}{k} - \frac{1}{k+1}\right)$, explain why most of the terms cancel, and find the sum. (b) Use the same idea to evaluate $\sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

Show answer

(a) $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$. Each subtracted fraction is added back in the next term, so only the first and last parts remain: $1 - \frac{1}{5} = \frac{4}{5}$. (b) Only $1 - \frac{1}{100}$ remains: $\frac{99}{100}$

COMMON MISTAKES

- Miscounting terms: $\sum_{k=3}^{10}$ has $10 - 3 + 1 = 8$ terms, not 10 or 7. Write out the first and last terms to be sure.
- Treating a constant term as a single number: $\sum_{k=1}^{6} 4 = 6 \times 4 = 24$, not 4, because the 4 is added once for each value of k.
- Writing a recursive rule with no starting value. "Each term is double the one before" is not a complete rule until you also give a_1.
- Getting the sign of $(-1)^n$ backwards: for $n = 1$ it is -1, so the first term is negative. Use $(-1)^{n+1}$ if the first term should be positive.

> **Make it a game · Chain Versus Jump.** Take turns calling terms of a sequence from its recursive rule, such as $a_1 = 3$ and $a_n = a_{n-1} + 4$, until the eighth term. Then the parent asks for the 25th term, and you must jump there with an explicit rule instead. Next, write the sigma notation for the sum of the first four terms and expand it.

9.2 Arithmetic Sequences and Series

OBJECTIVES

- Write the explicit rule $a_n = a_1 + (n-1)d$ for an arithmetic sequence and use it to find a term, the common difference or the number of terms.
- Find arithmetic means between two numbers.
- Derive the sum formula $S_n = \dfrac{n(a_1 + a_n)}{2}$ by adding the series forward and backward, and use it to find sums.
- Solve problems about rows of seats, stacks of logs, falling objects and steadily growing savings.

WATCH

Sum of an Arithmetic Progression (1 of 5: Visual derivation of formula)

Eddie Woo · 6:56

https://www.youtube.com/watch?v=ZZaUO_35hdk

> **WHEN TO WATCH**
> Before Example 2: the arithmetic series sum formula, shown visually.

Note: Eddie Woo teaches in Australia: he says 'arithmetic progression (AP)' for arithmetic sequence.

BACKUP VIDEO

Intro to Arithmetic Progressions (1 of 3)

Eddie Woo · 11:14

https://www.youtube.com/watch?v=0KReu2FKIto

Before Example 1: arithmetic sequences and the nth-term formula.

KEY TERMS

arithmetic sequence

A sequence in which each term is found by adding the same number to the term before it, such as 5, 8, 11, 14, ….

common difference

The fixed number d added to each term of an arithmetic sequence to get the next: $d = a_n - a_{n-1}$. It may be negative.

arithmetic means

The terms that fill in between two given terms so that all of them together form an arithmetic sequence. A single arithmetic mean of a and b is their average $\frac{a+b}{2}$.

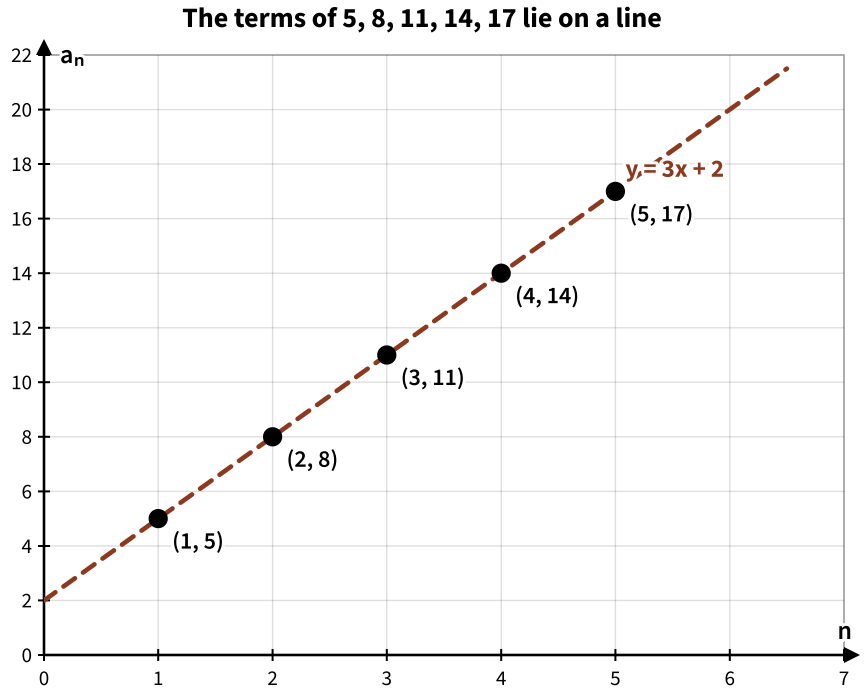
arithmetic series

The sum of the terms of an arithmetic sequence. S_n stands for the sum of the first n terms.

CORE IDEAS

An **arithmetic sequence** grows (or shrinks) by the same amount at every step. In 5, 8, 11, 14, ... each term is 3 more than the one before, so the **common difference** is $d = 3$. Recursively, $a_n = a_{n-1} + d$. In Algebra I you wrote explicit rules for these sequences; here is why the rule has the form it does.

To get from a_1 to a_n you take $n - 1$ steps (from term 1 to term 2 is one step, from term 1 to term 3 is two steps), and each step adds d. So $a_n = a_1 + (n-1)d$. For 5, 8, 11, ..., $a_n = 5 + 3(n-1) = 3n + 2$. This is a linear function of n with slope d, so the dots of an arithmetic sequence always lie on a line.



The rule $a_n = a_1 + (n-1)d$ links four quantities, so if you know three you can find the fourth. If you know two terms, the difference between them is a whole number of steps: from a_5 to a_{12} is 7 steps, so $d = \frac{a_{12} - a_5}{7}$. **Arithmetic means** fill the gap

between two numbers evenly. To place three means between 10 and 34, notice that 10 and 34 would be terms 1 and 5, which is 4 steps, so $d = \dfrac{34 - 10}{4} = 6$ and the means are 16, 22, 28. One mean between a and b is their average, since $a + d$ and $b - d$ must be equal.

Now add the terms. Write the sum of the first n terms forward, then backward: $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n$ and $S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1$. Add the two lines column by column. In every column the d's cancel and the column adds to $a_1 + a_n$. There are n columns, so $2S_n = n(a_1 + a_n)$, and $S_n = \dfrac{n(a_1 + a_n)}{2}$. In words: the sum is the number of terms times the average of the first and last terms.

If you know a_1 and d but not the last term, substitute $a_n = a_1 + (n - 1)d$ to get $S_n = \dfrac{n}{2}\,[2a_1 + (n - 1)d]$. The most important special case is the sum of the first n counting numbers: $a_1 = 1$ and $a_n = n$, so $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$. For example, $1 + 2 + \cdots + 100 = \dfrac{100 \times 101}{2} = 5050$.

Before using either formula, find n carefully. If a series runs from a first term to a last term in steps of d, the number of terms is $n = \dfrac{\text{last} - \text{first}}{d} + 1$. (From 6 to 24 in steps of 1 there are 19 numbers, not 18.)

KEY FORMULAS

$$a_n = a_1 + (n - 1)d$$

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}\,[2a_1 + (n - 1)d]$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

From history. A well-known story, perhaps embellished in the retelling, says that when Carl Friedrich Gauss (1777–1855) was a young schoolboy, his teacher told the class to add the whole numbers from 1 to 100. Gauss is said to have written 5050 almost at once by pairing 1 with 100, 2 with 99, and so on: 50 pairs, each adding to 101. Whether or not it happened just that way, the pairing idea is exactly the forward-and-backward argument in this lesson.

Practice habit. Know the sum of an arithmetic series in words: the number of terms times the average of the first and last terms. Always find the number of terms first, with $n = \dfrac{\text{last} - \text{first}}{d} + 1$.

WORKED EXAMPLES

Example 1. In an arithmetic sequence, $a_5 = 17$ and $a_{12} = 45$. Find the common difference d, the first term a_1, an explicit rule for a_n, and a_{30}.

Show solution

1. From a_5 to a_{12} is $12 - 5 = 7$ steps, so $7d = 45 - 17 = 28$ and $d = 4$.
2. From a_1 to a_5 is 4 steps: $a_1 = 17 - 4(4) = 1$.
3. $a_n = 1 + (n - 1)(4) = 4n - 3$.
4. Check: $a_5 = 20 - 3 = 17$ ✓ and $a_{12} = 48 - 3 = 45$ ✓.
5. $a_{30} = 4(30) - 3 = 117$.

Answer: $d = 4$, $a_1 = 1$, $a_n = 4n - 3$, $a_{30} = 117$

Example 2. Henry is saving for a used truck. He puts \$5 into savings in week 1, and each week after that he puts in \$2 more than the week before. (a) How much does he put in during week 26? (b) How much has he saved in all after 26 weeks? (c) After how many weeks will his total first reach at least \$1000?

Show solution

1. The weekly amounts form an arithmetic sequence with $a_1 = 5$ and $d = 2$.
2. (a) $a_{26} = 5 + 25(2) = 55$, so he puts in \$55.
3. (b) $S_{26} = \dfrac{26(5 + 55)}{2} = 13(60) = 780$, so he has saved \$780.
4. (c) In general $S_n = \dfrac{n}{2}\left[2(5) + (n - 1)(2)\right] = \dfrac{n}{2}(2n + 8) = n(n + 4)$.
5. Check the formula: $S_{26} = 26(30) = 780$ ✓.
6. Solve $n(n + 4) = 1000$: $n^2 + 4n - 1000 = 0$, so $n = -2 + \sqrt{1004} \approx 29.7$ (the negative root makes no sense).
7. Test whole numbers: $S_{29} = 29(33) = 957$, which is short; $S_{30} = 30(34) = 1020$, which is enough. So it takes 30 weeks.

Answer: (a) \$55; (b) \$780; (c) 30 weeks ($S_{29} = \$957$, $S_{30} = \$1020$)

PRACTICE

1. Easy The sequence 7, 11, 15, 19, … is arithmetic. Write an explicit rule for a_n and find a_{20}.

   Show answer

   $d = 4$, so $a_n = 7 + 4(n-1) = 4n + 3$; $a_{20} = 83$

2. Easy An arithmetic sequence has $a_1 = 50$ and $d = -6$. Find a_{15}.

   Show answer

   $a_{15} = 50 + 14(-6) = -34$

3. Easy Find the arithmetic mean of 18 and 46.

   Show answer

   $$\frac{18 + 46}{2} = 32$$

4. Easy Find the sum $1 + 2 + 3 + \cdots + 200$.

   Show answer

   $$\frac{200 \times 201}{2} = 20{,}100$$

5. Medium Which term of the arithmetic sequence 3, 10, 17, … is 220?

   Show answer

   $a_n = 3 + 7(n-1) = 7n - 4$; $7n - 4 = 220$ gives $n = 32$, so it is the 32nd term

6. Medium Insert three arithmetic means between 10 and 34.

   Show answer

   4 steps from 10 to 34, so $d = 6$: the means are 16, 22, 28

7. Medium Find the sum of the first 20 terms of the arithmetic series $2 + 9 + 16 + \cdots$.

   Show answer

   $a_{20} = 2 + 19(7) = 135$; $S_{20} = \dfrac{20(2 + 135)}{2} = 1370$

8. **Medium** A church's new sanctuary has curved rows of seats. The table shows the first four rows, and the pattern continues for 25 rows. (a) How many seats are in row 25? (b) How many seats are there in all?

Row	Seats
1	20
2	22
3	24
4	26

Show answer

(a) $a_{25} = 20 + 24(2) = 68$ seats; (b) $S_{25} = \dfrac{25(20 + 68)}{2} = 1100$ seats

9. **Medium** Evaluate $\sum_{k=1}^{40} (3k - 1)$.

Show answer

The first term is 2 and the last is 119, so the sum is $\dfrac{40(2 + 119)}{2} = 2420$

10. **Hard** Samuel stacks firewood logs in rows, with 24 logs on the bottom row, one fewer log in each row above, and 6 logs on the top row. (a) How many rows are there? (b) How many logs are in the stack?

Show answer

(a) $n = \dfrac{6 - 24}{-1} + 1 = 19$ rows; (b) $\dfrac{19(24 + 6)}{2} = 285$ logs

11. **Hard** In an arithmetic sequence, $a_4 = 10$ and $a_9 = -5$. Find a_1, d and the sum of the first 20 terms.

Show answer

$5d = -15$, so $d = -3$ and $a_1 = 10 + 3(3) = 19$; $a_{20} = 19 + 19(-3) = -38$, so $S_{20} = \dfrac{20(19 - 38)}{2} = -190$

12. Hard Find the sum of all the multiples of 7 between 100 and 300.

Show answer

The first is $105 = 7 \times 15$ and the last is $294 = 7 \times 42$, so there are $42 - 15 + 1 = 28$ terms; sum $= \dfrac{28(105 + 294)}{2} = 14(399) = 5586$

13. Hard A stone dropped from a high cliff falls 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on (ignore air resistance). (a) How far does it fall during the 10th second? (b) How far does it fall in the first 10 seconds in all?

Show answer

(a) $d = 32$, so $a_{10} = 16 + 9(32) = 304$ feet; (b) $\dfrac{10(16 + 304)}{2} = 1600$ feet, which matches $16t^2 = 16(10)^2$ ✓

14. Challenge How many terms of the arithmetic series $5 + 8 + 11 + \cdots$ must be added to get a sum of 1025?

Show answer

$S_n = \dfrac{n}{2}\left[10 + 3(n-1)\right] = \dfrac{n(3n+7)}{2} = 1025$, so $3n^2 + 7n - 2050 = 0$ and $n = \dfrac{-7 + 157}{6} = 25$ terms (check: $a_{25} = 77$ and $\dfrac{25(5 + 77)}{2} = 1025$ ✓)

15. Challenge Use the arithmetic sum formula to prove that the sum of the first n odd numbers is n^2: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Show answer

The odd numbers form an arithmetic sequence with $a_1 = 1$ and $d = 2$, so the nth odd number is $1 + 2(n-1) = 2n - 1$. Then $S_n = \dfrac{n(a_1 + a_n)}{2} = \dfrac{n(1 + 2n - 1)}{2} = \dfrac{n(2n)}{2} = n^2$.

COMMON MISTAKES

- Using n instead of $n - 1$ in $a_n = a_1 + (n-1)d$. The first term needs zero steps, so check your rule by substituting $n = 1$: it must give a_1.
- Miscounting the terms: the numbers 6 through 24 are 19 numbers, not 18. Use $n = \dfrac{\text{last} - \text{first}}{d} + 1$.
- Losing the sign of d when a sequence decreases. In 24, 23, 22, … the difference is $d = -1$.

- Using $S_n = \dfrac{n(a_1 + a_n)}{2}$ for a sequence that is not arithmetic. Check that the differences really are constant first.

> **Make it a game · Gauss Versus the Long Way.** Race the parent to add 4 + 7 + 10 + … + 40. One of you adds term by term. The other uses the explicit rule to count the terms, then pairs first with last. Compare the speed and the answer. Then insert three arithmetic means between 4 and 20, and state the common difference.

9.3 Geometric Sequences and Series

OBJECTIVES

- Write the explicit rule $a_n = a_1 r^{n-1}$ for a geometric sequence and use it to find a term, the common ratio or the number of terms.
- Find geometric means between two positive numbers.
- Derive the finite sum formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ by multiplying by r and subtracting, and use it to find sums.
- Model raises, doubling amounts, bouncing balls and ancestors with geometric sequences and series.

WATCH

Intro to Geometric Progressions (2 of 3: Algebraic derivation of sum formula)

Eddie Woo · 7:45

https://www.youtube.com/watch?v=2z248Y6Cvbc

> **WHEN TO WATCH**
> Before Example 1: deriving the sum of a geometric series.

Note: Eddie Woo teaches in Australia: he says 'geometric progression (GP)' for geometric sequence.

BACKUP VIDEO

Geometric Sequence Formula

Mario's Math Tutoring · 5:48

https://www.youtube.com/watch?v=3xbormMmuK4

Before the Core ideas, as a warm-up: the nth-term formula for a geometric sequence.

KEY TERMS

geometric sequence

A sequence in which each term is found by multiplying the term before it by the same nonzero number, such as 3, 6, 12, 24, ...

common ratio

The fixed number r that each term is multiplied by to get the next: $r = \dfrac{a_n}{a_{n-1}}$.

geometric means

The terms that fill in between two given terms so that all of them together form a geometric sequence. The single geometric mean of two positive numbers a and b is $\sqrt{ab}$.

geometric series

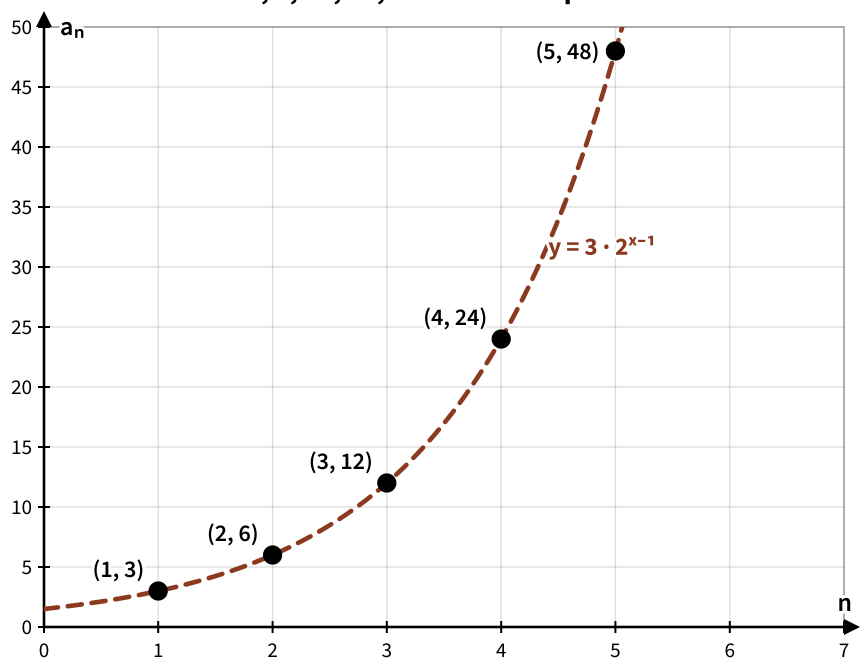
The sum of the terms of a geometric sequence. S_n stands for the sum of the first n terms.

CORE IDEAS

An arithmetic sequence adds the same amount at each step; a **geometric sequence** multiplies by the same amount. In 3, 6, 12, 24, ... each term is twice the one before, so the **common ratio** is $r = 2$. You find r by dividing any term by the term before it. Recursively, $a_n = r \cdot a_{n-1}$. The ratio can be a fraction (64, 32, 16, ... has $r = \frac{1}{2}$) or negative (3, −6, 12, −24, ... has $r = -2$, so the signs alternate).

To get from a_1 to a_n you take $n - 1$ steps, and each step multiplies by r, so $a_n = a_1 r^{n-1}$. For 3, 6, 12, ..., $a_n = 3 \cdot 2^{n-1}$. This is an exponential function of n, so the dots of a geometric sequence with a positive ratio other than 1 lie on an exponential curve, the discrete version of the growth and decay you studied in Units 7 and 8.

The terms of 3, 6, 12, 24, 48 lie on an exponential curve



If you know two terms, the steps between them tell you a power of r. If $a_2 = 12$ and $a_5 = 324$, there are 3 steps, so $r^3 = \frac{324}{12} = 27$ and $r = 3$. **Geometric means** fill a gap

the same way: to place two means between 5 and 135, use 3 steps, $r^3 = 27$, $r = 3$, giving 15 and 45. One geometric mean m between positive a and b must satisfy $\frac{m}{a} = \frac{b}{m}$ (the same ratio twice), so $m^2 = ab$ and $m = \sqrt{ab}$.

Now add the terms. Write $S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$. Multiply every term by r: $rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + a_1 r^n$. The two lists are the same except for the first term of S_n and the last term of rS_n, so subtracting wipes out everything in the middle: $S_n - rS_n = a_1 - a_1 r^n$. Factor: $S_n(1 - r) = a_1(1 - r^n)$, so $S_n = \frac{a_1(1 - r^n)}{1 - r}$ whenever $r \neq 1$. (When $r = 1$ every term equals a_1 and the sum is simply na_1.) When $r > 1$ it is often easier to use the equivalent form $S_n = \frac{a_1(r^n - 1)}{r - 1}$, found by multiplying top and bottom by -1.

Percentage changes are geometric. A 3% raise multiplies the salary by $r = 1.03$ each year (the old salary plus 3% of it); a ball that bounces back to 75% of its height has $r = 0.75$. Each person has 2 parents, 4 grandparents, 8 great-grandparents, ..., so if no one appears twice in the family tree, the number of ancestors n generations back is 2^n, and the total over n generations is a geometric series.

An old legend tells of a wise man who asked a king for one grain of wheat on the first square of a chessboard, two on the second, four on the third, doubling on each of the 64 squares. The king agreed, thinking it a small reward. The total is $1 + 2 + 4 + \cdots + 2^{63} = \frac{1(2^{64} - 1)}{2 - 1} = 2^{64} - 1$, about 1.8×10^{19} grains, far more wheat than the whole world grows in many centuries. Whether or not the story ever happened, it shows how quickly repeated doubling grows.

KEY FORMULAS

$$a_n = a_1 r^{n-1}$$

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1} \qquad (r \neq 1)$$

geometric mean of positive a and $b = \sqrt{ab}$

Practice habit. Know the powers of 2 up to $2^{10} = 1024$ and the powers of 3 up to $3^5 = 243$ cold. They let you spot geometric sequences at a glance and solve equations like $2^n = 1024$ by hand.

Example 1. For the geometric sequence 3, −6, 12, −24, …, find a_8 and the sum of the first 8 terms.

Show solution

1. $r = \dfrac{-6}{3} = -2$.
2. $a_8 = 3(-2)^7 = 3(-128) = -384$.
3. $S_8 = \dfrac{3\left(1-(-2)^8\right)}{1-(-2)} = \dfrac{3(1-256)}{3} = -255$.
4. Check by adding in pairs: $(3-6)+(12-24)+(48-96)+(192-384) = -3-12-48-192 = -255$ ✓.

Answer: $a_8 = -384$, $S_8 = -255$

Example 2. Ruth takes a job as a farm manager. The table shows her salary for the first three years; the pattern continues. (a) What is her salary in year 10, to the nearest cent? (b) What is the total she earns over the first 10 years, to the nearest cent?

Year	Salary (dollars)
1	40000
2	41200
3	42436

Show solution

1. Test for a common ratio: $\dfrac{41{,}200}{40{,}000} = 1.03$ and $\dfrac{42{,}436}{41{,}200} = 1.03$. It is geometric with $r = 1.03$ (a 3% raise each year).
2. (a) $a_{10} = 40{,}000(1.03)^{9} \approx 40{,}000(1.3047732) \approx 52{,}190.93$.
3. (b) $S_{10} = \dfrac{40{,}000\left(1.03^{10} - 1\right)}{1.03 - 1}$.
4. $1.03^{10} \approx 1.3439164$, so $S_{10} \approx \dfrac{40{,}000(0.3439164)}{0.03} \approx 458{,}555.17$.
5. Keep 1.03^{10} in the calculator until the last step so the cents come out right.
6. Reasonableness check: 10 years at an average of roughly \$46,000 is about \$460,000 ✓.

Answer: (a) \$52,190.93; (b) \$458,555.17

PRACTICE

1. Easy The sequence 5, 15, 45, 135, … is geometric. Write an explicit rule for a_n and find a_7.

   Show answer

   $r = 3$, so $a_n = 5 \cdot 3^{n-1}$; $a_7 = 5 \cdot 3^{6} = 5(729) = 3645$

2. **Easy** A geometric sequence has $a_1 = 64$ and $r = \dfrac{1}{2}$. Find a_7.

   Show answer

   $$a_7 = 64\left(\frac{1}{2}\right)^{6} = \frac{64}{64} = 1$$

3. **Easy** Find the geometric mean of 4 and 25.

   Show answer

   $\sqrt{4 \times 25} = \sqrt{100} = 10$ (4, 10, 25 has ratio 2.5)

4. **Easy** Is the sequence 2, 6, 18, 36 geometric? Explain.

   Show answer

   No: $\dfrac{6}{2} = 3$ and $\dfrac{18}{6} = 3$, but $\dfrac{36}{18} = 2$, so the ratio is not constant

5. **Medium** In a geometric sequence, $a_2 = 12$ and $a_5 = 324$. Find r and a_1.

   Show answer

   3 steps: $r^3 = \dfrac{324}{12} = 27$, so $r = 3$ and $a_1 = \dfrac{12}{3} = 4$

6. **Medium** Insert two geometric means between 5 and 135.

   Show answer

   $r^3 = 27$, $r = 3$: the means are 15 and 45

7. **Medium** Find the sum of the first 6 terms of the geometric series $2 + 10 + 50 + \cdots$.

   Show answer

   $$S_6 = \frac{2(5^6 - 1)}{5 - 1} = \frac{2(15{,}624)}{4} = 7812$$

8. **Medium** Evaluate $\sum_{k=1}^{8} 3 \cdot 2^{k-1}$.

   Show answer

   $a_1 = 3$, $r = 2$, $n = 8$: $\dfrac{3(2^8 - 1)}{2 - 1} = 3(255) = 765$

9. Medium For 15 days of extra chores, Samuel's father offers him a "doubling" payment: 1 cent on day 1, 2 cents on day 2, 4 cents on day 3, and so on. (a) How much is he paid on day 15? (b) How much is he paid in all? Give both answers in dollars.

Show answer

(a) $2^{14} = 16{,}384$ cents $= \$163.84$; (b) $2^{15} - 1 = 32{,}767$ cents $= \$327.67$

10. Hard A ball is dropped from a height of 12 feet. Each time it bounces, it rises to 75% of the height from which it fell. (a) How high does it rise on the 5th bounce, to the nearest tenth of a foot? (b) What is the total of the heights of the first 5 bounces, to the nearest tenth of a foot?

Show answer

The rebound heights are $9, 6.75, \ldots$ with $r = 0.75$. (a) $12(0.75)^5 \approx 2.8$ feet; (b) $\dfrac{9(1 - 0.75^5)}{1 - 0.75} \approx 27.5$ feet

11. Hard You have 2 parents, 4 grandparents, 8 great-grandparents, and so on (assuming no one appears twice in the family tree). How many ancestors in all are there in the 10 generations before you?

Show answer

$$2 + 4 + \cdots + 2^{10} = \frac{2(2^{10} - 1)}{2 - 1} = 2(1023) = 2046$$

12. Hard How many terms of the geometric series $3 + 6 + 12 + \cdots$ must be added to get a sum of 3069?

Show answer

$3(2^n - 1) = 3069$, so $2^n - 1 = 1023$, $2^n = 1024$, $n = 10$

13. Hard Evaluate $\sum_{k=1}^{6} 81\left(\dfrac{1}{3}\right)^{k-1}$ exactly.

Show answer

$$\frac{81\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}} = \frac{364}{3} \text{ (check: } 81 + 27 + 9 + 3 + 1 + \frac{1}{3} = 121\frac{1}{3} \text{ ✓)}$$

14. **Challenge** In the chessboard legend, the squares hold 1, 2, 4, 8, … grains of wheat, doubling across all 64 squares. (a) Use the geometric sum formula to show that the total number of grains is $2^{64} - 1$. (b) Using $2^{10} \approx 1000$, estimate the total in scientific notation.

Show answer

(a) $a_1 = 1$, $r = 2$, $n = 64$: $S_{64} = \dfrac{1(2^{64} - 1)}{2 - 1} = 2^{64} - 1$. (b) $2^{64} = 2^4 \cdot (2^{10})^6 \approx 16 \times 10^{18} = 1.6 \times 10^{19}$ grains (the exact value is about 1.84×10^{19})

15. **Challenge** Explain why the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ cannot be used when $r = 1$, and find S_n in that case.

Show answer

When $r = 1$ the denominator $1 - r$ is 0, and the step $S_n(1 - r) = a_1(1 - r^n)$ becomes $0 = 0$, which says nothing about S_n. But with $r = 1$ every term equals a_1, so $S_n = a_1 + a_1 + \cdots + a_1 = n a_1$.

COMMON MISTAKES

- Using r^n instead of r^{n-1} in the explicit rule. Check by substituting $n = 1$: the rule must give a_1.
- Using the percent instead of the multiplier: a 3% raise means $r = 1.03$, not $r = 0.03$, and a 20% loss means $r = 0.8$.
- Dropping parentheses with a negative ratio: $3(-2)^7 = -384$, but $3 \cdot -2^7$ typed into a calculator may be read differently. Put negative ratios in parentheses.
- Rounding r or r^n too early in money problems. Keep the full calculator value until the final step, then round to the nearest cent.

> **Make it a game · Doubling Pennies.** Pretend you receive one penny on day 1, two on day 2, four on day 3, and so on. Both of you guess how many pennies you would get on day 20, then use $a_n = a_1 r^{n-1}$ to find the answer, 524,288. Next, find the two geometric means between 3 and 24, and write the explicit rule.

9.4 Infinite Geometric Series

OBJECTIVES

- Explain what it means for an infinite series to have a sum, using partial sums.
- Derive the sum $S = \dfrac{a_1}{1-r}$ of an infinite geometric series with $|r| < 1$, and recognize when a geometric series diverges.
- Write repeating decimals as fractions, and explain why $0.999\ldots = 1$.
- Solve problems about bouncing balls, swings and Zeno's paradox with infinite geometric series.

WATCH

Limiting Sum Applications (1 of 3: Derivation of formula)

Eddie Woo · 8:52

https://www.youtube.com/watch?v=KNTYP9rE2ZE

> **WHEN TO WATCH**
> Before Example 1: deriving the sum of an infinite geometric series.

Note: Eddie Woo teaches in Australia: he says 'limiting sum' for the sum of an infinite series.

BACKUP VIDEO

Limiting Sum Applications (2 of 3: Visual interpretation & recurring decimals)

Eddie Woo · 8:39

https://www.youtube.com/watch?v=WX2kFWLd69Y

Before the practice set: repeating decimals as infinite series.

KEY TERMS

infinite series

A sum with infinitely many terms, such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

partial sum

The sum S_n of the first n terms of an infinite series.

converge

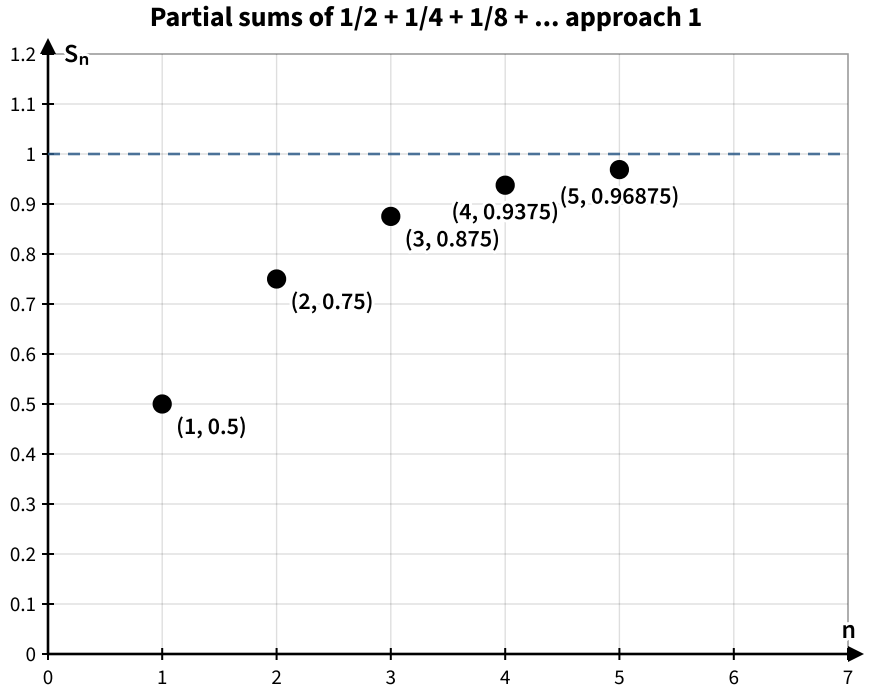
An infinite series converges if its partial sums get closer and closer to a single number S, as close as you like, as n grows. That number is called the sum of the series.

diverge

An infinite series diverges if its partial sums do not approach a single number; it has no sum.

CORE IDEAS

You cannot literally add infinitely many numbers one at a time, so we need to say what an infinite sum means. Add the terms in order and keep a running total: the **partial sums** $S_1, S_2, S_3, \ldots$. For $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ they are $0.5, 0.75, 0.875, 0.9375, \ldots$. Each partial sum is $1 - \left(\frac{1}{2}\right)^n$, so the gap to 1 is halved at every step and can be made as small as you like. When the partial sums close in on a single number like this, we say the series **converges** and call that number its sum: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. Picture walking toward a wall by covering half the remaining distance each step.



The same reasoning works for any geometric series. From Lesson 9.3, $S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1}{1 - r} r^n$. If $|r| < 1$, multiplying by r again and again shrinks r^n toward 0 (for example $0.5^{10} \approx 0.001$ and $0.5^{20} \approx 0.000001$). So the second part fades

away and the partial sums approach $\frac{a_1}{1-r}$. That is the sum of the infinite series: $S = \frac{a_1}{1-r}$ when $|r| < 1$.

If $|r| \geq 1$, the series **diverges**. With $r = 1$ the partial sums na_1 grow without bound. With $r = -1$ (as in $1 - 1 + 1 - 1 + \cdots$) the partial sums jump between 1 and 0 forever and never settle. With $|r| > 1$ the terms themselves grow larger, so the partial sums run off. The formula would give nonsense here: for $1 + 2 + 4 + \cdots$ it would say $\frac{1}{1-2} = -1$. So always check $|r| < 1$ first.

A repeating decimal is an infinite geometric series in disguise, which is why every repeating decimal is a fraction. For example, $0.\overline{36} = 0.36 + 0.0036 + 0.000036 + \cdots$ has $a_1 = 0.36$ and $r = 0.01$, so it equals $\frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$. In the same way, $0.999\ldots = 0.9 + 0.09 + 0.009 + \cdots = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$. This is not "almost 1": the partial sums in the table get within any distance of 1 you name, and the value of the infinite decimal is defined to be the number they approach. So $0.999\ldots$ and 1 are two names for the same number.

n	Partial sum S_n of 0.9 + 0.09 + 0.009 + …	Distance from 1
1	0.9	0.1
2	0.99	0.01
3	0.999	0.001
4	0.9999	0.0001
5	0.99999	0.00001

A bouncing ball gives a practical infinite series. If a ball is dropped from height h and each bounce rises to a fraction r of the height before, it travels down h once, and then for every bounce it travels up and back down the same height. The rebound heights $rh, r^2h, \ldots$ form a geometric series with sum $\frac{rh}{1-r}$, so the total distance is $h + 2 \cdot \frac{rh}{1-r}$. A real ball stops after a while, but the series gives an excellent estimate of the total distance it travels.

KEY FORMULAS

$$S = \sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r} \qquad \text{when } |r| < 1$$

If $|r| \geq 1$, the infinite geometric series diverges (has no sum).

$$0.\overline{9} = \frac{0.9}{1 - 0.1} = 1$$

From history. Zeno of Elea, a Greek thinker who lived around 450 BC, posed a famous paradox: swift Achilles can never catch a tortoise with a head start, because first he must reach the place where the tortoise was, and by then the tortoise has moved ahead, and so on without end. The answer is that the infinitely many stages take times that form a convergent geometric series, so their total is a finite time, after which Achilles is ahead (see practice problem 14).

Practice habit. Before writing $\frac{a_1}{1-r}$, check that $|r| < 1$ and say so. If $|r| \geq 1$, the answer is "no sum" no matter what the formula would give.

WORKED EXAMPLES

Example 1. Find the sum of the infinite geometric series $24 - 12 + 6 - 3 + \cdots$, or explain why it has no sum.

Show solution

1. $r = \dfrac{-12}{24} = -\dfrac{1}{2}$, and $\left|-\dfrac{1}{2}\right| < 1$, so the series converges.
2. $S = \dfrac{24}{1 - \left(-\frac{1}{2}\right)} = \dfrac{24}{\frac{3}{2}} = 16.$
3. Check with partial sums: 24, 12, 18, 15, 16.5, 15.75, ... They jump above and below 16 and close in on it ✓.

Answer: 16

Example 2. A rubber ball is dropped from a height of 20 feet onto a barn floor. Each time it bounces, it rises to 60% of the height from which it fell. Find the total vertical distance the ball travels before it comes to rest.

Show solution

1. The ball first falls 20 feet.
2. The rebound heights are $20(0.6) = 12$, then $12(0.6) = 7.2$, and so on: a geometric series with $a_1 = 12$ and $r = 0.6$.
3. Since $|0.6| < 1$, the rebound heights add to $\dfrac{12}{1 - 0.6} = \dfrac{12}{0.4} = 30$ feet.
4. Each rebound height is traveled twice, up and then down: $2(30) = 60$ feet.
5. Total: $20 + 60 = 80$ feet.

Answer: 80 feet

PRACTICE

1. Easy Find the sum of the infinite geometric series $6 + 3 + \dfrac{3}{2} + \cdots$.

   Show answer

   $r = \dfrac{1}{2}$: $S = \dfrac{6}{1 - \frac{1}{2}} = 12$

2. Easy Find the sum of the infinite geometric series $1 + \dfrac{1}{3} + \dfrac{1}{9} + \cdots$.

   Show answer

   $S = \dfrac{1}{1 - \frac{1}{3}} = \dfrac{3}{2}$

3. Easy Does the infinite geometric series $2 + 3 + 4.5 + \cdots$ have a sum? If so, find it.

   Show answer

   No sum: $r = 1.5$ and $|r| \geq 1$, so the series diverges

4. Easy Evaluate $\sum_{k=1}^{\infty} 5(0.2)^{k-1}$.

   Show answer

   $\dfrac{5}{1 - 0.2} = \dfrac{5}{0.8} = 6.25$

5. **Medium** Find the sum of the infinite geometric series $40 - 10 + 2.5 - \cdots$.

   Show answer

   $r = -\dfrac{1}{4}$: $S = \dfrac{40}{1.25} = 32$

6. **Medium** Write $0.777\ldots$ as a fraction in lowest terms by treating it as an infinite geometric series.

   Show answer

   $0.7 + 0.07 + \cdots = \dfrac{0.7}{1 - 0.1} = \dfrac{7}{9}$

7. **Medium** Write $0.\overline{45} = 0.454545\ldots$ as a fraction in lowest terms.

   Show answer

   $\dfrac{0.45}{1 - 0.01} = \dfrac{45}{99} = \dfrac{5}{11}$

8. **Medium** Evaluate $\sum_{k=1}^{\infty} \left(\dfrac{2}{3}\right)^{k}$.

   Show answer

   The first term is $\dfrac{2}{3}$ (not 1): $\dfrac{\frac{2}{3}}{1 - \frac{2}{3}} = 2$

9. **Medium** An infinite geometric series has first term 8 and sum 20. Find the common ratio.

   Show answer

   $\dfrac{8}{1 - r} = 20$, so $1 - r = 0.4$ and $r = 0.6$

10. **Hard** A ball is dropped from a height of 9 feet. Each bounce rises to $\dfrac{2}{3}$ of the height from which it fell. Find the total vertical distance it travels before coming to rest.

    Show answer

    Rebounds: $6 + 4 + \cdots = \dfrac{6}{1 - \frac{2}{3}} = 18$ feet; total $= 9 + 2(18) = 45$ feet

11. **Hard** Write $1.2333\ldots$ (the 3 repeats) as a fraction in lowest terms.

    Show answer

    $1.2 + (0.03 + 0.003 + \cdots) = \dfrac{6}{5} + \dfrac{0.03}{0.9} = \dfrac{6}{5} + \dfrac{1}{30} = \dfrac{37}{30}$

12. Hard Grace pulls her little brother back on a playground swing and lets go. On the first swing his seat travels along a 10-foot arc, and each later swing is 90% as long as the one before. About how far does the seat travel before the swing comes to rest?

Show answer

$$\frac{10}{1-0.9} = \frac{10}{0.1} = 100 \text{ feet}$$

13. Hard (a) For which values of x does the series $\sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = 1 + \frac{x}{3} + \frac{x^2}{9} + \cdots$ converge? Give the answer as an inequality. (b) Find its sum when $x = 1$.

Show answer

(a) $r = \frac{x}{3}$ and $\left|\frac{x}{3}\right| < 1$ gives $-3 < x < 3$; (b) $\frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$

14. Challenge Achilles runs at 10 meters per second and a tortoise crawls at 1 meter per second, with a 90-meter head start. Achilles takes 9 seconds to reach the tortoise's starting point; by then the tortoise has moved 9 meters, which takes Achilles 0.9 second to cover; and so on. (a) Add the times for all the stages to find when Achilles catches the tortoise. (b) How far has Achilles run at that moment? (c) Check your answer to (a) with the equation $10t = 90 + t$.

Show answer

(a) $9 + 0.9 + 0.09 + \cdots = \frac{9}{1 - 0.1} = 10$ seconds; (b) $10 \times 10 = 100$ meters; (c) $9t = 90$, so $t = 10$ ✓

15. Challenge A classmate says, "0.999 . . . is a tiny bit less than 1." Explain why $0.999\ldots = 1$ in two different ways.

Show answer

Way 1: $0.999\ldots = 0.9 + 0.09 + 0.009 + \cdots$ is geometric with $a_1 = 0.9$ and $r = 0.1$, so its sum is $\frac{0.9}{1 - 0.1} = 1$; the partial sums come within any distance of 1, and the infinite decimal names the number they approach. Way 2: $\frac{1}{3} = 0.333\ldots$, and multiplying by 3 gives $1 = 0.999\ldots$. (If 0.999... were less than 1, there would have to be a positive gap between them, but the gap is smaller than 0.1, 0.01, 0.001 and every other positive number, so it is 0.)

COMMON MISTAKES

- Using $\frac{a_1}{1-r}$ when $|r| \geq 1$. For $1 + 2 + 4 + \cdots$ the formula gives -1, which is nonsense; check $|r| < 1$ first.
- In bouncing-ball problems, forgetting that each rebound is traveled twice (up and down), or counting the first drop twice. The first drop happens once; every rebound height is doubled.
- Taking the wrong first term: $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ starts with $\frac{2}{3}$, not 1. Write out the first two terms before using the formula.
- With a decimal like $1.2333\ldots$, treating the whole decimal as one geometric series. Split off the part that does not repeat, then use the series only for the repeating part.

> **Make it a game · Strip Sum.** Take a paper strip as 1 unit. Tear off half, then half of what remains, and again, writing 1/2, 1/4, 1/8 and the partial sums. Say what total they are heading toward. Then apply S = $a_1/(1 - r)$ to confirm. Finally, decide whether 1 + 2 + 4 + 8 + … converges or diverges, and explain why.

9.5 Savings, Loans and Geometric Series

OBJECTIVES

- Explain why the value of regular deposits is a geometric series, and use $FV = P \cdot \dfrac{(1+i)^n - 1}{i}$ to plan savings.
- Derive the loan payment formula $M = \dfrac{L \cdot i}{1 - (1+i)^{-n}}$ and use it to find payments, total interest and how much can be borrowed.
- Build the first rows of an amortization table and explain how much of each payment is interest.
- Compare loans of different lengths and rates, including credit card debt, to make wise decisions about saving and borrowing.

WATCH

Annuities (1 of 2: Building the progression)

Eddie Woo · 12:20

https://www.youtube.com/watch?v=JnjNlEVEMFk

> **WHEN TO WATCH**
> Before Example 1: regular deposits as a geometric series.

Note: Eddie Woo teaches in Australia: he says 'annuity' and 'superannuation' (retirement savings); amounts are in dollars.

BACKUP VIDEO

Financial Series - Loan Repayments (2 of 3: How to determine the repayment)

Eddie Woo · 12:13

https://www.youtube.com/watch?v=NvbmRu6RhPU

Before Example 2: finding a loan repayment.

KEY TERMS

annuity

A series of equal payments or deposits made at equal time intervals, such as $300 at the end of every month.

future value

The total value of an account at a later date, including all deposits and all the interest they have earned.

periodic rate i

The interest rate for one period: the annual rate divided by the number of periods per year. At 6% compounded monthly, $i = \dfrac{0.06}{12} = 0.005$.

principal

The amount borrowed, or the part of the balance still owed, not counting interest.

amortization table

A table showing, for each loan payment, how much goes to interest, how much reduces the principal, and the balance that remains.

CORE IDEAS

Suppose you deposit the same amount P at the end of every period (month or year) into an account that earns interest at rate i per period, and you make n deposits. To find the **future value** right after the last deposit, ask how long each deposit has been in the account. The last deposit has earned nothing, so it is worth P. The one before it has earned one period of interest, so it is worth $P(1+i)$. The first deposit has earned interest for $n-1$ periods: $P(1+i)^{n-1}$. The table shows this for \$100 a year for 4 years at 5%.

Deposit	Years earning interest	Value at end of year 4 (dollars)
1st (end of year 1)	3	$100(1.05)^3 = 115.76$
2nd (end of year 2)	2	$100(1.05)^2 = 110.25$
3rd (end of year 3)	1	$100(1.05) = 105.00$
4th (end of year 4)	0	100.00
Total		431.01

Reading the deposits from last to first gives $FV = P + P(1+i) + P(1+i)^2 + \cdots + P(1+i)^{n-1}$, a geometric series with $a_1 = P$, ratio $r = 1+i$ and n terms. By the sum

formula from Lesson 9.3, $FV = \dfrac{P\left[(1+i)^n - 1\right]}{(1+i) - 1} = P \cdot \dfrac{(1+i)^n - 1}{i}$. Check with the table: $100 \cdot \dfrac{1.05^4 - 1}{0.05} = 100(4.310125) \approx \$431.01$ ✓.

A loan runs the other way. You borrow L, and each period the lender adds interest at rate i to what you owe, then subtracts your payment M. The balance after one payment is $L(1+i) - M$. After two payments it is $[L(1+i) - M](1+i) - M = L(1+i)^2 - M(1+i) - M$. After n payments it is $L(1+i)^n - M\left[(1+i)^{n-1} + \cdots + (1+i) + 1\right]$. The bracket is the same geometric series as before, so the balance after n payments is $L(1+i)^n - M \cdot \dfrac{(1+i)^n - 1}{i}$.

The loan is paid off when this balance is 0: $M \cdot \dfrac{(1+i)^n - 1}{i} = L(1+i)^n$. Solve for M: $M = \dfrac{L \cdot i\,(1+i)^n}{(1+i)^n - 1}$. Dividing the top and bottom by $(1+i)^n$ gives the usual form $M = \dfrac{L \cdot i}{1 - (1+i)^{-n}}$. In words: the payments, with interest, must exactly match what the loan would grow to if nothing were paid.

Each payment first covers the interest on the current balance, $i \times$ balance, and only the rest reduces the principal. Early in a loan the balance is large, so most of the payment is interest. As the balance falls, the interest part shrinks and the principal part grows. An **amortization table** tracks this row by row. Over the whole loan, the total interest is $nM - L$ (everything you pay, minus what you borrowed).

These formulas show why saving first and borrowing carefully matters. A longer loan has smaller payments but much more total interest: on a $200,000 home loan at 6%, a 30-year term costs over $100,000 more in interest than a 15-year term. A high-rate credit card with small payments can take years to pay off; if the payment is no more than $L \cdot i$, the interest alone uses it all and the balance never falls. When you save instead, the same geometric series works for you, whether you are saving for a tractor, a family car or a planned gift to your church.

KEY FORMULAS

$$i = \frac{\text{annual rate}}{\text{periods per year}}, \qquad n = \text{number of deposits or payments}$$

$$FV = P \cdot \frac{(1+i)^n - 1}{i}$$

$$M = \frac{L \cdot i}{1 - (1+i)^{-n}}$$

$$\text{total interest} = nM - L$$

Practice habit. Keep exact values until the last step: compute $(1+i)^n$ and leave it in the calculator, and round to the nearest cent only at the end. Rounding 1.005^{60} to 1.35 in Example 2 would make the payment \$385.71 instead of \$386.66.

WORKED EXAMPLES

Example 1. The Miller family is saving for a used tractor. They deposit \$300 at the end of each month for 5 years into an account paying 6% interest compounded monthly. (a) How much will the account hold after the last deposit? (b) How much of that is interest?

Show solution

1. $i = \dfrac{0.06}{12} = 0.005$ and $n = 12 \times 5 = 60$.
2. $FV = 300 \cdot \dfrac{1.005^{60} - 1}{0.005}$.
3. $1.005^{60} \approx 1.3488502$, so $\dfrac{0.3488502}{0.005} \approx 69.770031$.
4. (a) $FV \approx 300(69.770031) \approx 20{,}931.01$.
5. (b) They deposited $60 \times 300 = 18{,}000$, so the interest is $20{,}931.01 - 18{,}000 = 2931.01$.

Answer: (a) \$20,931.01; (b) \$2931.01 in interest

Example 2. Grace borrows \$20,000 for a family car at 6% interest compounded monthly, to be repaid in equal monthly payments over 5 years. (a) Find the monthly payment. (b) Find the total interest. (c) The table shows the first three rows of the amortization table. Explain how the first row is found.

Payment	Interest (dollars)	Principal (dollars)	Balance after payment (dollars)
1	100.00	286.66	19,713.34
2	98.57	288.09	19,425.25
3	97.13	289.53	19,135.72

Show solution

1. $L = 20{,}000$, $i = 0.005$, $n = 60$.
2. (a) $M = \dfrac{20{,}000(0.005)}{1 - 1.005^{-60}} = \dfrac{100}{1 - 0.7413722} = \dfrac{100}{0.2586278} \approx 386.66.$
3. (b) Total paid: $60 \times 386.66 = 23{,}199.60$. Total interest: $23{,}199.60 - 20{,}000 = 3199.60$.
4. (c) Interest for month 1: $20{,}000 \times 0.005 = 100.00$.
5. Principal repaid: $386.66 - 100.00 = 286.66$.
6. New balance: $20{,}000 - 286.66 = 19{,}713.34$. Month 2 repeats the steps with this balance: $19{,}713.34 \times 0.005 \approx 98.57$.
7. (Because the payment is rounded to the cent, the very last payment is adjusted by a few cents.)

Answer: (a) \$386.66 per month; (b) \$3199.60; (c) interest $20{,}000(0.005) = \$100$, principal $386.66 - 100 = \$286.66$, new balance \$19,713.34

PRACTICE

1. Easy A savings plan pays 4.8% annual interest compounded monthly, with a deposit at the end of each month for 3 years. Find the periodic rate i and the number of deposits n.

   Show answer

   $$i = \frac{0.048}{12} = 0.004,\ n = 36$$

2. **Easy** Henry deposits \$200 at the end of each year for 3 years in an account paying 6% compounded annually. Write the value of each deposit at the end of year 3, and add them to find the future value.

   Show answer

   $200(1.06)^2 + 200(1.06) + 200 = 224.72 + 212 + 200 = \$636.72$ (the formula gives $200 \cdot \dfrac{1.06^3 - 1}{0.06} = 636.72$ ✓)

3. **Easy** A loan has a balance of \$15,000 at 6% annual interest compounded monthly. How much interest is charged for the first month?

   Show answer

   $15{,}000 \times 0.005 = \$75$

4. **Easy** Samuel borrows \$10,800 and repays it with 48 monthly payments of \$250. Find the total he pays and the total interest.

   Show answer

   Total paid: $48 \times 250 = \$12{,}000$; interest: $12{,}000 - 10{,}800 = \$1200$

5. **Medium** A church sets aside \$2000 at the end of each year for 10 years toward a new roof, in an account paying 4% compounded annually. How much will be in the account after the 10th deposit, to the nearest cent?

   Show answer

   $2000 \cdot \dfrac{1.04^{10} - 1}{0.04} \approx 2000(12.006107) \approx \$24{,}012.21$

6. **Medium** A farmer deposits \$500 at the end of each month for 5 years, at 6% compounded monthly, to buy a new tractor. (a) Find the future value. (b) How much of it is interest?

   Show answer

   (a) $500 \cdot \dfrac{1.005^{60} - 1}{0.005} \approx \$34{,}885.02$; (b) $34{,}885.02 - 30{,}000 = \$4885.02$

7. **Medium** The Miller family wants \$15,000 for a car in 5 years. What equal deposit at the end of each month, at 6% compounded monthly, will reach this goal? Round to the nearest cent.

   Show answer

   $$P = \frac{15{,}000}{(1.005^{60} - 1)/0.005} = \frac{15{,}000}{69.770031} \approx \$214.99$$

8. Medium Ruth borrows \$10,000 for a delivery van for her bakery at 12% interest compounded monthly, repaid monthly over 3 years. (a) Find the monthly payment. (b) Find the total interest.

Show answer

(a) $i = 0.01$, $n = 36$: $M = \dfrac{10{,}000(0.01)}{1 - 1.01^{-36}} \approx \$332.14$; (b) $36(332.14) - 10{,}000 = \$1957.04$

9. Medium A loan of \$8000 at 6% compounded monthly has payments of \$200 per month. For (a) month 1 and (b) month 2, find the interest, the principal repaid and the new balance.

Show answer

(a) interest $8000(0.005) = \$40$, principal \$160, balance \$7840; (b) interest $7840(0.005) = \$39.20$, principal \$160.80, balance \$7679.20

10. Hard A family borrows \$200,000 for a home at 6% compounded monthly. Find (a) the monthly payment for a 30-year loan, (b) the monthly payment for a 15-year loan, (c) the total interest for the 30-year loan, and (d) the total interest for the 15-year loan.

Show answer

(a) $n = 360$: $M \approx \$1199.10$; (b) $n = 180$: $M \approx \$1687.71$; (c) $360(1199.10) - 200{,}000 = \$231{,}676.00$; (d) $180(1687.71) - 200{,}000 = \$103{,}787.80$. The shorter loan saves \$127,888.20 in interest.

11. Hard For the 30-year loan in problem 10 (payment \$1199.10), how much of the first payment is interest, and how much reduces the principal?

Show answer

Interest: $200{,}000 \times 0.005 = \$1000$; principal: $1199.10 - 1000 = \$199.10$ (more than 83% of the first payment is interest)

12. Hard Samuel can afford a payment of \$500 per month for 5 years. At 6% compounded monthly, what is the largest loan he can repay? Round to the nearest cent.

Show answer

Solve the payment formula for L: $L = \dfrac{M\left(1-(1+i)^{-n}\right)}{i} = \dfrac{500\left(1-1.005^{-60}\right)}{0.005} \approx \$25{,}862.78$

13. Challenge A credit card balance of \$3000 is charged 18% annual interest compounded monthly, and no new purchases are made. (a) If \$60 is paid each month, how many monthly payments are needed to pay off the balance? (Set the payment formula equal to 60, solve for n with logarithms, and round up to a whole month.) (b) Explain what would happen if only \$45 were paid each month.

Show answer

(a) $i = 0.015$: $1 - 1.015^{-n} = \dfrac{3000(0.015)}{60} = 0.75$, so $1.015^{-n} = 0.25$ and $n = \dfrac{\ln 4}{\ln 1.015} \approx 93.1$; 94 payments (almost 8 years). (b) The first month's interest is $3000(0.015) = \$45$, so a \$45 payment covers only the interest; the balance would never go down.

14. Challenge Show that the payment formula can be rewritten as $M = L \cdot i + \dfrac{L \cdot i}{(1+i)^n - 1}$, and explain what each of the two parts means.

Show answer

$M = \dfrac{Li}{1-(1+i)^{-n}} = \dfrac{Li(1+i)^n}{(1+i)^n - 1} = \dfrac{Li\left[((1+i)^n - 1) + 1\right]}{(1+i)^n - 1} = Li + \dfrac{Li}{(1+i)^n - 1}$. The first part, $L \cdot i$, is the interest on the full amount borrowed for one period. The second part is exactly the deposit P that, by the annuity formula $P \cdot \dfrac{(1+i)^n - 1}{i} = L$, would grow to L in n periods. So a loan payment is like paying the interest on L each period while saving up enough to pay back L at the end.

COMMON MISTAKES

- Using the annual rate and the number of years instead of the periodic rate and the number of periods. For monthly payments at 6% for 5 years, use $i = 0.005$ and $n = 60$, not 0.06 and 5.

- Rounding $(1+i)^n$ or $(1+i)^{-n}$ early. Keep every digit in the calculator and round only the final answer to the cent.
- Mixing up the two formulas. FV answers "how much will my deposits grow to?" and M answers "what payment pays off this loan?" Ask yourself which question is being asked.
- Thinking the interest part of a payment stays the same. Interest is figured on the current balance, so it is largest at the start and falls with every payment.

> **Make it a game · Pretend Bank Builder.** Your pretend bank pays 10 percent each period. Deposit 100 pretend dollars at the end of each of three periods. Compute the balance by adding $100(1.1)^2 + 100(1.1) + 100$, then check it with the formula $FV = P((1 + i)^n - 1)/i$. Then find the payment for borrowing a pretend 1000 over two periods at 10 percent, and compare with the loan formula.

9.6 Mathematical Induction

OBJECTIVES

- Explain why checking many cases does not prove a statement about every counting number.
- Explain how a base case and an inductive step together prove a statement for every counting number.
- Write induction proofs of sum formulas, including $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, $1 + 3 + \cdots + (2n - 1) = n^2$ and the geometric sum formula.
- Write induction proofs of divisibility statements such as "3 divides $4^n - 1$."

WATCH

Introduction to Mathematical Induction (1 of 2: Two Different kinds of Logic)

Eddie Woo · 6:53

https://www.youtube.com/watch?v=skDz6ichD4Y

> **WHEN TO WATCH**
> Before Example 1: the logic of proof by induction.

BACKUP VIDEO

Induction: Divisibility Proofs (1 of 2)

Eddie Woo · 6:14

https://www.youtube.com/watch?v=MthNLtZdFQ0

Before the practice set: induction proofs of divisibility.

KEY TERMS

mathematical induction

A method of proving that a statement $P(n)$ is true for every counting number n by proving a base case and an inductive step.

base case

The proof that the statement is true for the first value, usually $n = 1$.

inductive hypothesis

The assumption, made only inside the inductive step, that the statement is true for some particular counting number k.

inductive step

The proof that if the statement is true for k, then it must also be true for $k + 1$.

divides

A whole number a divides a whole number N if $N = a \cdot m$ for some whole number m. For example, 3 divides 15 because $15 = 3 \cdot 5$.

CORE IDEAS

In this unit you met formulas that claim to work for every counting number n, such as $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Checking $n = 1, 2, 3$, or even a thousand values, is good evidence, but it is not a proof, because there are infinitely many counting numbers left unchecked. Here is a warning. The expression $n^2 + n + 41$ gives a prime number for every n from 0 to 39, yet at $n = 40$ it gives $1600 + 40 + 41 = 1681 = 41 \times 41$, which is not prime.

n	$n^2 + n + 41$	Prime?
0	41	yes
1	43	yes
2	47	yes
3	53	yes
4	61	yes
5	71	yes
10	151	yes
20	461	yes

n	$n^2 + n + 41$	Prime?
39	1601	yes
40	1681	no: 41 × 41

Mathematical induction proves a statement $P(n)$ for every counting number in two parts. (1) **Base case:** show $P(1)$ is true. (2) **Inductive step:** show that whenever $P(k)$ is true for some counting number k, $P(k+1)$ must be true too. Think of an endless line of dominoes. The base case knocks over the first domino; the inductive step guarantees that each falling domino knocks over the next one. Together they knock down every domino.

Why is that a proof for every n? The base case gives $P(1)$. The inductive step with $k = 1$ then gives $P(2)$; with $k = 2$ it gives $P(3)$; and so on. Any counting number n is reached after $n - 1$ uses of the step, so $P(n)$ is true. Both parts are needed. A step with no base case proves nothing (if $k = k + 1$ were true, then adding 1 gives $k + 1 = k + 2$, yet no starting case is ever true), and a base case with no step proves only one case.

In the inductive step you do NOT assume what you want to prove. You assume $P(k)$ (the **inductive hypothesis**) and use it to reach $P(k+1)$. A good routine: write out $P(k+1)$ as your goal, start from its left side, find the part that $P(k)$ tells you about, replace it, and do algebra until you reach the right side of the goal. For example, to prove $1 + 3 + 5 + \cdots + (2n - 1) = n^2$: the base case is $1 = 1^2$. Assume $1 + 3 + \cdots + (2k - 1) = k^2$. The next odd number is $2(k+1) - 1 = 2k + 1$, so $1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k+1)^2$, which is the statement for $k + 1$.

Induction also proves divisibility statements. To show that 3 divides $4^n - 1$ for every n, the trick in the inductive step is to write the $(k+1)$ expression in terms of the k expression: $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(4^k - 1) + 3$. If 3 divides $4^k - 1$, then both parts of the sum are multiples of 3, so 3 divides their sum. Worked Example 2 writes this out in full.

The same pattern proves the geometric sum formula from Lesson 9.3, and you will use induction again in later courses whenever a statement must hold for every counting number. Induction does not discover formulas; you find a formula by looking at patterns (as in Lessons 9.1–9.3), and induction proves it is always right.

