

PLAYING THE ODDS

First edition

6 units

30 lessons + exams

A video for every lesson

FIRST EDITION

COURSE GUIDE

Playing the Odds is a one-semester elective for students in grades 8–12. It teaches the real mathematics of decisions under uncertainty: probability and counting, conditional probability, expected value, variance and risk, the law of large numbers, regression to the mean and simple game theory. The big idea is simple to say and hard to live: good players, coaches and generals rarely win by luck. They win by choosing, again and again, the option with the best expected result while keeping the chance of disaster small. They take the high-percentage shot, avoid the low-percentage gamble unless the situation demands it, and judge a decision by its process rather than by one outcome.

The mathematics is taught through the true stories of the people who found it and used it. Pascal and Fermat divide the prize of an unfinished game in their letters of 1654; Huygens writes the first printed book on probability; Jakob Bernoulli proves the law of large numbers; Henry Chadwick invents the box score; Branch Rickey and Allan Roth argue that getting on base matters more than batting average; Bill James and the analysts who followed him rebuild baseball, basketball, soccer, hockey, golf and tennis around expected value. Every lesson moves from **the story** to **how they figured it out**, then to **the mathematics** with worked examples, and finally **on the field**, where students see how players and families use the idea, and what a wise person does with it.

Students do the mathematics with their own hands. They roll dice hundreds of times and compare the results with the 36ths, simulate a thousand Monty Hall games in a spreadsheet, shoot a soft ball from two distances to measure their own points per attempt, build a run-expectancy table from a real game, chart shots in soccer and basketball, and score family games by expected value. Probability was born from questions about games of chance, and the course tells that history plainly, but it teaches probability for decisions, never for wagering: every project keeps score in

points. Scripture is woven in lightly, from “the lot is cast into the lap” to the king who sits down first and counts the cost.

Videos were chosen lesson by lesson, most often from Khan Academy and Numberphile, with 3Blue1Brown, Thinking Basketball, Golf Digest, Golf Channel, FIBA, WIRED, official league channels and smaller teaching channels run by careful analysts. Every video was previewed frame by frame and its description read; none promotes betting, and channels built around sports betting or opinion shows were excluded. Some of the smaller channels are run by individual analysts and coaches; they were chosen because they teach the lesson’s exact idea clearly. The “when” note says when to watch, and a parent note flags anything to know in advance. Many videos are short; pause them and work the numbers in the scorebook before the presenter does.

Start with Lesson 1.1 →

WHO IT IS FOR

Grades 8–12. The prerequisite is pre-algebra: fractions, decimals and percents. Algebra I helps, and some hard and challenge problems use it, so an 8th grader can follow every lesson while an 11th grader is still stretched. Students read the lessons and work the examples largely on their own; a parent checks the practice sets against the answers and discusses the essay questions. Nothing needs to be bought beyond ordinary household items: dice, coins, a deck of cards, a soft ball, graph paper and free spreadsheet software.

HOW TO USE THIS COURSE

1. **The weekly rhythm.** Each lesson takes three sessions of about 50 minutes. Session 1: the warm-up, the video, the core ideas and the worked examples. Session 2: the project and the scorebook. Session 3: the practice set, checked against the answers, with every missed problem reworked. Each unit ends with a test day, and the midterm and final each have a review day. At five sessions a week the course fills a semester of 20 weeks.
2. **Story, then mathematics, then the field.** Read the core ideas in order. *The story* gives a real moment with names, dates and places; *how they figured it out* shows what people counted or argued and what was wrong before; *the mathematics* states

the principle with a formula; *on the field* shows how players, coaches and families use it today. Every lesson names both the percentage (the expected gain) and the risk (what can go wrong, and how bad it would be). Cover the solution to each worked example, try it yourself, and compare line by line.

3. **Picture study.** Each lesson has one public-domain painting, portrait, engraving or historic photograph. Look at it quietly for two minutes, put it away and describe what you saw, then work through the “look for” items and the questions, which point back to the lesson’s idea of chance, measurement or decision.
4. **The scorebook habit.** Keep one bound graph-paper notebook for the semester: your scorebook. Every project follows the same four steps: *count* (tally real trials), *record* (a neat frequency table or chart), *compare* (your data against the mathematics) and *decide* (what you would do, and why). Record what actually happened, not what should have happened.
5. **Practice sets and corrections.** Problems are marked easy, medium, hard and challenge. Do all of the easy and medium problems and most of the hard ones; the challenge problem is for students who want more. Show fractions, decimals and percents where asked. Check against the answers the same day and rework every miss in the scorebook until you understand it.
6. **Unit tests, midterm and final.** Each unit test is worth 100 points and mixes calculations, concept questions and one short essay. The midterm (after Unit 3) and the final (after Unit 6) are each worth 100 points. Calculators are allowed, and all work must be shown.
7. **How we handle games of chance.** Probability began with questions about dice and cards, and the course tells that history honestly. But it teaches probability for decisions, never for wagering. No lesson, project or problem uses money stakes, betting or betting lines; every game keeps score in points. Casinos and lotteries appear in exactly one lesson (2.4), and only to prove a mathematical truth: their games are built to lose for the player over time, so the only winning move is not to play.
8. **Optional books and games.** Some units list free courses, free data sites, classic books and family games that fit the lessons. They are never required; every project works without them.
9. **Safety for shooting and throwing experiments.** Several projects use a soft ball and a basket or bucket. Warm up first, use soft balls indoors, and keep clear of

windows, lamps and people. A parent supervises any trip to a court, field or bowling alley.

SCOPE AND SEQUENCE

UNIT 1 • WEEKS 1–4

Chance: The Mathematics of Uncertainty

The mathematics of chance from its birth: Pascal and Fermat’s problem of points, counting outcomes with dice and Catan, independence and the gambler’s fallacy, the hot-hand debate, conditional probability with Monty Hall and Battleship, and Bernoulli’s law of large numbers tested by spreadsheet simulation.

- 1.1 The Problem of Points
- 1.2 Counting the Ways
- 1.3 Dice Have No Memory
- 1.4 What Do You Know Now?
- 1.5 The Law of Large Numbers

UNIT 2 • WEEKS 4–7

Expected Value: Weighing Reward and Risk

Expected value as the long-run average, from Huygens’s 1657 book to points per shot in basketball and football; variance and why the underdog should seek it; negative expected value in the house’s games and the wise mirror image of insurance; and Daniel Bernoulli’s answer to the risk of ruin.

- 2.1 Huygens and the Value of a Chance
- 2.2 Points per Shot
- 2.3 Same Average, Different Risk
- 2.4 The House Always Wins
- 2.5 Risk of Ruin

UNIT 3 • WEEKS 7–10

The High-Percentage Shot: Sports by the Numbers

Sports by the numbers: Chadwick's box score and on-base percentage, Bill James and the run-expectancy table, expected goals in soccer and hockey, the basketball shot chart and the underhand free throw, and course management in golf and the second serve in tennis.

- 3.1 Henry Chadwick's Box Score
- 3.2 Sabermetrics
- 3.3 Shot Quality
- 3.4 The Shot Chart
- 3.5 Course Management and the Second Serve

UNIT 5 • WEEKS 14–17

The Odds at the Game Table

The odds at the game table: backgammon's pip count and doubling decisions in tournament points, dice battles in Risk and the busiest squares of Monopoly, sound and speculative sacrifices in chess, hidden information in bridge and Battleship, and push-your-luck decisions in Can't Stop and Yahtzee.

- 5.1 Backgammon: The Race, the Blitz and the Cube
- 5.2 Dice Battles and Board Squares
- 5.3 Chess: Sound and Speculative Sacrifices
- 5.4 Hidden Information: Bridge, Stratego and Battleship
- 5.5 Push Your Luck: Can't Stop and Yahtzee

UNIT 4 • WEEKS 11–14

Decisions Under Pressure: Coaches and the Odds

Coaches deciding under pressure: fourth-down decisions and Romer's study, pulling the goalie, game theory and mixed strategies in penalty kicks and pitching, win probability and regression to the mean, and the difference between playing to win and playing not to lose.

- 4.1 Fourth Down
- 4.2 Pulling the Goalie
- 4.3 Keep Them Guessing
- 4.4 Win Probability, Momentum and Regression to the Mean
- 4.5 Playing to Win versus Playing Not to Lose

UNIT 6 • WEEKS 17–20

Wisdom, Risk and Providence

- 6.1 Life Tables and Seat Belts
- 6.2 Judge the Decision, Not the Result
- 6.3 Traps in Our Thinking
- 6.4 Time and Chance
- 6.5 Capstone: Your Odds Report

WEEKLY PACING CALENDAR

WEEK	PLAN
1	- Day 1: Lesson 1.1 The Problem of Points: video, teaching and worked examples - Day 2: Lesson 1.1: project and notebook - Day 3: Lesson 1.1: practice set and corrections - Day 4: Lesson 1.2 Counting the Ways: video, teaching and worked examples - Day 5: Lesson 1.2: project and notebook <i>Five sessions a week of about 50 minutes. Families on a four-day week can combine the practice day of one lesson with the teaching day of the next.</i>
2	- Day 1: Lesson 1.2: practice set and corrections - Day 2: Lesson 1.3 Dice Have No Memory: video, teaching and worked examples - Day 3: Lesson 1.3: project and notebook - Day 4: Lesson 1.3: practice set and corrections - Day 5: Lesson 1.4 What Do You Know Now?: video, teaching and worked examples
3	- Day 1: Lesson 1.4: project and notebook - Day 2: Lesson 1.4: practice set and corrections - Day 3: Lesson 1.5 The Law of Large Numbers: video, teaching and worked examples - Day 4: Lesson 1.5: project and notebook - Day 5: Lesson 1.5: practice set and corrections
4	- Day 1: Unit 1 test - Day 2: Lesson 2.1 Huygens and the Value of a Chance: video, teaching and worked examples - Day 3: Lesson 2.1: project and notebook - Day 4: Lesson 2.1: practice set and corrections - Day 5: Lesson 2.2 Points per Shot: video, teaching and worked examples
5	- Day 1: Lesson 2.2: project and notebook - Day 2: Lesson 2.2: practice set and corrections - Day 3: Lesson 2.3 Same Average, Different Risk: video, teaching and worked examples - Day 4: Lesson 2.3: project and notebook - Day 5: Lesson 2.3: practice set and corrections
6	- Day 1: Lesson 2.4 The House Always Wins: video, teaching and worked examples - Day 2: Lesson 2.4: project and notebook - Day 3: Lesson 2.4: practice set and corrections

WEEK	PLAN
	- Day 4: Lesson 2.5 Risk of Ruin: video, teaching and worked examples - Day 5: Lesson 2.5: project and notebook
7	- Day 1: Lesson 2.5: practice set and corrections - Day 2: Unit 2 test - Day 3: Lesson 3.1 Henry Chadwick's Box Score: video, teaching and worked examples - Day 4: Lesson 3.1: project and notebook - Day 5: Lesson 3.1: practice set and corrections
8	- Day 1: Lesson 3.2 Sabermetrics: video, teaching and worked examples - Day 2: Lesson 3.2: project and notebook - Day 3: Lesson 3.2: practice set and corrections - Day 4: Lesson 3.3 Shot Quality: video, teaching and worked examples - Day 5: Lesson 3.3: project and notebook
9	- Day 1: Lesson 3.3: practice set and corrections - Day 2: Lesson 3.4 The Shot Chart: video, teaching and worked examples - Day 3: Lesson 3.4: project and notebook - Day 4: Lesson 3.4: practice set and corrections - Day 5: Lesson 3.5 Course Management and the Second Serve: video, teaching and worked examples
10	- Day 1: Lesson 3.5: project and notebook - Day 2: Lesson 3.5: practice set and corrections - Day 3: Unit 3 test - Day 4: Review day for the midterm - Day 5: Midterm exam
11	- Day 1: Lesson 4.1 Fourth Down: video, teaching and worked examples - Day 2: Lesson 4.1: project and notebook - Day 3: Lesson 4.1: practice set and corrections - Day 4: Lesson 4.2 Pulling the Goalie: video, teaching and worked examples - Day 5: Lesson 4.2: project and notebook
12	- Day 1: Lesson 4.2: practice set and corrections - Day 2: Lesson 4.3 Keep Them Guessing: video, teaching and worked examples - Day 3: Lesson 4.3: project and notebook - Day 4: Lesson 4.3: practice set and corrections

WEEK	PLAN
	Day 5: Lesson 4.4 Win Probability, Momentum and Regression to the Mean: video, teaching and worked examples
13	- Day 1: Lesson 4.4: project and notebook - Day 2: Lesson 4.4: practice set and corrections - Day 3: Lesson 4.5 Playing to Win versus Playing Not to Lose: video, teaching and worked examples - Day 4: Lesson 4.5: project and notebook - Day 5: Lesson 4.5: practice set and corrections
14	- Day 1: Unit 4 test - Day 2: Lesson 5.1 Backgammon: The Race, the Blitz and the Cube: video, teaching and worked examples - Day 3: Lesson 5.1: project and notebook - Day 4: Lesson 5.1: practice set and corrections - Day 5: Lesson 5.2 Dice Battles and Board Squares: video, teaching and worked examples
15	- Day 1: Lesson 5.2: project and notebook - Day 2: Lesson 5.2: practice set and corrections - Day 3: Lesson 5.3 Chess: Sound and Speculative Sacrifices: video, teaching and worked examples - Day 4: Lesson 5.3: project and notebook - Day 5: Lesson 5.3: practice set and corrections
16	- Day 1: Lesson 5.4 Hidden Information: Bridge, Stratego and Battleship: video, teaching and worked examples - Day 2: Lesson 5.4: project and notebook - Day 3: Lesson 5.4: practice set and corrections - Day 4: Lesson 5.5 Push Your Luck: Can't Stop and Yahtzee: video, teaching and worked examples - Day 5: Lesson 5.5: project and notebook
17	- Day 1: Lesson 5.5: practice set and corrections - Day 2: Unit 5 test - Day 3: Lesson 6.1 Life Tables and Seat Belts: video, teaching and worked examples - Day 4: Lesson 6.1: project and notebook - Day 5: Lesson 6.1: practice set and corrections

WEEK	PLAN
18	- Day 1: Lesson 6.2 Judge the Decision, Not the Result: video, teaching and worked examples - Day 2: Lesson 6.2: project and notebook - Day 3: Lesson 6.2: practice set and corrections - Day 4: Lesson 6.3 Traps in Our Thinking: video, teaching and worked examples - Day 5: Lesson 6.3: project and notebook
19	- Day 1: Lesson 6.3: practice set and corrections - Day 2: Lesson 6.4 Time and Chance: video, teaching and worked examples - Day 3: Lesson 6.4: project and notebook - Day 4: Lesson 6.4: practice set and corrections - Day 5: Lesson 6.5 Capstone: Your Odds Report: video, teaching and worked examples
20	- Day 1: Lesson 6.5: project and notebook - Day 2: Lesson 6.5: practice set and corrections - Day 3: Unit 6 test - Day 4: Review day for the final - Day 5: Final exam

MATERIALS

Five ordinary dice and two coins

For nearly every probability experiment. Dice of two different colors make ordered outcomes easier to see.

A deck of ordinary playing cards

For counting and conditional-probability experiments. Families who prefer can use 52 index cards numbered 1–13 in four colors.

A calculator

Any calculator with a square-root key. A scientific calculator is helpful for powers like $(5/6)^{24}$.

A bound graph-paper notebook (the scorebook), pencil and ruler

For tallies, frequency tables, shot charts, expected-value tables and corrected practice problems.

Free spreadsheet software

Google Sheets or LibreOffice Calc. The functions =RAND() and =RANDBETWEEN() run every simulation in the course.

A soft ball and a basket, bucket or laundry hamper

For shooting experiments indoors or in the yard. A foam ball or a rolled pair of socks works well.

A tape measure and a stopwatch (or a phone)

For marking shooting distances and timing experiments.

Access to free statistics sites

Baseball-Reference, Basketball-Reference and Retrosheet, used in Unit 3 for real data. A parent should preview these sites, which carry advertising.

GRADING

Grades come from the practice sets, the projects, the scorebook, the six unit tests, the midterm and the final. A calculation earns full credit when the setup, the arithmetic and the answer (as a fraction, decimal or percent as asked) are all shown. A project earns full credit for honest data carefully counted and compared with the mathematics, even when the data disagree with theory, as long as the student explains why.

PART	%
Practice sets (completed and corrected)	15
Projects (data, analysis, decision)	25
Scorebook (organization, honesty, completeness)	10
Unit tests	30
Midterm exam	10
Final exam	10

Scale: A 90–100, B 80–89, C 70–79, D 60–69, F below 60.

- Practice sets are graded for completion and correction, not first-try accuracy: a problem done, checked and correctly reworked earns full credit.
- On tests, give partial credit for a correct setup (the right sample space, the right expected-value table) even if the arithmetic slips. An answer with no work earns

at most half credit.

- A project earns full marks when it has a prediction made before the trials, a tally of at least the required number of trials, a comparison with the mathematics, and a written decision with a reason.
- For a high-school transcript this course is one half credit of mathematics elective (probability and statistics). Keep the scorebook and the unit tests as the record of work.

A NOTE TO PARENTS

You do not need to be a mathematician to supervise this course well. Your jobs are to keep the rhythm (three sessions per lesson), check the practice sets against the answers, ask the student to explain one worked example aloud each week, look at the scorebook, and be present for the shooting and throwing experiments. The best question you can ask is “How do you know?” A student who can answer it from his own tallies is learning probability.

This course handles games of chance with care. It tells the history of probability honestly, including the dice and card questions that started it, but it never teaches or encourages wagering. Projects keep score in points, not money. Casinos and lotteries appear in only one lesson, which proves that their games are built to make the player lose over time and pairs that lesson with Proverbs and with the wise counterpart of insurance and saving. The course does not discuss sports betting, and its videos were checked to exclude betting sponsors.

The course is also a course in wisdom. Scripture teaches both that “the lot is cast into the lap; but the whole disposing thereof is of the LORD” (Proverbs 16:33) and that the prudent man counts the cost before he builds (Luke 14:28). Good decisions are diligent and humble at once: we do the percentages carefully, and we trust God with the outcome. Expect the student to be surprised often, because chance is surprising, and praise careful counting and honest records more than lucky results.

UNIT 1

CHANCE: THE MATHEMATICS OF UNCERTAINTY

Probability was born in 1654, when two Frenchmen who never met wrote letters about how to divide the prize of an unfinished game. This unit follows that story and the people who carried it forward: Pascal and Fermat with the problem of points, Galileo counting the falls of three dice, the dice players of Monte Carlo and the psychologists and economists who argued about the hot hand, Marilyn vos Savant and the thousands of readers who told her she was wrong about Monty Hall, and Jakob Bernoulli, whose law of large numbers Stanislaw Ulam turned into a computer method.

You will learn to list sample spaces, count outcomes with the multiplication principle, permutations and combinations, multiply probabilities of independent events, compute conditional probabilities with tables and trees, update with Bayes's rule, and judge how far a sample can stray with the standard error. Your projects are real experiments: playing out interrupted games, rolling two dice 360 times, testing your own shooting for a hot hand, simulating 1,000 Monty Hall games and watching the law of large numbers in a spreadsheet.

PREREQUISITES

- Pre-algebra: fractions, decimals and percents, and converting between them
- Order of operations, exponents and square roots on a calculator
- Reading and making a bar graph

MASTERY CHECKLIST

- ☐ I can list a sample space and find a probability as a fraction, a decimal and a percent.
- ☐ I can divide the prize of an interrupted game fairly by Fermat's or Pascal's method.
- ☐ I can count outcomes with the multiplication principle, permutations and combinations, and explain why 7 is the most common sum of two dice.
- ☐ I can multiply probabilities of independent events, use the complement for "at least one," and explain the gambler's fallacy.
- ☐ I can compute a conditional probability from a table or tree, explain the Monty Hall problem, and update a probability with Bayes's rule.
- ☐ I can compute a standard error and explain why small samples mislead.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Khan Academy: Statistics and Probability**: Free. Short videos and practice exercises on probability, counting and conditional probability that match Lessons 1.1–1.5. Good for extra practice after each practice set.
- **MIT OpenCourseWare: Probabilistic Systems Analysis and Applied Probability**: Free. MIT’s introductory probability course with lecture videos, recitations and problem sets. Far beyond this course, but the early lectures on counting, conditional probability and Bayes’s rule reward a strong 11th or 12th grader.
- **CATAN**: The publisher’s page for the family trading game. Its number tokens and pips make Lesson 1.2’s two-dice counting visible on every turn. Ages 10+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

1.1 The Problem of Points	3 sessions of 50 minutes
1.2 Counting the Ways	3 sessions of 50 minutes
1.3 Dice Have No Memory	3 sessions of 50 minutes
1.4 What Do You Know Now?	3 sessions of 50 minutes
1.5 The Law of Large Numbers	3 sessions of 50 minutes

UNIT 1 TEST

Calculators are allowed. Show your work for every calculation: write the sample space, count or formula you used, and give probabilities as fractions and as decimals or percents unless told otherwise. Answer the concept questions in complete sentences. The test is worth 100 points.

1. **5 pts**

A card is drawn from a standard deck. Find the probability that it is a face card (jack, queen or king), as a fraction, a decimal and a percent.

2. (7 pts)

Two players match to 4 wins. The game stops with A leading 3–2. Divide 100 points fairly, showing Fermat's method.

3. (7 pts)

Two players match to 5 wins; A leads 4–2. Divide 80 points fairly.

4. (7 pts)

Two dice are rolled. Find the probability that the sum is 5 or 9.

5. (6 pts)

A coach chooses 4 players from 10 for a relay team, then sets the running order. How many teams (ignoring order) are possible? How many teams with running orders?

6. (6 pts)

A Catan spot touches hexes numbered 6, 9 and 11. Another touches 8, 4 and 2. Which collects more often, and how many pips does each have?

7. (6 pts)

A 75% free-throw shooter takes 3 shots, independently. Find the probability she makes all three and the probability she makes at least one.

8. (6 pts)

Find the probability of at least one 6 in 4 rolls of a die, and explain how this answered part of de Méré's question.

9. (7 pts)

Explain the gambler's fallacy with the Monte Carlo example of 1913, and explain why streaks in sport are not proof that something has changed.

10. (7 pts)

Explain the bias Miller and Sanjurjo discovered, using the three-flip example, and what it meant for the 1985 hot-hand study.

11. (6 pts)

In the Monty Hall game with 3 doors, what is the probability of winning by switching? Explain in two sentences.

12. (7 pts)

In practice a shooter took 50 layups (made 40) and 50 long shots (made 15). Given that a shot was made, what is the probability it was a layup? Given that it was a layup, what is the probability it was made?

13. (7 pts)

A bag holds one fair coin and one two-headed coin. You pick one at random and flip it: heads. What is the probability you picked the two-headed coin?

14. (6 pts)

Find the standard error of the fraction of heads in 900 flips of a fair coin, and the range in which about 95% of results fall.

15. (10 pts)

Short essay. In about 150 words, explain how this unit's mathematics helps a person make better decisions in a game or sport. Use at least two of these ideas: counting equally likely outcomes, independence, conditional probability, the law of large numbers. Include one example of a mistake the mathematics helps you avoid.

ANSWER KEY

1. $\frac{12}{52} = \frac{3}{13} \approx 0.231 \approx 23.1\%$.
2. A needs 1, B needs 2: at most 2 rounds (AA, AB, BA, BB). B wins only BB. A gets 75 points, B 25.
3. A needs 1, B needs 3: B must win 3 straight, $\frac{1}{8}$. A gets $\frac{7}{8} \times 80 = 70$ points, B gets 10.
4. Sum 5: 4 ways; sum 9: 4 ways. $\frac{8}{36} = \frac{2}{9} \approx 22.2\%$.
5. Teams: $\binom{10}{4} = 210$. With orders: $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$.
6. First: $5 + 4 + 2 = 11$; second: $5 + 3 + 1 = 9$. The first collects more often.
7. All three: $0.75^3 \approx 0.422$. At least one: $1 - 0.25^3 = 1 - 0.015625 \approx 0.984$.
8. $1 - (5/6)^4 \approx 0.518$. It is slightly more than one half, while at least one double six in 24 rolls of two dice is only about 0.491, so the two games were not equal as de Méré had assumed.
9. When black came up 26 times, people believed red was due, but each spin was independent and red's chance never changed. In sport, a shooter with a fixed chance will still have long runs of makes and misses by chance alone, so a streak by itself does not show that skill has changed.
10. In 3-flip sequences, the average fraction of heads after a head is $\frac{5}{12}$, not $\frac{1}{2}$. So in short sequences a random shooter appears to shoot worse after makes. The 1985 study found equal rates after makes and misses; corrected for the bias, that equality is evidence of a modest hot hand.
11. $\frac{2}{3}$. Your first pick is wrong two times in three, and in those cases the host, who knows where the car is, must leave the car behind the other closed door, so switching wins.
12. $P(\text{layup} \mid \text{make}) = \frac{40}{55} \approx 72.7\%$; $P(\text{make} \mid \text{layup}) = \frac{40}{50} = 80\%$.
13. Two-headed path: $\frac{1}{2} \cdot 1 = \frac{1}{2}$; fair path: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. $P = \frac{1/2}{3/4} = \frac{2}{3}$.
14. $\sqrt{0.25/900} = \frac{0.5}{30} \approx 0.0167$; range about 0.467 to 0.533.
15. Answers vary. A strong essay uses two or more ideas correctly with a concrete example, such as choosing Catan spots by pips, not benching a good shooter after a slump that chance explains, switching in Monty Hall, or waiting for a large sample before judging a player, and it names a mistake such as the gambler's fallacy or trusting a small sample.

PORTFOLIO PIECE: A DICE LABORATORY REPORT

Choose a question about dice or coins that you cannot answer instantly, such as: How many rolls of one die does it take, on average, to see all six faces? How often do three dice show a sum of 10 or more? In 5 rolls of five dice, how often is there at least one five of a kind? First calculate (or estimate) the answer using this unit's counting rules. Then test it twice: with at least 200 real trials by hand, and with at least 2,000 simulated trials in a spreadsheet. Write a two-page report with your question, your calculation, your data tables, a bar graph or running-fraction graph, a comparison of your hand results, your simulation and the mathematics using the standard error, and a conclusion about what you learned and what you would do differently.

LOOK FOR	WHAT IT MEANS
Question and calculation	A clear question with a correct calculation or a careful explanation of why exact calculation is hard.
Real and simulated data	At least 200 hand trials and 2,000 spreadsheet trials, recorded honestly in neat tables.
Comparison with mathematics	Results compared with the calculation using relative frequency and the standard error; differences explained.
Graph	A labeled bar graph or running-fraction graph that shows the comparison clearly.
Conclusion	A thoughtful conclusion that states what was learned, its limits, and one improvement.

1.1 THE PROBLEM OF POINTS

OBJECTIVES

- Tell the story of the 1654 letters between Pascal and Fermat and the question that started them.
- List the sample space of a simple experiment and find a probability as favorable outcomes over equally likely outcomes, written as a fraction, a decimal and a percent.
- Divide the prize of an interrupted game fairly by counting every way the game could have finished.

Warm-up. Flip a coin 10 times and write the results in a row (H or T). Before you start, write down a guess: how many heads? After you finish, circle your actual count. Then answer: if two players were racing to 5 heads-or-tails wins and you had to stop after flip 6, who was ahead, and how sure are you that the leader would have won?

WATCH

The fairness puzzle that sparked probability

Chalk Talk · 13:14

https://www.youtube.com/watch?v=_YRfSe-f1FE

WHEN TO WATCH

After the story and “how they figured it out.” It retells the problem of points and shows Pascal’s and Fermat’s two methods side by side; pause at 6:16 and compare with the worked examples.

Note: The video divides a prize of \$100 to make the fractions concrete; in this course we divide points.

BACKUP VIDEO

Can You Solve The Problem That Inspired Probability Theory? (Problem Of The Points)

MindYourDecisions · 5:29

https://www.youtube.com/watch?v=C_nV3cVNjog

Instead of the main video, or after it as extra practice. It sets a harder version (first to 10 points at 8–7) and solves it; try it yourself before the presenter does.

KEY TERMS

outcome

One possible result of a chance experiment, such as “the die shows 4.”

event

A set of outcomes we care about, such as “an even number” = {2, 4, 6}.

sample space

The list of all possible outcomes. For one coin it is {H, T}; for two coins it is {HH, HT, TH, TT}.

probability

A number from 0 (impossible) to 1 (certain) that measures how likely an event is. With equally likely outcomes it is the number of favorable outcomes divided by the total.

complement

The event that something does *not* happen. Its probability is 1 minus the probability that it does.

problem of points

The old puzzle of how to divide the prize fairly when a game of several rounds is stopped before anyone has won.

CORE IDEAS

The story. In 1654 a French writer and courtier, Antoine Gombaud, who called himself the Chevalier de Méré, brought two puzzles about games of chance to the brilliant young mathematician Blaise Pascal in Paris. One was about dice. The other was very old: two players agree that whoever first wins a certain number of rounds takes the whole prize, but the game is interrupted before anyone has won. How should the prize be divided? Pascal wrote to Pierre de Fermat, a lawyer and judge in Toulouse who did mathematics in his spare hours. Through the summer and autumn of 1654 the two men, who never met, exchanged letters that solved the problem in two different ways. Those letters are usually called the birth of probability theory.

How they figured it out. The puzzle was at least 150 years old. In 1494 the Italian friar Luca Pacioli proposed dividing the prize in proportion to the rounds each player had already won. Niccolò Tartaglia objected that this cannot be right: if the game is to 100 wins and it stops at 1–0, Pacioli gives the leader the entire prize. Gerolamo Cardano, whose *Book on Games of Chance* (written about 1564, published in 1663) was the first to count equally likely outcomes, also struggled with it. Pascal and Fermat saw what everyone had missed: the past does not matter, only the *future* does. Fermat listed every way the remaining rounds could go and counted how many of them each player would win. Pascal worked backward from the end of the game, splitting the prize at each step. Both methods gave the same answer, and Pascal wrote to Fermat that he saw “the truth is the same at Toulouse and at Paris.”

The mathematics. When all outcomes in a sample space are equally likely, the probability of an event is $P(E) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$. A probability is always between 0 and 1, and it can be written three ways: $\frac{1}{4} = 0.25 = 25\%$. The **complement** rule says $P(\text{not } E) = 1 - P(E)$. For the problem of points, Fermat’s trick is to imagine that the game is always played to the *maximum* number of rounds that could still be needed, even if someone wins earlier. If player A needs a more wins and B needs b , at most $a + b - 1$ more rounds are needed, and with fair rounds

every sequence of results is equally likely. Count the sequences in which A gets *a* wins first; that fraction of the prize belongs to A.

On the field. The same reasoning is used whenever a contest is stopped early: a rained-out game, a tournament cut short, a board game the family must leave for supper. Coaches use it too, every time they ask “from here, how often do we win?” instead of “who is ahead right now?” A two-goal lead with 80 minutes left is not the same as a two-goal lead with 5 minutes left. **The risk side:** a leader who thinks the prize is already his has forgotten the rounds still to be played. Pascal’s insight is a habit of mind: judge a position by all the ways it can still turn out.

PICTURE STUDY



Pierre de Fermat, François de Poilly, 17th century · Engraving · Printed in France; many copies survive in libraries and print rooms

Image: [Wikimedia Commons](#), public domain

Fermat was a lawyer and a judge of the Parliament of Toulouse who did mathematics for joy. He published almost nothing in his lifetime; we know much of his work only from his letters, including the 1654 letters to Pascal that solved the problem of points.

Look closely

- Look at his clothing and the frame around the portrait. What do they tell you about his profession and his standing?
- Look at his face and expression. Does he look like a man in a hurry or a man who weighs things carefully?
- An engraving is cut into a copper plate so that many copies can be printed. Why might people have wanted many copies of this face?

Talk about it

1. Fermat's day job was judging disputes fairly. How is the problem of points a question of fairness as well as of mathematics?
2. Fermat solved the problem by listing every possible future. Why is that a good habit for a judge, a coach or a player?

PROJECT

Stop the Game and Play It Out

Fermat said a leader's fair share is the fraction of possible futures in which the leader wins. You will test that claim with a coin and real data: freeze a game at a set score, play it out many times, and compare your results with the mathematics.

You will need

- A coin
- The scorebook and a pencil
- A partner (optional; you can play both sides)

Steps

1. **Predict.** In a match to 3 wins, A leads 2–1. Write the fair split by Fermat's method (A: $\frac{3}{4}$). Then write your own guess for a match to 4 at 3–1 and at 2–2 before you calculate them.
2. Set up a tally table with three positions: match to 3 at 2–1, match to 4 at 3–1, match to 4 at 2–2. Each row has columns "A won" and "B won."
3. Play out the first position: flip the coin (heads = A wins a round, tails = B wins a round) until someone reaches 3. Record who won. That is one trial.
4. Repeat until you have 40 trials for the first position.
5. Do the same for 40 trials from 3–1 in a match to 4, and 40 trials from 2–2 in a match to 4.
6. For each position, compute A's fraction of wins as a fraction, a decimal and a percent.
7. Calculate the exact Fermat shares for all three positions and put them next to your data.
8. **Compare.** How far was each result from the exact share? Was any result off by more than 10 percentage points?
9. **Combine.** If a sibling or friend also did the project, add your tallies together. Is the combined result closer to the exact share?

10. **Decide.** Write a short paragraph: if a family game of yours were stopped at 3–1 in a match to 4, how would you divide 100 points, and why is that fairer than Pacioli’s rule?

Tips

- You do not need to keep flipping once someone has won; Fermat’s “play all the rounds” idea is only for counting.
- Record every trial, even the ones that surprise you. Surprises are data.

Check your work

- ☐ My tally has 40 trials for each of the three positions.
- ☐ I computed the exact Fermat shares and compared them with my results.
- ☐ I explained in writing why the future, not the past, decides a fair split.

SCOREBOOK

1. Draw a tree diagram for a match to 3 at 2–1: one branch for each possible next round, continued until someone wins. Write the probability on each branch and circle the leaves where A wins.
2. Copy your three-position tally table with columns: position, trials, A wins, A’s fraction, decimal, percent, exact share, difference.
3. **What I would decide and why:** a game of yours is stopped at a score of your choosing. Compute the fair split and explain it to someone who thinks the leader should get everything.

KEY FORMULAS

$$P(E) = \frac{\text{favorable outcomes}}{\text{possible outcomes}}$$

$$P(\text{not } E) = 1 - P(E)$$

$$\text{rounds still needed at most} = a + b - 1$$

From history. Pascal (1623–1662) built one of the first mechanical calculators at nineteen and later wrote the *Pensées*, a defense of the Christian faith. Fermat (1601–1665) is also famous for “Fermat’s Last Theorem,” a note in the margin of a book that was not proved until 1994. The probability work of their letters was carried forward by Christiaan Huygens in 1657 (Lesson 2.1).

WORKED EXAMPLES

Example 1. Roll one fair die. Find the probability of rolling a 5 or a 6. Write it as a fraction, a decimal and a percent, and find the probability of the complement.

Show solution

1. Sample space: $\{1, 2, 3, 4, 5, 6\}$, six equally likely outcomes.
2. Favorable outcomes: $\{5, 6\}$, so $P = \frac{2}{6} = \frac{1}{3}$.
3. As a decimal, $1 \div 3 \approx 0.333$; as a percent, about 33.3%.
4. Complement (a 1, 2, 3 or 4): $1 - \frac{1}{3} = \frac{2}{3} \approx 0.667 \approx 66.7\%$.

Answer: $\frac{1}{3} \approx 0.333 \approx 33.3\%$; the complement is $\frac{2}{3}$.

Example 2. Two evenly matched players match to 3 wins for a prize of 100 points. The game stops with A leading 2–1. Divide the prize by Fermat’s method.

Show solution

1. A needs $a = 1$ more win; B needs $b = 2$. At most $a + b - 1 = 2$ more rounds are needed.
2. List every way 2 rounds can go: AA, AB, BA, BB. Each has probability $\frac{1}{4}$.
3. A wins the match in AA, AB and BA (A gets one more win before B gets two). B wins only in BB.
4. $P(A) = \frac{3}{4} = 75\%$ and $P(B) = \frac{1}{4} = 25\%$.

Answer: A receives 75 points and B receives 25 points.

Example 3. Solve the same position (A needs 1, B needs 2) by Pascal's backward method, and compare with Pacioli's rule of dividing by rounds already won.

Show solution

1. If A wins the next round, A has won: A's share is 100 points.
2. If B wins the next round, the score is 2-2 and the players are equal: A's share would be 50 points.
3. Each is equally likely, so A's fair share now is $\frac{1}{2}(100) + \frac{1}{2}(50) = 75$ points, the same as Fermat's answer.
4. Pacioli's rule divides by wins so far, 2 to 1, giving A $\frac{2}{3} \times 100 \approx 66.7$ points. It undervalues A because it looks at the past instead of the rounds still to come.

Answer: Pascal: 75 to 25, agreeing with Fermat. Pacioli: about 66.7 to 33.3, which is not fair.

Example 4. In a match to 4 wins, the game stops with A leading 3-1. How should 80 points be divided?

Show solution

1. A needs 1 more win; B needs 3. At most $1 + 3 - 1 = 3$ more rounds.
2. Three rounds have $2 \times 2 \times 2 = 8$ equally likely sequences.
3. B wins the match only if B wins all three: BBB. So $P(B) = \frac{1}{8}$ and $P(A) = \frac{7}{8}$.
4. A gets $\frac{7}{8} \times 80 = 70$ points; B gets $\frac{1}{8} \times 80 = 10$ points.

Answer: A 70 points, B 10 points (87.5% and 12.5%).

PRACTICE

1. **Easy** A card is drawn from a well-shuffled deck of 52. What is the probability it is a heart? Give a fraction, a decimal and a percent.

Show answer

$$\frac{13}{52} = \frac{1}{4} = 0.25 = 25\%$$

2. **Easy** List the sample space for flipping two coins, and find the probability of exactly one head.

Show answer

{HH, HT, TH, TT}. Exactly one head: HT, TH, so $\frac{2}{4} = \frac{1}{2} = 50\%$.

3. **Easy** The probability of rolling a 6 on one die is $\frac{1}{6}$. What is the probability of *not* rolling a 6? Give it as a decimal and a percent.

Show answer

$$1 - \frac{1}{6} = \frac{5}{6} \approx 0.833 \approx 83.3\%$$

4. **Medium** In a match to 5 wins between equal players, the game stops at 4–3. Divide 100 points fairly.

Show answer

The leader needs 1, the other needs 2; at most 2 rounds: AA, AB, BA, BB. The trailer wins only BB, so the split is $\frac{3}{4}$ and $\frac{1}{4}$: 75 points and 25 points.

5. **Medium** Each player needs 2 more wins. Use Fermat's method to show that the prize should be split evenly.

Show answer

At most 3 rounds: 8 sequences. A wins in AAA, AAB, ABA, BAA (at least two A's): 4 of 8. So $P(A) = \frac{4}{8} = \frac{1}{2}$ and the prize splits evenly, as symmetry says it should.

6. **Medium** A needs 1 more win and B needs 4. How should 64 points be divided?

Show answer

B must win 4 rounds in a row: $P(B) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$. A gets $\frac{15}{16} \times 64 = 60$ points; B gets 4 points.

7. **Medium** Explain in two or three sentences why Pacioli's rule (divide by rounds already won) is unfair, using Tartaglia's example of a game to 100 that stops at 1–0.

Show answer

Pacioli's rule would give the whole prize to the player leading 1–0, even though 99 more wins are needed and the trailer still has an almost even chance. A fair division must depend on the chances of winning from the current position, that is, on the rounds still to be played, not on the rounds already played.

8. **Medium** In the story, why did Pascal write that “the truth is the same at Toulouse and at Paris”?

Show answer

Fermat in Toulouse (counting all future sequences) and Pascal in Paris (working backward from the end) used different methods but reached the same fair division, which convinced both that the answer was true and not a matter of opinion.

9. **Hard** A needs 2 more wins and B needs 3. Use Fermat’s method to find each player’s share of 160 points. (Hint: count the 16 sequences of 4 rounds in which B wins at least 3.)

Show answer

Sequences with at least 3 B’s: exactly 3 B’s (4 ways: AB BB, BA BB, BB AB, BB BA) plus BBBB (1 way) = 5. $P(B) = \frac{5}{16}$, $P(A) = \frac{11}{16}$. A gets 110 points and B gets 50 points.

10. **Hard** Check problem 9 by Pascal’s backward method. You may use: when A needs 1 and B needs 3, A’s share is $\frac{7}{8}$; when each needs 2, A’s share is $\frac{1}{2}$.

Show answer

If A wins the next round, A needs 1 and B needs 3: share $\frac{7}{8}$. If B wins, each needs 2: share $\frac{1}{2}$. A’s share is $\frac{1}{2} \cdot \frac{7}{8} + \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{16} + \frac{4}{16} = \frac{11}{16}$, which matches.

11. **Challenge** A needs 2 more wins and B needs 4. Find A’s share of the prize as a fraction and a percent, by either method.

Show answer

At most 5 rounds (32 sequences). B needs at least 4 B’s: exactly 4 (5 ways) or 5 (1 way) = 6. $P(B) = \frac{6}{32} = \frac{3}{16}$, so $P(A) = \frac{13}{16} = 81.25\%$. (Backward check: $\frac{1}{2} \cdot \frac{15}{16} + \frac{1}{2} \cdot \frac{11}{16} = \frac{26}{32}$.)

COMMON MISTAKES

- Dividing by the score so far (Pacioli’s rule) instead of by the chances of winning from here.
- Listing sequences that stop early (A, BA, BB) as if they were equally likely. They are not; Fermat’s method plays every sequence to the same length so that all are equally likely.
- Forgetting that probabilities of all outcomes in a sample space must add to 1.

Make it a game • Split the Pot Fairly. Play first to three wins with a die: even you score, odd your parent scores. Stop the game early when the score is 2 to 1, and divide ten paper chips fairly. Each writes a split on paper before revealing. Then use Pascal and Fermat's reasoning to list the possible remaining outcomes and find the fair split. Closest estimate wins a point.

Scripture. “A false balance is abomination to the LORD: but a just weight is his delight.” (Proverbs 11:1, KJV)

The problem of points began as a question of fairness: how to give each player what is truly his. A just division, like a just weight, delights the Lord.

1.2 COUNTING THE WAYS

OBJECTIVES

- Use the multiplication principle to count outcomes, and list the 36 outcomes of two dice to explain why 7 is the most common sum.
- Explain from counting why Catan's 6 and 8 are printed in red, and compare the value of settlement spots by their pips.
- Count arrangements with permutations and choices of groups with combinations, and use them to find probabilities.

Warm-up. Before rolling, write which sum of two dice you think comes up most often, and which least. Then roll two dice 20 times and tally the sums. Was your guess supported? Keep the tally; it becomes the start of your project.

WATCH

[Die rolling probability](#) | [Probability and combinatorics](#) | [Precalculus](#) | [Khan Academy](#)

[Khan Academy](#) • 5:14

<https://www.youtube.com/watch?v=O4Qnsubo2tg>

WHEN TO WATCH

After “the mathematics” paragraph. It builds the 6×6 grid of two-dice outcomes that the whole lesson depends on.

BACKUP VIDEO

[Permutation formula](#) | [Probability and combinatorics](#) | [Probability and Statistics](#) | [Khan Academy](#)

[Khan Academy](#) • 7:34

<https://www.youtube.com/watch?v=DROZVHObeko>

Before the third worked example, to see where the permutation formula comes from.

KEY TERMS

multiplication principle

If one choice can be made in m ways and a second in n ways, the two together can be made in $m \times n$ ways.

factorial

$n! = n \times (n - 1) \times \cdots \times 2 \times 1$, the number of ways to arrange n different things in a row.

combination

A choice of a group in which order does not matter, such as choosing 5 starters from a team of 12.

ordered outcome

An outcome in which the order matters, such as (2, 5) for a red die and a white die. (2, 5) and (5, 2) are different outcomes.

permutation

An arrangement in which order matters, such as first, second and third place.

pips

In Catan, the dots under a number token; each dot stands for one of the 36 ways two dice can make that number.

CORE IDEAS

The story. Around 1620 an Italian patron asked Galileo Galilei a puzzle that dice players had noticed. With three dice, the sum 9 and the sum 10 can each be made in six ways ($9 = 1+2+6$, $1+3+5$, $1+4+4$, $2+2+5$, $2+3+4$, $3+3+3$, and similarly for 10), yet long experience said 10 came up a little more often. Galileo's short paper *On a Discovery Concerning Dice* gave the answer. Three and a half centuries later, in 1995, the German designer Klaus Teuber published *The Settlers of Catan*. Its board has a number token on each land hex, and every time the two dice show that number, settlements on the hex collect a card. Two of the numbers, 6 and 8, are printed in red. Galileo would have known exactly why.

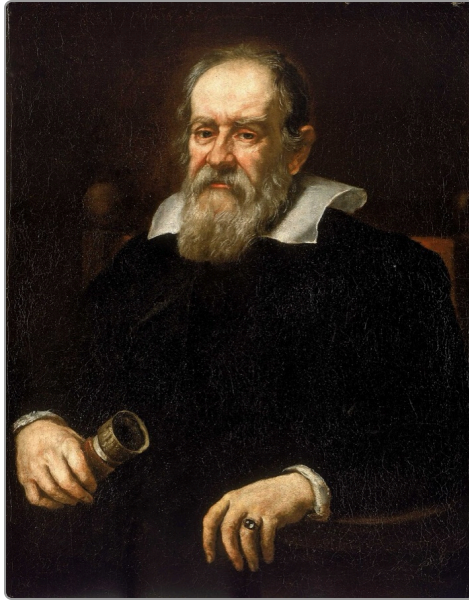
How they figured it out. The players had counted the wrong things. They counted ways to *write* a sum ($1+2+6$) instead of ways the dice can *fall*. Galileo imagined the three dice as different, say red, white and blue. Then $1+2+6$ can fall in 6 orders, $1+4+4$ in only 3, and $3+3+3$ in just 1. Counting this way, 10 can be made in 27 of the $6^3 = 216$ equally likely ways and 9 in only 25. The difference is small, $\frac{27}{216} = 12.5\%$ against about 11.6%, which is why it took long experience to notice it. The lesson has never changed: to find a probability, count equally likely outcomes, and the easiest way to make outcomes equally likely is to treat every die, coin or card as distinct.

The mathematics. The **multiplication principle** says that two dice have $6 \times 6 = 36$ ordered outcomes. The sum 7 can be made 6 ways (1-6, 2-5, 3-4, 4-3, 5-2, 6-1), more than any other, so $P(7) = \frac{6}{36} = \frac{1}{6}$. The counts for sums 2 through 12 rise and fall: 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. To arrange n different things in order there are $n!$ ways. To arrange r of n things in order (a **permutation**) there are ${}_nP_r = \frac{n!}{(n-r)!}$ ways. To choose r of n when order does not matter (a **combination**) we divide out the $r!$ orders of each group: ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$. These are the numbers in Pascal's arithmetical triangle.

On the field. In Catan the 6 and 8 each come up in 5 of the 36 outcomes, more than any number except the 7 (which moves the robber instead of producing). Each pip under a token is one of those 36 outcomes, so a good player adds up the pips around a spot: a settlement touching a 6, an 8 and a 5 collects on $5 + 5 + 4 = 14$ of 36 outcomes, about once every 2.6 rolls. That is the high-percentage spot. **The risk side:** it is also the spot everyone wants, it is often where the robber is placed, and having every hex on one or two numbers means a long dry stretch if those numbers do not come. Wise players weigh expected production against variety. Counting is how you see the board as it really is.

Think it through. The same counting tells a baseball manager that 9 hitters can be arranged in $9! = 362,880$ batting orders, and a coach that 5 starters can be chosen from 12 players in $\binom{12}{5} = 792$ ways. Large numbers come quickly; counting rules let us handle them without listing every case.

PICTURE STUDY



Portrait of Galileo Galilei, Justus Sustermans, 1636 · Oil on canvas · National Maritime Museum, Greenwich, London

Image: [Wikimedia Commons](#), public domain

Galileo is famous for the telescope, the pendulum and his experiments on falling bodies. But he also wrote a short paper on dice, and it is one of the first pieces of probability ever written. Sustermans painted him at seventy-two, near the end of his life.

Look closely

- Find the instrument in his hand. What does it tell you about how Galileo found things out?
- Look at his eyes and face. Sustermans painted him late in life, after years of trouble. What do you read in his expression?
- Notice the light falling on his face and beard against the dark background. What mood does it give the portrait?

Talk about it

1. Galileo answered the dice question by counting carefully instead of trusting what players said. How is that like using a telescope instead of trusting what people said about the sky?
2. The difference between 25 and 27 chances in 216 is small. Why does a small difference in probability matter more when a game is played many times?

PROJECT

360 Rolls of Two Dice

The counting says each sum of two dice has a fixed share of 36. Do real dice agree? You will roll two dice 360 times, which is enough to expect each 36th to come up about 10 times, and compare your frequencies with the mathematics.

You will need

- Two dice, preferably of different colors
- A cup or shallow box to roll into
- The scorebook, graph paper and a pencil
- A calculator
- Optional: a spreadsheet

Steps

1. **Predict.** Multiply each sum's count of 36ths by 10 to get the expected number of times in 360 rolls (sum 2: 10, sum 3: 20, ..., sum 7: 60, ..., sum 12: 10). Write these in a column labeled "expected."
2. Make a tally table with rows for sums 2 through 12.
3. Roll both dice into the cup and tally the sum. Roll in sets of 36 so you can rest your hand and keep count.
4. After every 72 rolls, count the tallies so far and compare informally with the predictions.
5. Finish all 360 rolls (or 10 sets of 36). Total each row and check that the totals add to 360.
6. Compute each sum's observed fraction ($\text{count} \div 360$) and the expected fraction ($\text{ways} \div 36$) as decimals.
7. Draw a bar graph of observed counts, and on the same axes mark the expected counts with a dot or a line.
8. Compute the difference (observed – expected) for each sum. Which sum was furthest from its prediction?
9. **Catan check.** Add your counts for 6 and 8. The expected total is 100. How close were you?

10. **Decide.** If you were placing your first Catan settlement using only your data, which numbers would you want? Would you trust your data or the counting more, and why?

Tips

- Different-colored dice make it easy to see that (2, 5) and (5, 2) are different outcomes.
- If you have a sibling, one rolls and one tallies, and you switch every 36 rolls.

Check your work

- ☐ My table has 360 tallies, and my totals add up.
- ☐ My bar graph shows observed and expected counts together.
- ☐ I explained which sum was furthest from its prediction and whether that worries me.

SCOREBOOK

1. Draw the 6×6 grid of two-dice outcomes. Color all the cells that sum to 7 in one color and those that sum to 6 or 8 in another.
2. Copy your frequency table: sum, ways out of 36, expected in 360, observed, difference.
3. Sketch a small Catan-style board of 7 hexes with numbers of your choice. Mark the best three-hex corner by pips, and write its pip total.

KEY FORMULAS

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

From history. Galileo's dice paper was written sometime between 1613 and 1623 but not printed until 1718. The arithmetical triangle that gives the combinations, known in the West as Pascal's triangle, was known centuries earlier in China, Persia and India; Pascal's *Treatise on the Arithmetical Triangle* (printed in 1665) was the first to use it to solve problems of chance.

WORKED EXAMPLES

Example 1. Two dice are rolled. Find the probability that the sum is 6 or 8, as a fraction, a decimal and a percent.

Show solution

1. There are $6 \times 6 = 36$ equally likely ordered outcomes.
2. Sum 6: (1,5), (2,4), (3,3), (4,2), (5,1): 5 outcomes. Sum 8: (2,6), (3,5), (4,4), (5,3), (6,2): 5 outcomes.
3. The events do not overlap, so add: $5 + 5 = 10$ favorable outcomes.
4. $P = \frac{10}{36} = \frac{5}{18} \approx 0.278 \approx 27.8\%$.

Answer: $\frac{5}{18} \approx 0.278 \approx 27.8\%$, more than a quarter of all rolls.

Example 2. Galileo's problem: with three dice, count the ways to roll a sum of 9 and a sum of 10, and find each probability.

Show solution

1. Total outcomes: $6^3 = 216$.
2. A set of three different numbers can fall in $3! = 6$ orders; a set with one pair, in 3 orders; three of a kind, in 1.
3. Sum 9: 1+2+6 (6), 1+3+5 (6), 1+4+4 (3), 2+2+5 (3), 2+3+4 (6), 3+3+3 (1). Total 25.
4. Sum 10: 1+3+6 (6), 1+4+5 (6), 2+2+6 (3), 2+3+5 (6), 2+4+4 (3), 3+3+4 (3). Total 27.
5. $P(9) = \frac{25}{216} \approx 0.116$ and $P(10) = \frac{27}{216} = 0.125$.

Answer: Sum 10 is slightly more likely: 12.5% against about 11.6%.

Example 3. A coach has 12 players. (a) In how many ways can she choose 5 starters? (b) In how many ways can she choose a captain, a vice-captain and a scorekeeper (three different jobs)?

Show solution

1. (a) Order does not matter: $\binom{12}{5} = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{95,040}{120} = 792$.
2. (b) Order matters, because the jobs are different: ${}_{12}P_3 = 12 \cdot 11 \cdot 10 = 1,320$.
3. Check: each group of 3 people can be given the 3 jobs in $3! = 6$ ways, and $\binom{12}{3} = 220$ groups $\times 6 = 1,320$.

Answer: (a) 792 ways; (b) 1,320 ways.

Example 4. Compare two Catan settlement spots: Spot A touches hexes numbered 6, 8 and 3; Spot B touches 5, 9 and 10. Which collects more often, on average?

Show solution

1. Ways to roll each number: $6 \rightarrow 5, 8 \rightarrow 5, 3 \rightarrow 2$; $5 \rightarrow 4, 9 \rightarrow 4, 10 \rightarrow 3$.
2. Spot A: $5 + 5 + 2 = 12$ of 36 outcomes, about 0.333 cards per roll.
3. Spot B: $4 + 4 + 3 = 11$ of 36 outcomes, about 0.306 cards per roll.
4. Over 60 rolls, A expects about $60 \times \frac{12}{36} = 20$ cards; B about $60 \times \frac{11}{36} \approx 18.3$.

Answer: Spot A (12 pips against 11), though B spreads its chances over three numbers instead of depending on two.

PRACTICE

1. **Easy** A team has 3 jersey colors and 4 sock colors. How many different uniforms (one jersey, one pair of socks) are possible?

Show answer

$$3 \times 4 = 12$$

2. **Easy** Two dice are rolled. Find $P(\text{sum} = 7)$ as a fraction, decimal and percent.

Show answer

$$\frac{6}{36} = \frac{1}{6} \approx 0.167 \approx 16.7\%$$

3. **Easy** Two dice are rolled. Find $P(\text{sum} = 2)$.

Show answer

Only (1,1): $\frac{1}{36} \approx 0.028 \approx 2.8\%$.

4. **Medium** In how many orders can 4 relay runners run their legs?

Show answer

$4! = 24$.

5. **Medium** Eight teams play a round robin in which every team plays every other team once. How many games are played?

Show answer

$$\binom{8}{2} = \frac{8 \cdot 7}{2} = 28.$$

6. **Medium** Two dice are rolled. Find the probability of doubles, and the probability of a sum of 10 or more.

Show answer

Doubles: $\frac{6}{36} = \frac{1}{6}$. Sum ≥ 10 : $3 + 2 + 1 = 6$ outcomes, so also $\frac{1}{6}$.

7. **Medium** A Catan spot touches 4, 9 and 10. Another touches 6, 5 and 3. How many pips does each have, and which collects more often?

Show answer

First: $3 + 4 + 3 = 10$; second: $5 + 4 + 2 = 11$. The second collects slightly more often ($\frac{11}{36}$ vs $\frac{10}{36}$ per roll).

8. **Medium** Explain in your own words why the dice players before Galileo were wrong to think 9 and 10 were equally likely with three dice.

Show answer

They counted the ways to write each sum as a list of numbers (six each), but those lists are not equally likely. A list of three different numbers can fall in 6 orders, a list with a pair in 3, and a triple in 1. Counting the equally likely ordered outcomes gives 27 for 10 and 25 for 9.

9. **Hard** Eight runners are in a final. In how many ways can the gold, silver and bronze medals be awarded? In how many ways can 3 runners be chosen to advance (order not mattering)?

Show answer

$$\text{Medals: } 8 \cdot 7 \cdot 6 = 336. \text{ Advancing: } \binom{8}{3} = \frac{336}{6} = 56.$$

10. **Hard** In a game with five dice, what is the probability of rolling five of a kind (all five the same) in one roll?

Show answer

Total outcomes $6^5 = 7,776$; favorable outcomes 6 (all 1s, ..., all 6s).

$$P = \frac{6}{7,776} = \frac{1}{1,296} \approx 0.00077, \text{ about } 0.08\%.$$

11. **Hard** With three dice, what is the probability of a sum of 3? Of a sum of 18? Of all three dice different?

Show answer

Sum 3: only (1,1,1): $\frac{1}{216}$. Sum 18: only (6,6,6): $\frac{1}{216}$. All different:

$$\frac{6 \cdot 5 \cdot 4}{216} = \frac{120}{216} = \frac{5}{9} \approx 55.6\%.$$

12. **Challenge** In one roll of five dice, what is the probability of a straight of five (1-2-3-4-5 or 2-3-4-5-6, in any order)?

Show answer

Each straight can appear in $5! = 120$ orders, and there are 2 straights: 240

$$\text{favorable outcomes. } P = \frac{240}{7,776} = \frac{5}{162} \approx 0.0309, \text{ about } 3.09\%.$$

COMMON MISTAKES

- Counting (2, 5) and (5, 2) as one outcome. With two dice there are 36 ordered outcomes, not 21.
- Using a permutation when order does not matter (choosing starters) or a combination when it does (awarding medals).
- Expecting the data to match the prediction exactly. Even 360 rolls scatter around the expected counts; Lesson 1.5 shows how much.

Make it a game • Dice Grid in Sixty. Draw a 6 by 6 grid and race against the clock to fill it with all thirty-six sums of two dice. Then count how many cells give each sum and predict the most common one. Roll two dice twenty times, mark each sum on a tally strip, and compare to the grid. Beat your own fill time on a second try.

1.3 DICE HAVE NO MEMORY

OBJECTIVES

- Multiply probabilities of independent events, and use the complement to find the probability of “at least one.”
- Explain the gambler’s fallacy and why long streaks are normal in random sequences.
- Tell how the 1985 hot-hand study was corrected in 2018, and explain the small-sample bias that Miller and Sanjurjo found.

Warm-up. Without flipping a coin, write a “random” list of 30 H’s and T’s from your head. Then flip a real coin 30 times and write the real list underneath. Circle the longest run of the same letter in each list. Which list has the longer run? Keep both lists for the project.

WATCH

Is the "hot hand" real? - Numberphile

[Numberphile](#) · 19:48

<https://www.youtube.com/watch?v=bPZFQ6i759g>

WHEN TO WATCH

After “how they figured it out.” Professor Lisa Goldberg explains the 1985 study and the 2018 correction; pause when she shows the table of short sequences and compare with Worked Example 3.

BACKUP VIDEO

The "Hot Hand" IS Real - Classic Fallacy Debunked

[MindYourDecisions](#) · 10:07

<https://www.youtube.com/watch?v=YFkJQyDPYmY>

Instead of the main video if time is short. It explains the same correction with a three-flip table.

KEY TERMS

independent events

Events where the outcome of one does not change the probability of the other, such as two flips of a fair coin.

multiplication rule

For independent events,
 $P(A \text{ and } B) = P(A) \times P(B)$.

gambler's fallacy

The false belief that a random outcome is “due” because it has not happened lately, as if dice or coins had a memory.

streak

A run of the same result in a row, such as five made shots or six heads.

hot hand

The belief that a player who has made several shots in a row is more likely than usual to make the next one.

CORE IDEAS

The story. On the evening of August 18, 1913, at the famous roulette tables of Monte Carlo, the ball fell on black again and again: 26 times in a row. As the streak grew, crowds pressed in and staked enormous sums on red, sure that red was “due.” They lost heavily. The wheel had no memory, and the 27th spin, like the first, was black or red with exactly the same chances as ever. This is the classic example of the **gambler's fallacy**. Seventy-two years later the opposite belief, that a streak should continue, was put on trial in basketball. In 1985 the psychologists Thomas Gilovich, Robert Vallone and Amos Tversky published “The Hot Hand in Basketball.” They studied the shots of the Philadelphia 76ers, the free throws of the Boston Celtics and a shooting experiment with Cornell University players, and they concluded that the hot hand was an illusion: a player was no more likely to make a shot after a make than after a miss.

How they figured it out. For thirty years the 1985 paper was a famous example of how people see patterns in randomness. Then two economists, Joshua Miller and Adam Sanjurjo, noticed something subtle. They published it in 2018 in *Econometrica* as “Surprised by the Hot Hand Fallacy? A Truth in Small Numbers.” Suppose you flip a fair coin only 3 times and, in each short sequence, compute the fraction of flips after a head that are heads. You would expect 50% on average, but the true average is

only $\frac{5}{12} \approx 41.7\%$. In short sequences, looking only at shots that follow a make builds in a bias against finding a streak. When Miller and Sanjurjo corrected for this bias, the Cornell data that had shown “no difference” actually showed a modest hot hand. Good scientists corrected good scientists, and both sides improved the science by checking the counting.

The mathematics. Two events are **independent** when one does not change the chances of the other. Then $P(A \text{ and } B) = P(A)P(B)$. Four heads in a row has probability $(\frac{1}{2})^4 = \frac{1}{16}$, but once four heads have happened, the next flip is still $\frac{1}{2}$. To find the chance of *at least one* success in n independent tries, use the complement: $P(\text{at least one}) = 1 - P(\text{none}) = 1 - (1 - p)^n$. This is how de Méré’s dice question of 1654 was answered: at least one six in 4 rolls of one die has probability $1 - (\frac{5}{6})^4 \approx 0.518$, but at least one double six in 24 rolls of two dice has only $1 - (\frac{35}{36})^{24} \approx 0.491$. Streaks are also far more common than people expect: in 100 flips of a fair coin, the chance of a run of 6 or more of the same face somewhere is about 81%.

On the field. A 40% three-point shooter will miss 5 in a row about $0.6^5 \approx 7.8\%$ of the time, so over a season of hundreds of attempts, cold streaks are certain to happen even when nothing is wrong. A wise coach does not bench a good shooter for a slump that chance explains, and a wise player does not force shots because he “feels hot.” The modest hot hand that the corrected studies found is real but small; it is not a reason to abandon good shot choice. **The risk side:** believing a result is “due” leads people to take bad risks, from staking more and more on red in 1913 to a pitcher who throws the same pitch because “he can’t hit it three times in a row.”

PICTURE STUDY



Roman Die, Unknown Roman craftsman, c. 1st–3rd century A.D. · Bone or ivory · The Metropolitan Museum of Art, New York

Image: [Wikimedia Commons](#), public domain

Dice are among the oldest game pieces in the world. This one was carved and drilled by hand nearly two thousand years ago. Roman soldiers, children and families threw dice like it; the Gospels tell that soldiers cast lots for Christ's garments.

Look closely

- Count the spots on each face you can see. A modern die puts opposite faces together to make 7. Can you tell from the visible faces whether this one may follow the same rule?
- Look at the edges and corners. Are they perfectly even? What might an uneven die do to the probabilities?
- Look at the drilled spots. Would removing more material on one face make the die lighter on that side?

Talk about it

1. This die has been still for nearly two thousand years. If it were rolled tomorrow, would its past rolls make any difference? Why not?
2. Why does every probability calculation in this lesson assume a fair die, and how could you test whether an old die is fair?

Safety first. Warm up before shooting. Use a soft ball indoors, and keep clear of windows, lamps and people.

PROJECT

Streaks, Real and Imagined

People believe random sequences should switch back and forth often, so they underestimate how long streaks get. You will compare a made-up sequence with a real one, then test your own shooting for a hot hand.

You will need

- A coin
- Your two lists of 30 from the warm-up
- A soft ball and a basket, bucket or hamper
- Tape measure and tape for a shooting mark
- The scorebook

Steps

1. **Predict.** Write how long you think the longest run will be in 100 real coin flips, and how many times you expect the result to switch from H to T or T to H.
2. Write a made-up sequence of 100 H's and T's as randomly as you can, without a coin.
3. Flip a real coin 100 times and record the sequence.
4. For each list, find the longest run and count the number of switches. (A truly random list of 100 has about 50 switches and usually a longest run of 6 or more.)
5. Show both lists to a family member without saying which is real. Can they tell? Record their guess and their reason.
6. **Shooting test.** Mark a spot about 2.5 m from a basket or bucket where you make roughly half your shots with a soft ball. Warm up with 10 shots.
7. Take 60 shots, one at a time, and record each as X (make) or O (miss) in order.
8. **Count:** makes after a make, attempts after a make, makes after a miss, attempts after a miss. Compute both rates.
9. Compare the two rates. Remember Miller and Sanjurjo: with short sequences, even a random shooter shows a lower rate after makes.

10. **Decide.** Write whether your data show a hot hand, a cold hand or neither, and whether 60 shots are enough to tell.

Tips

- Do not stop shooting after a miss or keep going after a make; take all 60 in order.
- Write your rates as fractions first (for example 14/27) and then as percents.

Check your work

- ☐ I recorded 100 real flips and a 100-flip made-up list and compared their longest runs and switches.
- ☐ I recorded 60 shots in order and computed the rates after makes and after misses.
- ☐ I explained why my conclusion is or is not trustworthy.

SCOREBOOK

1. Make a run-length frequency table for your real 100 flips: how many runs of length 1, 2, 3, 4, 5, 6 or more?
2. Make a table of your shots: after a make (attempts, makes, rate) and after a miss (attempts, makes, rate).
3. **What I would decide and why:** your team's best shooter has missed four in a row in a close game. Write what you would tell her as a coach, using this lesson.

KEY FORMULAS

$$P(A \text{ and } B) = P(A) P(B) \quad (\text{independent events})$$

$$P(\text{at least one success in } n) = 1 - (1 - p)^n$$

From history. The 1985 paper by Gilovich, Vallone and Tversky appeared in *Cognitive Psychology*. Amos Tversky later worked with Daniel Kahneman on the traps in our thinking (Lesson 6.3). Miller and Sanjurjo's 2018 paper is a model of how science corrects itself: by checking the counting, not by attacking the people.

WORKED EXAMPLES

Example 1. A fair coin has landed heads 4 times in a row. (a) What was the probability of that streak before it started? (b) What is the probability the next flip is heads?

Show solution

1. (a) The flips are independent: $\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625 = 6.25\%$.
2. (b) The coin has no memory. The next flip is heads with probability $\frac{1}{2}$.
3. Five heads in a row from the start would have been $\frac{1}{32}$; but given the first four, the fifth is still $\frac{1}{2}$.

Answer: (a) $\frac{1}{16} = 6.25\%$; (b) $\frac{1}{2}$. Thinking tails is “due” is the gambler’s fallacy.

Example 2. De Méré’s question: which is more likely, at least one six in 4 rolls of one die, or at least one double six in 24 rolls of two dice?

Show solution

1. One die, 4 rolls: $P(\text{no six}) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296} \approx 0.482$, so
 $P(\text{at least one}) \approx 1 - 0.482 = 0.518$.
2. Two dice, 24 rolls: $P(\text{no double six}) = \left(\frac{35}{36}\right)^{24} \approx 0.509$, so
 $P(\text{at least one}) \approx 0.491$.
3. De Méré had reasoned that both should be equal because $4/6 = 24/36$, but that rule ignores overlaps; the complement method is correct.

Answer: The first (about 51.8%) is more likely than the second (about 49%).

Example 3. Miller and Sanjurjo's bias: flip a fair coin 3 times. For each sequence, find the fraction of flips right after a head that are heads, and average over all sequences where that fraction exists.

Show solution

1. List the 8 sequences. TTT and TTH have no head in the first two flips, so no fraction exists. That leaves 6 sequences.
2. HHH: after-head flips are H, H $\rightarrow 1$. HHT: H, T $\rightarrow \frac{1}{2}$. HTH: T $\rightarrow 0$. HTT: T $\rightarrow 0$. THH: H $\rightarrow 1$. THT: T $\rightarrow 0$.
3. Average: $\frac{1 + \frac{1}{2} + 0 + 0 + 1 + 0}{6} = \frac{2.5}{6} = \frac{5}{12} \approx 0.417$.
4. So in short sequences, a perfectly random shooter appears to shoot about 42% after a make, not 50%. Finding 50% after a make would actually be evidence of a hot hand.

Answer: $\frac{5}{12} \approx 41.7\%$, not 50%: the bias that changed the hot-hand debate.

Example 4. A free-throw shooter makes 80% of her free throws, and each shot is independent. Find the probability that she makes both of two free throws, misses both, and makes at least one.

Show solution

1. Both: $0.8 \times 0.8 = 0.64$.
2. Both missed: $0.2 \times 0.2 = 0.04$.
3. At least one: $1 - 0.04 = 0.96$.

Answer: Both 64%; neither 4%; at least one 96%.

PRACTICE

1. Easy What is the probability of rolling two sixes in a row with one die?

Show answer

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \approx 2.8\%.$$

2. **Easy** A coin has landed tails 3 times in a row. What is the probability it lands tails next time?

Show answer

$\frac{1}{2}$; the coin has no memory.

3. **Easy** A player makes 70% of her shots, independently. What is the probability she makes 3 in a row?

Show answer

$$0.7^3 = 0.343 = 34.3\%.$$

4. **Medium** Find the probability of at least one six in 3 rolls of a die, as a fraction and a percent.

Show answer

$$1 - \left(\frac{5}{6}\right)^3 = 1 - \frac{125}{216} = \frac{91}{216} \approx 42.1\%.$$

5. **Medium** Find the probability of at least one head in 5 flips.

Show answer

$$1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32} \approx 96.9\%.$$

6. **Medium** A 90% free-throw shooter has missed two in a row. A fan says, “He’s due to make the next one.” Another says, “He’s cold; he’ll miss.” If his shots are independent, who is right?

Show answer

Neither. If the shots are independent, his chance on the next shot is still 90%, no more and no less. (In real life, a miss might come from fatigue or injury, which is worth checking, but not from being “due.”)

7. **Medium** Two cards are drawn from a 52-card deck without replacement. Are “first card is an ace” and “second card is an ace” independent? Find the probability that both are aces.

Show answer

Not independent: after one ace is taken, only 3 aces remain in 51 cards.

$$P = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221} \approx 0.45\%.$$

8. **Medium** Explain what happened at Monte Carlo on August 18, 1913, and what mistake the onlookers made.

Show answer

Black came up 26 times in a row at roulette. Onlookers thought red was “due” and staked heavily on it, but each spin was independent; the chance of red on each spin never changed, so they lost. This is the gambler’s fallacy.

9. **Hard** A 40% three-point shooter takes 5 threes. What is the probability she misses all five? About how often would that happen in 40 games of 5 threes each?

Show answer

$0.6^5 = 0.07776 \approx 7.8\%$; about $40 \times 0.078 \approx 3$ games. Cold streaks are normal.

10. **Hard** What did Gilovich, Vallone and Tversky conclude in 1985, and how did Miller and Sanjurjo change that conclusion in 2018?

Show answer

In 1985 they found that players shot no better after makes than after misses and called the hot hand a cognitive illusion. In 2018 Miller and Sanjurjo showed that in short sequences the proportion of makes after makes is biased downward, so “no difference” in the data actually means a small hot hand. Correcting for the bias revealed a modest hot-hand effect.

11. **Hard** How many rolls of two dice are needed for the probability of at least one double six to pass 50%? Test $n = 24$ and $n = 25$.

Show answer

$1 - (35/36)^{24} \approx 0.491$ and $1 - (35/36)^{25} \approx 0.506$. So 25 rolls are needed.

12. **Challenge** Repeat Worked Example 3 for 4 flips: list the 14 sequences in which a head appears in the first three flips, find the fraction of heads after heads in each, and average. (The answer is $\frac{17}{42}$; show the work.)

Show answer

The 14 fractions are: HHHH 1, HHHT $\frac{2}{3}$, HHTH $\frac{1}{2}$, HHTT $\frac{1}{2}$, HTHH $\frac{1}{2}$, HTHT 0, HTTH 0, HTTT 0, THHH 1, THHT $\frac{1}{2}$, THTH 0, THTT 0, TTHH 1, TTHT 0. Sum $= 5\frac{2}{3} = \frac{17}{3}$; average $\frac{17}{3} \div 14 = \frac{17}{42} \approx 0.405$.

COMMON MISTAKES

- Thinking a result is “due” after a streak (the gambler’s fallacy) or certain to continue (an exaggerated hot hand).
- Multiplying probabilities of events that are not independent, such as drawing cards without replacement.
- Computing “at least one” by adding probabilities. Use $1 - P(\text{none})$ instead.

Make it a game • Streak Spotter. Make a paper chip with H on one side and T on the other. Before flipping, guess the longest streak you will see in twenty flips. Flip, record, and measure. Closest guess wins. Then your parent plays a gambler who says tails is due after four heads, and you refute it. Finally, compute the chance of at least one six in three rolls.

Scripture. “Prove all things; hold fast that which is good.” (1 Thessalonians 5:21, KJV)

The hot-hand story shows scientists testing an accepted result and correcting it. Testing what we believe, and keeping what holds up, is a Christian habit as well as a scientific one.

1.4 WHAT DO YOU KNOW NOW?

OBJECTIVES

- Compute conditional probabilities from a table or a tree diagram using
$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}.$$
- Explain why switching wins the Monty Hall problem two times in three, and what assumption about the host makes it true.
- Use Bayes's idea to update a probability after new evidence, and apply it to deduction in Clue and searching in Battleship.

Warm-up. Take three playing cards: one ace and two other cards. Have a family member shuffle them face down and let you point to one. They then look at the other two and turn over a non-ace. Before you decide whether to switch, write your estimate of the chance your first card is the ace. Play 5 rounds, switching every time, and record the results.

WATCH

Monty Hall Problem - Numberphile

Numberphile · 5:30

<https://www.youtube.com/watch?v=4Lb-6rxZxx0>

WHEN TO WATCH

After the story. Professor Lisa Goldberg explains the problem with cards and a tree of cases; then read “how they figured it out.”

BACKUP VIDEO

Bayes theorem, the geometry of changing beliefs

3Blue1Brown · 15:11

<https://www.youtube.com/watch?v=HZGCoVF3YvM>

After “the mathematics” paragraph, for students ready for Bayes’s rule in full. The whole video rewards careful watching; pause at each diagram.

KEY TERMS

conditional probability

$P(A | B)$, the probability of A given that B is known to have happened. It is found by restricting the sample space to the outcomes in B .

Bayes's rule

A way to reverse a conditional probability:

$P(A | B) = \frac{P(B | A) P(A)}{P(B)}$. It updates a prior probability in light of evidence.

hunt and target

A Battleship search plan: fire spread-out shots to find a ship (hunt), then fire at neighboring squares once you hit (target), because a hit changes the probabilities around it.

tree diagram

A branching picture of a sequence of chance events, with the probability written on each branch. Multiply along a path; add the paths that make up an event.

prior and posterior

The probability before new evidence (prior) and after it (posterior).

CORE IDEAS

The story. On September 9, 1990, a reader named Craig Whitaker sent a puzzle to Marilyn vos Savant's column "Ask Marilyn" in *Parade* magazine. On a game show, you pick one of three doors. Behind one is a car; behind the others, goats. The host, who knows where the car is, opens another door to show a goat and offers you the chance to switch. Should you? Vos Savant answered: yes, switch, because switching wins two times in three. About ten thousand readers wrote to say she was wrong, among them nearly a thousand with doctorates, some of them mathematics professors. She was right. The puzzle, named for Monty Hall, host of *Let's Make a Deal*, had first been posed and solved by the statistician Steve Selvin in 1975.

How they figured it out. The readers reasoned, "Two doors are left, so each is 50–50." That would be true if the host opened a door at random. But he does not: he *knows* where the car is and always shows a goat. Your first pick is right with probability $\frac{1}{3}$, and nothing the host does changes that. In the other $\frac{2}{3}$ of games the car is behind one

of the two doors you did not pick, and the host, by opening the goat door, points you straight to it. Many doubters were convinced only when they played the game with cards or ran a computer simulation; the great mathematician Paul Erdős is said to have remained unconvinced until he watched one. The lesson is about **information**: what you know now depends on how the evidence was produced.

The mathematics. A **conditional probability** restricts the sample space to what you now know: $P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$. Thomas Bayes, an English minister, worked out

how to turn such a probability around; his essay was published in 1763, after his death, and the French mathematician Pierre-Simon Laplace developed the idea fully.

Bayes's rule is $P(A | B) = \frac{P(B | A) P(A)}{P(B)}$. In words: start with your prior belief

$P(A)$, weigh it by how well A explains the evidence, and divide by how likely the evidence was overall. A tree diagram makes it easy: multiply along each branch, then divide the branches you care about by all the branches that match what you saw.

On the field. Good game players update constantly. In *Clue*, one suspect, one item and one room are hidden in an envelope, which gives $6 \times 6 \times 9 = 324$ possible solutions; every card you see crosses some off. In *Battleship* a random shot hits a ship with probability $\frac{17}{100}$ (the five ships cover 17 of 100 squares), but once you hit, the neighboring squares become far more likely, so you *target* them. Because the smallest ship is two squares long, a *hunt* on a checkerboard pattern of 50 squares is sure to touch every ship. Coaches do the same: a pitcher who has thrown fastballs in every 3–1 count is more likely to throw one now. **The risk side:** updating on a single piece of evidence can mislead when the prior is small, so always ask how the evidence came to be.

PICTURE STUDY



Pierre-Simon Laplace, Paulin Guérin, 1838 · Oil on canvas · Palace of Versailles, France

Image: [Wikimedia Commons](#), public domain

Laplace (1749–1827) took the idea in Thomas Bayes’s essay and made it a working tool. In his *Philosophical Essay on Probabilities* (1814) he called probability theory “nothing but common sense reduced to calculation.”

Look closely

- Look at his expression. Does he look certain or thoughtful?
- Notice the gold embroidery and the medals. What do they suggest about the honor France gave its scientists?
- Look at the plain background. What does the painter want you to focus on?

Talk about it

1. Laplace called probability “common sense reduced to calculation.” How does the Monty Hall problem show that common sense sometimes needs the calculation to correct it?
2. Bayes’s rule changes a belief when new evidence arrives. Can you name a time when you changed your mind because of new evidence in a game?

PROJECT

A Thousand Doors

Play the Monty Hall game by hand, then simulate 1,000 games in a spreadsheet, and see whether switching really wins two times in three.

You will need

- Three playing cards (one ace and two others) and a helper
- A spreadsheet (Google Sheets or LibreOffice Calc)
- The scorebook

Steps

1. **Predict.** Write the fraction of games you expect to win by staying and by switching.
2. **By hand.** With a helper as host, play 20 games in which you always stay and 20 in which you always switch. The host must always turn over a non-ace that you did not pick. Tally wins.
3. **Spreadsheet.** In column A, put the car's door: `=RANDBETWEEN(1,3)`. In column B, your pick: `=RANDBETWEEN(1,3)`.
4. In column C, record whether staying wins: `=IF(A2=B2,1,0)`. Switching wins exactly when staying loses, so in column D enter `=1-C2`.
5. Fill the four formulas down to row 1001 to make 1,000 games.
6. Compute the totals: `=SUM(C2:C1001)` and `=SUM(D2:D1001)`. Divide each by 1,000.
7. Press recalculate (or change any cell) 5 times and record the five switching fractions. How much do they vary?
8. Write a short explanation of why column D is correct: why does switching win exactly when your first pick was wrong?
9. **Change the rules.** Explain (or simulate, if you can) what happens if the host opens a door at random and sometimes reveals the car.
10. **Decide.** Write the advice you would give to a contestant, in two sentences that a friend who has never studied probability would understand.

Tips

- In Google Sheets, press Ctrl+R or edit any cell to make RANDBETWEEN produce new numbers.
- Your hand games are few, so they may be far from two-thirds. That is exactly why the spreadsheet is useful.

Check your work

- ☐ I played 40 games by hand and recorded the results.
- ☐ My spreadsheet simulates 1,000 games and reports staying and switching win fractions.
- ☐ I can explain in plain words why switching wins about two-thirds of the time.

SCOREBOOK

1. Draw the full tree diagram for the Monty Hall game: car location, your pick, the host's door, and stay or switch. Write the probability on each leaf.
2. Make a two-way table of the shooting data from Worked Example 2 (close and long, make and miss) with totals, and write four different conditional probabilities from it.
3. Draw a 10×10 Battleship grid and shade the 50 squares of a checkerboard hunt pattern. Explain in one sentence why it works.

KEY FORMULAS

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(A | B) = \frac{P(B | A) P(A)}{P(B)}$$

From history. Steve Selvin posed the problem in letters to *The American Statistician* in 1975. Thomas Bayes (c. 1701–1761) was a Presbyterian minister in Tunbridge Wells; his “Essay towards solving a Problem in the Doctrine of Chances” was sent to the Royal Society by his friend Richard Price after Bayes died.

WORKED EXAMPLES

Example 1. The Monty Hall problem with three doors. Find the probability of winning the car if you always stay and if you always switch.

Show solution

1. Your first pick is the car with probability $\frac{1}{3}$ and a goat with probability $\frac{2}{3}$.
2. If you picked the car ($\frac{1}{3}$), the host opens either goat door; switching loses, staying wins.
3. If you picked a goat ($\frac{2}{3}$), the host must open the *other* goat door; the remaining door is the car, so switching wins.
4. Stay: $P(\text{win}) = \frac{1}{3}$. Switch: $P(\text{win}) = \frac{2}{3}$.

Answer: Staying wins $\frac{1}{3} \approx 33.3\%$; switching wins $\frac{2}{3} \approx 66.7\%$.

Example 2. In practice a player took 60 close shots and made 45, and 40 long shots and made 12. You are told a shot went in. What is the probability it was a close shot?

Show solution

1. Total makes: $45 + 12 = 57$.
2. We now know the shot was a make, so the sample space shrinks to the 57 makes.
3. $P(\text{close} \mid \text{make}) = \frac{45}{57} \approx 0.789$.
4. Compare: $P(\text{make} \mid \text{close}) = \frac{45}{60} = 0.75$. The two conditional probabilities are different questions.

Answer: $\frac{45}{57} \approx 78.9\%$.

Example 3. A bag holds two dice that look alike: one fair, and one weighted so that it shows a 6 half the time. You pick one at random and roll a 6. What is the probability you picked the weighted die?

Show solution

1. Prior: $P(W) = P(F) = \frac{1}{2}$.
2. Paths to a 6: weighted $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; fair $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.
3. $P(6) = \frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$.
4. Bayes: $P(W | 6) = \frac{1/4}{1/3} = \frac{3}{4}$.

Answer: The 6 raised the probability of the weighted die from $\frac{1}{2}$ to $\frac{3}{4}$.

Example 4. In Clue there are 6 suspects, 6 items and 9 rooms, and one of each is hidden in the envelope. You are dealt a hand that includes 1 suspect, 1 item and 1 room. If you guessed now, what would be your chance of being right, and how did your cards change it?

Show solution

1. Before seeing any cards: $6 \times 6 \times 9 = 324$ equally likely solutions; chance of a guess $\frac{1}{324}$.
2. Your cards cannot be in the envelope. Remaining: $5 \times 5 \times 8 = 200$ solutions.
3. Chance of a correct guess now: $\frac{1}{200} = 0.5\%$.

Answer: From $\frac{1}{324}$ to $\frac{1}{200}$; every card seen narrows the sample space.

PRACTICE

1. **Easy** In the three-door Monty Hall game, what is the probability of winning if you always stay?

Show answer

$$\frac{1}{3}.$$

2. **Easy** A card is drawn and you are told it is a face card (jack, queen or king). What is the probability it is a king?

Show answer

$$12 \text{ face cards, } 4 \text{ kings: } \frac{4}{12} = \frac{1}{3}.$$

3. **Easy** Two dice are rolled and the first shows 3. What is the probability the sum is 8?

Show answer

The second must be 5: $\frac{1}{6}$.

4. **Medium** Two dice are rolled and the sum is 8. What is the probability the first die shows 3?

Show answer

Sum 8: (2,6), (3,5), (4,4), (5,3), (6,2). One of the 5 has a first die of 3: $\frac{1}{5}$.

5. **Medium** A Monty Hall game has 100 doors. You pick one, and the host, who knows where the car is, opens 98 goat doors. What is the probability of winning if you switch?

Show answer

Your first pick is right only $\frac{1}{100}$ of the time, so switching wins $\frac{99}{100}$.

6. **Medium** In Clue, you have seen 3 suspects, 4 items and 5 rooms (none in the envelope). How many possible solutions remain?

Show answer

$$3 \times 2 \times 4 = 24.$$

7. **Medium** Use the shooting data from Worked Example 2 (close: 45 of 60; long: 12 of 40). Given that a shot missed, what is the probability it was a long shot?

Show answer

Misses: close 15, long 28, total 43. $P(\text{long} \mid \text{miss}) = \frac{28}{43} \approx 65.1\%$.

8. **Medium** Suppose the host does *not* know where the car is, opens one of the other two doors at random, and it happens to show a goat. Explain why switching now wins only half the time.

Show answer

Of the 6 equally likely (car, host-door) cases, the host reveals the car in 2, which are now ruled out. In the remaining 4, staying wins in 2 and switching in 2. Because the host's choice carried no knowledge, it gave no extra information about the other door.

9. **Hard** Three dice are in a bag; one is weighted to show a 6 one-third of the time and two are fair. You pick one at random and roll a 6. What is the probability you picked the weighted die?

Show answer

Weighted path: $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$; fair paths: $\frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$. $P(W | 6) = \frac{1/9}{2/9} = \frac{1}{2}$.

10. **Hard** Explain why a Battleship hunt that fires only at squares of one checkerboard color (50 squares) is sure to hit every ship at least once.

Show answer

Every ship is at least 2 squares long and lies in a straight row or column. Any 2 neighboring squares in a row or column have different checkerboard colors, so every ship covers at least one square of each color. Firing at all 50 squares of one color must hit every ship.

11. **Hard** Two dice are rolled and you are told the sum is even. What is the probability that both dice are even?

Show answer

Even sums happen when both are even (9 outcomes) or both are odd (9 outcomes): 18 outcomes. $P = \frac{9}{18} = \frac{1}{2}$.

12. **Challenge** A four-door Monty Hall game: one car, three goats. You pick a door; the host, who knows, opens one goat door; you then switch to one of the two remaining unopened doors at random. What is your probability of winning? Compare with staying.

Show answer

Your first pick is a goat with probability $\frac{3}{4}$; then the car is behind one of the 2 doors you may switch to, and you choose it with probability $\frac{1}{2}$.

$P(\text{win by switching}) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = 37.5\%$, better than staying ($\frac{1}{4} = 25\%$).

COMMON MISTAKES

- Assuming two remaining options are always 50–50. They are equally likely only if nothing about how they were left over favors one.
- Confusing $P(A | B)$ with $P(B | A)$. The probability that a made shot was close is not the probability that a close shot is made.
- Forgetting to divide by $P(B)$ in Bayes's rule, which leaves the answer too small.

Make it a game • Bag of Beans Tree. Fill a bag with 6 red beans and 4 white beans. Draw two without replacement, and sketch the tree diagram with probabilities. Predict the chance of a second red given the first was red. Then run twenty trials, tally them, and compare to your formula. Beat your parent's estimate with the closest prediction.

1.5 THE LAW OF LARGE NUMBERS

OBJECTIVES

- State the law of large numbers and explain why small samples can mislead while large samples settle down.
- Compute the standard error of a proportion, $\sqrt{p(1-p)/n}$, and use it to judge whether a result is surprising.
- Run a Monte Carlo simulation in a spreadsheet and compare the result with exact probability.

Warm-up. Flip a coin 10 times and write the fraction of heads. Then guess: if everyone in your family did the same, would all the fractions be close to 0.5? Would 100 flips each give fractions closer together? Write your guesses before you begin the lesson.

WATCH

Law of large numbers | Probability and Statistics | Khan Academy

Khan Academy · 8:59

<https://www.youtube.com/watch?v=VpuN8vCQ--M>

WHEN TO WATCH

After “the mathematics” paragraph. Sal Khan explains why the average of many trials approaches the expected value, and why the law of large numbers does not mean that results “even out” after a streak.

BACKUP VIDEO

Monte Carlo Simulation

MarbleScience · 10:05

<https://www.youtube.com/watch?v=7ESK5SaP-bc>

Before the project. It shows a Monte Carlo estimate of π with random points; stop at about 6:30, where the video turns to computer graphics.

KEY TERMS

relative frequency

The fraction of trials in which an event happened: $\text{count} \div \text{number of trials}$.

law of large numbers

As the number of independent trials grows, the relative frequency of an event gets closer and closer to its probability.

standard error

A measure of how much a sample proportion typically scatters around the true probability: $\sqrt{p(1-p)/n}$.
About 95% of samples land within 2 standard errors.

simulation

Imitating a chance process many times, by hand or by computer, to estimate a probability.

Monte Carlo method

Finding a quantity approximately by running many random trials and averaging, named in the 1940s at Los Alamos.

CORE IDEAS

The story. For about twenty years, the Swiss mathematician Jakob Bernoulli of Basel worked on a question that seemed obvious: if you draw stones from a jar with 3,000 white and 2,000 black, does the fraction of white you draw really settle near $3/5$ as you draw more? He proved that it does and called it his “golden theorem.” He died in 1705 before he could finish his book, and his nephew Nicolaus published *Ars Conjectandi* (*The Art of Conjecturing*) in 1713. Two and a half centuries later, in 1946, the mathematician Stanislaw Ulam was recovering from a serious illness and playing a card game of solitaire. He wondered what fraction of games could be won. The combinations were far too many to count, so he had a better idea: lay out a hundred games, count the wins, and let the law of large numbers do the rest. With John von Neumann he turned the idea into a method for the new electronic computers at Los Alamos, and a colleague, Nicholas Metropolis, named it the **Monte Carlo method**.

How they figured it out. People had always sensed that long experience is more trustworthy than short experience, but nobody knew how much more. Bernoulli proved that the chance of the relative frequency being far from the true probability shrinks toward zero as the number of trials grows, and he even estimated how many trials were needed for a given accuracy. Later mathematicians, especially Abraham de Moivre in 1733, found the exact shape of the scatter: it shrinks in proportion to $\frac{1}{\sqrt{n}}$. Ulam's contribution was practical: when exact counting is impossible, a computer that runs thousands of random trials can estimate any probability, and the law of large numbers guarantees the estimate gets better as the trials pile up.

The mathematics. If an event has probability p and we run n independent trials, the relative frequency $\hat{p} = \text{count}/n$ has a typical scatter, the **standard error**, of $SE = \sqrt{\frac{p(1-p)}{n}}$. About 95% of the time $\hat{p}$ lands within $2 SE$ of p . For a fair coin, 100 flips give $SE = \sqrt{0.25/100} = 0.05$, so anything from about 40 to 60 heads is ordinary; 10,000 flips give $SE = 0.005$, so the fraction should be within 0.49 to 0.51. Because of the square root, to cut the scatter in half you need *four times* as many trials. Notice what the law does *not* say: it does not say a streak of heads will be balanced by extra tails. It says the streak is gradually swamped by the many trials that follow.

On the field. Small samples lie. A hitter who goes 9 for 20 in April is batting .450, but with $n = 20$ his standard error is about 0.11, so his true ability could easily be .300. A three-point shooter who makes 4 of 7 in one game has told you almost nothing; one who makes 350 of 1,000 over three seasons has told you a lot. Wise coaches and scouts wait for the sample to be large before they decide, and they look at the size of the sample before they look at the percentage. **The risk side:** making a big decision from a small sample, such as trading a player after a hot week, is trusting noise.

PICTURE STUDY



Jakob Bernoulli, Niklaus Bernoulli, 1687 · Oil on canvas · Basel, Switzerland

Image: [Wikimedia Commons](#), public domain

Jakob Bernoulli (1655–1705) was the first of a remarkable family of Swiss mathematicians; his brother Johann and his nephew Daniel appear later in this course. His father wanted him to be a minister, and he studied theology before turning to mathematics. His portrait was painted by his brother Niklaus, an artist.

Look closely

- Look at the clothing and wig. What do they tell you about Bernoulli's place in Basel society?
- Look at his face. Does he seem patient or quick?
- Find the globe under his hand and the inscription ("Math. P.P." means professor of mathematics). What object would you add to show the law of large numbers?

Talk about it

1. Bernoulli worked on one theorem for about twenty years. What does that tell you about the kind of patience the law of large numbers itself teaches?
2. Bernoulli called his book *The Art of Conjecturing*. Why is conjecturing (making a careful guess from evidence) an art and not only a calculation?

PROJECT

Small Samples Lie, Big Samples Settle

You will watch the law of large numbers happen: first with your own coin, then with a spreadsheet that rolls dice thousands of times in a second.

You will need

- A coin
- A spreadsheet (Google Sheets or LibreOffice Calc)
- Graph paper and the scorebook
- A calculator

Steps

1. **Predict.** Write how far from 0.5 you expect the fraction of heads to be after 10, 50 and 200 flips.
2. Flip a coin 200 times. After every 10 flips, record the running total of heads and the running fraction (total heads $\div$ flips so far).
3. Graph the running fraction (vertical axis) against the number of flips (horizontal axis). Draw a horizontal line at 0.5 and dashed lines at $0.5 \pm 2\sqrt{0.25/n}$ for $n = 10, 50, 100$ and 200.
4. **Spreadsheet.** In column A, enter `=RANDBETWEEN(1,6)+RANDBETWEEN(1,6)` and fill down to 3,601 rows (3,600 rolls).
5. In a second column, count the 7s in the first 36, 360 and 3,600 rolls with `COUNTIF`, and compute each fraction.
6. Compare each fraction with $\frac{1}{6} \approx 0.167$ and with the 2-standard-error range for that n .
7. Recalculate 5 times and record the five fractions for each sample size. Which sample size gives fractions closest together?
8. Make a frequency table of all eleven sums from the 3,600 rolls and compare with your 360 hand rolls from Lesson 1.2.
9. **Decide.** A friend says a new shooter is “a 60% shooter” after one game of 10 shots. Using your graph, write a reply.

Tips

- Running fractions jump around early and calm down later; that is the whole lesson in one graph.
- Recalculate in Google Sheets by editing any empty cell.

Check your work

- ☐ My running-fraction graph shows 200 real flips with the 2-standard-error lines.
- ☐ My spreadsheet compares the fraction of 7s at 36, 360 and 3,600 rolls over 5 recalculations.
- ☐ I wrote a reply that uses sample size, not just the percentage.

SCOREBOOK

1. Table: sample size, number of 7s, fraction, $\frac{1}{6}$, difference, standard error, within 2 SE (yes or no), for five recalculations at each size.
2. Paste or redraw your running-fraction graph and label where the fraction first stays between 0.4 and 0.6.
3. **What I would decide and why:** a coach must choose between a player who is 6 for 10 this week and one who is 120 for 300 this season. Use standard errors to explain your choice.

KEY FORMULAS

$$\hat{p} = \frac{\text{count}}{n}$$
$$SE = \sqrt{\frac{p(1-p)}{n}}$$

about 95% of samples: $p - 2 SE \leq \hat{p} \leq p + 2 SE$

From history. Abraham de Moivre, a French Huguenot who fled to London, found the bell-shaped curve that describes the scatter of large samples in 1733. Ulam and Metropolis published “The Monte Carlo Method” in the *Journal of the American Statistical Association* in 1949. Today the same method is used in weather forecasting, engineering and the win-probability charts of Lesson 4.4.

WORKED EXAMPLES

Example 1. A fair coin is flipped 100 times, and then 10,000 times. Find the standard error of the fraction of heads each time, and the range in which about 95% of results fall.

Show solution

1. $n = 100$: $SE = \sqrt{\frac{0.5 \times 0.5}{100}} = \sqrt{0.0025} = 0.05$.

2. 95% range: 0.5 ± 0.10 , that is 0.40 to 0.60 (40 to 60 heads).

3. $n = 10,000$: $SE = \sqrt{\frac{0.25}{10,000}} = 0.005$; range 0.49 to 0.51.

4. 100 times as many flips makes the scatter $\sqrt{100} = 10$ times smaller.

Answer: SE 0.05 (40–60 heads) for 100 flips; SE 0.005 (4,900–5,100 heads) for 10,000 flips.

Example 2. A hitter starts the season 9 for 20 (.450). Suppose his true ability is .300. Is 9 for 20 surprising? What about 135 for 300?

Show solution

1. With $p = 0.3$ and $n = 20$: $SE = \sqrt{\frac{0.3 \times 0.7}{20}} = \sqrt{0.0105} \approx 0.102$.

2. Two standard errors: 0.3 ± 0.205 , about .095 to .505. So .450 is well within the ordinary range.

3. With $n = 300$: $SE = \sqrt{0.21/300} \approx 0.026$; range about .247 to .353.

4. 135 for 300 is .450, far outside that range. That would be strong evidence he is better than .300.

Answer: 9 for 20 is not surprising for a .300 hitter; 135 for 300 would be.

Example 3. Set up a spreadsheet simulation of 3,600 rolls of two dice, and find the expected number of 7s and its typical scatter.

Show solution

1. In A2 enter =RANDBETWEEN(1,6)+RANDBETWEEN(1,6) and fill down to A3601.
2. Count the 7s: =COUNTIF(A2:A3601,7).
3. Expected count: $3,600 \times \frac{1}{6} = 600$.
4. Standard error of the count:
$$n \cdot SE = \sqrt{np(1-p)} = \sqrt{3,600 \times \frac{1}{6} \times \frac{5}{6}} = \sqrt{500} \approx 22.4.$$
5. About 95% of runs should give between $600 - 45 = 555$ and $600 + 45 = 645$ sevens.

Answer: Expect about 600 sevens, usually between about 555 and 645.

PRACTICE

1. **Easy** You flip a coin 10 times and get 7 heads. What is the relative frequency of heads? Does this prove the coin is unfair?

Show answer

0.7. No: with only 10 flips, 7 heads is common (the standard error is about 0.16).

2. **Easy** Find the standard error of the fraction of heads in 400 flips of a fair coin.

Show answer

$$\sqrt{0.25/400} = 0.025.$$

3. **Easy** To cut the standard error in half, how many times as many trials do you need?

Show answer

Four times as many, because $SE \propto 1/\sqrt{n}$.

4. **Medium** Write a spreadsheet formula that rolls one die, and a formula that counts how many 6s appear in cells A2 through A601.

Show answer

=RANDBETWEEN(1,6) and =COUNTIF(A2:A601,6).

5. **Medium** In 3,600 rolls of two dice, how many sums of 2 do you expect? Find the standard error of that count.

Show answer

$$\text{Expected } 3,600 \times \frac{1}{36} = 100; \sqrt{3,600 \times \frac{1}{36} \times \frac{35}{36}} = \sqrt{97.2} \approx 9.9.$$

6. **Medium** A Monte Carlo estimate of π throws 1,000 random points into a square; 788 land inside the quarter circle. Since the fraction should be $\pi/4$, estimate π .

Show answer

$$\pi \approx 4 \times 0.788 = 3.152.$$

7. **Medium** State the law of large numbers in your own words, and explain why it does not mean that after 5 heads, tails is more likely.

Show answer

Over many independent trials, the fraction of times an event happens gets closer to its probability. It works not by making up for streaks but by swamping them: 5 extra heads hardly matter among thousands of flips. Each flip is still 50–50.

8. **Medium** Why did Ulam decide to lay out a hundred games of solitaire instead of calculating his chance of winning?

Show answer

The number of possible arrangements of the cards was far too large to count exactly, but by the law of large numbers the fraction of wins in many real (or simulated) games would estimate the probability, and more games would give a better estimate.

9. **Hard** A free-throw shooter's true rate is 75%. Find the standard error of her percentage over 20 attempts and over 500 attempts.

Show answer

$$\sqrt{0.75 \cdot 0.25/20} = \sqrt{0.009375} \approx 0.097 \text{ (about 9.7 percentage points);}$$
$$\sqrt{0.1875/500} \approx 0.019 \text{ (about 1.9 points).}$$

10. **Hard** How many flips of a fair coin are needed for the standard error of the fraction of heads to be 0.01 or less?

Show answer

$$\sqrt{0.25/n} \leq 0.01 \Rightarrow 0.25/n \leq 0.0001 \Rightarrow n \geq 2,500.$$

11. **Hard** Bernoulli's jar held 3,000 white and 2,000 black stones. If you draw 600 times with replacement, find the expected number of white stones and the 95% range of the fraction white.

Show answer

$$p = 0.6; \text{ expected } 360. SE = \sqrt{0.24/600} = 0.02; \text{ range about } 0.56 \text{ to } 0.64.$$

12. **Challenge** Shooter A made 18 of 40 threes (45%); shooter B made 160 of 400 (40%). Using standard errors, can you say A is the better shooter?

Show answer

A: $SE = \sqrt{0.45 \cdot 0.55/40} \approx 0.079$, so A's true rate could be anywhere from about 29% to 61%. B: $SE = \sqrt{0.4 \cdot 0.6/400} \approx 0.024$, about 35% to 45%. The ranges overlap widely, so A's small sample does not show A is better. B's 40% is far more reliable.

COMMON MISTAKES

- Believing the law of large numbers makes outcomes “even out” after a streak. It works by dilution, not correction.
- Judging a percentage without asking how many trials it is based on.
- Forgetting the square root: 4 times as many trials halves the scatter; it does not divide it by 4.

Make it a game • Settle Down Chart. Flip a paper chip ten times and record the share of heads. Then flip a hundred times together with your parent, taking turns, and plot the running fraction on a graph every ten flips. Watch it wobble, then settle. Before each stage, each of you predicts the final fraction. Closest to the real relative frequency wins a point.

UNIT 1

CHANCE: THE MATHEMATICS OF UNCERTAINTY

Probability was born in 1654, when two Frenchmen who never met wrote letters about how to divide the prize of an unfinished game. This unit follows that story and the people who carried it forward: Pascal and Fermat with the problem of points, Galileo counting the falls of three dice, the dice players of Monte Carlo and the psychologists and economists who argued about the hot hand, Marilyn vos Savant and the thousands of readers who told her she was wrong about Monty Hall, and Jakob Bernoulli, whose law of large numbers Stanislaw Ulam turned into a computer method.

You will learn to list sample spaces, count outcomes with the multiplication principle, permutations and combinations, multiply probabilities of independent events, compute conditional probabilities with tables and trees, update with Bayes's rule, and judge how far a sample can stray with the standard error. Your projects are real experiments: playing out interrupted games, rolling two dice 360 times, testing your own shooting for a hot hand, simulating 1,000 Monty Hall games and watching the law of large numbers in a spreadsheet.

PREREQUISITES

- Pre-algebra: fractions, decimals and percents, and converting between them
- Order of operations, exponents and square roots on a calculator
- Reading and making a bar graph

MASTERY CHECKLIST

- ☐ I can list a sample space and find a probability as a fraction, a decimal and a percent.
- ☐ I can divide the prize of an interrupted game fairly by Fermat's or Pascal's method.
- ☐ I can count outcomes with the multiplication principle, permutations and combinations, and explain why 7 is the most common sum of two dice.
- ☐ I can multiply probabilities of independent events, use the complement for "at least one," and explain the gambler's fallacy.
- ☐ I can compute a conditional probability from a table or tree, explain the Monty Hall problem, and update a probability with Bayes's rule.
- ☐ I can compute a standard error and explain why small samples mislead.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Khan Academy: Statistics and Probability**: Free. Short videos and practice exercises on probability, counting and conditional probability that match Lessons 1.1–1.5. Good for extra practice after each practice set.
- **MIT OpenCourseWare: Probabilistic Systems Analysis and Applied Probability**: Free. MIT’s introductory probability course with lecture videos, recitations and problem sets. Far beyond this course, but the early lectures on counting, conditional probability and Bayes’s rule reward a strong 11th or 12th grader.
- **CATAN**: The publisher’s page for the family trading game. Its number tokens and pips make Lesson 1.2’s two-dice counting visible on every turn. Ages 10+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

1.1	The Problem of Points	3 sessions of 50 minutes
1.2	Counting the Ways	3 sessions of 50 minutes
1.3	Dice Have No Memory	3 sessions of 50 minutes
1.4	What Do You Know Now?	3 sessions of 50 minutes
1.5	The Law of Large Numbers	3 sessions of 50 minutes

UNIT 1 TEST

Calculators are allowed. Show your work for every calculation: write the sample space, count or formula you used, and give probabilities as fractions and as decimals or percents unless told otherwise. Answer the concept questions in complete sentences. The test is worth 100 points.

1. **5 pts**

A card is drawn from a standard deck. Find the probability that it is a face card (jack, queen or king), as a fraction, a decimal and a percent.

2. (7 pts)

Two players match to 4 wins. The game stops with A leading 3–2. Divide 100 points fairly, showing Fermat’s method.

3. (7 pts)

Two players match to 5 wins; A leads 4–2. Divide 80 points fairly.

4. (7 pts)

Two dice are rolled. Find the probability that the sum is 5 or 9.

5. (6 pts)

A coach chooses 4 players from 10 for a relay team, then sets the running order. How many teams (ignoring order) are possible? How many teams with running orders?

6. (6 pts)

A Catan spot touches hexes numbered 6, 9 and 11. Another touches 8, 4 and 2. Which collects more often, and how many pips does each have?

7. (6 pts)

A 75% free-throw shooter takes 3 shots, independently. Find the probability she makes all three and the probability she makes at least one.

8. (6 pts)

Find the probability of at least one 6 in 4 rolls of a die, and explain how this answered part of de Méré’s question.

9. (7 pts)

Explain the gambler’s fallacy with the Monte Carlo example of 1913, and explain why streaks in sport are not proof that something has changed.

10. (7 pts)

Explain the bias Miller and Sanjurjo discovered, using the three-flip example, and what it meant for the 1985 hot-hand study.

11. (6 pts)

In the Monty Hall game with 3 doors, what is the probability of winning by switching? Explain in two sentences.

12. (7 pts)

In practice a shooter took 50 layups (made 40) and 50 long shots (made 15). Given that a shot was made, what is the probability it was a layup? Given that it was a layup, what is the probability it was made?

13. (7 pts)

A bag holds one fair coin and one two-headed coin. You pick one at random and flip it: heads. What is the probability you picked the two-headed coin?

14. (6 pts)

Find the standard error of the fraction of heads in 900 flips of a fair coin, and the range in which about 95% of results fall.

15. (10 pts)

Short essay. In about 150 words, explain how this unit's mathematics helps a person make better decisions in a game or sport. Use at least two of these ideas: counting equally likely outcomes, independence, conditional probability, the law of large numbers. Include one example of a mistake the mathematics helps you avoid.

ANSWER KEY

1. $\frac{12}{52} = \frac{3}{13} \approx 0.231 \approx 23.1\%$.
2. A needs 1, B needs 2: at most 2 rounds (AA, AB, BA, BB). B wins only BB. A gets 75 points, B 25.
3. A needs 1, B needs 3: B must win 3 straight, $\frac{1}{8}$. A gets $\frac{7}{8} \times 80 = 70$ points, B gets 10.
4. Sum 5: 4 ways; sum 9: 4 ways. $\frac{8}{36} = \frac{2}{9} \approx 22.2\%$.
5. Teams: $\binom{10}{4} = 210$. With orders: $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$.
6. First: $5 + 4 + 2 = 11$; second: $5 + 3 + 1 = 9$. The first collects more often.
7. All three: $0.75^3 \approx 0.422$. At least one: $1 - 0.25^3 = 1 - 0.015625 \approx 0.984$.
8. $1 - (5/6)^4 \approx 0.518$. It is slightly more than one half, while at least one double six in 24 rolls of two dice is only about 0.491, so the two games were not equal as de Méré had assumed.
9. When black came up 26 times, people believed red was due, but each spin was independent and red's chance never changed. In sport, a shooter with a fixed chance will still have long runs of makes and misses by chance alone, so a streak by itself does not show that skill has changed.
10. In 3-flip sequences, the average fraction of heads after a head is $\frac{5}{12}$, not $\frac{1}{2}$. So in short sequences a random shooter appears to shoot worse after makes. The 1985 study found equal rates after makes and misses; corrected for the bias, that equality is evidence of a modest hot hand.
11. $\frac{2}{3}$. Your first pick is wrong two times in three, and in those cases the host, who knows where the car is, must leave the car behind the other closed door, so switching wins.
12. $P(\text{layup} \mid \text{make}) = \frac{40}{55} \approx 72.7\%$; $P(\text{make} \mid \text{layup}) = \frac{40}{50} = 80\%$.
13. Two-headed path: $\frac{1}{2} \cdot 1 = \frac{1}{2}$; fair path: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. $P = \frac{1/2}{3/4} = \frac{2}{3}$.
14. $\sqrt{0.25/900} = \frac{0.5}{30} \approx 0.0167$; range about 0.467 to 0.533.
15. Answers vary. A strong essay uses two or more ideas correctly with a concrete example, such as choosing Catan spots by pips, not benching a good shooter after a slump that chance explains, switching in Monty Hall, or waiting for a large sample before judging a player, and it names a mistake such as the gambler's fallacy or trusting a small sample.

PORTFOLIO PIECE: A DICE LABORATORY REPORT

Choose a question about dice or coins that you cannot answer instantly, such as: How many rolls of one die does it take, on average, to see all six faces? How often do three dice show a sum of 10 or more? In 5 rolls of five dice, how often is there at least one five of a kind? First calculate (or estimate) the answer using this unit's counting rules. Then test it twice: with at least 200 real trials by hand, and with at least 2,000 simulated trials in a spreadsheet. Write a two-page report with your question, your calculation, your data tables, a bar graph or running-fraction graph, a comparison of your hand results, your simulation and the mathematics using the standard error, and a conclusion about what you learned and what you would do differently.

LOOK FOR	WHAT IT MEANS
Question and calculation	A clear question with a correct calculation or a careful explanation of why exact calculation is hard.
Real and simulated data	At least 200 hand trials and 2,000 spreadsheet trials, recorded honestly in neat tables.
Comparison with mathematics	Results compared with the calculation using relative frequency and the standard error; differences explained.
Graph	A labeled bar graph or running-fraction graph that shows the comparison clearly.
Conclusion	A thoughtful conclusion that states what was learned, its limits, and one improvement.

UNIT 2

EXPECTED VALUE: WEIGHING REWARD AND RISK

In 1657 Christiaan Huygens asked what a chance is worth and answered with the idea that runs through the rest of this course: expected value, the long-run average of a choice. This unit puts it to work. You will compute points per attempt for twos and threes, extra points and two-point tries; measure the spread of a strategy with variance and see why David chose a different contest and why a team that trails should seek variance; prove that the house's games are built to lose for the player and that insurance is their wise mirror image; and meet Daniel Bernoulli's answer to the St. Petersburg paradox, the risk of ruin and the wisdom of spreading risk.

The projects measure your own shooting percentages from two distances, test which strategy reaches a high target more often, simulate a house game and an insurance pool in points, and play the St. Petersburg game. Every project keeps score in points, never money.

PREREQUISITES

- Unit 1: probability as a fraction, decimal and percent; independence; the law of large numbers and the standard error
- Pre-algebra: weighted averages, square roots and exponents
- Optional (challenge problems): natural logarithms on a calculator

MASTERY CHECKLIST

- ☐ I can compute an expected value from a probability table and explain it as a long-run average.
- ☐ I can compute points per attempt and break-even percentages for shots and conversions.
- ☐ I can compute variance and standard deviation and explain when to increase or reduce variance.
- ☐ I can show that a casino or lottery game has negative expected value and explain why insurance is different.
- ☐ I can explain the St. Petersburg paradox, Bernoulli's utility, the risk of ruin and the value of spreading risk.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Khan Academy: Random variables and expected value**: Free. Videos and practice on expected value, variance and standard deviation that match Lessons 2.1, 2.3 and 2.4.
- **Basketball-Reference: NBA league averages**: Free data. Season-by-season league averages of three-point attempts, three-point percentage and free-throw percentage since 1946, used in Lesson 2.2. Contains advertising; a parent should preview it.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

2.1	Huygens and the Value of a Chance	3 sessions of 50 minutes
2.2	Points per Shot	3 sessions of 50 minutes
2.3	Same Average, Different Risk	3 sessions of 50 minutes
2.4	The House Always Wins	3 sessions of 50 minutes
2.5	Risk of Ruin	3 sessions of 50 minutes

UNIT 2 TEST

Calculators are allowed. Show your work for every calculation: write the probability table or formula, substitute, and give the answer with its unit (points, coins or percent). Answer the concept questions in complete sentences. The test is worth 100 points.

1. **5 pts**

A spinner scores 0, 3 or 6 points with probabilities 0.5, 0.3 and 0.2. Find the expected score.

2. **6 pts**

Use Huygens's rule: 4 equally likely chances of 9 points and 2 chances of 3 points. What is the chance worth?

3. (6 pts)

A player makes 46% of twos and 35% of threes. Compute points per attempt for each and say which is the higher-percentage shot.

4. (6 pts)

What three-point percentage equals a 51% two-point percentage?

5. (6 pts)

A kicker makes 95% of extra points. What two-point success rate breaks even? If the team converts 52%, which should it choose, other things equal?

6. (6 pts)

A 75% free-throw shooter goes to the line for two shots. Find the expected points and compare with a 34% three.

7. (8 pts)

Find the mean, variance and standard deviation of a shot worth 3 points made 40% of the time.

8. (7 pts)

Two strategies have the same expected points. Explain which a trailing team should choose and which a leading team should choose, and why.

9. (6 pts)

A game costs 3 points and pays 20 points with probability $\frac{1}{8}$. Find the expected net result per play and the house edge.

10. (8 pts)

Explain, using the law of large numbers, why a casino is nearly certain to profit even though some players leave as winners.

11. (8 pts)

A family's car is worth 20,000 dollars. Suppose the chance of a total loss in a year is 1 in 200, and insurance against it costs 150 dollars. Find the expected net cost of insuring and explain when it is wise.

12. (6 pts)

In the St. Petersburg game with a bank cap of 2^{15} , find the expected value.

13. 6 pts

A cargo is split among 5 ships, each independently lost with probability 0.1. Find the chance of losing everything and the chance of losing nothing.

14. 6 pts

A plan carries a 3% chance of ruin each time. What is the chance of surviving 30 independent uses of it?

15. 10 pts

Short essay. In about 150 words, explain the difference between the high-percentage shot and the safe shot. Use expected value, variance and risk of ruin, and give one example from a sport and one from family or business life.

ANSWER KEY

1. $0 + 0.9 + 1.2 = 2.1$ points.
2. $\frac{4(9) + 2(3)}{6} = \frac{42}{6} = 7$ points.
3. Two 0.92; three 1.05. The three.
4. $\frac{2}{3}(0.51) = 0.34$ (34%).
5. Break-even 47.5%. At 52% the two-point try (1.04 vs 0.95) has more expected points.
6. 1.5 against 1.02. The free throws are worth more.
7. $\mu = 1.2$; $\text{Var} = 9(0.4)(0.6) = 2.16$; $\sigma \approx 1.47$.
8. The trailing team should choose the higher-variance strategy, because it needs a large result that only a wide spread can give. The leading team should choose the lower-variance strategy, because it needs only to avoid a large swing against it.
9. $\frac{20}{8} - 3 = -0.5$ points per play; edge $0.5/3 \approx 16.7\%$.
10. Each play has a small negative expected value for the player. Over millions of plays the house's total gain grows with the number of plays while the spread grows only with its square root, so the house's average gain is nearly certain; individual players play few times, so luck dominates their results.
11. Expected loss $20,000/200 = 100$; net cost 50 dollars a year. It is wise when losing the car would be a heavy blow the family could not easily absorb: insurance trades a small expected cost for protection from ruin.
12. $15 + 1 = 16$ coins.
13. Everything: $0.1^5 = 0.00001$; nothing: $0.9^5 \approx 0.590$.
14. $0.97^{30} \approx 0.401$, about 40%.
15. Answers vary. A strong essay defines the high-percentage shot as the one with the highest expected value, explains that the safe shot has low variance and may have lower expected value, says the right choice depends on the goal (needing a big result versus protecting a lead) and on whether any outcome would be ruinous, and gives two concrete examples.