KEY FORMULAS

If $P(1)$ is true, and $P(k)$ true implies $P(k+1)$ true for every $k \geq 1,$ then $P(n)$ is true for every $n \geq 1$.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

From history. Arguments of this kind appear in the work of the Italian mathematician Francesco Maurolico, who used one in 1575 to prove facts about sums of odd numbers. Blaise Pascal used the same reasoning in his Treatise on the Arithmetical Triangle, written in 1654, to prove properties of the triangle of numbers you will study in Unit 10.

Practice habit. Write one step per line and label each part of an induction proof: "Base case," "Inductive hypothesis: assume P(k)," "Goal: P(k + 1)" written out in full, then the algebra that reaches the goal, and a closing sentence.

Example 1. Ruth builds a triangular display of canned goods for a church food drive: 1 can in the top row, 2 in the next row, 3 in the next, and so on, with n cans in the bottom row. Prove by induction that a display with n rows uses $\frac{n(n+1)}{2}$ cans; that is, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every counting number n.

Show solution

1. Base case ($n = 1$): the left side is 1 and the right side is $\frac{1 \cdot 2}{2} = 1$. True.
2. Inductive hypothesis: assume that for some counting number k, $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
3. Goal: $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$.
4. Start with the left side and use the hypothesis on the first k terms: $1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$.
5. Common denominator: $= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$, factoring out $k+1$.
6. This is the goal, so the statement for k implies the statement for $k+1$.
7. Conclusion: by the base case and the inductive step, the formula is true for every counting number n. (In the display, adding a new bottom row of $k+1$ cans to a k-row display gives a $(k+1)$-row display.)

Answer: Proved: the base case $1 = \frac{1 \cdot 2}{2}$ holds, and $\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$ gives the inductive step.

Example 2. Prove by induction that 3 divides $4^n - 1$ for every counting number n.

Show solution

1. Base case ($n = 1$): $4^1 - 1 = 3 = 3 \cdot 1$, so 3 divides it.
2. Inductive hypothesis: assume 3 divides $4^k - 1$ for some counting number k; that is, $4^k - 1 = 3m$ for some whole number m.
3. Goal: 3 divides $4^{k+1} - 1$.
4. Rewrite in terms of $4^k - 1$: $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4 \cdot 4^k - 4 + 3 = 4(4^k - 1) + 3$.
5. Use the hypothesis: $= 4(3m) + 3 = 3(4m + 1)$.
6. Since $4m + 1$ is a whole number, 3 divides $4^{k+1} - 1$.
7. Conclusion: by induction, 3 divides $4^n - 1$ for every counting number n. (Check: $4^2 - 1 = 15$ and $4^3 - 1 = 63$ are both multiples of 3 ✓.)

Answer: Proved: $4^1 - 1 = 3$, and $4^{k+1} - 1 = 4(4^k - 1) + 3 = 3(4m + 1)$.

PRACTICE

1. Easy For the statement $2 + 4 + 6 + \cdots + 2n = n(n + 1)$, compute the left side for $n = 1$, $n = 2$ and $n = 3$, and check that each equals the right side.

   Show answer

   2, 6, 12; and $1 \cdot 2 = 2$, $2 \cdot 3 = 6$, $3 \cdot 4 = 12$ ✓

2. Easy Let $P(n)$ be the statement $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$. Write out $P(1)$, $P(k)$ and $P(k + 1)$.

   Show answer

   $P(1)$: $1 = 2^1 - 1$. $P(k)$: $1 + 2 + \cdots + 2^{k-1} = 2^k - 1$. $P(k + 1)$: $1 + 2 + \cdots + 2^{k-1} + 2^k = 2^{k+1} - 1$.

3. Easy (a) Evaluate $n^2 + n + 41$ for $n = 41$. (b) Explain, without dividing, why the result cannot be prime.

   Show answer

   (a) $1681 + 41 + 41 = 1763$; (b) every term is a multiple of 41: $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43$

4. **Easy** Compute $4^n - 1$ for $n = 1, 2, 3, 4$ and check that each is divisible by 3.

Show answer

3, 15, 63, 255; these are 3×1, 3×5, 3×21, 3×85 ✓

5. **Medium** Prove by induction: $2 + 4 + 6 + \cdots + 2n = n(n+1)$ for every counting number n.

Show answer

Base: $2 = 1 \cdot 2$. Assume $2 + 4 + \cdots + 2k = k(k+1)$. Then $2 + 4 + \cdots + 2k + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$, which is the statement for $k+1$. By induction it holds for every n.

6. **Medium** Prove by induction: $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$ for every counting number n.

Show answer

Base: $1 = 2^1 - 1$. Assume $1 + 2 + \cdots + 2^{k-1} = 2^k - 1$. Then $1 + 2 + \cdots + 2^{k-1} + 2^k = (2^k - 1) + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$. By induction it holds for every n.

7. **Medium** At a church potluck, each person who arrives shakes hands once with every person already there. (a) How many handshakes are there in all when 30 people have arrived? (b) Prove by induction that when n people have arrived there have been $\dfrac{n(n-1)}{2}$ handshakes.

Show answer

(a) $\dfrac{30 \times 29}{2} = 435$. (b) Base: with 1 person there are 0 handshakes, and $\dfrac{1 \cdot 0}{2} = 0$. Assume k people have made $\dfrac{k(k-1)}{2}$ handshakes. Person $k+1$ shakes hands with all k people there, so the total becomes $\dfrac{k(k-1)}{2} + k = \dfrac{k^2 - k + 2k}{2} = \dfrac{(k+1)k}{2}$, the formula for $k+1$. By induction it holds for every n.

8. **Medium** Prove by induction: $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$ for every counting number n.

Show answer

Base: $3 = 1(3)$. Assume $3 + 7 + \cdots + (4k - 1) = k(2k + 1)$. The next term is $4(k + 1) - 1 = 4k + 3$, so the sum becomes $k(2k + 1) + 4k + 3 = 2k^2 + 5k + 3 = (k + 1)(2k + 3) = (k + 1)\,(2(k + 1) + 1)$. By induction it holds for every n.

9. **Medium** Prove by induction that 5 divides $6^n - 1$ for every counting number n.

Show answer

Base: $6 - 1 = 5$. Assume $6^k - 1 = 5m$ for a whole number m. Then $6^{k+1} - 1 = 6 \cdot 6^k - 6 + 5 = 6(6^k - 1) + 5 = 30m + 5 = 5(6m + 1)$, so 5 divides it. By induction the statement holds for every n.

10. **Hard** Prove by induction the geometric sum formula: for $r \neq 1$, $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$ for every counting number n.

Show answer

Base: the left side is a and the right side is $\dfrac{a(1 - r)}{1 - r} = a$. Assume the formula holds for k terms. Adding the next term ar^k: $\dfrac{a(1 - r^k)}{1 - r} + ar^k = \dfrac{a - ar^k + ar^k(1 - r)}{1 - r} = \dfrac{a - ar^k + ar^k - ar^{k+1}}{1 - r} = \dfrac{a(1 - r^{k+1})}{1 - r}$, the formula for $k + 1$ terms. By induction it holds for every n.

11. **Hard** In Lesson 9.1 a grocer stacked oranges in square layers, so a pyramid with n layers holds $1^2 + 2^2 + \cdots + n^2$ oranges. (a) Use the formula $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$ to find the number of oranges in a 6-layer pyramid. (b) Prove the formula by induction.

Show answer

(a) $\dfrac{6 \cdot 7 \cdot 13}{6} = 91$ oranges. (b) Base: $1 = \dfrac{1 \cdot 2 \cdot 3}{6}$. Assume the formula for k. Then $\dfrac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 = \dfrac{(k + 1)\,[k(2k + 1) + 6(k + 1)]}{6} = \dfrac{(k + 1)(2k^2 + 7k + 6)}{6} = \dfrac{(k + 1)(k + 2)(2k + 3)}{6}$, which is the formula with $k + 1$ in place of n. By induction it holds for every n.

12. Hard Prove by induction that 3 divides $n^3 + 2n$ for every counting number n.

Show answer

Base: $1 + 2 = 3$. Assume 3 divides $k^3 + 2k$. Then $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$. Both parts are multiples of 3, so 3 divides the sum. By induction the statement holds for every n.

13. Hard In the Tower of Hanoi puzzle, n wooden rings of different sizes must be moved from one peg to another, one ring at a time, never putting a larger ring on a smaller one. The least number of moves m_n satisfies $m_1 = 1$ and $m_{n+1} = 2m_n + 1$ (move the top n rings away, move the largest ring, then move the n rings back on top). (a) Use the recursive rule to find m_6. (b) Prove by induction that $m_n = 2^n - 1$.

Show answer

(a) 1, 3, 7, 15, 31, 63, so $m_6 = 63$. (b) Base: $m_1 = 1 = 2^1 - 1$. Assume $m_k = 2^k - 1$. Then $m_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$. By induction $m_n = 2^n - 1$ for every n.

14. Challenge (a) For which of $n = 1, 2, 3, 4, 5$ is $2^n > n^2$ true? (b) Prove by induction that $2^n > n^2$ for every whole number $n \geq 5$. (Use $n = 5$ as the base case.)

Show answer

(a) $n = 1$ ($2 > 1$) and $n = 5$ ($32 > 25$); it fails for 2, 3, 4 ($4 = 4$, $8 < 9$, $16 = 16$). (b) Base: $2^5 = 32 > 25$. Assume $2^k > k^2$ for some $k \geq 5$. Then $2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2 \geq k^2 + 5k$ (since $k \geq 5$) $= k^2 + 2k + 3k > k^2 + 2k + 1 = (k+1)^2$. So $2^{k+1} > (k+1)^2$, and by induction the inequality holds for every $n \geq 5$.

15. **Challenge** Find the flaw in this "proof" that all horses are the same color. Base case: in any group of 1 horse, all the horses are the same color. Inductive step: assume any group of k horses are all the same color. Take a group of $k+1$ horses. Leave out the first horse: the other k are the same color. Leave out the last horse: the other k are the same color. The two groups overlap, so all $k+1$ horses are the same color.

Show answer

The inductive step fails when $k = 1$. With 2 horses, leaving out the first leaves only the second, and leaving out the last leaves only the first; the two groups of one horse do not overlap, so nothing connects their colors. Since the step from $P(1)$ to $P(2)$ fails, the chain of dominoes breaks at the very first push, and nothing is proved. The step must work for every $k \geq 1$, not only for large k.

COMMON MISTAKES

- Skipping the base case. Without it, the inductive step alone can "prove" false statements, such as $n = n + 1$.
- Assuming the statement you are trying to prove. Assume $P(k)$, then show $P(k+1)$; never start by writing $P(k+1)$ as if it were already known and working on both sides.
- Writing $P(k+1)$ incorrectly. Replace n by $k+1$ everywhere, including the last term: in $1 + 3 + \cdots + (2n-1)$ the new last term is $2(k+1) - 1 = 2k + 1$, not $2k - 1 + 1$.
- Treating many checked cases as a proof. $n^2 + n + 41$ is prime for $n = 0$ to 39, but not for $n = 40$.

> **Make it a game · Domino Proof.** Stand playing cards on edge in a safe row and knock over the first one. Explain which two facts guarantee that every domino falls: the base case and the inductive step. Then test $n^2 + n + 41$ for n = 1, 2, 3 and so on. It gives primes until n = 40. Say why checking many cases proves nothing.

UNIT 9

Sequences and Series

This unit is about lists of numbers that follow a rule and what happens when you add them up. You will describe sequences with explicit and recursive rules and write sums compactly in sigma notation. Then you will study the two most important kinds of sequences, arithmetic (add the same amount each time) and geometric (multiply by the same amount each time), and derive formulas for their sums. You will see why some infinite sums have a finite value, and why $0.999\ldots$ is exactly 1.

These ideas tie together much of the course. Arithmetic sequences are linear functions on the counting numbers, and geometric sequences are exponential functions, so Units 1, 7 and 8 come back here. The geometric series also explains the formulas behind savings plans, car loans and home mortgages, so you will see how saving, borrowing and giving wisely look in numbers. The unit ends with mathematical induction, a method for proving that a statement is true for every counting number. Induction and sigma notation are used again in Unit 10 (the Binomial Theorem) and in every later mathematics course.

PREREQUISITES

- Function notation, domain and linear functions (Lessons 1.1 and 1.2)
- Exponent rules and exponential growth and decay, including $A = P(1 + r/n)^{nt}$ (Lessons 7.1–7.3)
- Solving exponential equations with logarithms (Lesson 8.4)
- Solving quadratic equations (Lesson 3.2)
- Simplifying algebraic expressions and fractions, and factoring out a common factor
- Reading and writing a short proof (Geometry)

MASTERY CHECKLIST

- ☐ I can find terms of a sequence from an explicit or recursive rule and write a rule from the first several terms.
- ☐ I can read, write and evaluate sums in sigma notation and use the properties of sums.
- ☐ I can use $a_n = a_1 + (n - 1)d$ to find a term, the common difference or the number of terms of an arithmetic sequence, and find arithmetic means.
- ☐ I can derive and use $S_n = \frac{n(a_1 + a_n)}{2}$ including $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- ☐ I can use $a_n = a_1 r^{n-1}$ to find terms, ratios and geometric means of a geometric sequence.
- ☐ I can derive and use $S_n = \frac{a_1(1 - r^n)}{1 - r}$ for raises, doubling amounts and similar problems.
- ☐ I can decide whether an infinite geometric series converges and find its sum $\frac{a_1}{1 - r}$ when $|r| < 1$.

- ☐ I can write a repeating decimal as a fraction and explain why $0.999\ldots = 1$.
- ☐ I can find the future value of regular deposits and the monthly payment and total interest of a loan.
- ☐ I can explain how the loan payment formula comes from a geometric series and build the first rows of an amortization table.
- ☐ I can write a proof by mathematical induction for a sum formula or a divisibility statement.
- ☐ I can explain why checking many cases is not a proof.

LESSONS IN THIS UNIT

9.1	**Sequences and Summation Notation**	2 sessions of 45 minutes
9.2	**Arithmetic Sequences and Series**	2 sessions of 45 minutes
9.3	**Geometric Sequences and Series**	2 sessions of 45 minutes
9.4	**Infinite Geometric Series**	2 sessions of 45 minutes
9.5	**Savings, Loans and Geometric Series**	2 sessions of 45 minutes
9.6	**Mathematical Induction**	2 sessions of 45 minutes

Unit 9 test

Time: 60 minutes. A scientific calculator is allowed, and you will need one for problems 8, 13 and 14. Show your work for every problem: write the formula you use and the values you substitute. Give exact answers unless a problem asks you to

round; round money to the nearest cent unless told otherwise. Write proofs in complete sentences, one step per line.

1. **5 pts**

   Write the first five terms of the sequence $a_1 = 3$, $a_n = 3a_{n-1} - 2$.

2. **5 pts**

   Evaluate $\sum_{k=2}^{5} (k^2 + 1)$.

3. **6 pts**

   Write an explicit rule for the sequence $-2, 4, -8, 16, \ldots$.

4. **6 pts**

   In an arithmetic sequence, $a_3 = 14$ and $a_{10} = 49$. Find the common difference d, the first term a_1, and a_{50}.

5. **7 pts**

   A rancher building a new pasture fence puts up 120 feet of fence on the first day, and each day after that puts up 15 feet more than the day before. (a) How many feet does the rancher put up on day 12? (b) How many feet in all are built in the first 12 days?

6. **6 pts**

   Find the sum of all the even numbers from 50 to 150, including 50 and 150.

7. **6 pts**

   A geometric sequence has $a_1 = 2$ and $r = 3$. Find a_6 and the sum of the first 6 terms.

8. **7 pts**

   A family bakery has sales of $50,000 in its first year, and its sales grow by 10% each year after that. Find the total sales over the first 8 years, to the nearest dollar.

9. **5 pts**

   Insert two geometric means between 3 and 192.

10. **6 pts**

(a) Find the sum of the infinite geometric series $45 + 15 + 5 + \cdots$. (b) Does the infinite series $0.5 - 1 + 2 - 4 + \cdots$ have a sum? Explain.

11. **5 pts**

Write $0.\overline{27} = 0.272727\ldots$ as a fraction in lowest terms.

12. **6 pts**

A church building-fund drive raises \$800 in its first week, and each week after that the gifts are $\dfrac{3}{4}$ of the previous week's. (a) If this pattern continued without end, what is the most the drive could raise in all? (b) Can the drive reach a goal of \$3500 this way?

13. **7 pts**

Henry deposits \$150 at the end of each month for 4 years in an account paying 3% interest compounded monthly, to buy a used car. (a) How much will he have after the last deposit? (b) How much of it is interest?

14. **8 pts**

Ruth borrows \$12,000 for a pickup truck for her landscaping business at 6% interest compounded monthly, repaid in equal monthly payments over 4 years. (a) Find the monthly payment. (b) Find the total interest she pays.

15. **10 pts**

Prove by mathematical induction that for every counting number n, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

16. **5 pts**

The expression $n^2 - n + 11$ is prime for $n = 1, 2, \ldots, 10$. (a) Explain why this does not prove that it is prime for every counting number n. (b) Find the first counting number n for which it is not prime.

ANSWER KEY

1. 3, 7, 19, 55, 163
2. $5 + 10 + 17 + 26 = 58$
3. $a_n = (-2)^n$ (check: $a_1 = -2$, $a_4 = 16$ ✓)
4. $7d = 35$, so $d = 5$; $a_1 = 14 - 2(5) = 4$; $a_{50} = 4 + 49(5) = 249$
5. (a) $120 + 11(15) = 285$ feet; (b) $\dfrac{12(120 + 285)}{2} = 2430$ feet
6. $n = \dfrac{150 - 50}{2} + 1 = 51$ terms; sum $= \dfrac{51(50 + 150)}{2} = 5100$
7. $a_6 = 2 \cdot 3^5 = 486$; $S_6 = \dfrac{2(3^6 - 1)}{3 - 1} = 728$
8. $r = 1.1$: $S_8 = \dfrac{50{,}000(1.1^8 - 1)}{0.1} \approx \$571{,}794$
9. $r^3 = \dfrac{192}{3} = 64$, $r = 4$: the means are 12 and 48
10. (a) $r = \dfrac{1}{3}$: $\dfrac{45}{1 - \frac{1}{3}} = 67.5$; (b) no: $r = -2$ and $|r| \geq 1$, so the partial sums never settle
11. $\dfrac{0.27}{1 - 0.01} = \dfrac{27}{99} = \dfrac{3}{11}$
12. (a) $\dfrac{800}{1 - \frac{3}{4}} = \$3200$; (b) no, the total can never exceed \$3200
13. (a) $i = 0.0025$, $n = 48$: $150 \cdot \dfrac{1.0025^{48} - 1}{0.0025} \approx \$7639.68$; (b) $7639.68 - 7200 = \$439.68$
14. (a) $M = \dfrac{12{,}000(0.005)}{1 - 1.005^{-48}} \approx \$281.82$; (b) $48(281.82) - 12{,}000 = \$1527.36$
15. Base case: for $n = 1$ the left side is $1 \cdot 2 = 2$ and the right side is $\dfrac{1 \cdot 2 \cdot 3}{3} = 2$. Inductive hypothesis: assume $1 \cdot 2 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$. Adding the next term $(k+1)(k+2)$: $\dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \dfrac{(k+1)(k+2)(k+3)}{3}$, which is the formula with $k+1$ in place of n. By induction the formula holds for every counting number n.
16. (a) Checking cases, however many, leaves infinitely many counting numbers unchecked; a proof must cover every n (for example, by induction). (b) $n = 11$: $121 - 11 + 11 = 121 = 11 \times 11$

UNIT 10

Counting, Probability and the Binomial Theorem

How many ways can a choir line up for a photo? How many different garden beds can be planted from a box of seed packets? Unit 10 teaches you to count large collections without listing them. Everything rests on one idea, the fundamental counting principle, and from it come factorials, permutations (arrangements where order matters) and combinations (selections where it does not).

Counting then pays off twice. In Lesson 10.3 the numbers $C(n, r)$ turn out to be the entries of Pascal's triangle and the coefficients in the expansion of $(a + b)^n$, finishing the work on binomial products you began in Lesson 4.2. In Lesson 10.4 counting becomes probability: the chance of an event is one count divided by another, and the Binomial Theorem gives the probability of exactly k successes in n tries.

These ideas reach well beyond this course. The Statistics course builds on binomial probability, and counting arguments run through computer science, where programmers must know how many cases a program could face. As always, the goal is not only to get the number but to be able to explain why the count is right.

PREREQUISITES

- Multiplying whole numbers by hand and reducing fractions
- Solving quadratic equations by factoring (Lesson 3.2)
- Multiplying polynomials and the cube of a binomial (Lesson 4.2)
- Compound interest $A = P(1 + r)^t$ with yearly compounding (Lesson 7.2)
- Sigma notation (Lesson 9.1)

MASTERY CHECKLIST

- ☐ I can use the fundamental counting principle and explain it with a tree diagram.
- ☐ I can evaluate factorials by hand by cancelling, and explain why $0! = 1$.
- ☐ I can count permutations with $P(n, r) = \frac{n!}{(n-r)!}$, including arrangements with restrictions.
- ☐ I can count arrangements of letters or objects with repeats using $\frac{n!}{a!\,b!\cdots}$.
- ☐ I can decide whether order matters and choose between permutations and combinations.
- ☐ I can derive $C(n, r)$ from $P(n, r)$ and explain why $C(n, r) \cdot r! = P(n, r)$.
- ☐ I can build Pascal's triangle and explain Pascal's rule by counting.
- ☐ I can expand $(a + b)^n$ with the Binomial Theorem, explain why the coefficients are $C(n, k)$, and find a single term.
- ☐ I can find probabilities by counting equally likely outcomes with permutations and combinations.

- ☐ I can use the complement for "at least one" problems and multiply probabilities of independent events.
- ☐ I can find binomial probabilities $C(n,k)p^k(1-p)^{n-k}$ and explain where the formula comes from.

LESSONS IN THIS UNIT

10.1	**The Counting Principle and Permutations**	2 sessions of 45 minutes
10.2	**Combinations**	2 sessions of 45 minutes
10.3	**Pascal's Triangle and the Binomial Theorem**	2 sessions of 45 minutes
10.4	**Probability with Counting**	2 sessions of 45 minutes

Unit 10 test

Time: 60 minutes. A scientific calculator is allowed, except on problems marked "by hand." Show your work: for each counting problem, say which rule you are using and why (for example, "order matters, so a permutation"). Give exact answers (whole numbers, or fractions in lowest terms) unless a problem asks you to round; round as directed.

1. **5 pts**

   By hand, evaluate (a) $\dfrac{7!}{5!}$ and (b) $\dfrac{10!}{8!\,2!}$.

2. **6 pts**

   The Carter family is planning a summer trip. They will choose one of 3 campgrounds, one of 5 open weekends and one of 2 ways to travel (car or train). How many different trip plans are possible?

3. **6 pts**

   A church pianist chooses 4 of her 15 favorite hymns and decides the order in which to play them before the service. How many different programs are possible?

4. **6 pts**

   In how many different ways can the letters of the word COOKBOOK be arranged?

5. **6 pts**

   A track coach has 12 runners. (a) In how many ways can she choose 4 of them for a relay team, if the running order is not yet decided? (b) In how many ways can she choose 4 runners and also assign the order in which they run?

6. **7 pts**

   A 4-H club will send a team of 5 members to the state fair. The club has 9 juniors and 6 seniors. How many teams have exactly 3 juniors and 2 seniors?

7. **6 pts**

   Solve $C(n, 2) = 45$ for the counting number n.

8. **7 pts**

   Expand $(x - 2)^5$.

9. **7 pts**

   Find the coefficient of x^4 in the expansion of $(3x + 2)^6$.

10. **6 pts**

    Given that $C(10, 3) = 120$ and $C(10, 4) = 210$, use Pascal's rule to find $C(11, 4)$.

11. **6 pts**

    Two number cubes are rolled. What is the probability that the sum is 5 or 6?

12. **8 pts**

    After Thanksgiving dinner, the names of 9 family members, 5 adults and 4 children, are put in a bowl, and 3 names are drawn at random for dish duty. (a) What is the probability that all three are children? (b) What is the probability that at least one adult is drawn? Give exact fractions.

13. **10 pts**

An orchard keeper finds that 85% of her apple grafts take (grow successfully), each independently of the others. She makes 6 grafts. (a) What is the probability that exactly 5 take? (b) What is the probability that at least 5 take? Round each answer to the nearest thousandth.

14. **6 pts**

A barn has 3 smoke detectors that work independently, and each one works with probability 0.95. What is the probability that at least one of them works?

15. **8 pts**

Without computing either number, explain why $P(8, 3)$ is exactly 6 times $C(8, 3)$. Then compute both to confirm.

ANSWER KEY

1. (a) $7 \cdot 6 = 42$; (b) $\dfrac{10 \cdot 9}{2} = 45$
2. $3 \times 5 \times 2 = 30$ plans
3. Order matters: $P(15, 4) = 15 \cdot 14 \cdot 13 \cdot 12 = 32{,}760$ programs
4. 8 letters with four O's and two K's: $\dfrac{8!}{4!\,2!} = \dfrac{40{,}320}{48} = 840$
5. (a) $C(12, 4) = \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{24} = 495$; (b) $P(12, 4) = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$
6. $C(9, 3) \cdot C(6, 2) = 84 \cdot 15 = 1260$ teams
7. $\dfrac{n(n-1)}{2} = 45$, so $n^2 - n - 90 = 0$ and $(n - 10)(n + 9) = 0$; $n = 10$. Check: $\dfrac{10 \cdot 9}{2} = 45$ ✓
8. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$ (check at $x = 1$: $(-1)^5 = -1$ and $1 - 10 + 40 - 80 + 80 - 32 = -1$ ✓)
9. The term is $C(6, k)(3x)^{6-k}(2)^k$, and $6 - k = 4$ gives $k = 2$: $C(6, 2)(3)^4(2)^2 = 15 \cdot 81 \cdot 4 = 4860$
10. $C(11, 4) = C(10, 3) + C(10, 4) = 120 + 210 = 330$
11. Sum 5 in 4 ways and sum 6 in 5 ways, out of 36: $\dfrac{9}{36} = \dfrac{1}{4}$
12. (a) $\dfrac{C(4, 3)}{C(9, 3)} = \dfrac{4}{84} = \dfrac{1}{21}$; (b) $1 - \dfrac{1}{21} = \dfrac{20}{21}$
13. (a) $C(6, 5)(0.85)^5(0.15) = 6 \times 0.4437053 \times 0.15 \approx 0.399$; (b) add $P(6) = (0.85)^6 \approx 0.3771495$: $0.3993348 + 0.3771495 \approx 0.776$
14. $1 - P(\text{none work}) = 1 - (0.05)^3 = 1 - 0.000125 = 0.999875$
15. Model answer: Choosing and ordering 3 of 8 objects can be done in two stages: first choose which 3 ($C(8, 3)$ ways), then put them in order ($3! = 6$ ways). By the counting principle, $P(8, 3) = C(8, 3) \cdot 6$. Check: $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ and $C(8, 3) = 56$, and $56 \times 6 = 336$ ✓

10.1 The Counting Principle and Permutations

OBJECTIVES

- Explain the fundamental counting principle with a tree diagram and use it to count outcomes.
- Evaluate factorials by hand and explain why $0! = 1$.
- Count permutations with $P(n, r) = \dfrac{n!}{(n-r)!}$, including arrangements with restrictions.
- Count arrangements of objects with repeats using $\dfrac{n!}{a!\,b!\cdots}$.

WATCH

Introduction to Permutations (Ordered Selections)

Eddie Woo · 11:21

https://www.youtube.com/watch?v=JVbbRCVBVRI

> **WHEN TO WATCH**
> Before Example 1: ordered selections and nPr.

Note: Eddie Woo teaches in Australia: he says 'arrangements' and 'ordered selections' for permutations.

BACKUP VIDEO

Overview of Permutations (1 of 4: Arranging n objects)

Eddie Woo · 6:59

https://www.youtube.com/watch?v=BKW4fE9urBc

Before Example 2: arranging n objects, including factorials.

KEY TERMS

fundamental counting principle

If one choice can be made in m ways and, after it, a second choice can be made in n ways, then the two choices together can be made in $m \cdot n$ ways; for more stages, multiply the number of options at every stage.

tree diagram

A picture that shows every outcome of a several-stage choice as a path of branches; each first-stage branch splits into the options for the next stage.

factorial

For a counting number n, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, the number of ways to arrange n different objects in a row; by definition $0! = 1$.

permutation

An arrangement of objects in which order matters. $P(n, r)$ is the number of ways to choose and arrange r of n different objects.

CORE IDEAS

Many questions in this unit begin "How many ways...?" Listing every possibility works for small cases, but it quickly becomes impossible. The **fundamental counting principle** says: if one choice can be made in m ways and, for each of those, a second choice can be made in n ways, then the two choices together can be made in $m \cdot n$ ways. The same rule works for three or more stages: multiply the number of options at each stage.

Why multiply? Picture a **tree diagram**. The first choice draws m branches. From the end of each of those branches, the second choice draws n more. Every complete path from the start to a tip is one outcome, and there are m groups of n tips, which is $m \cdot n$ tips in all. For example, a bakery lunch with 3 breads (white, wheat, rye) and 2 fillings (ham, turkey) gives $3 \times 2 = 6$ sandwiches; the table lists the six paths through the tree.

The 3 × 2 = 6 paths of the tree diagram

Bread (first choice)	Filling (second choice)	Sandwich
White	Ham	white with ham
White	Turkey	white with turkey

Bread (first choice)	Filling (second choice)	Sandwich
Wheat	Ham	wheat with ham
Wheat	Turkey	wheat with turkey
Rye	Ham	rye with ham
Rye	Turkey	rye with turkey

Arranging n different objects in a row is a counting problem in n stages: n choices for the first place, $n-1$ for the second (one object is used up), and so on down to 1 choice for the last place. The product $n(n-1)(n-2)\cdots 2\cdot 1$ is written $n!$ and read "n factorial." So 5 hymns can be sung in $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ different orders. Factorials grow very fast, as the table shows.

Factorials

n	n!
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880

n	n!
10	3628800

Why is $0! = 1$? Every factorial contains the one before it: $n! = n \cdot (n-1)!$. Dividing both sides by n gives $(n-1)! = \dfrac{n!}{n}$. With $n = 1$ this says $0! = \dfrac{1!}{1} = 1$. So $0! = 1$ is the only value that keeps the pattern working, and it also matches the count: there is exactly one way to arrange zero objects (leave the row empty).

A **permutation** is an arrangement in which order matters. To choose and arrange r of n different objects, there are n choices for the first place, $n-1$ for the second, and so on for r places, ending with $n-r+1$ choices: $P(n,r) = n(n-1)\cdots(n-r+1)$. Multiplying and dividing by $(n-r)!$ supplies the missing factors $(n-r)(n-r-1)\cdots 1$, so $P(n,r) = \dfrac{n!}{(n-r)!}$. For example, $P(10,3) = 10 \cdot 9 \cdot 8 = 720 = \dfrac{10!}{7!}$. When $r = n$, $P(n,n) = \dfrac{n!}{0!} = n!$, as it should be.

When some objects are identical, $n!$ counts each arrangement more than once. Take the letters of BANANA: 6 letters, with three A's and two N's. If the letters were all different (A_1, A_2, A_3, N_1, N_2, B) there would be $6! = 720$ arrangements. But each real word, such as NABANA, appears $3! = 6$ times from shuffling the labeled A's and $2! = 2$ times from shuffling the N's, so $3! \cdot 2! = 12$ times in all. The number of different arrangements is $\dfrac{6!}{3!\,2!} = \dfrac{720}{12} = 60$.

Two habits make counting problems manageable. First, ask whether order matters and whether repeats are allowed: a 4-digit gate code that may repeat digits has $10 \times 10 \times 10 \times 10$ possibilities, while a code with no repeated digit has $P(10,4)$. Second, fill any restricted place first (the person who must stand at the end, the digit that must be odd), then count the free places.

KEY FORMULAS

Counting principle: m ways, then n ways, gives $m \cdot n$ ways

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1, \qquad 0! = 1$$

$$P(n, r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$$\text{Arrangements of } n \text{ objects with groups of } a, b, \ldots \text{ identical objects} = \frac{n!}{a!\,b!\cdots}$$

Practice habit. Know the factorials from 0! to 7! cold: 1, 1, 2, 6, 24, 120, 720, 5040. Cancel before you multiply: $\frac{9!}{6!} = 9 \cdot 8 \cdot 7 = 504$, with no need to compute 9!.

WORKED EXAMPLES

Example 1. The Miller farm's gate has a keypad lock that takes a 4-digit code using the digits 0 through 9. (a) How many codes are possible if digits may repeat? (b) How many codes are possible if no digit may repeat?

Show solution

1. (a) There are 4 stages (the four positions), and each has 10 choices, because a digit may be used again.
2. By the counting principle: $10 \times 10 \times 10 \times 10 = 10^4 = 10{,}000$ codes.
3. (b) Now a digit that has been used is not available later: 10 choices for the first position, 9 for the second, 8 for the third, 7 for the fourth.
4. $10 \times 9 \times 8 \times 7 = 5040$, which is $P(10, 4) = \frac{10!}{6!}$.
5. Check: (b) must be smaller than (a), because it forbids some codes, and $5040 < 10{,}000$ ✓.

Answer: (a) 10,000 codes; (b) 5040 codes

Example 2. Eight members of a church youth choir, including Ruth and Samuel, stand in a single row for a photo. (a) How many different lineups are possible? (b) How many lineups have Ruth and Samuel at the two ends of the row?

Show solution

1. (a) All 8 singers are arranged in 8 places: $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$.
2. (b) Fill the restricted places first. The left end can be Ruth or Samuel: 2 choices. The right end must be the other one: 1 choice.
3. The remaining 6 singers fill the 6 middle places in $6! = 720$ ways.
4. By the counting principle: $2 \times 1 \times 720 = 1440$.
5. Check: $\dfrac{1440}{40{,}320} = \dfrac{1}{28}$, a small fraction of all lineups, which is sensible because only 2 of the 8 singers may stand at the ends.

Answer: (a) 40,320 lineups; (b) 1440 lineups

PRACTICE

1. Easy A bakery's lunch special lets you choose one of 4 breads, one of 5 fillings and one of 3 spreads. How many different sandwiches are possible?

   Show answer

   $4 \times 5 \times 3 = 60$ sandwiches

2. Easy By hand, evaluate $\dfrac{6!}{4!}$.

   Show answer

   $\dfrac{6 \cdot 5 \cdot 4!}{4!} = 6 \cdot 5 = 30$

3. Easy By hand, evaluate $P(7, 3)$.

   Show answer

   $7 \cdot 6 \cdot 5 = 210$

4. Easy The pastor has chosen 5 hymns for Sunday's service. In how many different orders can they be sung?

   Show answer

   $5! = 120$ orders

5. **Medium** Simplify $\dfrac{n!}{(n-2)!}$ for $n \geq 2$.

   Show answer

   $$\frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = n^2 - n$$

6. **Medium** A storage barn's keypad code is 2 letters (A to Z) followed by 3 digits (0 to 9). Letters and digits may repeat. How many codes are possible?

   Show answer

   $26 \times 26 \times 10 \times 10 \times 10 = 676{,}000$ codes

7. **Medium** How many different arrangements are there of the letters in the word ALFALFA?

   Show answer

   7 letters with three A's, two L's and two F's: $\dfrac{7!}{3!\,2!\,2!} = \dfrac{5040}{24} = 210$ arrangements

8. **Medium** A youth group of 12 members elects a chair, a vice-chair and a secretary. No one may hold two offices. In how many ways can the offices be filled?

   Show answer

   Order matters (the offices are different): $P(12, 3) = 12 \cdot 11 \cdot 10 = 1320$ ways

9. **Medium** Solve $P(n, 2) = 56$ for the counting number n.

   Show answer

   $n(n-1) = 56$, so $n^2 - n - 56 = 0$ and $(n-8)(n+7) = 0$; $n = 8$ (-7 is not a counting number). Check: $8 \cdot 7 = 56$ ✓

10. **Hard** How many three-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6 with no digit repeated, if the number must be odd?

    Show answer

    Fill the restricted place first: the last digit is 1, 3 or 5 (3 choices); then 5 choices remain for the first digit and 4 for the middle digit: $3 \times 5 \times 4 = 60$ numbers

11. Hard The six members of the Miller family sit in a church pew with exactly 6 seats. In how many ways can they sit if the two parents must sit next to each other?

Show answer

Treat the parents as one block: arrange 5 units in $5! = 120$ ways, then order the parents inside the block in $2! = 2$ ways: $120 \times 2 = 240$ ways

12. Hard Three roads connect Ashford to Bridgeton, and four roads connect Bridgeton to Carver. Henry drives from Ashford to Carver and back, passing through Bridgeton each way, and he does not want to drive on any road twice. How many different round trips are possible?

Show answer

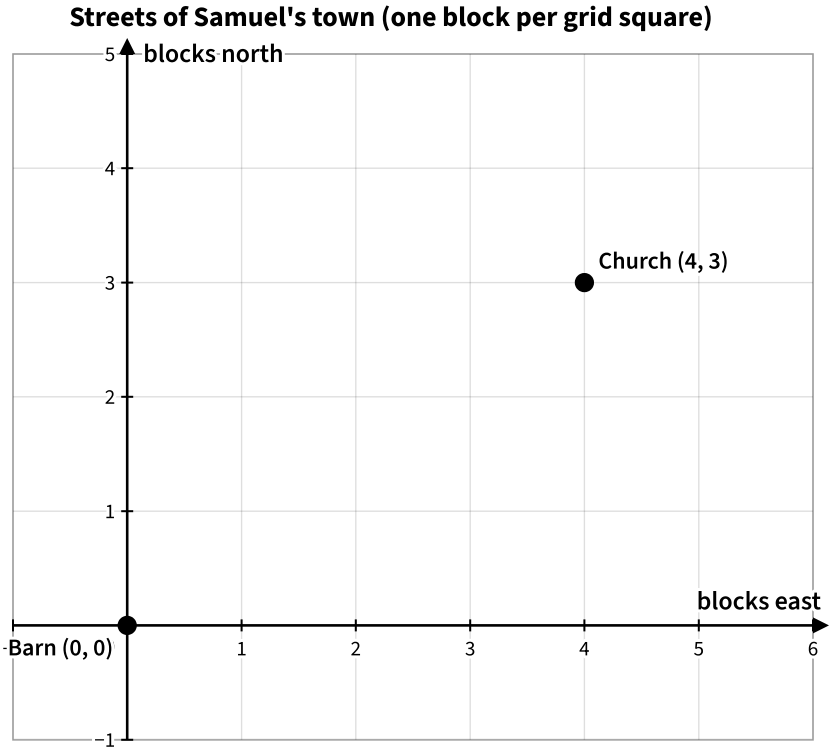
Going: $3 \times 4 = 12$ ways. Returning: 3 unused roads from Carver to Bridgeton and 2 unused roads from Bridgeton to Ashford: $3 \times 2 = 6$ ways. Total: $12 \times 6 = 72$ round trips

13. Hard Explain why mathematicians define $0! = 1$. Give at least two reasons.

Show answer

Model answer: (1) From $n! = n \cdot (n-1)!$ we get $(n-1)! = \dfrac{n!}{n}$; with $n = 1$ this gives $0! = \dfrac{1!}{1} = 1$, so the pattern forces $0! = 1$. (2) $0!$ counts the ways to arrange zero objects, and there is exactly one way: leave the row empty. (3) The formula $P(n, n) = \dfrac{n!}{0!}$ must equal $n!$, which works only if $0! = 1$.

14. **Challenge** Samuel's town is laid out in a grid of streets, shown below with one block per grid square. He walks from the barn at $(0, 0)$ to the church at $(4, 3)$, always moving one block east (right) or one block north (up). How many different routes can he take? (Hint: write each route as a string of letters E and N.)



Show answer

Every route is a string of 4 E's and 3 N's in some order, so count arrangements with repeats: $\dfrac{7!}{4!\,3!} = \dfrac{5040}{24 \cdot 6} = 35$ routes

COMMON MISTAKES

- Using $P(n, r)$ when repeats are allowed. A 4-digit code that may repeat digits has 10^4 possibilities, not $P(10, 4)$. Before you multiply, ask: can the same item be used again?
- Treating factorials like ordinary numbers: $\dfrac{6!}{3!}$ is $6 \cdot 5 \cdot 4 = 120$, not $2!$. Write out the factors and cancel.
- Forgetting to divide by the repeats when some objects are identical. In ALFALFA, swapping two A's gives the same word, so divide $7!$ by $3!\,2!\,2!$.
- Leaving a restriction until the end. Fill the restricted places first (the end seats, the last digit), then count the free places.

Make it a game · Menu Maker. Design a sandwich menu with 3 breads, 4 fillings and 2 sides. Draw part of the tree diagram, then use the fundamental counting principle to count the total, 24. Next, count the ways your family can line up for a photo using factorials. Finally, explain why 0! must equal 1, because there is exactly one way to arrange nothing.

10.2 Combinations

OBJECTIVES

- Decide whether order matters in a counting problem and choose permutations or combinations.
- Derive $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$ from $P(n,r) = C(n,r)\cdot r!$ and evaluate it by hand.
- Explain why $C(n,r) = C(n,n-r)$.
- Solve counting problems that combine the counting principle with combinations.

WATCH

Introduction to Combinations (Unordered Selections)

Eddie Woo · 8:22

https://www.youtube.com/watch?v=Ul4nl1S968E

> **WHEN TO WATCH**
> Before Example 1: unordered selections and nCr.

Note: Eddie Woo teaches in Australia: he says 'unordered selections' for combinations.

BACKUP VIDEO

Unordered Selections (1 of 3: Relation to permutations)

Eddie Woo · 10:02

https://www.youtube.com/watch?v=-4PpUzZU26M

After Example 1: why nCr = nPr ÷ r!.

KEY TERMS

combination
A selection of objects in which order does not matter, such as a committee or a set of pizza toppings.

$C(n,r)$ **or** $\binom{n}{r}$
Read "n choose r": the number of ways to choose r objects from n different objects when order does not matter.

permutation

From Lesson 10.1: an arrangement in which order matters; $P(n, r)$ counts the ways to choose and arrange r of n objects.

CORE IDEAS

In Lesson 10.1 order mattered: Ruth then Samuel was a different lineup from Samuel then Ruth. Often order does not matter. If a church picks 3 people for a planning committee, the committee of Ruth, Samuel and Grace is the same committee no matter whose name is called first. A selection in which order does not matter is a **combination**. The number of combinations of r objects chosen from n different objects is written $C(n, r)$ or $\binom{n}{r}$ and read "n choose r."

Here is the key idea. Take 5 people, A, B, C, D and E, and choose 3. The number of ordered choices is $P(5, 3) = 5 \cdot 4 \cdot 3 = 60$. But each committee, such as ABC, shows up in that count once for every way of ordering its 3 members, which is $3! = 6$ times. The table lists all 10 committees with the 6 orders of each: $10 \times 6 = 60$.

Choosing 3 of A, B, C, D, E: 10 committees, 6 orders each

Committee	Its 3! = 6 orders
ABC	ABC, ACB, BAC, BCA, CAB, CBA
ABD	ABD, ADB, BAD, BDA, DAB, DBA
ABE	ABE, AEB, BAE, BEA, EAB, EBA
ACD	ACD, ADC, CAD, CDA, DAC, DCA
ACE	ACE, AEC, CAE, CEA, EAC, ECA
ADE	ADE, AED, DAE, DEA, EAD, EDA
BCD	BCD, BDC, CBD, CDB, DBC, DCB
BCE	BCE, BEC, CBE, CEB, EBC, ECB

Committee	Its 3! = 6 orders
BDE	BDE, BED, DBE, DEB, EBD, EDB
CDE	CDE, CED, DCE, DEC, ECD, EDC

The same reasoning works for any n and r. Making an ordered choice of r objects is a two-stage task: first choose which r objects ($C(n, r)$ ways), then put them in order ($r!$ ways). By the counting principle, $P(n, r) = C(n, r) \cdot r!$. Dividing by $r!$ gives $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n!}{r!\,(n-r)!}$.

To compute by hand, use the short form: r factors counting down from n, divided by $r!$. For example, $C(10, 3) = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \dfrac{720}{6} = 120$. The result is always a whole number, because it counts something. Two special values: $C(n, 0) = 1$ and $C(n, n) = 1$ (one way to choose nobody, one way to choose everybody).

Choosing who is in is the same as choosing who is out. Picking 3 of 10 volunteers to stay home and cook also picks the 7 who go on the trip, so every choice of 3 matches exactly one choice of 7, and $C(10, 3) = C(10, 7)$. In general $C(n, r) = C(n, n - r)$. The formula agrees: replacing r by $n - r$ turns $\dfrac{n!}{r!\,(n-r)!}$ into $\dfrac{n!}{(n-r)!\,r!}$, the same fraction. This saves work: $C(20, 18) = C(20, 2) = \dfrac{20 \cdot 19}{2} = 190$.

To decide whether order matters, imagine swapping two of the chosen items. If the swap gives a different outcome (different jobs, places or codes), order matters: use permutations. If the swap gives the same outcome (the same group, the same set of toppings), order does not matter: use combinations.

Many problems have several stages. If Grace plants 3 of 8 vegetables and 2 of 4 herbs, she makes two choices one after the other, so multiply: $C(8, 3) \cdot C(4, 2) = 56 \cdot 6 = 336$. If instead a problem splits into separate cases ("exactly 2 adults or exactly 3 adults"), count each case and add, because each outcome falls in exactly one case. Stages that happen together multiply; separate cases add.

KEY FORMULAS

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{P(n,r)}{r!}$$

$$P(n,r) = C(n,r) \cdot r!$$

$$C(n,r) = C(n,n-r), \qquad C(n,0) = C(n,n) = 1$$

Practice habit. Before you compute, ask "Does order matter?" by imagining a swap of two chosen items. Then cancel before multiplying: $C(15,3) = \dfrac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 5 \cdot 7 \cdot 13 = 455$.

WORKED EXAMPLES

Example 1. Nine members of a church's building fund group will choose 4 of themselves. (a) In how many ways can they choose a 4-person committee to count the offering? (b) In how many ways can they choose a chair, a vice-chair, a secretary and a treasurer (four different people)?

Show solution

1. (a) Swapping two committee members gives the same committee, so order does not matter: a combination.
2. $C(9,4) = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{3024}{24} = 126.$
3. (b) Swapping the chair and the treasurer gives a different result, so order matters: a permutation.
4. $P(9,4) = 9 \cdot 8 \cdot 7 \cdot 6 = 3024.$
5. Check: $P(9,4)$ should equal $C(9,4) \cdot 4!$, and $126 \times 24 = 3024$ ✓.

Answer: (a) 126 committees; (b) 3024 ways

Example 2. Grace has seeds for 7 kinds of vegetables, including tomatoes, and 5 kinds of flowers. Her raised garden bed has room for 3 kinds of vegetables and 2 kinds of flowers. (a) How many different plantings can she choose? (b) How many of these plantings include tomatoes?

Show solution

1. (a) She makes two choices, the vegetables and then the flowers. Order within each group does not matter.
2. Vegetables: $C(7,3) = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$. Flowers: $C(5,2) = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$.
3. By the counting principle: $35 \times 10 = 350$ plantings.
4. (b) Tomatoes are already in, so she chooses the other 2 vegetables from the remaining 6: $C(6,2) = \dfrac{6 \cdot 5}{2} = 15$.
5. The flowers are chosen as before: $15 \times 10 = 150$ plantings.
6. Check: each planting uses 3 of the 7 vegetables, so tomatoes should appear in $\dfrac{3}{7}$ of all plantings, and $\dfrac{3}{7} \times 350 = 150$ ✓.

Answer: (a) 350 plantings; (b) 150 plantings

PRACTICE

1. Easy By hand, evaluate $C(6,2)$.

   Show answer

   $$\frac{6 \cdot 5}{2 \cdot 1} = 15$$

2. Easy By hand, evaluate $C(10,7)$. (Hint: use $C(n,r) = C(n,n-r)$.)

   Show answer

   $$C(10,7) = C(10,3) = \frac{10 \cdot 9 \cdot 8}{6} = 120$$

3. Easy At the county fair, 14 pies are entered in the baking contest. (a) In how many ways can the judges choose 3 pies to display on the front table? (b) In how many ways can they award first, second and third place ribbons?

   Show answer

   (a) Order does not matter: $C(14,3) = \dfrac{14 \cdot 13 \cdot 12}{6} = 364$; (b) order matters: $P(14,3) = 14 \cdot 13 \cdot 12 = 2184$

4. **Easy** A pizza shop offers 8 toppings. How many different pizzas with exactly 3 different toppings are possible?

   Show answer

   $C(8,3) = \dfrac{8 \cdot 7 \cdot 6}{6} = 56$ pizzas

5. **Medium** At a family reunion, each of 10 cousins shakes hands once with every other cousin. How many handshakes are there?

   Show answer

   Each handshake is a pair of cousins: $C(10,2) = \dfrac{10 \cdot 9}{2} = 45$ handshakes

6. **Medium** Compute $C(12,5)$. Then, without computing again, give $C(12,7)$.

   Show answer

   $C(12,5) = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{120} = 792$; $C(12,7) = 792$ also, because choosing 5 to include is the same as choosing 7 to leave out

7. **Medium** Henry may take 4 of the 10 library books on his shelf on a train trip. How many different sets of 4 books can he take?

   Show answer

   $C(10,4) = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{24} = 210$ sets

8. **Medium** A service-project team of 5 is chosen from 6 adults and 7 teenagers. How many teams have exactly 2 adults and 3 teenagers?

   Show answer

   $C(6,2) \cdot C(7,3) = 15 \cdot 35 = 525$ teams

9. **Medium** Nine points are marked on a circle. How many triangles have three of these points as vertices?

   Show answer

   Any 3 of the points form a triangle, and order does not matter: $C(9,3) = \dfrac{9 \cdot 8 \cdot 7}{6} = 84$ triangles

10. Hard Solve $C(n, 2) = 28$ for the counting number n.

Show answer

$\frac{n(n-1)}{2} = 28$, so $n^2 - n - 56 = 0$ and $(n-8)(n+7) = 0$; $n = 8$. Check: $C(8,2) = \frac{8 \cdot 7}{2} = 28$ ✓

11. Hard A team of 4 is chosen from 5 adults and 4 teenagers to run the church bake sale. How many teams have at least 2 adults?

Show answer

Add the separate cases. 2 adults: $C(5,2) \cdot C(4,2) = 10 \cdot 6 = 60$; 3 adults: $C(5,3) \cdot C(4,1) = 10 \cdot 4 = 40$; 4 adults: $C(5,4) = 5$. Total: 105 teams. (Check: all teams minus those with 0 or 1 adult: $126 - 1 - 20 = 105$ ✓)

12. Hard A farmer has 8 different fields. She chooses 3 of them for corn, then 2 of the remaining fields for soybeans, and leaves the rest in pasture. In how many ways can she do this?

Show answer

$C(8,3) \cdot C(5,2) = 56 \cdot 10 = 560$ ways

13. Hard Explain in two ways why $C(n, r) = C(n, n-r)$: once with a counting argument and once with the formula.

Show answer

Model answer: Counting: choosing the r objects that are in a group also decides the $n - r$ objects left out, and choosing the $n - r$ left out decides who is in. The two kinds of choices match one to one, so $C(n, r) = C(n, n-r)$. Formula: $C(n, n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!}$, which is the same fraction as $C(n, r)$.

14. **Challenge** A diagonal of a polygon is a segment joining two vertices that are not next to each other. (a) How many diagonals does a 12-sided polygon have? (b) Write a formula for the number of diagonals of an n-sided polygon, and simplify it.

Show answer

(a) Pairs of vertices: $C(12, 2) = 66$; 12 of these pairs are sides, so $66 - 12 = 54$ diagonals. (b) $C(n, 2) - n = \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$ (check: a pentagon has $\dfrac{5 \cdot 2}{2} = 5$ ✓)

COMMON MISTAKES

- Using a combination when the chosen people get different jobs. If the choices have different roles or places, order matters; use $P(n, r)$.
- Writing $C(n, r) = \dfrac{n!}{r!}$ and forgetting the $(n-r)!$. Use the short form: r factors counting down from n, divided by $r!$.
- Adding when you should multiply, or the reverse. Choices made in stages (3 vegetables and then 2 flowers) multiply; separate cases (exactly 2 adults or exactly 3 adults) add.
- Counting "at least one adult" by picking one adult first and then choosing the rest from everyone left. This counts many teams more than once. Count the separate cases, or subtract the "no adults" teams from the total.

> **Make it a game · Order Matters or Not.** The parent reads eight scenarios: picking a team of 3 from 8 people, choosing a president, vice president and treasurer, selecting pizza toppings, and so on. Say 'permutation' or 'combination' in two seconds and compute the answer. Show why P(8, 3) = 336 and C(8, 3) = 56 differ by exactly 3! orderings.

10.3 Pascal's Triangle and the Binomial Theorem

OBJECTIVES

- Build Pascal's triangle and explain why its entries are the numbers $C(n, r)$.
- Explain Pascal's rule $C(n, r) = C(n-1, r-1) + C(n-1, r)$ by counting.
- Expand $(a + b)^n$ with the Binomial Theorem and explain why the coefficients are $C(n, k)$.
- Find a single term of a binomial expansion without writing out the whole expansion.

WATCH

Binomial Coefficients & Pascal's Triangle

Eddie Woo · 7:08

https://www.youtube.com/watch?v=HVgiRck9WAA

> **WHEN TO WATCH**
> Before Example 1: binomial coefficients and Pascal's triangle.

BACKUP VIDEO

Binomial Expansion Find a Specific Term

Mario's Math Tutoring · 7:24

https://www.youtube.com/watch?v=Y32hGegCoPI

Before the practice set: finding one term of a binomial expansion.

KEY TERMS

Pascal's triangle

A triangle of numbers whose rows begin and end with 1, where each entry between the two 1s is the sum of the two entries just above it. Rows and positions are numbered starting from 0.

Pascal's rule

The fact $C(n, r) = C(n-1, r-1) + C(n-1, r)$, which is the adding rule that builds Pascal's triangle.

binomial

A polynomial with two terms, such as $a + b$, $x - 2$ or $2y - 1$.

binomial coefficient

The number $C(n, k) = \binom{n}{k}$, which is the coefficient of $a^{n-k}b^k$ in the expansion of $(a + b)^n$.

Binomial Theorem

The formula $(a + b)^n = \sum_{k=0}^{n} C(n, k)\, a^{n-k}b^k$ for multiplying out any power of a binomial.

CORE IDEAS

Pascal's triangle starts with a 1 at the top, called row 0. Each row begins and ends with 1, and each entry between those 1s is the sum of the two entries just above it: row 5 is 1, 5, 10, 10, 5, 1, and the 15 in row 6 is $5 + 10$. Rows are numbered from 0, and so are the positions within a row: row 4 is 1, 4, 6, 4, 1, and its entry in position 2 is 6.

Pascal's triangle, rows 0 to 7

Row n	Entries C(n, 0), C(n, 1), ..., C(n, n)	Row sum
0	1	1
1	1, 1	2
2	1, 2, 1	4
3	1, 3, 3, 1	8
4	1, 4, 6, 4, 1	16
5	1, 5, 10, 10, 5, 1	32
6	1, 6, 15, 20, 15, 6, 1	64
7	1, 7, 21, 35, 35, 21, 7, 1	128

Compare row 4 with Lesson 10.2: $C(4, 0) = 1$, $C(4, 1) = 4$, $C(4, 2) = 6$, $C(4, 3) = 4$, $C(4, 4) = 1$. In general, the entry in row n, position r, is $C(n, r)$. To see why, it is

enough to show that the numbers $C(n, r)$ are built by the same rule as the triangle. The ends match, since $C(n, 0) = C(n, n) = 1$. The inside entries match because of **Pascal's rule:** $C(n, r) = C(n-1, r-1) + C(n-1, r)$.

Here is why Pascal's rule is true, by counting. Choose r people from a group of n that includes Ruth. Every such group either includes Ruth or does not. If it includes Ruth, the other $r-1$ members come from the remaining $n-1$ people: $C(n-1, r-1)$ ways. If it leaves Ruth out, all r members come from the other $n-1$ people: $C(n-1, r)$ ways. The two cases do not overlap and together cover every group, so they add to $C(n, r)$. For example, $C(5, 2) = C(4, 1) + C(4, 2) = 4 + 6 = 10$.

In Lesson 4.2 you multiplied out $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; its coefficients 1, 3, 3, 1 are row 3. Here is why. Write $(a+b)^n$ as n factors, $(a+b)(a+b)\cdots(a+b)$. Multiplying out means choosing either a or b from each factor, in every possible way, and adding all the products. A product equals $a^{n-k}b^k$ exactly when b was chosen from k of the n factors and a from the rest. There are $C(n, k)$ ways to choose those k factors, so $a^{n-k}b^k$ appears $C(n, k)$ times.

That is the **Binomial Theorem:** $(a+b)^n = \sum_{k=0}^{n} C(n, k)\, a^{n-k}b^k$. In the terms, the powers of a go down from n to 0, the powers of b go up from 0 to n, and the two exponents always add to n. The coefficients come from row n of Pascal's triangle (for small n) or from the formula for $C(n, k)$ (for large n). When the binomial has a subtraction, such as $(x-2)^5$, use $b = -2$; the signs then alternate, because odd powers of -2 are negative.

To find one term without writing the whole expansion, use the general term $C(n, k)\, a^{n-k}b^k$. Decide which k gives the power you want, then compute. Because k starts at 0, the term containing b^k is term number $k+1$: the 5th term uses $k = 4$.

Each row of the triangle adds to a power of 2: $1 + 4 + 6 + 4 + 1 = 16 = 2^4$. To see why, put $a = b = 1$ in the Binomial Theorem: $2^n = (1+1)^n = C(n, 0) + C(n, 1) + \cdots + C(n, n)$. In counting language, both sides count all the ways to choose some of n objects (from none to all): the right side sorts the choices by size, and the left side decides "in or out" for each object, $2 \cdot 2 \cdots 2 = 2^n$ ways.

KEY FORMULAS

$$C(n, r) = C(n-1, r-1) + C(n-1, r)$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} b^n$$

$$\text{Term } k+1 \text{ of } (a+b)^n : \quad \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

From history. The number triangle is much older than Pascal. It was known to Persian and Indian mathematicians long before him, and the Chinese mathematician Yang Hui wrote about it in 1261. Blaise Pascal studied its patterns carefully in his Treatise on the Arithmetical Triangle (written 1654), and in the West it now carries his name.

Practice habit. Know rows 0 through 6 of Pascal's triangle cold, and build any later row by adding neighbors. After expanding, check that the exponents in every term add to n, then substitute $x = 1$ into both sides: the two values must agree.

WORKED EXAMPLES

Example 1. Expand $(x+2)^4$.

Show solution

1. Row 4 of Pascal's triangle gives the coefficients 1, 4, 6, 4, 1. Here $a = x$ and $b = 2$.
2. $(x+2)^4 = 1 \cdot x^4 + 4x^3(2) + 6x^2(2)^2 + 4x(2)^3 + 1 \cdot (2)^4$.
3. Simplify each term: $x^4 + 8x^3 + 6 \cdot 4\,x^2 + 4 \cdot 8\,x + 16$.
4. $= x^4 + 8x^3 + 24x^2 + 32x + 16$.
5. Check with $x = 1$: $(1+2)^4 = 81$ and $1 + 8 + 24 + 32 + 16 = 81$ ✓.

Answer: $x^4 + 8x^3 + 24x^2 + 32x + 16$

Example 2. Ruth deposits \$2,000 in a savings account that earns 5% interest compounded once a year, so after 4 years her balance is $2000(1+0.05)^4$ dollars. (a) Use the Binomial Theorem to write $(1+0.05)^4$ as a sum of five terms, and find its exact value. (b) Find her balance after 4 years, to the nearest cent.

Show solution

1. (a) Row 4 gives 1, 4, 6, 4, 1, with $a = 1$ and $b = 0.05$. Every power of 1 is 1.
2. $(1+0.05)^4 = 1 + 4(0.05) + 6(0.05)^2 + 4(0.05)^3 + (0.05)^4$.
3. Powers: $(0.05)^2 = 0.0025$, $(0.05)^3 = 0.000125$, $(0.05)^4 = 0.00000625$.
4. Terms: $1 + 0.2 + 0.015 + 0.0005 + 0.00000625 = 1.21550625$.
5. (b) Balance: $2000 \times 1.21550625 = 2431.0125$, which is \$2,431.01 to the nearest cent.
6. Notice how fast the terms shrink. The term $4(0.05) = 0.2$ is the simple interest (\$400); the later terms are interest earned on interest, and the last two together add only about \$1.01.
7. Check with a calculator: $1.05^4 = 1.21550625$ ✓.

Answer: (a) $1 + 0.2 + 0.015 + 0.0005 + 0.00000625 = 1.21550625$; (b) \$2,431.01

PRACTICE

1. Easy Row 7 of Pascal's triangle is 1, 7, 21, 35, 35, 21, 7, 1. Use it to write row 8.

   Show answer

   1, 8, 28, 56, 70, 56, 28, 8, 1 (for example, $21 + 35 = 56$ and $35 + 35 = 70$)

2. Easy Expand $(a+b)^5$.

   Show answer

   $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

3. Easy Expand $(x-1)^3$.

   Show answer

   $x^3 - 3x^2 + 3x - 1$

4. **Easy** Without expanding, (a) how many terms does $(x+y)^9$ have, and (b) what is the coefficient of x^8y?

   Show answer

   (a) 10 terms ($k = 0$ to 9); (b) $C(9,1) = 9$

5. **Medium** Expand $(x+3)^4$.

   Show answer

   $x^4 + 4(3)x^3 + 6(9)x^2 + 4(27)x + 81 = x^4 + 12x^3 + 54x^2 + 108x + 81$ (check at $x = 1$: 256 ✓)

6. **Medium** Expand $(2y-1)^5$.

   Show answer

   Use $a = 2y$, $b = -1$ and row 5: $32y^5 - 80y^4 + 80y^3 - 40y^2 + 10y - 1$ (check at $y = 1$: $1^5 = 1$ and $32 - 80 + 80 - 40 + 10 - 1 = 1$ ✓)

7. **Medium** Find the coefficient of x^2 in the expansion of $(x+5)^6$.

   Show answer

   $x^{6-k} = x^2$ gives $k = 4$: $C(6,4) \cdot 5^4 = 15 \cdot 625 = 9375$

8. **Medium** Row 9 of Pascal's triangle is 1, 9, 36, 84, 126, 126, 84, 36, 9, 1. Use Pascal's rule to write row 10, and check that it adds to 2^{10}.

   Show answer

   1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1; the sum is $1024 = 2^{10}$ ✓

9. **Medium** A sandwich shop lets customers add any of 6 different extras: cheese, lettuce, tomato, pickles, onions and peppers. (a) How many different ways are there to choose extras, from none at all up to all six? (b) How many of these choices use exactly 2 extras?

   Show answer

   (a) Each extra is in or out: $2^6 = 64$ (the sum of row 6); (b) $C(6,2) = 15$

10. **Hard** Find the 5th term of $(x-2y)^8$ when the expansion is written in descending powers of x.

    Show answer

    The 5th term has $k = 4$: $C(8,4)\,x^4(-2y)^4 = 70 \cdot 16\,x^4y^4 = 1120x^4y^4$

11. Hard Find the term containing x^6 in the expansion of $(x^2+3)^5$.

Show answer

$(x^2)^{5-k}=x^6$ gives $k=2$: $C(5,2)(x^2)^3(3)^2 = 10\cdot 9\,x^6 = 90x^6$

12. Hard Henry deposits \$5,000 at 2% interest compounded annually, so after 6 years his balance is $5000(1+0.02)^6$ dollars. (a) Use the first three terms of the Binomial Theorem to approximate $(1.02)^6$. (b) Use your approximation to estimate the balance to the nearest dollar.

Show answer

(a) $1+6(0.02)+15(0.02)^2 = 1+0.12+0.006 = 1.126$; (b) $5000\times 1.126 = 5630$, about \$5,630 (the exact balance is \$5,630.81)

13. Hard Explain by a counting argument why Pascal's rule $C(n,r)=C(n-1,r-1)+C(n-1,r)$ is true. Then check it for $n=6$, $r=2$.

Show answer

Model answer: Pick one person, Grace, from a group of n. A group of r either includes Grace or not. If it does, choose the other $r-1$ members from the remaining $n-1$ people: $C(n-1,r-1)$ ways. If it does not, choose all r from the other $n-1$: $C(n-1,r)$ ways. Every group is in exactly one case, so the two counts add to $C(n,r)$. Check: $C(5,1)+C(5,2)=5+10=15=C(6,2)$ ✓

14. Challenge Prove that for every $n\ge 1$, $C(n,0)-C(n,1)+C(n,2)-\cdots+(-1)^nC(n,n)=0$. (Hint: use the Binomial Theorem with $a=1$ and $b=-1$.) Then check the result on row 4.

Show answer

Model answer: By the Binomial Theorem, $(1+(-1))^n = \sum_{k=0}^{n} C(n,k)\,1^{n-k}(-1)^k = C(n,0)-C(n,1)+C(n,2)-\cdots+(-1)^nC(n,n)$. The left side is $0^n=0$ for $n\ge 1$, so the alternating sum is 0. Row 4: $1-4+6-4+1=0$ ✓

COMMON MISTAKES

- Forgetting to raise the coefficient inside the binomial to the power. In $(2y-1)^5$ the first term is $(2y)^5=32y^5$, not $2y^5$. Keep each part of the binomial in parentheses while you expand.
- Losing the negative sign. For $(x-2)^n$ use $b=-2$, so every term with an odd power of b is negative.

- Using the wrong k for a numbered term. The first term has $k = 0$, so the 5th term has $k = 4$.
- Starting the row count at 1. The row 1, 4, 6, 4, 1 is row 4, and it gives the coefficients of $(a + b)^4$.

> **Make it a game · Build the Triangle.** On a large sheet, build Pascal's triangle to row 8 as quickly as you can, adding adjacent entries. Time yourself. Pick any entry and show that it equals the sum of the two above it, such as C(5, 2) = C(4, 1) + C(4, 2). Then use row 4 to expand $(x + y)^4$, and explain what each coefficient counts.

10.4 Probability with Counting

OBJECTIVES

- Find the probability of an event with equally likely outcomes by counting the event and the sample space, using permutations and combinations where needed.
- Use the complement to find "at least one" probabilities.
- Multiply the probabilities of independent events.
- Derive and use the binomial probability formula $C(n, k)\, p^k (1 - p)^{n-k}$.

WATCH

Permutations and Combinations (2 of 2: Applications towards probability and counting)

Eddie Woo · 13:04

https://www.youtube.com/watch?v=fx3efwNp4H0

> **WHEN TO WATCH**
> Before Example 1: counting outcomes to find probabilities.

BACKUP VIDEO

Binomial Probability (1 of 2: What defines a Binomial Probability question)

Eddie Woo · 6:30

https://www.youtube.com/watch?v=A4BQ5Ji0w7k

Before Example 2: what makes a question a binomial probability question.

KEY TERMS

sample space

The list of all possible outcomes of an experiment, such as the 36 ordered results of rolling two number cubes.

event

A set of outcomes from the sample space, such as "the sum is 7."

probability (equally likely outcomes)

When all outcomes are equally likely, $P(E)$ is the number of outcomes in the event E divided by the number of outcomes in the sample space; it is always between 0 and 1.

complement

The event "E does not happen," written "not E"; its probability is $1 - P(E)$.

independent events

Events where one happening does not change the probability of the other, such as two separate free throws.

binomial probability

The probability of exactly k successes in n independent trials, each with success probability p: $C(n, k)\, p^k (1-p)^{n-k}$.

CORE IDEAS

An experiment is any action with an uncertain result, such as rolling a number cube or planting a seed. The **sample space** is the list of all possible outcomes, and an **event** is a set of outcomes. When all outcomes are equally likely, the probability of an event E is the fraction of outcomes that belong to it: $P(E) = \dfrac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$. A probability is always between 0 (impossible) and 1 (certain).

Counting the sample space carefully is the whole job. When two number cubes, one red and one blue, are rolled, the counting principle gives $6 \times 6 = 36$ equally likely outcomes. The table shows the sum for each. The sum 7 appears 6 times, so $P(\text{sum is } 7) = \dfrac{6}{36} = \dfrac{1}{6}$, while the sum 2 appears only once. Treat the two cubes as different even if they look alike; the 11 possible sums are not equally likely, but the 36 ordered pairs are.

Sums of two number cubes (36 equally likely outcomes)

Red ↓ Blue →	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9

Red ↓ Blue →	1	2	3	4	5	6
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

For larger sample spaces, count with the tools of this unit. If 2 names are drawn from 8 volunteers to drive the church van, there are $C(8,2) = 28$ equally likely pairs. If 5 of the 8 have driven the van before, the event "both drivers have driven it before" contains $C(5,2) = 10$ pairs, so its probability is $\frac{10}{28} = \frac{5}{14}$. Count the event and the sample space the same way: both with order or both without.

The **complement** of an event E is the event "E does not happen." Every outcome is in exactly one of the two, so their counts add to the size of the sample space, and $P(\text{not } E) = 1 - P(E)$. This is the easy way to handle "at least one": the opposite of "at least one" is "none," so $P(\text{at least one}) = 1 - P(\text{none})$. In the van example, the probability that at least one of the two drivers is new to the van is $1 - \frac{5}{14} = \frac{9}{14}$.

Two events are **independent** when one happening does not change the probability of the other, like two separate free throws or two seeds in different pots. For independent events, $P(A \text{ and } B) = P(A) \cdot P(B)$. This is the counting principle again: if A happens in the fraction $P(A)$ of all outcomes, and B happens in the fraction $P(B)$ of those, then both happen in the fraction $P(A) \cdot P(B)$. Drawing names without putting them back is not independent, because each draw changes what is left.

Now repeat an experiment n independent times, where each time it succeeds with probability p or fails with probability $1 - p$, as when each of several seeds sprouts or not. What is the probability of exactly k successes? One particular order, such as k successes followed by $n - k$ failures, has probability $p^k(1-p)^{n-k}$ by the multiplication rule. Every other order with k successes has the same probability, since it multiplies the same factors in a different order. The number of such orders is the number of ways to choose which k of the n trials succeed, $C(n,k)$. So $P(\text{exactly } k \text{ successes}) = C(n,k)\,p^k(1-p)^{n-k}$.

These probabilities are exactly the terms of the Binomial Theorem from Lesson 10.3, with $a = 1 - p$ and $b = p$. Adding all of them gives $\big((1-p)+p\big)^n = 1^n = 1$, just as the probabilities of every possible number of successes must add to 1. The table shows the case of 4 seeds, each sprouting with probability 0.8. The Statistics course goes much further with these binomial distributions.

4 seeds, each sprouting with probability 0.8

Seeds that sprout, k	C(4, k)	0.8^k × 0.2^(4 − k)	Probability
0	1	0.002	0.002
1	4	0.006	0.026
2	6	0.026	0.154
3	4	0.102	0.41
4	1	0.41	0.41
Total			1

KEY FORMULAS

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$

$$P(\text{not } E) = 1 - P(E), \qquad P(\text{at least one}) = 1 - P(\text{none})$$

$$P(A \text{ and } B) = P(A) \cdot P(B) \quad \text{for independent events}$$

$$P(\text{exactly } k \text{ successes in } n \text{ trials}) = C(n,k)\, p^k (1-p)^{n-k}$$

Practice habit. Write each probability as a fraction first and reduce it, then convert to a decimal if asked. Check that every answer is between 0 and 1, and that an event and its complement add to 1.

Example 1. A church youth group has 10 members: 6 girls and 4 boys. Three names are drawn at random to help serve the church supper. (a) What is the probability that all three are girls? (b) What is the probability that at least one boy is chosen?

Show solution

1. The order of the names does not matter, so the sample space is all groups of 3 from 10: $C(10,3) = \dfrac{10 \cdot 9 \cdot 8}{6} = 120$ equally likely groups.
2. (a) Groups of 3 girls: $C(6,3) = \dfrac{6 \cdot 5 \cdot 4}{6} = 20$.
3. $P(\text{all girls}) = \dfrac{20}{120} = \dfrac{1}{6} \approx 0.167$.
4. (b) "At least one boy" is the complement of "no boys," which is "all girls."
5. $P(\text{at least one boy}) = 1 - \dfrac{1}{6} = \dfrac{5}{6} \approx 0.833$.
6. Check (a) another way, by counting ordered draws instead: there are $P(10,3) = 10 \cdot 9 \cdot 8 = 720$ ordered draws, and $P(6,3) = 6 \cdot 5 \cdot 4 = 120$ of them are all girls, so $\dfrac{120}{720} = \dfrac{1}{6}$ ✓. (Counting both the event and the sample space with order gives the same answer as counting both without order.)

Answer: (a) $\dfrac{1}{6}$; (b) $\dfrac{5}{6}$

Example 2. A seed company states that 90% of its pepper seeds sprout. Grace plants 5 seeds, and each sprouts independently of the others. (a) What is the probability that exactly 4 sprout? (b) What is the probability that at least 4 sprout? Round each answer to the nearest thousandth.

Show solution

1. Each seed is a trial with success probability $p = 0.9$ and failure probability $1 - p = 0.1$; there are $n = 5$ trials.
2. (a) $P(\text{exactly } 4) = C(5, 4)(0.9)^4(0.1)^1 = 5 \times 0.6561 \times 0.1 = 0.32805$.
3. (b) "At least 4" means exactly 4 or exactly 5. These are separate cases, so add.
4. $P(\text{exactly } 5) = C(5, 5)(0.9)^5(0.1)^0 = 0.59049$.
5. $P(\text{at least } 4) = 0.32805 + 0.59049 = 0.91854$.
6. Rounded: (a) 0.328; (b) 0.919. Check: both are between 0 and 1, and (b) is larger than (a) because it includes more outcomes ✓.

Answer: (a) 0.328; (b) 0.919

PRACTICE

1. Easy A number cube is rolled once. What is the probability of rolling a number greater than 4?

   Show answer

   Two outcomes (5 and 6) out of 6: $\frac{2}{6} = \frac{1}{3}$

2. Easy A basket holds 12 eggs: 7 brown and 5 white. Ruth picks one egg at random. (a) What is the probability that it is white? (b) What is the probability that it is not white?

   Show answer

   (a) $\frac{5}{12}$; (b) $1 - \frac{5}{12} = \frac{7}{12}$

3. Easy Two number cubes, one red and one blue, are rolled. Use the table to find the probability that the sum is 10.

Sums of two number cubes (36 equally likely outcomes)

Red ↓ Blue →	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Show answer

The sum 10 appears 3 times: $\frac{3}{36} = \frac{1}{12}$

4. Easy Samuel tosses a coin and rolls a number cube. What is the probability that he gets heads and a 6?

Show answer

The two results are independent: $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

5. **Medium** Two number cubes are rolled. Use the table to find the probability that the sum is greater than 9.

Sums of two number cubes (36 equally likely outcomes)

Red ↓ Blue →	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Show answer

Sums 10, 11 and 12 appear $3 + 2 + 1 = 6$ times: $\dfrac{6}{36} = \dfrac{1}{6}$

6. **Medium** A machine chooses a 4-digit gate code at random, using the digits 0 through 9 with repeats allowed. (a) What is the probability that no digit is repeated? (b) What is the probability that at least one digit is repeated?

Show answer

(a) $\dfrac{P(10,4)}{10^4} = \dfrac{5040}{10{,}000} = 0.504$; (b) $1 - 0.504 = 0.496$

7. **Medium** Grace makes 80% of her free throws, and each shot is independent of the others. She shoots two free throws. (a) What is the probability that she makes both? (b) What is the probability that she makes at least one?

Show answer

(a) $0.8 \times 0.8 = 0.64$; (b) $1 - P(\text{misses both}) = 1 - 0.2 \times 0.2 = 0.96$

8. Medium Samuel guesses on all 5 questions of a true-false quiz. What is the probability that he gets exactly 3 right?

Show answer

$$C(5,3)\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{2} = \frac{10}{32} = \frac{5}{16} = 0.3125$$

9. Medium Four names are drawn at random from 7 adults and 5 teenagers to plan the church picnic. What is the probability that exactly 2 adults are chosen? Give an exact fraction.

Show answer

$$\frac{C(7,2)\cdot C(5,2)}{C(12,4)} = \frac{21\cdot 10}{495} = \frac{210}{495} = \frac{14}{33}$$

10. Hard Seven members of the Miller family, including the three children, line up in random order for a photo. What is the probability that the three children stand next to each other?

Show answer

All lineups: $7! = 5040$. Children together: treat them as one block, so $5!\cdot 3! = 120\cdot 6 = 720$. Probability: $\frac{720}{5040} = \frac{1}{7}$

11. Hard Henry plants 4 squash seeds in separate pots. Each seed sprouts with probability 0.7, independently of the others. What is the probability that at least one sprouts?

Show answer

$$1 - P(\text{none sprout}) = 1 - (0.3)^4 = 1 - 0.0081 = 0.9919$$

12. Hard The six letter tiles G, A, R, D, E, N are shuffled and laid in a row at random. What is the probability that the row begins with G and ends with N?

Show answer

All arrangements: $6! = 720$. With G first and N last, the middle 4 letters can be arranged in $4! = 24$ ways: $\frac{24}{720} = \frac{1}{30}$

13. Hard An archer hits the target 75% of the time, and each shot is independent. She shoots 6 arrows. (a) What is the probability that exactly 5 hit the target? (b) What is the probability that at least 5 hit? Round each answer to the nearest thousandth.

Show answer

(a) $C(6,5)(0.75)^5(0.25) = 6 \times 0.2373047 \times 0.25 \approx 0.356$; (b) add $P(6) = (0.75)^6 \approx 0.1779785$: $0.3559570 + 0.1779785 \approx 0.534$

14. Challenge Explain why the binomial probabilities $C(n,k)\,p^k(1-p)^{n-k}$ for $k = 0, 1, \ldots, n$ always add to 1.

Show answer

Model answer: By the Binomial Theorem with $a = 1-p$ and $b = p$, $\sum_{k=0}^{n} C(n,k)(1-p)^{n-k}p^k = \big((1-p)+p\big)^n = 1^n = 1$. This makes sense: in n trials the number of successes must be exactly one of $0, 1, \ldots, n$, so these probabilities cover every outcome exactly once.

15. Challenge Five members of a family were each born on one of the 7 days of the week; assume each day is equally likely and the birthdays are independent. What is the probability that at least two of them were born on the same day of the week? Round to the nearest thousandth.

Show answer

$P(\text{all different}) = \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \dfrac{2520}{16{,}807} \approx 0.150$, so $P(\text{at least two share a day}) = 1 - 0.150 = 0.850$

COMMON MISTAKES

- Using outcomes that are not equally likely. For two number cubes, the sum 7 is far more likely than the sum 2; count the 36 ordered pairs, not the 11 possible sums.
- Mixing order and no order. If you count the event with combinations, count the sample space with combinations too.
- Finding "at least one" by adding the chance of success on each try (this can even give more than 1). Use $1 - P(\text{none})$ instead.
- Leaving out the factor $C(n,k)$ in a binomial probability. $p^k(1-p)^{n-k}$ is the probability of ONE particular order; multiply by the number of orders.

Make it a game · At Least One Six. Roll two dice 36 times and tally how often at least one shows a six, scoring pretend points. First, predict by using the complement: $1 - (5/6)^2 = 11/36$, about 11 times. Compare your tally with the prediction. Then use C(n, r) to find the probability of drawing two even paper chips from a bag of ten numbered 1 to 10.

UNIT 10

Counting, Probability and the Binomial Theorem

How many ways can a choir line up for a photo? How many different garden beds can be planted from a box of seed packets? Unit 10 teaches you to count large collections without listing them. Everything rests on one idea, the fundamental counting principle, and from it come factorials, permutations (arrangements where order matters) and combinations (selections where it does not).

Counting then pays off twice. In Lesson 10.3 the numbers $C(n, r)$ turn out to be the entries of Pascal's triangle and the coefficients in the expansion of $(a + b)^n$, finishing the work on binomial products you began in Lesson 4.2. In Lesson 10.4 counting becomes probability: the chance of an event is one count divided by another, and the Binomial Theorem gives the probability of exactly k successes in n tries.

These ideas reach well beyond this course. The Statistics course builds on binomial probability, and counting arguments run through computer science, where programmers must know how many cases a program could face. As always, the goal is not only to get the number but to be able to explain why the count is right.

PREREQUISITES

- Multiplying whole numbers by hand and reducing fractions
- Solving quadratic equations by factoring (Lesson 3.2)
- Multiplying polynomials and the cube of a binomial (Lesson 4.2)
- Compound interest $A = P(1 + r)^t$ with yearly compounding (Lesson 7.2)
- Sigma notation (Lesson 9.1)

MASTERY CHECKLIST

- ☐ I can use the fundamental counting principle and explain it with a tree diagram.
- ☐ I can evaluate factorials by hand by cancelling, and explain why $0! = 1$.
- ☐ I can count permutations with $P(n, r) = \frac{n!}{(n-r)!}$, including arrangements with restrictions.
- ☐ I can count arrangements of letters or objects with repeats using $\frac{n!}{a!\,b!\cdots}$.
- ☐ I can decide whether order matters and choose between permutations and combinations.
- ☐ I can derive $C(n, r)$ from $P(n, r)$ and explain why $C(n, r) \cdot r! = P(n, r)$.
- ☐ I can build Pascal's triangle and explain Pascal's rule by counting.
- ☐ I can expand $(a + b)^n$ with the Binomial Theorem, explain why the coefficients are $C(n, k)$, and find a single term.
- ☐ I can find probabilities by counting equally likely outcomes with permutations and combinations.

- ☐ I can use the complement for "at least one" problems and multiply probabilities of independent events.
- ☐ I can find binomial probabilities $C(n,k)p^k(1-p)^{n-k}$ and explain where the formula comes from.

LESSONS IN THIS UNIT

10.1	**The Counting Principle and Permutations**	2 sessions of 45 minutes
10.2	**Combinations**	2 sessions of 45 minutes
10.3	**Pascal's Triangle and the Binomial Theorem**	2 sessions of 45 minutes
10.4	**Probability with Counting**	2 sessions of 45 minutes

Unit 10 test

Time: 60 minutes. A scientific calculator is allowed, except on problems marked "by hand." Show your work: for each counting problem, say which rule you are using and why (for example, "order matters, so a permutation"). Give exact answers (whole numbers, or fractions in lowest terms) unless a problem asks you to round; round as directed.

1. **5 pts**

   By hand, evaluate (a) $\dfrac{7!}{5!}$ and (b) $\dfrac{10!}{8!\,2!}$.

2. **6 pts**

   The Carter family is planning a summer trip. They will choose one of 3 campgrounds, one of 5 open weekends and one of 2 ways to travel (car or train). How many different trip plans are possible?

3. **6 pts**

   A church pianist chooses 4 of her 15 favorite hymns and decides the order in which to play them before the service. How many different programs are possible?

4. **6 pts**

   In how many different ways can the letters of the word COOKBOOK be arranged?

5. **6 pts**

   A track coach has 12 runners. (a) In how many ways can she choose 4 of them for a relay team, if the running order is not yet decided? (b) In how many ways can she choose 4 runners and also assign the order in which they run?

6. **7 pts**

   A 4-H club will send a team of 5 members to the state fair. The club has 9 juniors and 6 seniors. How many teams have exactly 3 juniors and 2 seniors?

7. **6 pts**

   Solve $C(n, 2) = 45$ for the counting number n.

8. **7 pts**

   Expand $(x - 2)^{5}$.

9. **7 pts**

   Find the coefficient of x^{4} in the expansion of $(3x + 2)^{6}$.

10. **6 pts**

    Given that $C(10, 3) = 120$ and $C(10, 4) = 210$, use Pascal's rule to find $C(11, 4)$.

11. **6 pts**

    Two number cubes are rolled. What is the probability that the sum is 5 or 6?

12. **8 pts**

    After Thanksgiving dinner, the names of 9 family members, 5 adults and 4 children, are put in a bowl, and 3 names are drawn at random for dish duty. (a) What is the probability that all three are children? (b) What is the probability that at least one adult is drawn? Give exact fractions.

13. **10 pts**

An orchard keeper finds that 85% of her apple grafts take (grow successfully), each independently of the others. She makes 6 grafts. (a) What is the probability that exactly 5 take? (b) What is the probability that at least 5 take? Round each answer to the nearest thousandth.

14. **6 pts**

A barn has 3 smoke detectors that work independently, and each one works with probability 0.95. What is the probability that at least one of them works?

15. **8 pts**

Without computing either number, explain why $P(8,3)$ is exactly 6 times $C(8,3)$. Then compute both to confirm.

ANSWER KEY

1. (a) $7 \cdot 6 = 42$; (b) $\dfrac{10 \cdot 9}{2} = 45$
2. $3 \times 5 \times 2 = 30$ plans
3. Order matters: $P(15, 4) = 15 \cdot 14 \cdot 13 \cdot 12 = 32{,}760$ programs
4. 8 letters with four O's and two K's: $\dfrac{8!}{4!\,2!} = \dfrac{40{,}320}{48} = 840$
5. (a) $C(12, 4) = \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{24} = 495$; (b) $P(12, 4) = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$
6. $C(9, 3) \cdot C(6, 2) = 84 \cdot 15 = 1260$ teams
7. $\dfrac{n(n-1)}{2} = 45$, so $n^2 - n - 90 = 0$ and $(n - 10)(n + 9) = 0$; $n = 10$. Check: $\dfrac{10 \cdot 9}{2} = 45$ ✓
8. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$ (check at $x = 1$: $(-1)^5 = -1$ and $1 - 10 + 40 - 80 + 80 - 32 = -1$ ✓)
9. The term is $C(6, k)(3x)^{6-k}(2)^k$, and $6 - k = 4$ gives $k = 2$: $C(6, 2)(3)^4(2)^2 = 15 \cdot 81 \cdot 4 = 4860$
10. $C(11, 4) = C(10, 3) + C(10, 4) = 120 + 210 = 330$
11. Sum 5 in 4 ways and sum 6 in 5 ways, out of 36: $\dfrac{9}{36} = \dfrac{1}{4}$
12. (a) $\dfrac{C(4, 3)}{C(9, 3)} = \dfrac{4}{84} = \dfrac{1}{21}$; (b) $1 - \dfrac{1}{21} = \dfrac{20}{21}$
13. (a) $C(6, 5)(0.85)^5(0.15) = 6 \times 0.4437053 \times 0.15 \approx 0.399$; (b) add $P(6) = (0.85)^6 \approx 0.3771495$: $0.3993348 + 0.3771495 \approx 0.776$
14. $1 - P(\text{none work}) = 1 - (0.05)^3 = 1 - 0.000125 = 0.999875$
15. Model answer: Choosing and ordering 3 of 8 objects can be done in two stages: first choose which 3 ($C(8, 3)$ ways), then put them in order ($3! = 6$ ways). By the counting principle, $P(8, 3) = C(8, 3) \cdot 6$. Check: $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ and $C(8, 3) = 56$, and $56 \times 6 = 336$ ✓

UNIT 11

Conic Sections

Slice a double cone with a flat plane and you get one of four curves: a circle, an ellipse, a parabola or a hyperbola. In this unit you define each one by a simple rule about distances (equal distances to a point and a line; distances to two points that add to a constant; distances that differ by a constant) and turn each rule into an equation with the distance formula. You will find the vertices, foci, directrices and asymptotes that describe each curve, and use them to sketch.

The unit builds on tools you already have: the distance formula and the circle from Geometry, completing the square and the quadratic formula from Algebra I and Unit 3, transformations from Unit 1, and solving systems from Unit 2. It ends by solving systems that mix lines and conics. The applications are old and new at once: satellite dishes and flashlights, whispering galleries, the orbits of planets and comets, and navigation by radio signals. Later courses in precalculus and calculus return to these curves often.

PREREQUISITES

- Use the distance and midpoint formulas (Geometry).
- Write and read the equation of a circle $(x-h)^2+(y-k)^2=r^2$ (Geometry).
- Complete the square and solve quadratic equations by factoring and the quadratic formula (Lessons 3.1 and 3.2).
- Translate graphs by replacing x with $x-h$ and y with $y-k$ (Lesson 1.2).
- Solve systems of equations by substitution and elimination (Lesson 2.1).
- Simplify square roots and solve equations in quadratic form (Lessons 4.5 and 6.2).

MASTERY CHECKLIST

- ☐ I can describe the conic sections as slices of a double cone.
- ☐ I can derive $x^2 = 4py$ from the focus-directrix definition of a parabola.
- ☐ I can find the vertex, focus and directrix of a parabola and write its equation from given information.
- ☐ I can explain why $c^2 = a^2 - b^2$ for an ellipse and find its center, vertices, co-vertices and foci.
- ☐ I can write the equation of an ellipse or circle from a description.
- ☐ I can explain why $c^2 = a^2 + b^2$ for a hyperbola and why its asymptotes are $y = \pm\frac{b}{a}x$.
- ☐ I can sketch a hyperbola using its central rectangle and asymptotes.
- ☐ I can identify a conic from $Ax^2 + Cy^2 + Dx + Ey + F = 0$ and complete the square to put it in standard form.
- ☐ I can recognize degenerate cases: a point, a pair of lines or no graph.
- ☐ I can solve a system of a line and a conic, or of two conics, by

substitution or elimination and check every solution.

- ☐ I can use conics and nonlinear systems to solve problems about reflectors, orbits, navigation and fields.

LESSONS IN THIS UNIT

11.1 Parabolas: Focus and Directrix	2 sessions of 45 minutes
11.2 Circles and Ellipses	2 sessions of 45 minutes
11.3 Hyperbolas	2 sessions of 45 minutes
11.4 Recognizing and Graphing Conic Sections	2 sessions of 45 minutes
11.5 Nonlinear Systems	2 sessions of 45 minutes

Unit 11 Test

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem. Give exact answers (fractions or simplest radical form) unless a problem asks you to round. Check solutions of systems in both original equations.

1. **5 pts**
   Find the vertex, focus and directrix of $(y-4)^2 = 12(x+1)$.

2. **6 pts**
   Write the equation of the parabola with focus $(3, -1)$ and directrix $y = 5$.

3. **7 pts**
   A tractor headlight has a parabolic reflector 8 inches across at its opening and 3 inches deep. How far from the vertex should the bulb be placed, at the focus? Give an exact answer and a decimal to the nearest hundredth.

4. **6 pts**

   Find the vertices and foci of $\dfrac{x^2}{49} + \dfrac{y^2}{24} = 1$.

5. **7 pts**

   Write the equation of the ellipse with center $(-2, 1)$, a vertex at $(-2, 6)$ and a focus at $(-2, 4)$.

6. **7 pts**

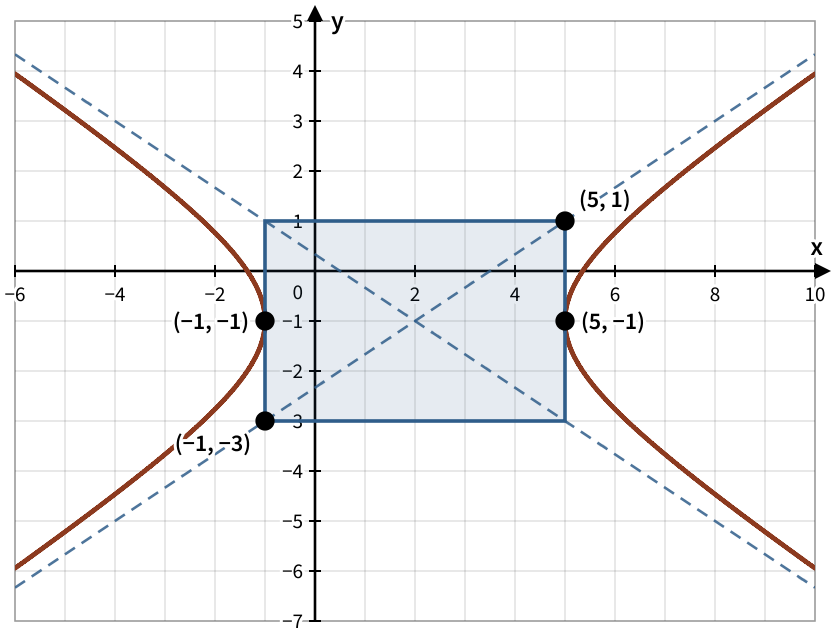
   A weather satellite moves in an elliptical orbit with the center of the Earth at one focus. Its closest distance to the Earth's center is 7 thousand km and its farthest is 9 thousand km. Using thousands of kilometers, with the center of the ellipse at the origin and the major axis on the x-axis, write the equation of the orbit.

7. **6 pts**

   Find the vertices, foci and asymptotes of $\dfrac{y^2}{36} - \dfrac{x^2}{64} = 1$.

8. **7 pts**

   The graph shows a hyperbola with its central rectangle (two corners labeled), its asymptotes and its vertices. Write its equation and give its foci in simplest radical form.



9. **6 pts**

   Identify each conic. (a) $5x^2 + 2y^2 - 10x + 3 = 0$ (b) $x^2 - 6x - y + 4 = 0$ (c) $4y^2 - x^2 + 8y + 2x - 1 = 0$

10. **7 pts**

    Write $4x^2 + y^2 + 24x - 4y + 24 = 0$ in standard form. Give the center, the vertices and the foci.

11. **6 pts**

    Show that the graph of $x^2 - 9y^2 - 2x + 1 = 0$ is a pair of lines, and give their equations.

12. **5 pts**

    Solve the system $x^2 + y^2 = 20$ and $y = 2x$.

13. **6 pts**

    Solve the system $x^2 + y^2 = 41$ and $y = x^2 - 11$. Give exact answers.

14. **7 pts**

    Ruth's family fences a rectangular hay field with 440 yards of fence, enclosing 12,000 square yards. Find the length and width of the field.

15. **6 pts**

    Explain why the equation $3x^2 + 3y^2 - 6x + 12y + 20 = 0$ has no graph, even though $A = C$.

16. **6 pts**

    Find the value of b for which the line $y = 2x + b$ is tangent to the parabola $y = x^2$, and find the point of tangency.

ANSWER KEY

1. Vertex $(-1, 4)$; $p = 3$; focus $(2, 4)$; directrix $x = -4$
2. Vertex $(3, 2)$, halfway between; the focus is 3 units below, so $p = -3$: $(x-3)^2 = -12(y-2)$
3. $x^2 = 4py$ through $(4, 3)$: $16 = 12p$, $p = \dfrac{4}{3} \approx 1.33$ inches
4. $a = 7$, $c^2 = 49 - 24 = 25$, $c = 5$: vertices $(\pm 7, 0)$, foci $(\pm 5, 0)$
5. Vertical; $a = 5$, $c = 3$, $b^2 = 25 - 9 = 16$: $\dfrac{(x+2)^2}{16} + \dfrac{(y-1)^2}{25} = 1$
6. $2a = 7 + 9 = 16$, $a = 8$; $c = 8 - 7 = 1$; $b^2 = 64 - 1 = 63$: $\dfrac{x^2}{64} + \dfrac{y^2}{63} = 1$
7. $a = 6$, $b = 8$, $c = 10$: vertices $(0, \pm 6)$, foci $(0, \pm 10)$, asymptotes $y = \pm \dfrac{3}{4}x$
8. Center $(2, -1)$, opens left and right, $a = 3$, $b = 2$: $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{4} = 1$; $c = \sqrt{13}$, foci $(2 \pm \sqrt{13}, -1)$
9. (a) ellipse (same signs, unequal; $5(x-1)^2 + 2y^2 = 2$) (b) parabola (only x squared) (c) hyperbola (opposite signs; $(y+1)^2 - \dfrac{(x-1)^2}{4} = 1$)
10. $4(x+3)^2 + (y-2)^2 = -24 + 36 + 4 = 16$, so $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{16} = 1$: center $(-3, 2)$, vertices $(-3, -2)$ and $(-3, 6)$, $c^2 = 12$, foci $(-3, 2 \pm 2\sqrt{3})$
11. $(x-1)^2 - 9y^2 = 0$, so $(x - 1 - 3y)(x - 1 + 3y) = 0$: the lines $y = \dfrac{x-1}{3}$ and $y = -\dfrac{x-1}{3}$
12. $5x^2 = 20$, $x = \pm 2$: $(-2, -4)$ and $(2, 4)$
13. $x^2 = y + 11$: $y^2 + y - 30 = 0$, $y = 5$ or -6. $y = 5$: $x = \pm 4$; $y = -6$: $x = \pm\sqrt{5}$. Solutions $(-4, 5)$, $(-\sqrt{5}, -6)$, $(\sqrt{5}, -6)$, $(4, 5)$
14. $x + y = 220$, $xy = 12000$: $x^2 - 220x + 12000 = 0$, $x = \dfrac{220 \pm 20}{2} = 120$ or 100. The field is 120 yards by 100 yards
15. Completing the square: $3(x-1)^2 + 3(y+2)^2 = -20 + 3 + 12 = -5$. The left side is a sum of squares times 3, so it is never negative, and no point (x, y) can make it equal -5. The equation has no graph.
16. $x^2 - 2x - b = 0$ has discriminant $4 + 4b = 0$, so $b = -1$; then $(x-1)^2 = 0$, and the point is $(1, 1)$

11.1 Parabolas: Focus and Directrix

OBJECTIVES

- Describe the conic sections as slices of a double cone and define a parabola by its focus and directrix.
- Derive the equation $x^2 = 4py$ from the definition using the distance formula.
- Find the vertex, focus, directrix and direction of opening of a parabola written as $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$.
- Write the equation of a parabola from its focus and directrix, or from its vertex and one point, and use it in reflector problems.

WATCH

Locus of a Parabola (2 of 3: Deriving the basic equation)

Eddie Woo · 6:03

https://www.youtube.com/watch?v=ELKmizNpb2Q

> **WHEN TO WATCH**
> Before Example 1: deriving the equation of a parabola from its focus and directrix.

Note: Eddie Woo teaches in Australia: he calls the parabola a 'locus'.

BACKUP VIDEO

Equation of Parabola Given Focus and Directrix

Mario's Math Tutoring · 3:43

https://www.youtube.com/watch?v=jUJEFAi3mP8

Before the practice set: writing the equation from a given focus and directrix.

KEY TERMS

conic section

A curve made by slicing a double cone (two cones joined tip to tip) with a flat plane: a circle, an ellipse, a parabola or a hyperbola.

parabola

The set of all points in a plane that are the same distance from a fixed point (the focus) and a fixed line (the directrix).

focus

The fixed point in the definition of a parabola. It lies inside the curve, on its axis of symmetry.

directrix

The fixed line in the definition of a parabola. It lies outside the curve, perpendicular to the axis of symmetry.

vertex

The turning point of a parabola, halfway between the focus and the directrix.

focal width

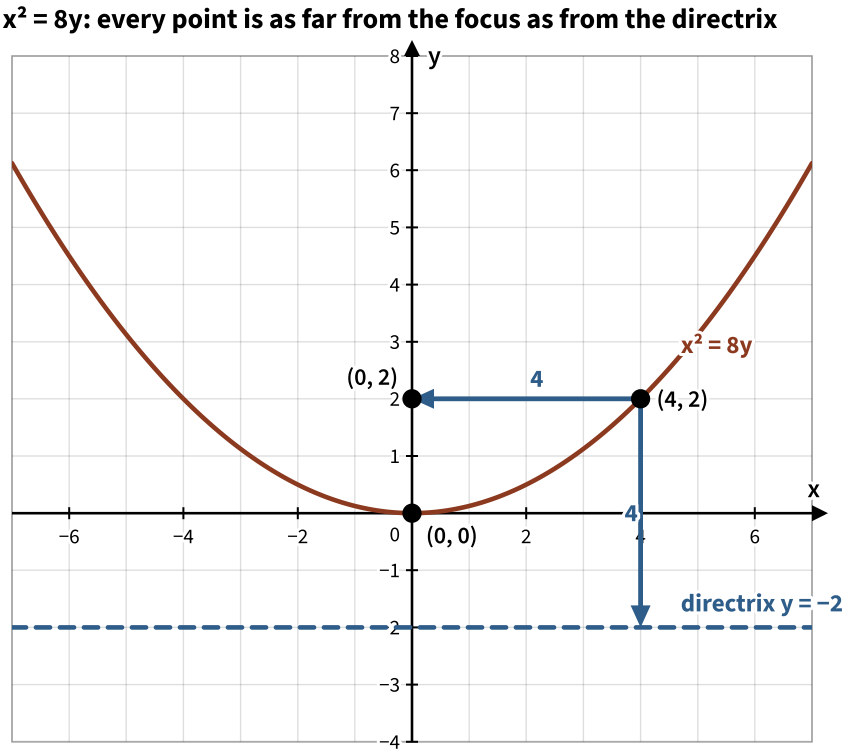
The length of the chord through the focus parallel to the directrix; it equals $|4p|$ and helps you sketch how wide the parabola is.

CORE IDEAS

Picture two ice-cream cones joined at their tips, one pointing up and one pointing down, and extending forever: a **double cone**. Slice it with a flat plane. A level cut gives a circle; a slightly tilted cut gives an ellipse; a cut parallel to the side of the cone gives a parabola; a steep cut through both halves gives a hyperbola with two branches. These four curves are the **conic sections**. In this unit we describe each one by a rule about distances and then turn that rule into an equation.

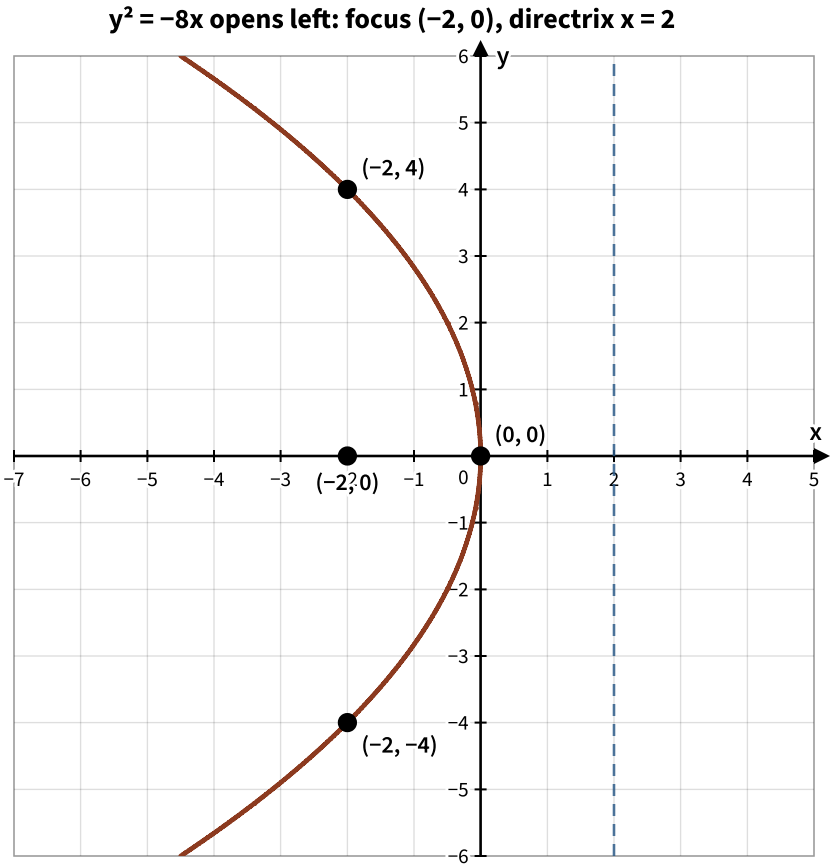
You met parabolas in Algebra I as graphs of $y = ax^2 + bx + c$. Here is the geometric definition: a **parabola** is the set of all points that are equally far from a fixed point, the **focus**, and a fixed line, the **directrix**. The **vertex** is itself on the parabola, so it is equally far from both: it sits exactly halfway between the focus and the directrix.

Now derive the equation. Put the vertex at the origin, the focus at $(0, p)$ and the directrix on the line $y = -p$. For any point (x, y), the distance to the focus is $\sqrt{x^2 + (y - p)^2}$ (distance formula), and the distance to the line $y = -p$ is $|y + p|$. Setting them equal and squaring both sides (both are non-negative, so nothing is gained or lost) gives $x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$. The y^2 and p^2 cancel, leaving $x^2 = 4py$. So every point of the parabola satisfies $x^2 = 4py$, and every point satisfying $x^2 = 4py$ is on the parabola.



In the picture, $x^2 = 8y$ means $4p = 8$, so $p = 2$: the focus is $(0, 2)$ and the directrix is $y = -2$. The point $(4, 2)$ is 4 units from the focus and 4 units above the directrix, just as the definition demands. The number p tells you everything: $|p|$ is the distance from the vertex to the focus (and from the vertex to the directrix); $p > 0$ means the parabola opens up toward the focus, and $p < 0$ means it opens down. Setting $y = p$ gives $x^2 = 4p^2$, so $x = \pm 2p$: the width of the parabola at the focus is $|4p|$. Plot those two points and your sketch will have the right shape.

If the directrix is vertical, the roles of x and y swap. With focus $(p, 0)$ and directrix $x = -p$ the same steps give $y^2 = 4px$. It opens right when $p > 0$ and left when $p < 0$. Quick test: the variable that is squared tells you the axis of symmetry runs along the *other* variable's direction; x^2 means it opens up or down, y^2 means it opens right or left.



To move the vertex to (h, k), replace x with $x - h$ and y with $y - k$; as in Lesson 1.2, this slides the whole picture (curve, focus and directrix) h units right and k units up. So $(x - h)^2 = 4p(y - k)$ has vertex (h, k), focus $(h, k + p)$ and directrix $y = k - p$, and $(y - k)^2 = 4p(x - h)$ has vertex (h, k), focus $(h + p, k)$ and directrix $x = h - p$.

Parabolas have a useful **reflecting property** (stated here, proved in calculus): every ray traveling parallel to the axis that strikes the inside of the parabola bounces off straight through the focus, and every ray leaving the focus bounces off parallel to the axis. That is why a satellite dish puts its receiver at the focus, a solar cooker puts the pot there, and a flashlight puts its bulb there to send out a straight beam.

KEY FORMULAS

$$x^2 = 4py: \quad \text{vertex } (0, 0),\ \text{focus } (0, p),\ \text{directrix } y = -p$$

$$y^2 = 4px: \quad \text{vertex } (0, 0),\ \text{focus } (p, 0),\ \text{directrix } x = -p$$

$$(x - h)^2 = 4p(y - k): \quad \text{focus } (h, k + p),\ \text{directrix } y = k - p$$

$$(y - k)^2 = 4p(x - h): \quad \text{focus } (h + p, k),\ \text{directrix } x = h - p$$

From history. The Greek geometer Apollonius of Perga wrote a great work called the Conics (about 200 BC), studying the curves made by slicing a cone. He gave the parabola, the ellipse and the hyperbola the names we still use. His results were studied for nearly two thousand years before anyone wrote them as equations.

Practice habit. Sketch before you compute: mark the vertex, then the focus and the directrix $|p|$ units away on opposite sides. The parabola always bends around the focus and away from the directrix.

WORKED EXAMPLES

Example 1. Find the vertex, focus and directrix of the parabola $(x+2)^2 = -12(y-1)$, and tell which way it opens.

Show solution

1. Match the form $(x-h)^2 = 4p(y-k)$: $x+2 = x-(-2)$, so $h=-2$ and $k=1$. The vertex is $(-2, 1)$.
2. $4p = -12$, so $p = -3$. Since x is squared and $p<0$, the parabola opens down.
3. Focus: $(h, k+p) = (-2, 1-3) = (-2, -2)$.
4. Directrix: $y = k - p = 1 + 3$, so $y = 4$.
5. Check with a point. Let $x = 4$: $36 = -12(y-1)$, so $y-1 = -3$ and $y = -2$. The point $(4, -2)$ is 6 units from the focus $(-2, -2)$ and 6 units below the line $y = 4$. ✓

Answer: Vertex $(-2, 1)$, focus $(-2, -2)$, directrix $y = 4$; opens down.

Example 2. Ruth is building a solar cooker for her family's camping trips. Its reflector is a parabolic dish 48 inches across and 9 inches deep. How far above the vertex (the bottom of the dish) should the pot be placed so that it sits at the focus?

Show solution

1. Place the vertex at the origin with the dish opening up, so its cross-section is $x^2 = 4py$.
2. The rim is 24 inches from the center (half of 48) and 9 inches up, so $(24, 9)$ is on the parabola.
3. Substitute: $24^2 = 4p(9)$, so $576 = 36p$.
4. Divide: $p = 16$. The equation is $x^2 = 64y$.
5. The focus is $(0, 16)$. Check: $(24, 9)$ is $\sqrt{24^2 + 7^2} = \sqrt{625} = 25$ from the focus and $9 + 16 = 25$ from the directrix $y = -16$. ✓

Answer: 16 inches above the vertex (the dish is $x^2 = 64y$).

PRACTICE

1. Easy Find the focus and directrix of $x^2 = 12y$.

   Show answer

   $4p = 12$, $p = 3$: focus $(0, 3)$, directrix $y = -3$

2. Easy Find the focus and directrix of $y^2 = -20x$. Which way does it open?

   Show answer

   $p = -5$: focus $(-5, 0)$, directrix $x = 5$; opens left

3. Easy Write the equation of the parabola with vertex $(0, 0)$ and focus $(0, -2)$.

   Show answer

   $p = -2$, so $x^2 = -8y$

4. Easy Write the equation of the parabola with vertex $(0, 0)$ and directrix $x = -4$.

   Show answer

   The focus is $(4, 0)$, so $p = 4$ and $y^2 = 16x$

5. **Medium** Find the vertex, focus and directrix of $(x-3)^2 = 8(y+1)$.

Show answer

Vertex $(3,-1)$, $p=2$: focus $(3,1)$, directrix $y=-3$

6. **Medium** Find the vertex, focus and directrix of $(y+2)^2 = -4(x-5)$.

Show answer

Vertex $(5,-2)$, $p=-1$ (opens left): focus $(4,-2)$, directrix $x=6$

7. **Medium** Write the equation of the parabola with focus $(2,5)$ and directrix $y=1$.

Show answer

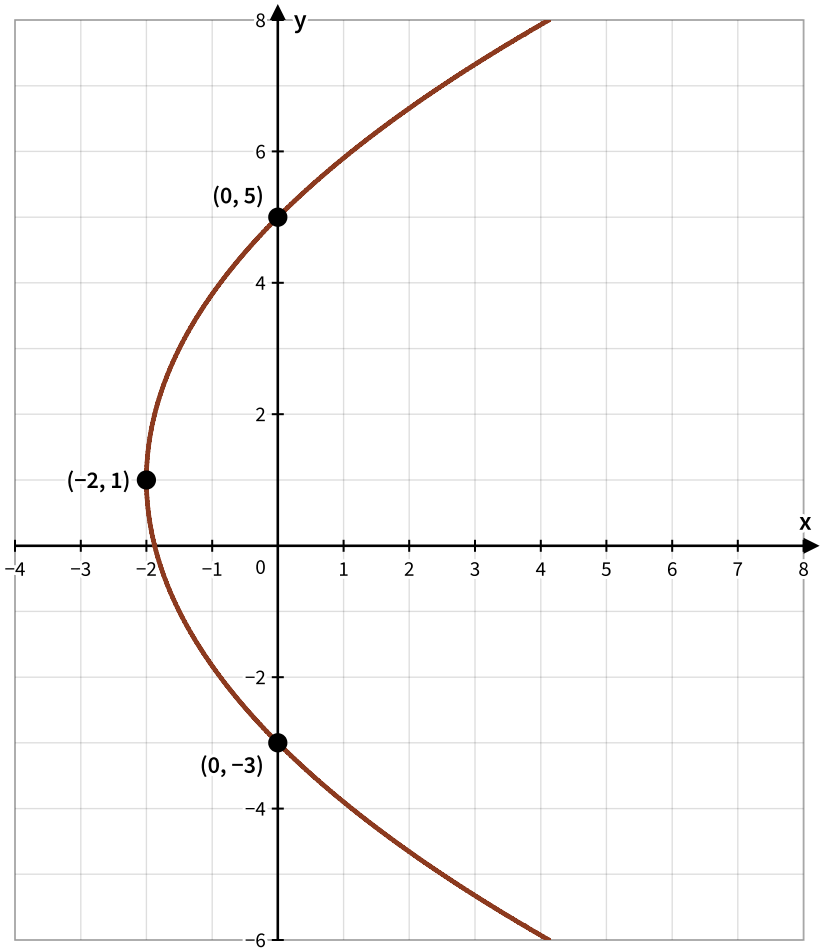
Vertex halfway: $(2,3)$; $p=5-3=2$; $(x-2)^2=8(y-3)$

8. **Medium** A radio telescope dish has the cross-section $x^2 = 160y$, measured in feet. (a) How far above the vertex is the receiver at the focus? (b) If the dish is 100 feet across, how deep is it?

Show answer

(a) $4p=160$, so 40 feet. (b) At $x=50$: $y=2500/160=15.625$ feet

9. **Medium** The parabola shown opens right, has vertex $(-2, 1)$ and passes through the other two labeled points. Write its equation and find its focus.



Show answer

$(y-1)^2 = 4p(x+2)$; using $(0,5)$: $16 = 8p$, $p = 2$. Equation $(y-1)^2 = 8(x+2)$; focus $(0, 1)$

10. **Hard** Write $x^2 - 6x - 12y + 33 = 0$ in the form $(x-h)^2 = 4p(y-k)$ and find the vertex, focus and directrix.

Show answer

$x^2 - 6x + 9 = 12y - 33 + 9$, so $(x-3)^2 = 12(y-2)$: vertex $(3, 2)$, focus $(3, 5)$, directrix $y = -1$

11. Hard Henry is making a flashlight reflector shaped like a parabola. The bulb sits at the focus, 1.5 inches from the vertex, and the reflector is 6 inches across at its opening. How deep is the reflector?

Show answer

$x^2 = 4(1.5)y = 6y$; at $x = 3$: $9 = 6y$, so $y = 1.5$ inches

12. Hard The main cable of a footbridge hangs in a parabola between two towers 400 feet apart. Where the cable attaches, the tops of the towers are 100 feet above the lowest point of the cable. Put the lowest point at the origin. (a) Write the equation of the cable. (b) How high above its lowest point is the cable 50 feet (horizontally) from the center?

Show answer

(a) $(200, 100)$ is on $x^2 = 4py$: $40000 = 400p$, $p = 100$, so $x^2 = 400y$. (b) $y = 2500/400 = 6.25$ feet

13. Challenge Find every point on the parabola $x^2 = 8y$ that is exactly 10 units from its focus. (Hint: use the definition, not the distance formula.)

Show answer

Distance to focus = distance to directrix $y = -2$, which is $y + 2$. So $y + 2 = 10$, $y = 8$, $x^2 = 64$: the points $(-8, 8)$ and $(8, 8)$

14. Challenge Prove that the parabola with focus $(p, 0)$ and directrix $x = -p$ has equation $y^2 = 4px$.

Show answer

A point (x, y) is on the parabola when its distance to $(p, 0)$ equals its distance to the line $x = -p$: $\sqrt{(x-p)^2 + y^2} = |x + p|$. Both sides are non-negative, so squaring keeps the same points: $x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$. Subtract $x^2 + p^2$ and add $2px$: $y^2 = 4px$.

COMMON MISTAKES

- Using $4p$ as the distance to the focus. In $x^2 = 12y$ the focus is 3 units away, not 12: always divide the coefficient by 4 first.
- Getting the direction wrong. Decide from the squared variable (x^2: up or down; y^2: right or left) and the sign of p, then check that the focus is inside the curve and the directrix outside.

- Putting the focus and the directrix on the same side of the vertex. They are always $|p|$ units away on opposite sides, so the vertex is their midpoint.
- Writing the directrix as a point. A directrix is a line, so its answer is an equation such as $y = -3$ or $x = 6$.

> **Make it a game · Fold a Parabola.** Draw a line near the bottom of a sheet of paper as the directrix, with a dot above as the focus. Fold so the focus lands on different points of the directrix, and crease each time. After a dozen folds, a parabola appears. Say what the vertex and focal distance p are, and write $x^2 = 4py$ for your sheet.

11.2 Circles and Ellipses

OBJECTIVES

- Write and read the equation of a circle $(x-h)^2+(y-k)^2=r^2$.
- Define an ellipse by its two foci and explain why $c^2=a^2-b^2$.
- Find the center, vertices, co-vertices, foci and axis lengths of an ellipse in standard form, with center at the origin or at (h,k).
- Write the equation of an ellipse from a description and use ellipses to solve problems about gardens, arches, whispering galleries and orbits.

WATCH

How to Graph Ellipses and Write the Equation

Mario's Math Tutoring · 7:32

https://www.youtube.com/watch?v=Rl1mE9b25_I

> **WHEN TO WATCH**
> Before Example 1: graphing an ellipse and writing its equation.

Note: Circles are taught from the Core ideas; neither video graphs one.

BACKUP VIDEO

Why slicing a cone gives an ellipse (beautiful proof)

3Blue1Brown · 12:51

https://www.youtube.com/watch?v=pQa_tWZmlGs

Optional, after the lesson: why slicing a cone gives an ellipse.

KEY TERMS

ellipse

The set of all points in a plane whose distances to two fixed points (the foci) add up to the same constant, called $2a$.

foci

The two fixed points in the definition of an ellipse (singular: focus). They lie on the major axis, c units from the center.

major axis

The longest chord of an ellipse, through both foci; its length is $2a$ and its endpoints are the vertices.

minor axis

The chord through the center perpendicular to the major axis; its length is $2b$ and its endpoints are the co-vertices.

center

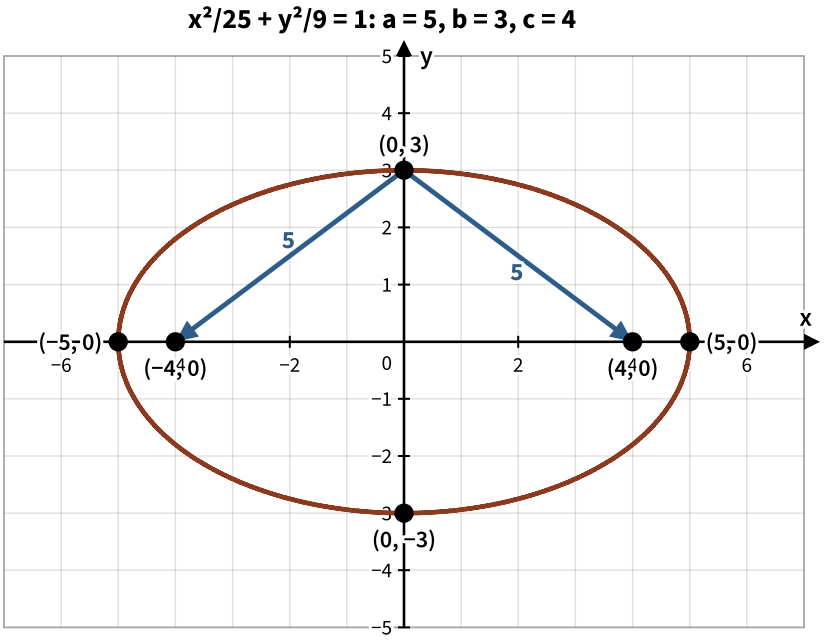
The midpoint of the segment joining the foci; it is also the midpoint of both axes.

CORE IDEAS

Recall from Geometry: a **circle** is the set of points at distance r from a center (h, k). The distance formula turns that sentence into $\sqrt{(x-h)^2 + (y-k)^2} = r$, and squaring gives $(x-h)^2 + (y-k)^2 = r^2$. For example, $(x+3)^2 + (y-1)^2 = 16$ is the circle with center $(-3, 1)$ and radius 4.

An **ellipse** uses two fixed points instead of one. It is the set of points P whose distances to two **foci** F_1 and F_2 add to a constant $2a$: $PF_1 + PF_2 = 2a$. You can draw one in a garden: drive two stakes into the ground, drop a loop of string around them, pull the loop tight with a stick and walk around. The part of the loop between the stakes never changes, so the two distances from the stick to the stakes always add to the same total, and the stick traces an ellipse.

Put the center at the origin with foci $(-c, 0)$ and $(c, 0)$. The **vertices** are $(\pm a, 0)$: from $(a, 0)$ the distances to the foci are $a - c$ and $a + c$, which add to $2a$. The **co-vertices** are $(0, \pm b)$ on the y-axis. Here is why $c^2 = a^2 - b^2$: the co-vertex $(0, b)$ is the same distance from both foci (by symmetry), and the two distances add to $2a$, so each is a. That distance is the hypotenuse of a right triangle with legs b and c, so by the Pythagorean theorem $b^2 + c^2 = a^2$.



Now the equation. Start from $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$. Move one root to the right side and square (as with radical equations in Lesson 6.5): after the x^2, y^2 and c^2 terms cancel, you get $a\sqrt{(x-c)^2+y^2}=a^2-cx$. Square again: $a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$, so $(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$. Replace a^2-c^2 by b^2 and divide by a^2b^2:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

In standard form, a always goes with the **larger** denominator, because $a>b$ (the hypotenuse is the longest side). If the larger denominator is under x^2, the major axis is horizontal; if it is under y^2, as in $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, the major axis is vertical and the foci are $(0,\pm c)$. A circle is the special ellipse with $a=b$: then $c=0$ and both foci sit at the center.

To move the center to (h,k), replace x by $x-h$ and y by $y-k$: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ (or with a^2 under the y-term). Find the center first, then count a, b and c units from it in the right directions.

Ellipses have their own reflecting property (stated): a ray or a sound wave leaving one focus bounces off the ellipse straight to the other focus. In an elliptical room, a **whispering gallery**, a whisper at one focus is heard clearly at the other. In the

heavens, planets, moons and comets travel in ellipses with the sun or planet at one focus.

KEY FORMULAS

$$(x-h)^2+(y-k)^2=r^2$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{(major axis horizontal, } a>b\text{)}$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad \text{(major axis vertical, } a>b\text{)}$$

$$c^2=a^2-b^2 \qquad PF_1+PF_2=2a$$

From history. For centuries astronomers assumed that planets move in circles. Working from careful observations of Mars, Johannes Kepler showed in 1609 that each planet moves in an ellipse with the sun at one focus. The curve Apollonius had studied as a slice of a cone turned out to be the shape of the heavens.

Practice habit. Know the conic standard forms cold. For an ellipse, find the center first, then let a be the square root of the larger denominator; a points along the major axis, and $c^2=a^2-b^2$ locates the foci.

Example 1. Find the center, vertices, co-vertices and foci of $\frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} = 1$.