PORTFOLIO PIECE: AN EXPECTED-VALUE STRATEGY GUIDE FOR A FAMILY GAME

Choose a family game with dice, cards or shooting that involves real choices (for example Yahtzee, Catan, a backyard shooting game or a card game played for points). Identify three decisions a player faces repeatedly. For each, list the options, estimate their probabilities (by counting, by simulation or by recording at least 30 real turns), and compute the expected points and the spread of each option. Then write a two-page strategy guide for a younger sibling: what to do when ahead, what to do when behind, and which risks are never worth taking. Include your tables, at least one graph, and a section on what your guide cannot predict.

LOOK FOR	WHAT IT MEANS
Decisions and options	Three real, recurring decisions clearly described, each with its options.
Probabilities and expected values	Probabilities found by counting, simulation or recorded data, with correct expected-value and spread calculations shown.
Ahead and behind	Advice that changes sensibly with the score, using the idea of increasing or reducing variance.
Clarity for a younger reader	Plain language, a clear table or chart, and examples a younger sibling could follow.
Honest limits	A thoughtful section on what the guide cannot predict and where the data were thin.

2.1 HUYGENS AND THE VALUE OF A CHANCE

OBJECTIVES

- Tell how Christiaan Huygens wrote the first printed book on probability in 1657 and what he meant by the value of a chance.
- Compute an expected value $E[X] = \sum p_i x_i$ from a table of outcomes and probabilities.
- Explain expected value as the long-run average, and use points per attempt to compare choices.

Warm-up. Roll one die 12 times and write the results. Add them and divide by 12. Before you finish, guess what the average will be close to. Is it possible for one roll to show the average you got?

WATCH

Mean (expected value) of a discrete random variable | AP Statistics | Khan Academy

Khan Academy · 4:31

<https://www.youtube.com/watch?v=qafPcWNUiM8>

WHEN TO WATCH

After “the mathematics” paragraph. It shows how to multiply each value by its probability and add, the same calculation Huygens made in 1657.

KEY TERMS

random variable

A number whose value depends on chance, such as the points scored on one shot or the number on a die.

points per attempt

The expected value of one attempt in a game: the points it is worth times the probability it succeeds.

expected value

The probability-weighted average of the possible values, $E[X] = \sum p_i x_i$. It is the long-run average per trial.

probability distribution

A table or list of every possible value of a random variable with its probability; the probabilities add to 1.

CORE IDEAS

The story. In the summer of 1655 a 26-year-old Dutch scientist, Christiaan Huygens, visited Paris. He heard that Pascal and Fermat had solved problems about games of chance, but he did not meet them and could not learn their methods. So he worked the problems out for himself at home in The Hague. His short book *De ratiociniis in ludo aleae* (*On Reasoning in Games of Chance*) was printed in 1657 as an appendix to his teacher Frans van Schooten's mathematics textbook. It was the first printed book on probability, and for half a century it was the textbook every student of chance learned from. Huygens later became famous for the pendulum clock, the wave theory of light and the discovery of Saturn's moon Titan.

How they figured it out. Pascal and Fermat had divided prizes; Huygens asked a more general question: what is a chance *worth*? His third proposition answers it. If I have p chances of getting a and q chances of getting b , all equally likely, then my chance is worth $\frac{pa + qb}{p + q}$. He justified it by imagining a fair arrangement among players that would give exactly those chances, so the value of a chance is what a player could fairly trade it for. His word, translated into Latin as *expectatio*, gives us the modern name **expected value**. Earlier writers had argued about fairness case by case; Huygens gave one rule that settles every case.

The mathematics. If a random variable X takes values $x_1, x_2, \dots$ with probabilities $p_1, p_2, \dots$, its expected value is $E[X] = \sum p_i x_i = p_1 x_1 + p_2 x_2 + \dots$. One die has $E = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$, even though no roll ever shows 3.5. Expected value is not what you expect on one try; it is the **long-run average** per try, which the law of large numbers (Lesson 1.5) guarantees you will approach over many tries. Expected values add: two dice have $E = 3.5 + 3.5 = 7$. In sport the most useful form is **points per attempt**: points if successful times the probability of success.

On the field. Points per attempt is how analysts compare shots, plays and strategies. A backyard shot worth 1 point that you make 80% of the time is worth 0.8 points per attempt; a harder shot worth 3 points that you make 30% of the time is worth 0.9. Over 100 attempts the harder shot should score about 10 more points, although on any one try it usually misses. The **high-percentage shot** is the one with the highest expected points, not the one that goes in most often. **The risk side:** expected value is a long-run idea. When you have only one try, or when one bad result would be a

disaster, you must also ask how widely the results spread, which is the subject of Lessons 2.3 and 2.5.

PICTURE STUDY



Portrait of Christiaan Huygens, Caspar Netscher, 1671 · Oil on canvas · Rijksmuseum Boerhaave, Leiden
Image: [Wikimedia Commons](#), public domain

Netscher painted Huygens at about forty-two, when he was the most celebrated scientist in Europe and a founding member of the French Academy of Sciences. The young man who wrote the first book of probability had become the inventor of the pendulum clock.

Look closely

- Look at his clothing. Is he dressed for work or for display?
- Look at the setting: a dark background, a stone ledge, a cushion. Why might a painter show a scientist at rest instead of in a workshop?
- Look at his face and posture. What kind of man does the painter want you to see?

Talk about it

1. Huygens built the most accurate clocks of his age. How is measuring time carefully like measuring the value of a chance carefully?
2. Huygens worked out the answers alone after hearing that others had solved them. What does that tell you about him as a learner?

PROJECT

Huygens's Table

You will design three small dice games, calculate the expected score of each before you play, and then play each game enough times to see whether the long-run average matches your calculation.

You will need

- Two dice
- The scorebook and a calculator

Steps

1. **Game A:** roll one die; a 1 scores 0, anything else scores its number. **Game B:** roll two dice; doubles score 10, anything else scores 1. **Game C:** roll one die, with the option of one reroll (keep a 4, 5 or 6).
2. **Predict.** Write each game's probability table and expected value (they are worked in this lesson: 3.33, 2.5 and 4.25).
3. Play Game A 60 times, recording each score. Compute the average after 10, 30 and 60 rounds.
4. Play Game B 60 times and Game C 60 times in the same way.
5. Make a table: game, expected value, average after 10, 30 and 60 rounds.
6. Graph the running average of one game against the number of rounds, with a horizontal line at the expected value.
7. Which game came closest to its expected value after 60 rounds? Which game's scores were most spread out? (Look at the highest and lowest single scores.)
8. **Design.** Invent a Game D of your own whose expected value is exactly 5 points per round, and prove it with a table.
9. **Decide.** If you could play only one round of A, B or C, which would you choose? If you could play 100 rounds? Explain why the answers might differ.

Tips

- Record every score, not just the totals, so you can see the spread.
- Game B is the most surprising: most rounds score 1, a few score 10.

Check your work

- ☐ I computed each game's expected value before playing.
- ☐ I played each game 60 times and compared running averages with the expected values.
- ☐ I designed a game with expected value 5 and proved it.

SCOREBOOK

1. Write Huygens's table for each game: value, probability, value $\times$ probability, and the sum.
2. Keep a running-average table for one game at rounds 10, 20, 30, 40, 50 and 60.
3. **What I would decide and why:** you are offered a sure 3 points or one roll of Game B. Compute both and write which you would take in a game you are winning, and in one you are losing.

KEY FORMULAS

$$E[X] = \sum_i p_i x_i$$

$$\text{Huygens: } \frac{pa + qb}{p + q}$$

$$\text{points per attempt} = (\text{points}) \times P(\text{success})$$

From history. Huygens first wrote his tract in Dutch as *Van Rekeningh in Spelen van Geluck*; van Schooten translated it into Latin for the 1657 printing, and the Dutch original appeared in 1660. Jakob Bernoulli reprinted the whole tract with his own commentary as Part I of *Ars Conjectandi* in 1713.

WORKED EXAMPLES

Example 1. Find the expected value of one roll of a fair die, and of the sum of two dice.

Show solution

1. Each face has probability $\frac{1}{6}$.
2. $E = \frac{1}{6}(1) + \frac{1}{6}(2) + \cdots + \frac{1}{6}(6) = \frac{21}{6} = 3.5$.
3. Expected values add, so two dice: $3.5 + 3.5 = 7$.

Answer: 3.5 for one die and 7 for two dice: the long-run averages, not results you can roll on one die.

Example 2. In a backyard shooting game, a close shot is worth 1 point and goes in 80% of the time; a far shot is worth 3 points and goes in 30% of the time. Find the points per attempt of each, and the expected total for 50 attempts.

Show solution

1. Close: $E = 0.8(1) + 0.2(0) = 0.8$ points per attempt.
2. Far: $E = 0.3(3) + 0.7(0) = 0.9$ points per attempt.
3. 50 attempts: close $50 \times 0.8 = 40$; far $50 \times 0.9 = 45$.

Answer: The far shot is the higher-percentage choice in expected points (0.9 against 0.8), even though it misses 70% of the time.

Example 3. Use Huygens's rule: I have 3 equally likely chances of 10 points and 2 chances of nothing. What is my chance worth?

Show solution

1. Here $p = 3$, $a = 10$, $q = 2$, $b = 0$.
2. $\frac{3(10) + 2(0)}{3 + 2} = \frac{30}{5} = 6$.
3. Check with probabilities: $\frac{3}{5}(10) + \frac{2}{5}(0) = 6$.

Answer: 6 points.

Example 4. A game: roll one die. You may keep the number as your score, or reroll once and keep the second number. When should you reroll, and what is the expected score with the best plan?

Show solution

1. A reroll is worth its expected value, 3.5. Keep any first roll larger than 3.5 (a 4, 5 or 6); reroll a 1, 2 or 3.
2. Probability of keeping: $\frac{3}{6} = \frac{1}{2}$, and then the average kept value is $\frac{4+5+6}{3} = 5$.
3. Probability of rerolling: $\frac{1}{2}$, and then the expected value is 3.5.
4. $E = \frac{1}{2}(5) + \frac{1}{2}(3.5) = 4.25$.

Answer: Reroll a 1, 2 or 3; the expected score is 4.25, better than 3.5 without the option.

PRACTICE

1. **Easy** A coin is flipped: heads scores 2 points, tails scores 0. What is the expected score?

Show answer

$$\frac{1}{2}(2) + \frac{1}{2}(0) = 1 \text{ point.}$$

2. **Easy** A spinner lands on 1 point with probability $\frac{1}{2}$, 2 points with probability $\frac{1}{4}$, and 4 points with probability $\frac{1}{4}$. Find the expected score.

Show answer

$$0.5 + 0.5 + 1 = 2 \text{ points.}$$

3. **Easy** A shooter makes 45% of her two-point shots. What are her points per attempt?

Show answer

$$0.45 \times 2 = 0.9.$$

4. **Medium** Use Huygens's rule: 2 chances of 12 points and 4 chances of 3 points. What is the chance worth?

Show answer

$$\frac{2(12) + 4(3)}{6} = \frac{36}{6} = 6 \text{ points.}$$

5. **Medium** A player makes 48% of two-pointers and 36% of three-pointers. Which shot has more points per attempt?

Show answer

Two: 0.96; three: 1.08. The three.

6. **Medium** If a shot is worth 0.9 points per attempt, about how many points should 50 attempts produce? Will it be exactly that?

Show answer

About 45 points. Not exactly: the actual total scatters around 45, more so over few attempts.

7. **Medium** The expected value of one die is 3.5, yet no die shows 3.5. Explain what the number means.

Show answer

It is the long-run average per roll. Over many rolls the total divided by the number of rolls gets close to 3.5.

8. **Medium** What did Huygens add to the work of Pascal and Fermat, and why was his book so important?

Show answer

He gave a general rule for the value (expectation) of any chance, not just the division of a prize, and his 1657 book was the first printed book on probability, used as the standard textbook for decades.

9. **Hard** Roll one die. A 1 scores 0 points; any other face scores its number. Find the expected score.

Show answer

$$\frac{0+2+3+4+5+6}{6} = \frac{20}{6} \approx 3.33 \text{ points.}$$

10. **Hard** Roll two dice. Doubles score 10 points; anything else scores 1 point. Find the expected score per roll.

Show answer

$$\frac{1}{6}(10) + \frac{5}{6}(1) = \frac{15}{6} = 2.5 \text{ points.}$$

11. **Hard** A two-point shot goes in 50% of the time. What make rate must a three-point shot have to be worth the same points per attempt?

Show answer

$$3p = 0.5 \times 2 = 1 \Rightarrow p = \frac{1}{3} \approx 33.3\%.$$

12. **Challenge** Extend Worked Example 4: you may reroll up to *two* times, keeping the last roll. Find the best plan and its expected score.

Show answer

On the last chance the value is 3.5. With one reroll left the value is 4.25 (keep 4, 5, 6). So on the first roll keep only a 5 or 6 (more than 4.25):

$$E = \frac{2}{6}(5.5) + \frac{4}{6}(4.25) = \frac{11}{6} + \frac{17}{6} = \frac{28}{6} \approx 4.67.$$

COMMON MISTAKES

- Averaging the possible values without weighting by their probabilities.
- Thinking the expected value is the most likely single result. In Game B the expected value is 2.5, but a single round never scores 2.5.
- Choosing the shot with the highest make rate instead of the highest points per attempt.

Make it a game • Name Your Price. Your parent offers you a chance: roll a die, and if it shows six you earn six paper chips, otherwise none. Write down the most you would pay to play, then compute the fair price with expected value. Huygens would be proud. Try three more chances with different payouts, and see who guessed closest to the fair price in each round.

2.2 POINTS PER SHOT

OBJECTIVES

- Compute points per attempt for twos, threes, free throws, extra points and two-point tries.
- Find break-even percentages, such as the three-point percentage that equals a given two-point percentage.
- Explain why the high-percentage shot is the one with the highest expected points, not the highest make rate.

Warm-up. Stand 1.5 m from a basket or bucket and take 10 shots with a soft ball; then take 10 from 3.5 m. Before shooting, decide that close makes are worth 2 points and far makes are worth 3. Which distance earned more points? Was it the one where you made more shots?

WATCH

How the 3-Pointer Is Changing The NBA | Data Attack

WIRED • 2:45

<https://www.youtube.com/watch?v=wWxDrL4Xctg>

WHEN TO WATCH

After “the mathematics” paragraph. Pause whenever a percentage appears and compute the points per attempt yourself.

KEY TERMS

make rate

The fraction of attempts that succeed, also called the shooting percentage.

break-even percentage

The make rate at which two choices have equal expected points.

points per attempt

Points for a success times the make rate: the expected points of one attempt.

conversion

In football, the try after a touchdown: a kick for 1 point or a play from close range for 2 points.

CORE IDEAS

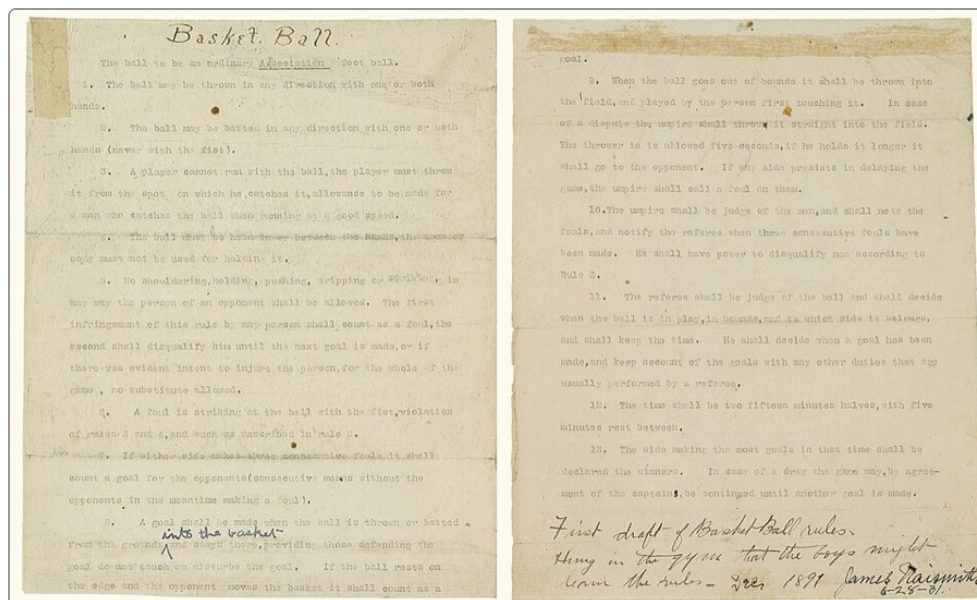
The story. On October 12, 1979, in the first game of the season, Chris Ford of the Boston Celtics made the first three-point shot in National Basketball Association history. The league had added the line that year, copying older leagues, but for a long time coaches treated the three as a stunt. In 1979–80 NBA teams averaged only 2.8 three-point attempts a game and made 28% of them. Football made a similar change: the National Football League allowed a two-point try after a touchdown in 1994, and in 2015 it moved the extra-point kick back to a 33-yard attempt, making it harder. Both changes turned the same question into arithmetic: is the bigger reward worth the lower chance?

How they figured it out. For decades many coaches judged a shot by whether it usually went in. A 45% jump shot looked better than a 35% three. Analysts asked a different question: how many points does each attempt produce, on average? $0.45 \times 2 = 0.90$ but $0.35 \times 3 = 1.05$. The three is the better shot even though it misses more often. As teams counted points per attempt, the three-point shot grew steadily: by 2024–25 NBA teams took 37.6 threes a game and made 36%. In football, analysts computed that a kick made about 94% of the time is worth about 0.94 points, while a two-point try that succeeds about half the time is worth about 1 point, so the two choices are nearly equal on average, and the score and time decide which is better.

The mathematics. For an attempt worth v points with make rate p , $E = vp$. Two choices break even when $v_1p_1 = v_2p_2$. For threes against twos, $3p_3 = 2p_2$, so $p_3 = \frac{2}{3}p_2$: a 33.3% three equals a 50% two, and a 30% three equals a 45% two. For a two-point try against a kick, $2p_2 = 1 \cdot p_k$, so $p_2 = \frac{1}{2}p_k$; with kicks at 94%, the break-even is 47%. For free throws, a trip to the line for two shots at rate p is worth $2p$. Expected points are additive, so a team's expected total is the sum over all its attempts.

On the field. Coaches use these numbers to design their offenses: plays that end in a layup, a free throw or an open three, and fewer long two-point jump shots. Football coaches carry a chart that tells when to go for two, and its first column is the break-even arithmetic you just did. **The risk side:** a make rate measured over 30 attempts is not the same as one measured over 3,000 (Lesson 1.5), and a shot that is only worth a little more on average may be a poor choice when one miss would end the game. A wise player knows her own percentages, not the league's.

PICTURE STUDY



The Original Rules of Basket Ball, James Naismith, December 1891 · Typescript on paper, two pages, with handwritten notes · Springfield, Massachusetts

Image: [Wikimedia Commons](#), public domain

In December 1891, James Naismith, a teacher at the YMCA training school in Springfield, Massachusetts, typed thirteen rules for a new indoor game and pinned them to the gym wall. A goal was made by throwing the ball into a basket, and the team with the most goals won. Every goal counted the same.

Look closely

- Find the rule that tells how a goal is made, and the rule that decides the winner.
- Look for any mention of points. How many points was a goal worth in the first game?
- Read Naismith's handwritten note at the bottom of the second page. Where were the rules hung, and why?

Talk about it

1. In Naismith's game every goal counted the same, so the best shot was simply the one most likely to go in. How did adding the three-point line change that?
2. If you could add one rule to make basketball reward a different kind of shot, what would it be, and how would teams change their play?

Safety first. Warm up before shooting. Use a soft ball indoors, and keep clear of windows, lamps and people.

PROJECT

Your Own Points per Shot

League numbers describe the average player. Your best shot depends on your own percentages. You will measure them from two distances, compute your points per attempt and find your own break-even point.

You will need

- A soft ball and a basket, bucket or laundry hamper
- A tape measure and tape to mark lines
- The scorebook and a calculator
- A partner to retrieve balls and tally (optional)

Steps

1. Mark a “two-point” line 1.5 m from the basket and a “three-point” line 3.5 m away.
2. **Predict.** Write your expected make rate from each line, and which line you expect to give more points per attempt.
3. Warm up with 10 easy shots.
4. Take 50 shots from the two-point line in sets of 10, recording makes in each set.
5. Take 50 shots from the three-point line in sets of 10, recording makes.
6. Compute each make rate and points per attempt ($2 \times \text{rate}$ and $3 \times \text{rate}$).
7. Compute your break-even three-point rate: $\frac{2}{3}$ of your two-point rate. Is your actual three-point rate above or below it?
8. Look at the sets of 10. How much did your rate vary from set to set? Is the difference between the two lines larger than that variation?
9. Move the far line in or out by 0.5 m and take 20 more shots to estimate where the two lines would break even for you.
10. **Decide.** Write the shot plan you would use in a game to 21 points, and explain it with your numbers.

Tips

- Shoot the same way every time (same ball, same style) so the only thing that changes is the distance.
- Fifty shots from each line is a small sample; Lesson 1.5's standard error tells you how much to trust it.

Check your work

- ☐ I took 50 shots from each line and recorded every set of 10.
- ☐ I computed points per attempt and my break-even three-point rate.
- ☐ I wrote a shot plan based on my own data.

SCOREBOOK

1. Shot table: line, attempts, makes, make rate, points per make, points per attempt.
2. Plot points per attempt against distance for the two (or three) distances you measured, and draw a smooth curve through them.
3. **What I would decide and why:** write a one-paragraph “go for two” rule for a football team whose two-point success rate is 45% and whose kicker makes 95%.

KEY FORMULAS

$$E = v \times p$$

$$\text{break-even: } v_1 p_1 = v_2 p_2$$

$$p_3 = \frac{2}{3} p_2 \quad p_{\text{two-point try}} = \frac{1}{2} p_{\text{kick}}$$

From history. The American Basketball League used a three-point line in 1961–62, and the American Basketball Association adopted it in 1967 before the NBA followed in 1979. College football adopted the two-point try in 1958, thirty-six years before the NFL. Naismith later coached at the University of Kansas, where he is honored as the founder of its basketball program.

WORKED EXAMPLES

Example 1. In 1979–80, NBA three-pointers went in 28.0% of the time, and all field goals together (almost all twos) went in 48.1 percent of the time. Compare the points per attempt.

Show solution

1. Threes: $3 \times 0.280 = 0.84$ points per attempt.
2. Field goals, treating them as twos: $2 \times 0.481 \approx 0.96$ points per attempt.
3. Break-even three-point rate for a 48.1 percent two: $\frac{2}{3}(0.481) \approx 0.321$.

Answer: In 1979–80 the average three (0.84) was worse than the average two (about 0.96); shooters needed about 32% from three to break even. By 2024–25 the league made 36.0% of threes, $3 \times 0.360 = 1.08$ points per attempt.

Example 2. A kicker makes 94% of extra points, and a team converts 48% of two-point tries. Which has more expected points? What two-point rate breaks even?

Show solution

1. Kick: $1 \times 0.94 = 0.94$. Two-point try: $2 \times 0.48 = 0.96$.
2. Break-even: $2p = 0.94 \Rightarrow p = 0.47$.

Answer: The two-point try is worth slightly more (0.96 against 0.94); 47% breaks even. The difference is so small that the score and the time left should decide.

Example 3. The NBA made 78.4% of free throws in 2023–24. What is a two-shot trip to the line worth? How does it compare with a 36.6% three?

Show solution

1. Two free throws: $2 \times 0.784 = 1.568$ points.
2. A 36.6% three: $3 \times 0.366 = 1.098$ points.
3. Getting fouled on a shot is worth about half a point more than a three-point attempt.

Answer: About 1.57 points per trip against about 1.10 per three.

Example 4. A player makes 40% of her long two-point shots and 34% of her threes, which are only a step farther away. Should she step back?

Show solution

1. Long two: $2 \times 0.40 = 0.80$. Three: $3 \times 0.34 = 1.02$.

2. Break-even three rate: $\frac{2}{3}(0.40) \approx 0.267$.

Answer: Yes: the three is worth about 0.22 more points per attempt, and she would need to fall below about 26.7% from three before the long two became better.

PRACTICE

1. **Easy** A shooter makes 52% of twos. What are her points per attempt?

Show answer

$$2 \times 0.52 = 1.04.$$

2. **Easy** A shooter makes 38% of threes. What are his points per attempt?

Show answer

$$3 \times 0.38 = 1.14.$$

3. **Easy** A player makes 70% of free throws. What is a two-shot trip to the line worth?

Show answer

$$2 \times 0.70 = 1.40 \text{ points.}$$

4. **Medium** What three-point percentage equals a 54% two-point percentage?

Show answer

$$\frac{2}{3}(0.54) = 0.36, \text{ that is } 36\%.$$

5. **Medium** If a team's kicker makes 96% of extra points, what two-point success rate breaks even?

Show answer

$$p = 0.96/2 = 0.48, 48\%.$$

6. **Medium** A team takes 30 threes at 35% and 50 twos at 50% in a game. Find its expected points from field goals.

Show answer

$$30(3)(0.35) + 50(2)(0.50) = 31.5 + 50 = 81.5.$$

7. **Medium** In 1979–80 NBA teams averaged 2.8 three-point attempts a game; in 2024–25 they averaged 37.6. How many times as many is that?

Show answer

$$37.6/2.8 \approx 13.4 \text{ times.}$$

8. **Medium** Explain in two sentences why “the high-percentage shot” should mean the shot with the most expected points, not the one that goes in most often.

Show answer

A shot that goes in often but is worth fewer points can produce fewer points over many attempts than a harder shot worth more. Expected points per attempt measures what each choice actually adds to the score in the long run.

9. **Hard** A “three-point play” happens when a player makes a two-point shot, is fouled, and then shoots one free throw. If she makes 75% of free throws, what is that free throw worth, and what is the whole play worth after the basket is made?

Show answer

Free throw 0.75; whole play $2 + 0.75 = 2.75$ expected points.

10. **Hard** A team converts two-point tries 50% of the time and its kicker makes 94%. Over 20 touchdowns, how many more points should always going for two produce than always kicking?

Show answer

Two: $20 \times 1.00 = 20$; kick: $20 \times 0.94 = 18.8$. About 1.2 more points, a small edge.

11. **Hard** A shooter makes 60% at the rim, 40% from midrange and 35% from three. Rank the three shots by points per attempt.

Show answer

Rim 1.20, three 1.05, midrange 0.80. Rim, then three, then midrange.

12. **Challenge** A team trails by 2 points with time for one last shot. Its two-point shot goes in 50% of the time and leads to overtime, where it wins 50% of the time. Its three-point shot goes in 35% of the time and wins the game outright. Which shot gives the higher chance of winning?

Show answer

Two: $0.50 \times 0.50 = 0.25$. Three: 0.35. The three, even though its make rate is lower. (Here the goal is the chance of winning, not points; Lesson 2.3 explains why that can change the answer.)

COMMON MISTAKES

- Comparing make rates instead of points per attempt.
- Forgetting that a two-shot trip to the line is worth two attempts' worth of points.
- Using league averages when your own (or your team's) percentages are different.

Make it a game • Paper Ball Points per Shot. Mark two lines at different distances from a bucket, one worth two points and one worth three. Take ten paper-ball shots from each. Compute your make rate and points per attempt for each line, and the break-even percentage that makes the three worth shooting. Choose where to shoot for ten more tries and compare your total to your prediction.

2.3 SAME AVERAGE, DIFFERENT RISK

OBJECTIVES

- Compute the variance and standard deviation of a random variable from its probability table.
- Show with numbers that two strategies can have the same expected value but very different spreads.
- Explain why a team that is behind should usually increase variance and a team that is ahead should reduce it.

Warm-up. Two players each take 4 shots. Player A makes exactly 2 every time. Player B makes 4 sometimes, 0 sometimes, and 2 sometimes. If both average 2 makes, which would you rather have when your team needs 4 makes to win? When your team needs only 1?

WATCH

Variance and standard deviation of a discrete random variable | AP Statistics | Khan Academy

Khan Academy • 6:25

https://www.youtube.com/watch?v=2egl_5c8i-g

WHEN TO WATCH

After “the mathematics” paragraph. It computes a variance and standard deviation from a probability table step by step; work along in your scorebook.

KEY TERMS

variance

The expected squared distance from the mean: $\text{Var}(X) = \sum p_i(x_i - \mu)^2$. It measures spread.

high-variance strategy

A plan whose results are widely spread: big successes and big failures, such as a long pass or a three-point shot.

standard deviation

The square root of the variance, σ , measured in the same units as the values.

underdog

The side less likely to win. An underdog often does best by choosing plans with more variance.

CORE IDEAS

The story. In the valley of Elah, Israel's army faced the Philistines and their champion Goliath, a giant armed with a bronze helmet, a coat of mail, a spear and a sword (1 Samuel 17). For forty days no one would meet him. Then the shepherd boy David volunteered. King Saul put his own armor and sword on David, but David took them off: "I cannot go with these; for I have not proved them." Instead he took his staff, his sling and five smooth stones from the brook. Scripture is plain that David's confidence was in the Lord, "for the battle is the LORD's." But the story also shows a wise choice: David refused to fight on Goliath's terms, in close combat where the giant was nearly certain to win, and chose a different contest in which a stone from a distance could decide everything at once.

How they figured it out. For a long time people judged strategies only by their average. In the 18th and 19th centuries mathematicians such as Abraham de Moivre and Carl Friedrich Gauss learned to measure *spread*, and in 1894 the statistician Karl Pearson named the **standard deviation**. In sport the idea took longer. Coaches knew in their bones that a team that was far behind had to "take chances," and in 1975 the Dallas quarterback Roger Staubach threw a desperate pass of about 50 yards to Drew Pearson to beat Minnesota in the last minute; Staubach later said he closed his eyes and said a prayer, and the long, last-chance pass has been called a "Hail Mary" ever since. Analysts later explained the instinct with mathematics: when you need a big result, a plan with the same or even a lower average but a wider spread gives you a better chance of getting it.

The mathematics. If X has mean $\mu = E[X]$, its **variance** is $\text{Var}(X) = \sum p_i(x_i - \mu)^2$, and its **standard deviation** is $\sigma = \sqrt{\text{Var}(X)}$. A shortcut is $\text{Var}(X) = E[X^2] - \mu^2$. A two-point shot at 50% and a three-point shot at $33\frac{1}{3}\%$ both average 1 point per attempt, but the two has $\sigma = 1$ while the three has $\sigma = \sqrt{2} \approx 1.41$. For independent attempts, variances add: n attempts have variance $n\sigma^2$ and standard deviation $\sigma\sqrt{n}$. When you need a total far above your average, the plan with the larger σ gives the better chance of reaching it.

On the field. A team that trails late should **increase variance**: shoot threes, throw deep, pull the goalie for an extra attacker (Lesson 4.2), press on defense. A team that leads should **reduce variance**: take safe, sure plays, run the clock, avoid turnovers. Both choices can lower the expected score and still raise the chance of winning,

because winning, not points, is the goal. **The risk side:** high variance cuts both ways. The underdog who takes wild chances may lose by more, which does not matter if a loss is a loss; but a favorite who plays recklessly turns a likely win into a coin flip. Choose your spread on purpose.

PICTURE STUDY



David Slings the Stone, James Tissot, c. 1896–1902 · Pen and ink · A sketch for the Old Testament illustrations published in Paris in 1904
Image: [Wikimedia Commons](#), public domain

The French artist James Tissot spent his last years drawing and painting the events of the Bible, after traveling to the Holy Land to study its landscapes and people. This is one of his sketches for 1 Samuel 17:49, the moment the stone leaves the sling.

Look closely

- Compare David and Goliath: their size, their armor, what each carries.
- Look at the distance between them. Why does the distance matter for a sling against a spear?
- Look at David's body. Where is all of his effort going?

Talk about it

1. If David had fought in Saul's armor with a sword, what would his chances have been? Why was choosing a different kind of contest wise?
2. David's confidence was in the Lord (1 Samuel 17:45–47), and yet he also chose his tools with care. How can trusting God and making careful plans go together?

Safety first. Warm up before shooting. Use a soft ball indoors, and keep clear of windows, lamps and people.

PROJECT

Measuring the Spread

You will measure both the average and the spread of your own shooting with two strategies, then test which strategy reaches a high target more often.

You will need

- A soft ball and a basket or bucket
- Tape measure and tape
- The scorebook and a calculator
- Optional: a spreadsheet

Steps

1. Use your lines from Lesson 2.2 (or mark lines at 1.5 m and 3.5 m). Close makes score 2; far makes score 3.
2. **Predict.** From your Lesson 2.2 rates, compute the mean and standard deviation of one close shot and one far shot using $\sigma = v\sqrt{p(1-p)}$.
3. Play 20 “rounds” of 5 close shots each. Record the points in each round (0, 2, 4, 6, 8 or 10).
4. Play 20 rounds of 5 far shots each. Record the points in each round (0, 3, 6, ..., 15).
5. For each strategy, compute the mean round score and the standard deviation of the 20 round scores (a spreadsheet’s =STDEV.P() or a careful hand calculation).
6. Draw a dot plot of the 20 round scores for each strategy, one above the other on the same scale.
7. **Target test.** Count how many rounds of each strategy reached 9 or more points. Which strategy reached the high target more often?
8. Count how many rounds of each strategy scored at least 2 points. Which strategy was safer for a low target?
9. **Decide.** Write a rule: “When my team needs a lot, I will ...; when my team needs only a little, I will” Support it with your counts.

Tips

- Standard deviation by hand: subtract the mean from each score, square, average the squares, and take the square root.
- If your far-shot rate is very low, move the far line in so that you make at least a quarter of your shots.

Check your work

- ☐ I recorded 20 rounds for each strategy and computed means and standard deviations.
- ☐ My dot plots show the difference in spread.
- ☐ I wrote a rule for choosing variance, supported by my target counts.

SCOREBOOK

1. Variance table for each shot: value, probability, deviation from the mean, deviation squared, probability $\times$ deviation squared.
2. Dot plots of your round scores for both strategies on one scale, with each mean marked.
3. **What I would decide and why:** your family-game team trails by 8 with two turns left. List your options and choose one by its spread, not only its average.

KEY FORMULAS

$$\text{Var}(X) = \sum p_i(x_i - \mu)^2 = E[X^2] - \mu^2$$

$$\sigma = \sqrt{\text{Var}(X)}$$

$$\sigma_{\text{total of } n} = \sigma\sqrt{n} \quad (\text{independent attempts})$$

From history. Roger Staubach's pass to Drew Pearson came in the 1975 NFC playoffs. Karl Pearson (no relation) introduced the term "standard deviation" in 1894. The idea that an underdog should seek variance appears in many games: a chess player who is losing seeks complicated positions, and a backgammon player who is far behind plays boldly (Unit 5).

WORKED EXAMPLES

Example 1. Shot A is a two-pointer made 50% of the time; Shot B is a three-pointer made one-third of the time. Find the mean, variance and standard deviation of the points from one attempt of each.

Show solution

1. A: $\mu = 0.5(2) = 1$. $\text{Var} = 0.5(2 - 1)^2 + 0.5(0 - 1)^2 = 1$, $\sigma = 1$.
2. B: $\mu = \frac{1}{3}(3) = 1$. $\text{Var} = \frac{1}{3}(3 - 1)^2 + \frac{2}{3}(0 - 1)^2 = \frac{4}{3} + \frac{2}{3} = 2$, $\sigma = \sqrt{2} \approx 1.41$.
3. Check B with the shortcut: $E[X^2] = \frac{1}{3}(9) = 3$, so $\text{Var} = 3 - 1^2 = 2$.

Answer: Same mean (1 point), but B's standard deviation is about 41% larger.

Example 2. A team trails by 3 points with time for one shot. A two-pointer goes in 55% of the time; a three goes in 35% of the time. Compare expected points and the chance of tying the game.

Show solution

1. Expected points: two $2(0.55) = 1.10$; three $3(0.35) = 1.05$.
2. Chance of tying (scoring at least 3): the two can never do it, so 0; the three does it 35% of the time.

Answer: The two has slightly higher expected points, but the three is the only shot that can save the game. When behind, choose variance.

Example 3. An underdog needs at least 15 points from its last 10 shots. Plan 1: ten two-pointers at 50%. Plan 2: ten threes at one-third. Both average 10 points. Which plan gives a better chance?

Show solution

1. Plan 1 reaches 15 only with 8 or more makes. Using combinations:

$$P = \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}} = \frac{45 + 10 + 1}{1024} \approx 0.055.$$

2. Plan 2 reaches 15 with 5 or more makes. Summing $\binom{10}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{10-k}$ for $k = 5$ to 10 gives about 0.213.

3. Standard deviations of the totals: Plan 1 $1 \times \sqrt{10} \approx 3.2$; Plan 2 $1.41 \times \sqrt{10} \approx 4.5$.

Answer: Plan 2 gives about a 21% chance against about 5.5%, nearly four times better, with the same average.

PRACTICE

1. **Easy** Find the variance and standard deviation of a variable that is always exactly 3.

Show answer

Variance 0 and standard deviation 0: there is no spread.

2. **Easy** A coin scores 4 points for heads and 0 for tails. Find the mean and the standard deviation.

Show answer

$$\mu = 2; \text{Var} = 0.5(4) + 0.5(4) = 4, \sigma = 2.$$

3. **Easy** Name one high-variance and one low-variance choice in a sport you know.

Show answer

Sample: in football, a deep pass (high) and a short run (low); in basketball, a three (high) and a layup or free throw (low).

4. **Medium** A spinner scores 0, 2 or 4 points, each with probability $\frac{1}{3}$. Find the mean, variance and standard deviation.

Show answer

$$\mu = 2; \text{Var} = \frac{1}{3}(4 + 0 + 4) = \frac{8}{3} \approx 2.67; \sigma \approx 1.63.$$

5. **Medium** One die: find the variance using $E[X^2] - \mu^2$.

Show answer

$$E[X^2] = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}; \text{Var} = \frac{91}{6} - 3.5^2 = \frac{35}{12} \approx 2.92; \sigma \approx 1.71.$$

6. **Medium** A team leads by 2 points with 30 seconds left and has the ball. Should it look for a quick, contested three or work the clock for a safe two? Explain in terms of variance.

Show answer

Work the clock for the safe shot. The leader wants to reduce variance and give the opponent fewer chances; a quick, contested three adds spread and gives the ball back sooner.

7. **Medium** Why did David refuse Saul's armor, according to 1 Samuel 17:39, and how was his choice like a coach's choice of strategy?

Show answer

He said he could not go with them because he had not proved (tested) them. Like a coach, he chose a plan he knew and that did not play to the opponent's strength: a sling from a distance instead of close combat.

8. **Medium** If the standard deviation of one attempt is 1.5 points, what is the standard deviation of the total of 16 independent attempts?

Show answer

$$1.5\sqrt{16} = 6 \text{ points.}$$

9. **Hard** A long pass gains 40 yards with probability 0.25 and 0 yards otherwise; a run gains 5 yards for sure. Find the mean and standard deviation of each. Which is better when a team needs 30 yards on the final play?

Show answer

Pass: $\mu = 10$, $\text{Var} = 0.25(900) + 0.75(100) = 300$, $\sigma \approx 17.3$. Run: $\mu = 5$, $\sigma = 0$.
Needing 30, the pass (25% chance) beats the run (0%).

10. **Hard** Compute the probability that ten 50% two-point shots produce at least 16 points, and that ten one-third threes produce at least 16 points (6 or more makes). Use $P(6 \text{ or more of } 10 \text{ at } \frac{1}{3}) \approx 0.077$ for the threes.

Show answer

Twos need 8 or more makes: $\frac{56}{1024} \approx 5.5\%$. Threes: about 7.7%. The threes are better for reaching a high target.

11. **Hard** A stronger team gains a small edge on every possession. Should it prefer a fast game with many possessions or a slow game with few? Explain in terms of averages and spread.

Show answer

A fast game with many possessions. Its expected margin grows in proportion to the number of possessions, while the spread grows only with the square root, so more possessions make its win more certain; fewer possessions bring the result closer to a coin flip. Underdogs often try to slow the game down for exactly this reason.

12. **Challenge** Show that the variance of a shot worth v points with make rate p is $v^2p(1-p)$. Then find the variance of a 40% two and a 30% three.

Show answer

$E[X^2] = v^2p$, $\mu = vp$, so $\text{Var} = v^2p - v^2p^2 = v^2p(1-p)$. Two: $4(0.4)(0.6) = 0.96$; three: $9(0.3)(0.7) = 1.89$.

COMMON MISTAKES

- Judging a strategy only by its expected value when the goal is to reach a target.
- Forgetting to square the deviations, so that positive and negative deviations cancel to zero.
- Adding standard deviations instead of variances when combining independent attempts.

Make it a game • Same Average, Wild Ride. Game A: take two beans every round. Game B: roll a die and take four beans on 4, 5 or 6, nothing on 1, 2 or 3. Both average two beans a round. Play ten rounds of each, recording every result. Compute the standard deviation of each, then say which one you would want as an underdog trailing late in a game.

Scripture. “And he took his staff in his hand, and chose him five smooth stones out of the brook” (1 Samuel 17:40, KJV)

David did not borrow another man’s plan. He chose tools he had proved, for a contest he could win, and he went in the name of the Lord.

2.4 THE HOUSE ALWAYS WINS

OBJECTIVES

- Compute the expected value of a game of chance and show that casino and lottery games have negative expected value for the player by design.
- Use the law of large numbers to explain why the house is nearly certain to win over many plays even when individual players sometimes win.
- Explain insurance as the wise mirror image: paying a small expected cost to avoid ruin, from Lloyd's coffee house and Halley's life table to today.

Warm-up. Play this made-up game 12 times with a die, keeping score in points: each play costs you 1 point; if you roll a 6, you get 5 points back. Start at 20 points. Before playing, guess whether you will finish above or below 20. Record your total after each play.

WATCH

[Expected profit from lottery ticket](#) | [Probability and Statistics](#) | [Khan Academy](#)

[Khan Academy](#) · 9:18

<https://www.youtube.com/watch?v=6vIBOHckmzU>

WHEN TO WATCH

After “the mathematics” paragraph. Sal Khan computes the expected profit of a lottery ticket and finds it negative; compare his method with Worked Example 2.

BACKUP VIDEO

[Comparing insurance with expected value](#) | [Probability and Statistics](#) | [Khan Academy](#)

[Khan Academy](#) · 9:16

<https://www.youtube.com/watch?v=BveA7JiRGS8>

With the insurance paragraph. It compares the expected cost of buying insurance with the risk of not buying it.

KEY TERMS

negative expected value

A game or choice whose long-run average result is a loss.

house edge

The fraction of each amount played that the operator of a game keeps on average; it is the player's negative expected value.

insurance

A promise, bought for a small regular payment (the premium), to cover a large loss if it happens.

risk pooling

Spreading the losses of a few among many people, so that each pays a small, predictable amount instead of risking a large one.

life table

A table showing, out of a large group born at the same time, how many are still living at each age; used to set fair prices for life insurance and pensions.

CORE IDEAS

The story. Every casino and every lottery is a working proof of the mathematics in this unit. Their games are designed so that the player's expected value is negative, and they offer them millions of times. The odds of winning the U.S. Powerball jackpot, for example, are 1 in 292,201,338, and the prizes are set so that, on average, the lottery pays out much less than it takes in. Three centuries earlier, in about 1688, merchants and ship captains began meeting at Edward Lloyd's coffee house on Tower Street in London to trade shipping news and to *underwrite* voyages: a merchant paid a small premium, and men of means signed their names under the policy, promising to pay if the ship was lost. In 1693 the astronomer Edmond Halley, using birth and death records from the city of Breslau, published one of the first reliable life tables so that annuities could be priced fairly.

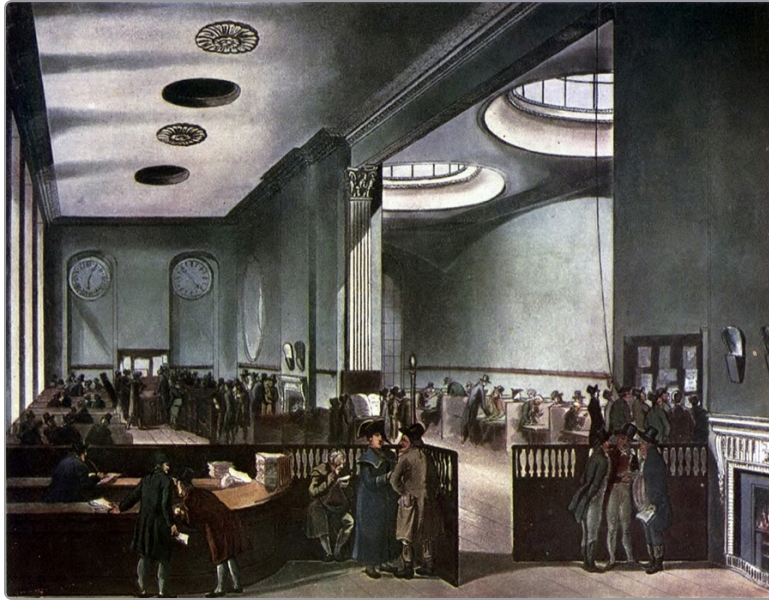
How they figured it out. The designers of games of chance did not need to be lucky; they needed only expected value and the law of large numbers. If each play earns the house a small amount on average, then over a million plays the house's total is almost exactly a million times that amount, while each player's result is dominated

by luck. The early insurers discovered the same law from the other side. No one could predict whether one particular ship would sink, but over hundreds of voyages the fraction lost was steady, so a premium a little above the expected loss would cover the losses and the costs. Halley did the same for human lives: no one knows the day of his own death, but out of a thousand people, the number living at each age can be counted.

The mathematics. A game in which you pay 1 point and receive 5 points back when a die shows 6 has expected net result $E = \frac{1}{6}(+4) + \frac{5}{6}(-1) = -\frac{1}{6} \approx -0.167$ points per play. The house edge is about 16.7%. The standard deviation of one play is about 1.86 points, so for one player over 10 plays the expected loss is only 1.7 points with a spread of about 5.9 points: he may well be ahead. But for the house over 1,000,000 plays the expected gain is about 166,667 points with a spread of only about 1,860: it is certain to win. The average grows with n ; the spread grows only with $\sqrt{n}$. Insurance runs the same mathematics in reverse: the buyer accepts a small *negative* expected value in money to remove a small chance of a ruinous loss.

On the field. The wise conclusion is simple: with a game designed to have negative expected value, the only winning move is not to play. Scripture warns that “wealth gotten by vanity shall be diminished” and that “he that hasteth to be rich hath an evil eye, and considereth not that poverty shall come upon him” (Proverbs 28:22). Steady work and saving are the positive counterpart: a small, sure gain repeated many times, like a high-percentage shot taken all season. Insurance belongs on the wise side too: a family pays a small premium so that a fire, a flood or a death does not ruin it. **The risk side:** do not confuse the two. Paying a small cost to avoid ruin is prudence; paying a small cost for a tiny chance of a fortune is the house’s game.

PICTURE STUDY



Lloyd's Subscription Room, Thomas Rowlandson and Augustus Charles Pugin, 1809 · Hand-colored aquatint, from *The Microcosm of London* · Royal Exchange, London

Image: [Wikimedia Commons](#), public domain

By 1809 the underwriters who had once met in Edward Lloyd's coffee house had moved to rooms in the Royal Exchange. Rowlandson drew the people and Pugin the architecture for this plate in *The Microcosm of London*, a famous book of views of the city.

Look closely

- Find the two great clocks. Why would time matter in a room where ships and their news were the business?
- Find the men reading papers and those talking in pairs. What might a merchant be asking an underwriter?
- Look at the long desks and the noticeboard. What kind of information would be posted there?

Talk about it

1. Each underwriter signed for only part of a ship's value. How did that spread the risk?
2. Why is insuring a ship a wise use of the mathematics of chance, while buying a lottery ticket is not?

PROJECT

Ten Players and One House

You will simulate a negative-expected-value game for many players, in points only, and watch the law of large numbers hand the house its profit. Then you will simulate an insurance pool and watch the same law protect its members.

You will need

- A die
- A spreadsheet (Google Sheets or LibreOffice Calc)
- The scorebook

Steps

1. **Predict.** In the die game (pay 1 point, receive 5 on a 6), write how many of 10 players you expect to be ahead after 10 plays, after 100 plays and after 1,000 plays.
2. **By hand.** Play 30 rounds of the die game, starting at 30 points. Record your total after each round.
3. **Spreadsheet.** In cell A2 enter `=IF(RANDBETWEEN(1,6)=6,4,-1)`, which gives one play's net result. Fill a block 1,000 rows down and 10 columns across: 10 players, 1,000 plays each.
4. Below each column, find each player's total after 10, 100 and 1,000 plays with `=SUM` over the first 10, 100 and 1,000 rows.
5. Count how many players are ahead (total above 0) at 10, 100 and 1,000 plays.
6. Add all 10,000 plays together for the house's total (the negative of the players' total). Compare with $10,000 \times \frac{1}{6} \approx 1,667$.
7. **Insurance pool.** In a new sheet, simulate 100 families for 1 year: each has a 1-in-50 chance of a 100-point loss (`=IF(RANDBETWEEN(1,50)=1,100,0)`). Total the losses. If each family paid a 3-point premium (300 in all), did the pool have enough?
8. Recalculate the pool 10 times and record the total losses each time. How often did 300 points cover them?

9. **Decide.** Write a paragraph comparing the players in the die game with the families in the pool, using the words expected value, spread and law of large numbers.

Tips

- Keep score in points only; the point of the project is the mathematics, not a stake.
- Recalculate several times: the players' results jump around, but the house's total barely moves in proportion.

Check your work

- ☐ I simulated 10 players for 1,000 plays and counted how many were ahead at each stage.
- ☐ I compared the house's total with the expected value.
- ☐ I simulated an insurance pool and explained why pooling protects families.

SCOREBOOK

1. Expected-value table for the die game and for the made-up lottery: outcome, probability, net value, product, sum.
2. Table: number of plays, number of the 10 players ahead, house total, expected house total.
3. **What I would decide and why:** a friend shows you an advertisement for a big prize. Write what you would tell him, using expected value and Proverbs 13:11.

KEY FORMULAS

$$E[\text{net}] = \sum p_i(\text{prize}_i) - \text{cost}$$

$$\text{house edge} = -\frac{E[\text{net}]}{\text{cost}}$$

$$\text{total after } n \text{ plays: mean } n\mu, \quad \text{spread } \sigma\sqrt{n}$$

From history. Lloyd's moved from the coffee house to the Royal Exchange in 1774. Halley's 1693 paper, "An Estimate of the Degrees of the Mortality of Mankind," appeared in the *Philosophical Transactions* of the Royal Society. Halley is better remembered for the comet that bears his name, and for paying to print Newton's *Principia*.

WORKED EXAMPLES

Example 1. The die game: pay 1 point; if the die shows 6, receive 5 points (a net gain of 4); otherwise lose the 1 point. Find the expected net result per play and the house edge.

Show solution

1. $P(6) = \frac{1}{6}$: net +4. $P(\text{not } 6) = \frac{5}{6}$: net -1 .
2. $E = \frac{1}{6}(4) + \frac{5}{6}(-1) = \frac{4-5}{6} = -\frac{1}{6}$.
3. House edge: $\frac{1/6}{1} \approx 16.7\%$. A fair version would pay back 6 points (net +5):
 $\frac{1}{6}(5) + \frac{5}{6}(-1) = 0$.

Answer: The player loses $\frac{1}{6}$ point per play on average; the house keeps about 16.7% of everything played.

Example 2. A made-up lottery sells tickets for \$1. One ticket in 1,000,000 wins \$500,000, and one ticket in 1,000 wins \$100. Find the expected net value of a ticket.

Show solution

1. Expected prize: $\frac{1}{1,000,000}(500,000) + \frac{1}{1,000}(100) = 0.50 + 0.10 = 0.60$ dollars.
2. Expected net: $0.60 - 1.00 = -0.40$ dollars per ticket.
3. Someone who buys 5 tickets a week for 40 years (10,400 tickets) should expect to lose about $10,400 \times 0.40 = 4,160$ dollars, and the chance he ever wins the top prize is only about one in a hundred.

Answer: $-\$0.40$ per ticket: 40 cents of every dollar is lost on average, by design.

Example 3. Use the die game (expected value $-\frac{1}{6}$, standard deviation 1.86 per play). Compare one player who plays 10 times with the house, which hosts 1,000,000 plays.

Show solution

1. Player: mean $10 \times (-\frac{1}{6}) \approx -1.7$, spread $1.86\sqrt{10} \approx 5.9$. Results from about -13 to $+10$ are ordinary, so many players finish ahead.
2. House: mean $1,000,000 \times \frac{1}{6} \approx 166,667$, spread $1.86 \times 1,000 = 1,860$. Two spreads below the mean is still about 163,000.
3. The house's profit is about 90 standard deviations above zero: not a matter of chance at all.

Answer: A single player's short run is mostly luck; the house's long run is certain.

Example 4. A family's house is worth \$300,000. Suppose (for illustration) the chance of its being destroyed by fire in a year is 1 in 1,000, and insurance costs \$400 a year. Find the expected loss without insurance and the expected cost of insuring.

Show solution

1. Expected loss uninsured: $\frac{1}{1,000} \times 300,000 = 300$ dollars a year.
2. Insurance costs 400, so its expected net cost is $400 - 300 = 100$ dollars a year.
3. In exchange, a 1-in-1,000 chance of losing everything becomes a sure, affordable payment.

Answer: Insurance has a small negative expected value (\$100 a year), but it removes the risk of ruin. That trade is wise; the lottery's trade is not.

PRACTICE

1. Easy A game costs 2 points to play and pays 10 points with probability $\frac{1}{10}$. Find the expected net result.

Show answer

$$\frac{1}{10}(10) - 2 = -1 \text{ point per play.}$$

2. **Easy** What does “the house always wins” mean mathematically?

Show answer

The game is designed with a negative expected value for the player, and over many plays the law of large numbers makes the house’s average gain nearly certain.

3. **Easy** Powerball’s jackpot odds are 1 in 292,201,338. Write that probability in scientific notation.

Show answer

$$\approx 3.4 \times 10^{-9}.$$

4. **Medium** In the die game (pay 1, receive 5 on a 6), what payout would make the game fair?

Show answer

$$\text{Receive 6 points (net +5): } \frac{1}{6}(5) - \frac{5}{6}(1) = 0.$$

5. **Medium** A made-up raffle sells 500 tickets at 2 points each and gives one prize of 600 points. Find the expected net value of a ticket and the organizer’s total gain.

Show answer

$$\text{Ticket: } \frac{600}{500} - 2 = -0.8 \text{ points. Organizer: } 1,000 - 600 = 400 \text{ points.}$$

6. **Medium** Explain why a player can be ahead after 10 plays of a negative-expected-value game while the house is still certain to profit.

Show answer

The player’s average loss over 10 plays is small compared with the spread (which grows with $\sqrt{n}$), so luck dominates. The house’s millions of plays make its average gain (which grows with n) far larger than the spread.

7. **Medium** What happened at Lloyd’s coffee house, and why did the insurers there not need to know which ships would sink?

Show answer

Merchants bought policies on voyages, and wealthy men underwrote them, promising to pay if a ship was lost. Over many voyages the fraction lost was steady, so premiums slightly above the expected loss covered the losses whichever ships sank.

8. **Medium** Why did Halley's life table matter for insurance?

Show answer

It showed, out of many people, how many survive to each age, so the price of an annuity or life policy could be set from real data instead of guesswork.

9. **Hard** In the made-up lottery of Worked Example 2, how many tickets would someone need to buy to have a 50% chance of winning the top prize at least once? (Use $1 - (1 - 10^{-6})^n = 0.5$; $n \approx 693,000$.) What would he expect to lose along the way?

Show answer

About 693,000 tickets, with an expected loss of about $693,000 \times 0.40 \approx 277,000$ dollars.

10. **Hard** A bicycle worth \$600 has a 5% chance of being stolen in a year. Insurance costs \$45 a year. Find the expected net cost of insuring, and explain whether a family that could easily replace the bicycle needs it.

Show answer

Expected loss $0.05 \times 600 = 30$; expected net cost $45 - 30 = 15$ dollars a year. A family that can easily replace the bicycle is not facing ruin, so the insurance mostly costs money; insurance is wisest for losses you could not bear.

11. **Hard** The house hosts 10,000 plays of the die game. Find the mean and the spread (standard deviation) of its total gain, and the range of about 95% of outcomes.

Show answer

Mean $10,000/6 \approx 1,667$; spread $1.86 \times 100 = 186$; range about 1,295 to 2,039. Still always positive.

12. **Challenge** Show that the standard deviation of one play of the die game is about 1.86. (Net values +4 with probability $\frac{1}{6}$ and -1 with probability $\frac{5}{6}$.)

Show answer

$$E[X^2] = \frac{1}{6}(16) + \frac{5}{6}(1) = \frac{21}{6} = 3.5; \text{Var} = 3.5 - \left(\frac{1}{6}\right)^2 = 3.5 - 0.0278 = 3.472; \\ \sigma \approx 1.86.$$

COMMON MISTAKES

- Thinking a long losing run makes a win “due” in a house game (the mistake of Lesson 1.3: dice have no memory).
- Counting the prize but forgetting to subtract the cost of playing.
- Calling insurance a bad deal because its expected value is negative. Its purpose is to remove the risk of ruin, not to make money.

Make it a game • Be the House. Design a pretend game with paper chips: pay one chip to roll a die, and win a number of chips on a particular roll. First make it fair, with expected value zero. Then adjust the payout so the house has a small edge. Play twenty rounds as the player and compute the average. Explain why the house wins in the long run.

Scripture. “Wealth gotten by vanity shall be diminished: but he that gathereth by labour shall increase.” (Proverbs 13:11, KJV)

The house’s games are built so that “wealth gotten by vanity” is diminished on average. Steady labor, like the high-percentage play repeated all season, is how wealth increases.

2.5 RISK OF RUIN

OBJECTIVES

- Explain the St. Petersburg paradox and Daniel Bernoulli's answer that the value of a gain depends on what you already have.
- Compute the chance of surviving many independent risks, and show how spreading risk reduces the chance of total loss.
- Explain why a sure small gain can beat a risky large one, and why the “safe” play is sometimes not safe.

Warm-up. Flip a coin until it lands heads, counting the flips. Your score is 2 points if heads comes on flip 1, 4 points on flip 2, 8 points on flip 3, and so on, doubling. Play 10 times and record the scores. Before you start, guess your average.

WATCH

A 300 year old probability paradox | St. Petersburg Paradox

OptWhiz · 11:58

<https://www.youtube.com/watch?v=inlha5mKETA>

WHEN TO WATCH

After the story. It explains why the expected value is infinite and walks through Bernoulli's answer with utility; compare it with Worked Example 2.