Show solution

1. Center: $(h, k) = (2, -1)$.
2. The larger denominator, 25, is under the x-term, so the major axis is horizontal with $a = 5$; and $b = 3$.
3. Vertices: $(2 \pm 5, -1)$, which are $(-3, -1)$ and $(7, -1)$.
4. Co-vertices: $(2, -1 \pm 3)$, which are $(2, -4)$ and $(2, 2)$.
5. $c^2 = a^2 - b^2 = 25 - 9 = 16$, so $c = 4$. Foci: $(2 \pm 4, -1)$, which are $(-2, -1)$ and $(6, -1)$.
6. Check: from the vertex $(7, -1)$ the distances to the foci are 9 and 1, which add to $10 = 2a$. ✓

Answer: Center $(2, -1)$; vertices $(-3, -1)$ and $(7, -1)$; co-vertices $(2, -4)$ and $(2, 2)$; foci $(-2, -1)$ and $(6, -1)$.

Example 2. Grace wants an elliptical flower bed 20 feet long and 12 feet wide in front of the church. She will mark it with two stakes and a loop of string. How far apart should the stakes be, and how long must the loop of string be?

Show solution

1. The length is the major axis: $2a = 20$, so $a = 10$. The width is the minor axis: $2b = 12$, so $b = 6$. (The bed is $\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$ with the center at the origin.)
2. $c^2 = a^2 - b^2 = 100 - 36 = 64$, so $c = 8$. The stakes go at the foci, 8 feet on each side of the center: 16 feet apart.
3. As the stick moves, the loop runs from the stick to each stake ($PF_1 + PF_2 = 2a = 20$ feet) plus the stretch between the stakes (16 feet).
4. Loop length: $20 + 16 = 36$ feet.
5. Check at a co-vertex $(0, 6)$: each stake is $\sqrt{8^2 + 6^2} = 10$ feet away, and $10 + 10 + 16 = 36$. ✓

Answer: Stakes 16 feet apart (8 feet each side of center); a loop 36 feet long.

PRACTICE

1. Easy Find the center and radius of the circle $(x + 3)^2 + (y - 4)^2 = 49$.

   Show answer

   Center $(-3, 4)$, radius 7

2. Easy Write the equation of the circle with center $(2, -5)$ and radius 6.

   Show answer

   $(x - 2)^2 + (y + 5)^2 = 36$

3. Easy For $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$, find the vertices, co-vertices and foci.

   Show answer

   $a = 13$, $b = 5$, $c^2 = 169 - 25 = 144$, $c = 12$: vertices $(\pm 13, 0)$, co-vertices $(0, \pm 5)$, foci $(\pm 12, 0)$

4. **Easy** For $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$, find the lengths of the major and minor axes and the foci.

Show answer

Vertical major axis, $a = 5$, $b = 3$, $c = 4$: major axis 10, minor axis 6, foci $(0, \pm 4)$

5. **Medium** Write the equation of the ellipse with vertices $(\pm 10, 0)$ and foci $(\pm 6, 0)$.

Show answer

$a = 10$, $c = 6$, $b^2 = 100 - 36 = 64$: $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$

6. **Medium** An ellipse centered at the origin has a vertical major axis of length 14 and a minor axis of length 10. Write its equation and give its foci in simplest radical form.

Show answer

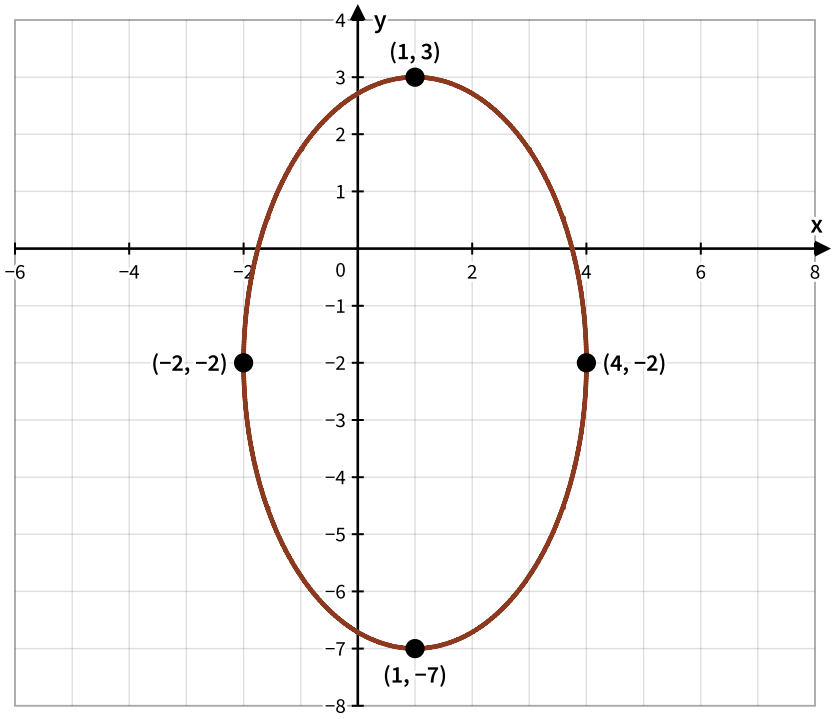
$a = 7$, $b = 5$: $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$; $c^2 = 24$, foci $(0, \pm 2\sqrt{6})$

7. **Medium** Find the center, vertices and foci of $\dfrac{(x+1)^2}{4} + \dfrac{(y-3)^2}{16} = 1$. Give the foci in simplest radical form.

Show answer

Center $(-1, 3)$; vertical, $a = 4$, $b = 2$; vertices $(-1, -1)$ and $(-1, 7)$; $c^2 = 12$, foci $(-1, 3 \pm 2\sqrt{3})$

8. **Medium** The graph shows an ellipse with its vertices and co-vertices labeled. Write its equation and find its foci.



Show answer

Center $(1, -2)$, $a = 5$ (vertical), $b = 3$: $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{25} = 1$; $c = 4$, foci $(1, -6)$ and $(1, 2)$

9. **Medium** Write $4x^2 + 25y^2 = 100$ in standard form and find its foci in simplest radical form.

Show answer

Divide by 100: $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$; $c^2 = 21$, foci $(\pm\sqrt{21}, 0)$

10. **Hard** Write the equation of the ellipse with foci $(2, 1)$ and $(2, 9)$ and a major axis of length 10.

Show answer

Center $(2, 5)$, $c = 4$, $a = 5$, $b^2 = 25 - 16 = 9$; vertical: $\dfrac{(x-2)^2}{9} + \dfrac{(y-5)^2}{25} = 1$

11. Hard A museum hall has an elliptical floor 60 feet long and 40 feet wide and works as a whispering gallery. Two visitors stand at the foci. (a) How far from the center does each stand? (b) How far is each from the nearer end wall? Round to the nearest tenth of a foot.

Show answer

$a = 30$, $b = 20$, $c^2 = 900 - 400 = 500$, $c = 10\sqrt{5} \approx 22.4$ feet; (b) $30 - 22.36 \approx 7.6$ feet

12. Hard A stone tunnel under a railway embankment has a semi-elliptical arch 40 feet wide at road level and 15 feet high at the center. (a) How high is the arch 6 feet from the center? Round to the nearest tenth. (b) Can a hay wagon loaded 12 feet wide and 13 feet tall pass through if it stays centered?

Show answer

(a) $\dfrac{x^2}{400} + \dfrac{y^2}{225} = 1$; at $x = 6$: $y^2 = 225(1 - 0.09) = 204.75$, $y \approx 14.3$ feet. (b) Yes, $13 < 14.3$

13. Challenge Halley's Comet moves in an ellipse with the sun at one focus. Use 0.6 AU for its closest distance to the sun and 35 AU for its farthest (1 AU is the average Earth-sun distance). Find (a) a, (b) c, and (c) b to the nearest tenth of an AU.

Show answer

(a) $2a = 0.6 + 35 = 35.6$, $a = 17.8$; (b) $c = a - 0.6 = 17.2$; (c) $b^2 = 17.8^2 - 17.2^2 = 316.84 - 295.84 = 21$, $b \approx 4.6$ AU

14. Challenge An ellipse has foci $(\pm c, 0)$ and $PF_1 + PF_2 = 2a$. Explain why the co-vertex $(0, b)$ is exactly a units from each focus, and use this to prove $c^2 = a^2 - b^2$.

Show answer

The point $(0, b)$ is on the perpendicular bisector of F_1F_2, so it is equally far from both foci. Those two equal distances add to $2a$, so each is a. The segment from $(0, b)$ to $(c, 0)$ is the hypotenuse of a right triangle with legs b and c, so $b^2 + c^2 = a^2$, that is, $c^2 = a^2 - b^2$.

COMMON MISTAKES

- Taking a from the x-term every time. a is the square root of the **larger** denominator, wherever it is; that tells you which way the major axis runs.

- Using $c^2 = a^2 + b^2$ (that is the hyperbola). For an ellipse the foci are inside the curve, so $c < a$ and $c^2 = a^2 - b^2$.
- Forgetting to square: in $\frac{x^2}{25} + \frac{y^2}{9} = 1$ the vertices are $(\pm 5, 0)$, not $(\pm 25, 0)$.
- Counting from the origin when the center is (h, k). Find the center first and measure a, b and c from it.

> **Make it a game · String and Pencil Ellipse.** Tape two coins to paper as foci. Loop a string around them and a pencil, keep it taut, and trace around. You have drawn an ellipse. Measure a, b and c with a ruler, then check $c^2 = a^2 - b^2$. Then write the equation of a circle with center (2, -1) and radius 3, and read aloud its center and radius.

11.3 Hyperbolas

OBJECTIVES

- Define a hyperbola by its two foci and explain why $c^2 = a^2 + b^2$.
- Find the center, vertices, foci and asymptotes of a hyperbola in standard form, with center at the origin or at (h, k).
- Explain why the asymptotes are $y = \pm\dfrac{b}{a}x$ and use the central rectangle to sketch a hyperbola.
- Write the equation of a hyperbola from a description, including a navigation problem based on differences in distance.

WATCH

Introduction to the Hyperbola

Eddie Woo · 13:47

https://www.youtube.com/watch?v=RH6tL-Lkzzo

> **WHEN TO WATCH**
> Before Example 1: what a hyperbola is.

BACKUP VIDEO

Hyperbolas - How to Graph

Mario's Math Tutoring · 8:19

https://www.youtube.com/watch?v=ehp6xDm4DO8

After Example 1: graphing a hyperbola with its asymptotes.

KEY TERMS

hyperbola

The set of all points in a plane whose distances to two fixed points (the foci) differ by the same constant, called $2a$. It has two separate branches.

transverse axis

The segment joining the two vertices; it has length $2a$ and lies on the line through the foci.

conjugate axis

The segment of length $2b$ through the center, perpendicular to the transverse axis.

central rectangle

The rectangle centered at the center of the hyperbola with sides $2a$ and $2b$; its diagonals, extended, are the asymptotes.

asymptotes of a hyperbola

The two lines through the center that the branches get closer and closer to as they move away from the center.

CORE IDEAS

An ellipse comes from a **sum** of distances; a **hyperbola** comes from a **difference**. It is the set of points P for which the distances to two foci differ by a constant: $|PF_1 - PF_2| = 2a$. Points closer to F_2 form one branch, and points closer to F_1 form the other.

Put the foci at $(\pm c, 0)$. The **vertices** are $(\pm a, 0)$: from $(a, 0)$ the distances to the foci are $c + a$ and $c - a$, which differ by $2a$. In triangle PF_1F_2 one side minus another is always less than the third side, so $2a < 2c$ and $a < c$. That means $c^2 - a^2$ is positive, so we may name it b^2:

$$c^2 = a^2 + b^2.$$

Notice the sign: for an ellipse c is the smallest of the three numbers, and for a hyperbola c is the largest.

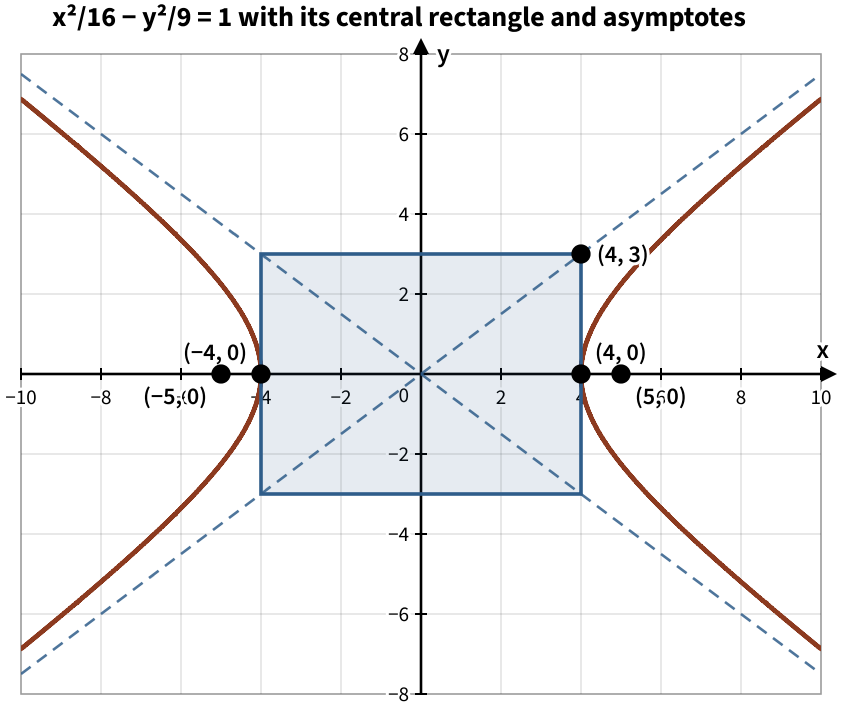
The derivation of the equation follows the same steps as the ellipse (move a root, square, simplify, square again) and again arrives at $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$. This time $a^2 - c^2 = -b^2$, so $-b^2x^2 + a^2y^2 = -a^2b^2$. Dividing by $-a^2b^2$ gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The curve never crosses the y-axis: at $x = 0$ the equation says $-y^2/b^2 = 1$, which is impossible.

Why the **asymptotes** are $y = \pm\dfrac{b}{a}x$: solve for y to get $y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$. When $|x|$ is large, $x^2 - a^2$ is almost exactly x^2, so y is almost $\pm\dfrac{b}{a}x$. In fact the gap is

$\frac{b}{a}\left(x-\sqrt{x^2-a^2}\right)=\frac{ab}{x+\sqrt{x^2-a^2}}$ (multiply by the conjugate), which shrinks toward 0 as x grows. Put another way: far from the center the 1 on the right side hardly matters, and $\frac{x^2}{a^2}=\frac{y^2}{b^2}$ is exactly the pair of lines $y=\pm\frac{b}{a}x$.



To sketch: (1) plot the center and the vertices a units away along the transverse axis; (2) draw the **central rectangle**, reaching a units from the center along the transverse axis and b units along the other; (3) draw its diagonals, extended, as dashed asymptotes; (4) draw each branch from a vertex, bending away from the center and toward the asymptotes. The foci are c units from the center, just beyond the vertices; c is the distance from the center to a corner of the rectangle, since $c^2=a^2+b^2$.

The **positive** term tells you the direction of opening, and a^2 is always under the positive term (it need not be the larger number). The hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ opens up and down, with vertices $(0,\pm a)$, foci $(0,\pm c)$ and asymptotes $y=\pm\frac{a}{b}x$. With center (h,k), replace x by $x-h$ and y by $y-k$; the asymptotes become $y-k=\pm(\text{slope})(x-h)$.

Hyperbolas guide ships and planes. Two radio stations send a signal at the same instant. A ship that receives one signal a little before the other can multiply the time

difference by the speed of radio waves to find how much closer it is to one station than to the other. That fixed difference puts the ship on a hyperbola with the stations as foci. A second pair of stations gives a second hyperbola, and the ship is where the two meet.

KEY FORMULAS

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1: \quad \text{opens left and right; asymptotes } y-k = \pm\frac{b}{a}(x-h)$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1: \quad \text{opens up and down; asymptotes } y-k = \pm\frac{a}{b}(x-h)$$

$$c^2 = a^2 + b^2 \qquad |PF_1 - PF_2| = 2a$$

Practice habit. Sketch before you compute: center, vertices, central rectangle, asymptotes, then the branches. Compare the two Pythagorean-looking facts: ellipse $c^2 = a^2 - b^2$, hyperbola $c^2 = a^2 + b^2$.

WORKED EXAMPLES

Example 1. Find the center, vertices, foci and asymptotes of $\frac{(y+1)^2}{16} - \frac{(x-3)^2}{9} = 1$.

Show solution

1. Center: $(h, k) = (3, -1)$.
2. The y-term is positive, so the hyperbola opens up and down, with $a^2 = 16$ and $b^2 = 9$: $a = 4$, $b = 3$.
3. Vertices: $(3, -1 \pm 4)$, which are $(3, -5)$ and $(3, 3)$.
4. $c^2 = a^2 + b^2 = 16 + 9 = 25$, so $c = 5$. Foci: $(3, -1 \pm 5)$, which are $(3, -6)$ and $(3, 4)$.
5. Asymptotes: slope $\pm\frac{a}{b} = \pm\frac{4}{3}$ through the center: $y + 1 = \pm\frac{4}{3}(x - 3)$.
6. Check: from the vertex $(3, 3)$ the distances to the foci are 9 and 1, which differ by $8 = 2a$. ✓

Answer: Center $(3, -1)$; vertices $(3, -5)$ and $(3, 3)$; foci $(3, -6)$ and $(3, 4)$; asymptotes $y + 1 = \pm\frac{4}{3}(x - 3)$.

Example 2. Two radio stations stand on a straight coastline 100 miles apart, at $A(-50, 0)$ and $B(50, 0)$ (miles). From the difference in arrival times of their signals, a fishing boat's captain finds that the boat is 60 miles closer to B than to A. (a) Write the equation of the hyperbola the boat is on. (b) The boat is also 30 miles out from the coast ($y = 30$). Find its position.

Show solution

1. The foci are the stations, so $c = 50$. The difference of distances is $2a = 60$, so $a = 30$.
2. $b^2 = c^2 - a^2 = 2500 - 900 = 1600$. The hyperbola is $\dfrac{x^2}{900} - \dfrac{y^2}{1600} = 1$; the boat is closer to B, so it is on the right branch ($x > 0$).
3. Substitute $y = 30$: $\dfrac{x^2}{900} = 1 + \dfrac{900}{1600} = \dfrac{25}{16}$.
4. $x^2 = 900 \times \dfrac{25}{16} = 1406.25$, so $x = 37.5$ (positive branch).
5. Check: the distance to B is $\sqrt{12.5^2 + 30^2} = \sqrt{1056.25} = 32.5$; the distance to A is $\sqrt{87.5^2 + 30^2} = \sqrt{8556.25} = 92.5$; the difference is 60. ✓

Answer: (a) $\dfrac{x^2}{900} - \dfrac{y^2}{1600} = 1$ (right branch); (b) the boat is at $(37.5, 30)$.

PRACTICE

1. Easy Find the vertices and foci of $\dfrac{x^2}{25} - \dfrac{y^2}{144} = 1$.

   Show answer

   $a = 5$, $c^2 = 25 + 144 = 169$, $c = 13$: vertices $(\pm 5, 0)$, foci $(\pm 13, 0)$

2. Easy Find the vertices and the asymptotes of $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$.

   Show answer

   Opens up and down, $a = 2$, $b = 4$: vertices $(0, \pm 2)$; asymptotes $y = \pm \dfrac{1}{2} x$

3. Easy For $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$, give the corners of the central rectangle and the equations of the asymptotes.

   Show answer

   Corners $(\pm 2, \pm 3)$; asymptotes $y = \pm \dfrac{3}{2} x$

4. **Easy** Write the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$.

Show answer

$b^2 = 25 - 9 = 16$: $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

5. **Medium** Write the equation of the hyperbola with vertices $(0, \pm 6)$ and asymptotes $y = \pm 2x$.

Show answer

Opens up and down, $a = 6$; slope $\dfrac{a}{b} = 2$ gives $b = 3$: $\dfrac{y^2}{36} - \dfrac{x^2}{9} = 1$

6. **Medium** Find the center, vertices, foci and asymptotes of $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$.

Show answer

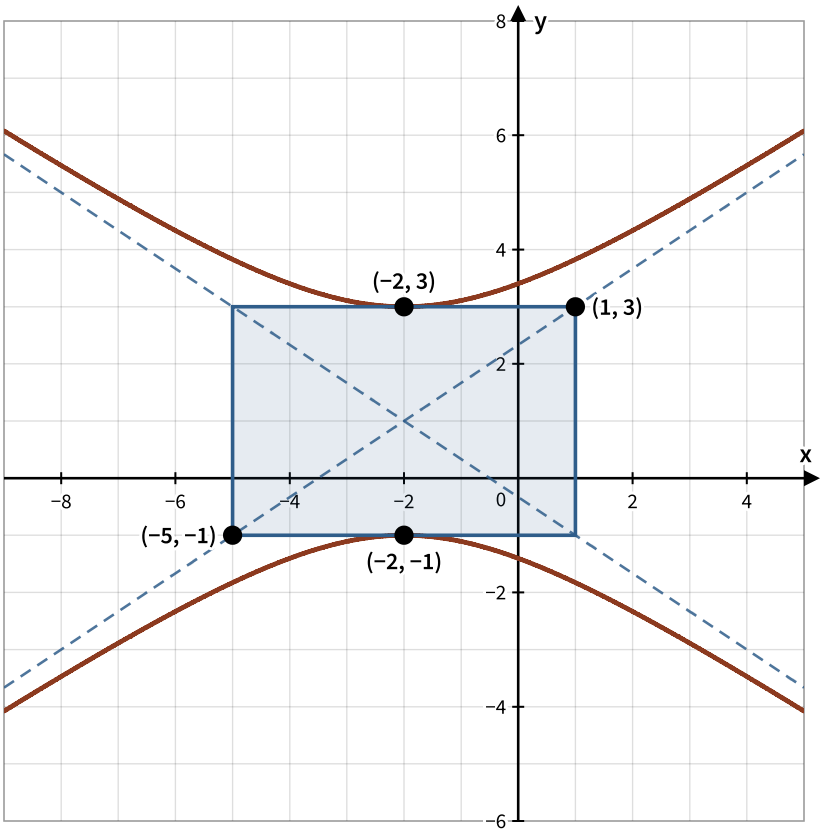
Center $(1, -2)$; $a = 4$, $b = 3$, $c = 5$; vertices $(-3, -2)$ and $(5, -2)$; foci $(-4, -2)$ and $(6, -2)$; asymptotes $y + 2 = \pm\dfrac{3}{4}(x - 1)$

7. **Medium** Write $9y^2 - 4x^2 = 36$ in standard form. Then give its vertices and asymptotes.

Show answer

Divide by 36: $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$; vertices $(0, \pm 2)$; asymptotes $y = \pm\dfrac{2}{3}x$

8. **Medium** The graph shows a hyperbola, its central rectangle (two corners labeled) and its asymptotes. The vertices are labeled. Write the equation of the hyperbola and of its asymptotes.



Show answer

Center $(-2, 1)$, opens up and down, $a = 2$, $b = 3$: $\dfrac{(y-1)^2}{4} - \dfrac{(x+2)^2}{9} = 1$; asymptotes $y - 1 = \pm\dfrac{2}{3}(x+2)$

9. **Medium** A comet that passes the sun only once travels along the right branch of $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ (in AU), with the sun at the focus on that side. (a) Where is the sun? (b) How close does the comet come to the sun?

Show answer

$c = 5$: (a) the sun is at $(5, 0)$; (b) the nearest point is the vertex $(4, 0)$, so $5 - 4 = 1$ AU

10. Hard Write the equation of the hyperbola with center $(-2, 3)$, a vertex at $(-2, 7)$ and a focus at $(-2, 8)$. Then give its asymptotes.

Show answer

Opens up and down, $a = 4$, $c = 5$, $b = 3$: $\dfrac{(y-3)^2}{16} - \dfrac{(x+2)^2}{9} = 1$; asymptotes $y - 3 = \pm\dfrac{4}{3}(x+2)$

11. Hard Two radio beacons stand beside a straight east-west highway, 200 miles apart, at $(-100, 0)$ and $(100, 0)$. A pilot finds that the small plane is 120 miles farther from the west beacon than from the east beacon, and that it is 60 miles north of the highway. Find the plane's position.

Show answer

$a = 60$, $c = 100$, $b = 80$: $\dfrac{x^2}{3600} - \dfrac{y^2}{6400} = 1$. At $y = 60$: $x^2 = 3600(1.5625) = 5625$, $x = 75$. Position $(75, 60)$; check: distances 185 and 65 differ by 120 ✓

12. Hard The sides of a power station's cooling tower follow the hyperbola $\dfrac{x^2}{900} - \dfrac{y^2}{2500} = 1$ in meters, with $y = 0$ at the narrowest point. (a) How wide is the tower at its narrowest? (b) How wide is it 60 meters above the narrowest point? Round to the nearest tenth of a meter.

Show answer

(a) $2a = 60$ meters. (b) $x^2 = 900\left(1 + \dfrac{3600}{2500}\right) = 2196$, $x \approx 46.86$, width ≈ 93.7 meters

13. Challenge For which slopes k does the line $y = kx$ never meet the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$? Explain how your answer relates to the asymptotes.

Show answer

Substitute: $x^2\left(\dfrac{1}{9} - \dfrac{k^2}{4}\right) = 1$ has no solution when $\dfrac{1}{9} - \dfrac{k^2}{4} \le 0$, that is $|k| \ge \dfrac{2}{3}$. Lines through the center at least as steep as the asymptotes $y = \pm\dfrac{2}{3}x$ miss the curve.

14. **Challenge** Explain why the branches of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ approach the lines $y = \pm\frac{b}{a}x$ but never touch them.

Show answer

Solving for y in the first quadrant gives $y = \frac{b}{a}\sqrt{x^2 - a^2}$, which is always less than $\frac{b}{a}x$ because $x^2 - a^2 < x^2$; so the curve never reaches the line. The gap is $\frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right) = \frac{ab}{x + \sqrt{x^2 - a^2}}$, and its denominator grows without bound as x grows, so the gap shrinks toward 0. By symmetry the same holds in the other quadrants.

COMMON MISTAKES

- Using $c^2 = a^2 - b^2$ from the ellipse. For a hyperbola the foci lie beyond the vertices, so c is the largest: $c^2 = a^2 + b^2$.
- Choosing a as the larger denominator. For a hyperbola a^2 is the denominator of the **positive** term, whichever is larger.
- Getting the asymptote slope upside down. Read it from the central rectangle: rise over run from the center to a corner. For $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ the slopes are $\pm\frac{a}{b}$.
- Forgetting the center in the asymptotes of a shifted hyperbola. They pass through (h, k): $y - k = \pm m(x - h)$, not $y = \pm mx$.

> **Make it a game · Rectangle Shortcut.** Beat the clock on three hyperbolas. For $x^2/16 - y^2/9 = 1$, find $c = 5$, the vertices $(\pm 4, 0)$, the foci and the asymptotes $y = \pm(3/4)x$ by drawing the central rectangle and its diagonals. Explain why $c^2 = a^2 + b^2$ this time. Then shift one hyperbola so its center is at (h, k).

11.4 Recognizing and Graphing Conic Sections

OBJECTIVES

- Identify a conic section from the coefficients A and C of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ and explain why the test works.
- Complete the square in x and in y to rewrite a general equation in standard form.
- Find the key features of the conic and sketch it.
- Recognize degenerate cases: a single point, a pair of lines, or no graph.

WATCH

Writing Conic Sections in Standard Form

Mario's Math Tutoring · 3:19

https://www.youtube.com/watch?v=Io-NoDECNNw

> **WHEN TO WATCH**
> Before Example 1: deciding which conic it is, then completing the square.

Note: Neither video explains why the test on A and C works or shows degenerate cases. Teach those from the Core ideas. Eddie Woo teaches in Australia: he says 'general form' for $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

BACKUP VIDEO

Further Work on the Ellipse (3 of 4: Completing the Square to find the general form of the Ellipse)

Eddie Woo · 11:36

https://www.youtube.com/watch?v=U0vuM4vtxFs

Before Example 2: completing the square in x and in y for an ellipse.

KEY TERMS

general form of a conic

An equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ with A and C not both zero. (Conics turned at an angle also need an xy-term; we leave those to later courses.)

standard form

The form of a conic's equation that shows its center or vertex and its size directly, such as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

completing the square

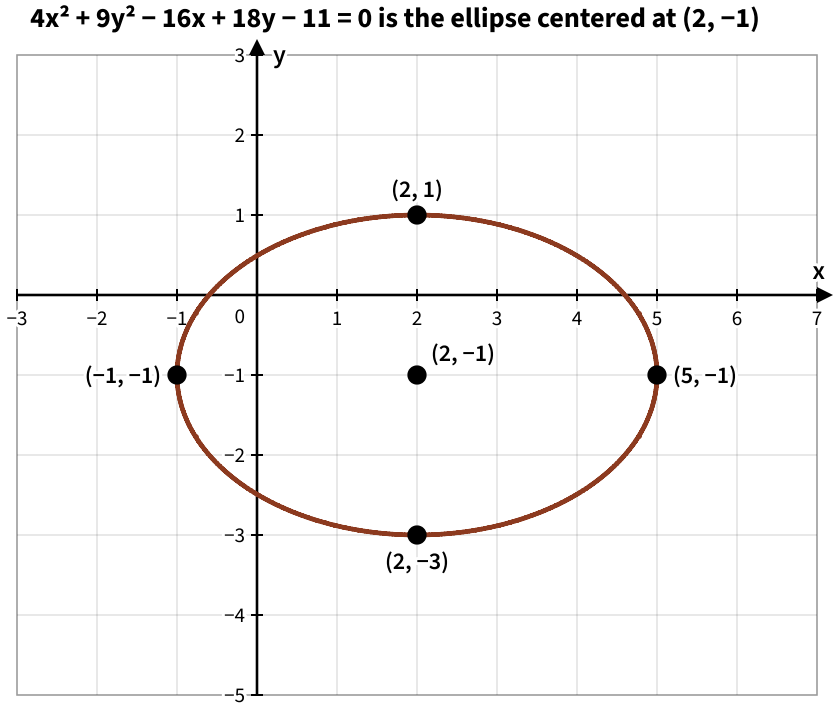
Adding the number that turns $x^2 + bx$ into the perfect square $\left(x + \frac{b}{2}\right)^2$, namely $\left(\frac{b}{2}\right)^2$, and adding the same amount to the other side.

degenerate conic

What you get when the plane slices through the tip of the double cone: a single point, a line, or a pair of crossing lines. Some equations in general form have no graph at all.

CORE IDEAS

Every conic you have met can be multiplied out into one shape. Take the ellipse $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$. Multiply by 36: $4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) = 36$. Expand and collect: $4x^2 + 9y^2 - 16x + 18y - 11 = 0$. This is the **general form** $Ax^2 + Cy^2 + Dx + Ey + F = 0$. It is tidy, but it hides the center and the size, so we need to know how to go back.



You can name the conic before doing any work by looking at A and C, the coefficients of x^2 and y^2. **Circle:** $A = C$. **Ellipse:** A and C have the same sign but are different. **Hyperbola:** A and C have opposite signs. **Parabola:** only one variable is squared ($A = 0$ or $C = 0$). Why this works: completing the square never changes the squared terms ($A(x-h)^2$ still begins with Ax^2), so A and C are exactly the coefficients you would see in standard form, multiplied through. A circle has equal

coefficients, an ellipse two positive (unequal) ones, a hyperbola a plus and a minus, and a parabola only one squared variable.

To reach standard form, complete the square in each variable. (1) Group the x-terms and the y-terms and move F to the right. (2) Factor A out of the x-group and C out of the y-group. (3) Inside each group add $\left(\frac{b}{2}\right)^2$, where b is the coefficient of the first-degree term inside the parentheses. (4) Add the same amounts to the right side, **multiplied by the factor you pulled out**. (5) Divide so that the right side is 1 (ellipse, hyperbola), or leave r^2 (circle) or $4p(\ldots)$ (parabola).

Step (4) is where most errors happen. In $4(x^2 - 4x + 4)$ you have really added $4 \times 4 = 16$ to the left side, not 4, so the right side must also get 16. Writing the amount you add under each group, one step per line, keeps the two sides balanced.

Sometimes the right side comes out as 0 or as a negative number. If $4(x-1)^2 + 9(y+2)^2 = 0$, both squares must be 0, so the graph is the single point $(1, -2)$. If the right side were -5, there would be no points at all, since a sum of squares is never negative. A hyperbola-type equation with 0 on the right, such as $x^2 - 4y^2 = 0$, factors as $(x - 2y)(x + 2y) = 0$ and is a pair of crossing lines (the asymptotes, with no hyperbola). The single point and the pair of lines are **degenerate** conics: they come from a plane passing through the tip of the cone. An equation with no graph does not come from any slice of the cone at all. So always look at the right side before you call something an ellipse or hyperbola.

Once the equation is in standard form, use what you know from Lessons 11.1 to 11.3: find the center or vertex, measure a, b, c or p from it, draw the asymptotes or the directrix if there are any, and sketch.

KEY FORMULAS

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

$A = C$: circle $\qquad$ $AC > 0,\ A \neq C$: ellipse $\qquad$ $AC < 0$: hyperbola $\qquad$ $AC = 0$: parabola

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Practice habit. Write one step per line when completing the square, and write the amount you add under each group, multiplied by the factor in front, so that you add exactly the same to both sides.

WORKED EXAMPLES

Example 1. Identify the conic $x^2 - 4y^2 + 6x + 8y - 11 = 0$, write it in standard form, and give its center, vertices and asymptotes.

Show solution

1. $A = 1$ and $C = -4$ have opposite signs, so it is a hyperbola (unless it turns out degenerate).
2. Group and move the constant: $(x^2 + 6x) - 4(y^2 - 2y) = 11$.
3. Complete the squares: add 9 in the x-group and 1 in the y-group. The y-group is multiplied by -4, so the right side gets $9 + (-4)(1)$: $(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = 11 + 9 - 4$.
4. $(x + 3)^2 - 4(y - 1)^2 = 16$. The right side is positive, so it is a true hyperbola.
5. Divide by 16: $\dfrac{(x+3)^2}{16} - \dfrac{(y-1)^2}{4} = 1$. Center $(-3, 1)$, opens left and right, $a = 4$, $b = 2$.
6. Vertices $(-3 \pm 4, 1)$: $(-7, 1)$ and $(1, 1)$. Asymptotes: $y - 1 = \pm\dfrac{2}{4}(x + 3) = \pm\dfrac{1}{2}(x + 3)$.
7. Check $(1, 1)$ in the original: $1 - 4 + 6 + 8 - 11 = 0$. ✓

Answer: Hyperbola $\dfrac{(x+3)^2}{16} - \dfrac{(y-1)^2}{4} = 1$; center $(-3, 1)$; vertices $(-7, 1)$ and $(1, 1)$; asymptotes $y - 1 = \pm\dfrac{1}{2}(x + 3)$.

Example 2. On the site plan for a church's new courtyard, the edge of a reflecting pool is $9x^2 + 16y^2 - 54x + 64y - 1151 = 0$, with x and y in feet. Identify the shape and find its center, its length and its width.

Show solution

1. $A = 9$ and $C = 16$: same sign, not equal, so it is an ellipse.
2. Group and move the constant: $9(x^2 - 6x) + 16(y^2 + 4y) = 1151$.
3. Complete the squares: inside, add 9 and 4. The right side gets $9 \times 9 = 81$ and $16 \times 4 = 64$: $9(x^2 - 6x + 9) + 16(y^2 + 4y + 4) = 1151 + 81 + 64$.
4. $9(x - 3)^2 + 16(y + 2)^2 = 1296$.
5. Divide by 1296: $\dfrac{(x-3)^2}{144} + \dfrac{(y+2)^2}{81} = 1$. Center $(3, -2)$, $a = 12$, $b = 9$.
6. Length $2a = 24$ feet; width $2b = 18$ feet.
7. Check the vertex $(15, -2)$ in the original: $2025 + 64 - 810 - 128 - 1151 = 0$. ✓

Answer: An ellipse $\dfrac{(x-3)^2}{144} + \dfrac{(y+2)^2}{81} = 1$ centered at $(3, -2)$, 24 feet long and 18 feet wide.

PRACTICE

1. Easy Identify each conic without completing the square. (a) $3x^2 + 3y^2 - 12x + 7 = 0$ (b) $2x^2 - 5y^2 + 4x - 10 = 0$

   Show answer

   (a) $A = C$: circle. (b) Opposite signs: hyperbola

2. Easy Identify each conic without completing the square. (a) $y^2 + 6x - 4y + 1 = 0$ (b) $4x^2 + y^2 - 8x + 2y = 0$

   Show answer

   (a) Only y is squared: parabola. (b) Same sign, unequal: ellipse

3. Easy Find the center and radius of $x^2 + y^2 - 6x + 10y + 9 = 0$.

   Show answer

   $(x-3)^2 + (y+5)^2 = -9 + 9 + 25 = 25$: center $(3, -5)$, radius 5

4. **Easy** Write $\dfrac{(x-1)^2}{4}+\dfrac{(y+2)^2}{9}=1$ in general form.

Show answer

Multiply by 36: $9(x^2-2x+1)+4(y^2+4y+4)=36$, so $9x^2+4y^2-18x+16y-11=0$

5. **Medium** Write $y^2-4y-8x+20=0$ in standard form and find the vertex, focus and directrix.

Show answer

$(y-2)^2=8x-16=8(x-2)$: vertex $(2,2)$, $p=2$, focus $(4,2)$, directrix $x=0$

6. **Medium** Write $x^2+4y^2+6x-8y-3=0$ in standard form and find the center and vertices.

Show answer

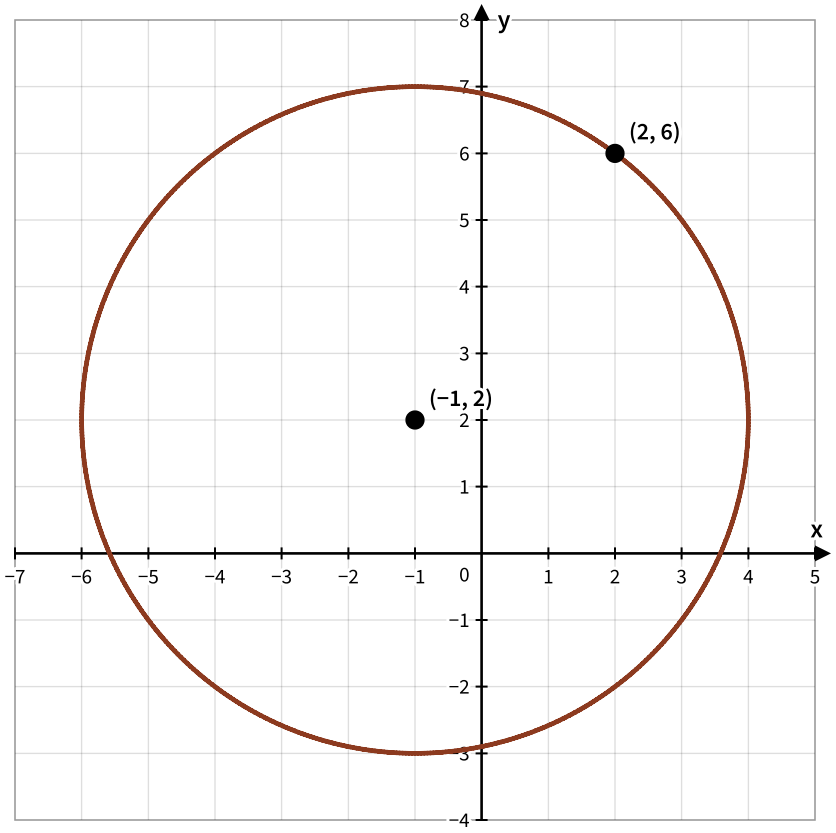
$(x+3)^2+4(y-1)^2=3+9+4=16$, so $\dfrac{(x+3)^2}{16}+\dfrac{(y-1)^2}{4}=1$: center $(-3,1)$, vertices $(-7,1)$ and $(1,1)$

7. **Medium** Write $9y^2-x^2-4x-36y+23=0$ in standard form. Give the center, vertices and asymptotes.

Show answer

$9(y-2)^2-(x+2)^2=-23+36-4=9$, so $(y-2)^2-\dfrac{(x+2)^2}{9}=1$: center $(-2,2)$, vertices $(-2,1)$ and $(-2,3)$, asymptotes $y-2=\pm\dfrac{1}{3}(x+2)$

8. **Medium** Samuel's plan of a round garden bed (in feet) shows its center and one point on its edge. Write the equation of the edge in general form.



Show answer

Radius: $\sqrt{3^2 + 4^2} = 5$. $(x + 1)^2 + (y - 2)^2 = 25$, so $x^2 + y^2 + 2x - 4y - 20 = 0$

9. **Medium** Write $x^2 + 4x + 4y - 8 = 0$ in standard form and find the vertex, focus and directrix.

Show answer

$(x + 2)^2 = -4y + 12 = -4(y - 3)$: vertex $(-2, 3)$, $p = -1$, focus $(-2, 2)$, directrix $y = 4$

10. **Hard** Describe the graph of $4x^2 + 9y^2 - 8x + 36y + 40 = 0$.

Show answer

$4(x - 1)^2 + 9(y + 2)^2 = -40 + 4 + 36 = 0$; both squares must be 0, so the graph is the single point $(1, -2)$ (a degenerate ellipse)

11. Hard On a farm map in yards, with the barn at the origin, the edge of a round pond is $x^2 + y^2 - 40x - 30y + 525 = 0$. (a) Find the center and radius of the pond. (b) How far is the barn from the nearest point of the pond's edge?

Show answer

(a) $(x-20)^2 + (y-15)^2 = -525 + 400 + 225 = 100$: center $(20, 15)$, radius 10 yards. (b) The barn is $\sqrt{20^2 + 15^2} = 25$ yards from the center, so $25 - 10 = 15$ yards

12. Hard In a design drawing measured in inches, the cross-section of a heater's parabolic reflector is $x^2 - 4x - 12y + 4 = 0$. The heating element belongs at the focus. Find the vertex and the focus.

Show answer

$(x-2)^2 = 12y$: vertex $(2, 0)$, $p = 3$, focus $(2, 3)$

13. Challenge For which values of F is the graph of $x^2 + y^2 - 6x + 2y + F = 0$ (a) a circle, (b) a single point, (c) empty?

Show answer

$(x-3)^2 + (y+1)^2 = 10 - F$: (a) $F < 10$; (b) $F = 10$, the point $(3, -1)$; (c) $F > 10$

14. Challenge Explain why the signs of A and C in $Ax^2 + Cy^2 + Dx + Ey + F = 0$ tell you whether a (non-degenerate) conic is an ellipse or a hyperbola.

Show answer

Completing the square changes $Ax^2 + Dx$ into $A(x-h)^2$ plus a constant, and $Cy^2 + Ey$ into $C(y-k)^2$ plus a constant, so the equation becomes $A(x-h)^2 + C(y-k)^2 = G$ with the same A and C. Dividing by G gives the standard form. If A and C have the same sign, both terms have the same sign after dividing, which is an ellipse (a circle if $A = C$). If they have opposite signs, one term is subtracted from the other, which is a hyperbola.

COMMON MISTAKES

- Adding to the right side only what you added inside the parentheses. If the group is $9(x^2 - 6x + 9)$, you added $9 \times 9 = 81$, so add 81 to the right side.
- Forgetting the minus sign when factoring a negative coefficient: $-4y^2 + 8y = -4(y^2 - 2y)$, not $-4(y^2 + 2y)$.

- Calling every equation with $A \neq C$ of the same sign an ellipse without checking the right side. If it is 0 the graph is a point; if it is negative there is no graph.
- Stopping before the right side is 1 for an ellipse or hyperbola, and then reading a and b from the wrong numbers.

> **Make it a game · Name That Conic.** The parent writes general-form equations on cards. Look only at the coefficients A and C and call the type in five seconds: same sign and equal means circle, same sign and unequal means ellipse, opposite signs means hyperbola, and one zero means parabola. Then complete the square on one equation to reach standard form. Score a point per correct call.

11.5 Nonlinear Systems

OBJECTIVES

- Solve systems made of a line and a conic, or of two conics, by substitution and by elimination.
- Explain why a line meets a conic in at most 2 points and two conics meet in at most 4, and connect the number of real solutions to the graph.
- Pair and check every solution in both original equations.
- Use nonlinear systems to solve problems about pastures, gardens, sprinklers and thrown balls.

WATCH

Simultaneous Linear/Quadratic Equations (1 of 2: Considering a line & parabola)

Eddie Woo · 11:22

https://www.youtube.com/watch?v=lg2FjYbhntQ

> **WHEN TO WATCH**
> Before Example 1: a line meeting a curve (here a parabola).

Note: Eddie Woo teaches in Australia: he says 'simultaneous equations' for a system.

BACKUP VIDEO

Intersection Between Lines & Curves (2 of 2: Circle and straight line)

Eddie Woo · 6:13

https://www.youtube.com/watch?v=3swPRGnDRbM

Before the practice set: a line meeting a circle.

KEY TERMS

nonlinear system

A system of equations in which at least one equation is not linear, such as a line and a circle.

solution of a system

An ordered pair (x, y) that makes every equation in the system true; on a graph, a point where the curves meet.

tangent line

A line that touches a curve at exactly one point without crossing it there; algebraically, substitution gives a quadratic with a repeated root (discriminant 0).

elimination

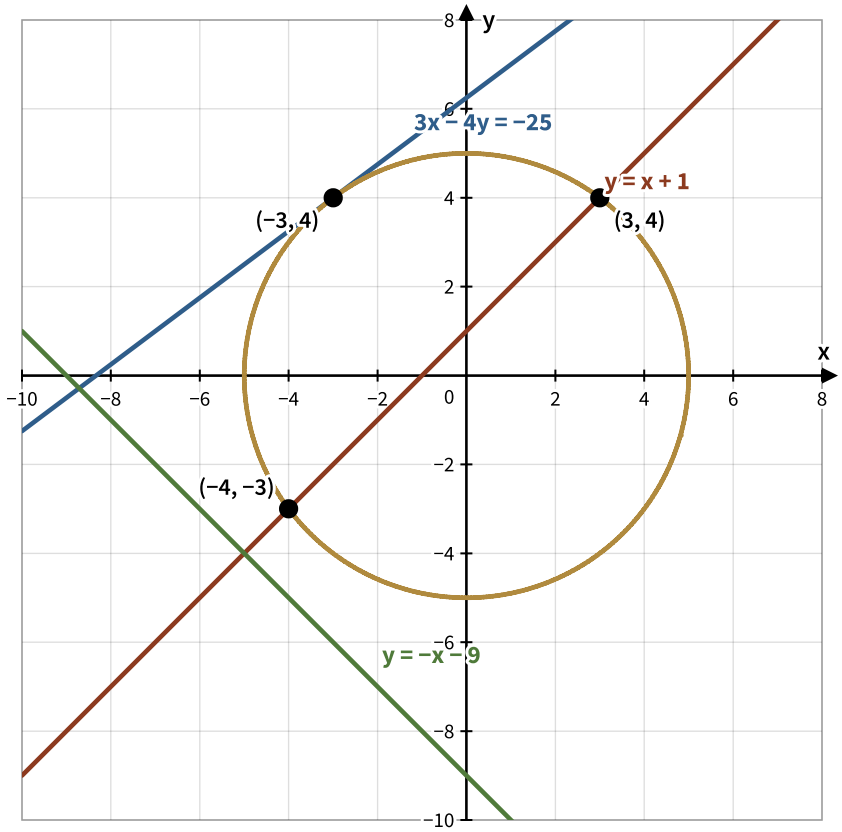
Adding or subtracting multiples of the equations to remove one variable (or one squared variable, such as y^2).

CORE IDEAS

A **nonlinear system** asks for the points that lie on two curves at once. You already solve linear systems by substitution and elimination (Unit 2); the same two methods work here, but now an equation can have two solutions, so there can be more than one intersection point.

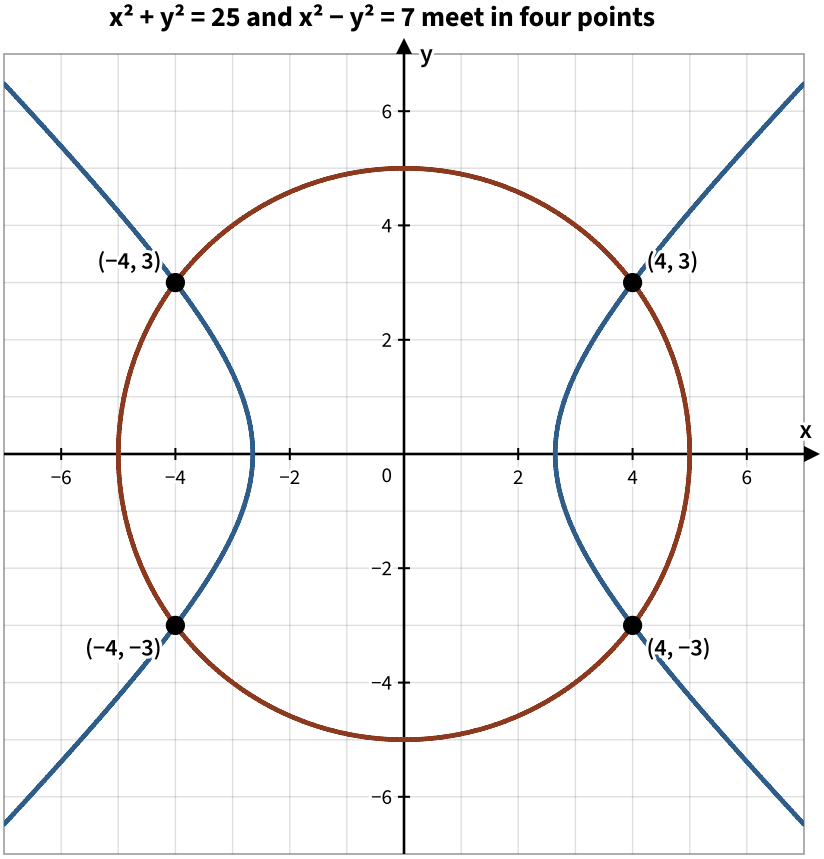
Substitution works best when one equation is linear. Take $x^2 + y^2 = 25$ and $y = x + 1$. Substitute: $x^2 + (x+1)^2 = 25$, so $2x^2 + 2x - 24 = 0$, or $x^2 + x - 12 = 0$. Factoring, $(x+4)(x-3) = 0$: $x = -4$ or $x = 3$. Each x gives one y from the **linear** equation: $(-4, -3)$ and $(3, 4)$. Both check in $x^2 + y^2 = 25$.

A circle and three lines: two points, one point (tangent), no points



Why a line meets a conic in at most two points: substituting the line into the conic always gives an equation of degree at most 2 in one variable, and a quadratic has at most two real roots. The discriminant tells the story. Positive: two crossing points. Zero: one repeated root, so the line is **tangent** (substituting $y = \dfrac{25 + 3x}{4}$ into $x^2 + y^2 = 25$ gives $x^2 + 6x + 9 = 0$, a double root at $x = -3$). Negative: the line misses the curve. In the picture these are the three lines.

Elimination works well when both equations contain only x^2 and y^2 (no x or y terms). For $x^2 + y^2 = 25$ and $x^2 - y^2 = 7$, add the equations: $2x^2 = 32$, so $x^2 = 16$. Then $y^2 = 25 - 16 = 9$. Now $x = \pm 4$ and $y = \pm 3$, and every combination works, because only the squares appear in the equations: four solutions $(\pm 4, \pm 3)$.



Two conics can meet in 0, 1, 2, 3 or 4 points. (Eliminating one variable leaves an equation of degree at most 4 in the other, and each point of intersection must solve it.) Be careful when pairing values. If you find y first and then solve $x^2 =$ (something), each y may give two x-values, one, or none; a negative value for x^2 means that y gives no real point. Squaring or taking square roots can also produce

pairs that satisfy one equation but not the other, so **check every pair in both original equations.**

A sketch is a good guide: it tells you how many solutions to expect and roughly where. Then let the algebra give exact answers. A graphing tool can confirm the picture, but it gives only decimals, while the algebra gives values like $\sqrt{7}$.

KEY FORMULAS

line and conic: at most 2 real solutions two conics: at most 4 real solutions

after substituting a line: $b^2 - 4ac > 0$ (2 points), $= 0$ (tangent), < 0 (no points)

Practice habit. Check every answer in the ORIGINAL equations, both of them. When you take a square root, write the $\pm$, then list the pairs one per line and test each.

WORKED EXAMPLES

Example 1. Solve the system $x^2 + 4y^2 = 20$ and $x + 2y = 6$.

Show solution

1. Solve the linear equation for x: $x = 6 - 2y$.
2. Substitute: $(6 - 2y)^2 + 4y^2 = 20$.
3. Expand: $36 - 24y + 4y^2 + 4y^2 = 20$, so $8y^2 - 24y + 16 = 0$.
4. Divide by 8 and factor: $y^2 - 3y + 2 = 0$, $(y - 1)(y - 2) = 0$, so $y = 1$ or $y = 2$.
5. Use the linear equation for x: $y = 1$ gives $x = 4$; $y = 2$ gives $x = 2$.
6. Check $(4, 1)$: $16 + 4 = 20$ ✓ and $4 + 2 = 6$ ✓. Check $(2, 2)$: $4 + 16 = 20$ ✓ and $2 + 4 = 6$ ✓.

Answer: $(2, 2)$ and $(4, 1)$: the line crosses the ellipse twice.

Example 2. The Miller family is fencing a rectangular pasture with an area of 4800 square yards. A diagonal path across it will be 100 yards long. Find the length and width of the pasture.

Show solution

1. Let the sides be x and y yards. Area: $xy = 4800$. Diagonal (Pythagorean theorem): $x^2 + y^2 = 10000$.
2. From the first equation, $y = \dfrac{4800}{x}$. Substitute: $x^2 + \dfrac{4800^2}{x^2} = 10000$.
3. Multiply by x^2: $x^4 - 10000x^2 + 23{,}040{,}000 = 0$, a quadratic in x^2.
4. Quadratic formula: $x^2 = \dfrac{10000 \pm \sqrt{100{,}000{,}000 - 92{,}160{,}000}}{2} = \dfrac{10000 \pm 2800}{2}$, so $x^2 = 6400$ or $x^2 = 3600$.
5. Lengths are positive, so $x = 80$ or $x = 60$; then $y = 60$ or $y = 80$. Both give the same rectangle.
6. Check: $80 \times 60 = 4800$ ✓ and $80^2 + 60^2 = 6400 + 3600 = 10000 = 100^2$ ✓.

Answer: The pasture is 80 yards by 60 yards.

PRACTICE

1. Easy Solve the system $y = x^2$ and $y = x + 2$.

   Show answer

   $x^2 - x - 2 = 0$, $x = -1$ or 2: $(-1, 1)$ and $(2, 4)$

2. Easy Solve the system $x^2 + y^2 = 10$ and $y = 3x$.

   Show answer

   $10x^2 = 10$, $x = \pm 1$: $(-1, -3)$ and $(1, 3)$

3. Easy Solve the system $x^2 + y^2 = 13$ and $y = x - 1$.

   Show answer

   $2x^2 - 2x - 12 = 0$, $x^2 - x - 6 = 0$, $x = -2$ or 3: $(-2, -3)$ and $(3, 2)$

4. Easy Show that the parabola $y = x^2 + 3$ and the line $y = 2x$ do not meet.

   Show answer

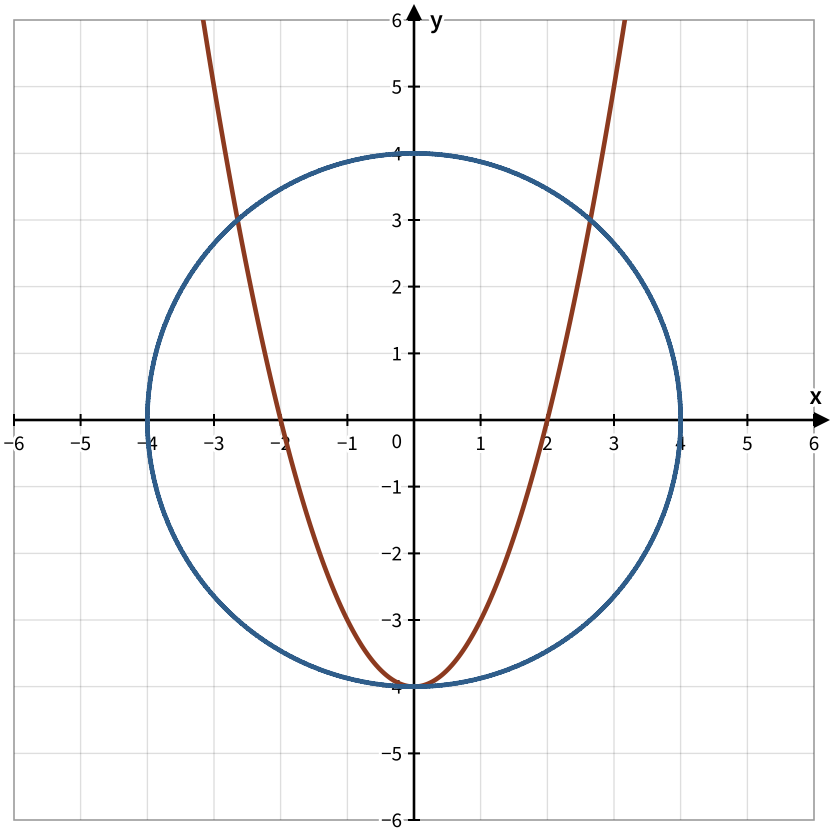
   $x^2 - 2x + 3 = 0$ has discriminant $4 - 12 = -8 < 0$, so there is no real solution

5. **Medium** Solve the system $x^2 + 2y^2 = 33$ and $2x^2 + y^2 = 18$ by elimination.

Show answer

Adding: $3x^2 + 3y^2 = 51$, so $x^2 + y^2 = 17$; subtracting gives $y^2 - x^2 = 15$. Then $y^2 = 16$, $x^2 = 1$: $(\pm 1, \pm 4)$, four solutions

6. **Medium** The graph shows the circle $x^2 + y^2 = 16$ and the parabola $y = x^2 - 4$. How many solutions does the system have? Find them exactly.



Show answer

Three. $x^2 = y + 4$, so $y + 4 + y^2 = 16$, $y^2 + y - 12 = 0$, $y = 3$ or -4. $y = 3$: $x = \pm\sqrt{7}$; $y = -4$: $x = 0$. Solutions $(-\sqrt{7}, 3)$, $(0, -4)$, $(\sqrt{7}, 3)$

7. **Medium** Solve the system $x^2 - y^2 = 16$ and $x + y = 8$. (Hint: factor the first equation.)

Show answer

$(x + y)(x - y) = 16$, so $8(x - y) = 16$ and $x - y = 2$. With $x + y = 8$: $(5, 3)$, the only solution

8. **Medium** Grace's rectangular vegetable garden has a perimeter of 34 meters and a diagonal of 13 meters. Find its length and width.

Show answer

$x + y = 17$ and $x^2 + y^2 = 169$: $x^2 - 17x + 60 = 0$, $x = 5$ or 12. The garden is 12 m by 5 m

9. **Hard** Henry throws a ball toward a hill. Measured in feet from the bottom of the hill, the ball's path is $y = -0.01x^2 + 0.8x + 6$ and the hillside is the line $y = 0.3x$. Where does the ball land on the hillside?

Show answer

$-0.01x^2 + 0.5x + 6 = 0$, so $x^2 - 50x - 600 = 0$, $(x - 60)(x + 10) = 0$, $x = 60$. The ball lands at $(60, 18)$: 60 feet out and 18 feet up

10. **Hard** Two lawn sprinklers on a church lawn are 12 meters apart, and each waters a circle of radius 10 meters. Put one at $(0, 0)$ and the other at $(12, 0)$. (a) Where do the edges of the two watered circles cross? (b) How far apart are those two crossing points?

Show answer

(a) $x^2 + y^2 = 100$ and $(x - 12)^2 + y^2 = 100$; subtracting gives $24x - 144 = 0$, $x = 6$, $y^2 = 64$: $(6, -8)$ and $(6, 8)$. (b) 16 meters

11. **Hard** For which values of k is the line $y = x + k$ tangent to the circle $x^2 + y^2 = 8$?

Show answer

$2x^2 + 2kx + k^2 - 8 = 0$; tangent when the discriminant $4k^2 - 8(k^2 - 8) = 64 - 4k^2 = 0$: $k = \pm 4$

12. **Challenge** How many real solutions does the system $x^2 + y^2 = r^2$, $x^2 - y^2 = 9$ have when (a) $r = 2$, (b) $r = 3$, (c) $r = 5$?

Show answer

Adding and subtracting: $x^2 = \dfrac{r^2 + 9}{2}$, $y^2 = \dfrac{r^2 - 9}{2}$. (a) $y^2 < 0$: 0 solutions. (b) $y = 0$, $x = \pm 3$: 2 solutions. (c) $x^2 = 17$, $y^2 = 8$: 4 solutions

13. **Challenge** Explain why a line and a parabola, an ellipse or a hyperbola can never meet in more than two points.

Show answer

Solve the line's equation for one variable (say $y = mx + b$, or $x =$ constant for a vertical line) and substitute into the conic's equation. Every term of the conic has degree at most 2, so the result is an equation of degree at most 2 in one variable. Such an equation (unless it is true for every value, which cannot happen since a conic contains no whole line) has at most two real roots, and each intersection point gives a root. So there are at most two intersection points.

COMMON MISTAKES

- Pairing values wrongly. After finding x, get y from the linear equation (or check both signs of a square root in both equations); not every $\pm$ combination works when the equations contain x or y to the first power.
- Forgetting the $\pm$ when taking a square root, and so missing half the solutions (for example, finding only $(4, 3)$ instead of $(\pm 4, \pm 3)$).
- Keeping a value that makes x^2 or y^2 negative. That value gives no real point, so it is not a solution.
- Skipping the check in the second original equation. Every solution must satisfy both equations.

> **Make it a game · Ruler and Plate.** Trace a plate to make a circle on paper. Lay a ruler across it, showing a line that meets the circle in two points, one point (tangent) and none. Then solve $x^2 + y^2 = 25$ with $y = x - 1$ by substitution, getting points (4, 3) and (-3, -4). Predict the maximum number of crossings for two conics, and test it with two traced outlines.

UNIT 11

Conic Sections

Slice a double cone with a flat plane and you get one of four curves: a circle, an ellipse, a parabola or a hyperbola. In this unit you define each one by a simple rule about distances (equal distances to a point and a line; distances to two points that add to a constant; distances that differ by a constant) and turn each rule into an equation with the distance formula. You will find the vertices, foci, directrices and asymptotes that describe each curve, and use them to sketch.

The unit builds on tools you already have: the distance formula and the circle from Geometry, completing the square and the quadratic formula from Algebra I and Unit 3, transformations from Unit 1, and solving systems from Unit 2. It ends by solving systems that mix lines and conics. The applications are old and new at once: satellite dishes and flashlights, whispering galleries, the orbits of planets and comets, and navigation by radio signals. Later courses in precalculus and calculus return to these curves often.

PREREQUISITES

- Use the distance and midpoint formulas (Geometry).
- Write and read the equation of a circle $(x-h)^2+(y-k)^2=r^2$ (Geometry).
- Complete the square and solve quadratic equations by factoring and the quadratic formula (Lessons 3.1 and 3.2).
- Translate graphs by replacing x with $x-h$ and y with $y-k$ (Lesson 1.2).
- Solve systems of equations by substitution and elimination (Lesson 2.1).
- Simplify square roots and solve equations in quadratic form (Lessons 4.5 and 6.2).

MASTERY CHECKLIST

- ☐ I can describe the conic sections as slices of a double cone.
- ☐ I can derive $x^2=4py$ from the focus-directrix definition of a parabola.
- ☐ I can find the vertex, focus and directrix of a parabola and write its equation from given information.
- ☐ I can explain why $c^2=a^2-b^2$ for an ellipse and find its center, vertices, co-vertices and foci.
- ☐ I can write the equation of an ellipse or circle from a description.
- ☐ I can explain why $c^2=a^2+b^2$ for a hyperbola and why its asymptotes are $y=\pm\frac{b}{a}x$.
- ☐ I can sketch a hyperbola using its central rectangle and asymptotes.
- ☐ I can identify a conic from $Ax^2+Cy^2+Dx+Ey+F=0$ and complete the square to put it in standard form.
- ☐ I can recognize degenerate cases: a point, a pair of lines or no graph.
- ☐ I can solve a system of a line and a conic, or of two conics, by

substitution or elimination and check every solution.

- ☐ I can use conics and nonlinear systems to solve problems about reflectors, orbits, navigation and fields.

LESSONS IN THIS UNIT

11.1 Parabolas: Focus and Directrix	2 sessions of 45 minutes
11.2 Circles and Ellipses	2 sessions of 45 minutes
11.3 Hyperbolas	2 sessions of 45 minutes
11.4 Recognizing and Graphing Conic Sections	2 sessions of 45 minutes
11.5 Nonlinear Systems	2 sessions of 45 minutes

Unit 11 Test

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem. Give exact answers (fractions or simplest radical form) unless a problem asks you to round. Check solutions of systems in both original equations.

1. **5 pts**
   Find the vertex, focus and directrix of $(y-4)^2 = 12(x+1)$.

2. **6 pts**
   Write the equation of the parabola with focus $(3, -1)$ and directrix $y = 5$.

3. **7 pts**
   A tractor headlight has a parabolic reflector 8 inches across at its opening and 3 inches deep. How far from the vertex should the bulb be placed, at the focus? Give an exact answer and a decimal to the nearest hundredth.

4. **6 pts**

Find the vertices and foci of $\frac{x^2}{49} + \frac{y^2}{24} = 1$.

5. **7 pts**

Write the equation of the ellipse with center $(-2, 1)$, a vertex at $(-2, 6)$ and a focus at $(-2, 4)$.

6. **7 pts**

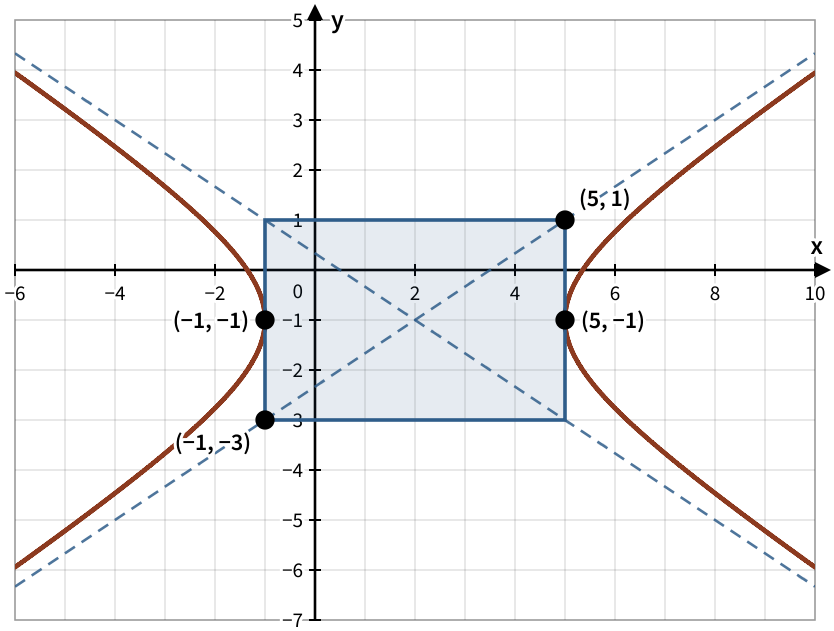
A weather satellite moves in an elliptical orbit with the center of the Earth at one focus. Its closest distance to the Earth's center is 7 thousand km and its farthest is 9 thousand km. Using thousands of kilometers, with the center of the ellipse at the origin and the major axis on the x-axis, write the equation of the orbit.

7. **6 pts**

Find the vertices, foci and asymptotes of $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

8. **7 pts**

The graph shows a hyperbola with its central rectangle (two corners labeled), its asymptotes and its vertices. Write its equation and give its foci in simplest radical form.



9. **6 pts**

   Identify each conic. (a) $5x^2 + 2y^2 - 10x + 3 = 0$ (b) $x^2 - 6x - y + 4 = 0$ (c) $4y^2 - x^2 + 8y + 2x - 1 = 0$

10. **7 pts**

    Write $4x^2 + y^2 + 24x - 4y + 24 = 0$ in standard form. Give the center, the vertices and the foci.

11. **6 pts**

    Show that the graph of $x^2 - 9y^2 - 2x + 1 = 0$ is a pair of lines, and give their equations.

12. **5 pts**

    Solve the system $x^2 + y^2 = 20$ and $y = 2x$.

13. **6 pts**

    Solve the system $x^2 + y^2 = 41$ and $y = x^2 - 11$. Give exact answers.

14. **7 pts**

    Ruth's family fences a rectangular hay field with 440 yards of fence, enclosing 12,000 square yards. Find the length and width of the field.

15. **6 pts**

    Explain why the equation $3x^2 + 3y^2 - 6x + 12y + 20 = 0$ has no graph, even though $A = C$.

16. **6 pts**

    Find the value of b for which the line $y = 2x + b$ is tangent to the parabola $y = x^2$, and find the point of tangency.

ANSWER KEY

1. Vertex $(-1, 4)$; $p = 3$; focus $(2, 4)$; directrix $x = -4$
2. Vertex $(3, 2)$, halfway between; the focus is 3 units below, so $p = -3$: $(x-3)^2 = -12(y-2)$
3. $x^2 = 4py$ through $(4, 3)$: $16 = 12p$, $p = \dfrac{4}{3} \approx 1.33$ inches
4. $a = 7$, $c^2 = 49 - 24 = 25$, $c = 5$: vertices $(\pm 7, 0)$, foci $(\pm 5, 0)$
5. Vertical; $a = 5$, $c = 3$, $b^2 = 25 - 9 = 16$: $\dfrac{(x+2)^2}{16} + \dfrac{(y-1)^2}{25} = 1$
6. $2a = 7 + 9 = 16$, $a = 8$; $c = 8 - 7 = 1$; $b^2 = 64 - 1 = 63$: $\dfrac{x^2}{64} + \dfrac{y^2}{63} = 1$
7. $a = 6$, $b = 8$, $c = 10$: vertices $(0, \pm 6)$, foci $(0, \pm 10)$, asymptotes $y = \pm \dfrac{3}{4}x$
8. Center $(2, -1)$, opens left and right, $a = 3$, $b = 2$: $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{4} = 1$; $c = \sqrt{13}$, foci $(2 \pm \sqrt{13}, -1)$
9. (a) ellipse (same signs, unequal; $5(x-1)^2 + 2y^2 = 2$) (b) parabola (only x squared) (c) hyperbola (opposite signs; $(y+1)^2 - \dfrac{(x-1)^2}{4} = 1$)
10. $4(x+3)^2 + (y-2)^2 = -24 + 36 + 4 = 16$, so $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{16} = 1$: center $(-3, 2)$, vertices $(-3, -2)$ and $(-3, 6)$, $c^2 = 12$, foci $(-3, 2 \pm 2\sqrt{3})$
11. $(x-1)^2 - 9y^2 = 0$, so $(x - 1 - 3y)(x - 1 + 3y) = 0$: the lines $y = \dfrac{x-1}{3}$ and $y = -\dfrac{x-1}{3}$
12. $5x^2 = 20$, $x = \pm 2$: $(-2, -4)$ and $(2, 4)$
13. $x^2 = y + 11$: $y^2 + y - 30 = 0$, $y = 5$ or -6. $y = 5$: $x = \pm 4$; $y = -6$: $x = \pm\sqrt{5}$. Solutions $(-4, 5)$, $(-\sqrt{5}, -6)$, $(\sqrt{5}, -6)$, $(4, 5)$
14. $x + y = 220$, $xy = 12000$: $x^2 - 220x + 12000 = 0$, $x = \dfrac{220 \pm 20}{2} = 120$ or 100. The field is 120 yards by 100 yards
15. Completing the square: $3(x-1)^2 + 3(y+2)^2 = -20 + 3 + 12 = -5$. The left side is a sum of squares times 3, so it is never negative, and no point (x, y) can make it equal -5. The equation has no graph.
16. $x^2 - 2x - b = 0$ has discriminant $4 + 4b = 0$, so $b = -1$; then $(x-1)^2 = 0$, and the point is $(1, 1)$

UNIT 12

Matrices and Vectors

This unit introduces two new kinds of mathematical objects: matrices, which are rectangular tables of numbers, and vectors, which are arrows or ordered lists of numbers. You will add, scale and multiply matrices, find determinants and inverses, and use them to solve systems of equations by a method that is really the elimination you already know, written more compactly. Every definition is explained by a practical question: why matrices are multiplied row by column (quantities times prices gives totals), why the inverse of a 2×2 matrix has the form it does, and why a dot product of 0 means a right angle.

The unit builds on the systems of equations from Units 2 and 3, the distance formula and Pythagorean theorem, and the transformations of the plane from Geometry. It ends by showing that a 2×2 matrix is itself a transformation: it can reflect, rotate, stretch or shear a figure, and its determinant tells how areas change. These ideas are the beginning of linear algebra, the mathematics behind computer graphics, engineering and modern computing, which later courses develop further.

PREREQUISITES

- Solving systems of two and three linear equations by elimination (Lessons 2.1 and 2.4)
- Evaluating expressions with negative numbers and fractions accurately
- The distance formula and the Pythagorean theorem and its converse (Geometry)
- Transformations of the plane as coordinate rules: reflections, rotations by 90° and 180°, dilations (Geometry)
- Finding the area of a triangle and a parallelogram (Geometry)

MASTERY CHECKLIST

- ☐ I can give the dimensions of a matrix and name an entry a_{ij} by its row and column.
- ☐ I can add and subtract matrices of the same size, multiply by a scalar, and solve simple matrix equations.
- ☐ I can multiply matrices row by column, decide whether a product is defined, and explain why AB and BA are usually different.
- ☐ I can use a matrix product to compute totals such as costs, hours or pounds of feed.
- ☐ I can find the determinant of a 2×2 or 3×3 matrix and use a determinant to find the area of a triangle.
- ☐ I can derive and use the formula for the inverse of a 2×2 matrix and explain why it exists only when the determinant is not 0.
- ☐ I can solve a system as $X = A^{-1}B$ and by Gaussian elimination on an augmented matrix, and recognize no solution or infinitely many solutions.
- ☐ I can add, subtract and scale vectors, find magnitudes and unit vectors, and compute dot products for weighted totals and work.
- ☐ I can prove and use the fact that two vectors are perpendicular exactly when their dot product is 0.

- ☐ I can write the matrix of a reflection, rotation, dilation, stretch or shear from the images of $(1, 0)$ and $(0, 1)$, and apply it to a figure.
- ☐ I can compose transformations by multiplying matrices in the right order and use the determinant to tell how areas change and whether a figure is flipped.

LESSONS IN THIS UNIT

12.1 Matrices and Matrix Operations	2 sessions of 45 minutes
12.2 Matrix Multiplication	2 sessions of 45 minutes
12.3 Determinants and Inverse Matrices	2 sessions of 45 minutes
12.4 Solving Systems with Matrices	2 sessions of 45 minutes
12.5 Vectors and the Dot Product	2 sessions of 45 minutes
12.6 Matrices as Transformations	2 sessions of 45 minutes

Unit 12 test

Time: 60 minutes. A scientific calculator is allowed; a graphing calculator or matrix tool is not needed and should not be used. Show your work: for products, show at least one row-times-column sum; for row reduction, name each row operation. Give exact answers unless a problem asks you to round. Check solutions of systems in the original equations.

1. **5 pts**

   Let $A = \begin{bmatrix} 5 & -2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 6 & 2 \end{bmatrix}$. Find $3A + B$.

2. **5 pts**

   Find a and b if $\begin{bmatrix} a+3 & 7 \\ -1 & 2b \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -1 & 10 \end{bmatrix}$.

3. **6 pts**

   The Hartley family's orchards recorded bushels picked. September: $S = \begin{bmatrix} 120 & 45 \\ 80 & 30 \end{bmatrix}$; October: $O = \begin{bmatrix} 150 & 20 \\ 95 & 10 \end{bmatrix}$ (rows: apples, pears; columns: north orchard, south orchard). (a) Find the matrix of total bushels for the two months. (b) How many more bushels of apples did the north orchard pick in October than in September?

4. **5 pts**

   A is a 3×2 matrix and B is a 2×4 matrix. (a) What are the dimensions of AB? (b) Is BA defined?

5. **6 pts**

   Find $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 5 \end{bmatrix}$.

6. **7 pts**

   A family sells bundles of firewood. On Friday they sold 30 oak and 20 pine bundles; on Saturday, 45 oak and 25 pine. Oak sells for \$7 a bundle and costs them \$3 to produce; pine sells for \$5 and costs \$2. (a) Multiply $Q = \begin{bmatrix} 30 & 20 \\ 45 & 25 \end{bmatrix}$ by $P = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$ (columns: price, cost) and say what the entries of QP mean. (b) Find the total profit for the two days.

7. **6 pts**

   Find $\det \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 4 \\ -2 & 5 & 1 \end{bmatrix}$ by expanding along the first row.

8. **7 pts**

   Find the inverse of $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and check that $AA^{-1} = I$.

9. **6 pts**

   A triangular pasture has corners at $(20, 10)$, $(100, 40)$ and $(50, 90)$ on a map measured in meters. Use a determinant to find its area.

10. **7 pts**

    Use your inverse from Problem 8 to solve the system $5x + 4y = 2$, $3x + 2y = 0$.

11. **8 pts**

    For a church garden, the youth group bought 30 plants: tomato plants at \$3 each, pepper plants at \$2 each and herb plants at \$4 each, for \$84 in all. They bought as many pepper plants as tomato and herb plants together. Write an augmented matrix and use Gaussian elimination to find how many of each they bought.

12. **5 pts**

    Each matrix is the result of row-reducing a system in x, y, z. State whether the system has one solution, no solution or infinitely many solutions. (a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

13. **6 pts**

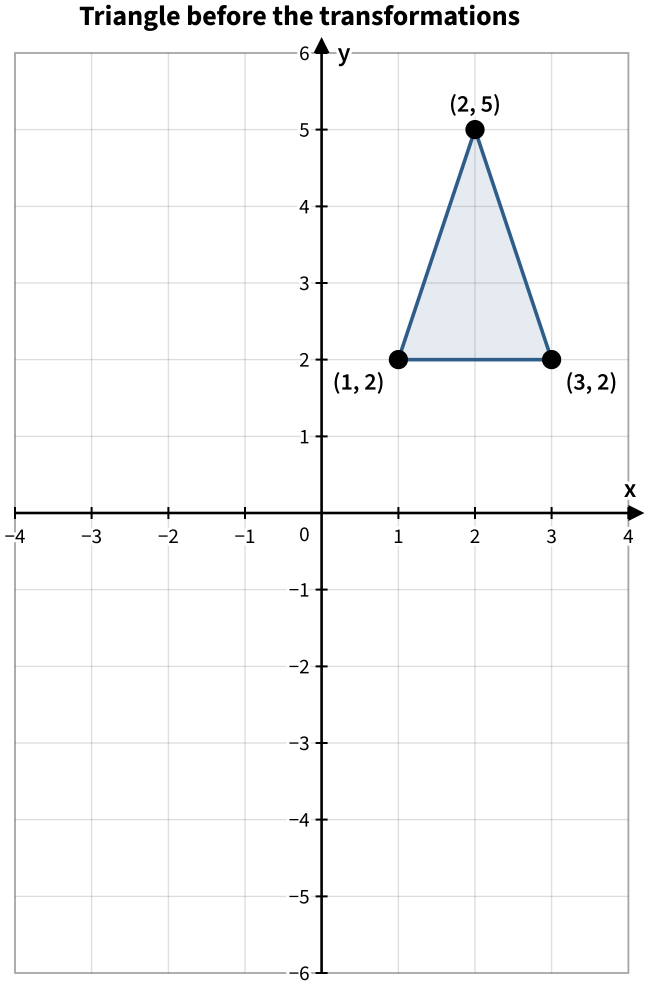
    Let $\mathbf{u} = \langle -2, 6 \rangle$ and $\mathbf{v} = \langle 4, 1 \rangle$. Find (a) $\mathbf{u} + 3\mathbf{v}$, (b) $|\mathbf{u} - \mathbf{v}|$ in exact form, and (c) $\mathbf{u} \cdot \mathbf{v}$.

14. **6 pts**

    A winch pulls a rowboat up a straight boat ramp. The boat moves along the displacement $\mathbf{d} = \langle 24, 10 \rangle$ feet while the cable pulls with a constant force $\mathbf{F} = \langle 120, 50 \rangle$ pounds. (a) How far does the boat move along the ramp? (b) How much work does the winch do, in foot-pounds?

15. **8 pts**

The triangle shown is first reflected in the y-axis and then rotated $180°$ about the origin. (a) Find the single matrix M for the combined transformation. (b) Find the vertices of the image of $(1, 2)$, $(3, 2)$ and $(2, 5)$, in that order. (c) Find $\det M$.



16. **7 pts**

(a) Prove that $\langle a, b\rangle$ and $\langle -b, a\rangle$ are perpendicular whenever a and b are not both 0. (b) Explain what the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does to every vector, and how part (a) agrees with that.

ANSWER KEY

1. $3A = \begin{bmatrix} 15 & -6 \\ 0 & 9 \end{bmatrix}$, so $3A + B = \begin{bmatrix} 14 & -2 \\ 6 & 11 \end{bmatrix}$
2. $a + 3 - 1 = 6$, so $a = 4$; $2b - 4 = 10$, so $b = 7$
3. (a) $S + O = \begin{bmatrix} 270 & 65 \\ 175 & 40 \end{bmatrix}$; (b) $150 - 120 = 30$ bushels
4. (a) 3×4; (b) no, because $(2 \times 4)(3 \times 2)$ has inside numbers 4 and 3, which do not match
5. Row 1: $2 + 2 + 3 = 7$ and $1 + 0 + 15 = 16$; row 2: $0 - 4 - 1 = -5$ and $0 + 0 - 5 = -5$. The product is $\begin{bmatrix} 7 & 16 \\ -5 & -5 \end{bmatrix}$
6. (a) $QP = \begin{bmatrix} 310 & 130 \\ 440 & 185 \end{bmatrix}$: Friday's sales were \$310 and costs \$130; Saturday's sales were \$440 and costs \$185. (b) $(310 - 130) + (440 - 185) = 180 + 255 = 435$, so \$435
7. $3(1 - 20) - 0 + (-1)(10 + 2) = -57 - 12 = -69$
8. $\det A = 10 - 12 = -2$, so $A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$. Check: $\begin{bmatrix} -5+6 & 10-10 \\ -3+3 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓
9. $\det \begin{bmatrix} 20 & 10 & 1 \\ 100 & 40 & 1 \\ 50 & 90 & 1 \end{bmatrix} = 20(40 - 90) - 10(100 - 50) + 1(9000 - 2000) = -1000 - 500 + 7000 = 5500$, so the area is $\frac{1}{2}(5500) = 2750$ square meters
10. $X = A^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, so $(x, y) = (-2, 3)$. Check: $-10 + 12 = 2$ ✓, $-6 + 6 = 0$ ✓
11. With $t + p + h = 30$, $3t + 2p + 4h = 84$, $-t + p - h = 0$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 3 & 2 & 4 & 84 \\ -1 & 1 & -1 & 0 \end{array}\right]$. $R_2 \to R_2 - 3R_1$ gives $\left[\begin{array}{ccc|c} 0 & -1 & 1 & -6 \end{array}\right]$; $R_3 \to R_3 + R_1$ gives $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 30 \end{array}\right]$, so $p = 15$; then $-15 + h = -6$, so $h = 9$, and $t = 30 - 15 - 9 = 6$. Check: $18 + 30 + 36 = 84$ ✓, $15 = 6 + 9$ ✓. They bought 6 tomato, 15 pepper and 9 herb plants.
12. (a) No solution: the last row says $0 = -3$. (b) Infinitely many solutions: the last row says $0 = 0$ and no row is impossible.
13. (a) $\langle -2 + 12, 6 + 3 \rangle = \langle 10, 9 \rangle$; (b) $\mathbf{u} - \mathbf{v} = \langle -6, 5 \rangle$, so $|\mathbf{u} - \mathbf{v}| = \sqrt{36 + 25} = \sqrt{61}$; (c) $-8 + 6 = -2$

14. (a) $\sqrt{576+100} = \sqrt{676} = 26$ feet; (b) $W = \mathbf{F}\cdot\mathbf{d} = 120(24) + 50(10) = 2880 + 500 = 3380$ foot-pounds

15. (a) The reflection is first, so it goes on the right: $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, a reflection in the x-axis. (b) $(1,-2)$, $(3,-2)$, $(2,-5)$. (c) $\det M = -1$: area unchanged, figure flipped.

16. Model answer: (a) $\langle a, b\rangle \cdot \langle -b, a\rangle = -ab + ba = 0$, and both vectors are nonzero, so they are perpendicular. (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$. Its columns are the images $(0,1)$ of $(1,0)$ and $(-1,0)$ of $(0,1)$, so it is the 90° counterclockwise rotation. A vector rotated by 90° is perpendicular to the original, which is exactly what part (a) shows: the image $\langle -b, a\rangle$ always has dot product 0 with $\langle a, b\rangle$.

12.1 Matrices and Matrix Operations

OBJECTIVES

- Identify the dimensions of a matrix and name any entry a_{ij} by its row and column.
- Add and subtract matrices of the same dimensions and multiply a matrix by a scalar.
- Solve simple matrix equations by comparing entries or by undoing addition and scalar multiplication.
- Organize real data, such as a farm stand's sales or a bakery's inventory, in a matrix and combine it.

WATCH

Introduction to Matrices (2 of 3: Notation & Basic Manipulation)

Eddie Woo · 8:22

https://www.youtube.com/watch?v=qa5vxEXIPTs

> **WHEN TO WATCH**
> Before Example 1: matrix notation, adding and scalar multiplication.

BACKUP VIDEO

Matrices - Dimensions, Add, Subtract Scalar Multiplication

Mario's Math Tutoring · 4:07

https://www.youtube.com/watch?v=lTIMC1wdj_U

Before the practice set: dimensions and matrix arithmetic.

KEY TERMS

matrix

A rectangular table of numbers arranged in rows and columns and enclosed in brackets. The plural is matrices.

dimensions

The size of a matrix, written rows × columns. A matrix with 2 rows and 3 columns is a 2 × 3 matrix (read "two by three").

entry

One number in a matrix. In matrix A, the entry in row i and column j is written a_{ij}; the row number always comes first.

equal matrices

Two matrices with the same dimensions whose corresponding entries are all equal.

scalar

An ordinary real number, used to multiply every entry of a matrix. The name comes from the idea of scaling.

zero matrix

A matrix whose entries are all 0, written O. Adding it to a matrix of the same size changes nothing.

CORE IDEAS

Much of the information in daily life already comes in rows and columns: a schedule, a price list, a record of sales. A **matrix** is such a table with the labels stripped off, leaving only the numbers. The table below records one weekend at the Miller family's farm stand: each row is a product and each column is a day.

Miller family farm stand, one weekend

Product	Friday	Saturday
Eggs (dozens)	18	25
Honey (jars)	6	9
Pies	4	7

As a matrix, the same information is $A = \begin{bmatrix} 18 & 25 \\ 6 & 9 \\ 4 & 7 \end{bmatrix}$. It has 3 rows and 2 columns, so its **dimensions** are 3×2. Always give rows first, then columns. The entry $a_{21} = 6$ sits in row 2, column 1: 6 jars of honey sold on Friday. Because the labels are gone, you must remember what each row and column means, just as you remember what x stands for in a word problem.

Two matrices are **equal** only when they have the same dimensions and every pair of matching entries is equal. So a single matrix equation such as $\begin{bmatrix} x & 2 \\ 5 & y \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & -1 \end{bmatrix}$ really says four things at once, and it tells you $x = 7$ and $y = -1$.

To **add or subtract** matrices, combine the entries in matching positions. If next weekend's sales are $B = \begin{bmatrix} 20 & 22 \\ 5 & 11 \\ 6 & 8 \end{bmatrix}$, then the total for both weekends is $A + B = \begin{bmatrix} 38 & 47 \\ 11 & 20 \\ 10 & 15 \end{bmatrix}$: eggs with eggs, Friday with Friday. This is why the matrices must have the same dimensions. Adding a 3×2 matrix to a 2×3 matrix would mean adding Friday's eggs to something that is not Friday's eggs, so the sum is simply not defined.

To multiply a matrix by a **scalar** k, multiply every entry by k. If the Millers expect to sell twice as much at a harvest festival, they compute $2A = \begin{bmatrix} 36 & 50 \\ 12 & 18 \\ 8 & 14 \end{bmatrix}$. Subtraction is adding the opposite: $A - B = A + (-1)B$.

Why do the familiar properties still hold? Each entry of $A + B$ is $a_{ij} + b_{ij}$, a sum of two real numbers. Real-number addition is commutative, so $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ in every position, and therefore $A + B = B + A$. The same entry-by-entry argument gives the associative property $(A + B) + C = A + (B + C)$ and the distributive property $k(A + B) = kA + kB$. The **zero matrix** O plays the role of 0, since $A + O = A$, and $-A$ plays the role of the opposite, since $A + (-A) = O$.

These properties let you solve matrix equations the way you solve linear equations. To solve $X + B = C$, add $-B$ to both sides: $X = C - B$. To solve $3X = C$, multiply both sides by the scalar $\frac{1}{3}$. You never divide one matrix by another; instead you undo each step with an opposite or a reciprocal scalar.

KEY FORMULAS

$$A \pm B = [\, a_{ij} \pm b_{ij} \,] \quad \text{(same dimensions only)}$$

$$kA = [\, k\, a_{ij} \,]$$

$$A + B = B + A, \qquad (A + B) + C = A + (B + C), \qquad k(A + B) = kA + kB$$

$$A + O = A, \qquad A + (-A) = O$$

From history. The English mathematician James Joseph Sylvester coined the word "matrix" in 1850 for a rectangular array of numbers. His friend Arthur Cayley developed the algebra of matrices in 1858, showing how to add, multiply and invert them as if they were single numbers. Their work laid the foundation for a branch of mathematics now used in engineering, physics and computing.

Practice habit. Write the dimensions (rows × columns) under every matrix before you combine it with another. If the sizes do not match, stop: the sum or difference is not defined.

WORKED EXAMPLES

Example 1. Let $A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix}$. Find $2A - 3B$.

Show solution

1. Both matrices are 2×2, so the expression is defined.
2. Multiply every entry of A by 2: $2A = \begin{bmatrix} 6 & -2 \\ 0 & 8 \end{bmatrix}$.
3. Multiply every entry of B by 3: $3B = \begin{bmatrix} -6 & 15 \\ 18 & 3 \end{bmatrix}$.
4. Subtract entry by entry: $\begin{bmatrix} 6-(-6) & -2-15 \\ 0-18 & 8-3 \end{bmatrix}$.
5. Simplify: $\begin{bmatrix} 12 & -17 \\ -18 & 5 \end{bmatrix}$.
6. Check one entry another way: row 1, column 1 is $2(3) - 3(-2) = 6 + 6 = 12$ ✓

Answer: $2A - 3B = \begin{bmatrix} 12 & -17 \\ -18 & 5 \end{bmatrix}$

Example 2. Grace's bakery has two shops, one on Main Street and one on Mill Road. The table shows what each shop sold in two weeks. (a) Write a 3×2 matrix for each week (rows: bread, muffins, pies; columns: Main St., Mill Rd.) and find the matrix of total sales for the two weeks. (b) Find the matrix Week 2 − Week 1 and say what its entry in row 2, column 1 means.

Loaves, muffins and pies sold by Grace's bakery

Item	Week 1 Main St.	Week 1 Mill Rd.	Week 2 Main St.	Week 2 Mill Rd.
Bread (loaves)	40	35	44	31
Muffins	60	48	55	52
Pies	12	15	10	18

Show solution

1. Week 1: $W_1 = \begin{bmatrix} 40 & 35 \\ 60 & 48 \\ 12 & 15 \end{bmatrix}$. Week 2: $W_2 = \begin{bmatrix} 44 & 31 \\ 55 & 52 \\ 10 & 18 \end{bmatrix}$. Both are 3×2.
2. (a) Add matching entries: $W_1 + W_2 = \begin{bmatrix} 40+44 & 35+31 \\ 60+55 & 48+52 \\ 12+10 & 15+18 \end{bmatrix} = \begin{bmatrix} 84 & 66 \\ 115 & 100 \\ 22 & 33 \end{bmatrix}$.
3. (b) Subtract matching entries: $W_2 - W_1 = \begin{bmatrix} 44-40 & 31-35 \\ 55-60 & 52-48 \\ 10-12 & 18-15 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -5 & 4 \\ -2 & 3 \end{bmatrix}$.
4. Row 2 is muffins and column 1 is Main Street, so the entry -5 means the Main Street shop sold 5 fewer muffins in week 2 than in week 1.
5. Check: adding the difference back to week 1 should give week 2; for example $60 + (-5) = 55$ ✓

Answer: (a) $\begin{bmatrix} 84 & 66 \\ 115 & 100 \\ 22 & 33 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & -4 \\ -5 & 4 \\ -2 & 3 \end{bmatrix}$; the -5 means Main Street sold 5 fewer muffins in week 2.

PRACTICE

1. **Easy** Give the dimensions of $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 3 & 7 \end{bmatrix}$ as rows × columns.

   Show answer

   2×3 (2 rows, 3 columns)

2. **Easy** Let $A = \begin{bmatrix} 4 & -2 & 9 \\ 1 & 0 & 6 \\ -3 & 8 & 5 \end{bmatrix}$. Find (a) a_{23} and (b) a_{31}.

   Show answer

   (a) $a_{23} = 6$ (row 2, column 3); (b) $a_{31} = -3$ (row 3, column 1)

3. **Easy** Find $\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ -7 & 0 \end{bmatrix}$.

   Show answer

   $$\begin{bmatrix} 6 & 2 \\ -4 & 2 \end{bmatrix}$$

4. **Easy** Find $-3 \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -6 \end{bmatrix}$.

   Show answer

   $$\begin{bmatrix} -6 & 3 & 0 \\ -12 & -15 & 18 \end{bmatrix}$$

5. **Medium** Find $\begin{bmatrix} 7 & 2 \\ -1 & 4 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 2 & -4 \\ -6 & 1 \end{bmatrix}$.

   Show answer

   $$\begin{bmatrix} 4 & -3 \\ -3 & 8 \\ 6 & 2 \end{bmatrix}$$

6. **Medium** Find x and y so that $\begin{bmatrix} 2x + 1 & 6 \\ 4 & y - 3 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 4 & -5 \end{bmatrix}$.

   Show answer

   $2x + 1 = 9$, so $x = 4$; $y - 3 = -5$, so $y = -2$

7. **Medium** Can a 2×3 matrix be added to a 3×2 matrix? Answer yes or no and explain.

Show answer

No. Matrices are added entry by entry in matching positions, and a 2×3 matrix has positions (such as row 1, column 3) that a 3×2 matrix does not have, so the sum is not defined.

8. **Medium** Solve for the matrix X: $X + \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -2 & 5 \end{bmatrix}$.

Show answer

$$X = \begin{bmatrix} 6 & 0 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \end{bmatrix}$$

9. **Medium** A farm stand's prices in dollars are $P = \begin{bmatrix} 3.50 & 4.00 \\ 6.25 & 7.00 \end{bmatrix}$, where row 1 is a dozen eggs, row 2 is a jar of jam, column 1 is the price at the stand and column 2 is the price delivered. Feed and jar costs rise, so every price goes up 20%. Find the new price matrix $1.2P$.

Show answer

$1.2P = \begin{bmatrix} 4.20 & 4.80 \\ 7.50 & 8.40 \end{bmatrix}$ (in dollars); for example $1.2 \times 6.25 = 7.50$

10. **Hard** Let $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & 6 \\ 1 & 1 \end{bmatrix}$. Find $3A - 2B + C$.

Show answer

$$3A - 2B = \begin{bmatrix} 3-8 & 0+2 \\ -6-0 & 9-4 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -6 & 5 \end{bmatrix}, \text{ so } 3A - 2B + C = \begin{bmatrix} -10 & 8 \\ -5 & 6 \end{bmatrix}$$

11. **Hard** Solve for the matrix X: $2X - \begin{bmatrix} 1 & 4 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 7 & 6 \end{bmatrix}$.

Show answer

$$2X = \begin{bmatrix} 6 & 2 \\ 4 & 6 \end{bmatrix}, \text{ so } X = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$$

12. Hard For a church bake sale, the starting stock is $S = \begin{bmatrix} 48 & 36 \\ 24 & 30 \end{bmatrix}$ (rows: cookies, muffins; columns: front table, back table). By noon the numbers sold are $T = \begin{bmatrix} 41 & 29 \\ 17 & 30 \end{bmatrix}$. (a) Find the matrix of items left, $S - T$. (b) How many items are left in all?

Show answer

(a) $S - T = \begin{bmatrix} 7 & 7 \\ 7 & 0 \end{bmatrix}$ (the back table's muffins sold out); (b) $7 + 7 + 7 + 0 = 21$ items

13. Challenge Find all values of x and y that make $\begin{bmatrix} x^2 & 3 \\ 2y & 5 \end{bmatrix} + \begin{bmatrix} -x & 1 \\ y & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 12 & 5 \end{bmatrix}$ true.

Show answer

Row 1, column 1: $x^2 - x = 6$, so $(x - 3)(x + 2) = 0$ and $x = -2$ or $x = 3$. Row 2, column 1: $3y = 12$, so $y = 4$. The other entries check: $3 + 1 = 4$ ✓, $5 + 0 = 5$ ✓

14. Challenge Prove that $A + B = B + A$ for any two matrices A and B with the same dimensions.

Show answer

Model proof: Both sums have the same dimensions as A and B. The entry in row i, column j of $A + B$ is $a_{ij} + b_{ij}$, and the entry in the same place of $B + A$ is $b_{ij} + a_{ij}$. These are real numbers, and real-number addition is commutative, so $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Since every pair of matching entries is equal, $A + B = B + A$.

COMMON MISTAKES

- Reading a_{ij} as column first. The row always comes first: a_{32} is in row 3, column 2. Say "row, column" to yourself as you count.
- Trying to add or subtract matrices of different dimensions by lining up whatever entries fit. Write each matrix's dimensions first; if they differ, the sum is not defined.
- Multiplying only the first row or first entry by a scalar. A scalar multiplies every entry of the matrix.

- Dropping signs when subtracting: $2 - (-4)$ is 6, not -2. Rewrite each subtraction of a negative as addition before you simplify.

> **Make it a game · Egg Carton Matrix.** Use an egg carton or a grid on paper as a matrix. Put a number slip in each cell. The parent calls an entry, such as a_{23}, and you read it aloud. State the dimensions. Then make a second matrix of the same size, add and subtract them, and multiply by a scalar. Try to do each operation in 60 seconds.

12.2 Matrix Multiplication

OBJECTIVES

- Multiply two matrices row by column and explain why the product is defined that way.
- Use the dimension rule to decide whether a product is defined and find its dimensions.
- Show with an example that matrix multiplication is not commutative, and use the identity matrix and powers of square matrices.
- Interpret a matrix times a column vector as a list of weighted sums in a real situation.

WATCH

Matrix Multiplication (1 of 3: Basic Principles)

Eddie Woo · 12:16

https://www.youtube.com/watch?v=dk-5hYrKsvY

> **WHEN TO WATCH**
> Before Example 1: how to multiply two matrices.

BACKUP VIDEO

Matrix Multiplication (3 of 3: Observations & Characteristics)

Eddie Woo · 11:15

https://www.youtube.com/watch?v=fSJmxXBYdsY

After Example 2: why AB and BA are usually different.

KEY TERMS

matrix product

For A (m × n) and B (n × p), the m × p matrix AB whose entry in row i, column j is found by multiplying row i of A by column j of B entry by entry and adding.

row-by-column rule

The way each entry of a product is computed: first entry of the row times first entry of the column, plus second times second, and so on.

square matrix

A matrix with the same number of rows and columns, such as a 2×2 or 3×3 matrix.

identity matrix

The square matrix I with 1s on the main diagonal (top left to bottom right) and 0s everywhere else. For any matrix A of the right size, $AI = A$ and $IA = A$.

column vector

A matrix with just one column, such as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

CORE IDEAS

Matrix multiplication is not done entry by entry. Its definition looks strange at first, but it comes straight from a very ordinary question: what does an order cost? Two families shop for eggs, bread and honey. Their orders and the prices at two stores are shown in the tables.

Orders

Family	Eggs (dozens)	Bread (loaves)	Honey (jars)
Samuel's family	2	3	1
Ruth's family	4	0	2

Prices in dollars

Item	Store A	Store B
Eggs (dozen)	3	4
Bread (loaf)	4	3
Honey (jar)	8	7

Write the orders as $Q = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ (rows: families; columns: items) and the prices as $P = \begin{bmatrix} 3 & 4 \\ 4 & 3 \\ 8 & 7 \end{bmatrix}$ (rows: items; columns: stores). Samuel's family at Store A pays $2(3) +$

$3(4) + 1(8) = 26$ dollars: each quantity times its price, then added. That uses **row 1 of Q** and **column 1 of P**. Doing the same for every family and every store gives $QP = \begin{bmatrix} 26 & 24 \\ 28 & 30 \end{bmatrix}$, a table of totals: the entry in row i, column j is what family i pays at store j. That is the row-by-column rule, and it is why the product is defined this way.

The rule also explains the **dimension rule**. A row of Q has to pair up entry for entry with a column of P, so the number of columns of the first matrix must equal the number of rows of the second (here, 3 items). The answer has one row for each row of the first matrix and one column for each column of the second: an m × n matrix times an n × p matrix is an m × p matrix. The inside numbers must match; the outside numbers give the size of the product. If the inside numbers differ, the product is not defined.

Order matters. With $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, you get $AB = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ but $BA = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. So matrix multiplication is **not commutative**: in general $AB \neq BA$, and the two products need not even be the same size (QP above is 2 × 2, but PQ is 3 × 3 and means nothing useful). Sometimes one product is defined and the other is not: a 2 × 3 matrix times a 3 × 4 matrix is defined, but the 3 × 4 matrix times the 2 × 3 matrix is not. The associative property $(AB)C = A(BC)$ and the distributive properties $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$ do hold whenever the sizes fit. Be careful: $(A + B)^2 = A^2 + AB + BA + B^2$, which is not $A^2 + 2AB + B^2$ unless $AB = BA$.

The **identity matrix** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts like the number 1. Why? Row $\begin{bmatrix} a & b \end{bmatrix}$ times column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $a \cdot 1 + b \cdot 0 = a$, and times $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is b, so each row comes through unchanged and $AI = A$. For a **square matrix** you can also take powers: $A^2 = AA$, $A^3 = AAA$. Only square matrices have powers, because AA needs the inside dimensions to match.

A matrix times a column vector is a list of weighted sums. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 26 \\ 28 \end{bmatrix}$ gives each family's total at Store A in one step. This kind of product, many weighted sums computed at once, is the basic operation inside spreadsheets, computer graphics and modern computing.

KEY FORMULAS

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$(m \times n)(n \times p) \to (m \times p)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$AI = IA = A$, $\quad (AB)C = A(BC)$, $\quad A(B+C) = AB + AC$, $\quad AB \neq BA$ in general

Practice habit. Write the dimensions side by side before you multiply, as in (2 × 3)(3 × 2): check that the inside numbers match and read the size of the answer from the outside numbers. Then compute one entry at a time, saying "row i, column j" aloud.

WORKED EXAMPLES

Example 1. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$. Find AB and BA.

Show solution

1. Both are 2×2, so both products are defined and are 2×2.
2. AB, row 1 times column 1: $2(0) + (-1)(1) = -1$. Row 1 times column 2: $2(5) + (-1)(-2) = 10 + 2 = 12$.
3. AB, row 2 times column 1: $3(0) + 4(1) = 4$. Row 2 times column 2: $3(5) + 4(-2) = 15 - 8 = 7$.
4. So $AB = \begin{bmatrix} -1 & 12 \\ 4 & 7 \end{bmatrix}$.
5. BA, row 1: $0(2) + 5(3) = 15$ and $0(-1) + 5(4) = 20$.
6. BA, row 2: $1(2) + (-2)(3) = -4$ and $1(-1) + (-2)(4) = -9$.
7. So $BA = \begin{bmatrix} 15 & 20 \\ -4 & -9 \end{bmatrix}$, which is different from AB: the order of the factors matters.

Answer: $AB = \begin{bmatrix} -1 & 12 \\ 4 & 7 \end{bmatrix}$, $BA = \begin{bmatrix} 15 & 20 \\ -4 & -9 \end{bmatrix}$

Example 2. Henry's woodshop builds benches, tables and bookshelves. The table gives the hours of cutting and finishing each one needs. In October the shop builds 5 benches, 2 tables and 3 bookshelves; in November, 4 benches, 3 tables and 1 bookshelf. (a) Use matrix multiplication to find the cutting and finishing hours for each month. (b) Cutting labor costs \$20 per hour and finishing labor \$15 per hour. Find the labor cost for each month.

Henry's woodshop

Item	Cutting hours	Finishing hours
Bench	2	1
Table	4	3
Bookshelf	3	2

Show solution

1. Orders (rows: months; columns: bench, table, bookshelf): $O = \begin{bmatrix} 5 & 2 & 3 \\ 4 & 3 & 1 \end{bmatrix}$, a 2×3 matrix.
2. Hours (rows: bench, table, bookshelf; columns: cutting, finishing): $H = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 3 & 2 \end{bmatrix}$, a 3×2 matrix. Inside dimensions 3 and 3 match, so OH is 2×2.
3. October cutting: $5(2) + 2(4) + 3(3) = 10 + 8 + 9 = 27$. October finishing: $5(1) + 2(3) + 3(2) = 5 + 6 + 6 = 17$.
4. November cutting: $4(2) + 3(4) + 1(3) = 8 + 12 + 3 = 23$. November finishing: $4(1) + 3(3) + 1(2) = 4 + 9 + 2 = 15$.
5. (a) $OH = \begin{bmatrix} 27 & 17 \\ 23 & 15 \end{bmatrix}$ (rows: months; columns: cutting, finishing hours).
6. (b) Multiply by the wage column $\begin{bmatrix} 20 \\ 15 \end{bmatrix}$, a 2×1 matrix, so the result is 2×1.
7. October: $27(20) + 17(15) = 540 + 255 = 795$. November: $23(20) + 15(15) = 460 + 225 = 685$.

Answer: (a) $\begin{bmatrix} 27 & 17 \\ 23 & 15 \end{bmatrix}$: October needs 27 cutting and 17 finishing hours, November 23 and 15. (b) $\begin{bmatrix} 795 \\ 685 \end{bmatrix}$: \$795 in October and \$685 in November.

PRACTICE

1. Easy A is a 2×3 matrix and B is a 3×4 matrix. (a) What are the dimensions of AB? (b) Is BA defined?

   Show answer

   (a) $(2 \times 3)(3 \times 4)$ gives 2×4. (b) No: $(3 \times 4)(2 \times 3)$ has inside numbers 4 and 2, which do not match.

2. Easy A family's bakery order is the row matrix $\begin{bmatrix} 3 & 5 & 2 \end{bmatrix}$ (loaves of bread, dozens of rolls, pies). The prices in dollars are the column $\begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$. Find the product and say what it means.

   Show answer

   $3(4) + 5(6) + 2(12) = 12 + 30 + 24 = 66$, so the product is $\begin{bmatrix} 66 \end{bmatrix}$: the order costs \$66.

3. Easy Find $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

   Show answer

   $$\begin{bmatrix} 2(5) + 0(-1) \\ 1(5) + 3(-1) \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

4. Easy Find $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

   Show answer

   $$\begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

5. **Medium** Find $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}$.

Show answer

Row 1: $2+0+0=2$ and $8+2+0=10$; row 2: $1+0-6=-5$ and $4-6-2=-4$. The product is $\begin{bmatrix} 2 & 10 \\ -5 & -4 \end{bmatrix}$

6. **Medium** Let $A=\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$. Find A^2.

Show answer

$A^2 = AA = \begin{bmatrix} 9-2 & 3+0 \\ -6+0 & -2+0 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ -6 & -2 \end{bmatrix}$ (not the entries squared)

7. **Medium** Let $A=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. Find (a) AB and (b) BA.

Show answer

(a) $AB=\begin{bmatrix} 2 & 8 \\ 0 & 3 \end{bmatrix}$; (b) $BA=\begin{bmatrix} 2 & 12 \\ 0 & 3 \end{bmatrix}$. They are not equal.

8. **Medium** For a choir fundraiser, candles sell for \$6, calendars for \$10 and cookbooks for \$15. Ruth sold 4 candles, 2 calendars and 1 cookbook; Samuel sold 0 candles, 5 calendars and 2 cookbooks; Grace sold 3 of each. (a) Write a 3×3 quantity matrix (rows: Ruth, Samuel, Grace) and multiply it by the price column to find what each person raised. (b) How much did the three raise in all?

Show answer

(a) $\begin{bmatrix} 4 & 2 & 1 \\ 0 & 5 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 59 \\ 80 \\ 93 \end{bmatrix}$: Ruth \$59, Samuel \$80, Grace \$93. (b) \$232

9. **Medium** Find x and y if $\begin{bmatrix} x & 2 \\ 1 & y \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$.

Show answer

Row 1: $3x+2=11$, so $x=3$. Row 2: $3+y=5$, so $y=2$.

10. Hard Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute (a) $A(B + C)$ and (b) $AB + AC$, and compare.

Show answer

(a) $B + C = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$ and $A(B + C) = \begin{bmatrix} 4 & -1 \\ 8 & 8 \end{bmatrix}$. (b) $AB = \begin{bmatrix} 5 & -2 \\ 5 & 6 \end{bmatrix}$, $AC = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$, so $AB + AC = \begin{bmatrix} 4 & -1 \\ 8 & 8 \end{bmatrix}$. They are equal, as the distributive property says.

11. Hard A farmer keeps 12 cows and 30 sheep. Each cow eats 20 pounds of hay and 4 pounds of grain a day; each sheep eats 3 pounds of hay and 1 pound of grain. (a) Multiply the row $\begin{bmatrix} 12 & 30 \end{bmatrix}$ by the matrix $\begin{bmatrix} 20 & 4 \\ 3 & 1 \end{bmatrix}$ to find the pounds of hay and grain needed each day. (b) Hay costs \$0.10 per pound and grain \$0.25 per pound. What is the daily feed cost?

Show answer

(a) $\begin{bmatrix} 12(20) + 30(3) & 12(4) + 30(1) \end{bmatrix} = \begin{bmatrix} 330 & 78 \end{bmatrix}$: 330 pounds of hay and 78 pounds of grain. (b) $330(0.10) + 78(0.25) = 33 + 19.50 = 52.50$, so \$52.50 per day.

12. Hard For square matrices A and B of the same size, expand $(A + B)^2 = (A + B)(A + B)$ with the distributive property. Explain why the result is usually not $A^2 + 2AB + B^2$.

Show answer

Model answer: $(A + B)(A + B) = A(A + B) + B(A + B) = A^2 + AB + BA + B^2$. The middle terms combine into $2AB$ only if $BA = AB$. Matrix multiplication is not commutative (for example $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$), so in general $(A + B)^2 \neq A^2 + 2AB + B^2$.

13. **Challenge** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find (a) A^2, (b) A^3, and (c) use the pattern to find A^{10}.

Show answer

(a) $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$; (b) $A^3 = A^2A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; (c) each multiplication by A adds 1 to the top right entry, so $A^{10} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$

14. **Challenge** Prove that $AI = A$ and $IA = A$ for every 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Show answer

Model proof: $AI = \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$. Also $IA = \begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$. In each sum the 0s remove every term except the one entry we want.

COMMON MISTAKES

- Multiplying matching entries, as in addition. A product entry is a whole row times a whole column, added up: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2$ is $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$, not $\begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$.
- Assuming $AB = BA$. Keep factors in the order given, and when you solve a problem decide which matrix goes first by checking that the inside dimensions match and that the labels line up (items with items).
- Getting the dimensions of a product backward. The product of an m × n and an n × p matrix is m × p: rows from the first matrix, columns from the second.
- Losing a negative sign in a row-times-column sum. Write each product in parentheses, such as $(-1)(-2)$, before adding.

> **Make it a game · Row Meets Column.** Write two matrices on paper. Use one finger to trace across a row of the first and another down a column of the second, saying the products and the sum aloud for each entry. Before multiplying, the parent flashes dimension pairs such as 2×3 and 3×4, and you say whether the product is defined and its size.

12.3 Determinants and Inverse Matrices

OBJECTIVES

- Evaluate the determinant of a 2×2 matrix and of a 3×3 matrix by expansion along the first row.
- Find the area of a triangle from the coordinates of its vertices using a determinant.
- Derive and use the formula for the inverse of a 2×2 matrix, and check it with $AA^{-1} = I$.
- Explain why a matrix has an inverse exactly when its determinant is not 0.

WATCH

Matrix Determinants (2 of 3: The Determinant's Geometric Meaning)

Eddie Woo · 10:35

https://www.youtube.com/watch?v=_L8roj-8HfM

> **WHEN TO WATCH**
> Before Example 1: what the determinant means geometrically.

BACKUP VIDEO

Finding the Inverse of a 2x2 Matrix

Eddie Woo · 5:01

https://www.youtube.com/watch?v=wD-1acVgbmY

After Example 1: a second look at the inverse of a 2×2 matrix.

KEY TERMS

determinant

A single number computed from a square matrix, written $\det A$ or with straight bars. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ it is $ad - bc$.

minor

The determinant of the smaller matrix left when you cross out one row and one column of a larger matrix.

inverse matrix

For a square matrix A, the matrix A^{-1} (if it exists) with $AA^{-1} = I$ and $A^{-1}A = I$. It undoes multiplication by A, just as $\frac{1}{5}$ undoes multiplication by 5.

invertible

Having an inverse. A square matrix is invertible exactly when its determinant is not 0; a matrix with determinant 0 is called singular.

CORE IDEAS

For real numbers, dividing by 5 is the same as multiplying by $\frac{1}{5}$, the number that satisfies $5 \cdot \frac{1}{5} = 1$. Matrices have no division, but some square matrices have an **inverse** A^{-1} with $AA^{-1} = I$, the identity matrix. Finding it leads naturally to a special number called the **determinant**.

The determinant of a 2×2 matrix is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$: the product down the main diagonal minus the product up the other diagonal. For a 3×3 matrix, expand along the first row. Each first-row entry is multiplied by its **minor** (the 2×2 determinant left after crossing out that entry's row and column), and the signs alternate $+, -, +$. For example, $\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} = 1(24 - 0) - 2(0 - 5) + 3(0 - 4) = 24 + 10 - 12 = 22$.

Determinants measure area. A triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) has area $\frac{1}{2}\left|\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$. Check it on a triangle you already know: for $(0,0)$, $(4,0)$, $(0,3)$ the determinant is $0 - 0 + 1(4 \cdot 3 - 0 \cdot 0) = 12$, and half of 12 is 6, which matches $\frac{1}{2} \cdot 4 \cdot 3$. The absolute value is needed because listing the vertices in the other direction around the triangle changes the sign of the determinant. You will see why determinants measure area in Lesson 12.6.

Now derive the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Call it $X = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and require $AX = I$. Multiplying row by column gives two linear systems: $aw + by = 1$ and $cw + dy = 0$ for the first column, and $ax + bz = 0$ and $cx + dz = 1$ for the second.

Solve the first system by elimination. Multiply the first equation by d and the second by b, then subtract: $(ad - bc)w = d$, so $w = \dfrac{d}{ad - bc}$. Multiply the first by c and the

second by a, then subtract: $(ad - bc)y = -c$, so $y = \dfrac{-c}{ad - bc}$. The second system works the same way and gives $x = \dfrac{-b}{ad - bc}$ and $z = \dfrac{a}{ad - bc}$. Collecting the four entries: **swap a and d, change the signs of b and c, and divide by the determinant.**

Every step divided by $ad - bc$, which is why the determinant decides everything. If $\det A \neq 0$, the formula gives the inverse, and you can check it: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$, and dividing by $ad - bc$ leaves I. If $\det A = 0$, the equations above become $0 = d$, $0 = -c$, $0 = -b$ and $0 = a$, so every entry of A would be 0, and then AX would be the zero matrix, never I. So **a matrix has an inverse exactly when its determinant is not 0.**

Larger matrices follow the same rule: a 3×3 matrix is invertible exactly when its determinant is not 0. Finding a 3×3 inverse by hand is long, so use a graphing calculator or an online matrix tool, then check a row or two of AA^{-1} by hand.

KEY FORMULAS

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$\text{Area of triangle} = \frac{1}{2}\left|\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{when } ad - bc \neq 0$$

Practice habit. Find the determinant first. If it is 0, stop: there is no inverse. If not, write the inverse and multiply AA^{-1} to check that you get I.

WORKED EXAMPLES

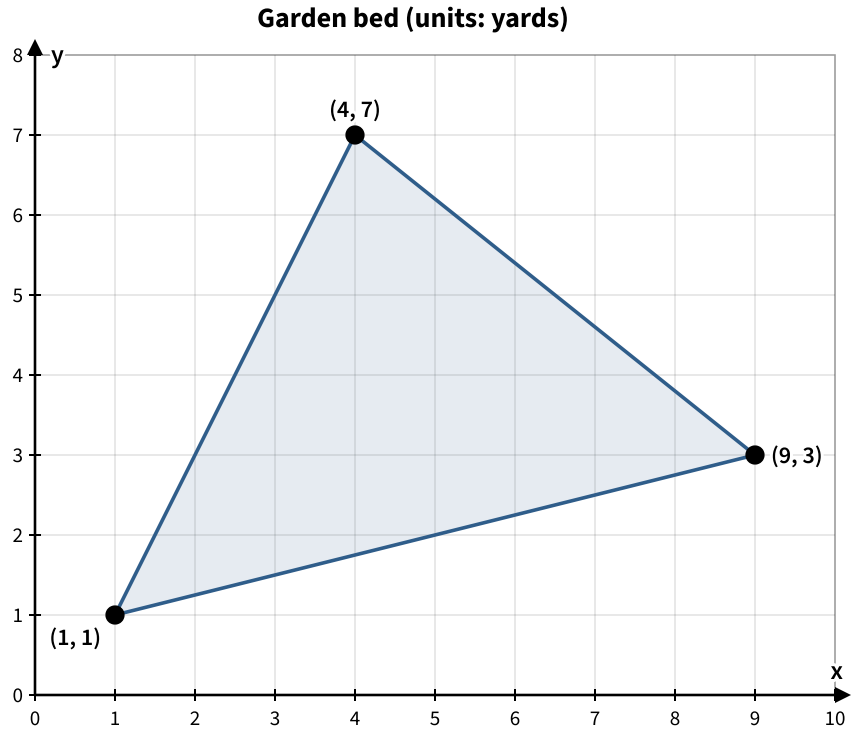
Example 1. Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$. Find $\det A$ and A^{-1}, and check your inverse.

Show solution

1. $\det A = 4(2) - 3(2) = 8 - 6 = 2$. It is not 0, so A^{-1} exists.
2. Swap the diagonal entries 4 and 2 and change the signs of 3 and 2: $\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$.
3. Divide by the determinant: $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{bmatrix}$.
4. Check: $AA^{-1} = \begin{bmatrix} 4(1)+3(-1) & 4(-\frac{3}{2})+3(2) \\ 2(1)+2(-1) & 2(-\frac{3}{2})+2(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Answer: $\det A = 2$; $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{bmatrix}$

Example 2. The Miller family is planning a triangular garden bed with corners at $(1, 1)$, $(9, 3)$ and $(4, 7)$ on a grid measured in yards. (a) Find the area of the bed. (b) One bag of mulch covers 3 square yards. How many bags do they need?



Show solution

1. Set up $\det \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 4 & 7 & 1 \end{bmatrix}$ and expand along the first row.
2. First term: $1 \cdot (3 \cdot 1 - 1 \cdot 7) = 1(-4) = -4$.
3. Second term (minus sign): $-1 \cdot (9 \cdot 1 - 1 \cdot 4) = -1(5) = -5$.
4. Third term: $1 \cdot (9 \cdot 7 - 3 \cdot 4) = 63 - 12 = 51$.
5. Determinant: $-4 - 5 + 51 = 42$.
6. (a) Area $= \dfrac{1}{2}|42| = 21$ square yards.
7. (b) $21 \div 3 = 7$ bags.
8. Check the area another way: the bed fits in the box from $x = 1$ to 9 and $y = 1$ to 7, of area $8 \times 6 = 48$. The three corner triangles outside the bed have areas $\dfrac{1}{2}(8)(2) = 8$, $\dfrac{1}{2}(5)(4) = 10$ and $\dfrac{1}{2}(3)(6) = 9$, and $48 - 8 - 10 - 9 = 21$ ✓

Answer: (a) 21 square yards; (b) 7 bags

PRACTICE

1. Easy Find $\det \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$.

   Show answer

   $5(4) - 2(3) = 20 - 6 = 14$

2. Easy Find $\det \begin{bmatrix} 7 & -2 \\ 1 & 3 \end{bmatrix}$.

   Show answer

   $7(3) - (-2)(1) = 21 + 2 = 23$

3. Easy (a) Find $\det \begin{bmatrix} -3 & 6 \\ 2 & -4 \end{bmatrix}$. (b) Does the matrix have an inverse?

   Show answer

   (a) $(-3)(-4) - 6(2) = 12 - 12 = 0$; (b) no, because the determinant is 0.

4. Easy Is $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ the inverse of $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$? Multiply to decide.

   Show answer

   Yes: $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. Medium Find the inverse of $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

   Show answer

   $\det = 6 - 5 = 1$, so the inverse is $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

6. Medium Find the inverse of $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$.

   Show answer

   $\det = 6 - 4 = 2$, so the inverse is $\frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$

7. **Medium** Find $\det \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 5 \end{bmatrix}$ by expanding along the first row.