Note: The video prices the game in dollars, as the 1738 paper priced it in ducats, and briefly mentions gambler's ruin as a name for the risk of ruin; in this course we keep score in points.

KEY TERMS

St. Petersburg paradox

A game with infinite expected value that no sensible person would trade much for, which shows that expected value alone cannot guide every decision.

utility

The real worth of an outcome to a person, which may not be proportional to its size; Daniel Bernoulli proposed that the utility of wealth grows like its logarithm.

risk of ruin

The chance of a loss so large that you cannot continue, recover or play again.

diversification

Spreading what you have across several independent risks, so that one loss cannot take everything.

CORE IDEAS

The story. In 1713 Nicolaus Bernoulli, a nephew of Jakob, sent a puzzle to the French mathematician Pierre Rémond de Montmort. A coin is tossed until it lands heads. If heads appears on the first toss, the player receives 2 coins; on the second, 4; on the third, 8; the prize doubles with every toss. What is a fair price for one turn? The expected value is $\frac{1}{2}(2) + \frac{1}{4}(4) + \frac{1}{8}(8) + \dots = 1 + 1 + 1 + \dots$, which is infinite. Yet nobody would give up even 20 coins for one turn. Nicolaus's cousin Daniel Bernoulli published the famous answer in 1738 in the journal of the Imperial Academy of Sciences at St. Petersburg, which gave the puzzle its name.

How they figured it out. Daniel Bernoulli argued that the value of a gain is not the same for everyone: a thousand coins mean much more to a poor man than to a rich one. He proposed that the **utility** of wealth grows like its logarithm, so each doubling of wealth adds the same amount of satisfaction. With that rule, the St. Petersburg game is worth only a few coins. There is a second answer, which Bernoulli also mentioned: no bank in the world can pay an unlimited prize. If the bank holds about a million coins (2^{20}), the game is worth only about 21 coins. Both answers teach the same lesson: when the rare huge outcomes are imaginary or the possible losses are ruinous, expected value alone misleads.

The mathematics. Repeated risks multiply. If a plan carries a 2% chance of ruin each time and you follow it 50 times independently, your chance of surviving is $0.98^{50} \approx 0.364$: ruin is more likely than not. Spreading risk works the other way: one cargo on one ship with a 10% chance of loss risks everything 10% of the time, but the same cargo split across four ships with independent risks loses *everything* only $0.1^4 = 0.0001$ of the time, with the same expected loss of 10%. Bernoulli's log utility explains why people prefer a sure gain: with 100 points in hand, a sure 50 more gives $\ln 150 \approx 5.01$, while a coin flip for 100 more or nothing gives $\frac{1}{2}(\ln 200 + \ln 100) \approx 4.95$, even though both have the same expected value.

On the field. Never risk what you cannot afford to lose. A team that leads should protect the lead (Lesson 2.3), but “protecting” can be done badly: a soccer team that

pulls every player back for twenty minutes may give the opponent so many chances that its lead is less safe than if it had kept playing. A climber, a sailor and a family budget all follow the same rule: choose plans in which no single bad result ends the game, then take the best expected value among them. **The risk side:** the most dangerous risks are small ones repeated many times, because each one feels safe.

PICTURE STUDY



Daniel Bernoulli, Unknown artist, c. 1750 · Oil on canvas (photographed for the ETH Library, Zürich) · ETH Library, Zürich (photograph)

Image: [Wikimedia Commons](#), public domain

Daniel Bernoulli (1700–1782), son of Johann and nephew of Jakob, taught in St. Petersburg and Basel. He is best known in physics for Bernoulli's principle of fluid flow, and in probability for his 1738 paper on the measurement of risk.

Look closely

- Look at his clothing and setting. Is he shown as a scholar, a gentleman or both?
- Look at his face. Does he seem cautious or bold?
- Nothing in the portrait shows his work. What object would you add to show the St. Petersburg paradox or the idea of utility?

Talk about it

1. Bernoulli said that the same gain is worth more to a poor man than to a rich one. Can you think of an example from your own life?
2. Why would a wise person turn down a game with infinite expected value?

PROJECT

The St. Petersburg Experiment

The St. Petersburg game has an infinite expected value, but what happens when people actually play it? You will play it with a coin, then simulate thousands of plays, and see why the average refuses to settle down.

You will need

- A coin
- A spreadsheet
- The scorebook and graph paper

Steps

1. **Predict.** Write what you think your average score will be over 50 plays.
2. Play 50 times with a coin: flip until heads; score 2^n points if heads comes on flip n . Record each score.
3. Compute your average after 10, 25 and 50 plays. Record your highest single score.
4. Make a frequency table of scores (2, 4, 8, 16, ...) and compare each frequency with $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ of 50.
5. **Spreadsheet.** One play: $=2^{(1+\text{INT}(\text{LOG}(1/\text{RAND}(),2)))}$. (This produces 2 with probability $1/2$, 4 with $1/4$, and so on.) Fill down 1,000 rows and 5 columns.
6. Find the average of each column of 1,000 plays. How different are the five averages?
7. Graph the running average of one column. Does it settle down like the dice in Lesson 1.5, or does it jump upward from time to time?
8. **Ruin test.** In another sheet, simulate 100 independent risks each with a 2% chance of ruin ($=\text{IF}(\text{RANDBETWEEN}(1,50)=1,1,0)$) and record whether any ruin occurred. Repeat 20 times and compare with $1 - 0.98^{100} \approx 0.87$.
9. **Decide.** How many sure points would you trade for one turn of the game? Explain with your data and Bernoulli's idea.

Tips

- If the spreadsheet formula is hard, use =IF(RAND()<0.5,2,IF(RAND()<0.5,4,...)) for a few levels, or flip coins with a partner.
- The running average jumps because rare huge prizes keep arriving; that is the paradox in action.

Check your work

- ☐ I played 50 real games and 5,000 simulated games and compared their averages.
- ☐ I graphed a running average and explained why it does not settle down.
- ☐ I stated a certainty equivalent for the game and justified it.

SCOREBOOK

1. Frequency table of your 50 real plays: first-head toss, prize, count, expected count.
2. Survival table: number of independent 2% risks (10, 25, 50, 100) and the chance of surviving all of them.
3. **What I would decide and why:** describe a decision in sport or family life where one bad result would be ruinous. Write the plan that keeps the risk of ruin small.

KEY FORMULAS

$$P(\text{survive } n \text{ independent risks}) = (1 - r)^n$$

$$P(\text{lose all } k \text{ independent shares}) = r^k$$

$$\text{Bernoulli's utility: } U(w) = \ln w$$

From history. The puzzle is named for the city where Daniel Bernoulli published his answer, not where it was first posed. Gabriel Cramer, a Swiss mathematician, had suggested a similar answer in a 1728 letter. Bernoulli's idea of diminishing utility became a foundation of economics and of the theory of insurance.

WORKED EXAMPLES

Example 1. In the St. Petersburg game, the prize is 2^n coins if the first head is on toss n . Show that the expected value is infinite, and find it if the bank can pay at most 2^{20} coins.

Show solution

1. Toss n has probability $\frac{1}{2^n}$ and prize 2^n , so it contributes $\frac{1}{2^n} \cdot 2^n = 1$ to the expected value.
2. There are infinitely many possible tosses, so $E = 1 + 1 + 1 + \dots$, which has no limit.
3. With a bank of 2^{20} : tosses 1 to 20 contribute 20. For $n > 20$, the prize is capped at 2^{20} , contributing $2^{20} \times P(n > 20) = 2^{20} \times 2^{-20} = 1$.
4. $E = 20 + 1 = 21$ coins.

Answer: Infinite in theory; about 21 coins with a bank of about a million coins.

Example 2. You have 100 points. Choice A: a sure gain of 50 points. Choice B: a coin flip that gains 100 points or nothing. Compare by expected value and by Bernoulli's log utility.

Show solution

1. Expected final points: A 150; B $\frac{1}{2}(200) + \frac{1}{2}(100) = 150$. Equal.
2. Utility of A: $\ln 150 \approx 5.011$.
3. Utility of B: $\frac{1}{2} \ln 200 + \frac{1}{2} \ln 100 \approx \frac{1}{2}(5.298) + \frac{1}{2}(4.605) = 4.952$.

Answer: Same expected value, but the sure gain has higher utility. A person who values what he has prefers the certain 50.

Example 3. A ship owner's cargo faces a 10% chance of loss on any one voyage. Compare sending it all on one ship with splitting it equally among four ships with independent risks.

Show solution

1. One ship: expected loss 10%; chance of losing everything 0.1.
2. Four ships: each quarter is lost with probability 0.1, so the expected loss is still $4 \times 0.25 \times 10\% = 10\%$.
3. Chance of losing everything: $0.1^4 = 0.0001$ (1 in 10,000). Chance of losing nothing: $0.9^4 \approx 0.656$.

Answer: The same expected loss, but the chance of ruin falls from 1 in 10 to 1 in 10,000.

Example 4. Illustration: a soccer team leads 1–0 with 20 minutes left. If it keeps playing normally, suppose the opponent scores (at least once) with probability 0.30 and the team scores again with probability 0.25. If it pulls everyone back, suppose the opponent scores with probability 0.45 and the team with only 0.05. Treating each side as scoring at most once and independently, which plan wins more often?

Show solution

1. Normal play wins if the opponent does not score (0.70), or if both score ($0.30 \times 0.25 = 0.075$): 0.775.
2. Pulling back wins with $0.55 + 0.45 \times 0.05 = 0.5725$.
3. Under these numbers the “safe” plan lowers the chance of winning by about 20 percentage points.

Answer: Normal play (about 78%) beats pulling back (about 57%). A plan that feels safe is not safe if it hands the opponent the variance.

PRACTICE

1. Easy In the St. Petersburg game, what is the probability that the first head comes on toss 3, and what is the prize?

Show answer

$\frac{1}{8}$; the prize is $2^3 = 8$ coins.

2. **Easy** A risk of ruin is 5% each time, independently. What is the chance of surviving 20 times?

Show answer

$0.95^{20} \approx 0.358$, about 36%.

3. **Easy** In your own words, what did Daniel Bernoulli mean by utility?

Show answer

The real worth of a gain to a person depends on what he already has; the same gain matters more to someone with little.

4. **Medium** If the St. Petersburg bank can pay at most $2^{10} = 1,024$ coins, what is the game's expected value?

Show answer

Tosses 1–10 contribute 10, and the capped tail contributes 1: $E = 11$ coins.

5. **Medium** A cargo is split equally among 3 ships, each with an independent 20% chance of loss. Find the expected fraction lost and the chance of losing everything.

Show answer

Expected loss 20%; everything lost with probability $0.2^3 = 0.008$.

6. **Medium** You have 50 points. Compare a sure gain of 30 points with a coin flip for 60 points or nothing, using expected value and $\ln$.

Show answer

Both have expected final value 80. Utility: $\ln 80 \approx 4.382$ against $\frac{1}{2}(\ln 110 + \ln 50) \approx \frac{1}{2}(4.700 + 3.912) = 4.306$. The sure gain is better.

7. **Medium** Explain why the most dangerous risks are often small ones repeated many times.

Show answer

Each one feels safe, but survival probabilities multiply; a 1-in-100 risk taken 100 times leaves only about a 37% chance of never meeting it.

8. **Medium** Ecclesiastes 11:2 says, “Give a portion to seven, and also to eight.” Explain this as advice about risk.

Show answer

Spread what you have among many independent places, so that no single misfortune can take everything: diversification.

9. **Hard** With a 1-in-100 chance of ruin each time, about how many independent tries does it take before the chance of surviving all of them falls below one-half? (Try $n = 68$ and $n = 69$.)

Show answer

$0.99^{68} \approx 0.505$ and $0.99^{69} \approx 0.4998$. About 69 tries.

10. **Hard** Use the soccer illustration of Worked Example 4, but suppose that pulling back lowers the opponent’s chance of scoring to 0.25 (while the team’s own chance falls to 0.05). Which plan is better now?

Show answer

Pulling back: $0.75 + 0.25 \times 0.05 = 0.7625$; normal play: 0.775. Normal play is still slightly better, but now the plans are close; the answer depends on real data.

11. **Hard** Why does a cap on the bank make the St. Petersburg game worth so little?

Show answer

All the infinite expected value comes from enormous prizes with tiny probabilities. When prizes above the cap are cut off, those terms vanish, and only a few coins of expected value remain.

12. **Challenge** Using $U = \ln w$, find the sure amount (the certainty equivalent) that a person with 100 points would accept instead of a coin flip for +100 or +0. Solve $\ln(100 + c) = \frac{1}{2}(\ln 200 + \ln 100)$.

Show answer

$100 + c = \sqrt{200 \times 100} = \sqrt{20,000} \approx 141.4$, so $c \approx 41.4$ points, less than the expected gain of 50.

COMMON MISTAKES

- Treating expected value as the only guide when an outcome could be ruinous.
- Adding risks instead of multiplying survival probabilities over repeated tries.

- Assuming that splitting a risk lowers the expected loss. It lowers the spread and the chance of total loss, not the average.

Make it a game · St. Petersburg Experiment. Flip a paper chip until it lands tails. If the first tail is on flip one, the payout is 2 chips; flip two, 4; flip three, 8; and so on. Play twenty rounds and compute your average payout. Then predict what a sensible person would pay to play, and explain Bernoulli's answer about how much a gain means depending on what you already have.

Scripture. “Give a portion to seven, and also to eight; for thou knowest not what evil shall be upon the earth.” (Ecclesiastes 11:2, KJV)

The Preacher’s counsel is diversification: because we do not know the future, we spread our portions so that no single evil can take everything.

UNIT 2

EXPECTED VALUE: WEIGHING REWARD AND RISK

In 1657 Christiaan Huygens asked what a chance is worth and answered with the idea that runs through the rest of this course: expected value, the long-run average of a choice. This unit puts it to work. You will compute points per attempt for twos and threes, extra points and two-point tries; measure the spread of a strategy with variance and see why David chose a different contest and why a team that trails should seek variance; prove that the house's games are built to lose for the player and that insurance is their wise mirror image; and meet Daniel Bernoulli's answer to the St. Petersburg paradox, the risk of ruin and the wisdom of spreading risk.

The projects measure your own shooting percentages from two distances, test which strategy reaches a high target more often, simulate a house game and an insurance pool in points, and play the St. Petersburg game. Every project keeps score in points, never money.

PREREQUISITES

- Unit 1: probability as a fraction, decimal and percent; independence; the law of large numbers and the standard error
- Pre-algebra: weighted averages, square roots and exponents
- Optional (challenge problems): natural logarithms on a calculator

MASTERY CHECKLIST

- ☐ I can compute an expected value from a probability table and explain it as a long-run average.
- ☐ I can compute points per attempt and break-even percentages for shots and conversions.
- ☐ I can compute variance and standard deviation and explain when to increase or reduce variance.
- ☐ I can show that a casino or lottery game has negative expected value and explain why insurance is different.
- ☐ I can explain the St. Petersburg paradox, Bernoulli's utility, the risk of ruin and the value of spreading risk.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Khan Academy: Random variables and expected value**: Free. Videos and practice on expected value, variance and standard deviation that match Lessons 2.1, 2.3 and 2.4.
- **Basketball-Reference: NBA league averages**: Free data. Season-by-season league averages of three-point attempts, three-point percentage and free-throw percentage since 1946, used in Lesson 2.2. Contains advertising; a parent should preview it.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

2.1	Huygens and the Value of a Chance	3 sessions of 50 minutes
2.2	Points per Shot	3 sessions of 50 minutes
2.3	Same Average, Different Risk	3 sessions of 50 minutes
2.4	The House Always Wins	3 sessions of 50 minutes
2.5	Risk of Ruin	3 sessions of 50 minutes

UNIT 2 TEST

Calculators are allowed. Show your work for every calculation: write the probability table or formula, substitute, and give the answer with its unit (points, coins or percent). Answer the concept questions in complete sentences. The test is worth 100 points.

1. **5 pts**

A spinner scores 0, 3 or 6 points with probabilities 0.5, 0.3 and 0.2. Find the expected score.

2. **6 pts**

Use Huygens's rule: 4 equally likely chances of 9 points and 2 chances of 3 points. What is the chance worth?

3. (6 pts)

A player makes 46% of twos and 35% of threes. Compute points per attempt for each and say which is the higher-percentage shot.

4. (6 pts)

What three-point percentage equals a 51% two-point percentage?

5. (6 pts)

A kicker makes 95% of extra points. What two-point success rate breaks even? If the team converts 52%, which should it choose, other things equal?

6. (6 pts)

A 75% free-throw shooter goes to the line for two shots. Find the expected points and compare with a 34% three.

7. (8 pts)

Find the mean, variance and standard deviation of a shot worth 3 points made 40% of the time.

8. (7 pts)

Two strategies have the same expected points. Explain which a trailing team should choose and which a leading team should choose, and why.

9. (6 pts)

A game costs 3 points and pays 20 points with probability $\frac{1}{8}$. Find the expected net result per play and the house edge.

10. (8 pts)

Explain, using the law of large numbers, why a casino is nearly certain to profit even though some players leave as winners.

11. (8 pts)

A family's car is worth 20,000 dollars. Suppose the chance of a total loss in a year is 1 in 200, and insurance against it costs 150 dollars. Find the expected net cost of insuring and explain when it is wise.

12. (6 pts)

In the St. Petersburg game with a bank cap of 2^{15} , find the expected value.

13. 6 pts

A cargo is split among 5 ships, each independently lost with probability 0.1. Find the chance of losing everything and the chance of losing nothing.

14. 6 pts

A plan carries a 3% chance of ruin each time. What is the chance of surviving 30 independent uses of it?

15. 10 pts

Short essay. In about 150 words, explain the difference between the high-percentage shot and the safe shot. Use expected value, variance and risk of ruin, and give one example from a sport and one from family or business life.

ANSWER KEY

1. $0 + 0.9 + 1.2 = 2.1$ points.
2. $\frac{4(9) + 2(3)}{6} = \frac{42}{6} = 7$ points.
3. Two 0.92; three 1.05. The three.
4. $\frac{2}{3}(0.51) = 0.34$ (34%).
5. Break-even 47.5%. At 52% the two-point try (1.04 vs 0.95) has more expected points.
6. 1.5 against 1.02. The free throws are worth more.
7. $\mu = 1.2$; $\text{Var} = 9(0.4)(0.6) = 2.16$; $\sigma \approx 1.47$.
8. The trailing team should choose the higher-variance strategy, because it needs a large result that only a wide spread can give. The leading team should choose the lower-variance strategy, because it needs only to avoid a large swing against it.
9. $\frac{20}{8} - 3 = -0.5$ points per play; edge $0.5/3 \approx 16.7\%$.
10. Each play has a small negative expected value for the player. Over millions of plays the house's total gain grows with the number of plays while the spread grows only with its square root, so the house's average gain is nearly certain; individual players play few times, so luck dominates their results.
11. Expected loss $20,000/200 = 100$; net cost 50 dollars a year. It is wise when losing the car would be a heavy blow the family could not easily absorb: insurance trades a small expected cost for protection from ruin.
12. $15 + 1 = 16$ coins.
13. Everything: $0.1^5 = 0.00001$; nothing: $0.9^5 \approx 0.590$.
14. $0.97^{30} \approx 0.401$, about 40%.
15. Answers vary. A strong essay defines the high-percentage shot as the one with the highest expected value, explains that the safe shot has low variance and may have lower expected value, says the right choice depends on the goal (needing a big result versus protecting a lead) and on whether any outcome would be ruinous, and gives two concrete examples.

PORTFOLIO PIECE: AN EXPECTED-VALUE STRATEGY GUIDE FOR A FAMILY GAME

Choose a family game with dice, cards or shooting that involves real choices (for example Yahtzee, Catan, a backyard shooting game or a card game played for points). Identify three decisions a player faces repeatedly. For each, list the options, estimate their probabilities (by counting, by simulation or by recording at least 30 real turns), and compute the expected points and the spread of each option. Then write a two-page strategy guide for a younger sibling: what to do when ahead, what to do when behind, and which risks are never worth taking. Include your tables, at least one graph, and a section on what your guide cannot predict.

LOOK FOR	WHAT IT MEANS
Decisions and options	Three real, recurring decisions clearly described, each with its options.
Probabilities and expected values	Probabilities found by counting, simulation or recorded data, with correct expected-value and spread calculations shown.
Ahead and behind	Advice that changes sensibly with the score, using the idea of increasing or reducing variance.
Clarity for a younger reader	Plain language, a clear table or chart, and examples a younger sibling could follow.
Honest limits	A thoughtful section on what the guide cannot predict and where the data were thin.

THE HIGH-PERCENTAGE SHOT: SPORTS BY THE NUMBERS

Every sport keeps score, but only in the last century and a half have people counted carefully enough to learn which plays really produce points. This unit follows the counters: Henry Chadwick and his box score, Branch Rickey and Allan Roth arguing for on-base percentage in 1954, Bill James and the sabermetricians with their run-expectancy table, Charles Reep with his notebooks and his famous misreading, the basketball analysts who drew shot charts of the whole league, and Mark Broadie, who measured every shot in professional golf.

You will compute batting average, on-base and slugging percentages, judge bunts and stolen bases by run expectancy, add up expected goals, rank basketball shots by points per shot, and choose golf targets and tennis serves by expected value and dispersion. The projects use real data: a box score you keep yourself, a run-expectancy table built from Retrosheet play-by-play, a shot chart of a real match, your own shooting from five spots, and a target game that measures your dispersion.

PREREQUISITES

- Units 1 and 2: probability, independence, expected value, variance and the standard error
- Comfort with percentages, decimals and simple formulas
- Optional: basic spreadsheet use for organizing data

MASTERY CHECKLIST

- ☐ I can compute BA, OBP and SLG and explain why OBP measures a hitter's value better than BA.
- ☐ I can use a run-expectancy table to judge a play and find the break-even rate for a stolen base.
- ☐ I can compute expected goals and the chance of scoring at least once, and explain the base-rate mistake in Reep's reasoning.
- ☐ I can rank basketball shots by points per shot and explain the value of free throws.
- ☐ I can compute strokes gained and the probability of winning a service point, and choose a target by expected value and dispersion.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Retrosheet**: Free. Box scores and play-by-play for major-league games going back more than a century, collected by volunteers; used in Lessons 3.1 and 3.2. No advertising.
- **Baseball-Reference**: Free data. Every player's and team's statistics, including OBP, SLG and Pythagorean records; used in Lessons 3.1 and 3.2. Contains advertising; a parent should preview it.
- **Society for American Baseball Research (SABR)**: Free articles and biographies on baseball history and research, including Henry Chadwick, Branch Rickey and Allan Roth. Fits Lessons 3.1 and 3.2.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

3.1	Henry Chadwick's Box Score	3 sessions of 50 minutes
3.2	Sabermetrics	3 sessions of 50 minutes
3.3	Shot Quality	3 sessions of 50 minutes
3.4	The Shot Chart	3 sessions of 50 minutes
3.5	Course Management and the Second Serve	3 sessions of 50 minutes

UNIT 3 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer to a sensible number of decimal places. Use the run-expectancy values printed in Lesson 3.2 when needed. Answer the concept questions in complete sentences. The test is worth 100 points.

1. **6 pts**

A player has 160 hits, 70 walks, 4 hit-by-pitches and 6 sacrifice flies in 540 at bats. Find his BA and OBP.

2. **6 pts**
The same player had 100 singles, 35 doubles, 5 triples and 20 home runs. Find his SLG and OPS.
3. **6 pts**
Explain what Branch Rickey and Allan Roth argued in 1954 and why it was important.
4. **6 pts**
Runners on 1st and 2nd, 1 out. The batter grounds into a double play (inning over). Find ΔRE .
5. **8 pts**
Runner on 2nd, 0 outs. Find the break-even rate for stealing third. (Success: 3rd, 0 outs = 1.35; caught: empty, 1 out = 0.25; stay = 1.10.)
6. **6 pts**
A team scores 700 runs and allows 800. Estimate its wins in 162 games by the Pythagorean formula.
7. **6 pts**
Why do modern teams bunt less often than teams did fifty years ago?
8. **6 pts**
A soccer team's shots have xG 0.45, 0.08, 0.08, 0.04 and 0.15. Find its xG and the probability it scores at least once (assume independence).
9. **8 pts**
Illustrative data: 5,000 moves had three or fewer passes and produced 60 goals; 500 moves had more passes and produced 15 goals. What share of goals came from short moves, and which kind of move scored more often per move? What does this teach about Reep's conclusion?
10. **6 pts**
Rank by points per shot: a 62% shot at the rim, a 43% midrange shot and a 37% three.

11. (6 pts)

A player who makes 55% of free throws overhand makes 70% underhand in practice. If she goes to the line for 6 two-shot trips a game, how many points per game could underhand add?

12. (6 pts)

Rick Barry made 90% of his free throws underhand, yet few players copy him. Give one mathematical and one human reason.

13. (6 pts)

A golf shot moves the ball from a spot worth 3.0 strokes to one worth 2.4. Find the strokes gained, and explain what the sign means.

14. (6 pts)

With $a_1 = 0.60$, $w_1 = 0.75$, $a_2 = 0.88$, $w_2 = 0.55$, find the probability of winning a service point and the probability of a double fault.

15. (12 pts)

Short essay. In about 150 words, choose one sport from this unit and explain how counting changed the way it is played. Name the person or people who did the counting, the number they discovered mattered most, and one way a player or coach should act differently because of it.

ANSWER KEY

1. $BA = 160/540 \approx .296$; $OBP = 234/620 \approx .377$.
2. $TB = 100 + 70 + 15 + 80 = 265$; $SLG = 265/540 \approx .491$; $OPS \approx .868$.
3. That on-base average mattered more than batting average for producing runs, based on Roth's detailed records. It began the shift from traditional statistics to evidence about what actually wins games; OBP became official in 1984.
4. $0 - 0.88 = -0.88$ runs.
5. $\frac{1.10 - 0.25}{1.35 - 0.25} = \frac{0.85}{1.10} \approx 77\%$.
6. $\frac{490,000}{490,000 + 640,000} \approx 0.434$; about 70 wins.
7. The run-expectancy table shows that a sacrifice usually lowers expected runs, because it gives up an out to gain a base; teams now save bunts for special situations where one run matters most.
8. $xG\ 0.80$; $P(0) = 0.55 \times 0.92^2 \times 0.96 \times 0.85 \approx 0.380$, so $P(\geq 1) \approx 0.62$.
9. Short moves produced $60/75 = 80\%$ of goals, but scored on 1.2% of moves against 3% for long moves. Reep looked at the share of goals without the base rate; per move, long moves scored more often.
10. Rim 1.24, three 1.11, midrange 0.86.
11. $12 \times (0.70 - 0.55) = 1.8$ points per game.
12. Mathematical: few players have tried it long enough to prove it works better for them. Human: many players feel it looks foolish and care more about appearance than percentage.
13. $3.0 - 2.4 - 1 = -0.4$: the shot was worse than an average tour shot from that spot by 0.4 strokes.
14. $0.45 + 0.40(0.484) = 0.6436$; double fault $0.40 \times 0.12 = 0.048$.
15. Answers vary. A strong essay names the right people (Chadwick, Rickey and Roth, James, Reep or the xG analysts, the NBA shot-chart analysts, Broadie), states the key measure (OBP, run expectancy, xG, points per shot, strokes gained), and gives a concrete change in decisions, with a sentence on the risk or limits of the number.

PORTFOLIO PIECE: A SPORTS DATA STUDY

Choose one sport and one question that data can answer, for example: Should my favorite team steal more or less? Which of my team's players should shoot more threes? Does our soccer team create better chances than it scores? Gather real data from a free source (Retrosheet, Baseball-Reference, Basketball-Reference, a league site) or from at least three games you chart yourself. Compute the appropriate measures from this unit (OBP and SLG, run expectancy, xG, points per shot, strokes gained or serve-win probability), make at least one chart, and write a two-page report with your question, data, calculations, chart, conclusion and its limits. End with a specific recommendation to a coach.

LOOK FOR	WHAT IT MEANS
Question and data	A clear question that the data can answer, with data from a named source or from at least three charted games.
Calculations	Correct use of at least two measures from the unit, with formulas and work shown.
Chart	A labeled chart or table (shot chart, run-expectancy grid, bar graph) that makes the main finding clear.
Honest limits	Discussion of sample size, base rates and what the data cannot show.
Recommendation	A specific, reasoned recommendation to a coach, with its expected gain and its risk.

3.1 HENRY CHADWICK'S BOX SCORE

OBJECTIVES

- Tell how Henry Chadwick invented the box score and why counting a game changed how it was understood.
- Compute batting average, on-base percentage and slugging percentage from a player's counting statistics.
- Explain, as Branch Rickey and Allan Roth argued in 1954, why on-base percentage measures a hitter's value better than batting average.

Warm-up. Watch or listen to one inning of a baseball or softball game (or recall the last inning you played). For every batter, write what happened: hit, walk, out, hit by pitch. How many batters reached base? How many got hits? Which number tells you more about how many runs the team could score?

WATCH

HOW TO READ A BOX SCORE - Baseball Basics

[Ashley MacLennan](#) · 5:08

<https://www.youtube.com/watch?v=Dh-VWsdU448>

WHEN TO WATCH

Before the project. It reads a modern box score column by column; keep a real box score from Retrosheet or a newspaper open beside you.

BACKUP VIDEO

Batting Average, On-Base Percentage, Slugging, and OPS+ - Baseball Basics

[Ashley MacLennan](#) · 8:37

<https://www.youtube.com/watch?v=nphT9Ztjgh0>

After “the mathematics” paragraph. It shows how batting average, on-base percentage and slugging are calculated; pause and compute each example yourself.

KEY TERMS

box score

A compact table that records each player's counting statistics for one game: at bats, runs, hits, runs batted in, walks and more.

on-base percentage (OBP)

The fraction of plate appearances in which the batter reaches base by hit, walk or hit-by-pitch:

$$\frac{H + BB + HBP}{AB + BB + HBP + SF}.$$

out

The scarce resource of baseball: each team has only 27 in a nine-inning game. Avoiding outs is the batter's first job.

batting average (BA)

Hits divided by at bats, H/AB . Walks and hit-by-pitches are not at bats and are ignored.

slugging percentage (SLG)

Total bases per at bat, TB/AB , where a single counts 1, a double 2, a triple 3 and a home run 4.

CORE IDEAS

The story. Henry Chadwick was born in Exeter, England, in 1824 and came to America as a boy. He grew up with cricket, whose newspapers printed a scorecard of every match. In the 1850s he began writing about the new game of baseball for New York newspapers and soon became its most important writer. Around 1859 he adapted the cricket scorecard into the baseball **box score**, a small grid that told, player by player, what had happened in a game. He also gave us the letter K for a strikeout. For decades he edited the annual baseball guides, campaigned for clean, honest play, and wrote the rules into shape. He was elected to the Baseball Hall of Fame in 1938, the only writer honored there as a builder of the game.

How they figured it out. Once games were counted, people began to rank players. The favorite number became the **batting average**, hits per at bat, and for a century it decided batting titles and contracts. But it ignores walks, as if a batter who reaches first base on four balls had done nothing. In the late 1940s Branch Rickey, president of the Brooklyn Dodgers, hired Allan Roth as baseball's first full-time team statistician. Roth tracked every pitch and every plate appearance. On August 2, 1954,

Life magazine printed Rickey's article "Goodby to Some Old Baseball Ideas," which argued from Roth's numbers that getting on base, not batting average, was what produced runs. Major League Baseball made on-base percentage an official statistic in 1984.

The mathematics. A rate statistic divides a count by the number of chances.

$$BA = \frac{H}{AB}, OBP = \frac{H + BB + HBP}{AB + BB + HBP + SF}, \text{ and}$$

$SLG = \frac{TB}{AB} = \frac{1B + 2(2B) + 3(3B) + 4(HR)}{AB}$. Each is a probability estimated from data, so the standard error of Lesson 1.5 applies: over 500 plate appearances, an OBP of .350 has a standard error of about $\sqrt{0.35 \times 0.65/500} \approx 0.021$. A common summary is OPS = OBP + SLG. The key idea is that each team gets only 27 outs; anything that avoids an out keeps the inning alive, so OBP measures the most important thing a batter does.

On the field. In 1941 Ted Williams of the Boston Red Sox batted .406, the last .400 season in the major leagues, and his on-base percentage was .553, because he also drew 147 walks. A manager who looks only at batting average sees a great hitter; one who looks at OBP sees a player who avoided an out more than half the time. Today coaches build lineups so that the best on-base hitters bat most often. **The risk side:** every out is a risk the team cannot take back, which is why the high-percentage hitter who waits for a good pitch and takes his walk is worth more than his batting average shows.

PICTURE STUDY



Henry Chadwick, G. Frank E. Pearsall, Late 19th century · Photograph · New York Public Library

Image: [Wikimedia Commons](#), public domain

Chadwick was called the “Father of Baseball” in his own lifetime, not because he invented the game, but because he did so much to shape its rules, its records and its reputation for honest play.

Look closely

- Look at his face and beard. About how old does he seem? What does that tell you about how long he served the game?
- Look at his clothing. Does he look like a player or a writer?
- Notice the plain studio background. What would you add to show what he gave baseball?

Talk about it

1. Chadwick counted what every player did in every game. Why does careful counting make a game fairer to the players?
2. If Chadwick had never invented the box score, could Rickey and Roth have discovered the value of on-base percentage? Why or why not?

PROJECT

Keep the Book

You will keep a real box score, then compute batting average, on-base percentage and slugging for real players, and decide whom you would want batting first.

You will need

- A real game to watch or listen to, or a box score from Retrosheet (retrosheet.org) or Baseball-Reference
- The scorebook, a ruler and a calculator

Steps

1. Rule a box-score grid in your scorebook: columns AB, R, H, 2B, 3B, HR, BB, HBP, SF, K; one row per batter.
2. **Predict.** Before the game, write which player you expect to have the best on-base percentage and why.
3. Watch or listen to at least five innings of a game (high-school, college or professional), and record every plate appearance for one team. If you cannot watch a game, copy a complete box score and play-by-play from Retrosheet.
4. Check your totals: hits in the box should equal the team's hits in the line score.
5. Now go to Baseball-Reference and choose five players from one team. Copy each player's season AB, H, 2B, 3B, HR, BB, HBP and SF.
6. Compute each player's BA, OBP and SLG by hand, then check against the website.
7. Rank the five players by BA and then by OBP. Do the rankings agree?
8. Find the player whose OBP is furthest above his BA. How many walks made the difference?
9. **Decide.** Write the batting order you would choose for these five, with a sentence for each about why.

Tips

- A plate appearance that ends in a walk, hit-by-pitch or sacrifice fly is not an at bat.
- Sports Reference sites carry advertising; a parent should preview them.

Check your work

- ☐ I kept a complete box score for at least five innings, and my totals check.
- ☐ I computed BA, OBP and SLG for five real players and checked them.
- ☐ I wrote a batting order with reasons based on OBP.

SCOREBOOK

1. Your box-score grid for the game, with team totals.
2. Table for five players: AB, H, BB, HBP, SF, TB, BA, OBP, SLG, OPS, with BA rank and OBP rank.
3. **What I would decide and why:** two players have the same batting average, but one walks twice as often. Which would you choose to bat first, and why?

KEY FORMULAS

$$BA = \frac{H}{AB}$$

$$OBP = \frac{H + BB + HBP}{AB + BB + HBP + SF}$$

$$SLG = \frac{1B + 2(2B) + 3(3B) + 4(HR)}{AB}$$

From history. Chadwick died in 1908 after catching a cold at a game on opening day. Branch Rickey is best remembered for signing Jackie Robinson to the Dodgers in 1945, and he also built baseball's first farm system in St. Louis. Allan Roth's pitch-by-pitch charts for the Dodgers are among the earliest detailed baseball data ever kept.

WORKED EXAMPLES

Example 1. In 1941 Ted Williams had 185 hits in 456 at bats, with 147 walks and 3 hit-by-pitches. (Sacrifice flies were not counted separately that year, so use SF = 0.) Find his BA and OBP.

Show solution

1. $BA = \frac{185}{456} \approx 0.406.$

2. $OBP = \frac{185 + 147 + 3}{456 + 147 + 3} = \frac{335}{606} \approx 0.553.$

3. He reached base in more than 55% of his plate appearances.

Answer: BA .406, OBP .553.

Example 2. Two hitters each have 150 hits in 500 at bats. Hitter A walked 20 times; Hitter B walked 90 times. Neither was hit by a pitch or hit a sacrifice fly. Compare their BA and OBP.

Show solution

1. Both: $BA = 150/500 = .300.$

2. A: $OBP = \frac{170}{520} \approx .327.$

3. B: $OBP = \frac{240}{590} \approx .407.$

4. Outs made (roughly AB – H): both 350, but B also came to the plate 70 more times without making an out.

Answer: Same batting average, but B reaches base about 41% of the time against A's 33%: B is the more valuable hitter.

Example 3. Ted Williams's 185 hits in 1941 were 112 singles, 33 doubles, 3 triples and 37 home runs, in 456 at bats. Find his slugging percentage and OPS.

Show solution

1. Total bases: $112(1) + 33(2) + 3(3) + 37(4) = 112 + 66 + 9 + 148 = 335.$

2. $SLG = 335/456 \approx 0.735.$

3. $OPS = 0.553 + 0.735 = 1.288.$

Answer: SLG .735; OPS 1.288.

Example 4. A box score line reads: AB 4, R 1, H 2, 2B 1, BB 1, HBP 0, SF 0. Find the player's BA, OBP and SLG for the game.

Show solution

1. $BA = 2/4 = .500$.

2. $OBP = (2 + 1)/(4 + 1) = 3/5 = .600$.

3. Total bases: one single (1) and one double (2) = 3, so $SLG = 3/4 = .750$.

Answer: BA .500, OBP .600, SLG .750 for the game (a tiny sample!).

PRACTICE

1. **Easy** A player has 45 hits in 150 at bats. What is his batting average?

Show answer

$$45/150 = .300.$$

2. **Easy** Why doesn't a walk count as an at bat?

Show answer

The batter did not have a chance to make a hit; the pitcher failed to throw strikes. Batting average counts only at bats, which is why it ignores walks.

3. **Easy** A player has 120 hits, 60 walks and 5 hit-by-pitches in 450 at bats, with 5 sacrifice flies. Find his OBP.

Show answer

$$\frac{185}{520} \approx .356.$$

4. **Medium** Find the slugging percentage of a player with 80 singles, 25 doubles, 3 triples and 20 home runs in 500 at bats.

Show answer

$$TB = 80 + 50 + 9 + 80 = 219; SLG = 219/500 = .438.$$

5. **Medium** What did Branch Rickey argue in his 1954 *Life* article, and who supplied the numbers?

Show answer

That on-base average, not batting average, better measured a hitter's contribution to scoring runs, and that many old baseball ideas did not survive careful counting. Allan Roth, the Dodgers' statistician, supplied the numbers.

6. **Medium** Can a player's OBP be lower than his batting average? (Hint: think about sacrifice flies.)

Show answer

Yes, rarely: a sacrifice fly adds to OBP's denominator but not to batting average's. A player with few walks and some sacrifice flies can have OBP slightly below BA.

7. **Medium** Explain in two sentences why avoiding outs is the batter's most important job.

Show answer

A team gets only 27 outs in a game, and when they are used up the game is over no matter the score. Every time a batter avoids an out, the inning continues and the team has another chance to score.

8. **Medium** Henry Chadwick adapted the box score from which sport's scorecard, and why did counting matter?

Show answer

Cricket. Counting each player's actions turned impressions into records that could be compared, which made it possible to study what actually produces runs.

9. **Hard** A hitter has OBP .350 over 600 plate appearances. Find the standard error of his OBP and the 95% range of his true OBP.

Show answer

$\sqrt{0.35 \times 0.65/600} \approx 0.0195$; range about .311 to .389.

10. **Hard** Player A: BA .310, OBP .340, SLG .430. Player B: BA .260, OBP .370, SLG .480. Which is more valuable by OPS? Which would batting average alone choose?

Show answer

OPS: A .770, B .850, so B. Batting average alone would choose A.

11. **Hard** Use the 1941 numbers: Ted Williams made about $456 - 185 = 271$ outs on at bats. If he had drawn 47 walks instead of 147 and made outs in the other 100 plate appearances, what would his OBP have been?

Show answer

Same plate appearances (606), on-base events $185 + 47 + 3 = 235$:

$OBP = 235/606 \approx .388$. The walks were worth about 165 points of OBP.

12. **Challenge** Show that if a player never walks, is never hit by a pitch and never hits a sacrifice fly, his OBP equals his BA. Then explain why a hitter with a lower BA but many walks can be more valuable.

Show answer

With $BB = HBP = SF = 0$, $OBP = H/AB = BA$. Walks add to OBP's numerator and denominator equally; since $OBP < 1$, each walk raises it. A high-walk hitter avoids outs more often, which produces more runs, even with a lower BA.

COMMON MISTAKES

- Counting walks as at bats, or leaving them out of OBP.
- Using hits instead of total bases for slugging.
- Trusting a rate from a few games. A .500 hitter after one week has told you very little (Lesson 1.5).

Make it a game · Scorekeeper Sprint. Your parent calls out twenty plate appearances: single, strikeout, walk, home run, and out. You record each in a box score on paper, then compute the batting average, on-base percentage and slugging percentage in under two minutes. Beat your time on a second set. Add one sentence explaining why Henry Chadwick's counting changed how people understood the game.

3.2 SABERMETRICS

OBJECTIVES

- Tell how Bill James and his followers built sabermetrics, and how the 2002 Oakland Athletics used it.
- Read a run-expectancy table and use it to judge plays such as the sacrifice bunt and the stolen base.
- Find the break-even success rate for a stolen base and apply Bill James's Pythagorean expectation.

Warm-up. A runner is on first base with nobody out. Before reading on, guess: on average, how many runs will the team score in the rest of the inning? Now guess for a runner on second with one out. Which situation is better for the offense? Write your guesses down.

WATCH

SHOULD YOU BUNT? Run Expectancy Matrix

[Simple Sabermetrics](#) • 8:31

<https://www.youtube.com/watch?v=quYXRCNQEQU>

WHEN TO WATCH

After “the mathematics” paragraph. It builds the run-expectancy table and tests the sacrifice bunt; compare its numbers with Worked Example 2.

BACKUP VIDEO

WHEN SHOULD YOU STEAL? RE24 Explained

[Simple Sabermetrics](#) • 7:28

<https://www.youtube.com/watch?v=DjY4Bq9Rz80>

Before Worked Example 3. It uses the same table to find when stealing a base is worth the risk.

KEY TERMS

sabermetrics

The search for objective knowledge about baseball through data, named by Bill James from SABR, the Society for American Baseball Research.

run expectancy

The average number of runs scored from a base-out state to the end of the inning.

Pythagorean expectation

Bill James's estimate of a team's winning percentage from runs scored (RS) and allowed (RA):
 $RS^2 / (RS^2 + RA^2)$.

base-out state

The situation in an inning: which bases are occupied and how many outs there are. There are $8 \times 3 = 24$ states.

break-even rate

The success rate at which a play neither adds nor subtracts expected runs.

CORE IDEAS

The story. In 1977 Bill James, working nights as a security guard at a pork-and-beans cannery in Kansas, self-published a booklet of baseball statistics and essays called *The Baseball Abstract*. Few people bought the first edition, but the questions in it were new: Which statistics really predict runs? Do some players help their teams more than their batting averages show? James named his method **sabermetrics** after SABR, the Society for American Baseball Research, founded in 1971. By the 1990s his readers were working for major-league teams. The most famous was the 2002 Oakland Athletics. With one of the lowest payrolls in baseball, they looked for players whose on-base percentage was undervalued by the market, won 103 games and set an American League record with 20 wins in a row. The story was told in Michael Lewis's 2003 book *Moneyball*.

How they figured it out. For a century, managers had played “by the book”: bunt the runner over, steal to make something happen. The sabermetricians asked what each play actually did to a team's chances of scoring. Using play-by-play records of thousands of games, they built the **run-expectancy table**: for each of the 24 base-out states, the average number of runs scored from that point to the end of the inning.

With it, any play can be judged by the change in run expectancy it causes. Volunteers led by Dave Smith founded Retrosheet in 1989 to collect the play-by-play of old games, which made such tables possible for anyone.

The mathematics. Here is the run-expectancy table for the major leagues, 2010–2015 (from Tom Tango’s published tables, rounded to two decimals). Bases empty: 0.48, 0.25, 0.10 (for 0, 1 and 2 outs). Runner on 1st: 0.86, 0.51, 0.22. On 2nd: 1.10, 0.66, 0.32. On 3rd: 1.35, 0.95, 0.35. On 1st and 2nd: 1.44, 0.88, 0.43. On 1st and 3rd: 1.78, 1.13, 0.48. On 2nd and 3rd: 1.96, 1.38, 0.58. Bases loaded: 2.29, 1.54, 0.75. The value of a play is $\Delta RE = RE_{\text{after}} - RE_{\text{before}} + \text{runs scored}$. A stolen base attempt with success rate p is worth trying when $p \cdot RE_{\text{success}} + (1 - p) \cdot RE_{\text{caught}} > RE_{\text{stay}}$, so the break-even rate is $p^* = \frac{RE_{\text{stay}} - RE_{\text{caught}}}{RE_{\text{success}} - RE_{\text{caught}}}$. Bill James also found that a team’s winning percentage is close to $\frac{RS^2}{RS^2 + RA^2}$.

On the field. The table taught teams that outs are precious. A successful sacrifice bunt with a runner on first and nobody out trades an out for a base and *lowers* run expectancy from 0.86 to 0.66. A stolen base attempt from first with nobody out needs to succeed about 72% of the time to break even, because being caught costs far more than success gains. So teams bunt much less than they once did and steal only with runners who succeed often. **The risk side:** the table gives averages. Late in a close game, a team that needs exactly one run may care more about the *chance of scoring at least one* than about expected runs, and a bunt can then make sense. Wise managers know which question they are asking.

PICTURE STUDY



Ty Cobb Stealing Third, Charles M. Conlon, July 23, 1910 · Photograph · Hilltop Park, New York

Image: [Wikimedia Commons](#), public domain

Charles Conlon, a newspaper proofreader who photographed baseball in his spare time, caught this famous moment in 1910: Detroit's Ty Cobb sliding into third base against the New York Highlanders. Cobb stole 897 bases in his career, the record for decades.

Look closely

- Look at the third baseman. Is he in position to make the tag, or has the throw pulled him off the base?
- Look at the runner's slide and the fielder's reach. Which of them is closer to the base?
- Look at the dust and the blur. What does it tell you about how fast this play happened?

Talk about it

1. A steal of third with one out is worth trying only if it succeeds often enough. What would a manager want to know about this runner and this catcher before sending him?
2. Stealing in Cobb's day was common because runs were scarce. How might the value of a stolen base change when teams hit many home runs?

PROJECT

Build Your Own Run-Expectancy Table

You will build a small run-expectancy table from the play-by-play of real games, the same way the sabermetricians built theirs, and compare it with the major-league table.

You will need

- Internet access to Retrosheet (retrosheet.org) box scores and play-by-play
- The scorebook, a ruler and a calculator (or a spreadsheet)

Steps

1. Choose three complete games from Retrosheet's box-score pages (they include play-by-play).
2. **Predict.** Before counting, write your guess for the run expectancy with bases empty and 0 outs, and with a runner on 1st and 0 outs.
3. Rule a table with 24 rows (8 base situations $\times$ 3 out counts) and two columns: "times this state occurred" and "runs scored from here to end of inning."
4. Go through each half-inning play by play. Each time a batter comes up, note the base-out state, then count the runs the team scored from that moment to the end of the inning.
5. Add each occurrence to the right row. (For a 1–2–3 inning, bases empty with 0, 1 and 2 outs each occur once and each adds 0 runs.)
6. After three games (about 54 half-innings), compute each state's run expectancy: total runs $\div$ occurrences.
7. Compare your values for the states that occurred at least 10 times with the major-league table in this lesson.
8. Find every stolen-base attempt and sacrifice bunt in your games and compute its ΔRE with the major-league table.
9. **Decide.** Write a short note to a coach: when should his team steal, and when should it bunt? Use your break-even calculations.

Tips

- Bases empty with 0 outs occurs at the start of every half-inning, so it will have the most data.
- Many states will occur only a few times in three games; that is why real tables use thousands of games.

Check your work

- ☐ My table records occurrences and runs for all 24 states from at least three games.
- ☐ I compared my most common states with the major-league table.
- ☐ I judged every steal and bunt in my games by its change in run expectancy.

SCOREBOOK

1. The 24-row table from your games, with the major-league value beside each row.
2. A list of every steal and bunt in your games with ΔRE for each.
3. **What I would decide and why:** your fastest runner succeeds 65% of the time. Write when (if ever) you would send him, using break-even rates.

KEY FORMULAS

$$\Delta RE = RE_{\text{after}} - RE_{\text{before}} + \text{runs scored}$$

$$p^* = \frac{RE_{\text{stay}} - RE_{\text{caught}}}{RE_{\text{success}} - RE_{\text{caught}}}$$

$$W\% \approx \frac{RS^2}{RS^2 + RA^2}$$

From history. Bill James joined the Boston Red Sox as an adviser in 2003, and the Red Sox won the World Series in 2004. The Society for American Baseball Research was founded by sixteen researchers at Cooperstown, New York, in August 1971. Michael Lewis's book tells the A's story; it is cited here as a source, not as recommended reading for students.

WORKED EXAMPLES

Example 1. Use the table: a batter hits a single with the bases empty and nobody out. What is the change in run expectancy?

Show solution

1. Before: bases empty, 0 outs: 0.48.
 2. After: runner on 1st, 0 outs: 0.86. No runs scored.
 3. $\Delta RE = 0.86 - 0.48 = +0.38$ runs.
-

Answer: The single is worth about 0.38 runs.

Example 2. Runner on 1st, nobody out. The batter lays down a successful sacrifice bunt: runner to 2nd, batter out. Find the change in run expectancy.

Show solution

1. Before: 1st, 0 outs: 0.86. After: 2nd, 1 out: 0.66.
 2. $\Delta RE = 0.66 - 0.86 = -0.20$ runs.
-

Answer: Even a successful bunt costs about 0.20 runs on average in this state.

Example 3. Runner on 1st, nobody out. Find the break-even success rate for a stolen base attempt.

Show solution

1. Stay: 0.86. Success (runner on 2nd, 0 outs): 1.10. Caught (bases empty, 1 out): 0.25.
 2. $p^* = \frac{0.86 - 0.25}{1.10 - 0.25} = \frac{0.61}{0.85} \approx 0.718$.
 3. Check at $p = 0.8$: $0.8(1.10) + 0.2(0.25) = 0.93 > 0.86$.
-

Answer: About 72%: a runner who succeeds less often than that costs his team runs by trying.

Example 4. The 2002 Oakland Athletics scored 800 runs and allowed 654. Use the Pythagorean expectation to estimate their wins in 162 games, and compare with their actual 103.

Show solution

1. $800^2 = 640,000$, $654^2 = 427,716$.

2. $W\% \approx \frac{640,000}{640,000 + 427,716} = \frac{640,000}{1,067,716} \approx 0.599$.

3. Wins $\approx 0.599 \times 162 \approx 97$.

Answer: About 97 wins expected; they won 103, about 6 more than their runs predicted, partly by winning close games, which tends not to repeat (regression to the mean, Lesson 4.4).

PRACTICE

1. **Easy** Using the table, which is better for the offense: runner on 2nd with 1 out, or runner on 1st with 0 outs?

Show answer

Runner on 1st, 0 outs (0.86 against 0.66).

2. **Easy** How many base-out states are there, and why?

Show answer

24: 8 arrangements of runners (each base empty or occupied, $2^3 = 8$) times 3 out counts.

3. **Easy** What does the word “sabermetrics” come from?

Show answer

From SABR, the Society for American Baseball Research, plus “-metrics” (measurement).

4. **Medium** Bases empty, 0 outs: the batter hits a home run. Find ΔRE .

Show answer

After: bases empty, 0 outs (0.48), plus 1 run scored: $0.48 - 0.48 + 1 = +1.00$.

5. **Medium** Runner on 1st, 1 out: a strikeout. Find ΔRE .

Show answer

After: 1st, 2 outs (0.22): $0.22 - 0.51 = -0.29$.

6. **Medium** Runner on 1st, 1 out. Find the break-even rate for a steal of second. (Success: 2nd, 1 out = 0.66; caught: empty, 2 outs = 0.10.)

Show answer

$$\frac{0.51 - 0.10}{0.66 - 0.10} = \frac{0.41}{0.56} \approx 73\%.$$

7. **Medium** A team scores 750 runs and allows 700. Estimate its winning percentage and wins in 162 games.

Show answer

$$\frac{562,500}{562,500 + 490,000} \approx 0.534; \text{ about 87 wins.}$$

8. **Medium** Explain why a successful sacrifice bunt can lower a team's expected runs.

Show answer

It gives up an out, the scarcest resource, to advance a runner one base. Outs cost more expected runs than the extra base gains, so run expectancy falls (0.86 to 0.66).

9. **Hard** Runner on 1st, 2 outs. Find the break-even rate for a steal. (Success: 2nd, 2 outs = 0.32; caught ends the inning = 0.)

Show answer

$$p^* = \frac{0.22 - 0}{0.32 - 0} \approx 69\%.$$

10. **Hard** Runners on 1st and 2nd, 0 outs: a successful bunt moves them to 2nd and 3rd with 1 out. Find ΔRE and explain why late in a tied game a manager might still bunt.

Show answer

$1.38 - 1.44 = -0.06$: a small loss in expected runs. But with runners on 2nd and 3rd and 1 out, a sacrifice fly or ground ball can score a run, so the chance of scoring at least one run may rise; when one run wins the game, that chance matters more than the average.

11. **Hard** A runner succeeds on 30 of 40 steal attempts from first with nobody out. Using Worked Example 3, about how many runs did his attempts add or cost in total?

Show answer

Each success: $1.10 - 0.86 = +0.24$; each caught: $0.25 - 0.86 = -0.61$. Total $30(0.24) + 10(-0.61) = 7.2 - 6.1 = +1.1$ runs: a small gain, since 75% is just above break-even.

12. **Challenge** Show that, by the Pythagorean formula, a team that scores 10% more runs than it allows ($RS = 1.1 RA$) should win about 54.75% of its games, whatever the scoring level.

Show answer

$$\frac{(1.1RA)^2}{(1.1RA)^2 + RA^2} = \frac{1.21}{2.21} \approx 0.5475; \text{ the } RA^2 \text{ cancels.}$$

COMMON MISTAKES

- Forgetting to add runs that score on the play when computing ΔRE .
- Treating the run-expectancy table as exact; it is an average over many games and changes from era to era.
- Using expected runs when the real goal late in a close game is the chance of scoring at least one run.

Make it a game • Rank the Base-Out States. Write eight base-out states on cards, such as runner on first with no outs, bases loaded with two outs, and nobody on with two outs. Race to rank them from most to fewest expected runs. Your parent ranks the same set secretly. Compare. Then name Bill James, the 2002 Athletics, and the idea behind run expectancy.

3.3 SHOT QUALITY

OBJECTIVES

- Explain expected goals (xG) as the sum of the scoring probabilities of a team's shots.
- Compare high-quality and low-quality shots in soccer and hockey, and compute the chance of scoring at least once from a set of shots.
- Tell the story of Charles Reep's data and explain why "most goals come from short moves" did not prove that long balls are the best strategy.

Warm-up. Imagine two soccer teams. Team A took 18 shots, mostly from far outside the penalty area. Team B took 6 shots, mostly from inside the six-yard box. Before reading on, write which team you think created better chances, and how you would measure it.

WATCH

The Mathematics of Expected Goals Explained

McKay Johns · 11:57

<https://www.youtube.com/watch?v=VcdFOiBWskM>

WHEN TO WATCH

After "the mathematics" paragraph. A soccer analyst works through how expected-goals probabilities are estimated from distance and angle on a whiteboard.

Note: The description advertises the presenter's paid course; it is not needed.

BACKUP VIDEO

Charles Reep: The First Football Data Analyst (?)

Armband Media · 5:33

<https://www.youtube.com/watch?v=EVcBV3Qswr8>

With "how they figured it out." It tells Charles Reep's story and what went wrong with his conclusion.

KEY TERMS

expected goals (xG)

The probability that a shot becomes a goal, estimated from thousands of similar shots; a team's xG is the sum over all its shots.

shot quality

How likely a shot is to score, depending on distance, angle, the type of chance and the defense.

the slot

In hockey, the area directly in front of the goal between the faceoff circles, where most goals are scored.

base rate

How common something is overall. Without it, counts like “most goals came from X” can mislead.

correlation and causation

Two things happening together (correlation) does not prove that one causes the other or that doing more of one will produce more of the other.

CORE IDEAS

The story. On March 18, 1950, at a match between Swindon Town and Bristol Rovers, an English accountant and former Royal Air Force officer named Charles Reep took out a notebook and began recording every pass and shot. He kept it up for decades, notating thousands of matches by hand. In 1968 he and the statistician Bernard Benjamin published “Skill and Chance in Association Football” in the *Journal of the Royal Statistical Society*. Reep found that most goals came from moves of three passes or fewer, and he became a champion of getting the ball forward quickly and directly. His ideas shaped English coaching for a generation.

How they figured it out. Later analysts saw the mistake in the reasoning. Almost all moves in soccer are short, because most possessions are lost within a few passes. So most goals came from short moves simply because most moves of *any* kind were short. The right question is not “what fraction of goals came from short moves?” but “what fraction of short moves, and of long moves, produced goals?” When later researchers asked it, longer possessions turned out to score at least as often per move. Reep counted carefully and honestly, but he read his counts without the **base rate**. Modern analysts took a better path: they recorded where each shot was taken

and whether it went in, over hundreds of thousands of shots, and from them estimated the probability that a shot from a given place becomes a goal. That probability is called **expected goals**, or xG.

The mathematics. If a team's shots have scoring probabilities $p_1, p_2, \dots, p_n$, its expected goals are $xG = p_1 + p_2 + \dots + p_n$. If the shots are roughly independent, the chance of scoring no goals is $(1 - p_1)(1 - p_2) \dots (1 - p_n)$, so $P(\text{at least one goal}) = 1 - \prod(1 - p_i)$. In many published models a penalty kick is worth about 0.76 xG, a close-range shot in front of goal a few tenths, and a shot from 25 meters only a few hundredths. The same idea works in hockey, where shots from the slot score far more often than shots from the blue line. A few good chances usually beat many poor ones.