Show answer

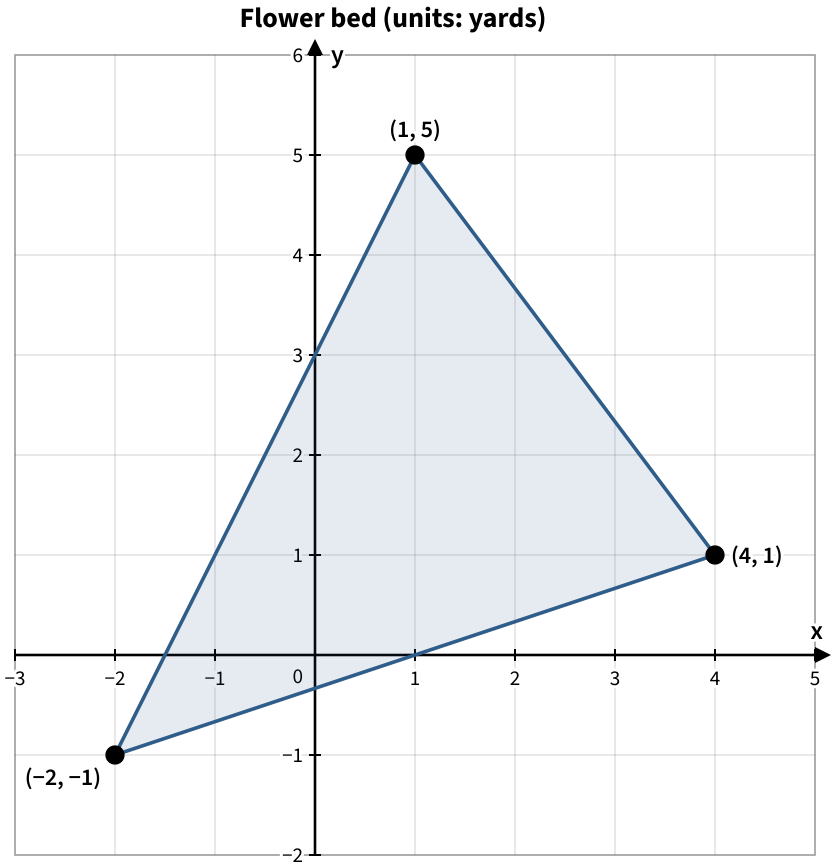
$2(-5-8) - 1(15-2) + 0 = -26 - 13 = -39$

8. **Medium** For what value of k does $\begin{bmatrix} k & 6 \\ 2 & 3 \end{bmatrix}$ have no inverse?

Show answer

No inverse when $3k - 12 = 0$, so $k = 4$

9. **Medium** A triangular flower bed in front of the church is drawn on a grid measured in yards, as shown. Find its area.



Show answer

$$\det\begin{bmatrix} -2 & -1 & 1 \\ 4 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} = -2(1-5) - (-1)(4-1) + 1(20-1) = 8 + 3 + 19 = 30,$$

so the area is $\frac{1}{2}(30) = 15$ square yards

10. **Hard** A surveyor maps a triangular hayfield with corners at $(0, 0)$, $(300, 50)$ and $(100, 250)$, in feet. (a) Find the area in square feet. (b) One acre is 43,560 square feet. Find the area in acres, to the nearest hundredth.

Show answer

(a) The determinant is $1(300 \cdot 250 - 50 \cdot 100) = 75{,}000 - 5{,}000 = 70{,}000$, so the area is 35,000 square feet. (b) $35{,}000 \div 43{,}560 \approx 0.80$ acre

11. Hard Find all values of x for which $\det \begin{bmatrix} x & 2 \\ 3 & x-1 \end{bmatrix} = 6$.

Show answer

$x(x-1) - 6 = 6$, so $x^2 - x - 12 = 0$ and $(x-4)(x+3) = 0$: $x = -3$ or $x = 4$. Check $x = 4$: $4(3) - 6 = 6$ ✓; $x = -3$: $(-3)(-4) - 6 = 6$ ✓

12. Hard Use a graphing calculator or matrix tool to find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ in decimal form. Then check row 1 of AA^{-1} by hand.

Show answer

$\det A = 5$ and $A^{-1} = \begin{bmatrix} 0.2 & 0.4 & -0.4 \\ -0.4 & 0.2 & 0.8 \\ 0.4 & -0.2 & 0.2 \end{bmatrix}$. Row 1 of A times the columns: $0.2 + 0.8 = 1$, $0.4 - 0.4 = 0$, $-0.4 + 0.4 = 0$ ✓

13. Hard Grace and Samuel send coded notes. Each letter becomes a number (A = 1, B = 2, ..., Z = 26), the numbers are taken in pairs as columns, and each column is multiplied by $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Samuel receives the coded columns $\begin{bmatrix} 31 \\ 23 \end{bmatrix}$ and $\begin{bmatrix} 37 \\ 21 \end{bmatrix}$. Use A^{-1} to decode the four letters.

Show answer

$\det A = 1$, so $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. $A^{-1}\begin{bmatrix} 31 \\ 23 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$ and $A^{-1}\begin{bmatrix} 37 \\ 21 \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix}$. The numbers 8, 15, 16, 5 spell HOPE.

14. Challenge Suppose A is a 2×2 matrix with $\det A = 5$. Find (a) $\det(3A)$ and (b) $\det(A^{-1})$.

Show answer

(a) Every entry is tripled, so $\det(3A) = (3a)(3d) - (3b)(3c) = 9(ad - bc) = 9 \times 5 = 45$. (b) $A^{-1} = \frac{1}{5}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, so $\det(A^{-1}) = \frac{1}{25}(da - bc) = \frac{5}{25} = \frac{1}{5}$

15. **Challenge** Prove that if $ad - bc = 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has no inverse. (Hint: suppose $AX = I$ for $X = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and use the eliminations from the lesson.)

Show answer

Model proof: Suppose $AX = I$. Then $aw + by = 1$, $cw + dy = 0$, $ax + bz = 0$ and $cx + dz = 1$. Eliminating as in the lesson gives $(ad - bc)w = d$, $(ad - bc)y = -c$, $(ad - bc)x = -b$ and $(ad - bc)z = a$. If $ad - bc = 0$, the left sides are all 0, so $a = b = c = d = 0$. But then A is the zero matrix and AX is the zero matrix, not I. This contradiction shows no inverse exists.

COMMON MISTAKES

- Computing $ad + bc$ or $bc - ad$. The determinant is the main diagonal product minus the other diagonal product: $ad - bc$.
- Forgetting the minus sign on the middle term of a 3 × 3 expansion. The signs along the first row are $+, -, +$; write them before you compute the minors.
- Swapping b and c instead of a and d in the inverse. Swap the main diagonal, negate the other diagonal, then divide by the determinant, and always check $AA^{-1} = I$.
- Forgetting to divide by the determinant, or dividing when it is 0. Compute $\det A$ first; if it is 0, there is no inverse.

> **Make it a game · Area Two Ways.** Draw a triangle on graph paper with vertices at whole-number points. Find its area by boxing it in a rectangle and subtracting corners, then find it again with a 3×3 determinant. Compare. Take turns choosing the triangle. Then compute the determinant of a 2×2 matrix and say whether it has an inverse.

12.4 Solving Systems with Matrices

OBJECTIVES

- Write a system of linear equations as a matrix equation $AX = B$ and as an augmented matrix.
- Solve a system with $X = A^{-1}B$, by hand for 2×2 systems and with technology for 3×3 systems, and explain why the inverse goes on the left.
- Use the three row operations and Gaussian elimination with back-substitution to solve systems of two and three equations.
- Recognize from the final rows of a reduced matrix when a system has no solution or infinitely many solutions.

WATCH

Solving Augmented Matrices (1 of 4: Worked Solution)

Eddie Woo · 8:24

https://www.youtube.com/watch?v=sza_BKR2KFc

> **WHEN TO WATCH**
> Before Example 2: solving a system with an augmented matrix.

BACKUP VIDEO

Inverse Matrices Solving Systems of Equations

Mario's Math Tutoring · 2:23

https://www.youtube.com/watch?v=Kn3Fk3uOR5Q

Before Example 1: solving a system with an inverse matrix.

KEY TERMS

coefficient matrix

The matrix A of the coefficients of the variables in a system, one row per equation and one column per variable.

matrix equation

A system written as $AX = B$, where X is the column of variables and B is the column of constants.

augmented matrix

The coefficient matrix with the column of constants attached on the right, usually set off by a vertical bar.

row operations

The three moves allowed on an augmented matrix: swap two rows; multiply a row by a nonzero number; add a multiple of one row to another row.

row-echelon form

A staircase shape in which each row starts with more zeros than the row above it, so the last equation has the fewest variables.

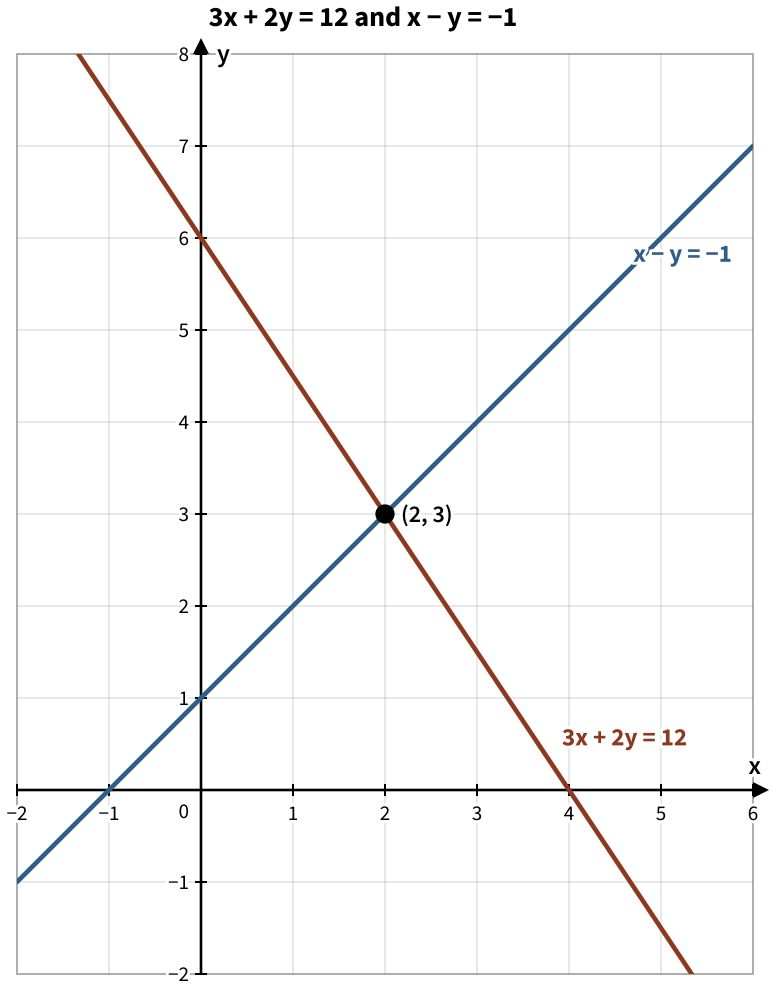
Gaussian elimination

Using row operations to bring an augmented matrix to row-echelon form, then solving by back-substitution from the bottom row up.

CORE IDEAS

Any linear system can be packed into one matrix equation. The system $3x + 2y = 12$ and $x - y = -1$ is $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$, because multiplying row by column gives back exactly the left sides $3x + 2y$ and $x - y$. We write this as $AX = B$.

To solve $5x = 15$ you multiply both sides by $\frac{1}{5}$. To solve $AX = B$, multiply both sides **on the left** by A^{-1}: $A^{-1}AX = A^{-1}B$, and since $A^{-1}A = I$ and $IX = X$, we get $X = A^{-1}B$. The side matters because matrix multiplication is not commutative; BA^{-1} is not even defined here, since a 2×1 matrix cannot be multiplied by a 2×2 matrix. For our system, $\det A = -3 - 2 = -5$, $A^{-1} = -\frac{1}{5}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$, and $A^{-1}B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The graph shows the two lines meeting at $(2, 3)$.



If $\det A = 0$, there is no inverse, and the system has either no solution (parallel lines or planes that never share a point) or infinitely many. For a 3×3 system, a graphing calculator or online matrix tool will compute $A^{-1}B$ quickly; check the answer in all three original equations.

The second method needs no inverse at all. Write the **augmented matrix**, which keeps only the numbers: $3x + 2y = 12$, $x - y = -1$ becomes $\left[\begin{array}{cc|c} 3 & 2 & 12 \\ 1 & -1 & -1 \end{array}\right]$. The three **row operations** are exactly the moves you already use in elimination: swapping two equations, multiplying an equation by a nonzero number, and adding a multiple of one equation to another. None of them changes the solutions, because each one can be undone by another row operation. Working with the matrix just saves writing x, y, z and the equals signs.

Gaussian elimination uses row operations to make zeros below the first entry of each row, producing a staircase (row-echelon form). Then the bottom row is a

one-variable equation; solve it and substitute upward (back-substitution). The goal for three equations is a matrix of the form $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$, where each $*$ is some number.

The final rows also tell you when something unusual happens. A row $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 5 \end{array}\right]$ says $0 = 5$, which is impossible, so the system has **no solution.** A row of all zeros, $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$, says $0 = 0$: that equation added no new information, and if no row is impossible, the system has **infinitely many solutions**. These are the same three cases you met in Lesson 2.4: planes that meet in one point, share no point, or share a line.

KEY FORMULAS

$$AX = B \quad \Rightarrow \quad X = A^{-1}B \quad (\text{when } \det A \neq 0)$$

$$\text{Row operations: } R_i \leftrightarrow R_j, \qquad R_i \to kR_i \ (k \neq 0), \qquad R_i \to R_i + kR_j$$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right]$, $c \neq 0$: no solution; $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$: possibly infinitely many

From history. The Chinese book Nine Chapters on the Mathematical Art, compiled by about the first century AD, solved systems of linear equations by arranging the coefficients in columns on a counting board and combining them with the same row operations used today. Centuries later the method came to carry the name of Carl Friedrich Gauss, who used it in his astronomical calculations.

Practice habit. Write one row operation per line and name it (for example, $R_2 \to R_2 - 3R_1$). When you finish, check the solution in every ORIGINAL equation, not in the reduced matrix.

WORKED EXAMPLES

Example 1. Solve the system $4x + 3y = 5$ and $2x + 2y = 4$ using an inverse matrix.

Show solution

1. Write $AX = B$: $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.
2. $\det A = 8 - 6 = 2$, so $A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1.5 \\ -1 & 2 \end{bmatrix}$.
3. Multiply on the left: $X = A^{-1}B = \begin{bmatrix} 1(5) + (-1.5)(4) \\ -1(5) + 2(4) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.
4. Check in the original equations: $4(-1) + 3(3) = -4 + 9 = 5$ ✓ and $2(-1) + 2(3) = -2 + 6 = 4$ ✓

Answer: $(x, y) = (-1, 3)$

Example 2. The church choir sold 200 tickets to its spring concert: adult tickets at \$8, student tickets at \$5 and child tickets at \$3. Ticket sales came to \$1,160, and twice as many adult tickets were sold as child tickets. Use Gaussian elimination to find how many of each kind were sold.

Show solution

1. Let a, s and c be the numbers of adult, student and child tickets. Equations: $a + s + c = 200$, $8a + 5s + 3c = 1160$, and $a = 2c$, written $a - 2c = 0$.
2. Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 8 & 5 & 3 & 1160 \\ 1 & 0 & -2 & 0 \end{array}\right]$.
3. $R_2 \to R_2 - 8R_1$: the new row 2 is $\left[\begin{array}{ccc|c} 0 & -3 & -5 & -440 \end{array}\right]$ (since $1160 - 1600 = -440$).
4. $R_3 \to R_3 - R_1$: the new row 3 is $\left[\begin{array}{ccc|c} 0 & -1 & -3 & -200 \end{array}\right]$.
5. Swap R_2 and R_3, then $R_2 \to -R_2$: row 2 is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 200 \end{array}\right]$ and row 3 is $\left[\begin{array}{ccc|c} 0 & -3 & -5 & -440 \end{array}\right]$.
6. $R_3 \to R_3 + 3R_2$: row 3 becomes $\left[\begin{array}{ccc|c} 0 & 0 & 4 & 160 \end{array}\right]$ (since $-5 + 9 = 4$ and $-440 + 600 = 160$).
7. Back-substitute. Row 3: $4c = 160$, so $c = 40$. Row 2: $s + 3(40) = 200$, so $s = 80$. Row 1: $a + 80 + 40 = 200$, so $a = 80$.
8. Check in the original equations: $80 + 80 + 40 = 200$ ✓; $8(80) + 5(80) + 3(40) = 640 + 400 + 120 = 1160$ ✓; $80 = 2(40)$ ✓

Answer: 80 adult, 80 student and 40 child tickets

PRACTICE

1. Easy Write the system $2x - y = 7$, $x + 4y = -1$ as a matrix equation $AX = B$. Give A and B.

   Show answer

   $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, so $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

2. **Easy** Write the augmented matrix for the system $x + 2y - z = 4$, $3x - y = 5$, $y + 2z = -3$.

Show answer

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & -1 & 0 & 5 \\ 0 & 1 & 2 & -3 \end{array}\right]$ (a missing variable gets a 0)

3. **Easy** Write the system represented by $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 2 \end{array}\right]$ and solve it by back-substitution.

Show answer

$x + 3y = 5$ and $y = 2$; then $x + 6 = 5$, so $x = -1$. Solution $(-1, 2)$

4. **Easy** Apply $R_2 \to R_2 - 2R_1$ to $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 7 & 12 \end{array}\right]$ and write the new matrix.

Show answer

$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & 6 \end{array}\right]$

5. **Medium** Solve $3x + 5y = 4$, $x + 2y = 1$ using an inverse matrix.

Show answer

$\det A = 6 - 5 = 1$, $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, and $A^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Check: $9 - 5 = 4$ ✓, $3 - 2 = 1$ ✓

6. **Medium** Solve $2x + 4y = 10$, $x + 3y = 7$ using an inverse matrix.

Show answer

$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$, so $X = \frac{1}{2}\begin{bmatrix} 30 - 28 \\ -10 + 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Check: $2 + 8 = 10$ ✓, $1 + 6 = 7$ ✓

7. **Medium** Solve $x - 2y = -4$, $3x + y = 9$ by Gaussian elimination on the augmented matrix.

Show answer

$\left[\begin{array}{cc|c} 1 & -2 & -4 \\ 3 & 1 & 9 \end{array}\right]$; $R_2 \to R_2 - 3R_1$ gives $\left[\begin{array}{cc|c} 0 & 7 & 21 \end{array}\right]$, so $y = 3$ and $x = -4 + 6 = 2$. Check: $3(2) + 3 = 9$ ✓

8. **Medium** Row reduction of a system gives $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 5 \end{array}\right]$. How many solutions does the system have? Explain.

Show answer

None. The second row says $0x + 0y = 5$, that is $0 = 5$, which is impossible.

9. **Medium** A bakery sold 120 muffins and scones on Saturday. Muffins cost \$2 and scones \$3, and the sales came to \$290. Write a matrix equation and solve it with an inverse matrix to find how many of each were sold.

Show answer

$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} m \\ s \end{bmatrix} = \begin{bmatrix} 120 \\ 290 \end{bmatrix}$; $\det = 1$, inverse $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, so $m = 360 - 290 = 70$ and $s = -240 + 290 = 50$: 70 muffins and 50 scones. Check: $140 + 150 = 290$ ✓

10. **Hard** Solve by Gaussian elimination: $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$.

Show answer

$R_2 \to R_2 - 2R_1$: $\left[\begin{array}{ccc|c} 0 & -3 & -1 & -9 \end{array}\right]$; $R_3 \to R_3 - R_1$: $\left[\begin{array}{ccc|c} 0 & 1 & -2 & -4 \end{array}\right]$. Swap them, then $R_3 \to R_3 + 3R_2$: $\left[\begin{array}{ccc|c} 0 & 0 & -7 & -21 \end{array}\right]$, so $z = 3$, $y = -4 + 6 = 2$, $x = 6 - 2 - 3 = 1$. Check: $2 - 2 + 3 = 3$ ✓, $1 + 4 - 3 = 2$ ✓

11. **Hard** Use row reduction to decide how many solutions the system $x + y - z = 2$, $2x + 3y - z = 7$, $3x + 4y - 2z = 9$ has.

Show answer

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ both give $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 3 \end{array}\right]$; then $R_3 \to R_3 - R_2$ gives $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$ and no row is impossible, so there are infinitely many solutions. (The third equation is the sum of the first two.)

12. **Hard** A farmer mixes 500 pounds of feed from corn, oats and barley (see the table). The mix must cost \$122.50 and contain 52.5 pounds of protein. Write the system as $AX = B$ and use a calculator or matrix tool to find $X = A^{-1}B$, the pounds of each grain.
Grains for the feed mix

Grain	Cost (dollars per pound)	Protein (pounds per pound of grain)
Corn	0.2	0.09
Oats	0.3	0.12
Barley	0.25	0.11

Show answer

$\begin{bmatrix} 1 & 1 & 1 \\ 0.20 & 0.30 & 0.25 \\ 0.09 & 0.12 & 0.11 \end{bmatrix} \begin{bmatrix} c \\ o \\ b \end{bmatrix} = \begin{bmatrix} 500 \\ 122.5 \\ 52.5 \end{bmatrix}$ gives 200 pounds of corn, 150 of oats and 150 of barley. Check: $40 + 45 + 37.50 = 122.50$ ✓, $18 + 18 + 16.5 = 52.5$ ✓

13. **Hard** The jet of water from a garden fountain follows a parabola $y = ax^2 + bx + c$ through the points $(0, 4)$, $(1, 7)$ and $(2, 6)$, where x is the horizontal distance and y the height, both in feet. Set up an augmented matrix for a, b, c, solve it, and write the equation.

Show answer

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 7 \\ 4 & 2 & 1 & 6 \end{array}\right]$ gives $c = 4$, $a + b = 3$, $4a + 2b = 2$, so $a = -2$, $b = 5$: $y = -2x^2 + 5x + 4$. Check $(2, 6)$: $-8 + 10 + 4 = 6$ ✓

14. **Challenge** Consider $x + 2y = 3$ and $2x + ky = 6$. (a) For what value of k does the coefficient matrix have no inverse? (b) For that k, how many solutions does the system have?

Show answer

(a) $\det = k - 4 = 0$, so $k = 4$. (b) With $k = 4$ the second equation is $2x + 4y = 6$, exactly twice the first, so every solution of the first works: infinitely many solutions.

15. **Challenge** Explain why the solution of $AX = B$ is $X = A^{-1}B$ and not $X = BA^{-1}$, for a 2×2 invertible matrix A and a 2×1 matrix B.

Show answer

Model answer: Multiply both sides of $AX = B$ on the left by A^{-1}: $A^{-1}(AX) = A^{-1}B$. By the associative property the left side is $(A^{-1}A)X = IX = X$, so $X = A^{-1}B$. Multiplying on the right would not cancel A, because AXA^{-1} cannot be rearranged when matrix multiplication is not commutative. Besides, BA^{-1} is $(2 \times 1)(2 \times 2)$, whose inside dimensions 1 and 2 do not match, so it is not even defined.

COMMON MISTAKES

- Writing $X = BA^{-1}$. The inverse must go on the left, $X = A^{-1}B$, because it has to sit right next to A to cancel it.
- Lining up the wrong numbers in an augmented matrix when a variable is missing. Write every equation with the variables in the same order, and put a 0 for any missing variable.
- Changing only part of a row in a row operation. Every entry in the row changes, including the constant after the bar; write the whole new row each time.
- Reading a row of zeros as "no solution." A row $\left[\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right]$ says $0 = 0$ and points to infinitely many solutions; only $\left[\begin{array}{cc|c} 0 & 0 & c \end{array}\right]$ with $c \neq 0$ means no solution.

> **Make it a game · Matrix Secret Code.** Write a secret message as number pairs, with A = 1, B = 2 and so on. Encode each pair with the matrix [[2, 1], [1, 1]]. Hand the coded pairs to the parent, who decodes using the inverse [[1, -1], [-1, 2]], multiplied on the left. Explain why the inverse goes on the left of the coded pair. Swap messages.

12.5 Vectors and the Dot Product

OBJECTIVES

- Find the components, magnitude and unit vector of a vector, and add, subtract and scale vectors both as arrows and as lists.
- Compute the dot product of two vectors with any number of components and use it for weighted totals and for work.
- Prove from the Pythagorean theorem that two vectors are perpendicular exactly when their dot product is 0, and use this test.

WATCH

3D Dot Product (2 of 3: Deriving the formula for component form)

Eddie Woo · 12:32

https://www.youtube.com/watch?v=PYaOSGDY1No

> **WHEN TO WATCH**
> Before the dot-product part of the Core ideas.

Note: Both videos use three-component vectors. Vector components, magnitude and unit vectors are taught from the Core ideas. Eddie Woo teaches in Australia: he writes vectors with i, j, k as well as column vectors.

BACKUP VIDEO

3D Dot Product (3 of 3: Application to perpendicular vectors)

Eddie Woo · 11:31

https://www.youtube.com/watch?v=CyXSZLElPpY

Before Example 2: perpendicular vectors and a zero dot product.

KEY TERMS

vector

A quantity with both size and direction, drawn as an arrow and written by its components, such as $\mathbf{u} = \langle 3, 1 \rangle$ (3 right, 1 up). More generally, any ordered list of numbers $\langle u_1, u_2, \ldots, u_n \rangle$.

component

One entry of a vector. In $\langle 3, 1 \rangle$ the x-component is 3 and the y-component is 1.

magnitude

The length of a vector, written $|\mathbf{u}|$. For $\langle a, b\rangle$ it is $\sqrt{a^2 + b^2}$.

unit vector

A vector of length 1. Dividing any nonzero vector by its magnitude gives the unit vector in the same direction.

dot product

The number $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$: multiply matching components and add.

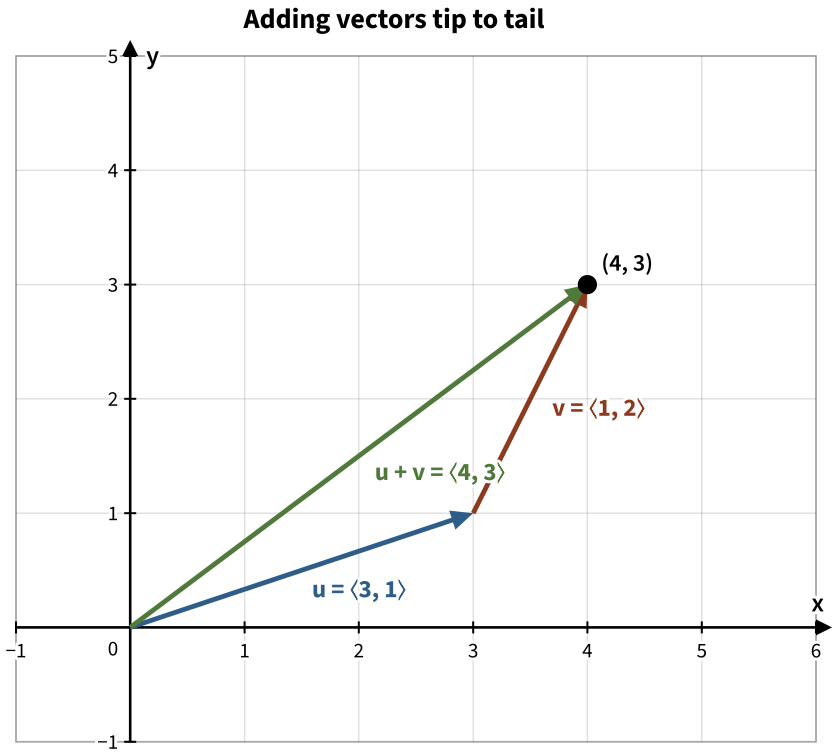
orthogonal

Another word for perpendicular. Two nonzero vectors are orthogonal when they meet at a right angle.

CORE IDEAS

In Geometry you translated figures by a vector $\langle a, b\rangle$: move a units right and b units up. That vector can be drawn as an arrow starting anywhere; only its length and direction matter. The arrow from (x_1, y_1) to (x_2, y_2) is the vector $\langle x_2 - x_1, y_2 - y_1\rangle$ (end minus start), and by the distance formula its **magnitude** is $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$.

To add vectors as arrows, place them **tip to tail**: draw $\mathbf{u}$, then start $\mathbf{v}$ where $\mathbf{u}$ ends. The sum $\mathbf{u} + \mathbf{v}$ runs from the start of $\mathbf{u}$ to the end of $\mathbf{v}$. As lists, you simply add components: $\langle 3, 1\rangle + \langle 1, 2\rangle = \langle 4, 3\rangle$, because the right moves add and the up moves add. Subtraction works the same way, $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2\rangle$, and a **scalar multiple** $k\mathbf{u}$ multiplies each component by k, stretching the arrow by a factor of $|k|$ (and reversing it if $k < 0$). Dividing by the length, $\dfrac{\mathbf{u}}{|\mathbf{u}|}$, gives a **unit vector** pointing the same way.



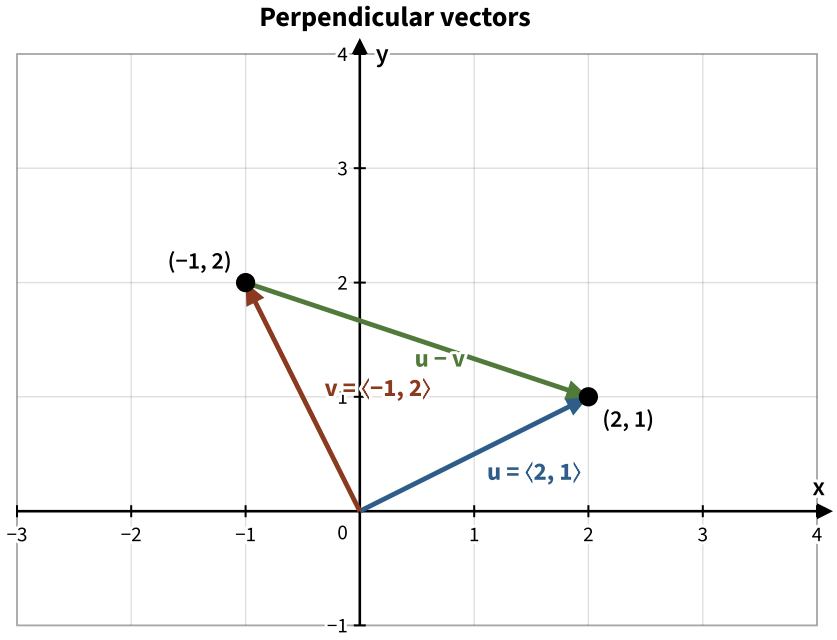
A vector does not have to live in the plane. Any ordered list of measurements is a vector: the eggs laid in one week by five hens might be $\langle 6, 5, 7, 4, 6 \rangle$. Vectors with the same number of components are added and scaled component by component, exactly like a matrix with one row. (In fact a vector is a $1 \times n$ or $n \times 1$ matrix.)

The **dot product** of two vectors multiplies matching components and adds: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$. The answer is a number, not a vector. You have used this idea already: if $\mathbf{q}$ lists the quantities in an order and $\mathbf{p}$ lists the prices, then $\mathbf{q} \cdot \mathbf{p}$ is the total cost. It is also exactly the row-times-column step of matrix multiplication. In physics, when a constant force $\mathbf{F}$ moves an object along a straight path $\mathbf{d}$, the **work** done is $W = \mathbf{F} \cdot \mathbf{d}$; only the part of the force pointing along the path does any work.

The dot product also detects right angles. Put $\mathbf{u}$ and $\mathbf{v}$ tail to tail at the origin. The third side of the triangle they form is $\mathbf{u} - \mathbf{v}$, from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$. Expand its squared length: $|\mathbf{u} - \mathbf{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 = (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - 2(u_1 v_1 + u_2 v_2)$, that is, $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,\mathbf{u} \cdot \mathbf{v}$.

By the Pythagorean theorem and its converse, the angle between $\mathbf{u}$ and $\mathbf{v}$ is a right angle exactly when $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. Comparing with the identity above, that

happens exactly when $2\,\mathbf{u}\cdot\mathbf{v}=0$. So **two nonzero vectors are perpendicular exactly when $\mathbf{u}\cdot\mathbf{v}=0$.** In the picture, $\langle 2,1\rangle\cdot\langle -1,2\rangle = -2+2=0$, and the vectors do meet at a right angle.



KEY FORMULAS

$$|\langle a,b\rangle| = \sqrt{a^2+b^2}$$

$$\mathbf{u}\pm\mathbf{v} = \langle u_1\pm v_1,\ u_2\pm v_2\rangle, \qquad k\mathbf{u} = \langle ku_1,\ ku_2\rangle, \qquad \text{unit vector} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,\mathbf{u}\cdot\mathbf{v}, \qquad \mathbf{u}\perp\mathbf{v} \iff \mathbf{u}\cdot\mathbf{v}=0$$

$$W = \mathbf{F}\cdot\mathbf{d}$$

Practice habit. Sketch before you compute: draw the arrows roughly to scale, then check that your algebra agrees with the picture (a sum that should point up and to the right should have positive components; perpendicular arrows should give a dot product of 0).

WORKED EXAMPLES

Example 1. Let $\mathbf{u} = \langle 3, -4 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$. Find (a) $2\mathbf{u} - \mathbf{v}$, (b) $|\mathbf{u}|$, and (c) the unit vector in the direction of $\mathbf{u}$.

Show solution

1. (a) $2\mathbf{u} = \langle 6, -8 \rangle$.
2. $2\mathbf{u} - \mathbf{v} = \langle 6 - (-1), -8 - 2 \rangle = \langle 7, -10 \rangle$.
3. (b) $|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.
4. (c) Divide each component by 5: $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \left\langle \dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$.
5. Check that it is a unit vector: $\sqrt{\dfrac{9}{25} + \dfrac{16}{25}} = \sqrt{1} = 1$ ✓

Answer: (a) $\langle 7, -10 \rangle$; (b) 5; (c) $\left\langle \dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$

Example 2. Samuel uses a tractor and chain to drag a fallen log along a straight path 40 feet east and 30 feet north, so the displacement is $\mathbf{d} = \langle 40, 30 \rangle$ feet. The chain pulls with a constant force $\mathbf{F} = \langle 250, 100 \rangle$ pounds. (a) How far does the log move? (b) How much work does the chain do, in foot-pounds? (c) A helper pushes sideways on the log with force $\mathbf{G} = \langle -30, 40 \rangle$ pounds. How much work does that push do?

Show solution

1. (a) $|\mathbf{d}| = \sqrt{40^2 + 30^2} = \sqrt{1600 + 900} = \sqrt{2500} = 50$ feet.
2. (b) $W = \mathbf{F} \cdot \mathbf{d} = 250(40) + 100(30)$.
3. $= 10{,}000 + 3{,}000 = 13{,}000$ foot-pounds.
4. (c) $\mathbf{G} \cdot \mathbf{d} = -30(40) + 40(30) = -1{,}200 + 1{,}200 = 0$.
5. The dot product is 0, so $\mathbf{G}$ is perpendicular to the path: a push straight across the direction of motion does no work in moving the log forward.

Answer: (a) 50 feet; (b) 13,000 foot-pounds; (c) 0 foot-pounds (the push is perpendicular to the path)

PRACTICE

1. Easy Let $\mathbf{u} = \langle 2, 5 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$. Find (a) $\mathbf{u} + \mathbf{v}$ and (b) $\mathbf{u} - \mathbf{v}$.

Show answer

(a) $\langle -1, 6 \rangle$; (b) $\langle 5, 4 \rangle$

2. Easy Find the magnitude of $\langle -6, 8 \rangle$.

Show answer

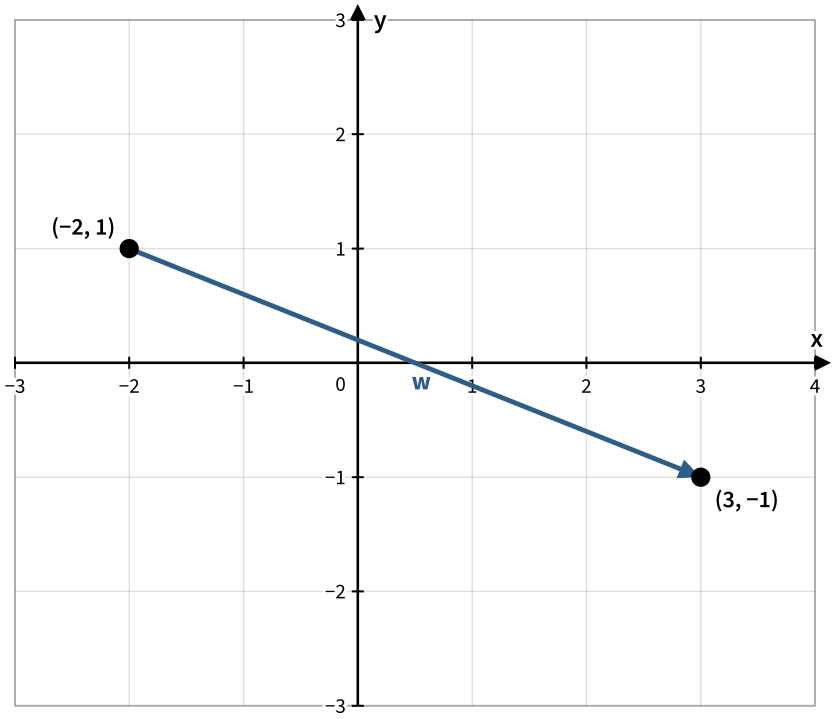
$\sqrt{36 + 64} = \sqrt{100} = 10$

3. Easy Find $\langle 4, -1 \rangle \cdot \langle 2, 3 \rangle$.

Show answer

$4(2) + (-1)(3) = 8 - 3 = 5$

4. Easy Write the vector $\mathbf{w}$ shown in the graph in component form.



Show answer

End minus start: $\langle 3 - (-2), -1 - 1 \rangle = \langle 5, -2 \rangle$

5. Medium Are $\langle 6, -4 \rangle$ and $\langle 2, 3 \rangle$ perpendicular? Use the dot product.

Show answer

Yes: $6(2) + (-4)(3) = 12 - 12 = 0$

6. **Medium** Find the unit vector in the direction of $\langle 5, 12\rangle$. Give exact components.

Show answer

$|\langle 5, 12\rangle| = \sqrt{25 + 144} = 13$, so the unit vector is $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

7. **Medium** Find k so that $\langle k, 3\rangle$ is perpendicular to $\langle 4, -2\rangle$.

Show answer

$4k - 6 = 0$, so $k = \frac{3}{2}$

8. **Medium** Let $\mathbf{u} = \langle 1, -2, 4\rangle$ and $\mathbf{v} = \langle 3, 0, -1\rangle$. Find $3\mathbf{u} - 2\mathbf{v}$.

Show answer

$\langle 3, -6, 12\rangle - \langle 6, 0, -2\rangle = \langle -3, -6, 14\rangle$

9. **Medium** Ruth records the eggs laid by her five hens. Last week's counts were $\mathbf{e} = \langle 6, 5, 7, 4, 6\rangle$ and this week's are $\mathbf{f} = \langle 5, 6, 6, 5, 7\rangle$ (one component per hen). (a) Find $\mathbf{e} + \mathbf{f}$. (b) How many eggs did the five hens lay in the two weeks?

Show answer

(a) $\langle 11, 11, 13, 9, 13\rangle$; (b) $11 + 11 + 13 + 9 + 13 = 57$ eggs

10. **Hard** A farm supply order lists quantities $\mathbf{q} = \langle 12, 8, 5, 20\rangle$ (fence posts, rolls of wire, bags of feed, boxes of staples) and prices in dollars $\mathbf{p} = \langle 4.50, 32.00, 18.75, 2.25\rangle$. Find the total cost $\mathbf{q} \cdot \mathbf{p}$.

Show answer

$12(4.50) + 8(32.00) + 5(18.75) + 20(2.25) = 54 + 256 + 93.75 + 45 = 448.75$, so \$448.75

11. **Hard** A boat's engine and rudder would move it with velocity $\langle 12, 5\rangle$ miles per hour (east, north) in still water, but a current adds $\langle -2, 3\rangle$ miles per hour. (a) Find the boat's actual velocity vector. (b) Find its speed to the nearest tenth of a mile per hour.

Show answer

(a) $\langle 12 + (-2), 5 + 3\rangle = \langle 10, 8\rangle$; (b) $\sqrt{100 + 64} = \sqrt{164} \approx 12.8$ miles per hour

12. Hard Use vectors to decide whether the triangle with vertices $A(1,1)$, $B(4,3)$ and $C(2,6)$ is a right triangle. If it is, name the vertex of the right angle.

Show answer

$\overrightarrow{BA} = \langle -3, -2\rangle$ and $\overrightarrow{BC} = \langle -2, 3\rangle$, and $\overrightarrow{BA} \cdot \overrightarrow{BC} = 6 - 6 = 0$. Yes, the right angle is at B.

13. Challenge Find both vectors of length 10 that are perpendicular to $\langle 3, 4\rangle$.

Show answer

$\langle -4, 3\rangle$ is perpendicular, since $-12 + 12 = 0$, and has length 5. Double it and also take its opposite: $\langle -8, 6\rangle$ and $\langle 8, -6\rangle$

14. Challenge Prove that two nonzero vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are perpendicular exactly when $\mathbf{u} \cdot \mathbf{v} = 0$.

Show answer

Model proof: Draw $\mathbf{u}$ and $\mathbf{v}$ from the origin; the third side of the triangle is $\mathbf{u} - \mathbf{v}$. Expanding, $|\mathbf{u} - \mathbf{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2(u_1 v_1 + u_2 v_2) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,\mathbf{u} \cdot \mathbf{v}$. By the Pythagorean theorem and its converse, the angle between $\mathbf{u}$ and $\mathbf{v}$ is right exactly when $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$, which by the identity happens exactly when $2\,\mathbf{u} \cdot \mathbf{v} = 0$, that is, when $\mathbf{u} \cdot \mathbf{v} = 0$.

COMMON MISTAKES

- Subtracting start minus end when writing a vector from two points. Always use end minus start: the arrow from $(-2, 1)$ to $(3, -1)$ is $\langle 5, -2\rangle$.
- Adding magnitudes: $|\mathbf{u} + \mathbf{v}|$ is usually not $|\mathbf{u}| + |\mathbf{v}|$. Add the components first, then find the length.
- Writing the dot product as a vector, such as $\langle 8, -3\rangle$. The dot product is a single number: multiply matching components and add the products.
- Forgetting to divide every component by the magnitude when making a unit vector. Check your answer: its length must be exactly 1.

> **Make it a game · Shopping Dot Product.** Write a shopping vector, such as 3 apples, 2 loaves and 4 jars, and a price vector of 2, 5 and 3 pretend dollars. Compute the dot product for the total cost. Then test whether (3, 4) and (-4, 3) are orthogonal. Find the magnitude and unit vector of (3, 4). Make your own list for the parent to total.

12.6 Matrices as Transformations

OBJECTIVES

- Apply a 2×2 matrix to points and to the vertices of a figure, and explain why the columns of the matrix are the images of $(1, 0)$ and $(0, 1)$.
- Write the matrices for reflections in the axes and in $y = x$, dilations, stretches, shears, and rotations by 90° and 180°.
- Compose transformations by multiplying matrices in the correct order and explain why the order matters.
- Use the determinant to find how a transformation changes areas and whether it flips a figure.

WATCH

Linear transformations and matrices | Chapter 3, Essence of linear algebra

3Blue1Brown · 10:58

https://www.youtube.com/watch?v=kYB8IZa5AuE

> **WHEN TO WATCH**
> Before the Core ideas: a matrix as a transformation of the plane.

BACKUP VIDEO

Matrices Quick Tip for Rotating and Reflecting

Mario's Math Tutoring · 3:11

https://www.youtube.com/watch?v=nNw48yWDEOk

Before Example 1: rotation and reflection matrices.

KEY TERMS

linear transformation

A rule that sends each point (x, y) to a new point by multiplying the column $\begin{bmatrix} x \\ y \end{bmatrix}$ by a fixed 2×2 matrix. It always sends the origin to itself.

image

The point (or figure) that a transformation produces from a given point (or figure).

standard basis vectors

The two unit vectors $\mathbf{e}_1 = \langle 1, 0 \rangle$ and $\mathbf{e}_2 = \langle 0, 1 \rangle$ along the axes.

shear

A transformation that slides each point parallel to an axis by an amount proportional to its distance from that axis, turning squares into slanted parallelograms, as in $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

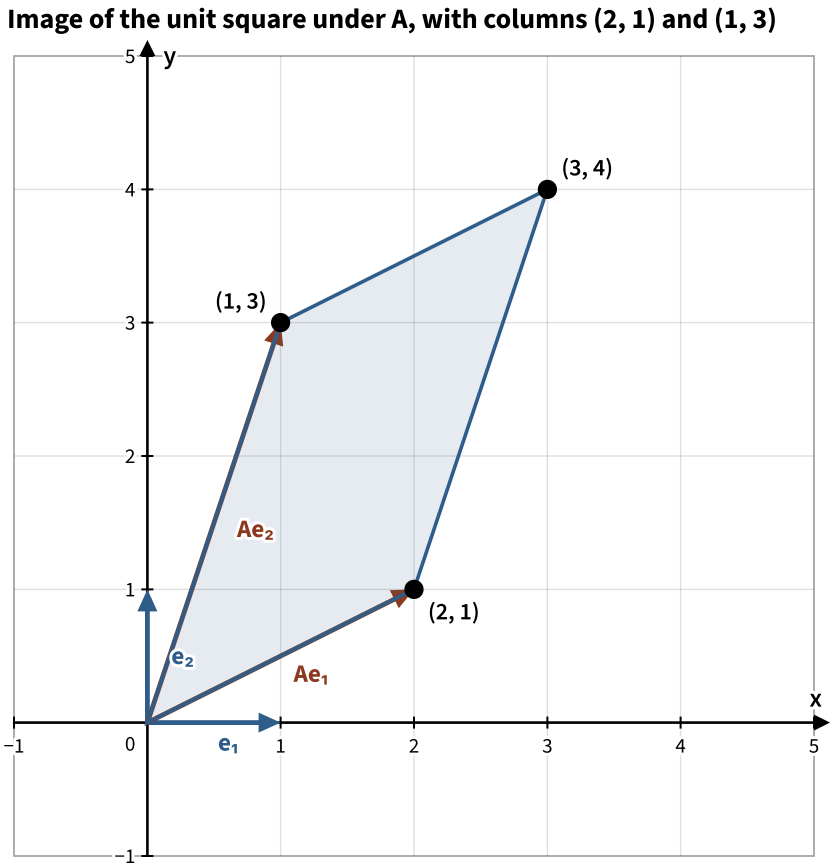
composition of transformations

Doing one transformation and then another. Its matrix is the product of the two matrices, with the first step on the right.

CORE IDEAS

In Geometry you wrote transformations as coordinate rules, such as $(x, y) \to (-y, x)$ for a 90° counterclockwise rotation. A 2×2 matrix does the same job: it sends the point (x, y) to the point given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. For example, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ is exactly the 90° rotation rule.

Here is the key fact: **the columns of the matrix are the images of $(1, 0)$ and $(0, 1)$.** Indeed, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, the first column, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, the second. And those two images decide everything, because any point is $(x, y) = x(1, 0) + y(0, 1)$, and the product rule gives $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$. So to build the matrix of any such transformation, just ask: where do $(1, 0)$ and $(0, 1)$ go? Write those images as the columns.



That one question produces every standard matrix. Reflection in the x-axis keeps $(1, 0)$ and sends $(0, 1)$ to $(0, -1)$, giving $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Reflection in $y = x$ swaps them, giving $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. A 90° counterclockwise rotation sends $(1, 0)$ to $(0, 1)$ and $(0, 1)$ to $(-1, 0)$, giving $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. A dilation by k gives $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$; a horizontal stretch by k only stretches $(1, 0)$, giving $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. The full list is in the formulas. To transform a whole figure, write its vertices as the columns of one matrix and multiply once.

To **compose** transformations, follow a point through both steps. If B is done first and A second, the point X goes to BX and then to $A(BX)$. By the associative property, $A(BX) = (AB)X$, so the single matrix for the combined transformation is AB: **the matrix of the first step goes on the right**, next to the point. Order matters. With $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflect in the x-axis) and $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotate 90°), reflecting first and then rotating gives $RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, a reflection in $y = x$, while rotating first gives $FR =$

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, a reflection in $y = -x$. This is the geometric reason matrix multiplication is not commutative.

The determinant tells how areas change. The unit square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ goes to the parallelogram built on the two columns (a,c) and (b,d). That parallelogram is two copies of the triangle with vertices $(0,0)$, (a,c), (b,d), whose area by Lesson 12.3 is $\frac{1}{2}|ad-bc|$. So the square of area 1 becomes a parallelogram of area $|ad-bc| = |\det A|$. Every figure can be filled with tiny squares, and each is scaled the same way, so **every area is multiplied by** $|\det A|$. In the picture, $\det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 6 - 1 = 5$, and the parallelogram has area 5.

The sign of the determinant carries meaning too. A **negative determinant means the figure is flipped**, like a reflection: going around the figure counterclockwise, the images go around clockwise. Every reflection matrix above has determinant -1 (flipped, area unchanged), rotations have determinant 1, and a determinant of 0 squashes the whole plane onto a line or a point, which is why such a matrix cannot be undone.

Looking ahead: vectors and matrices are the beginning of **linear algebra**. Every time a computer turns, tilts or resizes a picture on a screen, it multiplies the coordinates of many points by matrices just like these, and the same operations, done with very large matrices, lie at the heart of modern computing.

KEY FORMULAS

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\text{Reflect in } x\text{-axis: } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad y\text{-axis: } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y = x\text{: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Rotate } 90^\circ\text{: } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad 180^\circ\text{: } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{Dilate by } k\text{: } \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

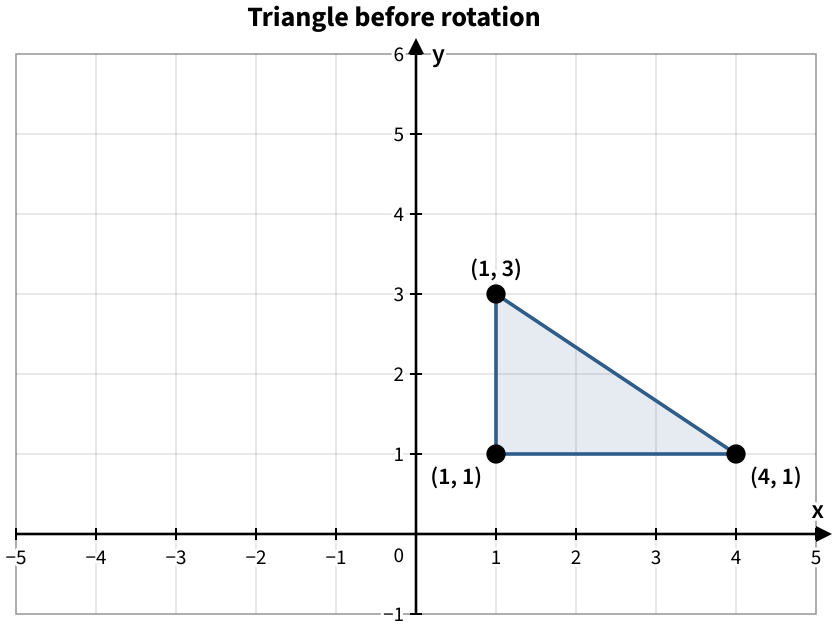
$$\text{Stretch: } \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \text{ (horizontal), } \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \text{ (vertical)} \qquad \text{Shear: } \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$\text{First } B, \text{ then } A: \ AB \qquad \text{New area} = |\det A| \times \text{old area}$$

Practice habit. To write any transformation matrix, ask "Where does $(1, 0)$ go? Where does $(0, 1)$ go?" and write the two answers as the columns. Then test the matrix on one more point you can check with a sketch.

WORKED EXAMPLES

Example 1. Use a matrix to rotate the triangle with vertices $(1, 1)$, $(4, 1)$ and $(1, 3)$ by $90°$ counterclockwise about the origin. Give the vertices of the image.



Show solution

1. The rotation sends $(1, 0)$ to $(0, 1)$ and $(0, 1)$ to $(-1, 0)$, so $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
2. Write the vertices as columns: $V = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ (2×3).
3. Multiply $(2 \times 2)(2 \times 3)$ to get a 2×3 matrix. Row 1 of R is $\begin{bmatrix} 0 & -1 \end{bmatrix}$, giving $-1, -1, -3$.
4. Row 2 of R is $\begin{bmatrix} 1 & 0 \end{bmatrix}$, giving $1, 4, 1$. So $RV = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 4 & 1 \end{bmatrix}$.
5. Read the columns: $(-1, 1)$, $(-1, 4)$, $(-3, 1)$.
6. Check with the Geometry rule $(x, y) \to (-y, x)$: $(4, 1) \to (-1, 4)$ ✓ and $(1, 3) \to (-3, 1)$ ✓

Answer: $(-1, 1)$, $(-1, 4)$, $(-3, 1)$

Example 2. Henry is designing a weather vane on a computer. The program first applies the shear $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to slant the arrow, then reflects it in the y-axis with $F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. (a) Find the single matrix M for the combined transformation. (b) Find the image of the point $(2, 3)$. (c) Find $\det M$ and say what it tells you about the arrow's area and orientation.

Show solution

1. (a) The shear is first, so it goes on the right: $M = FS$.
2. $FS = \begin{bmatrix} -1(1) + 0(0) & -1(1) + 0(1) \\ 0(1) + 1(0) & 0(1) + 1(1) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$.
3. (b) $M \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 - 3 \\ 0 + 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.
4. Check step by step: the shear sends $(2, 3)$ to $(2 + 3, 3) = (5, 3)$, and the reflection sends $(5, 3)$ to $(-5, 3)$ ✓
5. (c) $\det M = (-1)(1) - (-1)(0) = -1$.
6. $|\det M| = 1$, so the arrow's area does not change; the negative sign means the arrow is flipped into its mirror image (going around it counterclockwise, the image goes around clockwise).
7. Note: in the other order, $SF = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ sends $(2, 3)$ to $(1, 3)$, a different point, so the order of the steps matters.

Answer: (a) $M = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$; (b) $(-5, 3)$; (c) $\det M = -1$: area unchanged, figure flipped

PRACTICE

1. Easy Find the image of the point $(3, -2)$ under the matrix $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.

   Show answer

   $\begin{bmatrix} 2(3) + 1(-2) \\ 0(3) + (-1)(-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, the point $(4, 2)$

2. **Easy** Reflection in the y-axis sends (x, y) to $(-x, y)$. Find the images of $(1, 0)$ and $(0, 1)$ and write the matrix.

Show answer

$(1, 0) \to (-1, 0)$ and $(0, 1) \to (0, 1)$, so the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

3. **Easy** (a) Describe the transformation given by $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. (b) By what factor does it multiply areas?

Show answer

(a) A dilation by a factor of 3 centered at the origin; (b) $\det = 9$, so areas are multiplied by 9

4. **Easy** (a) Write the matrix for a rotation of $180°$ about the origin. (b) Use it to find the image of $(4, -1)$.

Show answer

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; (b) $(-4, 1)$

5. **Medium** Find the matrix of the transformation that sends $(1, 0)$ to $(2, 5)$ and $(0, 1)$ to $(-1, 3)$.

Show answer

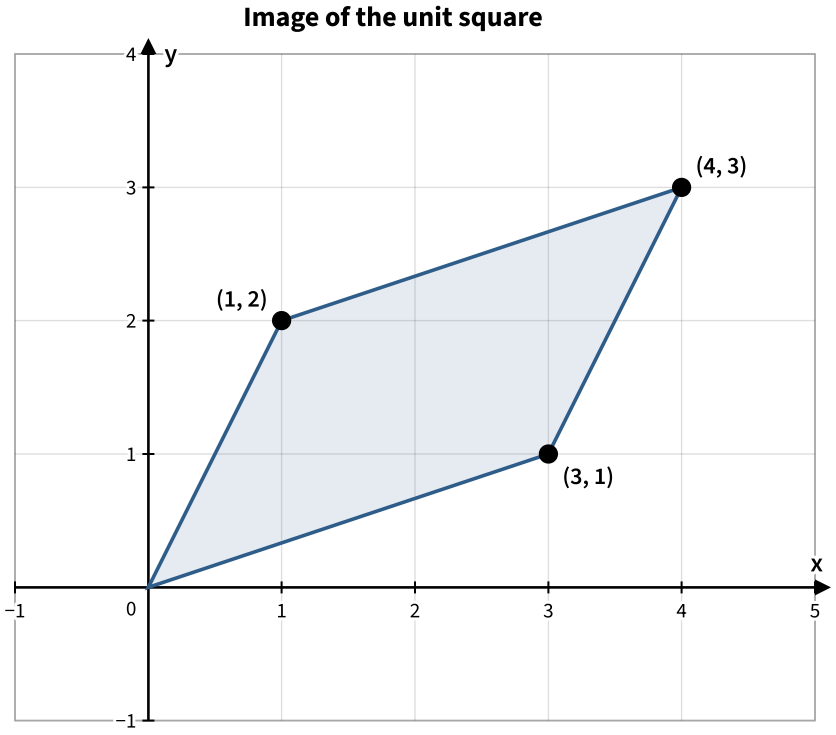
The images are the columns: $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$

6. **Medium** Apply the shear $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ to the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$. (a) Give the image vertices in the same order. (b) Find the area of the image.

Show answer

(a) $(0, 0)$, $(2, 0)$, $(6, 2)$, $(4, 2)$; (b) $\det = 1$, so the area stays 4 square units (a parallelogram with base 2 and height 2)

7. **Medium** The graph shows the image of the unit square (corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$) under a matrix A, where $(1,0)$ went to $(3,1)$ and $(0,1)$ went to $(1,2)$. (a) Write A. (b) Find the area of the parallelogram.



Show answer

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$; (b) $\det A = 6 - 1 = 5$, so the area is 5 square units

8. **Medium** Use the matrix for reflection in the line $y = x$ to find the image of the triangle with vertices $(1,2)$, $(5,2)$ and $(3,6)$.

Show answer

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ 2 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 6 \\ 1 & 5 & 3 \end{bmatrix}$: $(2,1)$, $(2,5)$, $(6,3)$

9. **Medium** A church banner design on a computer is a rectangle 4 units wide and 3 units tall. The designer applies $\begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}$. (a) Give the new width and height. (b) Find the new area. (c) By what factor did the area change, and what is the determinant?

Show answer

(a) $4 \times 1.5 = 6$ wide and $3 \times 2 = 6$ tall; (b) 36 square units; (c) $36 \div 12 = 3$, and $\det = 1.5 \times 2 - 0 = 3$, the same

10. **Hard** A figure is rotated 90° counterclockwise and then reflected in the x-axis. (a) Find the single matrix for the combined transformation. (b) Which single transformation is it?

Show answer

(a) Rotation first, so it goes on the right: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. (b) It sends (x, y) to $(-y, -x)$: a reflection in the line $y = -x$.

11. **Hard** Let $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the 90° rotation. Compute R^2 and explain why the answer makes sense geometrically.

Show answer

$R^2 = \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, the matrix of a 180° rotation: two quarter turns make a half turn.

12. **Hard** A sign maker designs a triangular pennant with vertices $(0, 0)$, $(4, 0)$ and $(0, 3)$, in inches, then transforms it with $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. (a) Find the area of the original pennant. (b) Use the determinant to find the area of the new one. (c) Check by finding the new vertices.

Show answer

(a) $\frac{1}{2}(4)(3) = 6$ square inches. (b) $\det = 6 - (-1) = 7$, so $7 \times 6 = 42$ square inches. (c) The vertices become $(0, 0)$, $(8, -4)$, $(3, 9)$, and $\frac{1}{2}|8(9) - 3(-4)| = \frac{1}{2}(84) = 42$ ✓

13. Hard Grace designs a quilt block that is turned a quarter turn (90° counterclockwise) each time it is repeated around a square. Use $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to compute R^4 and explain what it means for the fourth turn.

Show answer

$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, so $R^4 = R^2 R^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Four quarter turns make a full turn, so the block comes back to exactly its starting position.

14. Challenge Find the single matrix that first reflects in the line $y = x$ and then dilates by a factor of 2. Use its determinant to say how it changes areas and whether it flips figures.

Show answer

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, with $\det = 0 - 4 = -4$: areas are multiplied by 4 and figures are flipped.

15. Challenge Explain why the columns of a 2×2 matrix A are the images of $(1, 0)$ and $(0, 1)$, and why knowing those two images tells you the image of every point (x, y).

Show answer

Model answer: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \cdot 1 + b \cdot 0 \\ c \cdot 1 + d \cdot 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, the first column, and likewise $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, the second column. For any point, $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$, so the image of (x, y) is x times the image of $(1, 0)$ plus y times the image of $(0, 1)$. Once those two images are known, every image is known.

COMMON MISTAKES

- Writing the images of $(1, 0)$ and $(0, 1)$ as rows instead of columns. The image of $(1, 0)$ is the first column, top to bottom.
- Multiplying composition matrices in the order the steps are said. "First B, then A" is AB: the first step's matrix stands next to the point, on the right.
- Writing a point as a row and multiplying it on the left of the matrix. Write each point (or each vertex) as a column and put it on the right: $A \begin{bmatrix} x \\ y \end{bmatrix}$.

- Taking the determinant itself as the area factor when it is negative. Areas are multiplied by $|\det A|$; the negative sign only means the figure is flipped.