On the field. Coaches now design attacks to create high-xG chances: passes across the box, cut-backs to a teammate near the penalty spot, rebounds and screens in front of the hockey net. Defenders try the opposite: force shots from far out and wide. xG also helps judge a performance honestly: a team that lost 1–0 while creating 2.5 xG to 0.4 probably played well and was unlucky, and the law of large numbers says such luck evens out over a season. **The risk side:** xG is an average from many players and situations; a particular striker may finish better than average, and a model cannot see everything. Use it as evidence, not as a verdict.

PICTURE STUDY



Hockey at Engelberg, Switzerland, Bain News Service, c. 1910 · Glass-plate photograph · Library of Congress, Washington, D.C.

Image: [Wikimedia Commons](#), public domain

The Bain News Service of New York collected photographs from around the world for newspapers. This one shows an outdoor hockey match in the Swiss mountain resort of Engelberg, about the time organized hockey was spreading across Europe.

Look closely

- Find the goal and the goaltender. Where are most of the players gathered?
- Look at the size of the rink and the positions of the players. From where would a shot be most likely to score?
- Look at the hotels and the crowd. What does the setting tell you about how the game was played then?

Talk about it

1. If you could chart every shot in this match, where would you expect the goals to come from, and why?
2. How would you persuade a player who loves long shots that fewer, closer shots are better?

Safety first. Warm up before shooting or throwing. Use a soft ball indoors, and keep clear of windows, lamps and people. A parent supervises any trip to a court, field or park.

PROJECT

Chart the Chances

You will build your own expected-goals table from your own shots, then chart a real match and compare goals with xG.

You will need

- A soft ball and a goal (two cones, a garage door marked with tape, or a laundry basket on its side)
- Tape measure, chalk or tape for zone marks
- The scorebook
- A soccer or hockey match to watch (live, on television or online)

Steps

1. **Predict.** Mark three zones in front of your goal: close (2 m, straight on), medium (5 m, straight on) and wide (5 m, at a sharp angle). Write your guess for your scoring rate from each.
2. With a partner guarding the goal (soft ball only), take 20 shots from each zone and record makes.
3. Compute each zone's scoring rate. This is your personal xG table.
4. Play a short game to 5 goals in which you may shoot from any zone. Record each shot's zone and result, and total your xG after the game.
5. Compare your goals with your xG. Were you lucky, unlucky or about right?
6. **Chart a match.** Draw a half-pitch (or half-rink) in your scorebook. Watch one match and mark every shot by one team with a dot (an X for a goal).
7. Assign each shot an xG from this guide (illustrative): inside the goal area 0.35; central inside the penalty area 0.12; wide or edge of area 0.05; outside the area 0.03; penalty 0.76. For hockey: slot 0.15; inside the circles 0.07; point or blue line 0.03.
8. Add up each team's xG and compare with the score.
9. **Decide.** Write a short scouting report: which team created better chances, and what one change would help the other team most?

Tips

- Shoot with the same kind of kick or throw every time so that only the zone changes.
- In a televised match, the broadcast often shows the shot location on replay; pause to mark it.

Check your work

- ☐ I measured my scoring rate from three zones with 20 shots each.
- ☐ I charted every shot by one team in a real match and computed its xG.
- ☐ My scouting report uses xG and the shot chart, not just the score.

SCOREBOOK

1. Your personal xG table: zone, attempts, goals, rate.
2. Your shot chart of the match, with each team's xG and goals.
3. **What I would decide and why:** your team takes many long shots and scores rarely. Using this lesson, write the change you would make and the numbers that support it.

KEY FORMULAS

$$xG = \sum_i p_i$$

$$P(\text{at least one goal}) = 1 - \prod_i (1 - p_i)$$

$$P(\text{goal} \mid \text{move type}) = \frac{\text{goals from that type}}{\text{moves of that type}}$$

From history. Charles Reep (1904–2002) advised several English clubs, including Wolverhampton Wanderers in the 1950s. Expected-goals models became widespread in the 2010s as companies recorded the location of every shot. Hockey analysts had studied “shot quality” by location even earlier.

WORKED EXAMPLES

Example 1. Illustrative values: Team A takes 12 long shots (0.04 each) and 3 shots from the edge of the box (0.10 each). Team B takes 2 close-range shots (0.35 each) and 4 shots from inside the box (0.12 each). Compute each team's xG.

Show solution

1. A: $12(0.04) + 3(0.10) = 0.48 + 0.30 = 0.78$.
2. B: $2(0.35) + 4(0.12) = 0.70 + 0.48 = 1.18$.
3. B created about 0.4 more expected goals from 9 fewer shots.

Answer: Team A 0.78 xG from 15 shots; Team B 1.18 xG from 6 shots.

Example 2. For the same two teams, find the probability that each scores at least one goal, assuming independent shots.

Show solution

1. A: $P(0) = 0.96^{12} \times 0.90^3 \approx 0.613 \times 0.729 \approx 0.447$, so $P(\geq 1) \approx 0.553$.
2. B: $P(0) = 0.65^2 \times 0.88^4 \approx 0.4225 \times 0.600 \approx 0.253$, so $P(\geq 1) \approx 0.747$.

Answer: About 55% for A and 75% for B: shot quality wins.

Example 3. Illustrative data in the style of Reep: in a season there were 10,000 moves. 9,000 had three passes or fewer and produced 120 goals; 1,000 had four or more passes and produced 30 goals. What fraction of goals came from short moves? Which kind of move scored more often per move?

Show solution

1. Total goals 150; from short moves $120/150 = 80\%$.
2. Per move: short $120/9,000 \approx 1.3\%$; long $30/1,000 = 3.0\%$.
3. Most goals came from short moves only because most moves were short.

Answer: 80% of goals came from short moves, yet long moves scored more than twice as often per move. Reading the first number without the base rate leads to the wrong strategy.

PRACTICE

1. **Easy** A team takes shots with xG 0.3, 0.05, 0.1 and 0.55. What is its total xG?

Show answer

1.0.

2. **Easy** What does it mean for a shot to have an xG of 0.08?

Show answer

Shots like it, taken from that place in that way, become goals about 8% of the time.

3. **Easy** If a penalty kick has an xG of about 0.76, about how many goals should 50 penalties produce?

Show answer

$$50 \times 0.76 = 38.$$

4. **Medium** A hockey team takes 20 shots from the point (0.03 each) and 5 from the slot (0.15 each). Find its xG.

Show answer

$$0.60 + 0.75 = 1.35.$$

5. **Medium** A team takes two shots with xG 0.5 each. What is the probability it scores at least one? Compare with one shot of xG 1.0 (a certain goal).

Show answer

$1 - 0.5^2 = 0.75$. The same total xG (1.0) gives only a 75% chance of scoring when spread over two shots.

6. **Medium** Team X lost 1–0 but had 2.3 xG against 0.5. What would you tell the coach?

Show answer

The team created much better chances and was probably unlucky; over many games, performances like that usually win. Keep the approach and look for small improvements in finishing.

7. **Medium** Explain Reep's mistake in your own words.

Show answer

He noticed that most goals came from short moves, but most moves of every kind were short. He needed the scoring rate per move for short and long moves (the base rate), not the share of goals.

8. **Medium** Why do defenders try to force shots from wide angles and long distances?

Show answer

Those shots have low xG, so even if the opponent shoots often, it scores rarely.

9. **Hard** Ten shots each have xG 0.1. Find the expected goals and the probability of scoring at least once.

Show answer

xG 1.0; $1 - 0.9^{10} \approx 1 - 0.349 = 0.651$.

10. **Hard** A striker scored 18 goals from 12.0 xG in one season. Give two possible explanations, and say what you would expect next season.

Show answer

He may be an unusually good finisher, or he may have been lucky. Since finishing above xG usually shrinks toward the average (regression to the mean, Lesson 4.4), expect something between 12 and 18 next season unless several seasons show the same pattern.

11. **Hard** In practice you scored 6 of 40 shots from 8 m and 12 of 30 from 4 m. Estimate each zone's "xG" and the xG of a session with 20 shots from 8 m and 10 from 4 m.

Show answer

8 m: 0.15; 4 m: 0.40. Session: $20(0.15) + 10(0.40) = 7$ expected goals.

12. **Challenge** Show that for small p_i , $P(\text{at least one}) \approx xG$, but that for large values it is less. Use 10 shots of 0.02 and 2 shots of 0.5 (both xG = 1 or less) as examples.

Show answer

Ten shots of 0.02: xG 0.2, $P = 1 - 0.98^{10} \approx 0.183$, close to 0.2. Two shots of 0.5: xG 1.0, $P = 0.75$, well below 1. The approximation works only when chances are small, because it ignores the overlap of scoring twice.

COMMON MISTAKES

- Counting shots instead of weighing them by quality.
- Reading “most goals come from X” without asking how often X happens overall (the base rate).
- Treating one match’s goals-minus-xG as proof of skill rather than mostly luck.

Make it a game • Guess the Chance. Draw a soccer penalty area on paper and place ten beans on it as shots. Each of you writes a scoring probability for each shot, then adds them to find the expected goals. Compare totals and talk about why the slot is better than a wide angle. Flick a bean at a target once for each shot and see how close the total came.

Scripture. “The simple believeth every word: but the prudent man looketh well to his going.” (Proverbs 14:15, KJV)

A striking statistic, like “most goals come from short moves,” is a word to be tested, not simply believed. The prudent analyst looks at the base rate before he chooses his path.

3.4 THE SHOT CHART

OBJECTIVES

- Use a shot chart to compute points per shot by zone and explain why basketball shots moved to the rim, the free-throw line and the three-point line.
- Compute the value of free throws and explain why Rick Barry's underhand free throw was a high-percentage choice.
- Explain why players sometimes refuse a better percentage for reasons that are not mathematical.

Warm-up. Shoot 10 free throws with a soft ball the way you usually shoot. Then shoot 10 underhand, holding the ball with both hands between your knees and lifting it gently. Before you start, predict which way you will make more. Record both results.

WATCH

The surprising math behind basketball's least efficient shot

Thinking Basketball • 8:46

<https://www.youtube.com/watch?v=lZZry6Aed3k>

WHEN TO WATCH

After “the mathematics” paragraph. It compares the points per shot of layups, midrange jumpers and threes, and explains why the midrange shot declined.

BACKUP VIDEO

How To Master the Underhanded Free Throw - Canyon Barry brings it back! | FIBA 3x3 Tutorial

FIBA3x3 - The 3x3 Basketball Channel • 3:13

<https://www.youtube.com/watch?v=lKdPfTHa4Ko>

Before the project. Rick Barry's son Canyon, who shot free throws underhand in college and for USA 3x3, teaches the technique step by step.

KEY TERMS

shot chart

A map of a court with every shot marked where it was taken, often colored by make or miss or by points per shot.

midrange shot

A two-point jump shot taken well away from the basket but inside the three-point line, roughly 4.5 to 7 meters out.

three-point attempt rate

The fraction of a team's field-goal attempts that are threes.

underhand free throw

A free throw lifted from between the knees with both hands, giving a softer, more repeatable shot.

CORE IDEAS

The story. Rick Barry was one of the great scorers in basketball history, and he shot his free throws underhand, the way his father had taught him: ball between the knees, both hands, a soft lift. He made 90.0% of his NBA free throws (3,818 of 4,243) and 94.7% in the 1978–79 season. Wilt Chamberlain, the strongest player of his era, was a poor free-throw shooter. In the 1961–62 season he tried shooting underhand and made 61.3%, his best rate; on March 2, 1962, he scored 100 points in a game and made 28 of 32 free throws. Then he gave it up. He later said that shooting underhand made him feel foolish. His career rate fell back to 51.1 percent. Few players have copied Barry since, even though the method works.

How they figured it out. In the 2000s, NBA teams installed cameras and logged the location of every shot, and analysts drew shot charts for the whole league. The charts showed a clear pattern: shots at the rim and three-pointers produced far more points per shot than long two-point jumpers, and free throws were worth the most of all. Teams changed. In 1999–2000 about 17% of NBA field-goal attempts were threes; by 2024–25 it was about 42%. The midrange jumper, once the signature shot of many stars, became a shot of last resort. Barry's free throw shows the other side of the story: sometimes the numbers are clear, and people still choose the worse percentage because they care how they look.

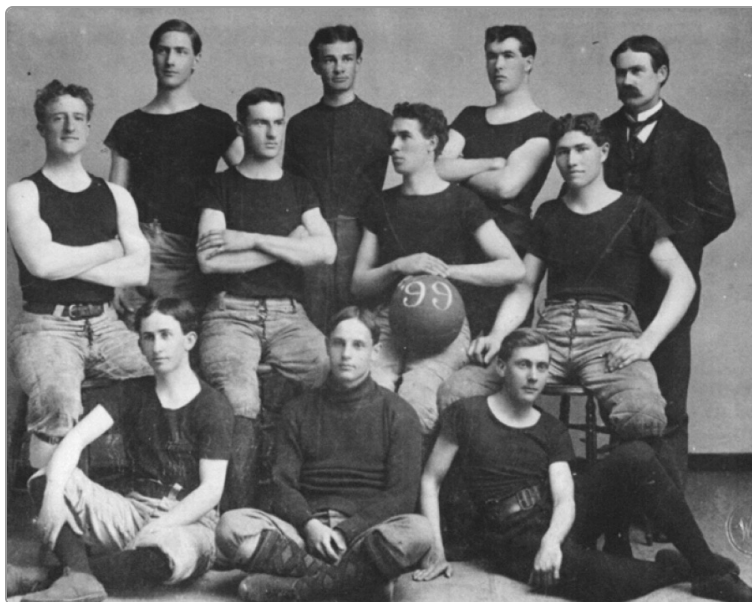
The mathematics. For each zone of a shot chart,
 $\text{points per shot} = (\text{point value}) \times (\text{make rate})$. Illustrative modern values: at the rim, about 65% for 2 points (1.30); midrange, about 41% for 2 (0.82); corner three, about 39% for 3 (1.17); other threes, about 36% for 3 (1.08). A trip to the line for two free

throws at rate p is worth $2p$: $2(0.90) = 1.80$ for Barry, $2(0.51) = 1.02$ for Chamberlain's career. A team's expected points are the sum over all its shots, so moving shots from the midrange to the rim or the three-point line raises the total even if the make rate overall goes down.

On the field. Good coaches design plays that end at the rim, at the free-throw line or in an open three, and they teach players to step back behind the line rather than shoot just inside it. A young player can use the same idea with her own shot chart.

The risk side: when every team chases the same shots, defenses adapt, and a skilled midrange shooter who makes 50% can still be valuable. And the deeper lesson of Barry and Chamberlain is about character: a wise player does what works even when it looks unusual, rather than what looks impressive and works less often.

PICTURE STUDY



The University of Kansas Basketball Team, Unknown photographer, 1899 · Photograph · Lawrence, Kansas

Image: [Wikimedia Commons](#), public domain

James Naismith, who invented basketball in 1891, became the first basketball coach at the University of Kansas in 1898. This is his first team. In their day every basket counted the same, and most shots were taken close to the basket.

Look closely

- Find Naismith. How is he dressed differently from the players?
- Look at the uniforms. How are they different from basketball uniforms today?
- Look at the players' faces and postures. What do they suggest about how seriously they took the new game?

Talk about it

1. In 1899 there was no three-point line. Where on the floor do you think these players took most of their shots, and why?
2. If you could show these players a modern shot chart, what one piece of advice would you give them?

Safety first. Warm up before shooting or throwing. Use a soft ball indoors, and keep clear of windows, lamps and people. A parent supervises any trip to a court, field or park.

PROJECT

Your Shot Chart

You will chart your own shots from five spots, compute your points per shot, and test whether the underhand free throw works better for you.

You will need

- A soft ball and a basket, bucket or hamper (or a real hoop with a parent present)
- Tape measure, chalk or tape
- Graph paper and the scorebook

Steps

1. Mark five spots: a layup spot right next to the basket, a short shot (1.5 m), a midrange spot (3 m), a “three-point” spot (4 m) and a free-throw spot (your choice of distance, about 2.5 m). Give the 4 m spot 3 points and all others 2, with free throws worth 1 each.
2. **Predict.** Write your expected make rate at each spot.
3. Take 20 shots from each spot, in sets of 5, and record makes.
4. Draw a shot chart on graph paper: the basket, the five spots, and each spot’s makes/attempts and points per shot.
5. Find your best and worst spots by points per shot.
6. **Free-throw test.** Take 30 free throws your usual way and 30 underhand, alternating sets of 10.
7. Compute both rates and the expected points of a two-shot trip each way.
8. Use the standard error to decide whether the difference is large enough to trust.
9. **Decide.** Write your personal shot plan and say honestly whether you would shoot underhand in front of others, and why.

Tips

- Underhand: feet shoulder-width apart, ball held with both hands at the sides, bend the knees, and lift with a soft follow-through toward the rim.

- Alternate methods so that tiredness affects both equally.

Check your work

- ☐ My shot chart shows 20 shots at each of five spots with points per shot.
- ☐ I compared overhand and underhand free throws over 30 attempts each with standard errors.
- ☐ I wrote a shot plan and an honest answer about the underhand shot.

SCOREBOOK

1. Your shot chart, colored by points per shot (for example red below 0.8, yellow 0.8–1.1, green above 1.1).
2. Table: method, attempts, makes, rate, standard error, value of a two-shot trip.
3. **What I would decide and why:** write a paragraph about a time you chose what looked good over what worked. What would you do now?

KEY FORMULAS

$$\text{points per shot} = v \times p$$

$$\text{free-throw trip} = 2p$$

$$\text{team expected points} = \sum (\text{shots in zone}) \times v \times p$$

From history. Rick Barry made 90.0% of his free throws in the NBA and 88.0% in the ABA. His son Canyon Barry also shot underhand, at the University of Florida and in international 3x3 basketball. Chamberlain's 100-point game, against the New York Knicks in Hershey, Pennsylvania, is still the NBA record.

WORKED EXAMPLES

Example 1. Use the illustrative zone values (rim 65% for 2, midrange 41% for 2, corner three 39% for 3, other three 36% for 3). Compute points per shot in each zone and rank them.

Show solution

1. Rim: $2 \times 0.65 = 1.30$.
2. Corner three: $3 \times 0.39 = 1.17$.
3. Other three: $3 \times 0.36 = 1.08$.
4. Midrange: $2 \times 0.41 = 0.82$.

Answer: Rim, corner three, other three, midrange. The midrange shot is worth about 0.5 points less per attempt than a shot at the rim.

Example 2. A team takes 85 shots: 25 at the rim, 25 midrange, 10 corner threes and 25 other threes. Using the values above, find its expected points from field goals. Then move 15 midrange shots to other threes and recompute.

Show solution

1. Original:
 $25(1.30) + 25(0.82) + 10(1.17) + 25(1.08) = 32.5 + 20.5 + 11.7 + 27.0 = 91.7$.
2. Moved: midrange 10 and other threes 40: $32.5 + 8.2 + 11.7 + 43.2 = 95.6$.
3. Gain: $95.6 - 91.7 = 3.9$ points per game.

Answer: About 91.7 points, rising to about 95.6: almost 4 points a game from shot choice alone.

Example 3. Compare the expected points from 10 trips to the line (20 free throws) for a 90% shooter like Rick Barry and a 51% shooter like Wilt Chamberlain's career rate. If Chamberlain had shot 61% underhand, how many points would 10 trips have added?

Show solution

1. Barry: $20 \times 0.90 = 18.0$ points.
2. Chamberlain at 51%: $20 \times 0.51 = 10.2$ points; at 61%: $20 \times 0.61 = 12.2$ points.
3. Difference: 2 points per 10 trips. Chamberlain often went to the line more than 10 times a game.

Answer: Barry 18.0 points, Chamberlain 10.2; staying underhand would have added about 2 points per 10 trips.

PRACTICE

1. **Easy** A player makes 58% of shots at the rim. What are his points per shot there?

Show answer

1.16.

2. **Easy** Rick Barry made 3,818 of 4,243 NBA free throws. Verify his percentage.

Show answer

$3,818/4,243 \approx 0.900$.

3. **Easy** What is a two-shot trip to the line worth for an 80% shooter?

Show answer

1.60 points.

4. **Medium** A midrange shooter makes 48% of her long twos. What three-point percentage would match her points per shot?

Show answer

$\frac{2}{3}(0.48) = 32\%$.

5. **Medium** In 1999–2000 about 17% of NBA shots were threes; in 2024–25 about 42%. If a team takes 88 shots a game, about how many threes is that in each season?

Show answer

About 15 and about 37.

6. **Medium** Why might a team that already shoots many threes still want a player who makes 50% from midrange?

Show answer

At 50% the midrange shot is worth 1.00 point, close to an average three; when defenses take away threes and the rim, a player who can score efficiently from midrange gives the offense another good option.

7. **Medium** Wilt Chamberlain made 61.3% of free throws in 1961–62 shooting underhand and 51.1 percent over his career. Give two reasons, one mathematical and one not, why he might have gone back to the overhand shot.

Show answer

Mathematical: a single season is a sample, and he may have thought 61% was luck. Not mathematical: he said the underhand shot made him feel foolish; concern for appearance outweighed the percentage.

8. **Medium** Explain why moving shots from midrange to the three-point line can raise a team's points even if its overall shooting percentage goes down.

Show answer

Threes go in less often but are worth 3 points; at 36% they are worth 1.08 points per shot against about 0.82 for midrange twos. Points depend on value times rate, not rate alone.

9. **Hard** A player shoots 70 free throws a season at 55% overhand. In practice she makes 68% underhand. How many more points a season would underhand produce if her practice rate held in games?

Show answer

$70(0.68 - 0.55) = 70(0.13) = 9.1$ points.

10. **Hard** A defense can allow either 10 rim attempts (at 65%) or 15 midrange attempts (at 41%). Which should it choose?

Show answer

Rim: $10(1.30) = 13.0$; midrange: $15(0.82) = 12.3$. Allow the midrange shots, by a small margin.

11. **Hard** Using the standard error, is a player who made 18 of 40 underhand free throws in practice (45%) clearly worse than his 50% overhand rate over 300 attempts?

Show answer

Underhand: $SE = \sqrt{0.45 \cdot 0.55/40} \approx 0.079$, so his true underhand rate could be anywhere from about 29% to 61%. The sample is too small to decide; he needs many more attempts.

12. **Challenge** A player has three options on a possession: a contested midrange shot (40% for 2), a pass that leads to a corner three (38% for 3) but is turned over 10% of the time (0 points), or a drive that draws a two-shot foul 30% of the time (80% free-throw shooter) and otherwise produces a rim shot at 55%. Find the expected points of each.

Show answer

Midrange: 0.80. Pass: $0.9 \times 1.14 = 1.026$. Drive:
 $0.3(1.60) + 0.7(1.10) = 0.48 + 0.77 = 1.25$. The drive is best.

COMMON MISTAKES

- Judging a shot by its make rate instead of points per shot.
- Forgetting that free throws come in pairs (or threes), so their value per trip is two or three times the rate.
- Deciding from 10 attempts that one method is better than another.

Make it a game • Hot Zone Map. Your parent draws a half court split into eight zones with a made percentage in each. In ninety seconds, compute the points per shot for every zone and rank them. Then identify the zone that explains why the midrange shot faded. Name the shape of the shot chart that shows where the league's shots moved, and the underhand free throw story.

3.5 COURSE MANAGEMENT AND THE SECOND SERVE

OBJECTIVES

- Explain how golfers manage risk by aiming away from hazards, and compute strokes gained for a shot.
- Compute the probability of winning a tennis service point for different serving strategies, and explain why the second serve is safer.
- Explain why a double fault or a ball in the water is the result of a risk calculation, and when to take more risk.

Warm-up. Put a coin on the floor 3 m away and draw a chalk or tape “water hazard” just beyond it. Toss 10 beanbags or rolled socks aiming at the coin, then 10 aiming at a spot 50 cm short of the coin, away from the hazard. Score 3 points for landing within 30 cm of the coin, 1 point for landing safely elsewhere, and -2 for landing in the hazard. Which aim scored more?

WATCH

The Clever Formula Tour Pros Use to Out-Smart Golf Courses | The Game Plan | Golf Digest

Golf Digest • 7:30

<https://www.youtube.com/watch?v=uWQpHrd7it8>

WHEN TO WATCH

After “the mathematics” paragraph. It explains the “short-side miss” and how professional golfers choose a target using Mark Broadie’s research.

BACKUP VIDEO

Strokes gained with "Godfather of Golf Analytics" | Golf Channel

Golf Channel • 7:15

<https://www.youtube.com/watch?v=MeNHbGhPFzU>

With “how they figured it out.” Mark Broadie explains how he developed strokes gained.

KEY TERMS

course management

Choosing targets and clubs in golf to keep score low over the whole round, including aiming away from hazards.

dispersion

How widely a player's shots scatter around the target: the golfer's and server's version of standard deviation.

strokes gained

Mark Broadie's measure of a shot's value: $SG = E_{\text{start}} - E_{\text{end}} - 1$, where E is the average number of strokes a tour player needs to hole out from a position.

double fault

In tennis, missing both the first and second serve, which loses the point.

CORE IDEAS

The story. Ben Hogan, one of the greatest golfers of the 20th century, was famous for planning every shot: where to miss, not just where to aim. Most golfers aim at the flag; Hogan often aimed at the safe side of the green, because a shot that drifted would still leave an easy chip, while a shot that drifted the other way would find water, sand or a hard “short side.” Half a century later a Columbia Business School professor, Mark Broadie, collected the location of hundreds of thousands of tour shots and turned that wisdom into numbers. His measure, **strokes gained**, was adopted by the PGA TOUR in 2011, starting with putting, and his 2014 book *Every Shot Counts* showed golfers how to use it.

How they figured it out. Old statistics like “greens in regulation” and “putts per round” mixed together the effects of different shots. Broadie started from expected value: from every spot on the course, how many strokes does an average tour player need to finish the hole? Call it E . A shot that moves the ball from a spot worth 3.0 strokes to a spot worth 1.8 strokes saved 1.2 strokes but used 1, so it gained 0.2. Shots could finally be compared fairly. The same data showed that every player's shots scatter in a predictable pattern, the **dispersion**, so the best target is the one that gives the best average result over the whole pattern, not the one that would be best if every shot went exactly where it was aimed. Tennis players face the same problem with the serve. A hard first serve wins the point more often when it goes in, but it goes in less often; if it misses, the player gets a second try. If the second also misses, it is a double fault.

The mathematics. Strokes gained is $SG = E_{\text{start}} - E_{\text{end}} - 1$ (with $E_{\text{end}} = 0$ when the ball is holed). For a tennis server, let a_1 be the chance a first serve goes in and w_1 the chance of winning the point when it does; a_2, w_2 for a second serve. With the usual plan (hard first serve, safe second serve), $P(\text{win}) = a_1 w_1 + (1 - a_1) a_2 w_2$. Illustrative values: $a_1 = 0.62, w_1 = 0.74, a_2 = 0.90, w_2 = 0.56$. Then

$P = 0.459 + 0.38(0.504) \approx 0.650$. Hitting two hard first serves gives

$0.459 + 0.38(0.459) \approx 0.633$, a little worse; the safer second serve trades power for a much lower chance of a double fault.

On the field. In golf, a player who aims at the middle of the green when the flag is tucked beside water loses a little when everything goes right and saves a great deal when it does not. In tennis, players hit the second serve with more spin and margin over the net. But the right amount of risk depends on the situation: a player far behind, or one facing an opponent who attacks weak second serves, may need to serve more boldly. **The risk side:** a double fault or a ball in the water is not always a mistake in execution; it can be the price of a reasonable risk. Judge the plan by its average over many shots, and the person by whether he chose the plan wisely. Self-control is part of course management: “every man that striveth for the mastery is temperate in all things” (1 Corinthians 9:25).

PICTURE STUDY



Old Tom Morris at St Andrews, Unknown photographer, 1901 · Photograph · The Old Course, St Andrews, Scotland

Image: [Wikimedia Commons](#), public domain

Tom Morris (1821–1908), called “Old Tom,” won the Open Championship four times, served as greenkeeper of the Old Course at St Andrews for nearly forty years and designed many courses. He knew every hollow and bunker of the Old Course and how to play around them.

Look closely

- Look at his clothing and club. How different are they from a golfer’s today?
- Look at the setting behind him: the sea and the rough ground of the links. How would wind and uneven ground change a golfer’s choice of target?
- Look at his posture and face. Does he look like a bold player or a careful one?

Talk about it

1. Old Tom designed courses with hazards placed to tempt golfers. How does a good course designer make the player weigh risk and reward?
2. If you played the Old Course with Old Tom, what do you think his first piece of advice would be?

Safety first. Warm up before shooting or throwing. Use a soft ball indoors, and keep clear of windows, lamps and people. A parent supervises any trip to a court, field or park.

PROJECT

Aim for the Fat Part

You will measure your own dispersion and use it to choose the best target, golf-style, and then test a serving strategy with a tossing game.

You will need

- Beanbags, rolled socks or soft balls (10 or more)
- Chalk or tape, a tape measure and a coin
- The scorebook

Steps

1. Mark a “green” (a circle about 1.5 m across) 4 m away, with a “flag” (coin) near its right edge, and a “water hazard” (a taped strip) just to the right of the green.
2. **Predict.** Write whether aiming at the flag or at the center of the green will give the better score.
3. Score each toss: within 30 cm of the flag 3 points; elsewhere on the green 2 points; off the green but dry 1 point; in the water –2 points.
4. Make 30 tosses aiming at the flag. Record each landing spot on a sketch and its score.
5. Make 30 tosses aiming at the center of the green. Record each.
6. Compute the average score for each aim and the fraction of tosses in the water.
7. Measure your dispersion: for the center tosses, find the average distance from the aim point.
8. **Serve test.** Make a target box 1 m square, 4 m away. A “hard serve” is a fast toss (wins 3 points if in); a “safe serve” is a soft, arcing toss (wins 2 points if in). Each point: try a hard serve; if it misses, try either a second hard or a safe serve. Play 20 points each way and compare the average score.
9. **Decide.** Write your strategy for both games and explain it with your averages.

Tips

- Use the same throwing motion every time, so that only the aim changes.

- Mark landing spots with small chalk dots to see your dispersion pattern.

Check your work

- ☐ I recorded 30 tosses for each aim and compared average scores.
- ☐ I measured my dispersion and used it to explain the better aim.
- ☐ I tested two serving plans over 20 points each and chose one with reasons.

SCOREBOOK

1. A sketch of your green, flag and hazard with every landing spot marked for both aims.
2. Expected-value table for each aim: landing zone, fraction, points, product, total.
3. **What I would decide and why:** a tennis player leads 5–2 in the final set and a 40–40 point comes. Would you advise a bold or safe second serve? What if she trailed 2–5?

KEY FORMULAS

$$SG = E_{\text{start}} - E_{\text{end}} - 1$$

$$P(\text{win serve point}) = a_1 w_1 + (1 - a_1) a_2 w_2$$

$$E[\text{score of an aim}] = \sum P(\text{landing zone}) \times (\text{strokes from there})$$

From history. Mark Broadie's research used the PGA TOUR's ShotLink data, which records the location of every shot in tour events. The first Open Championship was played at Prestwick in 1860; Old Tom Morris won four of the first eight.

WORKED EXAMPLES

Example 1. Illustrative baseline: a tour player averages 2.9 strokes to hole out from 150 yards in the fairway, 1.9 from 20 feet on the green, 1.5 from 8 feet, and 3.3 from a greenside bunker short-sided. Find the strokes gained of an approach shot from 150 yards that finishes (a) 20 feet away, (b) 8 feet away, (c) in the bunker.

Show solution

1. (a) $2.9 - 1.9 - 1 = 0.0$.
2. (b) $2.9 - 1.5 - 1 = +0.4$.
3. (c) $2.9 - 3.3 - 1 = -1.4$.

Answer: (a) 0.0, (b) +0.4, (c) -1.4: one bad result costs far more than one good result gains.

Example 2. Illustrative: aiming at a flag beside water, a golfer's shot ends 8 feet away 30% of the time, 20 feet away 45% of the time and in the water 25% of the time (water costs a penalty: count 4.0 strokes from the drop). Aiming at the center of the green, it ends 8 feet away 10%, 20 feet away 85% and in the bunker 5% (3.3 strokes). Compare the expected strokes after the shot.

Show solution

1. Flag: $0.30(1.5) + 0.45(1.9) + 0.25(4.0) = 0.45 + 0.855 + 1.0 = 2.305$.
2. Center: $0.10(1.5) + 0.85(1.9) + 0.05(3.3) = 0.15 + 1.615 + 0.165 = 1.93$.
3. Difference: about 0.38 strokes per shot in favor of the center.

Answer: Aiming at the center saves about 0.38 strokes per shot on average, even though it produces fewer close birdie chances.

Example 3. Use the illustrative tennis values $a_1 = 0.62$, $w_1 = 0.74$, $a_2 = 0.90$, $w_2 = 0.56$. Find the probability of winning the point with the usual plan, the probability of a double fault, and the probability with two hard first serves.

Show solution

1. Usual: $0.62(0.74) + 0.38(0.90)(0.56) = 0.4588 + 0.1915 \approx 0.650$.
2. Double fault (usual): $0.38 \times 0.10 = 0.038$ (3.8% of points).
3. Two first serves: $0.4588 + 0.38(0.4588) \approx 0.633$; double faults $0.38^2 \approx 0.144$.

Answer: Usual plan about 65.0% (3.8% double faults); two first serves about 63.3% (14.4% double faults).

PRACTICE

1. **Easy** A golfer's shot goes from a position worth 3.2 strokes to one worth 2.0. Find the strokes gained.

Show answer

$$3.2 - 2.0 - 1 = +0.2.$$

2. **Easy** A putt is holed from a position worth 1.5 strokes. What is its strokes gained?

Show answer

$$1.5 - 0 - 1 = +0.5.$$

3. **Easy** A server gets 60% of first serves in and 90% of second serves in. What is the probability of a double fault?

Show answer

$$0.40 \times 0.10 = 0.04, 4\%.$$

4. **Medium** With $a_1 = 0.65$, $w_1 = 0.72$, $a_2 = 0.92$, $w_2 = 0.52$, find the probability of winning a service point.

Show answer

$$0.468 + 0.35(0.4784) \approx 0.468 + 0.167 = 0.635.$$

5. **Medium** In Worked Example 2, what fraction of water balls would make aiming at the flag as good as aiming at the center, if the other two outcomes shared the rest in the same 30:45 ratio? Set up the equation (you may solve it by trying values).

Show answer

With water fraction x , 8 ft gets $0.4(1 - x)$ and 20 ft $0.6(1 - x)$. Expected $(1 - x)(0.4 \cdot 1.5 + 0.6 \cdot 1.9) + 4x = 1.74(1 - x) + 4x$. Set equal to 1.93:
 $1.74 + 2.26x = 1.93 \Rightarrow x \approx 0.084$, about 8%.

6. **Medium** Why is a golfer's dispersion like a standard deviation, and why does it matter when choosing a target?

Show answer

It measures how widely shots scatter around the aim. The best target is the one that gives the best average result over the whole scatter, so the wider the dispersion, the farther from hazards the aim should be.

7. **Medium** What did Mark Broadie measure, and when did the PGA TOUR adopt strokes gained?

Show answer

He measured the average strokes a tour player needs to hole out from every position, and used it to value each shot as strokes gained. The PGA TOUR adopted strokes gained (putting) in 2011.

8. **Medium** Explain why a double fault is not always a bad decision.

Show answer

A second serve hit with enough pace to win points will sometimes miss. If the chosen risk gives the best overall chance of winning points, an occasional double fault is the price of a good plan, not proof of a bad one.

9. **Hard** An opponent attacks weak second serves, so w_2 drops to 0.45 with the safe second serve. A harder second serve has $a_2 = 0.75$ and $w_2 = 0.60$. With $a_1 = 0.62$, $w_1 = 0.74$, which second serve is better?

Show answer

Safe: $0.4588 + 0.38(0.90)(0.45) = 0.4588 + 0.1539 = 0.613$. Hard: $0.4588 + 0.38(0.75)(0.60) = 0.4588 + 0.171 = 0.630$. The harder second serve is better against this opponent.

10. **Hard** Over 72 holes a golfer aims at the center instead of the flag on 20 approach shots like those in Worked Example 2. About how many strokes does he save?

Show answer

$20 \times 0.375 \approx 7.5$ strokes over four rounds.

11. **Hard** A server's first serve goes in 70% of the time and wins 70% when in; her second serve goes in 95% and wins 50% when in. Find her probability of winning a service point and of double-faulting.

Show answer

$0.49 + 0.30(0.475) = 0.49 + 0.1425 = 0.6325$; double faults $0.30 \times 0.05 = 0.015$.

12. **Challenge** Show that two first serves are better than the usual plan exactly when $a_1 w_1 > a_2 w_2$. Then explain why this is rarely true for professionals.

Show answer

Both plans share the first term $a_1 w_1$; the second terms are $(1 - a_1)a_1 w_1$ and $(1 - a_1)a_2 w_2$. So two first serves win exactly when $a_1 w_1 > a_2 w_2$. For most professionals $a_1 w_1$ is about 0.45–0.50 while $a_2 w_2$ is about 0.50, because the second serve goes in so much more often.

COMMON MISTAKES

- Choosing a target as if every shot went exactly where it was aimed.
- Judging a golfer by one round or a server by one match; the averages need many shots.
- Thinking the safest play is always best. When far behind, more risk can be wise (Lesson 2.3).

Make it a game • Aim Away from Trouble. Draw a target on paper with a shaded hazard to one side. Drop beans from shoulder height onto the paper, ten from each aim point: the center of the target, and a point away from the hazard. Count the landings in the target and the hazard. Choose your best aim and compute your points. Say how strokes gained would score a drop.

THE HIGH-PERCENTAGE SHOT: SPORTS BY THE NUMBERS

Every sport keeps score, but only in the last century and a half have people counted carefully enough to learn which plays really produce points. This unit follows the counters: Henry Chadwick and his box score, Branch Rickey and Allan Roth arguing for on-base percentage in 1954, Bill James and the sabermetricians with their run-expectancy table, Charles Reep with his notebooks and his famous misreading, the basketball analysts who drew shot charts of the whole league, and Mark Broadie, who measured every shot in professional golf.

You will compute batting average, on-base and slugging percentages, judge bunts and stolen bases by run expectancy, add up expected goals, rank basketball shots by points per shot, and choose golf targets and tennis serves by expected value and dispersion. The projects use real data: a box score you keep yourself, a run-expectancy table built from Retrosheet play-by-play, a shot chart of a real match, your own shooting from five spots, and a target game that measures your dispersion.

PREREQUISITES

- Units 1 and 2: probability, independence, expected value, variance and the standard error
- Comfort with percentages, decimals and simple formulas
- Optional: basic spreadsheet use for organizing data

MASTERY CHECKLIST

- ☐ I can compute BA, OBP and SLG and explain why OBP measures a hitter's value better than BA.
- ☐ I can use a run-expectancy table to judge a play and find the break-even rate for a stolen base.
- ☐ I can compute expected goals and the chance of scoring at least once, and explain the base-rate mistake in Reep's reasoning.
- ☐ I can rank basketball shots by points per shot and explain the value of free throws.
- ☐ I can compute strokes gained and the probability of winning a service point, and choose a target by expected value and dispersion.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Retrosheet**: Free. Box scores and play-by-play for major-league games going back more than a century, collected by volunteers; used in Lessons 3.1 and 3.2. No advertising.
- **Baseball-Reference**: Free data. Every player's and team's statistics, including OBP, SLG and Pythagorean records; used in Lessons 3.1 and 3.2. Contains advertising; a parent should preview it.
- **Society for American Baseball Research (SABR)**: Free articles and biographies on baseball history and research, including Henry Chadwick, Branch Rickey and Allan Roth. Fits Lessons 3.1 and 3.2.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

3.1	Henry Chadwick's Box Score	3 sessions of 50 minutes
3.2	Sabermetrics	3 sessions of 50 minutes
3.3	Shot Quality	3 sessions of 50 minutes
3.4	The Shot Chart	3 sessions of 50 minutes
3.5	Course Management and the Second Serve	3 sessions of 50 minutes

UNIT 3 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer to a sensible number of decimal places. Use the run-expectancy values printed in Lesson 3.2 when needed. Answer the concept questions in complete sentences. The test is worth 100 points.

1. **6 pts**

A player has 160 hits, 70 walks, 4 hit-by-pitches and 6 sacrifice flies in 540 at bats. Find his BA and OBP.

2. **6 pts**
The same player had 100 singles, 35 doubles, 5 triples and 20 home runs. Find his SLG and OPS.
3. **6 pts**
Explain what Branch Rickey and Allan Roth argued in 1954 and why it was important.
4. **6 pts**
Runners on 1st and 2nd, 1 out. The batter grounds into a double play (inning over). Find ΔRE .
5. **8 pts**
Runner on 2nd, 0 outs. Find the break-even rate for stealing third. (Success: 3rd, 0 outs = 1.35; caught: empty, 1 out = 0.25; stay = 1.10.)
6. **6 pts**
A team scores 700 runs and allows 800. Estimate its wins in 162 games by the Pythagorean formula.
7. **6 pts**
Why do modern teams bunt less often than teams did fifty years ago?
8. **6 pts**
A soccer team's shots have xG 0.45, 0.08, 0.08, 0.04 and 0.15. Find its xG and the probability it scores at least once (assume independence).
9. **8 pts**
Illustrative data: 5,000 moves had three or fewer passes and produced 60 goals; 500 moves had more passes and produced 15 goals. What share of goals came from short moves, and which kind of move scored more often per move? What does this teach about Reep's conclusion?
10. **6 pts**
Rank by points per shot: a 62% shot at the rim, a 43% midrange shot and a 37% three.

11. (6 pts)

A player who makes 55% of free throws overhand makes 70% underhand in practice. If she goes to the line for 6 two-shot trips a game, how many points per game could underhand add?

12. (6 pts)

Rick Barry made 90% of his free throws underhand, yet few players copy him. Give one mathematical and one human reason.

13. (6 pts)

A golf shot moves the ball from a spot worth 3.0 strokes to one worth 2.4. Find the strokes gained, and explain what the sign means.

14. (6 pts)

With $a_1 = 0.60$, $w_1 = 0.75$, $a_2 = 0.88$, $w_2 = 0.55$, find the probability of winning a service point and the probability of a double fault.

15. (12 pts)

Short essay. In about 150 words, choose one sport from this unit and explain how counting changed the way it is played. Name the person or people who did the counting, the number they discovered mattered most, and one way a player or coach should act differently because of it.

ANSWER KEY

1. $BA = 160/540 \approx .296$; $OBP = 234/620 \approx .377$.
2. $TB = 100 + 70 + 15 + 80 = 265$; $SLG = 265/540 \approx .491$; $OPS \approx .868$.
3. That on-base average mattered more than batting average for producing runs, based on Roth's detailed records. It began the shift from traditional statistics to evidence about what actually wins games; OBP became official in 1984.
4. $0 - 0.88 = -0.88$ runs.
5. $\frac{1.10 - 0.25}{1.35 - 0.25} = \frac{0.85}{1.10} \approx 77\%$.
6. $\frac{490,000}{490,000 + 640,000} \approx 0.434$; about 70 wins.
7. The run-expectancy table shows that a sacrifice usually lowers expected runs, because it gives up an out to gain a base; teams now save bunts for special situations where one run matters most.
8. $xG\ 0.80$; $P(0) = 0.55 \times 0.92^2 \times 0.96 \times 0.85 \approx 0.380$, so $P(\geq 1) \approx 0.62$.
9. Short moves produced $60/75 = 80\%$ of goals, but scored on 1.2% of moves against 3% for long moves. Reep looked at the share of goals without the base rate; per move, long moves scored more often.
10. Rim 1.24, three 1.11, midrange 0.86.
11. $12 \times (0.70 - 0.55) = 1.8$ points per game.
12. Mathematical: few players have tried it long enough to prove it works better for them. Human: many players feel it looks foolish and care more about appearance than percentage.
13. $3.0 - 2.4 - 1 = -0.4$: the shot was worse than an average tour shot from that spot by 0.4 strokes.
14. $0.45 + 0.40(0.484) = 0.6436$; double fault $0.40 \times 0.12 = 0.048$.
15. Answers vary. A strong essay names the right people (Chadwick, Rickey and Roth, James, Reep or the xG analysts, the NBA shot-chart analysts, Broadie), states the key measure (OBP, run expectancy, xG, points per shot, strokes gained), and gives a concrete change in decisions, with a sentence on the risk or limits of the number.

PORTFOLIO PIECE: A SPORTS DATA STUDY

Choose one sport and one question that data can answer, for example: Should my favorite team steal more or less? Which of my team's players should shoot more threes? Does our soccer team create better chances than it scores? Gather real data from a free source (Retrosheet, Baseball-Reference, Basketball-Reference, a league site) or from at least three games you chart yourself. Compute the appropriate measures from this unit (OBP and SLG, run expectancy, xG, points per shot, strokes gained or serve-win probability), make at least one chart, and write a two-page report with your question, data, calculations, chart, conclusion and its limits. End with a specific recommendation to a coach.

LOOK FOR	WHAT IT MEANS
Question and data	A clear question that the data can answer, with data from a named source or from at least three charted games.
Calculations	Correct use of at least two measures from the unit, with formulas and work shown.
Chart	A labeled chart or table (shot chart, run-expectancy grid, bar graph) that makes the main finding clear.
Honest limits	Discussion of sample size, base rates and what the data cannot show.
Recommendation	A specific, reasoned recommendation to a coach, with its expected gain and its risk.

UNIT 4

DECISIONS UNDER PRESSURE: COACHES AND THE ODDS

Units 1 through 3 taught you to compute probabilities and expected values and to recognize the high-percentage shot. This unit puts you on the sideline, where a coach must decide in seconds, with a crowd watching and his job on the line. You will price fourth-down decisions with David Romer's expected points, find the best time to pull a goalie, solve the penalty kick with von Neumann's and Nash's game theory, separate skill from luck with Galton's regression to the mean, and ask when the "safe" play is safe at all.

Each lesson pairs a real decision with real data: Romer's 1,068 fourth downs, Art Ross's 1931 goalie pull, Palacios-Huerta's 1,417 penalty kicks, Galton's 928 children. You will audit real fourth downs from play-by-play, simulate empty-net minutes, take penalty kicks in the backyard, study first-half and second-half statistics, and test "safe" and "bold" strategies in a dice game.

PREREQUISITES

- Expected value as a probability-weighted average (Unit 2)
- Variance and the rule that the team behind should increase it (Lesson 2.3)
- The run-expectancy table (Lesson 3.2)
- Solving a linear equation for one unknown

MASTERY CHECKLIST

- ☐ I can compute the expected points of going for it, punting and kicking, and find a break-even probability.
- ☐ I can explain why a trailing team should pull its goalie early, using a per-minute model.
- ☐ I can find a mixed-strategy equilibrium in a 2×2 game by making the opponent indifferent.
- ☐ I can explain regression to the mean, compute the spread caused by luck, and shrink an observed result toward the average.
- ☐ I can use a run-expectancy table to price an intentional walk and explain the difference between calculated courage and playing not to lose.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Game Theory 101 (William Spaniel)**: Free. A full course of short video lectures on game theory by the author of the Lesson 4.3 video, including mixed strategies and the indifference method. Ages 14+.
- **Hockey-Reference**: Free. Game logs and scoring summaries showing when goals and empty-net goals were scored, for extending the Lesson 4.2 simulator with real data. Ages 13+.
- **Baseball-Reference**: Free. Player splits by half and by month for the regression study in Lesson 4.4, and situational data for the intentional-walk questions in Lesson 4.5. Ages 13+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

4.1	Fourth Down	3 sessions of 50 minutes
4.2	Pulling the Goalie	3 sessions of 50 minutes
4.3	Keep Them Guessing	3 sessions of 50 minutes
4.4	Win Probability, Momentum and Regression to the Mean	3 sessions of 50 minutes
4.5	Playing to Win versus Playing Not to Lose	3 sessions of 50 minutes

UNIT 4 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer with a sentence explaining what it means. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

Going for it on fourth down succeeds with probability p , giving $S = 2.5$; failure gives $F = -1.5$; a punt gives $K = 0.5$. Find the break-even probability.

2. (6 pts)

In problem 1, find the expected value of going for it if $p = 0.62$, and state the better choice.

3. (6 pts)

Romer found teams kicked on 959 of the 1,068 fourth downs where going for it was better. Compute the percent and give one explanation for the pattern.

4. (7 pts)

Fourth-and-goal at the 2: a field goal nets about 2.4 points; a touchdown nets about 6.4; a failed try is worth +1.5 to you. Find the break-even probability.

5. (7 pts)

Trailing by one goal, with the goalie pulled you score with probability 0.08 per minute and the opponent with probability 0.20. Find the chance you tie within 2 minutes (scoring before they do).

6. (6 pts)

Explain why pulling the goalie can be wise even though it makes the expected goal difference worse.

7. (5 pts)

Who pulled a goalie for an extra attacker in the 1931 playoffs, and what happened?

8. (8 pts)

A kicker scores 90% if the keeper guesses wrong, 60% kicking left if the keeper guesses left, and 75% kicking right if the keeper guesses right. Let g = probability the keeper dives left. Find g at equilibrium and the scoring rate.

9. (6 pts)

What did Palacios-Huerta (2003) find when he compared real penalty kicks with the minimax prediction?

10. (7 pts)

A 30% three-point shooter takes 100 threes. Find the standard deviation of his percentage due to luck, and decide whether 40% over those 100 shots would be surprising if his true rate is 30%.

11. (7 pts)

With $r = n/(n + k)$, $k = 200$, a league mean of .260 and a hitter at .360 over 50 at-bats, predict his future average.

12. (7 pts)

Win probability moves from 44% to 70% on one play. What is the play's win probability added?

13. (7 pts)

Runner on 2nd, 1 out: 0.66 expected runs. Runners on 1st and 2nd, 1 out: 0.88. What does an intentional walk cost, and name one reason a manager might still order it?

14. (7 pts)

Explain in two or three sentences why a prevent defense can lower a team's chance of winning.

15. (8 pts)

Short essay (one paragraph): Using one example from this unit and the parable of the talents (Matthew 25:14-30), explain the difference between calculated courage, recklessness and playing not to lose.

ANSWER KEY

1. $p^* = (0.5 + 1.5)/(2.5 + 1.5) = 2.0/4.0 = 0.5$.
2. $0.62(2.5) + 0.38(-1.5) = 1.55 - 0.57 = 0.98$; better than 0.5, so go for it.
3. $959/1068 \approx 89.8\%$. Explanation: fear of blame (a failed try is second-guessed, a punt is normal), or reliance on custom instead of data.
4. $p^* = (2.4 - 1.5)/(6.4 - 1.5) = 0.9/4.9 \approx 0.18$.
5. $0.08 + 0.72(0.08) = 0.1376$, about 13.8%.
6. Late in a one-goal game only tying matters; losing by two is no worse than losing by one. The pull raises the chance of tying while the extra goals against cost almost nothing.
7. Art Ross of the Boston Bruins pulled Tiny Thompson in the final minute against Montreal; Boston did not score and lost 1–0, but the idea endured.
8. Kick left: $60g + 90(1 - g) = 90 - 30g$. Kick right: $90g + 75(1 - g) = 75 + 15g$. Equal: $15 = 45g$, $g = 1/3$. Scoring rate $90 - 10 = 80\%$.
9. Kickers chose their natural side about 60.0% of the time (predicted 61.5%) and keepers about 57.7% (predicted 58.0%): professionals play very close to the equilibrium.
10. $\sqrt{0.3 \times 0.7/100} = \sqrt{0.0021} \approx 0.046$. 40% is $0.10/0.046 \approx 2.2$ standard deviations above: unusual, but it will happen to some shooters by luck.
11. $r = 50/250 = 0.2$; $0.260 + 0.2(0.100) = 0.280$.
12. +26 percentage points (+0.26).
13. 0.22 runs on average. Reasons: a much weaker next hitter, setting up a double play, or caring late in a game about any run rather than the average.
14. It reduces the chance of a long completion but gives the offense easy short gains that move the ball and stop the clock. What matters is the chance the opponent scores, and if that rises, the defense has protected against the wrong risk.
15. Answers vary. A strong answer defines each: calculated courage takes a risk after counting the cost because it gives the best chance (going for it on fourth-and-1 at midfield, pulling the goalie with several minutes left); recklessness takes risks without counting (fourth-and-15 from your own 10); playing not to lose avoids visible failure at the expense of the real goal (the punt that lowers win probability, the servant who buried his talent because he “was afraid”).

PORTFOLIO PIECE: A COACH'S DECISION GUIDE

Write a four-page decision guide for one sport you play or follow, the kind a coach could carry to the sideline. It must include (1) a decision chart for one recurring choice (fourth down, when to pull the goalie, when to walk a hitter, when to go for two) built from expected values or win probabilities, with the calculations shown for at least four cells; (2) a mixed-strategy recommendation for one guessing situation (penalty kicks, serve direction, run or pass) with the equilibrium computed from real or carefully measured data; (3) a page warning about regression to the mean, with an example from real statistics; and (4) a one-paragraph statement of the guide's philosophy that explains when to be bold and when to protect a lead.

LOOK FOR	WHAT IT MEANS
Decision chart	Every cell filled; at least four cells supported by worked break-even or expected-value calculations; assumptions stated.
Game theory	A 2×2 table with sources for the numbers, the equilibrium mixes found by indifference, and the value of the game.
Skill and luck	A real example of regression to the mean with a standard-deviation calculation and a shrunken prediction.
Judgment and honesty	The philosophy paragraph distinguishes calculated courage from recklessness and from playing not to lose, and the guide admits what its numbers cannot capture.

4.1 FOURTH DOWN

OBJECTIVES

- Explain expected points and how Virgil Carter, Robert Machol and David Romer measured the value of field position.
- Compute the expected points of going for it, punting and kicking a field goal, and find the break-even success rate for going for it.
- Build a fourth-down decision chart and explain why coaches have kicked far more often than the numbers justify.

Warm-up. Your team faces fourth-and-1 at the opponent's 40-yard line in the first quarter. Before any mathematics, write your choice: go for it, punt, or try a 57-yard field goal. Ask two family members the same question and tally the votes. Keep the tally; at the end of the lesson you will see whether the numbers agree with the family.

WATCH

The NFL's Fourth-Down Revolution

NFL Films • 7:25

<https://www.youtube.com/watch?v=d60F61zVOJY>

WHEN TO WATCH

After the story paragraph, before the mathematics. Watch for the chart of fourth-down attempts and listen to how the coaches describe the fear of being second-guessed.

BACKUP VIDEO

Next Gen Stats: What is the 4th Down Decision Guide?

NFL • 2:20

<https://www.youtube.com/watch?v=DzmPv-9NjcA>

Optional, after worked example 1. It shows a real decision guide comparing the chance of winning if a team goes for it or kicks.

KEY TERMS

fourth down

An offense has four plays (downs) to gain 10 yards. On fourth down it must choose: try for the first down, punt the ball away, or kick a field goal.

break-even probability

The success rate at which two choices have equal expected value. If your real chance of success is higher, the bolder choice is better on average.

blame avoidance

Choosing the option that is less likely to be criticized rather than the option most likely to win.

expected points (EP)

The average number of points, net of what the opponent scores next, that a team can expect from a given situation, such as first-and-10 on its own 30. Measured from thousands of real plays.

decision chart

A table, made in advance, that says what to do for each combination of field position and yards to go.

CORE IDEAS

The story. In 1969 Virgil Carter was a quarterback in the NFL and a student of operations research at Northwestern University. With his professor Robert Machol he went through the play-by-play of the 1969 season and asked a simple question: how many points is a first down worth at each spot on the field? Their paper, “Operations Research on Football,” appeared in 1971. Thirty years later David Romer, an economist at the University of California, Berkeley, pushed the question all the way to fourth down. His first draft carried the title “It’s Fourth Down and What Does the Bellman Equation Say?” The finished paper, “Do Firms Maximize? Evidence from Professional Football,” was published in the *Journal of Political Economy* in 2006.

How they figured it out. Romer downloaded the play-by-play of nearly every NFL regular-season game from 1998 to 2000 and used only the first quarters, so that the score and the clock would not distort the values. From 11,112 situations he estimated the value of a first-and-10 at every yard line. A first-and-10 on your own 1 was worth about -1.6 points; on your own 15 it was worth zero; after that the value rose roughly one point every 18 yards, reaching 5.55 points for first-and-goal on the 1. With those

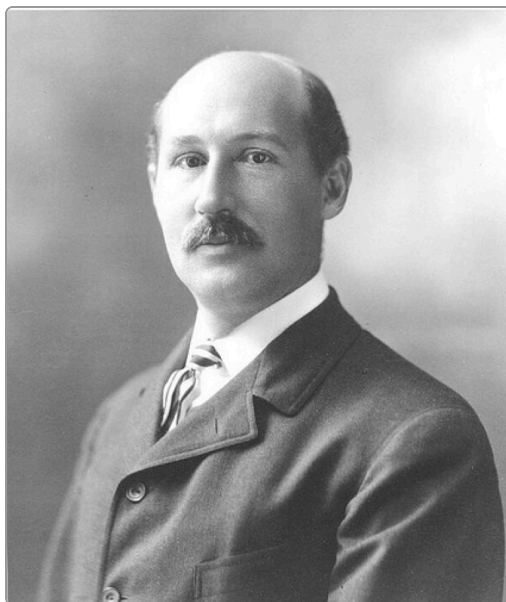
values he could price every fourth-down choice. The result was startling. On the 1,068 fourth downs where his analysis said going for it was better on average, teams kicked 959 times. On the 1,604 where kicking was better, they went for it only 9 times. Every mistake ran in the same direction: too cautious.

The mathematics. Each choice is a probability-weighted average. Going for it succeeds with probability p and leaves you with the value of a new first down, S ; it fails with probability $1 - p$ and hands the opponent the ball at that spot, worth F (a negative number) to you. Kicking is worth K . Setting $pS + (1 - p)F = K$ and solving gives the **break-even probability** $p^* = \frac{K - F}{S - F}$. If your real chance of converting is higher than p^* , going for it is the percentage play. Romer's break-even line was surprising: on its own half of the field, a team should go for it with less than about 4 yards to go; at the opponent's 33 the line rose to about 10 yards; and inside the opponent's 5 going for it was always better on average.