> **Make it a game · Cut-Out Transformer.** Cut a small triangle from paper and plot its vertices on graph paper. Choose a 2×2 matrix, such as [[0, -1], [1, 0]]. Before multiplying, predict the images of (1, 0) and (0, 1) by reading the columns. Then apply it to the triangle, and move your cut-out to match. Name the transformation and try a reflection next.

UNIT 12

Matrices and Vectors

This unit introduces two new kinds of mathematical objects: matrices, which are rectangular tables of numbers, and vectors, which are arrows or ordered lists of numbers. You will add, scale and multiply matrices, find determinants and inverses, and use them to solve systems of equations by a method that is really the elimination you already know, written more compactly. Every definition is explained by a practical question: why matrices are multiplied row by column (quantities times prices gives totals), why the inverse of a 2×2 matrix has the form it does, and why a dot product of 0 means a right angle.

The unit builds on the systems of equations from Units 2 and 3, the distance formula and Pythagorean theorem, and the transformations of the plane from Geometry. It ends by showing that a 2×2 matrix is itself a transformation: it can reflect, rotate, stretch or shear a figure, and its determinant tells how areas change. These ideas are the beginning of linear algebra, the mathematics behind computer graphics, engineering and modern computing, which later courses develop further.

PREREQUISITES

- Solving systems of two and three linear equations by elimination (Lessons 2.1 and 2.4)
- Evaluating expressions with negative numbers and fractions accurately
- The distance formula and the Pythagorean theorem and its converse (Geometry)
- Transformations of the plane as coordinate rules: reflections, rotations by 90° and 180°, dilations (Geometry)
- Finding the area of a triangle and a parallelogram (Geometry)

MASTERY CHECKLIST

- ☐ I can give the dimensions of a matrix and name an entry a_{ij} by its row and column.
- ☐ I can add and subtract matrices of the same size, multiply by a scalar, and solve simple matrix equations.
- ☐ I can multiply matrices row by column, decide whether a product is defined, and explain why AB and BA are usually different.
- ☐ I can use a matrix product to compute totals such as costs, hours or pounds of feed.
- ☐ I can find the determinant of a 2×2 or 3×3 matrix and use a determinant to find the area of a triangle.
- ☐ I can derive and use the formula for the inverse of a 2×2 matrix and explain why it exists only when the determinant is not 0.
- ☐ I can solve a system as $X = A^{-1}B$ and by Gaussian elimination on an augmented matrix, and recognize no solution or infinitely many solutions.
- ☐ I can add, subtract and scale vectors, find magnitudes and unit vectors, and compute dot products for weighted totals and work.
- ☐ I can prove and use the fact that two vectors are perpendicular exactly when their dot product is 0.

- ☐ I can write the matrix of a reflection, rotation, dilation, stretch or shear from the images of $(1, 0)$ and $(0, 1)$, and apply it to a figure.
- ☐ I can compose transformations by multiplying matrices in the right order and use the determinant to tell how areas change and whether a figure is flipped.

LESSONS IN THIS UNIT

12.1 Matrices and Matrix Operations	2 sessions of 45 minutes
12.2 Matrix Multiplication	2 sessions of 45 minutes
12.3 Determinants and Inverse Matrices	2 sessions of 45 minutes
12.4 Solving Systems with Matrices	2 sessions of 45 minutes
12.5 Vectors and the Dot Product	2 sessions of 45 minutes
12.6 Matrices as Transformations	2 sessions of 45 minutes

Unit 12 test

Time: 60 minutes. A scientific calculator is allowed; a graphing calculator or matrix tool is not needed and should not be used. Show your work: for products, show at least one row-times-column sum; for row reduction, name each row operation. Give exact answers unless a problem asks you to round. Check solutions of systems in the original equations.

1. **5 pts**

   Let $A = \begin{bmatrix} 5 & -2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 6 & 2 \end{bmatrix}$. Find $3A + B$.

2. **5 pts**

   Find a and b if $\begin{bmatrix} a+3 & 7 \\ -1 & 2b \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -1 & 10 \end{bmatrix}$.

3. **6 pts**

   The Hartley family's orchards recorded bushels picked. September: $S = \begin{bmatrix} 120 & 45 \\ 80 & 30 \end{bmatrix}$; October: $O = \begin{bmatrix} 150 & 20 \\ 95 & 10 \end{bmatrix}$ (rows: apples, pears; columns: north orchard, south orchard). (a) Find the matrix of total bushels for the two months. (b) How many more bushels of apples did the north orchard pick in October than in September?

4. **5 pts**

   A is a 3×2 matrix and B is a 2×4 matrix. (a) What are the dimensions of AB? (b) Is BA defined?

5. **6 pts**

   Find $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 5 \end{bmatrix}$.

6. **7 pts**

   A family sells bundles of firewood. On Friday they sold 30 oak and 20 pine bundles; on Saturday, 45 oak and 25 pine. Oak sells for \$7 a bundle and costs them \$3 to produce; pine sells for \$5 and costs \$2. (a) Multiply $Q = \begin{bmatrix} 30 & 20 \\ 45 & 25 \end{bmatrix}$ by $P = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$ (columns: price, cost) and say what the entries of QP mean. (b) Find the total profit for the two days.

7. **6 pts**

   Find $\det \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 4 \\ -2 & 5 & 1 \end{bmatrix}$ by expanding along the first row.

8. **7 pts**

   Find the inverse of $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and check that $AA^{-1} = I$.

9. **6 pts**

   A triangular pasture has corners at $(20, 10)$, $(100, 40)$ and $(50, 90)$ on a map measured in meters. Use a determinant to find its area.

10. **7 pts**

    Use your inverse from Problem 8 to solve the system $5x + 4y = 2$, $3x + 2y = 0$.

11. **8 pts**

    For a church garden, the youth group bought 30 plants: tomato plants at \$3 each, pepper plants at \$2 each and herb plants at \$4 each, for \$84 in all. They bought as many pepper plants as tomato and herb plants together. Write an augmented matrix and use Gaussian elimination to find how many of each they bought.

12. **5 pts**

    Each matrix is the result of row-reducing a system in x, y, z. State whether the system has one solution, no solution or infinitely many solutions. (a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

13. **6 pts**

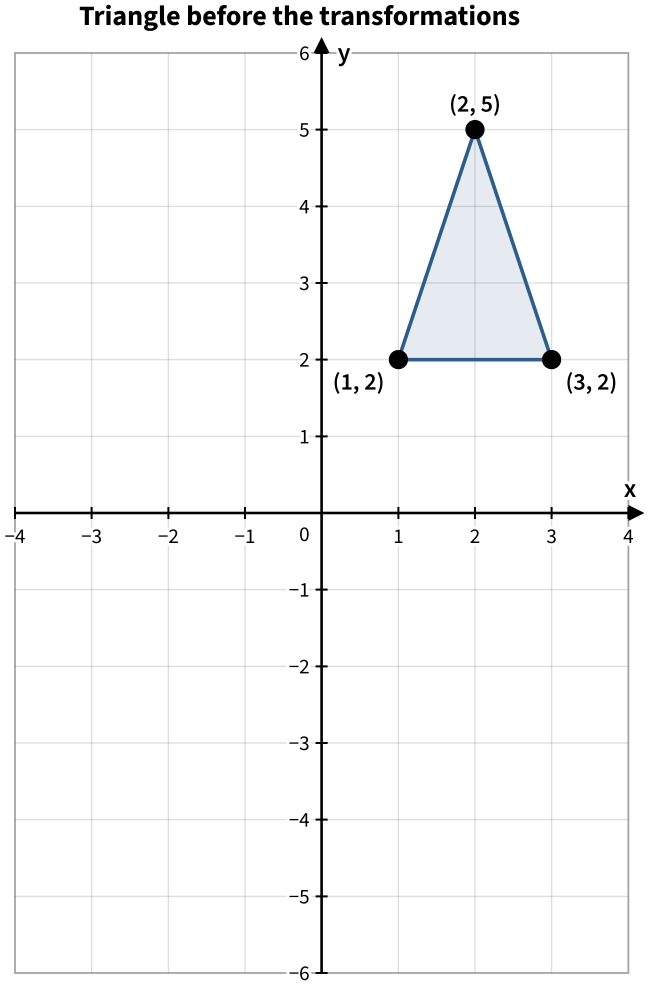
    Let $\mathbf{u} = \langle -2, 6 \rangle$ and $\mathbf{v} = \langle 4, 1 \rangle$. Find (a) $\mathbf{u} + 3\mathbf{v}$, (b) $|\mathbf{u} - \mathbf{v}|$ in exact form, and (c) $\mathbf{u} \cdot \mathbf{v}$.

14. **6 pts**

    A winch pulls a rowboat up a straight boat ramp. The boat moves along the displacement $\mathbf{d} = \langle 24, 10 \rangle$ feet while the cable pulls with a constant force $\mathbf{F} = \langle 120, 50 \rangle$ pounds. (a) How far does the boat move along the ramp? (b) How much work does the winch do, in foot-pounds?

15. **8 pts**

The triangle shown is first reflected in the y-axis and then rotated $180°$ about the origin. (a) Find the single matrix M for the combined transformation. (b) Find the vertices of the image of $(1, 2)$, $(3, 2)$ and $(2, 5)$, in that order. (c) Find $\det M$.



16. **7 pts**

(a) Prove that $\langle a, b\rangle$ and $\langle -b, a\rangle$ are perpendicular whenever a and b are not both 0. (b) Explain what the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does to every vector, and how part (a) agrees with that.

ANSWER KEY

1. $3A = \begin{bmatrix} 15 & -6 \\ 0 & 9 \end{bmatrix}$, so $3A + B = \begin{bmatrix} 14 & -2 \\ 6 & 11 \end{bmatrix}$
2. $a + 3 - 1 = 6$, so $a = 4$; $2b - 4 = 10$, so $b = 7$
3. (a) $S + O = \begin{bmatrix} 270 & 65 \\ 175 & 40 \end{bmatrix}$; (b) $150 - 120 = 30$ bushels
4. (a) 3×4; (b) no, because $(2 \times 4)(3 \times 2)$ has inside numbers 4 and 3, which do not match
5. Row 1: $2 + 2 + 3 = 7$ and $1 + 0 + 15 = 16$; row 2: $0 - 4 - 1 = -5$ and $0 + 0 - 5 = -5$. The product is $\begin{bmatrix} 7 & 16 \\ -5 & -5 \end{bmatrix}$
6. (a) $QP = \begin{bmatrix} 310 & 130 \\ 440 & 185 \end{bmatrix}$: Friday's sales were \$310 and costs \$130; Saturday's sales were \$440 and costs \$185. (b) $(310 - 130) + (440 - 185) = 180 + 255 = 435$, so \$435
7. $3(1 - 20) - 0 + (-1)(10 + 2) = -57 - 12 = -69$
8. $\det A = 10 - 12 = -2$, so $A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$. Check: $\begin{bmatrix} -5+6 & 10-10 \\ -3+3 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓
9. $\det \begin{bmatrix} 20 & 10 & 1 \\ 100 & 40 & 1 \\ 50 & 90 & 1 \end{bmatrix} = 20(40 - 90) - 10(100 - 50) + 1(9000 - 2000) = -1000 - 500 + 7000 = 5500$, so the area is $\frac{1}{2}(5500) = 2750$ square meters
10. $X = A^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, so $(x, y) = (-2, 3)$. Check: $-10 + 12 = 2$ ✓, $-6 + 6 = 0$ ✓
11. With $t + p + h = 30$, $3t + 2p + 4h = 84$, $-t + p - h = 0$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 3 & 2 & 4 & 84 \\ -1 & 1 & -1 & 0 \end{array}\right]$. $R_2 \to R_2 - 3R_1$ gives $\left[\begin{array}{ccc|c} 0 & -1 & 1 & -6 \end{array}\right]$; $R_3 \to R_3 + R_1$ gives $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 30 \end{array}\right]$, so $p = 15$; then $-15 + h = -6$, so $h = 9$, and $t = 30 - 15 - 9 = 6$. Check: $18 + 30 + 36 = 84$ ✓, $15 = 6 + 9$ ✓. They bought 6 tomato, 15 pepper and 9 herb plants.
12. (a) No solution: the last row says $0 = -3$. (b) Infinitely many solutions: the last row says $0 = 0$ and no row is impossible.
13. (a) $\langle -2 + 12, 6 + 3 \rangle = \langle 10, 9 \rangle$; (b) $\mathbf{u} - \mathbf{v} = \langle -6, 5 \rangle$, so $|\mathbf{u} - \mathbf{v}| = \sqrt{36 + 25} = \sqrt{61}$; (c) $-8 + 6 = -2$

14. (a) $\sqrt{576+100} = \sqrt{676} = 26$ feet; (b) $W = \mathbf{F}\cdot\mathbf{d} = 120(24) + 50(10) = 2880 + 500 = 3380$ foot-pounds

15. (a) The reflection is first, so it goes on the right: $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, a reflection in the x-axis. (b) $(1,-2)$, $(3,-2)$, $(2,-5)$. (c) $\det M = -1$: area unchanged, figure flipped.

16. Model answer: (a) $\langle a, b\rangle \cdot \langle -b, a\rangle = -ab + ba = 0$, and both vectors are nonzero, so they are perpendicular. (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$. Its columns are the images $(0,1)$ of $(1,0)$ and $(-1,0)$ of $(0,1)$, so it is the 90° counterclockwise rotation. A vector rotated by 90° is perpendicular to the original, which is exactly what part (a) shows: the image $\langle -b, a\rangle$ always has dot product 0 with $\langle a, b\rangle$.

UNIT 13

A Bridge to Trigonometry

This short closing unit gathers what you know about right triangles from Geometry and builds the bridge to the Trigonometry course. In Lesson 13.1 you review sine, cosine and tangent, see why they depend only on the angle (similar triangles), find exact values from the special right triangles, solve right triangles with inverse functions, and derive the identity $\sin^2\theta + \cos^2\theta = 1$ from the Pythagorean theorem. You also use these ratios to split a vector into its components, tying back to the vectors of Unit 12.

In Lesson 13.2 an angle becomes an amount of turning: positive or negative, and as large as you like. You meet the radian, the natural way to measure angles by arc length, and the unit circle, which defines sine and cosine for every angle, not just acute ones. Turning wheels, from a Ferris wheel to a water wheel to Earth itself, show why these functions repeat. The Trigonometry course picks up exactly here, with the full unit circle, the graphs of sine and cosine, and identities.

PREREQUISITES

- Sine, cosine and tangent of acute angles and angles of elevation and depression (Geometry)
- Similar triangles, the AA similarity criterion and the special right triangles (Geometry)
- The Pythagorean theorem, simplifying square roots and rationalizing denominators
- The circumference of a circle, arc length and the equation $x^2 + y^2 = r^2$ (Geometry)
- Inverse functions (Lesson 1.5)
- Vectors, components and magnitude (Lesson 12.5) and dilations (Lesson 12.6)

MASTERY CHECKLIST

- ☐ I can explain, using similar triangles, why sine, cosine and tangent depend only on the angle.
- ☐ I can give the exact sine, cosine and tangent of 30°, 45° and 60° by sketching the special right triangles.
- ☐ I can find missing sides of a right triangle and use $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to find missi

ng an gle s.

- ☐ I can derive $\sin^2\theta + \cos^2\theta = 1$ and use it to find one ratio from another.
- ☐ I can solve problems about roof pitch, ramps, ladders and angles of elevation and depression.
- ☐ I can write a vector of length r at angle θ as $\langle r\cos\theta, r\sin\theta\rangle$.
- ☐ I can draw positive, negative and large angles in standard position and find coterminal angles.
- ☐ I can explain why π radians $= 180°$, convert between degrees and radians, and use $s = r\theta$.
- ☐ I can use the unit circle to find sine and cosine of multiples of $90°$ and tell their signs in each quadrant.
- ☐ I can model a point on a turning wheel with $(r\cos\theta, r\sin\theta)$ and find its height.

LESSONS IN THIS UNIT

13.1 Right-Triangle Trigonometry: A Review

2 sessions of 45 minutes

Unit 13 test

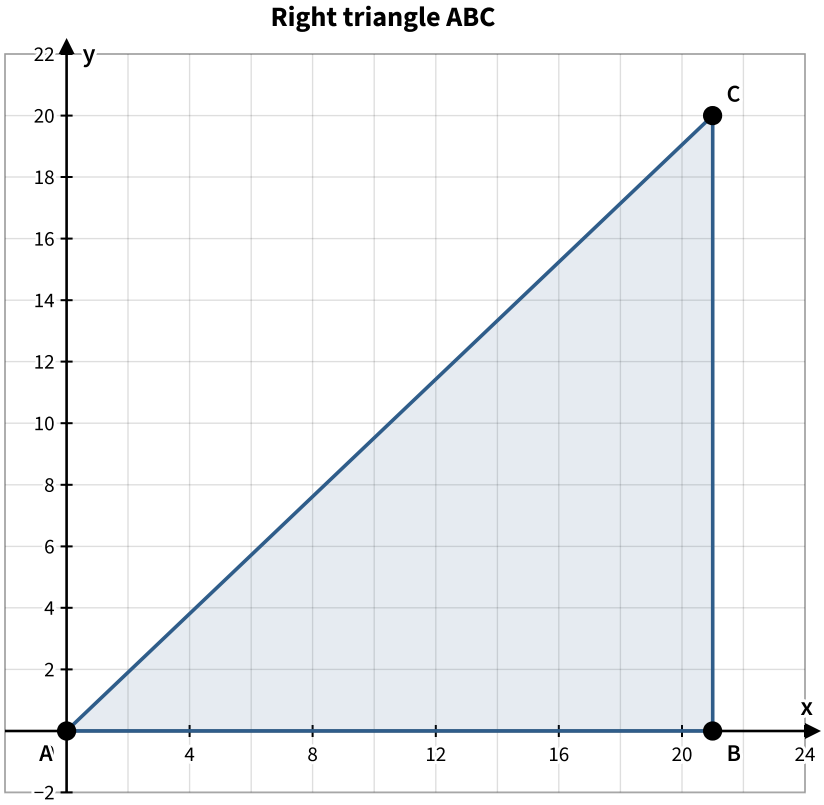
You have 50 minutes. A scientific calculator is allowed, set to degree mode unless a problem works in radians, except on parts marked "by hand". Show your work, including the ratio or formula you use. Give exact answers when the problem asks for them; otherwise round as each problem directs. The test is worth 100 points.

1. **6 pts**

   By hand, give the exact value of (a) $\cos 30^\circ$ and (b) $\tan 60^\circ$.

2. **6 pts**

   In right triangle ABC shown, the right angle is at B, $AB = 21$, $BC = 20$ and $AC = 29$. Let θ be the angle at A. Find (a) $\sin\theta$, (b) $\cos\theta$ and (c) $\tan\theta$ as fractions.



3. **6 pts**

   One acute angle of a right triangle is 52°, and the leg adjacent to it is 15 meters long. Find the length of the leg opposite the 52° angle, to the nearest tenth of a meter.

4. **6 pts**

   A right triangle has a hypotenuse of 14 and a leg of 9. Find the angle opposite the leg of length 9, to the nearest tenth of a degree.

5. **7 pts**

   θ is an acute angle with $\cos\theta = \dfrac{7}{25}$. Use $\sin^2\theta + \cos^2\theta = 1$ to find the exact values of (a) $\sin\theta$ and (b) $\tan\theta$.

6. **6 pts**

   A hay elevator (a conveyor that carries bales up into a loft) is 30 feet long and set at an angle of 28° with the level ground. How much higher than its lower end is its upper end, to the nearest tenth of a foot?

7. **7 pts**

   On a hiking trip, the Miller family walks 5 miles in a straight line in a direction 35° north of east. Write their displacement as a vector $\langle \text{east}, \text{north} \rangle$ in miles, with each component rounded to the nearest tenth.

8. **6 pts**

   (a) Convert 225° to radians, in exact form. (b) Convert $\dfrac{5\pi}{3}$ radians to degrees.

9. **6 pts**

   (a) Find the angle that is at least 0° and less than 360° and coterminal with -500°. (b) Find the angle that is at least 0 and less than 2π radians and coterminal with $\dfrac{7\pi}{2}$ radians.

10. **7 pts**

    By hand, use the unit circle to find (a) $\cos 270^\circ$, (b) $\sin 180^\circ$ and (c) $\sin(-90^\circ)$.

11. **7 pts**

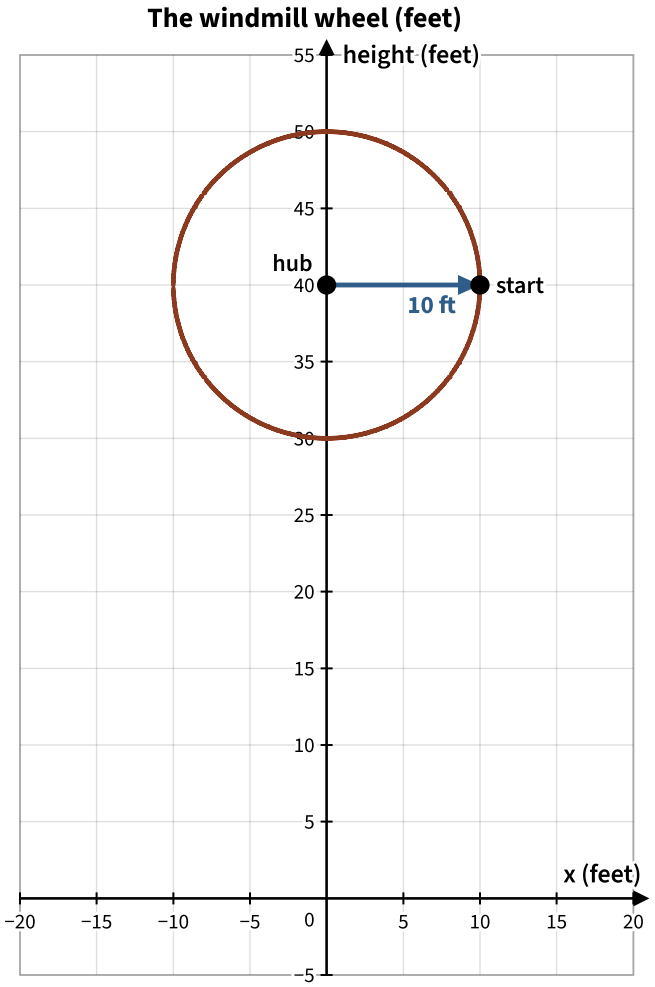
The terminal side of θ lies in Quadrant III, and $\sin\theta = -\frac{8}{17}$. Find the exact values of (a) $\cos\theta$ and (b) $\tan\theta$.

12. **8 pts**

A carousel horse rides 18 feet from the center of the carousel. How far does the horse travel when the carousel turns through $\frac{5\pi}{3}$ radians? Give (a) the exact distance and (b) the distance to the nearest tenth of a foot.

13. **8 pts**

The wheel of a windmill has blades 10 feet long, and its hub is 40 feet above the ground. The tip of one blade starts level with the hub, on the right side, and the wheel turns counterclockwise. Find the height of the tip above the ground after the wheel turns (a) $180°$, (b) $270°$ and (c) $450°$.



14. **8 pts**

From a point on level ground, the angle of elevation to the top of a grain silo is $40°$. After walking 30 feet straight toward the silo, the angle of elevation is $55°$. How tall is the silo, to the nearest foot?

15. **6 pts**

(a) Explain why $\sin 50°$ has the same value in every right triangle that has a $50°$ angle. (b) Explain why $\sin\theta < 1$ for every acute angle θ in a right triangle.

ANSWER KEY

1. (a) $\dfrac{\sqrt{3}}{2}$ (b) $\sqrt{3}$
2. (a) $\dfrac{20}{29}$ (b) $\dfrac{21}{29}$ (c) $\dfrac{20}{21}$
3. $\tan 52^\circ = \dfrac{x}{15}$, so $x = 15\tan 52^\circ \approx 19.2$ m
4. $\sin\theta = \dfrac{9}{14}$, so $\theta = \sin^{-1}\left(\dfrac{9}{14}\right) \approx 40.0^\circ$
5. (a) $\sin^2\theta = 1 - \dfrac{49}{625} = \dfrac{576}{625}$, so $\sin\theta = \dfrac{24}{25}$ (b) $\tan\theta = \dfrac{24/25}{7/25} = \dfrac{24}{7}$
6. $h = 30\sin 28^\circ \approx 14.1$ ft
7. $\langle 5\cos 35^\circ, 5\sin 35^\circ\rangle \approx \langle 4.1, 2.9\rangle$
8. (a) $225 \cdot \dfrac{\pi}{180} = \dfrac{5\pi}{4}$ (b) $\dfrac{5\pi}{3} \cdot \dfrac{180^\circ}{\pi} = 300^\circ$
9. (a) $-500^\circ + 720^\circ = 220^\circ$ (b) $\dfrac{7\pi}{2} - 2\pi = \dfrac{3\pi}{2}$
10. (a) 270° lands at $(0, -1)$, so 0 (b) 180° lands at $(-1, 0)$, so 0 (c) -90° lands at $(0, -1)$, so -1
11. $\cos^2\theta = 1 - \dfrac{64}{289} = \dfrac{225}{289}$, and $x < 0$ in Quadrant III. (a) $\cos\theta = -\dfrac{15}{17}$ (b) $\tan\theta = \dfrac{-8/17}{-15/17} = \dfrac{8}{15}$
12. (a) $s = r\theta = 18 \cdot \dfrac{5\pi}{3} = 30\pi$ ft (b) about 94.2 ft
13. $h = 40 + 10\sin\theta$. (a) $40 + 10(0) = 40$ ft (b) $40 + 10(-1) = 30$ ft (c) 450° is coterminal with 90°, so $40 + 10(1) = 50$ ft
14. Let d be the second distance and h the height: $h = d\tan 55^\circ = (d + 30)\tan 40^\circ$. So $d(\tan 55^\circ - \tan 40^\circ) = 30\tan 40^\circ$, giving $d \approx 42.7$ ft and $h \approx 42.7\tan 55^\circ \approx 61$ ft. Check: $(42.7 + 30)\tan 40^\circ \approx 61$ ✓
15. (a) Any two right triangles with a 50° angle also share a right angle, so they are similar by AA. Corresponding sides of similar triangles are proportional: if one triangle is k times the other, the opposite side and the hypotenuse are both multiplied by k, and $\dfrac{k \cdot \text{opp}}{k \cdot \text{hyp}} = \dfrac{\text{opp}}{\text{hyp}}$. So the ratio is the same in every such triangle. (b) The hypotenuse is the longest side, since $\text{hyp}^2 = \text{opp}^2 + \text{adj}^2 > \text{opp}^2$. So $\dfrac{\text{opp}}{\text{hyp}} < 1$.

13.1 Right-Triangle Trigonometry: A Review

OBJECTIVES

- Explain, using similar triangles, why sine, cosine and tangent depend only on the angle and not on the size of the triangle.
- Find exact values of sine, cosine and tangent for 30°, 45° and 60° from the special right triangles.
- Find missing sides and angles of right triangles, using inverse functions on a calculator for angles.
- Derive $\sin^2\theta + \cos^2\theta = 1$ and use it, along with the components $\langle r\cos\theta, r\sin\theta\rangle$ of a vector, to solve problems.

WATCH

Trigonometry in Right Triangles (4 of 5: Exact values & reciprocals)

Eddie Woo · 13:44

https://www.youtube.com/watch?v=y6vJVM9euAo

> **WHEN TO WATCH**
> After Example 1: exact values from the special triangles.

Note: Neither video covers the similar-triangles argument or the vector components. Teach those from the Core ideas.

BACKUP VIDEO

Special Right Triangles 30-60-90 & 45-45-90 (3 Easy Methods Explained)

Mario's Math Tutoring · 9:41

https://www.youtube.com/watch?v=-l1xwn7zNZ4

Before the practice set: solving 30-60-90 and 45-45-90 triangles.

KEY TERMS

Trigonometric ratio

One of the ratios $\sin\theta$, $\cos\theta$ or $\tan\theta$ of two sides of a right triangle; its value depends only on the acute angle θ.

Inverse sine, cosine and tangent

$\sin^{-1}x$ is the acute angle whose sine is x (and likewise $\cos^{-1}x$ and $\tan^{-1}x$). They undo sine, cosine and tangent, the way any inverse function undoes its function.

Angle of elevation (depression)

The angle between a horizontal line and a line of sight to an object above (below) the horizontal.

Pitch

The steepness of a roof given as rise over run, such as 6 in 12 (6 inches up for each 12 inches across); it is the tangent of the roof's angle.

Pythagorean identity

The equation $\sin^2\theta + \cos^2\theta = 1$, true for every acute angle θ (Lesson 13.2 extends it to all angles); $\sin^2\theta$ means $(\sin\theta)^2$.

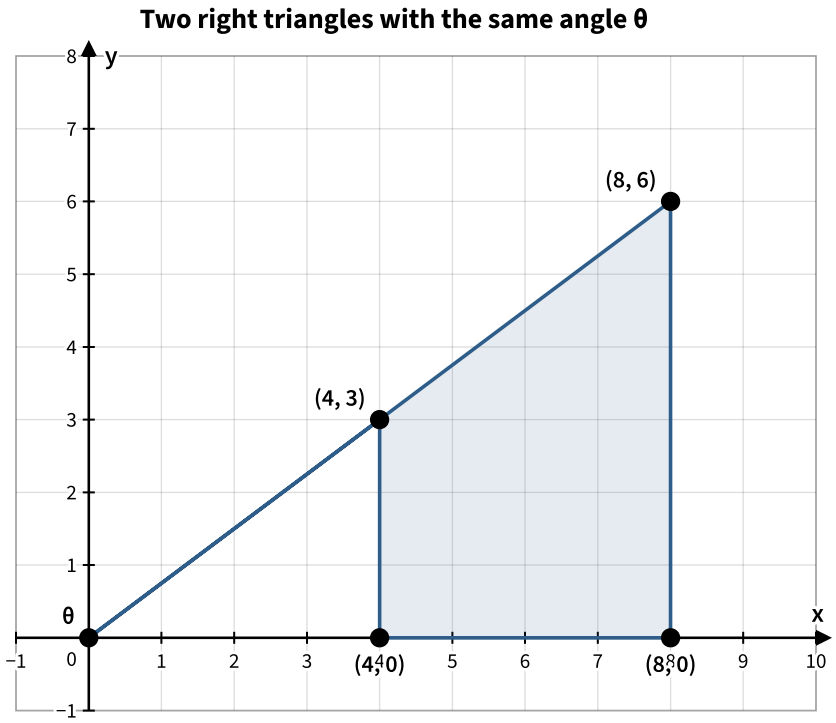
Components of a vector

The horizontal and vertical parts of a vector; a vector of length r at angle θ above the positive x-axis has components $r\cos\theta$ and $r\sin\theta$.

CORE IDEAS

In Geometry you met three ratios of sides in a right triangle. Pick one of the acute angles and call it θ (the Greek letter theta). The **hypotenuse** is the side across from the right angle, the **opposite** side is the leg across from θ, and the **adjacent** side is the leg that touches θ. Then $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$. The words opposite and adjacent always depend on which angle you are looking at: the leg opposite one acute angle is adjacent to the other.

Why can a calculator give one value for $\sin 37^\circ$ when there are right triangles of every size with a 37° angle? Because all of them are similar. Two right triangles that share an acute angle θ have two pairs of equal angles (the right angles and θ), so they are similar by AA, and corresponding sides of similar triangles are proportional. If the larger triangle is k times the smaller, every side is multiplied by k, and in each ratio the k cancels: $\dfrac{k\cdot\text{opp}}{k\cdot\text{hyp}} = \dfrac{\text{opp}}{\text{hyp}}$. In the picture, the small triangle has legs 4 and 3 and hypotenuse 5, the large one has legs 8 and 6 and hypotenuse 10, and $\sin\theta = \dfrac{3}{5} = \dfrac{6}{10}$. So the ratios depend only on the angle, which is what makes sine, cosine and tangent *functions* of the angle: one input θ, one output.



Two special triangles give exact values. Cut a square with side 1 along a diagonal: you get a $45°$-$45°$-$90°$ triangle with legs 1 and 1 and, by the Pythagorean theorem, hypotenuse $\sqrt{2}$. So $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ and $\tan 45° = 1$. Cut an equilateral triangle with side 2 in half by an altitude: you get a $30°$-$60°$-$90°$ triangle with hypotenuse 2, short leg 1 (half of 2) and long leg $\sqrt{2^2 - 1^2} = \sqrt{3}$. The short leg is opposite the $30°$ angle, so $\sin 30° = \dfrac{1}{2}$, $\cos 30° = \dfrac{\sqrt{3}}{2}$ and $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$; looking from the $60°$ angle, $\sin 60° = \dfrac{\sqrt{3}}{2}$, $\cos 60° = \dfrac{1}{2}$ and $\tan 60° = \sqrt{3}$. Notice that $\sin 30° = \cos 60°$. The two acute angles of a right triangle add to $90°$, and the side opposite one is adjacent to the other, so $\sin\theta = \cos(90° - \theta)$ always.

θ	sin θ	cos θ	tan θ
30°	1/2	√3/2	√3/3
45°	√2/2	√2/2	1
60°	√3/2	1/2	√3

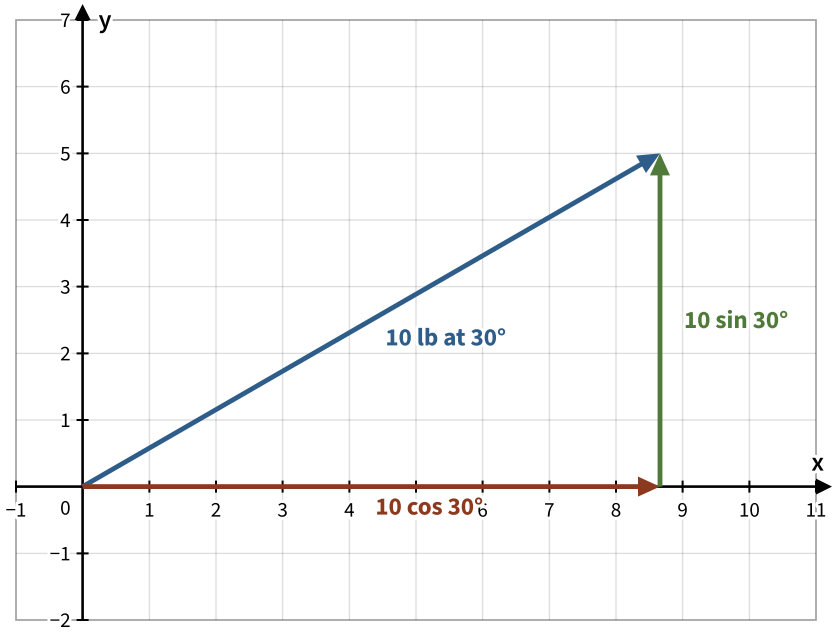
To find a missing side, choose the ratio that uses the side you know and the side you want, write an equation and solve. If the hypotenuse is 20 and you want the side x opposite a 40° angle, then $\sin 40^\circ = \dfrac{x}{20}$, so $x = 20 \sin 40^\circ \approx 12.9$. To find a missing angle, use an inverse function: if $\tan\theta = 0.5$, then $\theta = \tan^{-1}(0.5) \approx 26.6^\circ$. The inverse answers the question "which acute angle has this ratio?" It undoes tangent the way f^{-1} undoes f in Lesson 1.5, and just as $f^{-1}(x)$ is not $\dfrac{1}{f(x)}$, $\tan^{-1} x$ is not $\dfrac{1}{\tan x}$. Set your calculator to degree mode first (check: $\sin 30^\circ$ should give 0.5).

The Pythagorean theorem turns into an identity. In a right triangle, $\text{opp}^2 + \text{adj}^2 = \text{hyp}^2$. Divide every term by hyp^2: $\left(\dfrac{\text{opp}}{\text{hyp}}\right)^2 + \left(\dfrac{\text{adj}}{\text{hyp}}\right)^2 = 1$, which says $\sin^2\theta + \cos^2\theta = 1$. Dividing the two ratios gives $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opp}/\text{hyp}}{\text{adj}/\text{hyp}} = \dfrac{\text{opp}}{\text{adj}} = \tan\theta$. So one ratio of an acute angle gives you all of them: if $\sin\theta = \dfrac{5}{13}$, then $\cos^2\theta = 1 - \dfrac{25}{169} = \dfrac{144}{169}$, so $\cos\theta = \dfrac{12}{13}$ (positive, since it is a ratio of lengths), and $\tan\theta = \dfrac{5/13}{12/13} = \dfrac{5}{12}$.

Right triangles are everywhere in building and travel. A roof's **pitch** gives rise over run: a 6-in-12 roof rises 6 inches for every 12 inches across, so its angle is $\tan^{-1}\left(\dfrac{6}{12}\right) \approx 26.6^\circ$. A ramp, a ladder against a wall, and a line of sight up to a steeple (an angle of elevation) or down from a tower (an angle of depression) are solved the same way: sketch the triangle, label the sides from the angle, and pick the ratio.

Trigonometry also takes a vector apart. A vector of length r pointing at an angle θ above the positive x-axis is the hypotenuse of a right triangle whose horizontal leg is $r\cos\theta$ and whose vertical leg is $r\sin\theta$ (from $\cos\theta = \dfrac{\text{adj}}{r}$ and $\sin\theta = \dfrac{\text{opp}}{r}$). So the vector is $\langle r\cos\theta, r\sin\theta\rangle$. A rope pulling with 10 pounds of force at 30° above level is $\langle 10\cos 30^\circ, 10\sin 30^\circ\rangle = \langle 5\sqrt{3}, 5\rangle \approx \langle 8.66, 5\rangle$: about 8.66 pounds of forward pull and 5 pounds of lift. Check with Lesson 12.5: its magnitude is $\sqrt{(5\sqrt{3})^2 + 5^2} = \sqrt{75 + 25} = 10$ ✓

A 10-pound pull at 30° and its components



KEY FORMULAS

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta = \cos(90^\circ - \theta)$$

$$\text{a vector of length } r \text{ at angle } \theta: \quad \langle r\cos\theta,\ r\sin\theta\rangle$$

Practice habit. Know the two special triangles cold: legs 1 and 1 with hypotenuse $\sqrt{2}$, and sides 1, $\sqrt{3}$, 2. Sketch them in the margin and read the exact values for 30°, 45° and 60° from the sketch instead of memorizing a table.

Example 1. A 20-foot ladder leans against the wall of a barn, making a 70° angle with the level ground. (a) How high up the wall does the top of the ladder reach? (b) How far is the foot of the ladder from the wall? Round each answer to the nearest tenth of a foot.

Show solution

1. Sketch: the ladder is the hypotenuse, 20 ft. The height h on the wall is opposite the 70° angle; the distance d along the ground is adjacent to it.
2. (a) Opposite and hypotenuse go with sine: $\sin 70^\circ = \dfrac{h}{20}$.
3. Multiply by 20: $h = 20 \sin 70^\circ \approx 20(0.93969) \approx 18.79$, so about 18.8 ft.
4. (b) Adjacent and hypotenuse go with cosine: $\cos 70^\circ = \dfrac{d}{20}$, so $d = 20 \cos 70^\circ \approx 20(0.34202) \approx 6.84$, about 6.8 ft.
5. Check with the Pythagorean theorem: $18.794^2 + 6.840^2 \approx 353.2 + 46.8 = 400 = 20^2$ ✓

Answer: (a) about 18.8 ft (b) about 6.8 ft

Example 2. A barn roof has a pitch of 5 in 12: it rises 5 inches for every 12 inches of horizontal run. (a) Find the angle the roof makes with the horizontal, to the nearest tenth of a degree. (b) The horizontal run from the outside wall to the ridge is 15 feet. Find the exact length of a rafter from the wall to the ridge (ignore any overhang).

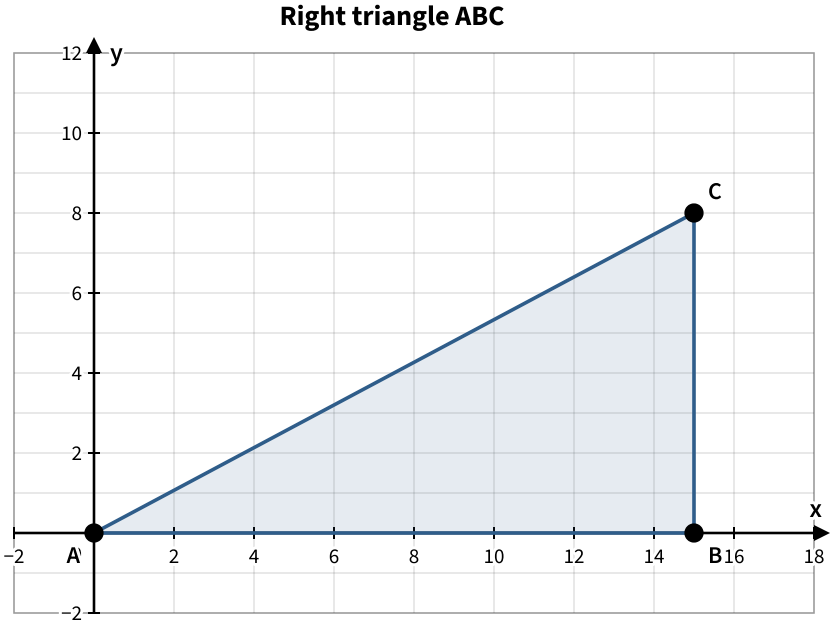
Show solution

1. A right triangle along the roof with legs 5 (rise) and 12 (run) has the roof's angle θ at its lower corner, so $\tan\theta = \dfrac{5}{12}$.
2. (a) $\theta = \tan^{-1}\left(\dfrac{5}{12}\right) \approx 22.62^\circ$, so about 22.6°.
3. (b) The hypotenuse of the 5-by-12 triangle is $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$, so $\sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{12}{13}$.
4. Check the identity: $\left(\dfrac{5}{13}\right)^2 + \left(\dfrac{12}{13}\right)^2 = \dfrac{25 + 144}{169} = 1$ ✓
5. The rafter R is the hypotenuse of a similar triangle with run 15 ft: $\cos\theta = \dfrac{15}{R}$, so $\dfrac{12}{13} = \dfrac{15}{R}$.
6. Cross-multiply: $12R = 195$, so $R = 16.25$ ft.
7. Check: the rise is $15 \times \dfrac{5}{12} = 6.25$ ft, and $6.25^2 + 15^2 = 39.0625 + 225 = 264.0625 = 16.25^2$ ✓

Answer: (a) about 22.6° (b) 16.25 ft (16 ft 3 in)

PRACTICE

1. Easy In right triangle ABC shown, the right angle is at B, $AB = 15$, $BC = 8$ and $AC = 17$. Let θ be the angle at A. Find (a) $\sin\theta$, (b) $\cos\theta$ and (c) $\tan\theta$ as fractions.



Show answer

(a) $\frac{8}{17}$ (b) $\frac{15}{17}$ (c) $\frac{8}{15}$

2. Easy By hand, give the exact value of (a) $\sin 60°$, (b) $\tan 45°$ and (c) $\cos 45°$.

Show answer

(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\frac{\sqrt{2}}{2}$

3. Easy A right triangle has a hypotenuse of 12 cm and an acute angle of $35°$. Find the length of the leg opposite the $35°$ angle, to the nearest tenth of a centimeter.

Show answer

$x = 12\sin 35° \approx 6.9$ cm

4. Easy The legs of a right triangle are 7 and 10. Find the angle opposite the leg of length 7, to the nearest tenth of a degree.

Show answer

$\tan\theta = \frac{7}{10}$, so $\theta = \tan^{-1}(0.7) \approx 35.0°$

5. **Medium** A 30°-60°-90° triangle has a hypotenuse of 10. By hand, find the exact length of (a) the leg opposite the 30° angle and (b) the leg opposite the 60° angle.

Show answer

(a) $10 \sin 30^\circ = 10 \cdot \dfrac{1}{2} = 5$ (b) $10 \sin 60^\circ = 10 \cdot \dfrac{\sqrt{3}}{2} = 5\sqrt{3}$; check: $25 + 75 = 100$ ✓

6. **Medium** θ is an acute angle with $\sin\theta = 0.6$. Without finding θ, use $\sin^2\theta + \cos^2\theta = 1$ to find (a) $\cos\theta$ and (b) $\tan\theta$.

Show answer

(a) $\cos^2\theta = 1 - 0.36 = 0.64$, so $\cos\theta = 0.8$ (b) $\tan\theta = \dfrac{0.6}{0.8} = 0.75$

7. **Medium** A building guideline for wheelchair ramps allows a rise of at most 1 inch for every 12 inches of horizontal run. (a) What is the steepest allowed angle with the ground, to the nearest tenth of a degree? (b) The entrance of a church fellowship hall is 30 inches above the sidewalk. How much horizontal run, in feet, does a ramp at this steepest slope need?

Show answer

(a) $\tan^{-1}\left(\dfrac{1}{12}\right) \approx 4.8^\circ$ (b) $30 \times 12 = 360$ inches, which is 30 ft

8. **Medium** Standing on level ground 150 feet from the base of a church steeple, Grace measures the angle of elevation to the top of the steeple as 32°. Ignoring her eye height, how tall is the steeple, to the nearest foot?

Show answer

$\tan 32^\circ = \dfrac{h}{150}$, so $h = 150 \tan 32^\circ \approx 93.7$; about 94 ft

9. **Medium** Henry pulls a sled of firewood with a rope, using a force of 80 pounds at 25° above the horizontal. Write the force as a vector $\langle r\cos\theta, r\sin\theta\rangle$, with each component rounded to the nearest tenth of a pound.

Show answer

$\langle 80\cos 25^\circ, 80\sin 25^\circ\rangle \approx \langle 72.5, 33.8\rangle$

10. Hard Ruth's kite string is 120 feet long, pulled straight, and makes a 60° angle with the horizontal at her hand. How high is the kite above her hand? Give (a) the exact height in simplest radical form, by hand, and (b) the height to the nearest tenth of a foot.

Show answer

(a) $120 \sin 60^\circ = 120 \cdot \dfrac{\sqrt{3}}{2} = 60\sqrt{3}$ ft (b) about 103.9 ft

11. Hard θ is an acute angle with $\tan\theta = 2$. Find the exact values of (a) $\sin\theta$ and (b) $\cos\theta$, with rationalized denominators. (Hint: draw a right triangle with legs 2 and 1.)

Show answer

The hypotenuse is $\sqrt{2^2 + 1^2} = \sqrt{5}$. (a) $\dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$ (b) $\dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$; check: $\dfrac{4}{5} + \dfrac{1}{5} = 1$ ✓

12. Hard A ranger in a fire lookout tower 80 feet above level ground sees two campfires straight ahead, one behind the other. The angle of depression is 20° to the nearer fire and 12° to the farther fire. How far apart are the two fires, to the nearest foot?

Show answer

The angle of depression equals the angle of elevation from each fire, so the ground distances are $\dfrac{80}{\tan 20^\circ} \approx 219.8$ ft and $\dfrac{80}{\tan 12^\circ} \approx 376.4$ ft. The fires are about $376.4 - 219.8 \approx 157$ ft apart.

13. Challenge Prove that for every acute angle θ, $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$.

Show answer

Start from $\sin^2\theta + \cos^2\theta = 1$. Since $\cos\theta$ is a ratio of positive lengths, $\cos\theta \neq 0$, so divide every term by $\cos^2\theta$: $\dfrac{\sin^2\theta}{\cos^2\theta} + 1 = \dfrac{1}{\cos^2\theta}$. Because $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$, the first term is $\tan^2\theta$, so $\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$. ∎ (Another way: divide $\text{opp}^2 + \text{adj}^2 = \text{hyp}^2$ by adj^2.)

COMMON MISTAKES

- Using a calculator in radian mode. Before you start, check that $\sin 30^\circ$ gives 0.5; if it gives about -0.988, switch to degree mode.

- Mixing up opposite and adjacent. Label the sides from the angle you are using, every time: the adjacent leg touches the angle, the opposite leg does not, and the hypotenuse is never called adjacent.
- Reading $\sin^{-1} x$ as $\dfrac{1}{\sin x}$. The -1 means inverse function: $\sin^{-1}(0.5) = 30^\circ$ is an angle, while $\dfrac{1}{\sin 30^\circ} = 2$.
- Rounding too early. Keep the full calculator value of $\sin 70^\circ$ or $\tan 20^\circ$ through the computation and round only the final answer.

> **Make it a game · Big Triangle, Small Triangle.** Draw two right triangles with the same acute angle, one small and one large, on paper. Measure the sides with a ruler and compute opposite over hypotenuse for each. The ratios match. Say why using similar triangles. Then recite from memory the sine, cosine and tangent of 30, 45 and 60 degrees, and score yourself.

13.2 Angles, Radians and the Unit Circle: A First Look

OBJECTIVES

- Draw positive, negative and larger-than-360° angles in standard position and find coterminal angles.
- Explain what a radian is and why π radians $= 180°$, convert between degrees and radians, and use the arc length formula $s = r\theta$.
- Use the unit circle to define $\sin\theta$ and $\cos\theta$ for any angle, give their values at multiples of 90°, and state their signs in each quadrant.
- Model circular motion, such as a Ferris wheel or water wheel, with the point $(r\cos\theta, r\sin\theta)$.

WATCH

Introduction to Radians (2 of 3: Developing an initial sense)

Eddie Woo · 11:13

https://www.youtube.com/watch?v=sdeJAtfDHPE

> **WHEN TO WATCH**
> Before Example 1: what a radian is.

Note: Neither video defines sine and cosine on the unit circle or models a Ferris wheel. Teach those from the Core ideas.

BACKUP VIDEO

Radians Intro (What are they? Converting Units, Draw Standard Position)

Mario's Math Tutoring · 7:38

https://www.youtube.com/watch?v=zs_FyLTpW9w

Before Example 1: converting units and drawing angles in standard position.

KEY TERMS

Standard position

An angle drawn with its vertex at the origin and its initial side along the positive x-axis.

Initial side and terminal side

The ray where the turning starts (the positive x-axis) and the ray where it stops. Counterclockwise turning gives a positive angle; clockwise gives a negative angle.

Coterminal angles

Angles in standard position with the same terminal side; they differ by a whole number of full turns (360° or 2π radians).

Radian

The measure of a central angle whose arc is exactly as long as the circle's radius; π radians $= 180^\circ$, so 1 radian is about 57.3°.

Unit circle

The circle of radius 1 centered at the origin, $x^2 + y^2 = 1$. The terminal side of an angle θ meets it at the point $(\cos\theta, \sin\theta)$.

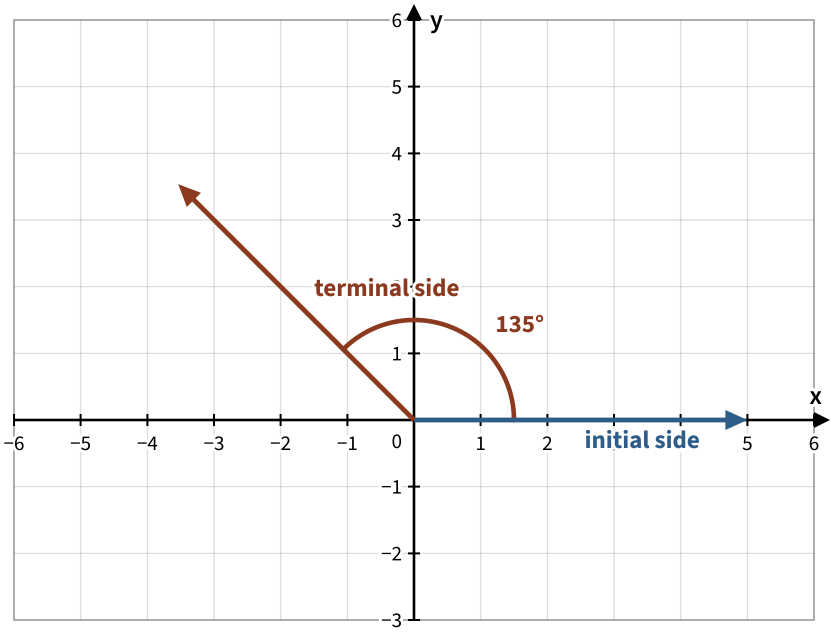
Periodic

Repeating its values at regular intervals; sine and cosine repeat every 360° (every 2π radians).

CORE IDEAS

In a right triangle an angle is always between 0° and 90°. Wheels, gears and planets turn much farther than that, so now think of an angle as an amount of turning. An angle is in **standard position** when its vertex is at the origin and its **initial side** lies along the positive x-axis; the **terminal side** is where the ray stops after turning. Turning counterclockwise gives a positive angle and clockwise a negative one. One full turn is 360°, so 450° is a full turn plus 90° more, and -90° is a quarter turn clockwise. The axes divide the plane into Quadrants I, II, III and IV, numbered counterclockwise starting from the upper right.

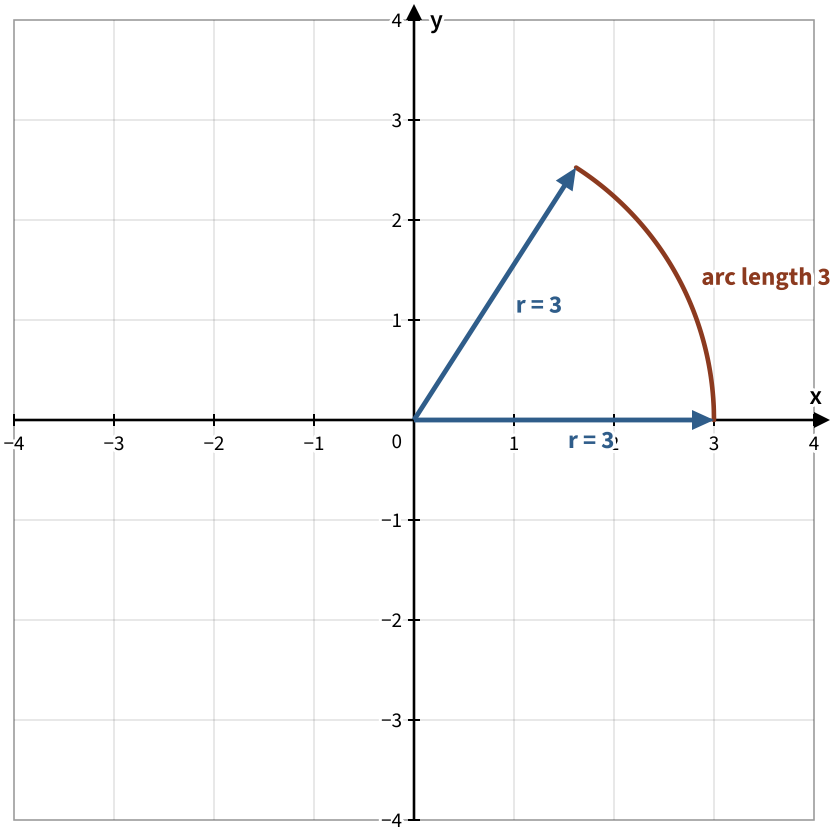
An angle of 135° in standard position



Angles that end on the same terminal side are **coterminal**. Adding or subtracting a full turn brings the ray back to the same place, so the angles coterminal with θ are $\theta + 360^\circ k$ for every integer k. For example, 60°, 420° and -300° are coterminal. To find the angle between 0° and 360° that is coterminal with 800°, subtract full turns: $800^\circ - 2(360^\circ) = 80^\circ$.

Degrees are an old and convenient choice (360 divides evenly in many ways), but a more natural way to measure an angle is by arc length. Draw a circle of radius r centered at the vertex. The angle measures **1 radian** when the arc it cuts off is exactly as long as the radius. Arc length grows in proportion to the angle, so an angle cutting off an arc of length s measures $\theta = \dfrac{s}{r}$ radians, which gives the arc length formula $s = r\theta$. A full turn cuts off the whole circumference, $2\pi r$, so a full turn is $\dfrac{2\pi r}{r} = 2\pi$ radians. A full turn is also 360°, so 2π radians $= 360^\circ$, that is, π radians $= 180^\circ$. One radian is $\dfrac{180^\circ}{\pi} \approx 57.3^\circ$, a little less than 60°: a 60° central angle cuts off a chord of length r (the triangle it makes with the two radii is equilateral), and its arc is longer than that chord, so the 60° arc is longer than r. An arc of exactly r therefore needs a slightly smaller angle.

An angle of 1 radian: the arc equals the radius



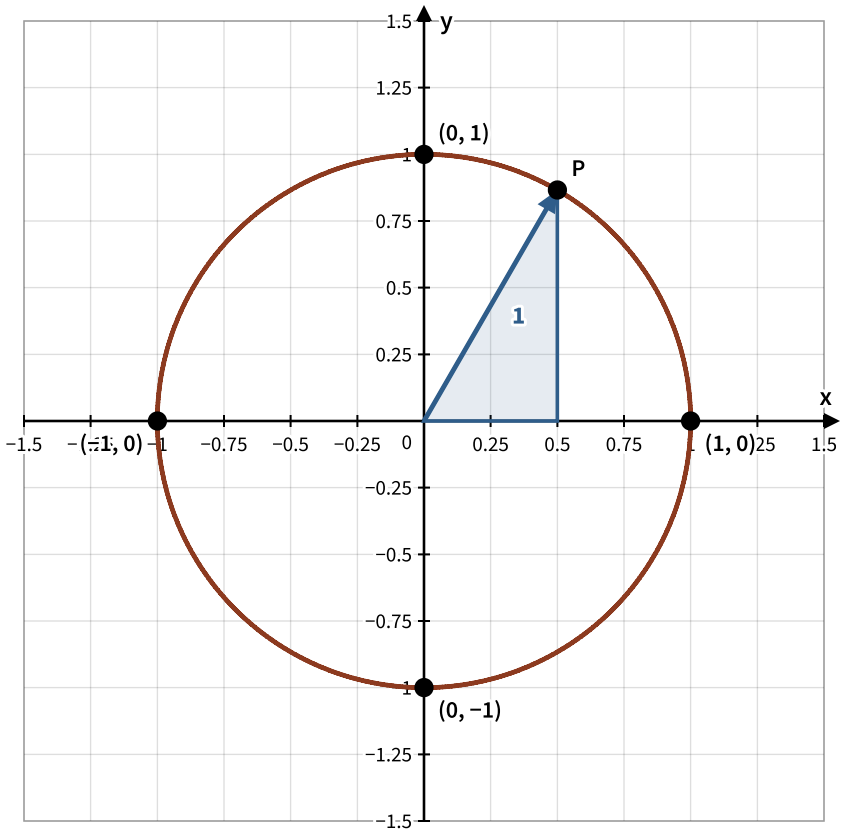
To convert, multiply by a fraction equal to 1. Since $180^\circ = \pi$ radians, multiplying degrees by $\frac{\pi}{180^\circ}$ gives radians, and multiplying radians by $\frac{180^\circ}{\pi}$ gives degrees. So $90^\circ = 90 \cdot \frac{\pi}{180} = \frac{\pi}{2}$ and $\frac{3\pi}{4} = \frac{3\pi}{4} \cdot \frac{180^\circ}{\pi} = 135^\circ$. Radian measures are usually left as exact multiples of π, and an angle written with no degree sign is understood to be in radians. In radians, coterminal angles differ by multiples of 2π. The table lists the angles you will meet most often.

Degrees	Radians
0°	0
30°	π/6
45°	π/4
60°	π/3

Degrees	Radians
90°	π/2
180°	π
270°	3π/2
360°	2π

The **unit circle** is the circle of radius 1 centered at the origin, $x^2 + y^2 = 1$. Draw an acute angle θ in standard position and let $P = (x, y)$ be the point where its terminal side meets the unit circle. Dropping a perpendicular from P to the x-axis makes a right triangle with hypotenuse 1, so $\cos\theta = \dfrac{x}{1} = x$ and $\sin\theta = \dfrac{y}{1} = y$. In other words, $P = (\cos\theta, \sin\theta)$. For $\theta = 60^\circ$, the point P in the picture is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

The unit circle and the point at 60°



This gives a definition that works for every angle, not only acute ones: **for any angle θ, $\cos\theta$ is the x-coordinate and $\sin\theta$ is the y-coordinate of the point where the terminal side meets the unit circle**, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ whenever $\cos\theta \neq 0$. For acute angles it agrees with the triangle definition, so nothing from Lesson 13.1 is lost. The angles on the axes are now easy: 0° lands at $(1, 0)$, 90° at $(0, 1)$, 180° at $(-1, 0)$ and 270° at $(0, -1)$. So $\cos 180^\circ = -1$, $\sin 270^\circ = -1$, $\sin 0^\circ = 0$, and $\tan 90^\circ$ is undefined because $\cos 90^\circ = 0$. The signs come from the quadrant: in Quadrant II, $x < 0$ and $y > 0$, so cosine is negative and sine is positive. And since every point on the circle satisfies $x^2 + y^2 = 1$, the identity $\sin^2\theta + \cos^2\theta = 1$ now holds for every angle.

Quadrant	θ between	sin θ (y)	cos θ (x)	tan θ (y/x)
I	0° and 90°	+	+	+
II	90° and 180°	+	−	−
III	180° and 270°	−	−	+
IV	270° and 360°	−	+	−

Coterminal angles land on the same point, so $\sin(\theta + 360^\circ) = \sin\theta$ and $\cos(\theta + 360^\circ) = \cos\theta$: the values repeat every full turn. Quantities that repeat like this are **periodic**, and turning wheels are the classic example. For a circle of radius r centered at the origin, stretch the unit circle by the factor r (a dilation, as in Lesson 12.6): the point at angle θ is $(r\cos\theta, r\sin\theta)$, the same components you found for a vector in Lesson 13.1. So a seat on a Ferris wheel of radius r whose center is c feet above the ground, starting level with the center on the right and turning counterclockwise, is at height $c + r\sin\theta$ after turning through θ. As the wheel keeps turning, the height rises and falls in a repeating wave; the Trigonometry course graphs these waves and continues from here.