On the field. For years almost nobody followed the numbers. A failed fourth-down try is a visible, second-guessed "mistake"; a punt is the normal thing, and nobody is fired for doing the normal thing. That is **blame avoidance**, and Romer himself suggested it as an explanation. In the last decade the league has changed: teams now carry decision guides to the sideline, and NFL Films reports that fourth-down attempts nearly doubled between 2015 and 2025. A wise coach knows his own team's numbers (a strong running game, a weak kicker, a wet field), decides before the game what he will do in each situation, and then has the courage to do it.

The risk side. Going for it has a wider spread of results. When it fails, the opponent takes over in good field position, and everyone sees it. Punting feels safe because its bad outcome is invisible: a slightly lower chance of winning, spread quietly over the rest of the game. Romer estimated that better fourth-down choices would add about 2 percentage points to a team's chance of winning each game, roughly one extra win every three seasons. That is a modest edge, but it is the kind of edge that good teams collect, game after game.

PICTURE STUDY



Walter Camp, Unknown photographer, about 1910 · Photograph · Yale University Manuscripts & Archives
Image: [Wikimedia Commons](#), public domain

Walter Camp played and coached at Yale and sat on the committees that wrote the rules of American football. In 1882 he introduced the idea of downs: a team had three tries to gain five yards or give up the ball. In 1912 the rule became four downs to gain ten yards. Every fourth-down decision you study in this lesson exists because of a rule this man wrote.

Look closely

- Look at Camp's face and posture. Does he look like a man who would follow custom or question it?
- Notice the plain, formal dress. Football rule-makers of his day were mostly college men and coaches. What would they have wanted the new down rule to do?
- Imagine the photograph as a portrait of a decision-maker. What would you need to know to give him advice on fourth down?

Talk about it

1. Before the down rule, one team could keep the ball for a whole half without trying to advance it. How does a down rule force teams to take risks?
2. Camp kept careful records of games and players. How would records have helped him judge whether a rule change worked?

3. If you could show Camp Romer's chart, what do you think he would say about how teams use their fourth down?

PROJECT

Fourth-Down Audit and Decision Chart

You will audit real fourth-down decisions from real games, compare them with a break-even chart you build yourself, and then write the chart your own team would carry to the sideline.

You will need

- Free play-by-play from Pro-Football-Reference (each game's box score page has a full play-by-play table) or a college football statistics site
- Graph paper or a spreadsheet
- Calculator
- The expected-points values from this lesson

Steps

1. Choose three complete games (NFL or college). A parent can help you find the play-by-play pages.
2. List every fourth down in the first three quarters: quarter, score, yard line, yards to go, the decision (go, punt, field goal) and the result.
3. Add up: how many fourth downs, how many punts, how many attempts, how many field goals. Compute the percent of each.
4. For each fourth down, estimate S , F and K from the expected-points values (round to the nearest 10 yards), and compute the break-even probability.
5. Use the illustrative conversion rates from worked example 3 (64% for 1 yard down to 34% for 10 yards) to decide what the numbers recommend.
6. Mark each real decision as agree or disagree with your calculation. Count how many disagreements were too cautious and how many too bold.
7. Build a decision chart: rows for field position (own 20, own 40, midfield, opponent's 40, opponent's 30, opponent's 20, opponent's 10) and columns for 1, 2, 3, 4, 5, 7 and 10 yards to go. Fill each cell with GO, PUNT or FG.
8. Return to your warm-up tally. Did your family agree with the chart?
9. Write one paragraph: which cell of your chart would be hardest for a real coach to follow, and why?

Tips

- Field-goal tries from beyond about 50 yards miss often, so treat them carefully.
- If a team was trailing by a lot late, its choices are about winning probability, not expected points. That is why you use the first three quarters.

Check your work

- ☐ I recorded every fourth down in three real games and counted the decisions.
- ☐ I computed at least six break-even probabilities with $p^* = (K - F)/(S - F)$.
- ☐ My decision chart has a GO, PUNT or FG in every cell, and I can explain one cell with a calculation.

SCOREBOOK

1. Copy the expected-points values from this lesson into a table, and draw a line graph of value against yard line. Mark where the value crosses zero.
2. Draw a decision tree for fourth-and-1 at the opponent's 40: two branches (go, punt), with the probabilities and point values on each branch, and the expected value at the end.
3. "What I would decide and why": write the choice you would make on fourth-and-3 at midfield, with the numbers that support it and the risk you are accepting.

KEY FORMULAS

$$E[\text{go}] = pS + (1 - p)F$$

$$p^* = \frac{K - F}{S - F}$$

$$E[\text{field goal}] = q \cdot 3 + (1 - q)F_{\text{miss}}$$

From history. Carter and Machol's 1971 paper is one of the first serious statistical studies of football strategy. They divided the field into 10-yard strips and found, much as Romer did later, that a first down deep in your own territory is worth less than nothing, because the next score is more likely to be the opponent's.

WORKED EXAMPLES

Example 1. Use these values, rounded from Romer's estimates: a first-and-10 at your own 10 is worth -0.6 points, at your own 40 is worth $+1.4$, and at the opponent's 35 is worth $+2.8$. It is fourth-and-1 at the opponent's 40. If you convert, the play typically carries you to about the opponent's 35. If you fail, the opponent takes over at its own 40. A punt typically leaves the opponent at its own 10. Find the break-even probability, and the expected points of each choice if you convert fourth-and-1 64% of the time (Romer's figure for third-and-1).

Show solution

1. Success: $S = +2.8$.
2. Failure: the opponent has first-and-10 at its own 40, worth 1.4 to them, so $F = -1.4$.
3. Punt: the opponent has the ball at its own 10, worth -0.6 to them, so $K = +0.6$.
4. Break-even: $p^* = \frac{0.6 - (-1.4)}{2.8 - (-1.4)} = \frac{2.0}{4.2} = 0.476$.
5. With $p = 0.64$: $E[\text{go}] = 0.64(2.8) + 0.36(-1.4) = 1.792 - 0.504 = 1.29$.

Answer: Break-even about 48%. Going for it is worth about 1.29 points; punting is worth 0.6. Going for it gains about 0.7 points on average.

Example 2. Fourth-and-goal at the 2. A field goal is nearly certain: 3 points, but then you kick off, and Romer valued a kickoff at about 0.6 points for the receiving team, so a field goal nets about $3 - 0.6 = 2.4$. A touchdown is worth about 7 points (6 plus an almost-certain extra point), also followed by a kickoff, so about 6.4. If the try fails, the opponent starts at its own 2, which is worth about -1.5 to them, so $+1.5$ to you. Find the break-even probability and the expected value of going for it if teams score from the 2 about 45% of the time.

Show solution

1. Kick: $K = 2.4$. Go: $S = 6.4$ and $F = +1.5$. (Failing here is not very bad: the opponent is pinned against its goal line.)
2. Break-even: $p^* = \frac{2.4 - 1.5}{6.4 - 1.5} = \frac{0.9}{4.9} = 0.18$.
3. With $p = 0.45$: $E[\text{go}] = 0.45(6.4) + 0.55(1.5) = 2.88 + 0.825 = 3.7$.

Answer: Going for it pays whenever the chance of scoring is above about 18%. At 45% it is worth about 3.7 points against 2.4 for the kick. These match Romer's own figures.

Example 3. Build a one-row decision chart for fourth down at the opponent's 40, using the break-even rate of 48% from worked example 1 and these illustrative conversion rates: 1 yard 64%, 2 yards 58%, 3 yards 53%, 4 yards 49%, 5 yards 44%, 7 yards 39%, 10 yards 34%.

Show solution

1. Compare each conversion rate with 48%.
2. 1, 2, 3 and 4 yards: 64%, 58%, 53% and 49% are all above 48%, so go for it (4 yards is nearly a toss-up).
3. 5, 7 and 10 yards: 44%, 39% and 34% are below 48%, so punt.
4. Refinement: on longer conversions the offense usually gains more than the line to gain, which raises S and lowers the break-even rate, so the true line sits a little farther out. Romer's full analysis put it near 7 yards at this spot.

Answer: At the opponent's 40: go for it on fourth-and-1 through fourth-and-4; punt on fourth-and-5 or longer (by this simple model). The chart is made calmly in advance so that it can be followed under pressure.

PRACTICE

1. **Easy** Using the values in worked example 1, what is a first-and-10 at your own 40 worth to you? What is it worth to you if the *opponent* has first-and-10 at its own 40?

Show answer

+1.4 points; -1.4 points.

2. **Easy** Explain in one or two sentences what a break-even probability is.

Show answer

It is the success rate at which two choices have the same expected value.

Above it, the riskier choice is better on average; below it, the safer choice is better.

3. **Easy** Going for it succeeds with probability 0.5. Success is worth +3 points and failure -1 point. What is the expected value?

Show answer

$0.5(3) + 0.5(-1) = 1.0$ point.

4. **Medium** Fourth-and-2 at midfield. Success is worth $S = 2.1$, failure $F = -1.9$, and a punt that leaves the opponent at its own 15 is worth $K = 0$. Find the break-even probability, and decide what to do if you convert 58% of the time.

Show answer

$p^* = (0 + 1.9)/(2.1 + 1.9) = 1.9/4.0 = 0.475$. Since $0.58 > 0.475$, go for it:

$E = 0.58(2.1) + 0.42(-1.9) = 1.218 - 0.798 = 0.42$, better than 0.

5. **Medium** A 50-yard field goal is made 70% of the time (illustrative). A make is worth about 2.4 points after the kickoff; a miss gives the opponent the ball at its own 40, worth -1.4 to you. Find the expected value of the kick.

Show answer

$0.7(2.4) + 0.3(-1.4) = 1.68 - 0.42 = 1.26$.

6. **Medium** In Romer's sample, teams kicked on 959 of the 1,068 fourth downs where going for it was better on average. What percent is that? On the 1,604 fourth downs where kicking was better, they went for it 9 times. What percent is that?

Show answer

$959/1068 \approx 89.8\%$ kicked when they should have gone; $9/1604 \approx 0.6\%$ went when they should have kicked. The errors were almost all on the cautious side.

7. **Medium** Give two reasons, one about people and one about information, why a coach might punt even when going for it has the higher expected value.

Show answer

People: fear of blame, since a failed try is visible and second-guessed while a punt is "normal." Information: he may believe (rightly or wrongly) that his own offense converts less often than the league average, or he may not have done the calculation.

8. **Medium** In worked example 1, $E[\text{go}] = 4.2p - 1.4$. What conversion rate makes going for it exactly one point better than punting (worth 0.6)?

Show answer

$$4.2p - 1.4 = 1.6 \Rightarrow p = 3.0/4.2 \approx 0.714, \text{ about } 71\%.$$

9. **Hard** Your offense is weak near the goal line and scores from the 2 only 15% of the time. Using the values in worked example 2, should you go for it or kick? Show the comparison.

Show answer

$E[\text{go}] = 0.15(6.4) + 0.85(1.5) = 0.96 + 1.275 = 2.24$, less than 2.4, so kick. The break-even was 18%, and this offense is below it. Knowing your own team's numbers matters.

10. **Hard** Expected points are measured in points, but games are won by finishing ahead. Late in the fourth quarter, leading by 4 points, you face fourth-and-1 at the opponent's 30 with 1:50 left and the opponent has no timeouts. Explain why a coach might go for it even if a field goal had slightly higher expected points.

Show answer

A field goal makes the lead 7, but the opponent gets the ball back with time to tie. Converting the first down lets you run out the clock and wins almost for certain. Late in a game, the right measure is the probability of winning, not expected points.

11. **Challenge** Start from $pS + (1 - p)F = K$ and derive $p^* = \frac{K - F}{S - F}$. Then use it to show what happens to the break-even rate in worked example 1 if the punt goes into the end zone for a touchback, so the opponent starts at its own 20 (worth +0.3 to them).

Show answer

$pS + F - pF = K \Rightarrow p(S - F) = K - F \Rightarrow p = (K - F)/(S - F)$. With a touchback, $K = -0.3$, so $p^* = (-0.3 + 1.4)/(2.8 + 1.4) = 1.1/4.2 \approx 0.26$. A punt from the 40 that bounces into the end zone is so poor that going for it is right at only a 26% success rate.

COMMON MISTAKES

- Comparing the chance of success with 50% instead of with the break-even probability.
- Forgetting that a failed attempt gives the opponent the ball at that spot, so F is negative.
- Treating a punt as worth zero. A punt is worth something (it moves the opponent back), and that value is K .
- Using expected points in the last minutes of a close game, when the chance of winning is what matters.

Make it a game • Coach's Call Clock. Your parent plays an announcer and presents a fourth-down scenario: the yard line, yards to go, and the score. You have twenty seconds to say whether to go for it, kick, or punt, using expected points and the break-even probability. Your parent asks one follow-up. Then talk about blame avoidance. Score a point for each reasoned call, five rounds.

4.2 PULLING THE GOALIE

OBJECTIVES

- Tell the story of Art Ross's first goalie pull in 1931 and why the idea took decades to be used well.
- Use a simple per-minute model to compare the chance of tying a game with and without the goalie in net.
- Explain why a trailing team should accept a worse average result in exchange for a better chance of winning, and why analysts say to pull the goalie earlier.

Warm-up. Flip a coin for each minute of a pretend final three minutes: heads means your team scores, tails means nothing happens. Do it five times and count how often you tie the game (at least one head). Now change the rule: roll a die each minute; a 1 means you score, a 6 means they score into your empty net. Predict which plan ties more often before you calculate.

WATCH

When should you pull the goalie in hockey?

Wage World • 9:59

<https://www.youtube.com/watch?v=vypFm5HTTlk>

WHEN TO WATCH

After the mathematics paragraph. The video builds a table of win chances by score and time, exactly the kind of model this lesson uses.

KEY TERMS

extra attacker

A sixth skater sent on in place of the goaltender, leaving the net empty.

scoring rate

The average number of goals a team scores per minute in a given situation.

empty-net goal

A goal scored into a net with no goaltender in it.

win probability

The chance of winning the game from the current score and time, estimated from a model or from many past games.

asymmetric payoff

A situation in which one bad outcome costs little extra (losing by two instead of one) while one good outcome is worth a great deal (tying the game).

CORE IDEAS

The story. On March 26, 1931, the Boston Bruins trailed the Montreal Canadiens 1–0 in the final minute of a Stanley Cup semifinal game at Boston Garden. Boston’s coach, Art Ross, called his goaltender, Tiny Thompson, to the bench and sent out an extra skater, Red Beattie. Boston did not score, and the game ended 1–0. But hockey had a new idea. Within a year Ross tried it again against the New York Rangers, and this time the Rangers shot the puck into the empty net, one of the first empty-net goals on record. For the next eighty years, almost every team pulled its goalie, if at all, in the final minute.

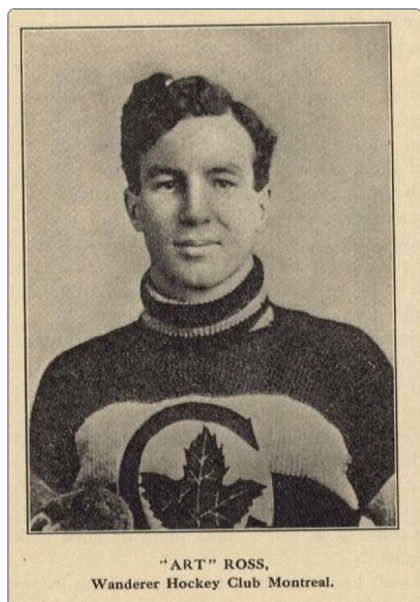
How they figured it out. In the 1970s the statistician Donald Morrison treated goals as random events, each team scoring at a steady average rate, and asked when the goalie should come out. The answer was much earlier than coaches believed. In 2018 Clifford Asness and Aaron Brown repeated the analysis with modern NHL data and concluded that a team down by one goal should pull its goalie with about five or six minutes left, and earlier still when down by two. Their numbers showed why: an empty net makes the opponent several times more likely to score, but the extra attacker raises your own scoring rate by less than double. That sounds like a bad trade, until you remember what a goal against actually costs.

The mathematics. When you trail by one goal late, there are only two results that matter: you tie the game or you do not. Losing 3–1 counts exactly the same as losing 2–1. So a goal against, which costs a great deal in the first period, costs almost nothing in the last minutes. That is an **asymmetric payoff**. In a simple model, each minute you score with probability a and they score with probability b . Your chance to tie within n minutes, scoring before they do, is $a + (1 - a - b)a + (1 - a - b)^2 a + \dots$, a geometric series. Pulling the goalie makes your expected goal difference worse, yet it raises the chance of the only outcome that matters. This is Lesson 2.3’s rule, live on the ice: the team that is behind should increase variance.

On the field. In recent seasons NHL coaches have been pulling their goalies noticeably earlier, sometimes with three or four minutes left, and occasionally even earlier when down by two. Fans still groan when the empty-netter goes in, because they see the goal and do not see the comebacks that never had a chance under the old timing. A wise coach decides his pull times before the season, tells his players, and does not let one empty-net goal change a sound plan.

The risk side. Pulling the goalie is not a reckless risk. It risks only a bigger margin of defeat in a game that is probably lost already. When the score is tied, or when goal difference decides a tiebreaker in the standings, the payoffs change and the goalie should stay in. Every risk must be measured against what is actually at stake.

PICTURE STUDY



Art Ross of the Montreal Wanderers, Unknown photographer, about 1909 · Photograph, printed as a player portrait · Montreal, Canada

Image: [Wikimedia Commons](#), public domain

Before he coached the Boston Bruins, Art Ross was one of the best defensemen in early professional hockey. He was also a restless inventor: he helped design an improved puck and a better goal net. The man who first pulled a goalie for an extra attacker had spent years thinking about how the game could be played differently.

Look closely

- Look at the heavy wool sweater with its high roll neck. What does it tell you about where and how hockey was played in his day?
- Read the caption. Ross played for the Wanderers before he ever coached in Boston. How might years as a player shape a coach's ideas?
- Look at his expression. Does he look like a cautious man or a bold one? Can you really tell from a face?

Talk about it

1. Ross was a defenseman. Why might a defender be the first to see the value of adding an extra attacker?
2. When Ross first pulled his goalie, Boston lost 1–0 anyway. Should one result have convinced anyone that the idea was good or bad? Why?

3. Ross is remembered for an idea that failed the first time he tried it. What does that say about judging decisions by single results?

Safety first. If you act out a pulled-goalie drill in the driveway with a soft ball, keep clear of cars and windows, and have a parent watch.

PROJECT

The Empty-Net Simulator

You will play out hundreds of final minutes with dice and with a spreadsheet, and find for yourself the best time to pull the goalie in a simple model.

You will need

- Two ordinary dice (or a 20-sided die substitute: a spreadsheet)
- Spreadsheet software with $= RAND()$
- Graph paper

Steps

1. **Dice model.** Roll two dice for each minute. With the goalie in: a total of 2 means you score and 12 means they score (each $1/36$, about 0.03). With the goalie pulled: 3 or 4 means you score ($5/36$, about 0.14) and 6, 7 or 8 means they score ($16/36$, about 0.44). These rules exaggerate the real rates so that the effect shows up in a few dozen games; write them in your scorebook before you start.
2. Play 30 “final three minutes” with the goalie in all the way. Record tie or no tie.
3. Play 30 more with the goalie pulled for all three minutes. Record tie or no tie.
4. **Spreadsheet model.** In a spreadsheet, set a and b in two cells for goalie in (0.05, 0.05) and two cells for goalie pulled (0.08, 0.20).
5. In each row (one game), use $= RAND()$ once per minute to decide whether you score, they score or nobody scores, stopping at the first goal.
6. Simulate 1,000 games for each pull time: 0, 1, 2, 3, 4, 5 and 6 minutes left.
7. Record the tie rate for each pull time in a table and graph it, with pull time on the horizontal axis.
8. Compare the spreadsheet’s tie rates with the exact formula from this lesson for at least two pull times.
9. Write your recommendation: in this model, when should the goalie be pulled, and how much does it matter?

Tips

- Keep a tally sheet for the dice games: one column for the minute, one for the roll, one for the result.
- If your simulated results differ from the formula by a few percentage points, that is the small-sample noise from Lesson 1.5. More games will bring them closer.

Check your work

- ☐ I played at least 60 dice games and simulated at least 1,000 games for each pull time.
- ☐ My graph shows tie rate against pull time, with labelled axes.
- ☐ My recommendation is supported by both the simulation and the formula.

SCOREBOOK

1. Draw a probability tree for two minutes with the goalie pulled: three branches each minute (you score, they score, nothing), with probabilities, and circle the paths that end in a tie.
2. Make a table of tie chance against pull time from your simulation, and a second column from the formula.
3. “What I would decide and why”: your team is down 2–1 with 4:00 left and has just won a face-off in the offensive zone. Pull or wait? Give your numbers and your reason.

KEY FORMULAS

$$P(\text{tie within } n \text{ minutes}) = a \frac{1 - (1 - a - b)^n}{a + b}$$

$$E[\text{goal difference per minute}] = a - b$$

From history. In 1931 a goal against an empty net did not have a special name. By the 1960s the empty-net goal was common enough to be tracked as its own statistic, and today the NHL records empty-net goals for every player.

WORKED EXAMPLES

Example 1. Use these illustrative per-minute scoring chances, in line with the 2018 study's finding that an empty net multiplies the opponent's rate by about four while the extra attacker raises yours by less than double. Goalie in: you 0.05, opponent 0.05. Goalie pulled: you 0.08, opponent 0.20. Assume at most one goal per minute. Find the chance of tying within the last 2 minutes each way.

Show solution

1. Goalie in: $a + (1 - a - b)a = 0.05 + 0.90(0.05) = 0.05 + 0.045 = 0.095$.
2. Goalie pulled:
 $0.08 + (1 - 0.08 - 0.20)(0.08) = 0.08 + 0.72(0.08) = 0.08 + 0.0576 = 0.1376$.
3. Expected goal difference per minute: goalie in $0.05 - 0.05 = 0$; pulled
 $0.08 - 0.20 = -0.12$.

Answer: About 9.5% with the goalie in and 13.8% with the goalie pulled. The pulled goalie gives a worse average goal difference but a better chance to tie.

Example 2. With the same rates, compare two plans over the final 3 minutes: (A) pull the goalie with 3 minutes left; (B) keep the goalie in for 2 minutes and pull him for the last minute. If the opponent scores first, count the game as lost.

Show solution

1. Plan A, three pulled minutes:
 $0.08 + 0.72(0.08) + 0.72^2(0.08) = 0.08 + 0.0576 + 0.0415 = 0.179$.
2. Plan B, first two minutes with the goalie in: tie first with probability 0.095; nobody scores with probability $0.90^2 = 0.81$.
3. Then the last minute pulled: $0.81 \times 0.08 = 0.0648$.
4. Plan B total: $0.095 + 0.0648 = 0.160$.

Answer: Pulling at 3 minutes ties about 17.9% of the time; pulling at 1 minute ties about 16.0%. The earlier pull is better, which is what Morrison and later Asness and Brown found.

Example 3. Use the formula $P = a \frac{1 - (1 - a - b)^n}{a + b}$ to find the chance of tying with the goalie pulled for 6 minutes, with $a = 0.08$ and $b = 0.20$.

Show solution

1. $1 - a - b = 0.72$, and $0.72^6 = 0.139$.

2. $P = 0.08 \times \frac{1 - 0.139}{0.28} = 0.08 \times 3.075 = 0.246$.

3. With the goalie in for the same 6 minutes:

$$0.05 \times \frac{1 - 0.90^6}{0.10} = 0.05 \times \frac{1 - 0.531}{0.10} = 0.234.$$

Answer: About 24.6% pulled against 23.4% with the goalie in. In this model, pulling with 6 minutes left still helps a little, which is close to what the 2018 study found with real data.

PRACTICE

1. **Easy** In what year, and by which coach, was a goaltender first pulled for an extra attacker in an NHL playoff game, according to the usual account?

Show answer

1931, by Art Ross of the Boston Bruins (goalie Tiny Thompson).

2. **Easy** Trailing 2–1 with one minute left, why does it matter so little if the opponent scores an empty-net goal?

Show answer

Losing 3–1 and losing 2–1 are the same result: a loss. The only outcome that changes anything is a tying goal.

3. **Easy** With the goalie pulled for one minute, you score with probability 0.08 and they score with probability 0.20. What is the expected goal difference for that minute?

Show answer

$$0.08 - 0.20 = -0.12 \text{ goals.}$$

4. **Medium** With the goalie in, $a = 0.05$ and $b = 0.05$ per minute. Find the chance of tying within 3 minutes (you must score before they do).

Show answer

$$0.05 + 0.90(0.05) + 0.81(0.05) = 0.05 + 0.045 + 0.0405 = 0.1355.$$

5. **Medium** Compare your answer to problem 4 with Plan A in worked example 2 (17.9%). Which is better, and by how many percentage points?

Show answer

Pulling for all 3 minutes is better: $17.9\% - 13.6\% \approx 4.3$ percentage points.

6. **Medium** If a tied game goes to overtime and each team then wins half the time, what is the chance of winning under Plan A and Plan B of worked example 2?

Show answer

$$\text{Plan A: } 0.179 \times 0.5 \approx 0.090; \text{ Plan B: } 0.160 \times 0.5 = 0.080.$$

7. **Medium** Explain why a strategy can have a worse expected goal difference but a higher chance of winning.

Show answer

Goals do not all count the same. Late in a one-goal game, an extra goal against costs nothing, while a goal for is worth everything. A plan that trades many harmless goals against for a few valuable goals for raises the chance of winning even though the average difference falls.

8. **Medium** Give one situation in which a coach should *not* pull his goalie even though his team trails late.

Show answer

When goal difference decides a tiebreaker in the standings, when the team trails by so many that a comeback is nearly impossible and losing by more harms the team, or in a two-game aggregate series where conceding again would eliminate the team.

9. **Hard** Using the formula, find the chance of tying within 4 minutes with the goalie pulled ($a = 0.08$, $b = 0.20$).

Show answer

$$0.72^4 = 0.2687; P = 0.08 \times (1 - 0.2687)/0.28 = 0.08 \times 2.612 = 0.209, \text{ about } 20.9\%.$$

10. **Hard** Down by two goals, you must score twice before they score once (a simplification). With the goalie pulled for the last 2 minutes ($a = 0.08$, $b = 0.20$, at most one goal per minute), what is the chance? What does the answer tell you about when to pull when down by two?

Show answer

You must score in both minutes: $0.08 \times 0.08 = 0.0064$, less than one percent. Two goals need much more time, so a team down by two should pull its goalie much earlier.

11. **Challenge** Show that the sum $a + ra + r^2a + \cdots + r^{n-1}a$ with $r = 1 - a - b$ equals $a \frac{1 - r^n}{1 - r}$, and find its limit as n grows very large. Interpret the limit.

Show answer

The geometric series gives $a \frac{1 - r^n}{1 - r}$, and $1 - r = a + b$. As $n \rightarrow \infty$, $r^n \rightarrow 0$, so $P \rightarrow a/(a + b)$: with unlimited time, the chance you score first is your share of all the goals scored, $0.08/0.28 \approx 0.29$ when pulled.

COMMON MISTAKES

- Judging the decision by the one empty-net goal you remember rather than by the many games.
- Adding a over several minutes ($0.08 + 0.08 + 0.08$) without first checking that nobody has scored yet.
- Forgetting that a goal against costs almost nothing only when the game is nearly lost anyway; it is not a general rule.

Make it a game • Goalie Pull Two Truths. Your parent tells three statements about pulling the goalie. Two are true, such as Art Ross's 1931 move, the extra attacker, or the empty-net goal. One is a tall tale. You name the tall tale and correct it. Swap roles. Then estimate how early in a one-goal game you would pull the goalie, and explain your reasoning with win probability.

4.3 KEEP THEM GUESSING

OBJECTIVES

- Describe von Neumann’s minimax theorem and Nash’s equilibrium in plain words and give their dates.
- Find a mixed-strategy equilibrium in a 2×2 game by making the opponent indifferent.
- Use Palacios-Huerta’s penalty-kick data to check whether professional players actually play the equilibrium, and explain why predictability is costly.

Warm-up. Play twenty rounds of “penalty kick” with a family member using two coins. The kicker secretly turns a coin to heads (kick left) or tails (kick right); the keeper does the same for the dive. Uncover together. If they match, the keeper saves; if they differ, it is a goal. Before you start, predict how many goals the kicker will score. Keep a tally; you will use it in the project.

WATCH

Game Theory 101: Soccer Penalty Kicks

William Spaniel • 6:03

<https://www.youtube.com/watch?v=OTs5JX6Tut4>

WHEN TO WATCH

After the mathematics paragraph, before worked example 1. It solves the penalty-kick game step by step with the same method this lesson uses.

BACKUP VIDEO

Game Theory VI: Mixed Strategy vs. Pure Strategy (Soccer Penalty Kicks)

Economics in Many Lessons • 4:44

<https://www.youtube.com/watch?v=k8x5Gmgrnp8>

Optional, for a second short explanation of pure and mixed strategies with a penalty-kick table.

KEY TERMS

zero-sum game

A contest in which one side's gain is exactly the other side's loss, such as a goal scored or saved.

mixed strategy

Choosing among options at random with fixed probabilities, such as kicking left 60% of the time.

Nash equilibrium

A pair of strategies in which neither player can do better by changing only his own choice.

pure strategy

Always making the same choice, such as always kicking to the left.

minimax

Choosing the strategy that makes your worst case as good as possible. Von Neumann proved in 1928 that in every two-player zero-sum game, the best mixed strategies of the two players meet at one value.

indifference

In a mixed equilibrium, each player mixes so that the opponent gains nothing by choosing one option over the other.

CORE IDEAS

The story. In 1928 the young Hungarian mathematician John von Neumann published a paper on parlor games such as matching pennies and proved what he called the minimax theorem: in any two-player game where one side's gain is the other's loss, each player has a best way to mix his choices, and the two best mixtures meet at a single value of the game. In 1944 he and the economist Oskar Morgenstern expanded the idea into a book, *Theory of Games and Economic Behavior*. In 1950 a Princeton graduate student, John Nash, published a one-page paper showing that every game with a finite number of choices, zero-sum or not, has at least one equilibrium in which no player wants to change his strategy alone.

How they figured it out. For decades game theory was a theory without much evidence. Then the economist Ignacio Palacios-Huerta found the perfect laboratory: the penalty kick. The ball takes about three-tenths of a second to reach the goal, so the keeper must choose a side before he sees the kick; both players are effectively choosing at the same moment. In "Professionals Play Minimax" (*Review of Economic*

Studies, 2003) he studied 1,417 penalty kicks from top European leagues between 1995 and 2000. He sorted each kick by the kicker's *natural side* (the side a right-footed player kicks to most easily) and measured how often kickers scored in each combination.

The mathematics. Palacios-Huerta's scoring rates were: kicker to natural side, keeper the same way, 69.9%; kicker natural, keeper the other way, 92.9%; kicker to the other side, keeper to the natural side, 95.0%; both to the other side, 58.3%. If a keeper always dived to the natural side, kickers would go the other way and score 95% of the time. If kickers always went natural, keepers would follow and hold them to 70%. Neither pure strategy is safe. The equilibrium is found by **indifference**: the keeper mixes so that the kicker scores equally often either way, and the kicker mixes so that the keeper saves equally often either way. The theory predicted that kickers should go to the natural side 61.5% of the time and keepers 58.0%. The players actually did so 60.0% and 57.7% of the time. Professionals, without any formulas, had found von Neumann's minimax.

On the field. The same logic runs through sport. Mark Walker and John Wooders studied serves in Wimbledon finals and found that top servers won about equally often serving wide or down the middle, just as minimax predicts (though they switched direction a little too often to be truly random). Pitchers mix fastballs and off-speed pitches; quarterbacks mix runs and passes; a tennis player at the net guesses which way the passing shot will come. Christy Mathewson's 1912 book *Pitching in a Pinch* is full of the art of not letting the batter know what is coming.

The risk side. The cost of a pattern is that an opponent who reads it will punish it. A mixed strategy is not indecision; it is a deliberate defense against being exploited. The skill is in being unpredictable on purpose, with the right proportions, and in watching for opponents who are predictable, because a player who always does the same thing has given you free points.

PICTURE STUDY



Christy Mathewson Pitching in the 1913 World Series, Bain News Service, 1913 · Glass negative photograph · Shibe Park, Philadelphia; Library of Congress

Image: [Wikimedia Commons](#), public domain

Christy Mathewson of the New York Giants won 373 games, and in this game, Game 2 of the 1913 World Series, he pitched a ten-inning shutout. The year before, he had published *Pitching in a Pinch*, a book about outguessing batters. He threw a famous “fadeaway” pitch, but he used it sparingly, so that hitters could never sit and wait for it.

Look closely

- Find the pitcher and the catcher. What must the two of them agree on before every pitch?
- Find the batter. He must decide in a fraction of a second whether to swing. What is he guessing?
- Look at the crowd. They can see the whole field. Can they tell what pitch is coming?

Talk about it

1. If Mathewson threw his best pitch every time, what would the batters do?
2. Why does a pitcher hide his grip and use the same motion for every pitch?
3. How is the battle between pitcher and batter like the penalty kick?

Safety first. Warm up before kicking, use a soft ball, kick away from windows, and keep the keeper's face out of the line of a hard kick. A parent should supervise.

PROJECT

Penalty Kicks in the Backyard

You will measure your own scoring rates from each side, compute the equilibrium for your own “penalty-kick game,” and then test whether mixing really protects you against a thinking opponent.

You will need

- A soft soccer ball or playground ball
- A goal: two cones or two shoes about 3 meters (10 feet) apart, against a fence or wall
- A sibling or parent as goalkeeper
- A die (for randomizing), tape measure, scorebook

Steps

1. Mark the goal and a kicking spot about 6 meters (20 feet) away. Warm up with a few easy kicks.
2. Decide which side is your natural side (a right-footed kicker usually finds the keeper’s right easiest).
3. **Measure the four rates.** Before each kick, you and the keeper each write a side on a card and show the cards after the kick. Take 40 kicks in all. Record the side you kicked, the side the keeper dived and whether it was a goal.
4. Make a 2×2 table of your scoring percentages for the four combinations. If any cell has fewer than 5 kicks, take extra kicks with that combination.
5. Use the indifference method to compute your equilibrium mix and the keeper’s.
6. **Test the theory.** For 20 kicks, kick to your natural side every time while the keeper tries to read you. Record your scoring rate.
7. For 20 more kicks, use a die to choose your side: roll and kick natural if the result falls within your equilibrium proportion (for 60%, natural on 1–4 of a six-sided die is 67%, close enough; or use a spreadsheet’s $RAND()$). Record your scoring rate.

8. Compare the two scoring rates with each other and with the predicted equilibrium value.
9. Write a half-page report: did mixing protect you, and how close did you come to the theory?

Tips

- Keep the ball low and soft. This is about choices, not power.
- The keeper should commit to a side as the ball is struck, as real keepers do, rather than waiting.

Check your work

- ☐ I recorded at least 80 kicks with sides and results.
- ☐ I computed both equilibrium mixes from my own table.
- ☐ I can explain, with my numbers, why a predictable kicker scores less.

SCOREBOOK

1. Draw the 2×2 penalty table with Palacios-Huerta's numbers, and shade the cell each player wants most.
2. Tally your warm-up coin game and compare the goal rate with 50%.
3. "What I would decide and why": you are the keeper and the kicker has gone to his natural side in his last five penalties. Dive natural or not? Use this lesson's ideas.

KEY FORMULAS

$$g \pi_{NN} + (1 - g) \pi_{NO} = g \pi_{ON} + (1 - g) \pi_{OO}$$

$$\text{value} = g \pi_{NN} + (1 - g) \pi_{NO}$$

From history. Von Neumann also helped design some of the first electronic computers. His 1928 paper began from simple parlor games of guessing and bluffing, but its real legacy is in economics, military planning and sport, wherever two sides must choose without knowing what the other will do.

WORKED EXAMPLES

Example 1. Find the keeper's equilibrium mix using Palacios-Huerta's scoring rates. Let g be the probability the keeper dives to the kicker's natural side. Scoring rates: kicker natural & keeper natural 69.92%, kicker natural & keeper other 92.91%, kicker other & keeper natural 94.97%, kicker other & keeper other 58.30%.

Show solution

1. The keeper wants the kicker to score equally often either way.
2. Kicker goes natural: $69.92g + 92.91(1 - g) = 92.91 - 22.99g$.
3. Kicker goes other: $94.97g + 58.30(1 - g) = 58.30 + 36.67g$.
4. Set equal: $92.91 - 22.99g = 58.30 + 36.67g$, so $34.61 = 59.66g$ and $g = 0.580$.

Answer: The keeper should dive to the kicker's natural side 58.0% of the time. Real keepers did so 57.7% of the time.

Example 2. Find the kicker's equilibrium mix, k = probability of kicking to the natural side, and the value of the game (the scoring rate at equilibrium).

Show solution

1. The kicker wants the keeper to be equally successful either way.
2. Keeper dives natural: scoring rate $69.92k + 94.97(1 - k) = 94.97 - 25.05k$.
3. Keeper dives other: $92.91k + 58.30(1 - k) = 58.30 + 34.61k$.
4. Set equal: $36.67 = 59.66k$, so $k = 0.615$.
5. Value: $94.97 - 25.05(0.615) = 94.97 - 15.41 = 79.6\%$.

Answer: The kicker should go to the natural side 61.5% of the time (real kickers: 60.0%), and at equilibrium about 79.6% of penalties are scored, close to the real rate of about 80%.

Example 3. A tennis server (illustrative numbers) wins 60% of points serving wide when the receiver guesses middle, 30% serving wide when the receiver guesses wide, 70% serving middle when the receiver guesses wide, and 40% serving middle when the receiver guesses middle. Find the server's best mix and the value.

Show solution

1. Let w = probability of serving wide. Receiver guesses wide: server wins $30w + 70(1 - w) = 70 - 40w$.
2. Receiver guesses middle: server wins $60w + 40(1 - w) = 40 + 20w$.
3. Set equal: $70 - 40w = 40 + 20w$, so $30 = 60w$, $w = 0.5$.
4. Value: $40 + 20(0.5) = 50\%$.

Answer: Serve wide half the time; the server then wins 50% of points no matter what the receiver does. Serving only wide would let a smart receiver hold him to 30%.

PRACTICE

1. Easy What did von Neumann prove in 1928, and what did Nash add in 1950?

Show answer

Von Neumann: every two-player zero-sum game has a value, reached when both players use their best (minimax) mixed strategies. Nash: every finite game, zero-sum or not, has at least one equilibrium in which no player can gain by changing only his own strategy.

2. Easy In matching pennies, each player shows heads or tails; one wins on a match, the other on a mismatch. What is the equilibrium, and what is each player's chance of winning?

Show answer

Each player shows heads with probability $1/2$; each wins 50% of the time.

3. Easy Why is a penalty kick a good test of game theory?

Show answer

Both players must choose at essentially the same moment, the stakes are clear (goal or no goal), the choices are simple (left, right, middle), and there are thousands of recorded examples.

4. Medium Using the scoring rates in worked example 1, what percent would a kicker score if the keeper always dived to the natural side and the kicker knew it?

Show answer

He would kick to the other side every time and score 94.97%, about 95%.

5. Medium Using the same rates, what would a keeper hold a kicker to if the kicker always went to his natural side and the keeper knew it?

Show answer

The keeper would always dive natural, and the kicker would score 69.92%, about 70%.

6. Medium In a game, a striker scores 90% if the keeper guesses wrong. If the keeper guesses right, the striker scores 50% to his left and 70% to his right. Let g = probability the keeper dives left. Find g at equilibrium.

Show answer

Kick left: $50g + 90(1 - g) = 90 - 40g$. Kick right: $90g + 70(1 - g) = 70 + 20g$.
Equal: $20 = 60g$, $g = 1/3$.

7. Medium In problem 6, what is the striker's scoring rate at equilibrium?

Show answer

$90 - 40(1/3) = 76.7\%$.

8. Medium Explain why, at a mixed equilibrium, a player is indifferent between his options even though he is mixing them in unequal proportions.

Show answer

The opponent's mix is chosen exactly so that each option gives him the same expected result. If one option were better, he would play it all the time, and the opponent would then change his mix. Only at equal payoffs can both keep mixing.

9. **Hard** Walker and Wooders found that Wimbledon servers won about equally often serving left or right, but switched direction too often. Why is “switching too often” a weakness?

Show answer

If a server rarely serves the same way twice in a row, an alert receiver can predict the next serve from the last one. True randomness means each serve is independent of the previous one, as with the dice in Lesson 1.3.

10. **Hard** In the tennis example (worked example 3), suppose the receiver mistakenly guesses wide 80% of the time. What should the server do, and what does he win?

Show answer

Against wide-guessing, serving middle wins $0.8(70) + 0.2(40) = 64\%$; serving wide wins $0.8(30) + 0.2(60) = 36\%$. Serve middle every time and win 64%. A predictable opponent is exploited by a pure strategy.

11. **Challenge** For a general 2×2 zero-sum game with scoring rates A, B (kicker's first option against keeper's first and second) and C, D (kicker's second option against keeper's first and second), show that the keeper's equilibrium probability of choosing his first option is $g = \frac{D - B}{A - B - C + D}$, and check it with Palacios-Huerta's numbers.

Show answer

Set $gA + (1 - g)B = gC + (1 - g)D$: $g(A - B - C + D) = D - B$. With $A = 69.92$, $B = 92.91$, $C = 94.97$, $D = 58.30$:

$$g = (58.30 - 92.91) / (69.92 - 92.91 - 94.97 + 58.30) = -34.61 / -59.66 = 0.580.$$

COMMON MISTAKES

- Choosing your own mix to make *your* options equal. In a mixed equilibrium, each player mixes to make the *opponent* indifferent.
- Thinking that mixing means 50–50. The right mix depends on the payoffs and is often unequal.
- Alternating sides in a pattern (left, right, left, right). That is not random, and a good opponent will notice.

Make it a game • Be Unreadable. Play a penalty-kick game. The kicker secretly picks left or right by writing it on a slip of paper, and the keeper dives left or right. Play twenty rounds, then switch. Try to be unreadable by rolling a die to decide. Tally the results and explain what von Neumann's minimax theorem and Nash's equilibrium say about mixing choices.

4.4 WIN PROBABILITY, MOMENTUM AND REGRESSION TO THE MEAN

OBJECTIVES

- Explain regression to the mean using Galton's 1886 study of heights and apply it to sports statistics such as early-season batting averages.
- Separate skill from luck with the binomial standard deviation and predict next season's performance with a shrinkage formula.
- Read a win-probability chart and explain why most streaks and swings of "momentum" are what randomness looks like.

Warm-up. Flip a coin 20 times and write the sequence of H and T. Circle the longest run. Before you look at anyone else's sequence, predict: what is the chance that a sequence of 20 flips contains a run of 4 or more heads? Write your guess as a percent.

WATCH

How We're Fooled By Statistics

Veritasium · 7:37

<https://www.youtube.com/watch?v=1tSqSMOyNFE>

WHEN TO WATCH

After the story paragraph. It tells the flight-instructor story and shows how regression to the mean fools people about praise, punishment and streaks in sport.

KEY TERMS

regression to the mean

When a measurement is extreme partly because of luck, the next measurement of the same thing is likely to be closer to the average.

true talent

The long-run average a player would show over a very large number of attempts; what we are trying to estimate.

standard deviation of a proportion

For a success rate p over n attempts, chance alone produces a spread of about $\sqrt{p(1-p)/n}$.

win probability

The estimated chance a team wins from the current score, time, possession and field position, built from many past games.

momentum (in the statistical sense)

The claim that recent success makes further success more likely. Most apparent momentum disappears when compared with what random sequences produce.

CORE IDEAS

The story. In 1886 the Victorian scientist Francis Galton published a paper titled “Regression towards Mediocrity in Hereditary Stature.” He had gathered the heights of 205 pairs of parents and their 928 grown children. Tall parents did have tall children, but on average the children were less extreme than the parents: about two-thirds as far from the average. Very short parents had short children who were, again, closer to the average. Galton called this “regression,” and he showed it with a diagram of dots and ellipses that became the ancestor of every scatterplot and line of best fit.

How they figured it out. Galton’s insight is that any measurement is part lasting cause and part chance. A player who hits .400 over his first 50 at-bats is probably good, but he has probably also been lucky, because the most extreme results always collect the luckiest players. When the luck does not repeat, his average falls back toward the league. Nothing has “gone wrong.” The same thing explains the famous “*Sports Illustrated* cover jinx”: athletes reach the cover after an extraordinary run, and extraordinary runs usually include good fortune that does not last. Daniel Kahneman told of Israeli flight instructors who believed that scolding worked and praise backfired, because cadets who were praised after a superb landing usually did worse next time, and those scolded after a poor one usually did better. Both were simply regressing to their own average.

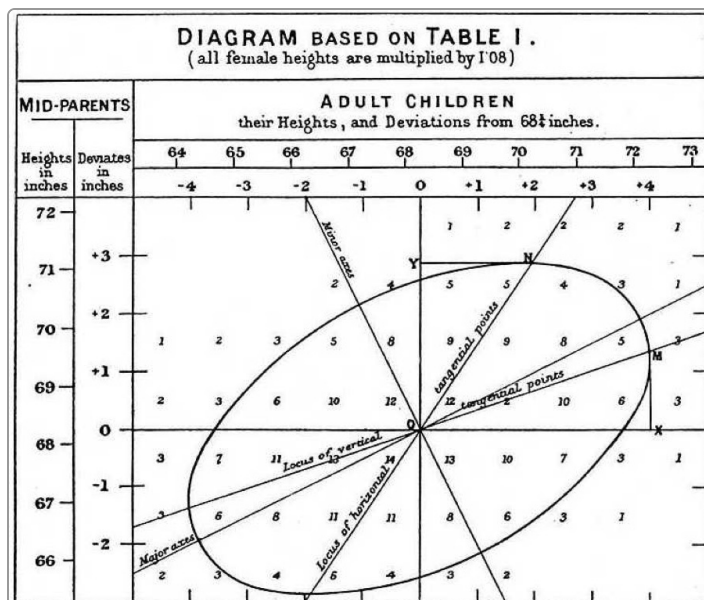
The mathematics. Chance alone spreads a success rate by about $\sqrt{p(1-p)/n}$ around a player’s true talent. For a .250 hitter over 50 at-bats that is

$\sqrt{0.25 \times 0.75/50} \approx 0.061$, so a true .250 hitter will sometimes bat .310 or .190 over 50 at-bats by luck alone. A good prediction **shrinks** the observed result toward the average: $\text{prediction} = \text{mean} + r(\text{observed} - \text{mean})$, where r is between 0 and 1 and grows with the number of attempts. One simple model is $r = n/(n + k)$, where k measures how noisy the statistic is.

On the field. Win-probability charts, now shown during many broadcasts, turn the score, the time and the field position into a single percentage built from thousands of past games. They swing a great deal, and it is tempting to read every swing as “momentum.” But a fair coin flipped 20 times produces a run of four or more heads almost half the time (47.8%). Most runs of scoring are what randomness looks like. Lesson 1.3 showed that the “hot hand” in shooting is real but small; the same is true of momentum. A wise coach does not panic after a bad stretch or celebrate a lucky one; he asks what the underlying numbers say.

The risk side. Regression to the mean misleads in both directions. Teams trade for a player after his best season and pay for luck. Coaches change a strategy after one bad week and credit the change when results return to normal. The defense is patience and larger samples: judge with many attempts, and expect extremes to be followed by something more ordinary.

PICTURE STUDY



Galton's Diagram of Heights, Francis Galton, 1886 · Printed diagram · From “Regression towards Mediocrity in Hereditary Stature,” *Journal of the Anthropological Institute*

Image: [Wikimedia Commons](#), public domain

This is the diagram from Galton's 1886 paper. Down the side are the parents' average heights (“mid-parents”); across the top are their grown children's heights. Each number counts how many children fell in that cell. Galton noticed that equal counts formed a tilted ellipse, and that the line through the middle of each row was less steep than a line of perfect copying. That gentler slope is regression to the mean, and out of drawings like this came the correlation and regression that every sports analyst uses today.

Look closely

- Find the mid-parent heights down the left side and the children's heights across the top. Find the average, $68\frac{1}{4}$ inches, marked as 0.
- Find the ellipse. It joins cells with about the same count. Why is it tilted rather than upright?
- Read the note under the title: female heights were multiplied by 1.08. Why did Galton need to adjust them?
- Look for where the largest numbers sit. Are they near the middle or near the edges?

Talk about it

1. If the ellipse were a perfect circle, what would that say about parents' and children's heights? If it were a long, thin ellipse along a diagonal?
2. Galton counted and drew all of this by hand. What would a spreadsheet make easier, and what might he have noticed by doing it himself?
3. How could you make a grid like this for a sport: for example, a player's shooting percentage in the first and second halves of the season?

PROJECT

Who Regresses? A First-Half, Second-Half Study

You will gather real statistics for many players, split each season into halves, and measure how much the best and worst first-half performers fall back toward average in the second half.

You will need

- Baseball-Reference or Basketball-Reference (free player splits by month or half)
- Spreadsheet software
- Graph paper for a scatterplot

Steps

1. Choose one statistic: batting average (baseball) or three-point percentage (basketball).
2. From one past season, record the first-half and second-half values for at least 30 regular players with plenty of attempts in each half.
3. Compute the overall average of the first-half values.
4. Sort the players by first-half value. Mark the top 5 and the bottom 5.
5. Compute the average first-half and second-half values for the top 5 and for the bottom 5.
6. Make a scatterplot with first half on the horizontal axis and second half on the vertical axis. Draw the line $y = x$.
7. Estimate r for your data: for the top 5, divide (second-half average – overall mean) by (first-half average – overall mean).
8. Repeat the estimate for the bottom 5.
9. Write a paragraph explaining your results to a coach who thinks his top first-half players are “slumping.”

Tips

- Players with few attempts regress the most. Set a minimum (for example 150 at-bats per half) so your numbers are steady.

- Points above the line $y = x$ improved; points below declined. Look at where the extreme players fall.

Check your work

- ☐ I recorded first-half and second-half values for at least 30 players.
- ☐ My scatterplot has labelled axes and the line

$y = x$.

- ☐ I computed an estimate of r and explained what it means.

SCOREBOOK

1. Copy your warm-up coin sequence and mark every run of 3 or more. Compare your guess with the true chance of a run of 4 or more heads in 20 flips (47.8%).
2. Draw a sketch of a win-probability chart for a game you watch, marking the three biggest swings and their WPA.
3. “What I would decide and why”: a player hits .410 in his first month. Your coach wants to give him a big role. Write your advice using this lesson.

KEY FORMULAS

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

$$\text{prediction} = \bar{x} + r(x - \bar{x}), \quad r = \frac{n}{n+k}$$

$$\text{WPA} = WP_{\text{after}} - WP_{\text{before}}$$

From history. Galton also built the “bean machine,” now called the Galton board, in which balls bounce through pins and pile up in a bell shape. He used it to show how many small chances add up to a predictable pattern, the same pattern that appears in batting averages and shooting percentages.

WORKED EXAMPLES

Example 1. A hitter bats .400 (20 hits in 50 at-bats) in April. The league average is .250. How many standard deviations above .250 is he, if he were really a .250 hitter?

Show solution

1. Spread from luck: $\sqrt{0.25 \times 0.75/50} = \sqrt{0.00375} = 0.0612$.

2. Distance: $0.400 - 0.250 = 0.150$.

3. Number of standard deviations: $0.150/0.0612 = 2.45$.

Answer: About 2.45 standard deviations. That is unusual for a true .250 hitter, so he is probably better than average, but among hundreds of hitters a few will reach this by luck alone.

Example 2. Predict the same hitter's rest-of-season average with $r = n/(n + k)$ and an illustrative $k = 200$ at-bats. Then repeat for a hitter who batted .300 over 500 at-bats.

Show solution

1. April hitter: $r = 50/250 = 0.2$; prediction $0.250 + 0.2(0.150) = 0.280$.

2. Full-season hitter: $r = 500/700 = 0.714$; prediction
 $0.250 + 0.714(0.050) = 0.286$.

Answer: The .400 April hitter is predicted at .280 and the .300 full-season hitter at .286. A long, modest record can deserve more trust than a short, spectacular one.

Example 3. Galton found that children's heights are on average about two-thirds as far from the mean as their parents' average height. In his data the mean was about $68\frac{1}{4}$ inches. If two parents' heights (with the mother's multiplied by 1.08, as Galton did) average $71\frac{1}{4}$ inches, predict their grown child's height .

Show solution

1. Parents' deviation: $71.25 - 68.25 = 3$ inches.
2. Child's predicted deviation: $\frac{2}{3} \times 3 = 2$ inches.
3. Prediction: $68.25 + 2 = 70.25$ inches.

Answer: About $70\frac{1}{4}$ inches: taller than average, but closer to the average than the parents.

Example 4. A team's win probability is 35% before a long touchdown pass and 62% after it. What is the win probability added (WPA) by the play? Later, an interception drops it from 62% to 41%. What is that play's WPA?

Show solution

1. Touchdown: $0.62 - 0.35 = +0.27$.
2. Interception: $0.41 - 0.62 = -0.21$.

Answer: +27 percentage points and -21 percentage points. Win probability added lets you rank plays by how much they changed the chance of winning.

PRACTICE

1. Easy In your own words, what is regression to the mean?

Show answer

When a result is extreme partly because of luck, the next result for the same player or team tends to be closer to average, because the luck does not repeat.

2. Easy Galton studied 928 grown children. Did tall parents tend to have children who were taller than average, shorter than average, or exactly as tall as the parents?

Show answer

Taller than average, but on average not as tall as the parents: closer to the mean.

3. Easy Explain the “cover jinx” without any curse.

Show answer

Athletes reach the cover after an extraordinary run that usually includes good luck. When the luck fades, their performance falls back toward their true level, which looks like a jinx.

4. Medium Find the standard deviation from luck for a 40% shooter taking 100 shots.

Show answer

$$\sqrt{0.4 \times 0.6/100} = \sqrt{0.0024} \approx 0.049, \text{ about 5 percentage points.}$$

5. Medium A 40% shooter makes 52 of 100 in a week. How many standard deviations above his true rate is that?

Show answer

$$(0.52 - 0.40)/0.049 \approx 2.4 \text{ standard deviations.}$$

6. Medium With $r = n/(n + k)$, $k = 200$, a league mean of .250 and a hitter at .330 over 100 at-bats, predict his future average.

Show answer

$$r = 100/300 = 1/3; 0.250 + \frac{1}{3}(0.080) \approx 0.277.$$

7. Medium A flight instructor notices that cadets who make terrible landings usually improve after being scolded. Give a statistical explanation.

Show answer

A terrible landing is partly bad luck. On the next attempt the luck is likely to be more ordinary, so the landing is likely to be better whether or not he is scolded: regression to the mean.

8. Medium A team trails, then scores on three straight possessions, and an announcer says it “has all the momentum.” If the team scores on half of its possessions in general, what is the chance of three straight scores in a given three possessions?

Show answer

$0.5^3 = 0.125$. Over a whole game with many possessions, a run of three is very likely to happen somewhere by chance.

9. **Hard** Win probability goes $50\% \rightarrow 58\% \rightarrow 52\% \rightarrow 71\% \rightarrow 66\%$ over four plays. Find the WPA of each play and the total. What do you notice about the total?

Show answer

+8, -6, +19, -5 percentage points; total +16. The total equals the last value minus the first ($66 - 50$), because the changes add up.

10. **Hard** A coach benches his team's worst-performing player each week. The next week the replacement usually plays worse than the benched player's usual level, and the benched player, when he returns, does better. Is benching working? Explain.

Show answer

Not necessarily. The worst performer each week was often unlucky that week, so he would have improved anyway when he returned (regression to the mean). The coach is crediting his benching for what chance would have done.

11. **Challenge** Explain why $r = n/(n + k)$ makes sense when n is very small and very large, and estimate k if a statistic is half luck and half skill after 300 attempts.

Show answer

If n is near 0, r is near 0 and we predict the league mean, since we know almost nothing. If n is huge, r is near 1 and we trust the record. Half skill after 300 attempts means $r = 0.5$ at $n = 300$, so $300/(300 + k) = 0.5$ and $k = 300$.

COMMON MISTAKES

- Explaining every rise and fall with a story (confidence, pressure, a jinx) instead of asking how much is luck.
- Thinking regression means everyone becomes average. Good players stay good; they simply become less extreme.
- Forgetting that small samples regress much more than large ones.

Make it a game • Lucky Leader. Four players each roll two dice ten times and total their scores. The leader is the hot start. Then everyone rolls ten more times. Predict before rolling whether the leader will stay on top. Compare their averages. Name the idea that explains the slide, and connect Galton's heights from 1886 to early-season batting averages.

4.5 PLAYING TO WIN VERSUS PLAYING NOT TO LOSE

OBJECTIVES

- Explain the prevent defense and the intentional walk as choices that reduce one risk while increasing another.
- Use a run-expectancy table to compute the cost of an intentional walk and judge when it can still be wise.
- Connect calculated courage in sport with the parable of the talents, and tell the difference between courage and recklessness.

Warm-up. You lead a family dice game by 6 points with 3 turns left. On each turn you may roll one die (add the number) or play safe (add 2 points). Your opponent must roll two dice each turn. Before calculating, write which you would choose and why. Then think: does it matter more to score many points or to keep your opponent from catching up?

WATCH

Why Teams Come Back - The Rubber Band Effect

Michael MacKelvie · 18:09

<https://www.youtube.com/watch?v=8By4b6cTVFU>

WHEN TO WATCH

After the story paragraph. It studies whether teams that are behind really play better late in games, and what the prevent defense costs.

Note: Skip from 8:45 to 9:44, a sponsor segment for electric bicycles.

KEY TERMS

prevent defense

A late-game football defense that rushes few players and drops many deep to stop long passes, while giving up short completions.

intentional walk

A baseball tactic in which the defense deliberately puts a batter on first base.

run expectancy

The average number of runs scored from a given base-and-out situation until the end of the inning (from Lesson 3.2).

playing not to lose

Choosing to avoid a visible, dramatic failure rather than to maximize the chance of winning.

calculated courage

Accepting a risk because the numbers show it gives the best chance, after counting the cost.

CORE IDEAS

The story. Every football fan has watched it happen: a team leads by a touchdown with two minutes left, its defense backs off to keep anyone from getting behind it, and the opponent marches down the field with short completions, ten and fifteen yards at a time. The prevent defense prevents the big play, but it can also prevent the win. Baseball has its own version. With a dangerous hitter at the plate and a base open, a manager signals four fingers and the batter jogs to first. The intentional walk feels safe, because nobody can blame the manager for the hit that never happened.

How they figured it out. Analysts compared what these choices actually produce. In football, studies of late-game drives (like the one in this lesson's video) found that trailing offenses do better against soft, deep coverage, because they can take what is given underneath and stop the clock. In baseball, the run-expectancy table from Lesson 3.2 shows that putting a runner on base almost always increases the number of runs the opponent can expect. In 2017 Major League Baseball let managers simply signal for an intentional walk without throwing four pitches. The rule made the walk quicker, but it did not make it cheaper.

The mathematics. Each choice reduces one risk and raises another. An intentional walk removes the chance of a home run by this hitter but adds a base runner and may bring a run home later. With a runner on second and one out, the average major-league inning yields about 0.66 more runs; with runners on first and second and one out, about 0.88. The walk costs about $0.88 - 0.66 = 0.22$ runs on average, unless the next batter is much weaker or the extra force play and double play matter more. Late in a close game, the right measure is not expected runs but the *chance of allowing at least one run*, which can change the answer.

On the field. Our Lord's parable of the talents tells of three servants given money to manage. Two put it to work and doubled it. The third said, "I was afraid, and went and hid thy talent in the earth" (Matthew 25:25). His master did not praise his caution. The parable is not about sports, but it names a temptation every competitor knows: playing so as not to be blamed instead of using what you have been given. Calculated courage counts the cost first (Luke 14:28), then acts. Recklessness acts without counting. Fear refuses to act at all.

Think it through. "Safe" plays are not safe when they reduce your chance to win. The question is never "Which play has the scariest possible outcome?" but "Which play gives us the best chance to win, counting every outcome and how likely it is?" A team ahead should reduce variance (Lesson 2.3), but it should do it without handing the opponent easy yards or easy runs.

PICTURE STUDY



The Parable of the Talents, Willem de Poorter, 17th century · Oil on panel · Dutch Republic

Image: [Wikimedia Commons](#), public domain

Willem de Poorter, a Dutch painter who probably studied with Rembrandt, painted the moment of reckoning in the parable of the talents, when the master asks each servant what he has done with what he was given. Dutch merchants of his day, who sent ships around the world and counted their risks carefully, knew this story well.

Look closely

- Find the master on the raised platform. How is he receiving the account?
- Find the table between the servant and the master, where the account is laid out. It is a record, like a scorebook.
- Look at the servant leaning forward. Does he look confident or anxious?
- Find the figures watching from the staircase and the doorway. Why might others care about this reckoning?

Talk about it

1. The servant who buried his talent avoided every visible risk. Why was he still judged unfaithful?
2. How is a decision record, like the account laid out on this table, useful to a coach or player?

3. Where in your own games or sports have you been tempted to play “not to lose”?

PROJECT

The Safe-Play Audit

You will play a dice game against a family member with two strategies, one bold and one “safe,” and record which actually wins more often from different positions.

You will need

- Two dice
- Scorebook and pencil
- A family member as opponent
- Optional: spreadsheet with $= \text{RANDBETWEEN}(1, 6)$

Steps

1. Rules of the game: first to 30 points wins. On your turn choose SAFE (add 4 points) or BOLD (roll two dice and add the total, except a roll of doubles scores 0). Your opponent always rolls BOLD.
2. Find the expected value of BOLD: the average of two dice is 7, but doubles (6 of the 36 rolls) score 0. Compute it and compare with SAFE's 4.
3. Play 10 games in which you always choose SAFE. Record wins and losses.
4. Play 10 games in which you always choose BOLD.
5. Play 10 games with a smart rule: SAFE when you are ahead by 8 or more, BOLD otherwise.
6. Record each game's result and the score at the end.
7. Compare the win rates of the three strategies in a table.
8. If you have a spreadsheet, simulate 500 games of each strategy to check your family results.
9. Write a paragraph: when is the “safe” play really safe, and when does it hand the game away?

Tips

- The expected value of BOLD is $(252 - 42)/36 = 210/36 \approx 5.83$ points per turn. Check this yourself before you play.

- Keep the games quick: the point is to collect results, not to agonize over every turn.

Check your work

- ☐ I played 30 games and recorded every result.
- ☐ I computed the expected value of BOLD and compared it with SAFE.
- ☐ My conclusion is supported by my win rates.

SCOREBOOK

1. Copy the run-expectancy values from worked example 1 into a small grid and mark the change caused by an intentional walk with arrows.
2. Draw a decision tree for the prevent-or-normal choice in worked example 3, with the probability of losing at each end.
3. “What I would decide and why”: describe a moment in a game you have played when you chose the safe option. Was it really the percentage play?

KEY FORMULAS

$$\text{cost of a walk} = RE_{\text{after}} - RE_{\text{before}}$$

$$P(\text{lose}) = \sum P(\text{opponent outcome}) \times P(\text{lose} \mid \text{outcome})$$

WORKED EXAMPLES

Example 1. Use these values from the run-expectancy table of Lesson 3.2 (major leagues, 2010–2015): runner on 2nd, 1 out: 0.66; runners on 1st and 2nd, 1 out: 0.88; runner on 2nd, 2 outs: 0.32; runners on 1st and 2nd, 2 outs: 0.43. What does an intentional walk cost with a runner on 2nd and 1 out? With a runner on 2nd and 2 outs?

Show solution

1. One out: $0.88 - 0.66 = +0.22$ runs.
2. Two outs: $0.43 - 0.32 = +0.11$ runs.

Answer: The walk gives up about 0.22 runs on average with one out and about 0.11 with two outs. It is less costly with two outs, but it is never free.

Example 2. When can the walk still be right? Suppose the batter is an excellent hitter: with a runner on 2nd and 1 out against him, the inning yields 0.85 runs on average. The next batter is weak: the inning after a walk to the star yields 0.80 runs (illustrative). Compare.

Show solution

1. Pitch to the star: expected 0.85 runs.
2. Walk the star and face the weak hitter: expected 0.80 runs.
3. Difference: $0.85 - 0.80 = 0.05$ runs saved by walking.

Answer: In this case the walk saves about 0.05 runs, a small gain. The ordinary table assumes average hitters; a big gap between hitters can reverse the answer.

Example 3. Prevent or normal? You lead by 6 with 1:30 left, and the opponent has no timeouts at its own 25. A touchdown plus the extra point would beat you. With a normal defense the opponent scores a touchdown on 12% of such drives; with a prevent defense, which gives up easy short passes, it scores on 15% (illustrative numbers). The prevent defense allows a play of 40 yards or more on only 2% of drives, against 5% for the normal defense. Which defense should you choose?

Show solution

1. What matters is the chance of losing, not the chance of a long play.
2. The opponent wins only with a touchdown (and nearly certain extra point): normal 12%, prevent 15%.
3. The long plays are part of the touchdown chance already; counting them again would be double counting.

Answer: Choose the normal defense: about an 88% chance to win against 85%. The prevent defense reduces the visible risk (a long play) but increases the risk that matters.

PRACTICE

1. **Easy** What does the servant in Matthew 25:25 say was the reason he hid his talent?

Show answer

“I was afraid.”

2. **Easy** What is a prevent defense trying to stop, and what does it give up?

Show answer

It tries to stop long passes and big plays; it gives up short completions, which let the offense move steadily and stop the clock.

3. **Easy** Using the table in worked example 1, how many runs on average are expected with runners on 1st and 2nd and 1 out?

Show answer

About 0.88 runs.

4. **Medium** Using the values in worked example 1, an intentional walk with a runner on 2nd and 2 outs raises run expectancy from 0.32 to 0.43. Express the cost as a percentage increase.

Show answer

$0.11/0.32 \approx 34\%$.

5. **Medium** Explain the difference between calculated courage and recklessness, with one sports example of each.

Show answer

Calculated courage takes a risk after counting the cost because it offers the best chance (going for it on fourth-and-1 at midfield with a 64% conversion rate). Recklessness ignores the numbers (going for it on fourth-and-15 from your own 10 in the first quarter).

6. **Medium** In worked example 3, suppose the prevent defense allowed touchdowns on only 10% of such drives. Which defense would you choose then?

Show answer

The prevent defense: $10\% < 12\%$. The choice depends on which defense actually lowers the chance of losing, which must be measured, not assumed.

7. **Medium** Why might a manager choose an intentional walk even when it raises expected runs?

Show answer

To face a much weaker hitter, to set up a force or double play, or because, late in a game, he cares about allowing any run at all rather than the average number of runs.

8. **Medium** A football team leads by 3 points at midfield, 4th-and-1, with 2:00 left. Explain the difference between “playing not to lose” and “playing to win” in this situation.

Show answer

Playing not to lose: punt because a failed try looks bad. Playing to win: compare the chance of winning after each choice; converting would let the team run out the clock, which may be the better percentage even though failure is more visible.

9. **Hard** With a runner on 3rd and 1 out (run expectancy 0.95), a manager walks the batter to put runners on 1st and 3rd and 1 out (1.13). Find the cost in expected runs. Then explain why the double-play possibility might make this walk sensible late in a tie game.

Show answer

Cost: $1.13 - 0.95 = 0.18$ runs. With runners on 1st and 3rd, a ground ball can become an inning-ending double play before the runner scores. Late in a tie game, what matters is preventing that one run, and the double play makes that more likely even though the average number of runs rises.

10. **Hard** The third servant in the parable thought he was choosing the safe option. Using this lesson’s language, explain why his choice was not actually safe.

Show answer

He avoided the visible risk of losing the money but guaranteed the result he most needed to avoid: an unfaithful stewardship that gained nothing. Like the prevent defense, his “safe” play reduced one risk while ensuring a worse outcome.

11. **Challenge** A team ahead by 7 with 3:00 left can run the ball (40% chance to make a first down and effectively end the game; otherwise punt and the opponent scores a tying touchdown 25% of the time and wins half the overtimes) or pass (55% chance to make a first down; 3% chance of an interception that gives the opponent a short field, after which they tie 50% of the time; otherwise punt as before). Compute the chance of losing for each plan. (Illustrative numbers.)

Show answer

Run: lose $0.60 \times 0.25 \times 0.5 = 0.075$. Pass: $0.03 \times 0.5 \times 0.5 = 0.0075$ from interceptions, plus $0.42 \times 0.25 \times 0.5 = 0.0525$ from punts; total 0.060. Passing is the better percentage (6.0% against 7.5%), even though it carries a visible risk of an interception.

COMMON MISTAKES

- Judging a defense by the scariest outcome (a long touchdown) instead of by the chance of losing.
- Thinking an intentional walk is free because no hit was allowed.
- Confusing courage with ignoring the numbers. Courage acts on the numbers; recklessness ignores them.

Make it a game • Risk Seesaw. Your parent names a choice: a prevent defense, an intentional walk, or a conservative play. You state in ten seconds which risk it lowers and which risk it raises. A correct pair earns a point. After five rounds, switch roles. Finish by giving one example from your own life where playing not to lose cost more than playing to win.

Scripture. “And I was afraid, and went and hid thy talent in the earth: lo, there thou hast that is thine.” (Matthew 25:25, KJV)

The servant chose what looked safe and was judged for it. Faithful use of what we are given sometimes requires calculated risk.

UNIT 4

DECISIONS UNDER PRESSURE: COACHES AND THE ODDS

Units 1 through 3 taught you to compute probabilities and expected values and to recognize the high-percentage shot. This unit puts you on the sideline, where a coach must decide in seconds, with a crowd watching and his job on the line. You will price fourth-down decisions with David Romer's expected points, find the best time to pull a goalie, solve the penalty kick with von Neumann's and Nash's game theory, separate skill from luck with Galton's regression to the mean, and ask when the "safe" play is safe at all.

Each lesson pairs a real decision with real data: Romer's 1,068 fourth downs, Art Ross's 1931 goalie pull, Palacios-Huerta's 1,417 penalty kicks, Galton's 928 children. You will audit real fourth downs from play-by-play, simulate empty-net minutes, take penalty kicks in the backyard, study first-half and second-half statistics, and test "safe" and "bold" strategies in a dice game.

PREREQUISITES

- Expected value as a probability-weighted average (Unit 2)
- Variance and the rule that the team behind should increase it (Lesson 2.3)
- The run-expectancy table (Lesson 3.2)
- Solving a linear equation for one unknown

MASTERY CHECKLIST

- ☐ I can compute the expected points of going for it, punting and kicking, and find a break-even probability.
- ☐ I can explain why a trailing team should pull its goalie early, using a per-minute model.
- ☐ I can find a mixed-strategy equilibrium in a 2×2 game by making the opponent indifferent.
- ☐ I can explain regression to the mean, compute the spread caused by luck, and shrink an observed result toward the average.
- ☐ I can use a run-expectancy table to price an intentional walk and explain the difference between calculated courage and playing not to lose.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Game Theory 101 (William Spaniel)**: Free. A full course of short video lectures on game theory by the author of the Lesson 4.3 video, including mixed strategies and the indifference method. Ages 14+.
- **Hockey-Reference**: Free. Game logs and scoring summaries showing when goals and empty-net goals were scored, for extending the Lesson 4.2 simulator with real data. Ages 13+.
- **Baseball-Reference**: Free. Player splits by half and by month for the regression study in Lesson 4.4, and situational data for the intentional-walk questions in Lesson 4.5. Ages 13+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

4.1 Fourth Down	3 sessions of 50 minutes
4.2 Pulling the Goalie	3 sessions of 50 minutes
4.3 Keep Them Guessing	3 sessions of 50 minutes
4.4 Win Probability, Momentum and Regression to the Mean	3 sessions of 50 minutes
4.5 Playing to Win versus Playing Not to Lose	3 sessions of 50 minutes

UNIT 4 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer with a sentence explaining what it means. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

Going for it on fourth down succeeds with probability p , giving $S = 2.5$; failure gives $F = -1.5$; a punt gives $K = 0.5$. Find the break-even probability.

2. (6 pts)

In problem 1, find the expected value of going for it if $p = 0.62$, and state the better choice.

3. (6 pts)

Romer found teams kicked on 959 of the 1,068 fourth downs where going for it was better. Compute the percent and give one explanation for the pattern.

4. (7 pts)

Fourth-and-goal at the 2: a field goal nets about 2.4 points; a touchdown nets about 6.4; a failed try is worth +1.5 to you. Find the break-even probability.

5. (7 pts)

Trailing by one goal, with the goalie pulled you score with probability 0.08 per minute and the opponent with probability 0.20. Find the chance you tie within 2 minutes (scoring before they do).

6. (6 pts)

Explain why pulling the goalie can be wise even though it makes the expected goal difference worse.

7. (5 pts)

Who pulled a goalie for an extra attacker in the 1931 playoffs, and what happened?

8. (8 pts)

A kicker scores 90% if the keeper guesses wrong, 60% kicking left if the keeper guesses left, and 75% kicking right if the keeper guesses right. Let g = probability the keeper dives left. Find g at equilibrium and the scoring rate.

9. (6 pts)

What did Palacios-Huerta (2003) find when he compared real penalty kicks with the minimax prediction?

10. (7 pts)

A 30% three-point shooter takes 100 threes. Find the standard deviation of his percentage due to luck, and decide whether 40% over those 100 shots would be surprising if his true rate is 30%.

11. (7 pts)

With $r = n/(n + k)$, $k = 200$, a league mean of .260 and a hitter at .360 over 50 at-bats, predict his future average.

12. (7 pts)

Win probability moves from 44% to 70% on one play. What is the play's win probability added?

13. (7 pts)

Runner on 2nd, 1 out: 0.66 expected runs. Runners on 1st and 2nd, 1 out: 0.88. What does an intentional walk cost, and name one reason a manager might still order it?

14. (7 pts)

Explain in two or three sentences why a prevent defense can lower a team's chance of winning.

15. (8 pts)

Short essay (one paragraph): Using one example from this unit and the parable of the talents (Matthew 25:14-30), explain the difference between calculated courage, recklessness and playing not to lose.

ANSWER KEY

1. $p^* = (0.5 + 1.5)/(2.5 + 1.5) = 2.0/4.0 = 0.5$.
2. $0.62(2.5) + 0.38(-1.5) = 1.55 - 0.57 = 0.98$; better than 0.5, so go for it.
3. $959/1068 \approx 89.8\%$. Explanation: fear of blame (a failed try is second-guessed, a punt is normal), or reliance on custom instead of data.
4. $p^* = (2.4 - 1.5)/(6.4 - 1.5) = 0.9/4.9 \approx 0.18$.
5. $0.08 + 0.72(0.08) = 0.1376$, about 13.8%.
6. Late in a one-goal game only tying matters; losing by two is no worse than losing by one. The pull raises the chance of tying while the extra goals against cost almost nothing.
7. Art Ross of the Boston Bruins pulled Tiny Thompson in the final minute against Montreal; Boston did not score and lost 1–0, but the idea endured.
8. Kick left: $60g + 90(1 - g) = 90 - 30g$. Kick right: $90g + 75(1 - g) = 75 + 15g$. Equal: $15 = 45g$, $g = 1/3$. Scoring rate $90 - 10 = 80\%$.
9. Kickers chose their natural side about 60.0% of the time (predicted 61.5%) and keepers about 57.7% (predicted 58.0%): professionals play very close to the equilibrium.
10. $\sqrt{0.3 \times 0.7/100} = \sqrt{0.0021} \approx 0.046$. 40% is $0.10/0.046 \approx 2.2$ standard deviations above: unusual, but it will happen to some shooters by luck.
11. $r = 50/250 = 0.2$; $0.260 + 0.2(0.100) = 0.280$.
12. +26 percentage points (+0.26).
13. 0.22 runs on average. Reasons: a much weaker next hitter, setting up a double play, or caring late in a game about any run rather than the average.
14. It reduces the chance of a long completion but gives the offense easy short gains that move the ball and stop the clock. What matters is the chance the opponent scores, and if that rises, the defense has protected against the wrong risk.
15. Answers vary. A strong answer defines each: calculated courage takes a risk after counting the cost because it gives the best chance (going for it on fourth-and-1 at midfield, pulling the goalie with several minutes left); recklessness takes risks without counting (fourth-and-15 from your own 10); playing not to lose avoids visible failure at the expense of the real goal (the punt that lowers win probability, the servant who buried his talent because he “was afraid”).

PORTFOLIO PIECE: A COACH'S DECISION GUIDE

Write a four-page decision guide for one sport you play or follow, the kind a coach could carry to the sideline. It must include (1) a decision chart for one recurring choice (fourth down, when to pull the goalie, when to walk a hitter, when to go for two) built from expected values or win probabilities, with the calculations shown for at least four cells; (2) a mixed-strategy recommendation for one guessing situation (penalty kicks, serve direction, run or pass) with the equilibrium computed from real or carefully measured data; (3) a page warning about regression to the mean, with an example from real statistics; and (4) a one-paragraph statement of the guide's philosophy that explains when to be bold and when to protect a lead.

LOOK FOR	WHAT IT MEANS
Decision chart	Every cell filled; at least four cells supported by worked break-even or expected-value calculations; assumptions stated.
Game theory	A 2×2 table with sources for the numbers, the equilibrium mixes found by indifference, and the value of the game.
Skill and luck	A real example of regression to the mean with a standard-deviation calculation and a shrunken prediction.
Judgment and honesty	The philosophy paragraph distinguishes calculated courage from recklessness and from playing not to lose, and the guide admits what its numbers cannot capture.

THE ODDS AT THE GAME TABLE

The family game table is a probability laboratory. In this unit you will count your way through five classic games: the race and the blot in backgammon, the dice battles of *Risk* and the board squares of *Monopoly*, the sacrifices of chess and the duel between Kasparov and Deep Blue, the hidden cards of bridge, and the push-your-luck decisions of *Can't Stop* and *Yahtzee*. Along the way you will meet Hoyle's dice tables, Markov chains, Elo ratings, combinations of cards, and the stopping rule.

Every game in this unit is played for points, never for stakes. Your projects will produce real data: hundreds of dice rolls and deals, battle tables, a census of board squares, sacrifice experiments and stopping-rule tournaments. By the end you will be the player at your table who knows not only what to do, but why.

PREREQUISITES

- The 36 outcomes of two dice and the multiplication principle (Lesson 1.2)
- Independence: dice have no memory (Lesson 1.3)
- Conditional probability (Lesson 1.4)
- Expected value and variance (Unit 2)
- Combinations $\binom{n}{k}$ on a calculator

MASTERY CHECKLIST

- ☐ I can count pips, count the rolls that hit a blot, and explain the 25% acceptance point of the doubling cube in tournament points.
- ☐ I can compute
Ris battle odds and expected losses,
k and use a simple Markov chain and its steady state.
- ☐ I can compute expected scores in chess and Elo expected scores, and explain when a speculative sacrifice is the percentage play.
- ☐ I can compute suit-split and finesse probabilities and update them from evidence.
- ☐ I can derive a stopping rule for a push-your-luck game and explain how it changes near the end of a game.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Lichess:** Free and without advertising. A nonprofit chess site with an analysis board and engine evaluations, useful for the sacrifice lab in Lesson 5.3. A parent should set up any account and supervise online play. Ages 13+.
- **American Contract Bridge League: Learn to Play:** Free lessons and resources for beginners from bridge's governing body in North America, for families who want to play the deals in Lesson 5.4. Ages 12+.
- **Can't Stop (Eagle-Gryphon Games):** The current edition of Sid Sackson's push-your-luck dice game from Lesson 5.5. Not required: the lesson's math and project work with ordinary dice. Ages 9+.
- **Yahtzee (Hasbro):** The publisher's page for the classic five-dice game in Lesson 5.5. Five ordinary dice and a homemade score sheet work just as well. Ages 8+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

5.1	Backgammon: The Race, the Blitz and the Cube	3 sessions of 50 minutes
5.2	Dice Battles and Board Squares	3 sessions of 50 minutes
5.3	Chess: Sound and Speculative Sacrifices	3 sessions of 50 minutes
5.4	Hidden Information: Bridge, Stratego and Battleship	3 sessions of 50 minutes
5.5	Push Your Luck: Can't Stop and Yahtzee	3 sessions of 50 minutes

UNIT 5 TEST

Calculators are allowed. Show your work for every calculation: list or count the outcomes, write the formula, and give the answer as a fraction, decimal or percent with a sentence explaining it. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

Show that the average roll in backgammon moves $8\frac{1}{6}$ pips.

2. (6 pts)

How many rolls hit a blot 5 pips away (no blocked points)? List the combinations that do not contain a 5.

3. (7 pts)

In a tournament game worth 1 point you are doubled with a 28% chance of winning. Accept or decline? Show the comparison.

4. (6 pts)

In a one-against-one *Risk* battle (ties to the defender), find the attacker's chance of winning.

5. (7 pts)

In three-against-two battles, the attacker loses 0, 1 or 2 armies with probabilities 0.372, 0.336 and 0.293. Find the expected armies lost by each side.

6. (6 pts)

A player leaves Jail with an ordinary roll. Find the chance he lands on square 16, 18 or 19.

7. (7 pts)

After a win a team wins its next game 70% of the time; after a loss, 40%. Find its long-run winning percentage.

8. (6 pts)

Find the expected score of a chess move that wins 35%, draws 40% and loses 25%.

9. (7 pts)

Find the Elo expected score of a 1450 player against a 1650 player, and his new rating ($K = 20$) if he wins.

10. (6 pts)

Describe the decisive moment of Game 6 of Kasparov versus Deep Blue (1997).

11. (7 pts)

You need two finesses to both succeed, each 50% and independent. Find the chance. Find the chance at least one succeeds.

12. **7 pts**

You and partner hold 26 HCP. Your left-hand opponent opened showing 12 to 14. Can the right-hand opponent hold a king? Explain.

13. **6 pts**

In Pig, show that rolling again has positive expected value only while the turn total is below 20.

14. **6 pts**

Find the chance of completing a Yahtzee from three of a kind with two rolls left.

15. **10 pts**

Short essay (one paragraph): Choose two games from this unit. For each, describe one decision where the high-percentage play and the low-risk play are different, and explain how a wise player decides between them.

ANSWER KEY

1. Non-doubles total 210 pips over 30 rolls; doubles move $4(1 + \dots + 6) = 84$.
 $(210 + 84)/36 = 294/36 = 8\frac{1}{6}$.
2. 15 rolls: the 11 containing a 5, plus 4-1, 1-4, 3-2, 2-3.
3. Accept: $4(0.28) - 2 = -0.88$; decline -1. Accept.
4. $15/36 \approx 41.7\%$.
5. Attacker: $0.336 + 2(0.293) = 0.922$; defender: $2 - 0.922 = 1.078$. (0.921 and 1.079 with unrounded values.)
6. Needs 6, 8 or 9: $(5 + 5 + 4)/36 = 14/36 \approx 38.9\%$.
7. $\pi_W(0.3) = (1 - \pi_W)(0.4) \Rightarrow 0.7\pi_W = 0.4 \Rightarrow \pi_W \approx 0.571$.
8. $0.35 + 0.20 = 0.55$.
9. $E = 1/(1 + 10^{0.5}) = 0.240$; new rating $1450 + 20(0.760) = 1465.2$.
10. In a Caro-Kann, Deep Blue sacrificed a knight on e6 on move 8; Kasparov's position collapsed and he resigned after 19 moves, losing the match $3\frac{1}{2}$ - $2\frac{1}{2}$.
11. Both: 25%. At least one: $1 - 0.25 = 75\%$.
12. Opponents hold 14. The left one has at least 12, so the right one has at most 2. A king is worth 3, so he cannot hold one.
13. $E = \frac{5}{6}(4) - \frac{1}{6}t = (20 - t)/6 > 0 \iff t < 20$.
14. $\frac{1}{36} + \frac{10}{36} \cdot \frac{1}{6} + \frac{25}{36} \cdot \frac{1}{36} = \frac{121}{1296} \approx 9.3\%$.
15. Answers vary. A strong essay names specific decisions (accepting a double at 30%, a speculative sacrifice when only a win counts, playing for a 3-2 split rather than a finesse, holding at 20 in Pig except when far behind), computes or cites the relevant probabilities, and explains that the right choice depends on the goal (expected points, chance of winning) and on what can be afforded to lose.

PORTFOLIO PIECE: A FAMILY GAME STRATEGY GUIDE

Write a strategy guide of four to six pages for one family game (backgammon, *Risk*, *Monopoly*, chess, bridge, *Stratego*, *Battleship*, *Can't Stop*, *Yahtzee*, *Catan* or another approved by a parent). It must include (1) the key probabilities of the game, computed by counting and checked by at least 200 rolls, deals or simulated turns; (2) at least three specific decisions with the expected value or winning chance of each option; (3) one decision where the best play changes depending on the score or the stage of the game; and (4) a short section, “How to play this game well and graciously,” about honest play, good sportsmanship, and keeping score in points only.

LOOK FOR	WHAT IT MEANS
Probabilities	Key probabilities are computed correctly with the counting shown, and compared with real or simulated data of at least 200 trials.
Decisions	At least three decisions are analyzed with expected values or winning chances, and the recommendation follows from the numbers.
Situation and risk	The guide shows how the right play changes when ahead or behind, and names the risk side of each recommendation.
Clarity and character	Clear writing, labelled tables or charts, and a thoughtful section on honest, gracious play.

5.1 BACKGAMMON: THE RACE, THE BLITZ AND THE CUBE

OBJECTIVES

- Count pips, compute the average roll ($8\frac{1}{6}$ pips), and use the pip count to judge who leads a race.
- Count the rolls that hit a blot at a given distance and weigh a safe play against a bold one.
- Explain the doubling cube as a decision about whether you win often enough, and derive the 25% acceptance point in tournament points.

Warm-up. Roll two dice 36 times and, for each roll, write whether it contains a 6 or adds to 6 (for example 4–2 or 3–3). Before you start, predict how many of the 36 rolls will “hit” a target 6 pips away. Keep your tally for worked example 3.

WATCH

Pip Counting Made Easy - The Secret to Fast and Precise Pip Counting

Backgammon Galaxy · 8:01

<https://www.youtube.com/watch?v=rv1DW1SbVw0>

WHEN TO WATCH

After the mathematics paragraph, before worked example 1. It shows quick ways to count pips at the board.

BACKUP VIDEO

Backgammon for complete beginners. Part 12 - The pip count.

Ensor42 · 4:13

<https://www.youtube.com/watch?v=Nj0RTq-GeSs>

Optional, for a slower beginner’s demonstration of the pip count on a real board.

KEY TERMS

pip

One point (triangle) of movement on a backgammon board. A checker on your 13-point needs 13 pips to bear off.

pip count

The total number of pips all your checkers must still travel. The lower count leads the race.

blot

A single checker alone on a point, which an opponent can hit and send back to the start.

blitz

An all-out attack that hits blots and builds points in your home board to shut the opponent out.

doubling cube

A die marked 2, 4, 8, 16, 32 and 64. A player may offer to double the value of the game; the opponent must accept or concede the game at its current value.

CORE IDEAS

The story. Backgammon belongs to the oldest family of board games in the world, the race games played with dice on boards like the Royal Game of Ur. The Romans played a version called *tabula*, and King Alfonso X of Castile included “tables” games in his *Book of Games* of 1283. In 1743 Edmond Hoyle published *A Short Treatise on the Game of Back-Gammon*, which included tables of the chances of the dice, so that players could reckon how likely a hit was. In the 1920s players in New York clubs added the doubling cube, and the game changed from a race of luck to a contest of judgment.

How they figured it out. Every backgammon decision begins with counting. There are 36 equally likely rolls of two dice (Lesson 1.2). A double moves four times, so the average roll is worth more than 7 pips: it is $294/36 = 8\frac{1}{6}$ pips. A player who knows the pip count can tell who is ahead in the race, and roughly by how many rolls. A player who knows the number of hitting rolls can tell how dangerous a blot is. In the 1990s computer programs that taught themselves by playing millions of games against themselves (TD-Gammon, and later the free program GNU Backgammon) showed that even strong human players had misjudged many positions, especially by playing too safe.

The mathematics. A blot 6 pips away can be hit by any roll containing a 6 (11 rolls) or by 5-1, 1-5, 4-2, 2-4, 3-3 and 2-2 (6 more): 17 rolls of 36, or 47%. A blot 8 pips away can be hit only by combinations (6-2, 2-6, 5-3, 3-5, 4-4, 2-2): 6 rolls, or 17%. Direct shots (1 to 6 away) are far more dangerous than indirect shots (7 to 12 away). The doubling cube asks a different question. If your opponent doubles and you decline, you concede the game at its current value, say 1 point. If you accept, the game is

played for 2 points. With winning chance p , accepting is worth $2p - 2(1 - p) = 4p - 2$, which beats -1 whenever $p > 0.25$. You should accept a double if you win at least one game in four.

On the board. In this course we play backgammon for tournament points only: each game is worth points toward a match total, and the cube changes how many points the game is worth. The percentage player leads when ahead in the race by running and avoiding contact; when behind in the race, he keeps checkers back and looks for a shot, because a player who is losing the race must increase the variance (Lesson 2.3). A blitz is right when the opponent has blots and your home board is strong; it is reckless when it leaves your own blots facing direct shots.

The risk side. Most players decline too many doubles. A position that wins only 30% of the time feels hopeless, but conceding loses 1 point every time, while accepting loses 2 points 70% of the time and wins 2 points 30% of the time, for an average of -0.8 . The cost of a mistake is measured against the alternative, not against a perfect result.

PICTURE STUDY



A Table Game in the Book of Games, Workshop of Alfonso X of Castile, 1283 · Ink and tempera on parchment · Libro de los juegos, Royal Library of El Escorial, Spain

Image: [Wikimedia Commons](#), public domain

King Alfonso X of Castile, called “the Wise,” had his scribes and painters produce a *Book of Games* in 1283 covering chess, dice and “tables,” the family of race games to which backgammon belongs. Each problem is illustrated with players at the board, and the text explains the position and how it should be played.

Look closely

- Find the board. It is painted standing up so that the reader can see the position. Is it laid out like a modern backgammon board?
- Find the counters. How are they grouped? Which side seems to be further along in the race?
- Look at the players’ hands. One points toward the board; one waits. Who seems to be deciding?
- Notice the writing above the picture. The book explains positions in words, like a strategy guide.

Talk about it

1. Why would a king think a book of game problems was worth the cost of a hand-painted manuscript?

2. The players in 1283 knew nothing of probability theory. How might they still have learned which moves were wise?
3. How is a page like this similar to the strategy guide you will write in this unit's portfolio?

PROJECT

The Safe Play and the Bold Play

You will test the hit table with real dice, then play out a set of practice positions both safely and boldly and keep score in tournament points.

You will need

- A backgammon set, or a board drawn on paper with 24 points and 15 coins or counters per side
- Two dice and a dice cup
- Scorebook

Steps

1. Use your warm-up tally (36 rolls). Compare the number that hit a target 6 away with the expected 17.
2. Roll 108 more times (three sets of 36) and count hits on targets 6 away and 8 away. Compute the percentages and compare with 47% and 17%.
3. Set up a simple race: each side has its 15 checkers in its home board in the same arrangement. Count the pip counts.
4. Play 10 races with a family member, recording the pip counts at the start and who won. Does the leader on roll usually win?
5. Set up a contact position in which you must choose between leaving a blot 6 away or 9 away. Play it out 10 times each way (20 games in all), recording wins and losses.
6. Play a short match to 5 points with the doubling cube. Record every double, your estimate of your winning chance at the time, whether it was accepted, and the result.
7. Compute how many tournament points the accepting player won or lost on average when he accepted with an estimated chance above 25%, and below it.
8. Write a one-page summary: what did the dice teach you about safe and bold play?

Tips

- Count pips by groups: a stack of five on the 6-point is 30 pips at a glance.
- Your winning-chance estimates will be rough. Writing them down is the point; you will learn to calibrate them.

Check your work

- ☐ I rolled at least 144 times and compared my hit rates with $17/36$ and $6/36$.
- ☐ I recorded pip counts and results for at least 10 races.
- ☐ I recorded every double in my match, with my estimate and the result.

SCOREBOOK

1. Draw the backgammon board with the starting position and write the pip count beside each stack.
2. Make a table of hitting rolls for distances 1 through 12, and shade the direct shots.
3. “What I would decide and why”: you are doubled with about 30% winning chances in a tournament game. Accept or decline? Show the arithmetic.

KEY FORMULAS

$$\text{average roll} = \frac{210 + 84}{36} = \frac{294}{36} = 8\frac{1}{6} \text{ pips}$$

$$E[\text{accept}] = 2p - 2(1 - p) = 4p - 2$$

$$\text{accept if } 4p - 2 > -1 \iff p > \frac{1}{4}$$

From history. In 1992 Gerald Tesauro of IBM published TD-Gammon, a program that learned backgammon by playing itself and reached the level of the world’s best players. Its preferences changed how experts played certain opening rolls, a striking case of a machine teaching people about percentages.

WORKED EXAMPLES

Example 1. At the start of a game each player has 2 checkers on his 24-point, 5 on his 13-point, 3 on his 8-point and 5 on his 6-point. Find the starting pip count, and estimate how many rolls it takes to bear off from there if no checker were ever hit.

Show solution

1. $2 \times 24 + 5 \times 13 + 3 \times 8 + 5 \times 6 = 48 + 65 + 24 + 30 = 167.$

2. At $8\frac{1}{6}$ pips per roll: $167 \div 8.17 \approx 20.4$ rolls, before allowing for wasted pips at the end.

Answer: 167 pips each, about 20 to 21 rolls of pure racing (in practice a few more, because some pips are wasted while bearing off).

Example 2. Show that the average roll moves $8\frac{1}{6}$ pips.

Show solution

1. The 36 rolls of two dice have totals that add to $36 \times 7 = 252.$

2. The six doubles (1-1 through 6-6) contribute $2 + 4 + 6 + 8 + 10 + 12 = 42$ of that, so the 30 non-doubles contribute $252 - 42 = 210.$

3. Doubles move four times the die, not twice: $4(1 + 2 + 3 + 4 + 5 + 6) = 84.$

4. Average: $(210 + 84)/36 = 294/36 = 8\frac{1}{6}.$

Answer: $8\frac{1}{6} \approx 8.17$ pips per roll.

Example 3. You must leave one blot. Option A leaves it 6 pips in front of an enemy checker; option B leaves it 9 pips away. Count the hitting rolls for each, assuming no points are blocked.

Show solution

1. 6 away: any roll with a 6 (11 rolls) plus 5-1, 1-5, 4-2, 2-4, 3-3, 2-2 (6 rolls): 17 rolls.
2. 9 away: 6-3, 3-6, 5-4, 4-5, 3-3 (three 3s): 5 rolls.
3. Probabilities: $17/36 \approx 47\%$ and $5/36 \approx 14\%$.

Answer: Option B is far safer: 14% against 47%. Indirect shots (7 or more away) are much less dangerous than direct ones.

Example 4. In a tournament game worth 1 point, your opponent doubles. You estimate your winning chance at 30%. Should you accept? What if it were 20%?

Show solution

1. Decline: you lose 1 point for certain, so the value is -1 .
2. Accept at 30%: $4(0.30) - 2 = -0.8$. Since $-0.8 > -1$, accept.
3. Accept at 20%: $4(0.20) - 2 = -1.2$. Since $-1.2 < -1$, decline.

Answer: Accept at 30% (you save 0.2 points on average); decline at 20%. The break-even is 25%. (This simple rule ignores gammons and who owns the cube afterward, which strong players also count.)

PRACTICE

1. Easy How many rolls of two dice contain at least one 6?

Show answer

11 of 36.

2. Easy Who leads the race if you have a pip count of 112 and your opponent has 124? By about how many rolls?

Show answer

You lead by 12 pips, about $12/8.17 \approx 1.5$ rolls.

3. **Easy** Using the hit table (1 away: 11, 2: 12, 3: 14, 4: 15, 5: 15, 6: 17, 7: 6, 8: 6, 9: 5, 10: 3, 11: 2, 12: 3), what is the chance a blot 4 pips away is hit?

Show answer

$$15/36 \approx 42\%.$$

4. **Medium** List the 6 rolls that hit a blot exactly 8 pips away with no blocked points.

Show answer

6-2, 2-6, 5-3, 3-5, 4-4 and 2-2.

5. **Medium** Find the pip count of a player with checkers on these points: two on the 18, three on the 9, four on the 6, three on the 4 and three on the 2.

Show answer

$$36 + 27 + 24 + 12 + 6 = 105.$$

6. **Medium** You are doubled in a 1-point tournament game with a 25% chance to win. Show that accepting and declining are equal.

Show answer

Accept: $4(0.25) - 2 = -1$; decline: -1 . They are equal.

7. **Medium** Explain why a player far behind in the race should keep checkers back in the opponent's home board instead of running.

Show answer

He cannot win a pure race, so he must create chances to hit a blot. Keeping checkers back increases the variance of the game, giving him a chance to come back, as the team behind should do.

8. **Medium** From the story: what did Hoyle's 1743 treatise add to backgammon, and why does it matter for this course?

Show answer

Tables of the chances of the dice, such as how likely a blot was to be hit, so that players could decide by reckoning rather than hope.

9. **Hard** You have two blots, one 3 pips and one 11 pips in front of an enemy checker. Counting a roll once even if it could hit both, how many rolls hit at least one? (List rolls for the 11-blot that do not already hit the 3-blot.)

Show answer

3 away: 14 rolls. 11 away: 6–5 and 5–6; neither contains a 3 or adds to 3 with 1–2, so both are new. Total $14 + 2 = 16$ rolls, about 44%.

10. **Hard** With a 35% winning chance, by how many points per game on average does accepting a double beat declining?

Show answer

Accept: $4(0.35) - 2 = -0.6$; decline -1 . Accepting is better by 0.4 points per game.

11. **Challenge** When you accept a double, you own the cube and can redouble later. Explain in words why owning the cube makes accepting slightly better than the 25% rule says, so that strong players accept with somewhat less than 25%.

Show answer

Owning the cube lets you redouble when the game turns in your favor and force your opponent to concede at the higher value, or to play on for even more points when you are a big favorite. That extra option has value, so the true acceptance point is a little below 25%.

COMMON MISTAKES

- Forgetting that doubles move four times, so the average roll is $8\frac{1}{6}$, not 7.
- Counting 3–3 as hitting a blot 9 away only once; it hits because three 3s make 9 (it is one roll of 36).
- Declining doubles that should be accepted because the position “looks bad.” The question is whether you win more than one game in four.

Make it a game • Pip Count Sprint. Set up two sides of a backgammon-style layout using beans on a sheet of paper, with each stack standing for a point and its distance home. Race to count each side's pip count. Compute the difference and state who leads. Then compute the average roll of 8 and one sixth pips and estimate how many turns the race needs.

5.2 DICE BATTLES AND BOARD SQUARES

OBJECTIVES

- Compute the attacker's odds in a *Risk* battle by listing outcomes, and find the expected armies lost on each side.
- Explain why some *Monopoly* squares are landed on more often than others, using Jail and Illinois Avenue.
- Set up and use a simple Markov chain with a transition table and find its long-run (steady-state) probabilities.

Warm-up. Roll one die for an “attacker” and one for a “defender” 20 times. The attacker wins only if his number is strictly higher; ties go to the defender. Predict the attacker's wins out of 20 before you start, then tally.

WATCH

The Game of Risk - Numberphile

Numberphile · 10:31

<https://www.youtube.com/watch?v=RdooKXXcWWc>

WHEN TO WATCH

After the story paragraph. Marcus du Sautoy of Oxford explains the dice battles in *Risk* and the history of the game.

BACKUP VIDEO

The Mathematics of Winning Monopoly

Stand-up Maths · 18:40

<https://www.youtube.com/watch?v=ubQXz5RBBtU>

After worked example 3, for the Markov-chain section. It compares two mathematicians' simulations of *Monopoly* and lists the landing chance of every square.

KEY TERMS

outcome table

A list or grid of all equally likely results, used to count favorable cases.

expected losses

The average number of armies each side loses per battle, found by weighting each result by its probability.

Markov chain

A process that moves between states, where the chance of the next state depends only on the current state, not on the history.

transition probability

The chance of moving from one state to another in one step.

steady state

The long-run share of time a Markov chain spends in each state.

CORE IDEAS

The story. In 1957 the French filmmaker Albert Lamorisse invented a board game called *La Conquête du Monde* (“The Conquest of the World”). Parker Brothers published it in America in 1959 as *Risk*. Players attack with up to three dice and defend with up to two; the highest dice are compared, and ties go to the defender. A generation earlier, in 1935, Parker Brothers had published *Monopoly*, descended from Lizzie Magie’s *Landlord’s Game*, patented in 1904. In 1906, half a world away, the Russian mathematician Andrey Markov had begun studying chains of chance events in which each step depends only on the one before. It turned out to be exactly the mathematics a *Monopoly* board needs.

How they figured it out. For *Risk*, the method is the one from Lesson 1.2: list every outcome. One attacking die against one defending die gives 36 outcomes; the attacker wins in only 15 of them (because ties go to the defender), so the defender is the favorite, 21 to 15. But three attacking dice against two defending dice give $6^5 = 7,776$ outcomes, and counting them shows the attacker wins both armies 37.2% of the time, splits 33.6% and loses both 29.3%. For *Monopoly*, the method is simulation and Markov chains. Matt Parker’s simulation of 100,000,000 rolls (in the backup video) found that a player spends about 6.3% of his time in Jail, more than twice the 2.5% of an average square, and that Illinois Avenue, at about 3.2%, is the most-landed property.

The mathematics. Expected losses in a three-against-two *Risk* battle are $0(0.372) + 1(0.336) + 2(0.293) = 0.921$ armies for the attacker and $2 - 0.921 = 1.079$ for the defender. The attacker gains about 0.16 armies per battle on average, so attacking with three dice is a high-percentage play when you have the armies. In a Markov chain, a **transition table** gives the chance of moving from each state to each

other state. On a *Monopoly* board, the states are the 40 squares (plus “in Jail”), and the transitions come from the dice, the cards and the “Go to Jail” square. The long-run share of time on each square is the chain’s **steady state**.

On the board. Jail collects players from three sources: landing on “Go to Jail,” drawing a card that sends you there, and rolling three doubles in a row. Players leaving Jail then roll the dice, and the most common totals (6, 7, 8, 9) land them on St. James Place, Community Chest, Tennessee Avenue and New York Avenue: the orange group, which is landed on more often than any other color group. Illinois Avenue gains extra visits from a Chance card that sends players straight to it. In *Risk*, the percentage player attacks with full dice against weak positions and avoids long chains of attacks that leave his own territories thin.

The risk side. Even a favorable battle can go badly several times in a row. An attack with a small edge per roll is safe only when you have enough armies to absorb a run of bad dice (Lesson 2.5, risk of ruin). The best players count not only the odds of each battle but how many battles they can afford to lose.

PICTURE STUDY



Andrey Markov, Unknown photographer, 1870s · Photograph · St. Petersburg, Russia

Image: [Wikimedia Commons](#), public domain

Andrey Andreyevich Markov taught mathematics at the University of St. Petersburg. In 1906 he began studying “chains” of chance events, and in 1913 he tested his idea on the 20,000 letters of Pushkin’s poem *Eugene Onegin*, counting how often a vowel followed a vowel and a consonant followed a consonant. The same mathematics now describes everything from board games to weather forecasts.

Look closely

- Look at his expression and posture. What qualities would a mathematician need to count 20,000 letters by hand?
- Notice the plain, formal dress. Photographs were expensive and sitters held still for several seconds. How does that show?
- Imagine Markov looking at a *Monopoly* board. Which square would he study first?

Talk about it

1. Markov’s chains have “no memory beyond the present step.” How is that like the dice in Lesson 1.3?
2. Why is a board game, where your next square depends only on where you are and what you roll, a perfect Markov chain?

3. What other part of a game or sport could you describe as a chain of states?

PROJECT

Battle Tables and the Board Census

You will measure *Risk* battle results with real dice and compare them with the exact counts, then run a census of a *Monopoly*-style board to see which squares are visited most.

You will need

- Five ordinary dice (three of one color for the attacker, two of another for the defender)
- A *Monopoly* board if you have one, or a 40-square loop drawn on paper with square 10 as Jail and square 30 as Go to Jail
- Spreadsheet software with $= \text{RANDBETWEEN}(1, 6)$
- Scorebook

Steps

1. Roll 100 three-against-two battles. Sort the dice, compare highest with highest and second with second, ties to the defender. Tally: attacker wins both, split, defender wins both.
2. Compare your percentages with 37.2%, 33.6% and 29.3%. Compute your observed expected losses.
3. Roll 50 one-against-one battles and compare with 41.7%.
4. **Board census by hand.** Move a token around your 40-square loop with two dice for 200 rolls. Whenever you land on 30, move to 10. Tally every landing.
5. **Board census by spreadsheet.** In a spreadsheet, simulate 10,000 rolls: each new square is $(\text{old square} + \text{two dice}) \bmod 40$, and 30 becomes 10. Count the landings on each square.
6. Graph the landing percentages for all 40 squares as a bar chart. Mark the 2.5% average line.
7. Find the three most-visited squares in your simulation, and explain why each is high.
8. Add one Chance-card rule (for example, landing on 7 sends you to 24 one time in eight) and rerun. How does square 24 change?

9. Write a short strategy note: which properties would you try hardest to buy, and why, using your census?

Tips

- In the spreadsheet, the mod function is
 $= MOD(A2 + RANDBETWEEN(1, 6) + RANDBETWEEN(1, 6), 40).$
- Your hand census of 200 rolls will be noisy (Lesson 1.5). The 10,000-roll census will be much steadier.

Check your work

- ☐ I rolled 100 three-against-two battles and compared them with the exact percentages.
- ☐ My bar chart shows all 40 squares with the 2.5% line.
- ☐ I can explain why the square after “Go to Jail” matters to the squares 6 to 9 beyond Jail.

SCOREBOOK

1. Draw a 6×6 grid for one-against-one *Risk* battles and shade the 15 attacker wins.
2. Draw the two-state win-loss chain as two circles with arrows labelled 0.6, 0.4, 0.45 and 0.55.
3. “What I would decide and why”: you have 5 armies next to an enemy territory with 2. Attack now or wait a turn to build up? Use expected losses.

KEY FORMULAS

$$P(\text{attacker wins, 1 vs 1}) = \frac{15}{36}$$

$$\pi_W = \frac{P(L \rightarrow W)}{P(L \rightarrow W) + P(W \rightarrow L)}$$

$$P(\text{state after 2 steps}) = \sum_k P(i \rightarrow k) P(k \rightarrow j)$$

From history. Markov’s 1913 study of *Eugene Onegin* was one of the first statistical analyses of language. Today the same idea of predicting the next item from the current one is at the heart of the text-prediction on a phone keyboard.

WORKED EXAMPLES

Example 1. One attacking die against one defending die; ties go to the defender. Find the attacker's chance of winning.

Show solution

1. List the 36 outcomes. For each defender number d , the attacker wins with any higher number: $6 - d$ ways.
2. Total: $5 + 4 + 3 + 2 + 1 + 0 = 15$.
3. $P = 15/36 \approx 0.417$.

Answer: 41.7%: with one die each, the defender is the favorite.

Example 2. Three attacking dice against two defending dice. Counting all 7,776 outcomes gives: attacker loses 0 armies in 2,890, loses 1 (and defender loses 1) in 2,611, and loses 2 in 2,275. Find the expected losses for each side.

Show solution

1. Probabilities: $2890/7776 = 0.372$, $2611/7776 = 0.336$, $2275/7776 = 0.293$.
2. Attacker's expected loss: $0(0.372) + 1(0.336) + 2(0.293) = 0.921$.
3. Defender's expected loss: $2(0.372) + 1(0.336) + 0(0.293) = 1.079$.

Answer: The attacker loses about 0.92 armies and the defender about 1.08 per battle: a small but real edge to the attacker.

Example 3. In *Monopoly*, a player leaves Jail (square 10) with an ordinary roll of two dice. The orange properties are on squares 16, 18 and 19. Ignoring cards and doubles, find the chance he lands on an orange property.

Show solution

1. He needs 6, 8 or 9.
2. Ways: 6 can be rolled 5 ways, 8 can be rolled 5 ways, 9 can be rolled 4 ways: 14 ways.
3. $P = 14/36 \approx 0.389$.

Answer: About 39% on one roll, which is why the orange group is so valuable.

Example 4. A simple two-state Markov chain for a team: after a win, it wins the next game 60% of the time; after a loss, it wins the next game 45% of the time (illustrative). Find (a) the chance it wins two games from now if it won today, and (b) its long-run winning percentage.

Show solution

1. (a) Two paths: win then win, $0.6 \times 0.6 = 0.36$; lose then win, $0.4 \times 0.45 = 0.18$. Total 0.54.
2. (b) In the steady state, the flow from W to L equals the flow from L to W:

$$\pi_W(0.4) = (1 - \pi_W)(0.45).$$
3. $0.85 \pi_W = 0.45 \Rightarrow \pi_W = 0.529$.

Answer: (a) 54%; (b) about 52.9% of games in the long run, whatever happened today.

PRACTICE

1. **Easy** In a one-die-each *Risk* battle, what is the defender's chance of winning?

Show answer

$$21/36 \approx 58.3\%.$$

2. **Easy** How many equally likely outcomes are there when 3 attacking dice and 2 defending dice are rolled?

Show answer

$$6^5 = 7,776.$$

3. **Easy** On a 40-square board, what share of landings would each square get if every square were equally likely? Compare with Jail's 6.3% and Illinois Avenue's 3.2%.

Show answer

$$1/40 = 2.5\%. \text{ Jail gets about 2.5 times the average and Illinois about 1.3 times.}$$

4. **Medium** Two attacking dice against one defending die: the attacker wins 125 of 216 outcomes. Find the percent and compare with one die against one.

Show answer

$$125/216 \approx 57.9\%, \text{ up from } 41.7\% \text{ with one die.}$$

5. **Medium** Three attacking dice against one defending die: the attacker wins 855 of 1,296 outcomes. Find the percent.

Show answer

$$855/1296 \approx 66.0\%.$$

6. **Medium** Two attacking dice against two defending dice: the attacker wins both armies 22.8% of the time, the two sides split 32.4%, and the defender wins both 44.8%. Find the attacker's expected army loss and decide whether 2-against-2 favors the attacker.

Show answer

Attacker loses $0(0.228) + 1(0.324) + 2(0.448) = 1.22$; the defender loses $2 - 1.22 = 0.78$. It favors the defender.

7. **Medium** A player leaves Jail with an ordinary roll. What is the chance he lands on New York Avenue (square 19)?

Show answer

He needs 9: $4/36 \approx 11.1\%$.

8. **Medium** Explain in words why Jail is the most-visited square on the *Monopoly* board.

Show answer

Players arrive there from several sources: the Go to Jail square, cards that send them there, and three doubles in a row, besides landing on it normally (Just Visiting). It also holds players for several turns.

9. **Hard** In Jail you may try to roll doubles to get out; after the third failed try you leave anyway. Find the expected number of turns spent trying.

Show answer

Leave on turn 1 with probability $1/6$, on turn 2 with probability $(5/6)(1/6) = 5/36$, otherwise on turn 3 ($25/36$).

$$E = 1(6/36) + 2(5/36) + 3(25/36) = 91/36 \approx 2.53 \text{ turns.}$$

10. **Hard** In the two-state chain of worked example 4, find the chance of winning three games from now if the team won today.

Show answer

Two steps: win 0.54, loss 0.46. Third step:

$0.54(0.6) + 0.46(0.45) = 0.324 + 0.207 = 0.531$. The chain is already close to its steady state of 0.529.

11. **Challenge** An attacker has 4 armies and must leave 1 behind, so he attacks with 3 dice against a defender with 2 armies (2 dice). Find the chance he wins the territory in one battle, and the chance he is reduced to attacking with 1 die after one battle.

Show answer

He wins both in one battle with probability 0.372. If he loses both (0.293), he has 2 armies left and can attack with only 1 die next time. So 37.2% immediate conquest and 29.3% chance of being reduced to one attacking die.

COMMON MISTAKES

- Forgetting that ties go to the defender in *Risk*, which is why one die against one favors the defender.
- Assuming every board square is equally likely. The rules (Go to Jail, cards) bend the chain.
- Adding probabilities along a path in a Markov chain instead of multiplying them.

Make it a game • Battle Table Tally. Use beans of two colors as armies. The attacker rolls three dice, the defender rolls two, and they compare the highest dice. Play twenty battles and tally the armies lost on each side. Before you start, each of you predicts the expected losses. Compare with your outcome table from the lesson, and see who guessed closest.

5.3 CHESS: SOUND AND SPECULATIVE SACRIFICES

OBJECTIVES

- Treat a chess result as a score (1, $\frac{1}{2}$ or 0) and compute the expected score of a move from estimated chances of winning, drawing and losing.
- Use the Elo formula to compute the expected score between two rated players and update a rating after a game.
- Tell the story of Kasparov versus Deep Blue (1997) and explain the difference between a sound sacrifice and a speculative one.

Warm-up. Play one quick game of chess (or watch one online) and, every five moves, write a guess: “White wins __%, draw __%, Black wins __%.” At the end, compare your guesses with the result. Keep the page for the scorebook.

WATCH

Garry Kasparov's Most Memorable Moments | Part 3 | 19 Move Loss Against IBM's Deep Blue

agadmator's Chess Channel • 9:14

<https://www.youtube.com/watch?v=5l7-Rt7lVPQ>

WHEN TO WATCH

After the story paragraph. It plays through the sixth game of the 1997 match, including Deep Blue's knight sacrifice on move 8.

BACKUP VIDEO

Deep Blue vs Kasparov: How a computer beat best chess player in the world - BBC News

BBC News • 3:05

<https://www.youtube.com/watch?v=KF6sLCeBj0s>

Optional, before the story paragraph. Deep Blue's designers describe the match and what it meant.

KEY TERMS

expected score

The average result of a game or move, counting a win as 1, a draw as $\frac{1}{2}$ and a loss as 0.

sacrifice

Deliberately giving up material (a pawn or piece) for an advantage in time, position or attack.

sound sacrifice

A sacrifice that, with best play, gives at least as good an expected score as declining it, as far as can be calculated.

speculative sacrifice

A sacrifice whose soundness cannot be calculated, played in the hope that the opponent will go wrong.

Elo rating

A number measuring playing strength, invented by Arpad Elo; the difference in ratings predicts the expected score.

CORE IDEAS

The story. In February 1996 in Philadelphia, world champion Garry Kasparov beat IBM's chess computer Deep Blue 4–2. A year later, in May 1997 in New York, he played a rematch against an improved machine that could examine about 200 million positions per second. The match was tied $2\frac{1}{2}$ – $2\frac{1}{2}$ entering the sixth game on May 11. Kasparov chose the Caro-Kann Defense, and on move 8 Deep Blue sacrificed a knight on e6, a sacrifice well known to human masters. Kasparov's position collapsed, and he resigned after 19 moves. The machine had won the match $3\frac{1}{2}$ – $2\frac{1}{2}$.

How they figured it out. A chess program cannot know who will win from most positions. Instead it searches ahead and gives each position an **evaluation**: a number, measured roughly in pawns, saying how good it is. In 1960 the U.S. Chess Federation adopted a rating system designed by the physics professor Arpad Elo, and the international federation, FIDE, adopted it in 1970. Elo's system connected ratings to **expected score**: a player rated 200 points higher is expected to score about 76%. Modern engines report something similar: not just “+1.5” but the chances of winning, drawing and losing.

The mathematics. If a move leads to a win with probability w , a draw with probability d and a loss with probability ℓ , its expected score is $E = w + \frac{1}{2}d$. Elo's formula for the expected score of player A against player B is

$E_A = \frac{1}{1 + 10^{(R_B - R_A)/400}}$. After a game, the rating changes by $K(S - E)$, where S is the actual score and K is a constant, often 20.

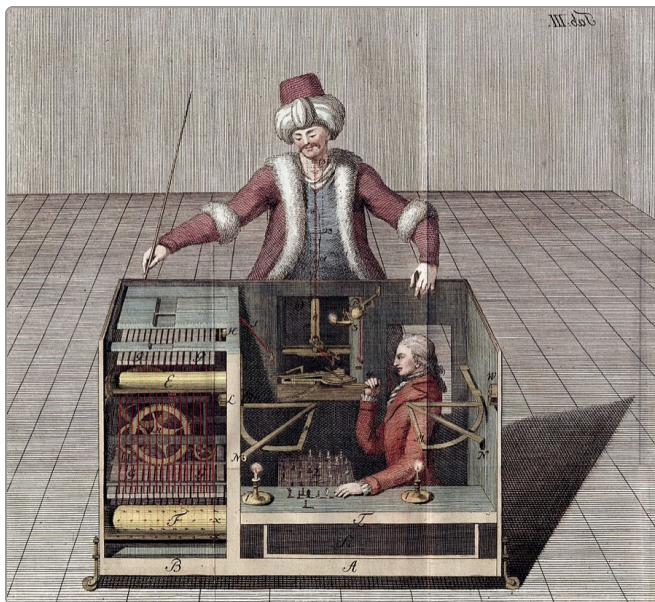
On the board. A **sound sacrifice** gives up material because the calculation shows the expected score goes up: the attack is worth more than the knight. A **speculative sacrifice** is a hope that the opponent will not find the defense. Against a weaker

opponent in a game you must win, a speculative sacrifice can be the percentage play, because it raises your chance of winning even if it lowers your expected score.