KEY FORMULAS

$$\pi \text{ radians} = 180^\circ$$

$$\text{radians} = \text{degrees} \times \frac{\pi}{180^\circ}, \qquad \text{degrees} = \text{radians} \times \frac{180^\circ}{\pi}$$

$$s = r\theta \quad (\theta \text{ in radians})$$

$$\text{on the unit circle, the point at angle } \theta \text{ is } (\cos\theta, \sin\theta)$$

$$\text{coterminal angles: } \theta + 360^\circ k \text{ (degrees) or } \theta + 2\pi k \text{ (radians)}, \ k \text{ an integer}$$

Practice habit. Know cold that π radians $= 180^\circ$ and that 0°, 90°, 180° and 270° land at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Sketch every angle in standard position before you compute, so you can see its quadrant and catch sign errors.

WORKED EXAMPLES

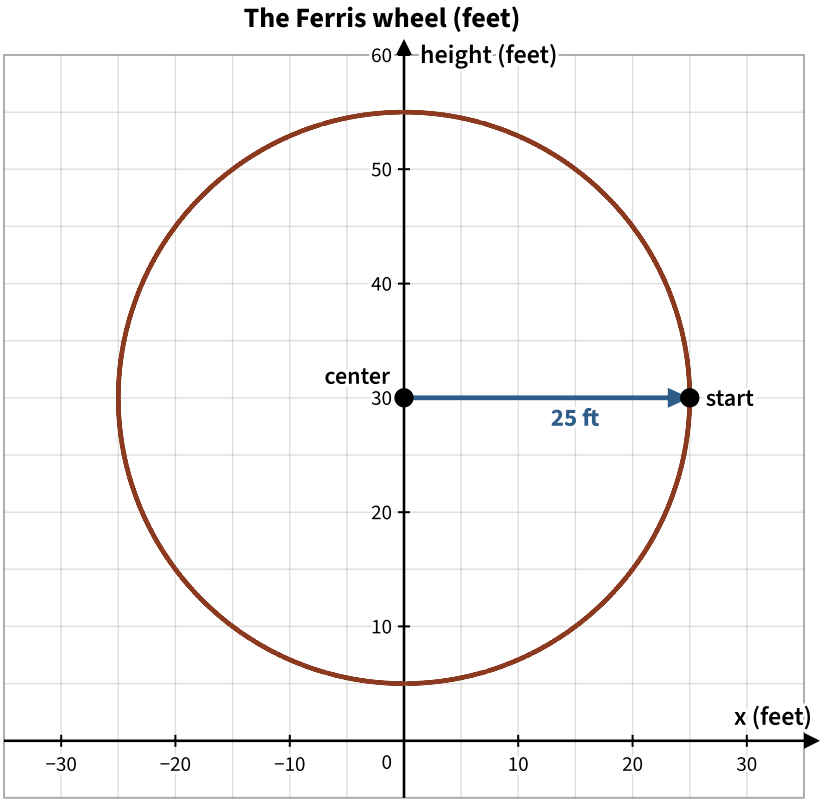
Example 1. (a) Convert 150° to radians, in exact form. (b) Convert $\dfrac{7\pi}{4}$ radians to degrees. (c) Find the angle between 0° and 360° that is coterminal with -405°, and name the quadrant of its terminal side.

Show solution

1. (a) Multiply by $\dfrac{\pi}{180^\circ}$: $150 \cdot \dfrac{\pi}{180} = \dfrac{150\pi}{180} = \dfrac{5\pi}{6}$.
2. Size check: 150° is a little less than $180^\circ = \pi$, and $\dfrac{5\pi}{6}$ is a little less than π ✓
3. (b) Multiply by $\dfrac{180^\circ}{\pi}$: $\dfrac{7\pi}{4} \cdot \dfrac{180^\circ}{\pi} = \dfrac{7 \times 180^\circ}{4} = 7 \times 45^\circ = 315^\circ$.
4. (c) Add full turns until the angle is between 0° and 360°: $-405^\circ + 360^\circ = -45^\circ$, still negative.
5. $-45^\circ + 360^\circ = 315^\circ$.
6. 315° is between 270° and 360°, so the terminal side is in Quadrant IV. Check: turning 405° clockwise is one full turn plus 45° more, which ends 45° below the positive x-axis, the same ray as 315° ✓

Answer: (a) $\dfrac{5\pi}{6}$ (b) 315° (c) 315°, in Quadrant IV

Example 2. A Ferris wheel at a county fair has a radius of 25 feet, and its center is 30 feet above the ground. Samuel's seat starts level with the center, on the right side, and the wheel turns counterclockwise. (a) How high is Samuel after the wheel turns 270°? (b) How high is he after it turns 750°? (c) How far has his seat traveled along the circle after 750°, to the nearest tenth of a foot?



Show solution

1. Put the center at $(0, 30)$. Relative to the center, the seat at angle θ is at $(25\cos\theta, 25\sin\theta)$, so its height is $h = 30 + 25\sin\theta$.
2. (a) $270°$ lands at $(0, -1)$ on the unit circle, so $\sin 270° = -1$ and $h = 30 + 25(-1) = 5$ ft.
3. Check: the bottom of the wheel is $30 - 25 = 5$ ft above the ground ✓
4. (b) $750° - 2(360°) = 30°$, so $750°$ is coterminal with $30°$ and $\sin 750° = \sin 30° = \dfrac{1}{2}$.
5. $h = 30 + 25 \cdot \dfrac{1}{2} = 42.5$ ft.
6. (c) The arc length formula needs radians: $750 \cdot \dfrac{\pi}{180} = \dfrac{25\pi}{6}$ radians.

7. $s = r\theta = 25 \cdot \dfrac{25\pi}{6} = \dfrac{625\pi}{6} \approx 327.2$ ft.

8. Check another way: two full turns are $2(2\pi \cdot 25) = 100\pi$ ft, and the extra 30° is $\dfrac{1}{12}$ of the circumference 50π; $100\pi + \dfrac{50\pi}{12} = \dfrac{600\pi + 25\pi}{6} = \dfrac{625\pi}{6}$ ✓

Answer: (a) 5 ft (b) 42.5 ft (c) $\dfrac{625\pi}{6} \approx 327.2$ ft

PRACTICE

1. Easy Convert to radians, in exact form: (a) 120° (b) 300° (c) -45°.

   Show answer

   (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{5\pi}{3}$ (c) $-\dfrac{\pi}{4}$

2. Easy Convert to degrees: (a) $\dfrac{\pi}{9}$ (b) $\dfrac{5\pi}{6}$ (c) $\dfrac{5\pi}{4}$.

   Show answer

   (a) 20° (b) 150° (c) 225°

3. Easy For each angle, find the coterminal angle that is at least 0° and less than 360°: (a) 400° (b) -120° (c) 1000°.

   Show answer

   (a) $400^\circ - 360^\circ = 40^\circ$ (b) $-120^\circ + 360^\circ = 240^\circ$ (c) $1000^\circ - 720^\circ = 280^\circ$

4. Easy Use the unit circle to find, by hand: (a) $\cos 180^\circ$ (b) $\sin 270^\circ$ (c) $\sin 450^\circ$.

   Show answer

   (a) 180° lands at $(-1, 0)$, so -1 (b) 270° lands at $(0, -1)$, so -1 (c) 450° is coterminal with 90°, which lands at $(0, 1)$, so 1

5. Medium Give the exact coordinates of the point where the terminal side of the angle meets the unit circle: (a) -90° (b) 540°.

   Show answer

   (a) a quarter turn clockwise ends at $(0, -1)$ (b) $540^\circ - 360^\circ = 180^\circ$, so $(-1, 0)$

6. **Medium** An angle θ has $\sin\theta > 0$ and $\tan\theta < 0$. In which quadrant does its terminal side lie? Give the quadrant number and explain using the coordinates (x, y) of the point on the unit circle.

Show answer

Quadrant II: $\sin\theta = y > 0$, and $\tan\theta = \dfrac{y}{x} < 0$ with $y > 0$ forces $x < 0$. The point has $x < 0$ and $y > 0$.

7. **Medium** The blades of a farm windmill are 12 feet long, measured from the hub to the tip. How far does a blade tip travel when the wheel turns through $\dfrac{2\pi}{3}$ radians? Give (a) the exact distance and (b) the distance to the nearest tenth of a foot.

Show answer

(a) $s = r\theta = 12 \cdot \dfrac{2\pi}{3} = 8\pi$ ft (b) about 25.1 ft

8. **Medium** A central angle in a circle of radius 12 inches cuts off an arc 30 inches long. Find the angle (a) in radians and (b) in degrees, to the nearest tenth of a degree.

Show answer

(a) $\theta = \dfrac{s}{r} = \dfrac{30}{12} = 2.5$ radians (b) $2.5 \cdot \dfrac{180^\circ}{\pi} \approx 143.2^\circ$

9. **Medium** The terminal side of θ meets the unit circle at a point in Quadrant II whose x-coordinate is $-\dfrac{3}{5}$. Find the exact values of (a) $\sin\theta$, (b) $\cos\theta$ and (c) $\tan\theta$.

Show answer

$y^2 = 1 - \dfrac{9}{25} = \dfrac{16}{25}$, and $y > 0$ in Quadrant II, so $y = \dfrac{4}{5}$. (a) $\dfrac{4}{5}$ (b) $-\dfrac{3}{5}$ (c) $\dfrac{4/5}{-3/5} = -\dfrac{4}{3}$

10. Hard A water wheel at a gristmill has a radius of 8 feet, and its center is 6 feet above the surface of the millstream. A bucket starts level with the center, on the right side, and the wheel turns counterclockwise. Find the bucket's height above the water after the wheel turns (a) 90°, (b) 180° and (c) 270°. What does the answer to (c) mean?

Show answer

$h = 6 + 8\sin\theta$. (a) $6 + 8(1) = 14$ ft (b) $6 + 8(0) = 6$ ft (c) $6 + 8(-1) = -2$ ft: the bucket is 2 feet below the surface, under the water.

11. Hard Earth turns once on its axis (360°) in about 24 hours. (a) Through how many radians does Earth turn in 1 hour? Give an exact answer. (b) A town on the equator is about 3960 miles from Earth's center. About how far does the town travel in 1 hour as Earth turns, to the nearest mile?

Show answer

(a) $\dfrac{2\pi}{24} = \dfrac{\pi}{12}$ radians (b) $s = 3960 \cdot \dfrac{\pi}{12} = 330\pi \approx 1037$ miles

12. Hard Grace's bicycle wheels have a radius of 13 inches. She rides 100 feet in a straight line, and the wheels roll without slipping, so the distance along the ground equals the arc length rolled. (a) Through how many radians does each wheel turn, to the nearest tenth? (b) How many revolutions is that, to the nearest tenth?

Show answer

100 ft $= 1200$ in. (a) $\theta = \dfrac{s}{r} = \dfrac{1200}{13} \approx 92.3$ radians (b) $\dfrac{1200/13}{2\pi} \approx 14.7$ revolutions

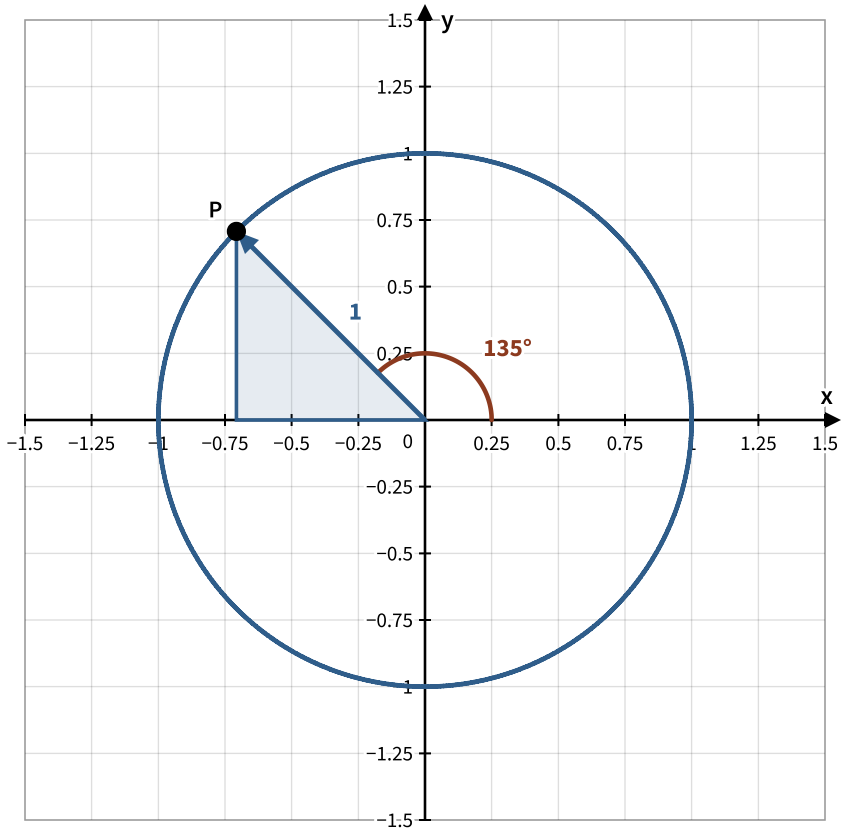
13. Challenge Explain why $\sin^2\theta + \cos^2\theta = 1$ is true for every angle θ, not only acute angles, and why $\sin(\theta + 360^\circ) = \sin\theta$.

Show answer

By definition, $(\cos\theta, \sin\theta)$ is the point where the terminal side of θ meets the unit circle. Every point (x, y) on that circle satisfies $x^2 + y^2 = 1$, so $\cos^2\theta + \sin^2\theta = 1$ for every angle. The angle $\theta + 360^\circ$ is θ followed by one more full turn, so it has the same terminal side and meets the circle at the same point; its y-coordinate is the same, so $\sin(\theta + 360^\circ) = \sin\theta$.

14. **Challenge** The figure shows the angle 135° in standard position; its terminal side meets the unit circle at P. The right triangle drawn from P to the x-axis has an angle of $180^\circ - 135^\circ = 45^\circ$ at the origin. Use it to find the exact values of (a) $\cos 135^\circ$ and (b) $\sin 135^\circ$.

The angle 135° on the unit circle



Show answer

The triangle is a 45°-45°-90° triangle with hypotenuse 1, so each leg is $\frac{\sqrt{2}}{2}$. P is in Quadrant II, so $x < 0$ and $y > 0$: $P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. (a) $\cos 135^\circ = -\frac{\sqrt{2}}{2}$ (b) $\sin 135^\circ = \frac{\sqrt{2}}{2}$; check: $\frac{1}{2} + \frac{1}{2} = 1$ ✓

COMMON MISTAKES

- Multiplying by the wrong conversion fraction. Check the size: $\frac{\pi}{180}$ makes the number much smaller (degrees to radians), and $\frac{180}{\pi}$ makes it much larger (radians to degrees). Since $\pi \approx 3.14$ is 180°, an answer like $\frac{\pi}{2} = 270^\circ$ cannot be right.

- Using $s = r\theta$ with θ in degrees. The formula comes from the definition of the radian, so convert to radians first: a 90° turn on a 10-foot circle is $10 \cdot \frac{\pi}{2} \approx 15.7$ feet, not 900 feet.
- Swapping sine and cosine on the unit circle. The point is $(\cos\theta, \sin\theta)$: cosine comes first just as x does, so cosine is the x-coordinate and sine is the y-coordinate.
- Turning the wrong way for negative angles. Positive angles turn counterclockwise from the positive x-axis and negative angles clockwise; sketch the turn before you name the quadrant.

> **Make it a game · Wrap the Radius.** Take a round lid and a piece of string as long as its radius. Wrap the string around the rim and count how many radius lengths fit: a little more than six. That is 2π radians, or 360 degrees. Then the parent calls angles in degrees, and you answer in radians, using s = rθ to find an arc length.

UNIT 13

A Bridge to Trigonometry

This short closing unit gathers what you know about right triangles from Geometry and builds the bridge to the Trigonometry course. In Lesson 13.1 you review sine, cosine and tangent, see why they depend only on the angle (similar triangles), find exact values from the special right triangles, solve right triangles with inverse functions, and derive the identity $\sin^2\theta + \cos^2\theta = 1$ from the Pythagorean theorem. You also use these ratios to split a vector into its components, tying back to the vectors of Unit 12.

In Lesson 13.2 an angle becomes an amount of turning: positive or negative, and as large as you like. You meet the radian, the natural way to measure angles by arc length, and the unit circle, which defines sine and cosine for every angle, not just acute ones. Turning wheels, from a Ferris wheel to a water wheel to Earth itself, show why these functions repeat. The Trigonometry course picks up exactly here, with the full unit circle, the graphs of sine and cosine, and identities.

PREREQUISITES

- Sine, cosine and tangent of acute angles and angles of elevation and depression (Geometry)
- Similar triangles, the AA similarity criterion and the special right triangles (Geometry)
- The Pythagorean theorem, simplifying square roots and rationalizing denominators
- The circumference of a circle, arc length and the equation $x^2 + y^2 = r^2$ (Geometry)
- Inverse functions (Lesson 1.5)
- Vectors, components and magnitude (Lesson 12.5) and dilations (Lesson 12.6)

MASTERY CHECKLIST

- ☐ I can explain, using similar triangles, why sine, cosine and tangent depend only on the angle.
- ☐ I can give the exact sine, cosine and tangent of 30°, 45° and 60° by sketching the special right triangles.
- ☐ I can find missing sides of a right triangle and use $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to find missi

ng an gle s.

- ☐ I can derive $\sin^2\theta + \cos^2\theta = 1$ and use it to find one ratio from another.
- ☐ I can solve problems about roof pitch, ramps, ladders and angles of elevation and depression.
- ☐ I can write a vector of length r at angle θ as $\langle r\cos\theta, r\sin\theta\rangle$.
- ☐ I can draw positive, negative and large angles in standard position and find coterminal angles.
- ☐ I can explain why π radians $= 180°$, convert between degrees and radians, and use $s = r\theta$.
- ☐ I can use the unit circle to find sine and cosine of multiples of $90°$ and tell their signs in each quadrant.
- ☐ I can model a point on a turning wheel with $(r\cos\theta, r\sin\theta)$ and find its height.

LESSONS IN THIS UNIT

13.1 Right-Triangle Trigonometry: A Review

2 sessions of 45 minutes

Unit 13 test

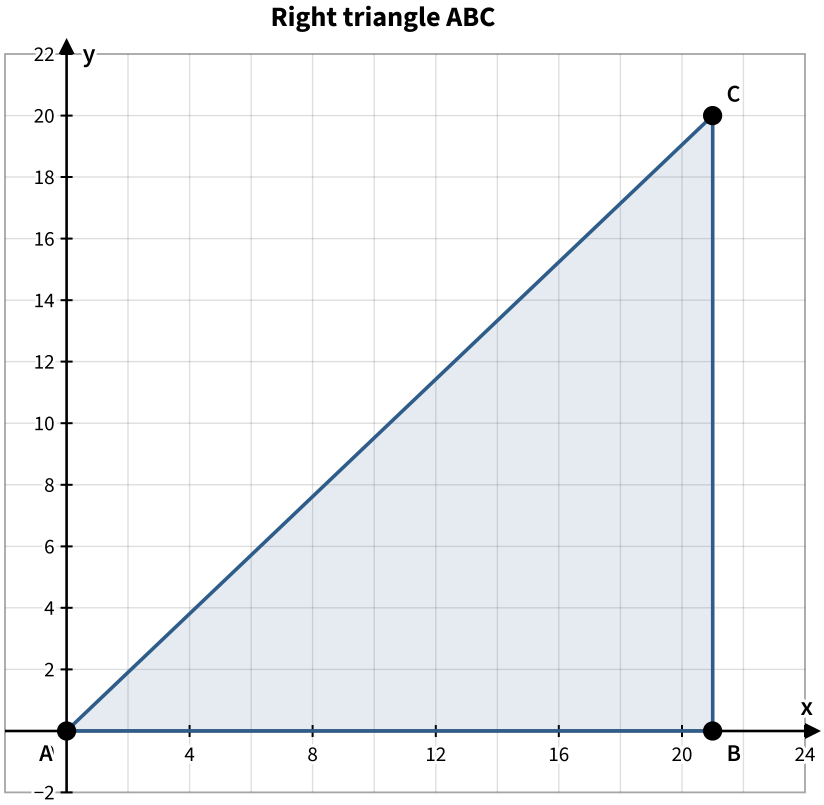
You have 50 minutes. A scientific calculator is allowed, set to degree mode unless a problem works in radians, except on parts marked "by hand". Show your work, including the ratio or formula you use. Give exact answers when the problem asks for them; otherwise round as each problem directs. The test is worth 100 points.

1. **6 pts**

   By hand, give the exact value of (a) $\cos 30^\circ$ and (b) $\tan 60^\circ$.

2. **6 pts**

   In right triangle ABC shown, the right angle is at B, $AB = 21$, $BC = 20$ and $AC = 29$. Let θ be the angle at A. Find (a) $\sin\theta$, (b) $\cos\theta$ and (c) $\tan\theta$ as fractions.



3. **6 pts**

   One acute angle of a right triangle is 52°, and the leg adjacent to it is 15 meters long. Find the length of the leg opposite the 52° angle, to the nearest tenth of a meter.

4. **6 pts**

   A right triangle has a hypotenuse of 14 and a leg of 9. Find the angle opposite the leg of length 9, to the nearest tenth of a degree.

5. **7 pts**

   θ is an acute angle with $\cos\theta = \dfrac{7}{25}$. Use $\sin^2\theta + \cos^2\theta = 1$ to find the exact values of (a) $\sin\theta$ and (b) $\tan\theta$.

6. **6 pts**

   A hay elevator (a conveyor that carries bales up into a loft) is 30 feet long and set at an angle of 28° with the level ground. How much higher than its lower end is its upper end, to the nearest tenth of a foot?

7. **7 pts**

   On a hiking trip, the Miller family walks 5 miles in a straight line in a direction 35° north of east. Write their displacement as a vector $\langle \text{east}, \text{north} \rangle$ in miles, with each component rounded to the nearest tenth.

8. **6 pts**

   (a) Convert 225° to radians, in exact form. (b) Convert $\dfrac{5\pi}{3}$ radians to degrees.

9. **6 pts**

   (a) Find the angle that is at least 0° and less than 360° and coterminal with -500°. (b) Find the angle that is at least 0 and less than 2π radians and coterminal with $\dfrac{7\pi}{2}$ radians.

10. **7 pts**

    By hand, use the unit circle to find (a) $\cos 270^\circ$, (b) $\sin 180^\circ$ and (c) $\sin(-90^\circ)$.

11. **7 pts**

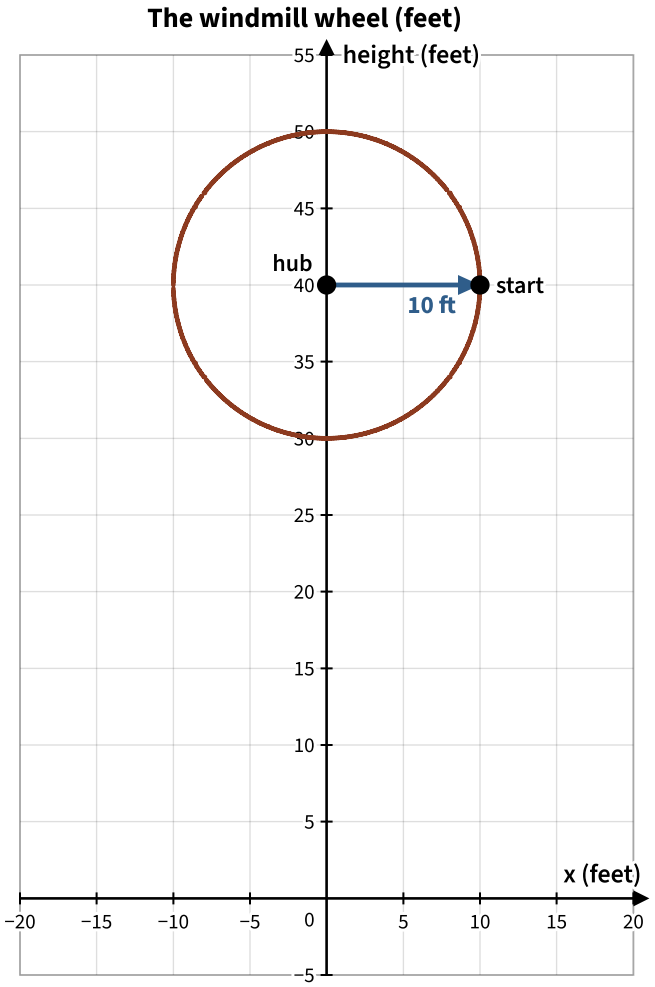
The terminal side of θ lies in Quadrant III, and $\sin\theta = -\frac{8}{17}$. Find the exact values of (a) $\cos\theta$ and (b) $\tan\theta$.

12. **8 pts**

A carousel horse rides 18 feet from the center of the carousel. How far does the horse travel when the carousel turns through $\frac{5\pi}{3}$ radians? Give (a) the exact distance and (b) the distance to the nearest tenth of a foot.

13. **8 pts**

The wheel of a windmill has blades 10 feet long, and its hub is 40 feet above the ground. The tip of one blade starts level with the hub, on the right side, and the wheel turns counterclockwise. Find the height of the tip above the ground after the wheel turns (a) $180°$, (b) $270°$ and (c) $450°$.



14. **8 pts**

From a point on level ground, the angle of elevation to the top of a grain silo is $40°$. After walking 30 feet straight toward the silo, the angle of elevation is $55°$. How tall is the silo, to the nearest foot?

15. **6 pts**

(a) Explain why $\sin 50°$ has the same value in every right triangle that has a $50°$ angle. (b) Explain why $\sin\theta < 1$ for every acute angle θ in a right triangle.

ANSWER KEY

1. (a) $\dfrac{\sqrt{3}}{2}$ (b) $\sqrt{3}$
2. (a) $\dfrac{20}{29}$ (b) $\dfrac{21}{29}$ (c) $\dfrac{20}{21}$
3. $\tan 52^\circ = \dfrac{x}{15}$, so $x = 15\tan 52^\circ \approx 19.2$ m
4. $\sin\theta = \dfrac{9}{14}$, so $\theta = \sin^{-1}\left(\dfrac{9}{14}\right) \approx 40.0^\circ$
5. (a) $\sin^2\theta = 1 - \dfrac{49}{625} = \dfrac{576}{625}$, so $\sin\theta = \dfrac{24}{25}$ (b) $\tan\theta = \dfrac{24/25}{7/25} = \dfrac{24}{7}$
6. $h = 30\sin 28^\circ \approx 14.1$ ft
7. $\langle 5\cos 35^\circ, 5\sin 35^\circ\rangle \approx \langle 4.1, 2.9\rangle$
8. (a) $225 \cdot \dfrac{\pi}{180} = \dfrac{5\pi}{4}$ (b) $\dfrac{5\pi}{3} \cdot \dfrac{180^\circ}{\pi} = 300^\circ$
9. (a) $-500^\circ + 720^\circ = 220^\circ$ (b) $\dfrac{7\pi}{2} - 2\pi = \dfrac{3\pi}{2}$
10. (a) 270° lands at $(0, -1)$, so 0 (b) 180° lands at $(-1, 0)$, so 0 (c) -90° lands at $(0, -1)$, so -1
11. $\cos^2\theta = 1 - \dfrac{64}{289} = \dfrac{225}{289}$, and $x < 0$ in Quadrant III. (a) $\cos\theta = -\dfrac{15}{17}$ (b) $\tan\theta = \dfrac{-8/17}{-15/17} = \dfrac{8}{15}$
12. (a) $s = r\theta = 18 \cdot \dfrac{5\pi}{3} = 30\pi$ ft (b) about 94.2 ft
13. $h = 40 + 10\sin\theta$. (a) $40 + 10(0) = 40$ ft (b) $40 + 10(-1) = 30$ ft (c) 450° is coterminal with 90°, so $40 + 10(1) = 50$ ft
14. Let d be the second distance and h the height: $h = d\tan 55^\circ = (d + 30)\tan 40^\circ$. So $d(\tan 55^\circ - \tan 40^\circ) = 30\tan 40^\circ$, giving $d \approx 42.7$ ft and $h \approx 42.7\tan 55^\circ \approx 61$ ft. Check: $(42.7 + 30)\tan 40^\circ \approx 61$ ✓
15. (a) Any two right triangles with a 50° angle also share a right angle, so they are similar by AA. Corresponding sides of similar triangles are proportional: if one triangle is k times the other, the opposite side and the hypotenuse are both multiplied by k, and $\dfrac{k \cdot \text{opp}}{k \cdot \text{hyp}} = \dfrac{\text{opp}}{\text{hyp}}$. So the ratio is the same in every such triangle. (b) The hypotenuse is the longest side, since $\text{hyp}^2 = \text{opp}^2 + \text{adj}^2 > \text{opp}^2$. So $\dfrac{\text{opp}}{\text{hyp}} < 1$.

COURSE EXAM

Midterm Exam

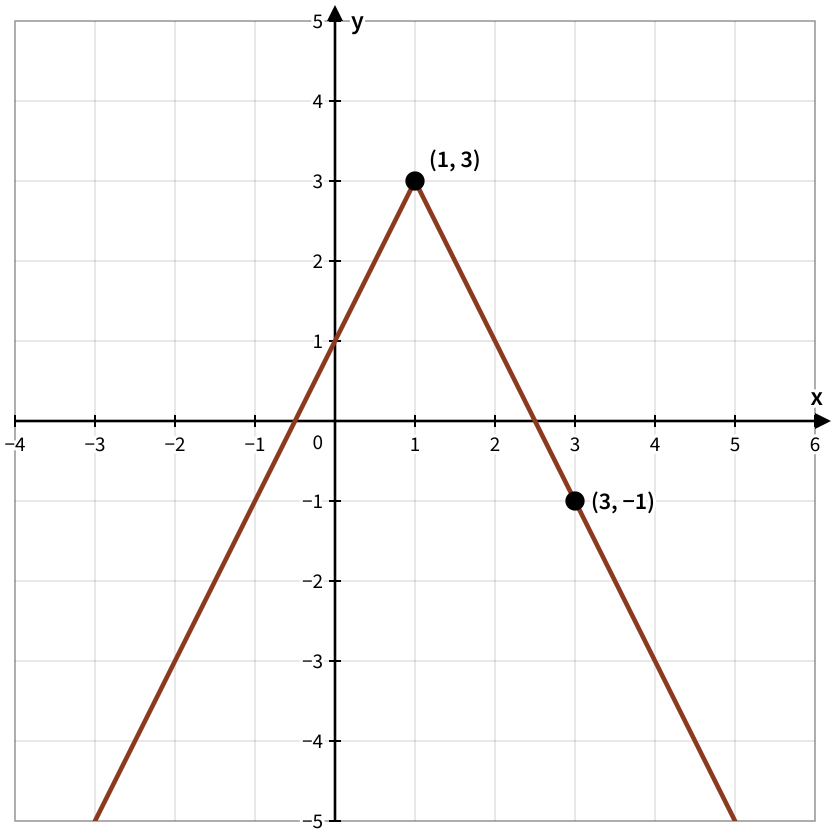
Covers Units 1–6. Time: 90 minutes.

You have 90 minutes. A scientific calculator is allowed except on problems marked "by hand." Show all of your work: answers without supporting work earn little credit, and correct work earns partial credit. Give exact answers (fractions, radicals, i) unless a problem asks you to round, and include units in answers to word problems. Use graph paper for any graph you draw. Check solutions in the original equation, and state excluded values where they matter.

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1. **4 pts** **Lesson 1.2**

   The graph shows a transformation of $y = |x|$. Write its equation in the form $y = a|x - h| + k$.



2. **3 pts** **Lesson 1.3**

   Let $f(x) = \begin{cases} x^2 - 1, & x < 2 \\ 3x - 4, & x \geq 2 \end{cases}$. Find $f(-3)$, $f(2)$ and $f(5)$.

3. **4 pts** **Lesson 1.4**

   A hardware store is giving \$5 off any purchase, and this week every tool is also 20% off. Let $f(x) = x - 5$ and $g(x) = 0.8x$, where x is a price in dollars. Samuel buys a saw marked \$40. (a) Find $(f \circ g)(40)$. (b) Find $(g \circ f)(40)$.

4. **4 pts** **Lesson 1.5**

   Find the inverse of $f(x) = \dfrac{3x + 1}{x - 2}$.

5. **3 pts** **Lesson 1.6**

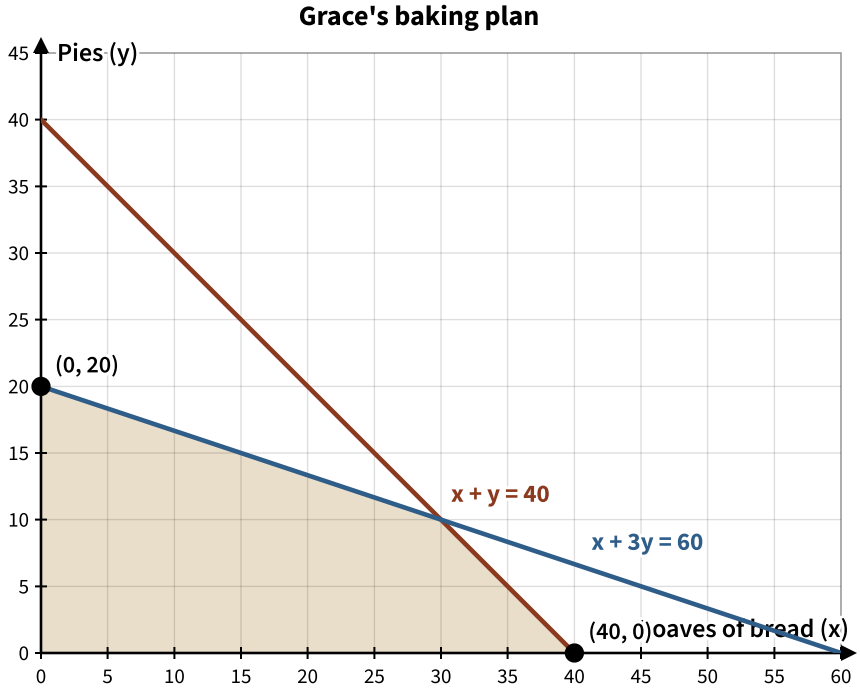
   (a) Find the average rate of change of $f(x) = x^2 - 4x$ from $x = 1$ to $x = 5$. (b) Is $g(x) = x^3 - 4x$ even, odd or neither? Show the test.

6. **4 pts** **Lesson 2.2**

   Find all the vertices (corner points) of the region defined by $x \geq 0$, $y \geq 0$, $x + 2y \leq 10$ and $3x + y \leq 15$.

7. **5 pts** **Lesson 2.3**

Grace bakes loaves of bread (x) and pies (y) for a Saturday market. Her oven space limits her to at most 40 items, so $x + y \le 40$. Each loaf takes 10 minutes of hands-on work and each pie 30 minutes, and she has at most 600 minutes, so $x + 3y \le 60$. She earns a profit of \$3 per loaf and \$5 per pie. The feasible region is shown. Find the missing corner point, then find how many loaves and pies give the greatest profit, and that profit.



8. **5 pts** **Lesson 2.4**

A church choir sold 200 tickets to its spring concert: adult tickets at \$8, student tickets at \$5 and child tickets at \$2, for a total of \$1,120. The number of student and child tickets together was 20 more than the number of adult tickets. How many of each kind were sold?

9. **3 pts** **Lesson 3.1**

Write $y = 2x^2 - 12x + 13$ in vertex form, and give the vertex.

10. **4 pts** **Lesson 3.3**

By hand, write each in the form $a + bi$. (a) $(2 + 3i)(4 - i)$ (b) $\dfrac{5 + i}{2 - i}$

11. **4 pts** **Lesson 3.4**

Solve $x^2 - 4x + 29 = 0$. Give exact answers.

12. **4 pts** **Lesson 3.5**

Henry throws a ball straight up from a height of 6 feet at 56 feet per second, so its height after t seconds is $h = -16t^2 + 56t + 6$ (feet). During what time interval is the ball at least 46 feet high? Answer in interval notation.

13. **5 pts** **Lesson 3.6**

Challenge. Ruth sells apple cider from her family's orchard at \$6 a gallon and sells 300 gallons a week. For every 25 cents she raises the price, she sells 10 fewer gallons (assume this holds for any price in between). (a) Show that the weekly revenue at a price of p dollars is $R(p) = p(540 - 40p)$. (b) What price gives the greatest revenue, and what is that revenue?

14. **4 pts** **Lesson 4.3**

Use long division to divide $2x^3 - 3x^2 + 4x - 5$ by $x^2 + 1$. Give the quotient and remainder.

15. **3 pts** **Lesson 4.4**

Use synthetic division to divide $f(x) = x^4 - 3x^3 + 2x - 7$ by $x + 2$. Give the quotient and remainder, and state what the remainder tells you about $f(-2)$.

16. **4 pts** **Lesson 4.5**

Factor completely over the integers. (a) $3x^4 - 24x$ (b) $x^4 - 5x^2 - 36$

17. **5 pts** **Lesson 4.6**

Challenge. Use the Rational Root Theorem to list the possible rational zeros of $f(x) = 2x^3 - 3x^2 - 11x + 6$, then find all of its zeros.

18. **5 pts** **Lesson 4.7**

Let $f(x) = -(x + 1)^2(x - 2)^3$. (a) Give each zero with its multiplicity, and say whether the graph crosses the x-axis or touches and turns there. (b) Explain why, using the signs of the factors. (c) Describe the end behavior.

19. **4 pts** **Lesson 5.2**

Simplify $\dfrac{3}{x - 2} - \dfrac{x + 7}{x^2 - x - 2}$ and state the excluded values.

20. **5 pts** **Lesson 5.4**

For $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$, find the hole, the vertical asymptote, the horizontal asymptote, the x-intercept and the y-intercept.

21. **4 pts** **Lesson 5.5**

Solve $\dfrac{x}{x-3} + \dfrac{6}{x+3} = \dfrac{18}{x^2-9}$. Check for extraneous solutions.

22. **4 pts** **Lesson 5.6**

Two hoses together fill a stock tank in 4 hours. The first hose alone takes 6 hours. How long would the second hose take alone?

23. **3 pts** **Lesson 6.1**

Evaluate by hand. (a) $64^{2/3}$ (b) $243^{-2/5}$ (c) $\left(\dfrac{16}{81}\right)^{-3/4}$

24. **3 pts** **Lesson 6.3**

Rationalize the denominator and simplify: $\dfrac{10}{4 - \sqrt{6}}$.

25. **4 pts** **Lesson 6.5**

Solve $\sqrt{2x + 7} = x + 2$. Check every answer.

ANSWER KEY

1. $y = -2|x - 1| + 3$ (vertex $(1, 3)$; from $(1, 3)$ to $(3, -1)$ the graph falls 4 in 2 units, so $a = -2$)
2. $f(-3) = 8$, $f(2) = 2$, $f(5) = 11$
3. (a) $f(g(40)) = f(32) = \$27$ (discount first, then \$5 off); (b) $g(f(40)) = g(35) = \$28$. Taking the percent off first saves Samuel one more dollar.
4. Swap: $x = \dfrac{3y + 1}{y - 2}$, so $xy - 2x = 3y + 1$, $xy - 3y = 2x + 1$, and $f^{-1}(x) = \dfrac{2x + 1}{x - 3}$
5. (a) $\dfrac{f(5) - f(1)}{5 - 1} = \dfrac{5 - (-3)}{4} = 2$; (b) $g(-x) = -x^3 + 4x = -g(x)$, so g is odd
6. $(0, 0)$, $(0, 5)$, $(4, 3)$, $(5, 0)$; the corner $(4, 3)$ comes from solving $x + 2y = 10$ and $3x + y = 15$
7. Corner: solve $x + y = 40$ and $x + 3y = 60$: $2y = 20$, so $(30, 10)$. $P = 3x + 5y$: $P(0, 0) = 0$, $P(40, 0) = 120$, $P(0, 20) = 100$, $P(30, 10) = 140$. Bake 30 loaves and 10 pies for a profit of \$140.
8. $a + s + c = 200$, $8a + 5s + 2c = 1120$, $s + c = a + 20$. Then $2a + 20 = 200$, so $a = 90$; $5s + 2c = 400$ with $c = 110 - s$ gives $s = 60$, $c = 50$. 90 adult, 60 student and 50 child tickets (check: $720 + 300 + 100 = 1120$ ✓).
9. $y = 2(x^2 - 6x + 9) + 13 - 18 = 2(x - 3)^2 - 5$; vertex $(3, -5)$
10. (a) $8 - 2i + 12i - 3i^2 = 11 + 10i$; (b) $\dfrac{(5 + i)(2 + i)}{(2 - i)(2 + i)} = \dfrac{9 + 7i}{5} = \dfrac{9}{5} + \dfrac{7}{5}i$
11. $x = \dfrac{4 \pm \sqrt{16 - 116}}{2} = \dfrac{4 \pm 10i}{2} = 2 \pm 5i$
12. $-16t^2 + 56t - 40 \geq 0$, so $2t^2 - 7t + 5 \leq 0$, $(2t - 5)(t - 1) \leq 0$: $[1, 2.5]$ seconds (check: $h(1) = h(2.5) = 46$)
13. (a) Each \$1 increase loses 40 gallons, so gallons sold $= 300 - 40(p - 6) = 540 - 40p$ and $R = p(540 - 40p)$. (b) $R = -40p^2 + 540p$ has its vertex at $p = \dfrac{540}{80} = 6.75$; $R(6.75) = 6.75 \times 270 = \$1822.50$.
14. Quotient $2x - 3$, remainder $2x - 2$ (check: $(x^2 + 1)(2x - 3) + 2x - 2 = 2x^3 - 3x^2 + 4x - 5$ ✓)
15. Coefficients $1, -3, 0, 2, -7$ with $c = -2$: quotient $x^3 - 5x^2 + 10x - 18$, remainder 29, so $f(-2) = 29$ by the Remainder Theorem
16. (a) $3x(x^3 - 8) = 3x(x - 2)(x^2 + 2x + 4)$; (b) $(x^2 - 9)(x^2 + 4) = (x - 3)(x + 3)(x^2 + 4)$

17. Candidates: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$. $f(3) = 0$; synthetic division leaves $2x^2 + 3x - 2 = (2x - 1)(x + 2)$. Zeros: $-2, \frac{1}{2}, 3$.

18. (a) $x = -1$ (multiplicity 2): touches and turns; $x = 2$ (multiplicity 3): crosses. (b) $(x + 1)^2$ is never negative, so it does not change sign as x passes -1 and the graph stays on the same side; $(x - 2)^3$ is negative for $x < 2$ and positive for $x > 2$, so f changes sign at 2. (c) The leading term is $-x^5$: as $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to \infty$.

19. LCD $(x - 2)(x + 1)$: $\dfrac{3(x + 1) - (x + 7)}{(x - 2)(x + 1)} = \dfrac{2x - 4}{(x - 2)(x + 1)} = \dfrac{2}{x + 1}$, with $x \neq 2$, $x \neq -1$

20. $f(x) = \dfrac{(x - 3)(x + 2)}{(x - 2)(x + 2)} = \dfrac{x - 3}{x - 2}$ for $x \neq -2$. Hole $\left(-2, \dfrac{5}{4}\right)$; vertical asymptote $x = 2$; horizontal asymptote $y = 1$; x-intercept 3; y-intercept $\dfrac{3}{2}$.

21. Multiply by $(x - 3)(x + 3)$: $x(x + 3) + 6(x - 3) = 18$, so $x^2 + 9x - 36 = 0$, $(x + 12)(x - 3) = 0$. $x = 3$ is excluded (extraneous), so $x = -12$ (check: $\dfrac{12}{15} - \dfrac{6}{9} = \dfrac{2}{15} = \dfrac{18}{135}$ ✓).

22. $\dfrac{1}{6} + \dfrac{1}{t} = \dfrac{1}{4}$, so $\dfrac{1}{t} = \dfrac{1}{12}$: 12 hours

23. (a) $(\sqrt[3]{64})^2 = 16$; (b) $\dfrac{1}{(\sqrt[5]{243})^2} = \dfrac{1}{9}$; (c) $\left(\dfrac{81}{16}\right)^{3/4} = \left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

24. $\dfrac{10(4 + \sqrt{6})}{16 - 6} = 4 + \sqrt{6}$

25. Square: $2x + 7 = x^2 + 4x + 4$, so $x^2 + 2x - 3 = 0$, $(x + 3)(x - 1) = 0$. Check: $x = 1$ gives $\sqrt{9} = 3$ ✓; $x = -3$ gives $\sqrt{1} = 1 \neq -1$, extraneous. Solution: $x = 1$.

COURSE EXAM

Final Exam

Covers Units 1–13. Time: 2 hours.

You have 2 hours. A scientific calculator is allowed except on problems marked "by hand." Show all of your work: answers without supporting work earn little credit, and correct work earns partial credit. Give exact answers (fractions, radicals, logarithms, i) unless a problem asks you to round, and include units in answers to word problems. Use graph paper for any graph you draw. Check solutions in the original equation.

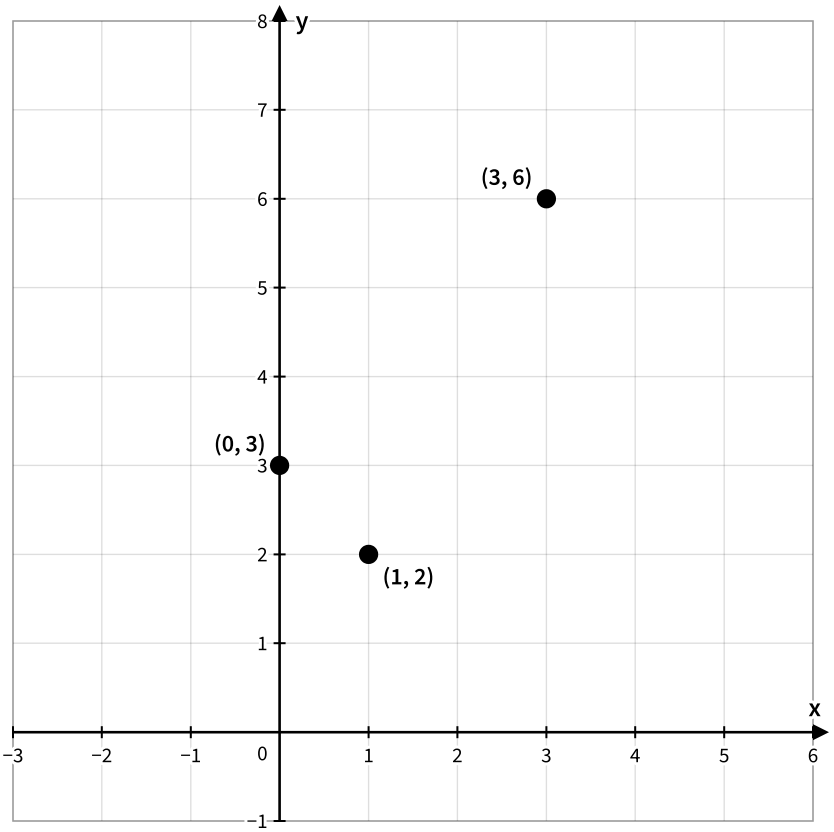
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1. **3 pts** **Lesson 1.4**

   Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 3}$. Find $(f \circ g)(x)$ in simplest form and give its domain.

2. **3 pts** **Lesson 2.4**

   Find the equation $y = ax^2 + bx + c$ of the parabola through the three points shown.



3. **3 pts** **Lesson 3.4**

   Write a quadratic function $f(x) = x^2 + bx + c$ whose zeros are $2 + i$ and $2 - i$.

4. **3 pts** **Lesson 4.7**

   Solve $x^3 - x^2 - 6x < 0$. Answer in interval notation.

5. **3 pts** **Lesson 5.3**

   Henry builds wooden birdhouses. His tools and setup cost \$600, and each birdhouse costs \$8 in materials, so his average cost per birdhouse is $\bar{C}(x) = \dfrac{600 + 8x}{x}$. (a) How many birdhouses must he build to bring the average cost down to \$10? (b) What value does the average cost approach as x grows large?

6. **4 pts** **Lesson 5.5**

   Challenge. A boat travels 24 miles upstream and 24 miles back in a total of 5 hours. Its speed in still water is 10 miles per hour. Find the speed of the current.

7. **3 pts** **Lesson 6.2**

   By hand, simplify $(8x^{6}y^{-3})^{2/3}$, assuming $x > 0$ and $y > 0$. Write the answer with positive exponents.

8. **3 pts** **Lesson 6.5**

   Solve $(x - 2)^{2/3} = 16$.

9. **3 pts** **Lesson 6.4**

   The period of a pendulum is $T = 2\pi\sqrt{\dfrac{L}{32}}$ seconds, where L is its length in feet. A rope swing in the barn is 8 feet long. Find its period to the nearest hundredth of a second.

10. **4 pts** **Lesson 7.2**

    Ruth deposits \$5,000 at 4% annual interest for 10 years. Find the balance to the nearest cent if the interest is (a) compounded continuously, (b) compounded quarterly.

11. **4 pts** **Lesson 7.3**

A tractor is worth \$48,000 two years after it is bought and \$24,576 five years after it is bought. (a) Write its value y after t years in the form $y = a \cdot b^t$. (b) By what percent does it lose value each year?

12. **4 pts** **Lesson 7.4**

The Miller family stocks a pond with fish. The number of fish after t years is $P(t) = \dfrac{1200}{1 + 11e^{-0.5t}}$. (a) How many fish were stocked? (b) What is the carrying capacity? (c) When will the pond hold 600 fish? Round to the nearest tenth of a year.

13. **4 pts** **Lesson 7.4**

The standard logistic function is $\sigma(x) = \dfrac{1}{1 + e^{-x}}$. (a) Prove that $\sigma(-x) = 1 - \sigma(x)$ for every real x. (b) Given $\sigma(2) \approx 0.881$, find $\sigma(-2)$.

14. **3 pts** **Lesson 8.1**

Evaluate by hand. (a) $\log_3 81$ (b) $\log_5 \dfrac{1}{125}$ (c) $\log_4 8$

15. **3 pts** **Lesson 8.4**

Grace deposits \$2,000 in an account paying 5% interest compounded continuously. How long until the balance reaches \$3,000? Give the exact answer and the answer to the nearest tenth of a year.

16. **3 pts** **Lesson 8.5**

Solve $\log_2 x + \log_2 (x - 2) = 3$. Explain why any rejected value must be rejected.

17. **3 pts** **Lesson 8.6**

Use $\text{pH} = -\log[\text{H}^+]$. (a) A sample of garden soil has $[\text{H}^+] = 3.2 \times 10^{-6}$. Find its pH to the nearest tenth. (b) How many times greater is $[\text{H}^+]$ in soil of pH 5 than in soil of pH 7?

18. **3 pts** **Lesson 9.2**

Samuel stacks firewood logs in rows. The bottom row has 20 logs, each row above has one fewer, and the top row has 6 logs. (a) How many rows are there? (b) How many logs are in the stack?

19. **3 pts** **Lesson 9.4**

A ball is dropped from a height of 12 feet. Each bounce rises to $\frac{3}{4}$ of the height of the one before. Find the total distance the ball travels before it comes to rest.

20. **4 pts** **Lesson 9.5**

Samuel borrows \$18,000 for a used truck at 6% annual interest, paid monthly for 5 years. Use $M = \dfrac{L \cdot i}{1 - (1+i)^{-n}}$. (a) Find the monthly payment to the nearest cent. (b) Find the total interest paid.

21. **3 pts** **Lesson 10.2**

A church is forming a committee for a service project: 3 adults chosen from 8 adult volunteers and 2 teenagers chosen from 6 teen volunteers. How many different committees are possible?

22. **3 pts** **Lesson 10.3**

Find the term containing x^3 in the expansion of $(2x - 1)^6$.

23. **4 pts** **Lesson 10.4**

Each bean seed in a packet germinates with probability 0.8, independently of the others. Grace plants 6 seeds. Round to the nearest thousandth. (a) Find the probability that exactly 5 germinate. (b) Find the probability that at least one does not germinate.

24. **3 pts** **Lesson 11.1**

For the parabola $(x - 2)^2 = 8(y + 1)$, find the vertex, the focus and the directrix.

25. **4 pts** **Lesson 11.4**

Consider $4x^2 + 9y^2 - 16x + 54y + 61 = 0$. (a) Identify the conic. (b) Write the equation in standard form by completing the square. (c) Give its center.

26. **4 pts** **Lesson 11.5**

Challenge. A rectangular pasture has an area of 1,200 square yards and a diagonal of 50 yards. Find its length and width.

27. **3 pts** **Lesson 12.2**

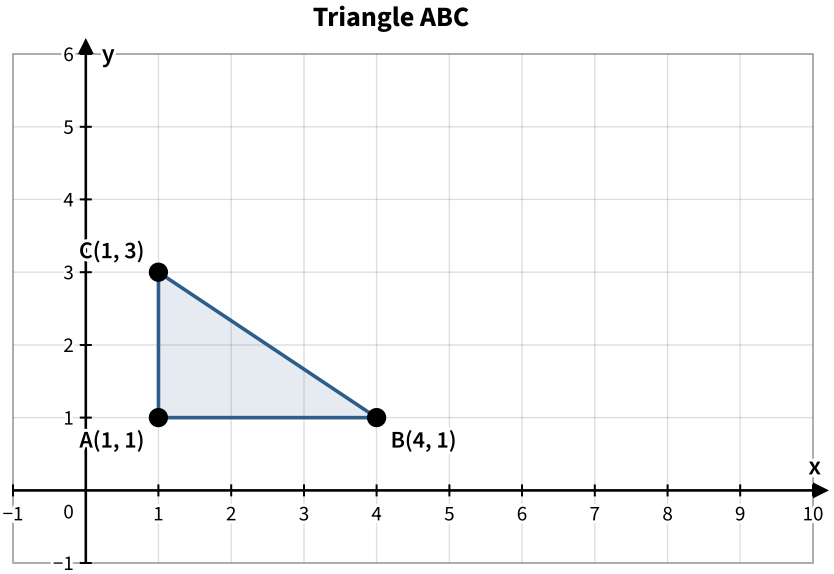
A farm stand sold 10 dozen eggs and 4 jars of honey on Friday, and 6 dozen eggs and 8 jars of honey on Saturday. Eggs are \$4 a dozen and honey is \$9 a jar. Find QP for $Q = \begin{bmatrix} 10 & 4 \\ 6 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, and say what its entries mean.

28. **4 pts** **Lesson 12.4**

Write the system $3x + 2y = 7$, $5x + 4y = 11$ as $AX = B$. (a) Find A^{-1}. (b) Use it to solve the system.

29. **3 pts** **Lesson 12.6**

Triangle ABC is shown. (a) Find the images of A, B and C under the transformation with matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. (b) Use the determinant to find the area of the image triangle.



30. **3 pts** **Lesson 13.1**

Round to the nearest tenth. (a) A 24-foot ladder leans against a barn and makes a 65° angle with the ground. How high up the wall does it reach? (b) A shed roof rises 4 feet over a horizontal run of 12 feet. What angle does it make with the horizontal?

ANSWER KEY

1. $f(g(x)) = (\sqrt{x-3})^2 + 1 = x - 2$, domain $[3, \infty)$ (the input must first be allowed in g)
2. $c = 3$; $a + b = -1$; $9a + 3b = 3$, so $3a + b = 1$. Then $a = 1$, $b = -2$: $y = x^2 - 2x + 3$
3. Sum of roots 4, product $(2+i)(2-i) = 5$, so $f(x) = x^2 - 4x + 5$
4. $x(x-3)(x+2) < 0$; a sign chart on the zeros $-2, 0, 3$ gives $(-\infty, -2) \cup (0, 3)$
5. (a) $600 + 8x = 10x$, so $x = 300$ birdhouses; (b) the horizontal asymptote is $y = 8$: the average cost approaches \$8 per birdhouse
6. $\dfrac{24}{10-c} + \dfrac{24}{10+c} = 5$, so $480 = 5(100 - c^2)$, $c^2 = 4$, and $c = 2$ mph (check: $3 + 2 = 5$ hours ✓)
7. $8^{2/3} x^4 y^{-2} = \dfrac{4x^4}{y^2}$
8. $\left(\sqrt[3]{x-2}\right)^2 = 16$, so $\sqrt[3]{x-2} = \pm 4$ and $x - 2 = \pm 64$: $x = -62$ or $x = 66$ (both check: $(\pm 64)^{2/3} = 16$ ✓)
9. $T = 2\pi\sqrt{\dfrac{8}{32}} = 2\pi \cdot \dfrac{1}{2} = \pi \approx 3.14$ seconds
10. (a) $5000e^{0.4} \approx \$7459.12$; (b) $5000(1.01)^{40} \approx \$7444.32$
11. (a) $b^3 = \dfrac{24576}{48000} = 0.512$, so $b = 0.8$; $a = \dfrac{48000}{0.64} = 75000$: $y = 75000(0.8)^t$. (b) 20% per year.
12. (a) $P(0) = \dfrac{1200}{12} = 100$ fish; (b) 1200 fish; (c) $11e^{-0.5t} = 1$, so $t = 2\ln 11 \approx 4.8$ years
13. (a) $1 - \sigma(x) = \dfrac{1 + e^{-x} - 1}{1 + e^{-x}} = \dfrac{e^{-x}}{1 + e^{-x}}$. Multiply top and bottom by e^{x}: $\dfrac{1}{e^{x} + 1} = \dfrac{1}{1 + e^{-(-x)}} = \sigma(-x)$. (b) $\sigma(-2) \approx 1 - 0.881 = 0.119$
14. (a) 4, since $3^4 = 81$; (b) -3, since $5^{-3} = \dfrac{1}{125}$; (c) $\dfrac{3}{2}$, since $4^{3/2} = 8$
15. $2000e^{0.05t} = 3000$, $e^{0.05t} = 1.5$, $t = 20\ln 1.5 \approx 8.1$ years
16. $\log_2[x(x-2)] = 3$, so $x^2 - 2x = 8$, $(x-4)(x+2) = 0$. $x = -2$ is rejected because $\log_2(-2)$ is undefined (the argument must be positive). Solution $x = 4$ (check: $2 + 1 = 3$ ✓).
17. (a) $-\log(3.2 \times 10^{-6}) = 6 - \log 3.2 \approx 5.5$; (b) $\dfrac{10^{-5}}{10^{-7}} = 100$ times
18. (a) $6 = 20 + (n-1)(-1)$, so $n = 15$ rows; (b) $S_{15} = \dfrac{15(20+6)}{2} = 195$ logs

19. Rebound heights $9 + 6.75 + \cdots = \dfrac{9}{1 - 3/4} = 36$; each rebound is traveled up and down, so total $= 12 + 2(36) = 84$ feet

20. (a) $i = 0.005$, $n = 60$: $M = \dfrac{18000(0.005)}{1 - 1.005^{-60}} \approx \$347.99$; (b) $60 \times 347.99 - 18000 = \$2879.40$

21. $C(8,3) \cdot C(6,2) = 56 \times 15 = 840$ committees

22. $\binom{6}{3}(2x)^3(-1)^3 = 20 \cdot 8x^3 \cdot (-1) = -160x^3$

23. (a) $C(6,5)(0.8)^5(0.2) = 6(0.32768)(0.2) \approx 0.393$; (b) $1 - 0.8^6 = 1 - 0.262144 \approx 0.738$

24. $4p = 8$, so $p = 2$; vertex $(2, -1)$, focus $(2, 1)$, directrix $y = -3$

25. (a) Ellipse (A and C have the same sign and differ). (b) $4(x-2)^2 + 9(y+3)^2 = -61 + 16 + 81 = 36$, so $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{4} = 1$. (c) $(2, -3)$

26. $x^2 + y^2 = 2500$ and $xy = 1200$. Then $(x+y)^2 = 2500 + 2400 = 4900$, so $x + y = 70$, and $(x-y)^2 = 2500 - 2400 = 100$, so $x - y = \pm 10$. The pasture is 30 yards by 40 yards.

27. $QP = \begin{bmatrix} 10(4) + 4(9) \\ 6(4) + 8(9) \end{bmatrix} = \begin{bmatrix} 76 \\ 96 \end{bmatrix}$: the stand took in \$76 on Friday and \$96 on Saturday

28. (a) $\det A = 12 - 10 = 2$, so $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2.5 & 1.5 \end{bmatrix}$. (b) $X = A^{-1}B = \begin{bmatrix} 14 - 11 \\ -17.5 + 16.5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$: $(3, -1)$ (check: $9 - 2 = 7$, $15 - 4 = 11$ ✓)

29. (a) $(x, y) \to (2x + y, y)$: $A'(3,1)$, $B'(9,1)$, $C'(5,3)$. (b) The original area is $\dfrac{1}{2}(3)(2) = 3$ and $\det = 2$, so the image has area 6 square units.

30. (a) $24 \sin 65^\circ \approx 21.8$ feet; (b) $\tan\theta = \dfrac{4}{12}$, so $\theta \approx 18.4^\circ$