Against a computer, which always finds the defense, it is reckless. Since the engines arrived, players have learned that many famous old attacks were unsound and some quiet defensive moves were stronger than anyone believed.

The risk side. A sacrifice raises the variance of the game. That is good when you need a win and bad when a draw is enough. A wise player asks two questions before every sacrifice: “Is it sound?” and “What result do I need?”

PICTURE STUDY



The Turk, a Chess-Playing Machine, Joseph Friedrich zu Racknitz, 1789 · Engraving · From Racknitz, *Über den Schachspieler des Herrn von Kempelen*, Leipzig and Dresden

Image: [Wikimedia Commons](#), public domain

In 1770 Wolfgang von Kempelen built “the Turk,” a cabinet with a mechanical figure that seemed to play chess, and it toured Europe and America for decades, defeating many challengers. It was a hoax: a strong human player hid inside and worked the arm. In 1789 Racknitz published engravings guessing how the trick was done. Two hundred years later, Deep Blue really did what the Turk only pretended to do.

Look closely

- Find the hidden player inside the cabinet and the small chessboard in front of him. How could he follow the game on the board above?
- Find the gears and rollers. Why would the showman open the cabinet to show them before each game?
- Find the candles. What problem would a hidden player have without them, and what might give him away?

Talk about it

1. The Turk fooled audiences for decades. What kind of evidence would have exposed it sooner?

2. How is a hidden human player different from a real chess engine, which searches millions of positions?
3. Why is honest play (2 Timothy 2:5, “except he strive lawfully”) part of what makes a game worth winning?

PROJECT

The Sacrifice Lab

You will play the same sacrificial position many times against family members and a free engine, record the results as scores, and decide whether the sacrifice is sound in practice.

You will need

- A chess set
- A free chess site or app with an analysis board (for example, lichess.org), used with a parent's permission
- Scorebook

Steps

1. Set up the position after Deep Blue's 8.Nxe6 from Game 6 (1.e4 c6 2.d4 d5 3.Nc3 dxe4 4.Nxe4 Nd7 5.Ng5 Ngf6 6.Bd3 e6 7.N1f3 h6 8.Nxe6).
2. Count the material: how many points has White given up (knight = 3, pawn = 1)?
3. Play the position from here as White 5 times against family members or friends, and 5 times against a free engine on a modest level. Record each result as 1, $\frac{1}{2}$ or 0.
4. Compute your expected score from the human games and from the engine games separately.
5. Now set up the position after 8.Ne4 instead (the quiet retreat of the attacked knight) and play it 5 times against the same opponents. Record the scores.
6. Compare the expected scores of the sacrifice and the quiet move against each kind of opponent.
7. Ask the analysis board for its evaluation of both positions. Record it.
8. Write a paragraph: is 8.Nxe6 sound, speculative, or does it depend on the opponent?

Tips

- Use a clock with a few minutes per side so that the games are quick and you can play many.

- Write every result down before you start the next game, even the embarrassing ones.

Check your work

- ☐ I recorded at least 15 games with scores.
- ☐ I computed expected scores for each move and each type of opponent.
- ☐ My conclusion uses both my results and the engine's evaluation.

SCOREBOOK

1. Copy your warm-up guesses and compare them with the result. Were you too confident?
2. Make a table of Elo expected scores for rating differences of 0, 100, 200, 300 and 400 points.
3. “What I would decide and why”: you are White in the last round and need only a draw. Your opponent offers a pawn that opens your king. Take it or not?

KEY FORMULAS

$$E = w + \frac{1}{2}d$$

$$E_A = \frac{1}{1 + 10^{(R_B - R_A)/400}}$$

$$R_{\text{new}} = R + K(S - E)$$

From history. In December 2017 the program AlphaZero, which learned chess only by playing itself, defeated one of the strongest traditional engines in a match. Its games were full of long-term sacrifices that human masters found beautiful, a reminder that soundness is something to be tested, not assumed.

WORKED EXAMPLES

Example 1. You may sacrifice a knight. Your estimates: after the sacrifice you win 50%, draw 25%, lose 25%. After the quiet move you win 30%, draw 50%, lose 20%. Which move has the higher expected score?

Show solution

1. Sacrifice: $0.50 + \frac{1}{2}(0.25) = 0.625$.

2. Quiet move: $0.30 + \frac{1}{2}(0.50) = 0.55$.

Answer: The sacrifice (0.625) beats the quiet move (0.55) by 0.075 points per game, even though it loses slightly more often.

Example 2. Find the expected score of a player rated 1700 against a player rated 1500, and the reverse.

Show solution

1. $E_{1700} = \frac{1}{1 + 10^{(1500-1700)/400}} = \frac{1}{1 + 10^{-0.5}} = \frac{1}{1 + 0.316} = 0.760$.

2. $E_{1500} = 1 - 0.760 = 0.240$.

Answer: The higher-rated player is expected to score 76%; the lower-rated, 24%.

Example 3. The 1500 player beats the 1700 player. With $K = 20$, find both new ratings.

Show solution

1. 1500 player: $1500 + 20(1 - 0.240) = 1500 + 15.2 = 1515.2$.

2. 1700 player: $1700 + 20(0 - 0.760) = 1700 - 15.2 = 1684.8$.

Answer: About 1515 and 1685. An upset moves both ratings a lot; an expected result moves them a little.

Example 4. In the last round of a tournament you must win to take first place; a draw is no better than a loss. Move A: win 30%, draw 60%, lose 10%. Move B (a speculative sacrifice): win 40%, draw 5%, lose 55%. Which move has the higher expected score, and which should you play?

Show solution

1. Expected score A: $0.30 + 0.30 = 0.60$; B: $0.40 + 0.025 = 0.425$.
2. But your goal is first place, which only a win gives: A wins 30%, B wins 40%.

Answer: A has the higher expected score, but B gives the better chance of first place. When only a win counts, the higher-variance move is the percentage play.

PRACTICE

1. **Easy** What are the scores for a win, a draw and a loss in chess?

Show answer

1, $\frac{1}{2}$ and 0.

2. **Easy** In what year did Deep Blue defeat Kasparov in a match, and where?

Show answer

1997, in New York.

3. **Easy** Find the expected score of a move that wins 40%, draws 40% and loses 20%.

Show answer

$0.40 + 0.20 = 0.60$.

4. **Medium** Find the expected score of a player rated 100 points higher than his opponent.

Show answer

$1/(1 + 10^{-0.25}) = 1/(1 + 0.562) = 0.640$.

5. **Medium** Find the expected score of a player rated 400 points higher.

Show answer

$1/(1 + 10^{-1}) = 1/1.1 = 0.909$.

6. **Medium** A 1600 player draws a 1600 player. With $K = 20$, how do their ratings change?

Show answer

Each expected 0.5 and scored 0.5, so neither rating changes.

7. **Medium** Explain the difference between a sound and a speculative sacrifice.

Show answer

A sound sacrifice is justified by calculation: with best defense it still gives at least as good an expected score. A speculative sacrifice cannot be fully calculated and depends on the opponent making a mistake.

8. **Medium** Why is a speculative sacrifice more dangerous against a computer than against a human?

Show answer

A computer searches millions of positions and almost always finds the best defense, so the hope that the opponent will go wrong rarely comes true.

9. **Hard** A 1400 player beats a 1800 player. With $K = 20$, find the 1400 player's new rating.

Show answer

$E = 1/(1 + 10^1) = 1/11 = 0.091$; new rating $1400 + 20(1 - 0.091) = 1418.2$.

10. **Hard** You lead a tournament by half a point going into the last round, so a draw guarantees first place. Move A: win 20%, draw 75%, lose 5%. Move B (a sacrifice): win 55%, draw 5%, lose 40%. Which should you play? Compare expected scores and the chance of first place.

Show answer

A: $E = 0.575$; chance of first place (win or draw) 95%. B: $E = 0.575$; chance of first place 60%. Same expected score, but A is far better for your goal: the leader should reduce variance.

11. **Challenge** Show that two players' Elo expected scores always add to 1, and find the rating difference at which the stronger player is expected to score 90%.

Show answer

$$\frac{1}{1 + 10^x} + \frac{1}{1 + 10^{-x}} = \frac{1}{1 + 10^x} + \frac{10^x}{10^x + 1} = 1. \text{ For 90\%: } 1 + 10^{-D/400} = 1/0.9, \\ \text{so } 10^{-D/400} = 1/9, D = 400 \log_{10} 9 \approx 382 \text{ points.}$$

COMMON MISTAKES

- Counting a draw as zero. A draw is worth $\frac{1}{2}$, and it often decides tournaments.
- Judging a sacrifice by one game. One win does not prove it sound, and one loss does not prove it unsound.
- Forgetting the goal. The move with the highest expected score is not always the move with the best chance of the result you need.

Make it a game • Score the Sacrifice. Your parent presents a pretend position with two options. The safe move has a 25 percent win, 60 percent draw, 15 percent loss chance. The sacrifice has 45 win, 5 draw, 50 loss. Compute each expected score with 1, one half and 0, and decide. Then swap, with your own invented numbers. Check your parent's math within sixty seconds.

5.4 HIDDEN INFORMATION: BRIDGE, STRATEGO AND BATTLESHIP

OBJECTIVES

- Explain the bridge finesse as a 50% play and compute the chances of one finesse, two finesses, and common suit splits.
- Use what an opponent has shown (bidding, high-card points, play) to update a probability, in the spirit of Bayes.
- Distinguish honest inference from cheating, and explain why hidden information makes games of skill richer.

Warm-up. Shuffle a deck and deal two hands of 13 cards face down. Before looking, write the chance that the king of spades is in the left hand. Now look at the right hand only. If it holds the king of spades, what is the chance it is in the left hand? If it does not, what is the chance? Write all three numbers.

WATCH

Which Finesse To Take - Free Bridge Lesson

Peter Hollands · 17:36

<https://www.youtube.com/watch?v=M1EMpwkxAjs>

WHEN TO WATCH

After the mathematics paragraph. A bridge teacher walks through deals where you must decide which finesse to take, using what the opponents have shown.

KEY TERMS

finesse

In bridge, an attempt to win a trick with a lower card by playing toward it, hoping a missing higher card is held by the opponent who plays before it.

suit split

How the cards of a suit that you do not hold are divided between the two opponents, such as 3–2 or 4–1.

high-card points (HCP)

A count of strength in bridge: ace 4, king 3, queen 2, jack 1, for 40 points in the deck.

inference

Drawing a conclusion about hidden information from what has been seen, such as an opponent's bid or play.

conditional probability

The probability of an event once you know something else has happened, written $P(A \mid B)$.

CORE IDEAS

The story. In 1925, on a cruise through the Panama Canal, the American sportsman Harold Vanderbilt worked out the scoring of a new card game, contract bridge, and within a few years it was played in homes and clubs around the world. Bridge is a game of hidden information: you see your own 13 cards and, after the first lead, your partner's, but the 26 cards held by the opponents are hidden. Every decision is a judgment about where the missing cards probably lie. Other games share this character. In *Stratego*, the ranks of the opponent's pieces are hidden until they fight; in *Battleship*, the ships are hidden until they are hit (Lesson 1.4).

How they figured it out. Bridge players were among the first to use combinatorics at the card table. If you and your partner hold 8 cards of a suit, the other 5 are split between two 13-card hands, and counting the ways shows the split is 3–2 about 68% of the time, 4–1 about 28% and 5–0 about 4%. The classic **finesse** is the simplest hidden-information play: with the ace and queen of a suit and the king missing, you lead toward your ace–queen and, if the player before you plays low, play the queen. It wins when the king is held by the opponent who plays before your queen, which is 50% before you know anything else.

The mathematics. Two independent finesses both work with probability $0.5 \times 0.5 = 0.25$, and at least one works with probability $1 - 0.25 = 0.75$. But a good player rarely stays at 50%. The deck holds 40 high-card points. If an opponent has opened the bidding with a hand showing 15 to 17 points, and you and your partner hold 24, the other opponent can hold at most 1 point, so he cannot hold a king (worth 3). The finesse that “works half the time” now works every time. This is conditional probability, $P(A \mid B) = P(A \text{ and } B)/P(B)$, and Bayes's idea from Lesson 1.4: let evidence change your estimate.

On the table. Expert players count everything: each opponent's points as the bidding and play reveal them, the number of cards each has shown in each suit, and what a player would have done with different cards. The same habit wins at *Stratego* (a piece that charges aggressively is likely high-ranked, or a bluff) and at *Battleship* (a miss on

one square changes the chances on every square nearby). Our scripture reminds us that inference must be honest. Studying what an opponent has fairly shown is skill; peeking at his cards is cheating.

The risk side. A finesse that fails usually loses a trick, and sometimes the contract. Before taking a 50% chance, a good declarer asks whether a safer line exists, such as a play that wins whenever the suit splits 3–2 (68%), or whether he can try one chance and then fall back on another.

PICTURE STUDY



The Cardsharps, Caravaggio (Michelangelo Merisi da Caravaggio), about 1594 · Oil on canvas · Kimbell Art Museum, Fort Worth, Texas

Image: [Wikimedia Commons](#), public domain

Caravaggio painted a card game in which hidden information is stolen. The innocent young man studies his hand, while the older man reads his cards over his shoulder and signals to an accomplice, who is drawing an extra card from his belt. The painting is a lesson in what a game becomes when someone refuses to strive lawfully.

Look closely

- Find the older man's eyes and his gloved hand. What is he doing with his fingers?
- Find the card hidden behind the young cheat's back. Where is he keeping it?
- Look at the face of the honest player. Does he suspect anything?
- Notice the light. Who is lit clearly, and who is partly in shadow?

Talk about it

1. In bridge, a player may learn about his opponents' cards from their bids and plays. What makes that honest and this dishonest?
2. Why do rules against signalling and peeking protect the skill of a game, not only its fairness?

3. How does 2 Timothy 2:5 apply to this scene, and to games you play?

PROJECT

Where Is the King?

You will deal hundreds of real hands to check the suit-split percentages and the finesse, then practice inference by having a family member bid or describe a hand and deciding where the missing cards lie.

You will need

- A standard deck of 52 cards (or 52 numbered index cards in four colors)
- Scorebook
- Spreadsheet software (optional)

Steps

1. Set aside eight spades, including the ace and the queen: these are the spades you and your partner hold. The other five spades, including the king, are the outstanding cards.
2. Shuffle the 5 outstanding spades with 21 cards from the other suits (26 cards in all) and deal two hands of 13.
3. Record how the 5 spades split (5–0, 4–1, 3–2). Repeat 50 times.
4. Compare your percentages with 67.8%, 28.3% and 3.9%.
5. Record also which hand held the king of spades in each deal. In what percent of deals would a finesse against the left hand have won?
6. **Inference round.** A family member deals himself 13 cards from a full deck and tells you only his high-card points. Before he shows his hand, write your estimate of the chance that he holds the king of spades. Repeat 20 times.
7. Compare your estimates with the results. Did hands with more points hold the king more often?
8. Write a short paragraph: how much did the point count help you, and why?

Tips

- Shuffle thoroughly (at least seven riffle shuffles) between deals, or the results will be biased.
- In the inference round, a hand with more points has more of the 16 honors, so it is more likely to hold any particular king.

Check your work

- ☐ I dealt at least 50 hands and compared the splits with the exact percentages.
- ☐ I estimated the king's location 20 times from point counts and checked the results.
- ☐ I can explain how evidence changes a 50% guess.

SCOREBOOK

1. Make a table of suit splits for 4, 5 and 6 outstanding cards from this lesson.
2. Draw a tree for two finesses: four branches (win-win, win-lose, lose-win, lose-lose), each with probability 0.25.
3. “What I would decide and why”: you can take a finesse or play for a 3-2 split. Which, and why?

KEY FORMULAS

$$P(\text{split}) = \frac{\binom{n}{k} \binom{26-n}{13-k}}{\binom{26}{13}} \times 2 \quad (k \neq n - k)$$

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(\text{at least one of two finesses}) = 1 - (0.5)(0.5) = 0.75$$

From history. Ely Culbertson made bridge a national craze in the early 1930s, and Charles Goren later popularized the 4-3-2-1 high-card point count, first proposed by Milton Work. Bridge columns in newspapers taught a generation of readers practical probability.

WORKED EXAMPLES

Example 1. You and your partner hold 8 cards of a suit; the other 5 are divided between the two opponents' unknown hands of 13 cards each. Show that a 3–2 split (either way) has probability about 68%.

Show solution

1. Choose which 3 of the 5 missing cards go to the left-hand opponent: $\binom{5}{3} = 10$ ways; his other 10 cards come from the 21 non-suit cards: $\binom{21}{10} = 352,716$.
 2. All ways to deal him 13 of the 26 unknown cards: $\binom{26}{13} = 10,400,600$.
 3. Left opponent has exactly 3: $10 \times 352,716 / 10,400,600 = 0.339$.
 4. Either opponent can have the 3: $2 \times 0.339 = 0.678$.
-

Answer: About 67.8%. (The same method gives 4–1 about 28.3% and 5–0 about 3.9%.)

Example 2. Declarer needs two tricks from two separate finesses, each a fresh 50% chance. What is the chance both work? At least one?

Show solution

1. Both: $0.5 \times 0.5 = 0.25$.
 2. Neither: $0.5 \times 0.5 = 0.25$. At least one: $1 - 0.25 = 0.75$.
-

Answer: Both work 25% of the time; at least one works 75%.

Example 3. You and your partner hold 23 high-card points. Your left-hand opponent opened the bidding showing 15 to 17 points. The king of hearts is missing, and a heart finesse wins only if the king is with your left-hand opponent. What is the chance it wins?

Show solution

1. Total points: 40. The opponents hold $40 - 23 = 17$.
2. Left-hand opponent: at least 15. So the right-hand opponent has at most $17 - 15 = 2$ points.
3. A king is worth 3, so the right-hand opponent cannot hold it.

Answer: The king must be with the left-hand opponent: the finesse wins 100% of the time. The bidding turned a 50% guess into a certainty.

Example 4. With 9 cards of a suit missing the queen (4 cards outstanding), you cash the ace and both opponents follow with small cards. You can play for the queen to drop or finesse against the left-hand opponent. Splits of 4 cards: 2-2 40.7%, 3-1 49.7%, 4-0 9.6%. Compare the two plays only in the cases where they differ.

Show solution

1. Playing for the drop wins and the finesse loses when the split is 2-2 with the queen on your right: half of 40.7%, or 20.35%.
2. The finesse wins and the drop loses when the left opponent began with the queen and two small cards and the right opponent with one small card: half of 49.7% is 24.9% for “left has three,” and the queen is one of those three cards three times in four: $24.9\% \times \frac{3}{4} = 18.65\%$.
3. In all other cases, the two plays succeed or fail together.

Answer: The drop wins in 20.35% of deals where the finesse fails; the finesse wins in 18.65% where the drop fails. Play for the drop. This is the reason for the old rule, “eight ever, nine never.”

PRACTICE

1. **Easy** How many high-card points are there in a deck, and how are they counted?

Show answer

40: ace 4, king 3, queen 2, jack 1, in each of four suits.

2. **Easy** Before any bidding or play, what is the chance that a particular missing king is held by your left-hand opponent?

Show answer

50%.

3. **Easy** Explain the difference between honest inference and cheating in a card game.

Show answer

Inference uses what the opponent has fairly shown (bids, plays, discards). Cheating gains hidden information by unfair means, such as seeing his cards or using signals outside the rules.

4. **Medium** With 7 cards of a suit (6 outstanding), the splits are 4-2 48.4%, 3-3 35.5%, 5-1 14.5% and 6-0 1.5%. What is the chance the suit splits 3-3 or 4-2?

Show answer

$35.5\% + 48.4\% = 83.9\%$.

5. **Medium** You need three finesses to all succeed, each 50%. What is the chance?

Show answer

$0.5^3 = 0.125, 12.5\%$.

6. **Medium** You may either take a finesse (50%) or play for a 3-2 split of a suit (67.8%). Which is the percentage play, and by how much?

Show answer

Play for the 3-2 split: $67.8\% - 50\% = 17.8$ percentage points better.

7. **Medium** You hold 25 HCP with your partner. The left-hand opponent opened showing 12 to 14. A missing queen is the card you are guessing about. Can the right-hand opponent hold it? Explain.

Show answer

The opponents hold 15. If the left one has at least 12, the right one has at most 3. A queen is worth 2, so the right-hand opponent may hold it; the inference does not settle the question.

8. **Medium** In *Battleship*, why does a miss give you information even though nothing was hit?

Show answer

It rules out every ship placement that would have covered that square, which changes the probabilities of the squares around it.

9. **Hard** Using the counting method of worked example 1, find the probability that 4 outstanding cards split 2-2.

Show answer

$$\binom{4}{2} \binom{22}{11} / \binom{26}{13} = 6 \times 705,432 / 10,400,600 \approx 0.407.$$

10. **Hard** A *Stratego* player has two unknown pieces that could be his flag. One has never moved; the other has moved. Using the rule that the flag can never move, what is the chance each is the flag?

Show answer

The piece that moved cannot be the flag (probability 0), so the piece that never moved is the flag with probability 1, if one of them must be the flag. Inference from the rules can settle a question completely.

11. **Challenge** In bridge, two finesses are not always independent. Suppose you will take finesse A first; if it wins, finesse B succeeds 60% of the time; if it loses, B succeeds 40% (for instance because A tells you which opponent is stronger). What is the chance at least one wins? Compare with 75%.

Show answer

$P = 0.5 + 0.5 \times 0.4 = 0.70$. Less than 75%, because information from the first finesse makes the second less likely to rescue you.

COMMON MISTAKES

- Treating every finesse as exactly 50% after the bidding and play have given you information.
- Adding the chances of two finesses (50% + 50% = 100%) instead of using the complement for “at least one.”
- Forgetting that hidden information is discovered gradually; each card played changes the probabilities.

Make it a game · Find the Queen. Your parent secretly decides, by rolling a die, which of two hidden hands holds the queen. You guess left or right twenty times, recording the hits. Compare to the fifty percent you expect from a finesse. Then compute the chance of winning with one finesse and with two finesses, and name one suit split you would hope for.

Scripture. “And if a man also strive for masteries, yet is he not crowned, except he strive lawfully.” (2 Timothy 2:5, KJV)

Inference from what an opponent has fairly shown is skill; stealing hidden information is not. A crown won by cheating is no crown.

5.5 PUSH YOUR LUCK: CAN'T STOP AND YAHTZEE

OBJECTIVES

- Use expected value to decide when to stop rolling in a push-your-luck game, and derive the “hold at 20” rule for the dice game Pig.
- Compute the chance of making progress in *Can't Stop* for a set of columns, and the chance of completing a Yahtzee from a given start.
- Explain how the right stopping rule changes when you are behind or ahead near the end of a game.

Warm-up. Play three turns of Pig: roll one die as many times as you like, adding the numbers, but if you roll a 1 your turn scores 0. Before each roll, write your running total and whether you will roll again. After three turns, write the total at which you think a player should stop.

WATCH

1,296 and Yahtzee - Numberphile

Numberphile · 6:27

<https://www.youtube.com/watch?v=fiTwar7mFws>

WHEN TO WATCH

After the story paragraph. It shows why a Yahtzee on a single roll has a chance of 1 in 1,296, and what happens when you try for one.

BACKUP VIDEO

How to Play Can't Stop | Board Game Rules & Instructions

Boardgaimz · 2:35

<https://www.youtube.com/watch?v=L6Zk7j1mJDE>

Before the project, if you do not know the rules of *Can't Stop*.

KEY TERMS

push-your-luck game

A game in which each roll can add to your score or wipe out what you have gained this turn, so you must decide when to stop.

marginal expected gain

The expected change in your score from taking one more risk, compared with stopping now.

stopping rule

A rule, decided in advance, for when to stop taking risks, such as “hold when the turn total reaches 20.”

bust

A roll that ends the turn and loses the turn’s progress.

sequential decision

A series of choices in which each choice is made after seeing the result of the last.

CORE IDEAS

The story. Sid Sackson was one of the great American game designers, and in 1980 Parker Brothers published his *Can’t Stop*. Each turn a player rolls four dice, pairs them into two sums, and moves markers up the columns for those sums. He may roll again as often as he likes, but if a roll gives no usable sum, every gain of that turn is lost. *Yahtzee* is older: it was invented in Canada by a couple who played it on their yacht, and the toy maker Edwin Lowe began selling it in 1956. In *Yahtzee* you roll five dice up to three times, keeping any you like, and you must judge whether to chase a big score or take a sure small one.

How they figured it out. The simplest push-your-luck game is the old dice game Pig. On each roll, a 2, 3, 4, 5 or 6 adds to your turn total, but a 1 wipes it out. Should you roll again? Compare what you expect to gain with what you expect to lose. Five times in six you add, on average, 4 points (the average of 2 through 6). One time in six you lose your whole turn total t . The expected change is $\frac{5}{6} \cdot 4 - \frac{1}{6}t = \frac{20-t}{6}$, which is positive until $t = 20$. So to score the most points per turn, keep rolling until your turn total reaches 20, then stop. In 2004 the computer scientists Todd Neller and Clifton Presser computed the perfect strategy for winning Pig, and found it close to “hold at 20” early in the game but very different near the end.

The mathematics. In *Can’t Stop*, the question is the chance a roll of four dice can make at least one of your columns. Counting all $6^4 = 1,296$ rolls shows that if your columns are 6, 7 and 8 you can move 92.0% of the time, but if they are 2, 3 and 12 only 43.8%. In *Yahtzee*, a *Yahtzee* in one roll has chance $6/6^5 = 1/1296$, but with three rolls and the best keeping strategy it rises to about 4.6%. From three of a kind with two rolls left, the chance is $121/1296 \approx 9.3\%$; from four of a kind, $1 - (5/6)^2 = 11/36 \approx 30.6\%$.

At the table. The percentage player sets a stopping rule before the turn begins and keeps to it, instead of deciding by feel, which usually means stopping too early after a scare or pushing too far after a hot streak. Near the end of a game the rule changes: a player far behind must push harder, because a small sure score will not catch up (the trailing team increases variance, Lesson 2.3), while a player close to winning should stop as soon as he has enough.

The risk side. In push-your-luck games the danger grows with success: the more you have gained this turn, the more one bad roll costs. That is why the stopping point is reached by the size of the turn total, not by the number of rolls. A family that learns this at the game table has learned something about many decisions in life.

PICTURE STUDY



The House of Cards, Jean Siméon Chardin, probably 1737 · Oil on canvas · National Gallery of Art, Washington, D.C.

Image: [Wikimedia Commons](#), public domain

Chardin painted quiet scenes of ordinary people absorbed in small tasks. Here a boy is building a house of cards. Each new card makes the house taller and more impressive, and each one risks bringing the whole house down. It is a perfect picture of pushing your luck.

Look closely

- Find the boy's hands. How carefully is he holding the next cards?
- Look at the row of standing cards. What happens to all of it if one card slips?
- Look at his face. Is he enjoying the risk, or simply concentrating?
- Find the open drawer. He has plenty of cards left. Does having more cards make it wise to keep building?

Talk about it

1. When should he stop building? What would make one more story worth the risk?
2. How is the house of cards like a turn in Pig or *Can't Stop*, where everything gained this turn can be lost at once?

3. Chardin painted children and young people at play many times. Why do you think he found games worth painting?

PROJECT

The Stopping-Rule Tournament

You will test several stopping rules for Pig against each other, then track your own Yahtzee decisions over 20 games.

You will need

- One die for Pig and five dice for Yahtzee
- A Yahtzee score sheet (or a homemade one with the 13 categories)
- Spreadsheet software with $= \text{RANDBETWEEN}(1, 6)$
- Scorebook

Steps

1. **Pig by hand.** Play 20 turns with the rule “hold at 10” and 20 turns with “hold at 20.” Record each turn’s score (0 if you bust).
2. Compute the average points per turn for each rule.
3. **Pig by spreadsheet.** Simulate 1,000 turns for each rule: hold at 10, 15, 20, 25 and 30. Record the average points per turn for each.
4. Graph average points per turn against the holding number. Where is the peak?
5. Play 5 games of Pig to 100 against a family member, using hold at 20 except when you are far behind near the end. Record when and why you changed the rule.
6. **Yahtzee tracking.** Play 20 solitaire games of Yahtzee. Each time you have three of a kind after the first roll, record whether you went for the Yahtzee and whether you got it.
7. Compare your Yahtzee success rate from three of a kind with the predicted 9.3%.
8. Record your average final score over the 20 games.
9. Write a one-page report: which stopping rule won, and what did your Yahtzee records teach you about chasing big scores?

Tips

- The spreadsheet Pig turn can be built row by row: each row rolls a die, adds to the total or resets to 0, and stops when the total reaches the holding number.
- Keep score in points only. The point of the game is the decision, not a prize.

Check your work

- ☐ I compared at least five holding numbers by simulation and graphed the results.
- ☐ I recorded my three-of-a-kind decisions in 20 Yahtzee games and compared them with 9.3%.
- ☐ I can explain why the best stopping rule changes near the end of a game.

SCOREBOOK

1. Draw the graph of expected gain $(20 - t)/6$ against the turn total t from 0 to 30, and mark where it crosses zero.
2. Make a table of Yahtzee chances: one roll (1/1296), from three of a kind with two rolls (9.3%), from four of a kind with one roll (16.7%) and with two (30.6%).
3. “What I would decide and why”: in *Can’t Stop* you have made good progress in columns 2 and 12 this turn. Roll again or stop? Use the numbers.

KEY FORMULAS

$$E[\text{gain from one more roll}] = \frac{5}{6} \cdot 4 - \frac{1}{6}t = \frac{20 - t}{6}$$

$$P(\text{Yahtzee from four of a kind, 2 rolls}) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$$

$$E[\text{successful rolls before a bust}] = \frac{p}{1 - p}$$

From history. The dice game Pig is a folk game whose rules were written down by the game expert John Scarne in 1945. It is now a favorite example in computer-science courses on decision-making, because its perfect strategy can be computed but is surprisingly complicated.

WORKED EXAMPLES

Example 1. In Pig, your turn total is 14. Should you roll again? What if it is 26?

Show solution

1. At $t = 14$: expected change $(20 - 14)/6 = +1$ point. Roll.

2. At $t = 26$: expected change $(20 - 26)/6 = -1$ point. Hold.

Answer: Roll at 14, hold at 26. The break-even is a turn total of 20.

Example 2. You hold three dice showing 5 and have two rolls left. Find the chance of a Yahtzee (five 5s).

Show solution

1. Roll the other two dice. Both are 5s: $1/36$. Exactly one is a 5: $10/36$. Neither: $25/36$.

2. Last roll: if one die is left, it must be a 5 ($1/6$); if two are left, both must be 5s ($1/36$).

$$3. P = \frac{1}{36} + \frac{10}{36} \cdot \frac{1}{6} + \frac{25}{36} \cdot \frac{1}{36} = \frac{36+60+25}{1296} = \frac{121}{1296}.$$

Answer: $121/1296 \approx 9.3\%$.

Example 3. In *Can't Stop*, your three columns are 6, 7 and 8, and the chance of making at least one of them on a roll is 92.0%. If you keep rolling until you bust, how many successful rolls do you expect first? Repeat for columns 2, 3 and 12 (43.8%).

Show solution

1. The number of successes before the first failure has mean $p/(1 - p)$.

2. 6, 7, 8: $0.920/0.080 = 11.5$ rolls.

3. 2, 3, 12: $0.438/0.562 \approx 0.78$ rolls.

Answer: About 11.5 successful rolls with the middle columns but fewer than one with the edge columns. Choose middle columns when you can, and stop early when your columns are on the edges.

PRACTICE

1. **Easy** What is the chance of a Yahtzee on a single roll of five dice?

Show answer

$$6/7776 = 1/1296.$$

2. **Easy** In Pig, what is the chance that one roll busts your turn?

Show answer

$$1/6.$$

3. **Easy** Who designed *Can't Stop*, and when was it published?

Show answer

Sid Sackson; Parker Brothers published it in 1980.

4. **Medium** In Pig, your turn total is 18. Find the expected change from one more roll and decide.

Show answer

$$(20 - 18)/6 = +\frac{1}{3}. \text{ Roll, though the edge is small.}$$

5. **Medium** You hold four of a kind in Yahtzee with one roll left. What is the chance of a Yahtzee?

Show answer

$$1/6 \approx 16.7\%.$$

6. **Medium** You hold four of a kind with two rolls left. Find the chance of a Yahtzee two ways: directly, and by the complement.

Show answer

$$\text{Directly: } \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{11}{36}. \text{ Complement: } 1 - (5/6)^2 = 11/36 \approx 30.6\%.$$

7. **Medium** In *Can't Stop*, the chance of making the single column 7 with four dice is 64.4%. How many successful rolls do you expect before a bust if 7 is your only column?

Show answer

$$0.644/0.356 \approx 1.8 \text{ rolls.}$$

8. **Medium** Explain why, in push-your-luck games, the decision to stop should depend on how much you have gained this turn, not on how many times you have rolled.

Show answer

The risk of the next roll is losing the turn total, so the cost of a bust grows with the total. The number of rolls already made does not change the odds of the next roll (dice have no memory).

9. **Hard** In a variant of Pig, a 1 or a 2 busts the turn, and 3–6 add their value. Find the turn total at which you should stop.

Show answer

Average of 3–6 is 4.5. Expected change: $\frac{4}{6}(4.5) - \frac{2}{6}t = 3 - \frac{t}{3}$, zero at $t = 9$. Stop at 9.

10. **Hard** Near the end of a game of Pig to 100, you have 60 and your opponent has 98. Explain why “hold at 20” is wrong for you on this turn.

Show answer

Your opponent will almost certainly win on his next turn, so a turn total of 20 is nearly worthless to you. You should keep rolling until you reach 100 or bust: the trailing player must take more risk.

11. **Challenge** Show that from three of a kind with two rolls left, the chance of at least four of a kind (four or five matching) by the end is $1 - (25/36)^2$, and compute it. Assume you keep any matches and reroll the rest.

Show answer

You fail only if neither of the two dice matches on either roll: each roll of two dice has no match with probability $(5/6)^2 = 25/36$, and you need that twice. So $P = 1 - (25/36)^2 = 1 - 625/1296 = 671/1296 \approx 51.8\%$.

COMMON MISTAKES

- Stopping or continuing because of how many times you have already rolled, instead of how much you have to lose.
- Thinking that after several good rolls a bust is “due.” Dice have no memory (Lesson 1.3).
- Using the points-per-turn rule at the end of a game, when only winning matters.

Make it a game • Pig Hold Showdown. Play Pig with a die: roll as many times as you like, adding the numbers to a turn total, but a one wipes the turn out. Hold at 20 for five turns, then play five turns of your own choice. Compare totals. Explain how the marginal expected gain tells you when to stop. First to 100 pretend points wins the match.

UNIT 5

THE ODDS AT THE GAME TABLE

The family game table is a probability laboratory. In this unit you will count your way through five classic games: the race and the blot in backgammon, the dice battles of *Risk* and the board squares of *Monopoly*, the sacrifices of chess and the duel between Kasparov and Deep Blue, the hidden cards of bridge, and the push-your-luck decisions of *Can't Stop* and *Yahtzee*. Along the way you will meet Hoyle's dice tables, Markov chains, Elo ratings, combinations of cards, and the stopping rule.

Every game in this unit is played for points, never for stakes. Your projects will produce real data: hundreds of dice rolls and deals, battle tables, a census of board squares, sacrifice experiments and stopping-rule tournaments. By the end you will be the player at your table who knows not only what to do, but why.

PREREQUISITES

- The 36 outcomes of two dice and the multiplication principle (Lesson 1.2)
- Independence: dice have no memory (Lesson 1.3)
- Conditional probability (Lesson 1.4)
- Expected value and variance (Unit 2)
- Combinations $\binom{n}{k}$ on a calculator

MASTERY CHECKLIST

- ☐ I can count pips, count the rolls that hit a blot, and explain the 25% acceptance point of the doubling cube in tournament points.
- ☐ I can compute
Ris battle odds and expected losses,
k and use a simple Markov chain and its steady state.
- ☐ I can compute expected scores in chess and Elo expected scores, and explain when a speculative sacrifice is the percentage play.
- ☐ I can compute suit-split and finesse probabilities and update them from evidence.
- ☐ I can derive a stopping rule for a push-your-luck game and explain how it changes near the end of a game.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Lichess**: Free and without advertising. A nonprofit chess site with an analysis board and engine evaluations, useful for the sacrifice lab in Lesson 5.3. A parent should set up any account and supervise online play. Ages 13+.
- **American Contract Bridge League: Learn to Play**: Free lessons and resources for beginners from bridge's governing body in North America, for families who want to play the deals in Lesson 5.4. Ages 12+.
- **Can't Stop (Eagle-Gryphon Games)**: The current edition of Sid Sackson's push-your-luck dice game from Lesson 5.5. Not required: the lesson's math and project work with ordinary dice. Ages 9+.
- **Yahtzee (Hasbro)**: The publisher's page for the classic five-dice game in Lesson 5.5. Five ordinary dice and a homemade score sheet work just as well. Ages 8+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

5.1	Backgammon: The Race, the Blitz and the Cube	3 sessions of 50 minutes
5.2	Dice Battles and Board Squares	3 sessions of 50 minutes
5.3	Chess: Sound and Speculative Sacrifices	3 sessions of 50 minutes
5.4	Hidden Information: Bridge, Stratego and Battleship	3 sessions of 50 minutes
5.5	Push Your Luck: Can't Stop and Yahtzee	3 sessions of 50 minutes

UNIT 5 TEST

Calculators are allowed. Show your work for every calculation: list or count the outcomes, write the formula, and give the answer as a fraction, decimal or percent with a sentence explaining it. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

Show that the average roll in backgammon moves $8\frac{1}{6}$ pips.

2. (6 pts)

How many rolls hit a blot 5 pips away (no blocked points)? List the combinations that do not contain a 5.

3. (7 pts)

In a tournament game worth 1 point you are doubled with a 28% chance of winning. Accept or decline? Show the comparison.

4. (6 pts)

In a one-against-one *Risk* battle (ties to the defender), find the attacker's chance of winning.

5. (7 pts)

In three-against-two battles, the attacker loses 0, 1 or 2 armies with probabilities 0.372, 0.336 and 0.293. Find the expected armies lost by each side.

6. (6 pts)

A player leaves Jail with an ordinary roll. Find the chance he lands on square 16, 18 or 19.

7. (7 pts)

After a win a team wins its next game 70% of the time; after a loss, 40%. Find its long-run winning percentage.

8. (6 pts)

Find the expected score of a chess move that wins 35%, draws 40% and loses 25%.

9. (7 pts)

Find the Elo expected score of a 1450 player against a 1650 player, and his new rating ($K = 20$) if he wins.

10. (6 pts)

Describe the decisive moment of Game 6 of Kasparov versus Deep Blue (1997).

11. (7 pts)

You need two finesses to both succeed, each 50% and independent. Find the chance. Find the chance at least one succeeds.

12. **7 pts**

You and partner hold 26 HCP. Your left-hand opponent opened showing 12 to 14. Can the right-hand opponent hold a king? Explain.

13. **6 pts**

In Pig, show that rolling again has positive expected value only while the turn total is below 20.

14. **6 pts**

Find the chance of completing a Yahtzee from three of a kind with two rolls left.

15. **10 pts**

Short essay (one paragraph): Choose two games from this unit. For each, describe one decision where the high-percentage play and the low-risk play are different, and explain how a wise player decides between them.

ANSWER KEY

1. Non-doubles total 210 pips over 30 rolls; doubles move $4(1 + \dots + 6) = 84$.
 $(210 + 84)/36 = 294/36 = 8\frac{1}{6}$.
2. 15 rolls: the 11 containing a 5, plus 4-1, 1-4, 3-2, 2-3.
3. Accept: $4(0.28) - 2 = -0.88$; decline -1. Accept.
4. $15/36 \approx 41.7\%$.
5. Attacker: $0.336 + 2(0.293) = 0.922$; defender: $2 - 0.922 = 1.078$. (0.921 and 1.079 with unrounded values.)
6. Needs 6, 8 or 9: $(5 + 5 + 4)/36 = 14/36 \approx 38.9\%$.
7. $\pi_W(0.3) = (1 - \pi_W)(0.4) \Rightarrow 0.7\pi_W = 0.4 \Rightarrow \pi_W \approx 0.571$.
8. $0.35 + 0.20 = 0.55$.
9. $E = 1/(1 + 10^{0.5}) = 0.240$; new rating $1450 + 20(0.760) = 1465.2$.
10. In a Caro-Kann, Deep Blue sacrificed a knight on e6 on move 8; Kasparov's position collapsed and he resigned after 19 moves, losing the match $3\frac{1}{2}$ - $2\frac{1}{2}$.
11. Both: 25%. At least one: $1 - 0.25 = 75\%$.
12. Opponents hold 14. The left one has at least 12, so the right one has at most 2. A king is worth 3, so he cannot hold one.
13. $E = \frac{5}{6}(4) - \frac{1}{6}t = (20 - t)/6 > 0 \iff t < 20$.
14. $\frac{1}{36} + \frac{10}{36} \cdot \frac{1}{6} + \frac{25}{36} \cdot \frac{1}{36} = \frac{121}{1296} \approx 9.3\%$.
15. Answers vary. A strong essay names specific decisions (accepting a double at 30%, a speculative sacrifice when only a win counts, playing for a 3-2 split rather than a finesse, holding at 20 in Pig except when far behind), computes or cites the relevant probabilities, and explains that the right choice depends on the goal (expected points, chance of winning) and on what can be afforded to lose.

PORTFOLIO PIECE: A FAMILY GAME STRATEGY GUIDE

Write a strategy guide of four to six pages for one family game (backgammon, *Risk*, *Monopoly*, chess, bridge, *Stratego*, *Battleship*, *Can't Stop*, *Yahtzee*, *Catan* or another approved by a parent). It must include (1) the key probabilities of the game, computed by counting and checked by at least 200 rolls, deals or simulated turns; (2) at least three specific decisions with the expected value or winning chance of each option; (3) one decision where the best play changes depending on the score or the stage of the game; and (4) a short section, “How to play this game well and graciously,” about honest play, good sportsmanship, and keeping score in points only.

LOOK FOR	WHAT IT MEANS
Probabilities	Key probabilities are computed correctly with the counting shown, and compared with real or simulated data of at least 200 trials.
Decisions	At least three decisions are analyzed with expected values or winning chances, and the recommendation follows from the numbers.
Situation and risk	The guide shows how the right play changes when ahead or behind, and names the risk side of each recommendation.
Clarity and character	Clear writing, labelled tables or charts, and a thoughtful section on honest, gracious play.

WISDOM, RISK AND PROVIDENCE

The last unit turns from the field and the game table to the player himself. You will read Halley's life table and learn why small risks repeated thousands of times matter more than dramatic rare ones; judge decisions by their quality rather than their results, using Super Bowl XLIX's final pass; meet the thinking traps that catch even experts; and consider how probability fits with providence, through Pascal's faith and Eric Liddell's choice in 1924. The course ends with your own odds report.

This unit asks for maturity. The mathematics is the same (probability, expected value, conditional probability, the law of large numbers), but now it is applied to habits, to judgment and to character. The goal of the whole course is a person who does the percentages diligently, keeps honest records, plays lawfully and graciously, and trusts God with the outcome.

PREREQUISITES

- Expected value, variance and risk of ruin (Unit 2)
- Conditional probability and Bayes's idea (Lesson 1.4)
- Regression to the mean and the spread from luck (Lesson 4.4)
- A spreadsheet with $= RAND()$ for simulations

MASTERY CHECKLIST

- ☐ I can compute survival chances from a life table and the cumulative risk of a repeated small risk.
- ☐ I can separate decision quality from outcome and keep an honest decision journal.
- ☐ I can explain loss aversion, sunk cost, overconfidence and base-rate neglect, and correct base-rate neglect with Bayes's rule.
- ☐ I can compute series and season probabilities and explain how Scripture speaks of chance and providence.
- ☐ I can produce an odds report with data, expected values, a margin of error, a sensitivity check and a clear chart.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Pascal's *Pensées* (Project Gutenberg)**: Free. Pascal's notes on faith and reason in English translation, including the wager discussed in Lesson 6.4. Best read with a parent; some passages are difficult. Ages 15+.
- **Laplace, *A Philosophical Essay on Probabilities* (Project Gutenberg)**: Free. A classic 1814 essay that explains probability in words, with chapters on life tables, testimony and errors in judgment that fit Lessons 6.1 and 6.3. Ages 15+.
- **Retrosheet**: Free. Play-by-play baseball data going back more than a century, a rich source for a capstone odds report in Lesson 6.5. Ages 13+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

6.1	Life Tables and Seat Belts	3 sessions of 50 minutes
6.2	Judge the Decision, Not the Result	3 sessions of 50 minutes
6.3	Traps in Our Thinking	3 sessions of 50 minutes
6.4	Time and Chance	3 sessions of 50 minutes
6.5	Capstone: Your Odds Report	3 sessions of 50 minutes

UNIT 6 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer with a sentence explaining it. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

From Halley's table (age 20: 598; age 40: 445), find the chance a 20-year-old would reach 40.

2. (7 pts)

A risk of 1 in 2,000 per occasion is repeated 1,000 times. Find the chance of at least one occurrence.

3. (6 pts)

A seat belt reduces the risk of fatal injury to front-seat occupants by about 45%. If a trip carried a risk p without a belt, what is it with one? Why does the habit matter even on short trips?

4. (6 pts)

Explain the difference between decision quality and outcome, using the four-box grid.

5. (8 pts)

From the 1-yard line, a run scores 55% and a pass scores 45%, falls incomplete 53% and is intercepted 2%. Passing first allows three plays; running first allows two. Compare the chance of scoring.

6. (6 pts)

What is hindsight bias? How does a decision journal guard against it?

7. (7 pts)

With $\lambda = 2$, find the expected value and the felt value of a 50–50 chance of +120 or –80 points.

8. (8 pts)

3% of prospects become starters. A tryout drill flags 80% of future starters and 15% of non-starters. If a prospect is flagged, what is the chance he becomes a starter?

9. (6 pts)

Define a sunk cost and explain why it should not affect the next decision, with one sports example.

10. (7 pts)

A team wins each game with probability 0.6. Find its chance of winning a best-of-three series.

11. (6 pts)

Find the spread of wins from luck for a 0.6 team over 162 games.

12. (6 pts)

Why did Eric Liddell switch from the 100 meters to the 400 meters in 1924, and what happened?

13. (7 pts)

A student makes 70 of 100 shots. Find the margin of error and the likely range of his true rate.

14. (6 pts)

How many attempts are needed for a margin of error of 4 percentage points when the rate is near 50%?

15. (8 pts)

Short essay (one paragraph): Using Proverbs 16:33 or Ecclesiastes 9:11 and one example from this course, explain how a Christian athlete can both play the percentages diligently and trust God with the outcome.

ANSWER KEY

1. $445/598 \approx 0.744$.
2. $1 - (0.9995)^{1000} \approx 1 - e^{-0.5} = 0.393$.
3. $0.55p$. The habit is repeated thousands of times, and the cumulative risk depends on every trip.
4. Decision quality is judged by the reasoning and information at the time; the outcome includes luck. The grid: good decision/good outcome, good decision/bad outcome, bad decision/good outcome, bad decision/bad outcome.
5. Run, run: $1 - 0.45^2 = 0.7975$. Pass first: $0.45 + 0.53(0.7975) = 0.873$. Passing first is better on these numbers.
6. The belief, after learning a result, that it was predictable all along. A journal records probabilities and reasons before the result, so they cannot be rewritten afterward.
7. EV: $0.5(120) - 0.5(80) = 20$; felt: $60 - 80 = -20$. A loss-averse person refuses a favorable choice.
8. $0.024/(0.024 + 0.1455) = 0.024/0.1695 \approx 0.142$, about 14%.
9. A cost already paid that cannot be recovered. It is the same whichever choice is made next, so only future costs and benefits matter. Example: continuing a failing game plan because of practice time already spent.
10. $0.36 + 2(0.36)(0.4) = 0.648$.
11. $\sqrt{162(0.6)(0.4)} \approx 6.2$ wins.
12. The 100-meter heats were on a Sunday, which he would not race on. He trained for the 400 meters and won the gold medal in a world-record 47.6 seconds.
13. $2\sqrt{0.7 \times 0.3/100} \approx 0.092$; about 61% to 79%.
14. $4(0.25)/0.0016 = 625$.
15. Answers vary. A strong essay explains that probability describes our knowledge and the long-run pattern while God governs every outcome; uses a specific example (taking the high-percentage shot, judging a decision by its quality, Liddell's conviction); and concludes that diligence and humility belong together.

PORTFOLIO PIECE: THE ODDS REPORT AND DECISION JOURNAL

Submit your capstone odds report from Lesson 6.5 (2–4 pages with at least 200 observations, expected values, margins of error, a sensitivity check and one clear chart), together with your two-week decision journal from Lesson 6.2 and a one-page reflection. The reflection should name one decision in your journal that turned out badly but was sound, one that turned out well but was not, and one way this course has changed how you play, practice or decide.

LOOK FOR	WHAT IT MEANS
Data and computation	At least 200 recorded observations; correct success rates, expected values and margins of error for each option.
Uncertainty and risk	A sensitivity check is done and reported honestly; the risk side of each option is stated.
Presentation	One clear chart with a title that states the finding; a well-organized report and a confident five-minute talk.
Decision journal and reflection	At least 20 journal entries with probabilities recorded before results; a thoughtful reflection that separates decision quality from outcome.
Character	Honest records with no discarded trials, points only, and a gracious, humble conclusion.

6.1 LIFE TABLES AND SEAT BELTS

OBJECTIVES

- Read Edmond Halley's 1693 Breslau life table and compute survival probabilities from it.
- Compute the chance of at least one accident when a small risk is repeated many times, using $1 - (1 - p)^n$.
- Explain why seat belts, helmets and safety rules in sport are high-percentage habits, using base rates and the size of the risk they remove.

Warm-up. Estimate how many car trips your family has taken this week, and multiply by 52 for a year. Now suppose (an illustrative number) each trip carries a 1-in-100,000 chance of a serious accident. Before calculating, guess: over 20 years of trips like this, is the chance of at least one serious accident closer to one percent, ten percent or fifty percent?

WATCH

What are micromorts & microlives, & how are they linked to risk?

Risk Bites · 3:45

<https://www.youtube.com/watch?v=Oi6RE3nrqul>

WHEN TO WATCH

After the mathematics paragraph. It introduces the micromort, a one-in-a-million chance, as a way to compare everyday risks.

BACKUP VIDEO

The Surprising Origin of Life Tables

The Deliberate Engineer · 6:36

<https://www.youtube.com/watch?v=QjM4d7ISqfk>

After the story paragraph, for a closer look at the pages of Halley's 1693 paper and its life table.

KEY TERMS

life table

A table showing, out of a starting group, how many people are still living at each age.

base rate

How often something happens in a whole population, before any particular information is considered.

micromort

A one-in-a-million chance of a fatal accident, a unit invented by Ronald Howard of Stanford to compare small risks.

relative risk reduction

The fraction by which a safety measure cuts a risk; a 45% reduction multiplies the risk by 0.55.

cumulative risk

The chance that a small risk, repeated many times, happens at least once.

CORE IDEAS

The story. In the 1680s a Lutheran pastor in the city of Breslau (now Wrocław, Poland), Caspar Neumann, kept careful records of the births and burials in his city, together with the age at which each person died. His tables reached the Royal Society in London, and in 1693 Edmond Halley, the astronomer who later predicted the return of the comet that bears his name, used them to build the first scientific life table: “An Estimate of the Degrees of the Mortality of Mankind.” Out of 1,000 children alive at age 1, Halley estimated, 598 would reach age 20, 346 age 50 and 142 age 70. He used the table to compute fair prices for life annuities, and life insurance was put on a mathematical footing.

How they figured it out. Halley’s insight was to treat a whole city as a sample and to count, not guess. A single family’s experience tells you little; a city’s records over several years give a **base rate** you can trust (Lesson 1.5). Two centuries later, engineers and physicians did the same for accidents. When crash records showed how much of the harm in collisions came from occupants striking the inside of the car or being thrown out, the case for seat belts became overwhelming. The U.S. National Highway Traffic Safety Administration reports that lap and shoulder belts reduce the risk of fatal injury to front-seat car occupants by 45%.

The mathematics. A risk that is tiny on any one occasion becomes large when it is repeated. If the chance of a mishap on one trip is p , the chance of at least one mishap in n independent trips is $1 - (1 - p)^n$, which is close to $1 - e^{-np}$ when p is small. A habit that cuts p by 45% on every trip cuts the lifetime risk by nearly the same fraction. That is why the percentage player buckles his seat belt on a two-minute trip to the store: the trip is short, but the habit is repeated thousands of times.

On the field. Sport has its own life tables. Helmets in football, hockey, cycling and baseball, spotters in gymnastics, warm-ups before sprinting, and the rule “when thunder roars, go indoors” all remove a little risk from each game or practice, and a little risk removed thousands of times is a great deal of harm prevented. Leagues changed rules after counting injuries: the National Weather Service advises waiting 30 minutes after the last thunder before going back outside. A wise athlete takes the high-percentage shot on the field and the high-percentage habit off it.

The risk side. People tend to fear dramatic, rare dangers and to ignore ordinary, repeated ones. A life table and a base rate correct that error. “A prudent man foreseeth the evil, and hideth himself” (Proverbs 22:3): prudence is not fear, but the habit of looking at the numbers and acting on them.

PICTURE STUDY



Edmond Halley, Thomas Murray, about 1690 · Oil on canvas · The Royal Society, London

Image: [Wikimedia Commons](#), public domain

Edmond Halley was painted by Thomas Murray around the time he wrote his paper on the Breslau tables. Halley was a man of many sciences: he mapped the stars of the southern sky, charted the earth's magnetism for sailors, paid for the printing of Newton's *Principia*, and predicted the return of the comet now named for him. To him, a table of births and burials was as worthy of careful study as the motion of a comet.

Look closely

- Find the paper in Halley's hand. What does it suggest about how he wished to be remembered?
- Look at his face. Does he look like a man who trusted guesses or numbers?
- Read the inscription at the top. It names him a doctor of laws, professor of geometry and secretary of the Royal Society. How did he earn those titles?

Talk about it

1. Halley used the same habits of careful measurement for comets and for human lives. What are those habits?
2. Why might people in 1693 have been surprised that the length of human life could be studied with numbers?

3. How does a life table help a person plan wisely without making him fearful?

Safety first. If your project includes outdoor sports, follow the 30-minute lightning rule, wear the proper protective equipment, and have a parent supervise any trip to a field or court.

PROJECT

A Family Risk Audit

You will gather real counts of how often your family does certain everyday activities, combine them with clearly labelled illustrative risk numbers, and find which habits remove the most risk over a year.

You will need

- A week-long tally sheet
- Calculator or spreadsheet
- Halley's table values from this lesson

Steps

1. For one week, tally how often family members do five everyday activities: car trips, bicycle rides, sports practices or games, use of stairs or ladders, and outdoor play during storms (hopefully zero).
2. Multiply each weekly count by 52 to estimate a year.
3. For each activity, choose an illustrative per-occasion risk of a minor injury (for example 1 in 5,000 for a bicycle ride) and write it down clearly marked as illustrative.
4. Compute the yearly chance of at least one injury for each activity with $1 - (1 - p)^n$.
5. For each activity, name one safety habit (helmet, seat belt, warm-up, spotter, the 30-minute lightning rule) and an illustrative risk reduction, and recompute.
6. Rank the habits by how much yearly risk each removes.
7. From Halley's table, compute the chance that a 20-year-old in 1693 reached 30, 40, 50 and 60, and graph those chances.
8. Write a one-page family safety summary with your top three habits and the reasoning behind them.

Tips

- The point is not the exact risk numbers, which you have marked illustrative, but how repetition and habits change them.

- Keep the tone sensible: the purpose is wise habits, not worry.

Check your work

- ☐ I tallied real counts for a week and scaled them to a year.
- ☐ I computed yearly risks with and without a safety habit for at least four activities.
- ☐ I graphed Halley's survival chances and explained one thing they show.

SCOREBOOK

1. Copy Halley's values (ages 1, 10, 20, 30, 40, 50, 60, 70, 80: 1000, 661, 598, 531, 445, 346, 242, 142, 41) and draw his survival curve.
2. Make a table of $1 - (1 - p)^n$ for $p = 0.0001$ and $n = 100, 1,000, 10,000$.
3. "What I would decide and why": your team wants to finish practice although thunder was heard five minutes ago. What do you advise, and why?

KEY FORMULAS

$$P(\text{survive from age } a \text{ to age } b) = \frac{\ell_b}{\ell_a}$$

$$P(\text{at least one in } n) = 1 - (1 - p)^n \approx 1 - e^{-np}$$

$$p_{\text{with safety}} = (1 - r)p$$

From history. Halley's paper also showed how to price an annuity, a yearly payment for life, by adding up the chance of being alive each year. Insurance companies and pension funds still use the same method, with modern life tables built from national records.

WORKED EXAMPLES

Example 1. Halley's table gives the number living at each age, out of 1,000 alive at age 1: age 20, 598; age 30, 531; age 50, 346; age 60, 242. Find the chance that a 20-year-old in Breslau would live to 50, and to 60.

Show solution

1. To 50: $\ell_{50}/\ell_{20} = 346/598 = 0.579$.

2. To 60: $\ell_{60}/\ell_{20} = 242/598 = 0.405$.

Answer: About 58% would reach 50 and about 40% would reach 60.

Example 2. Suppose (an illustrative number) each car trip carries a 1-in-10,000 chance of an injury accident, and a person makes 2 trips a day for 10 years. Find the chance of at least one such accident. Then suppose a safety habit cuts the risk on each trip by 45%.

Show solution

1. Trips: $n = 2 \times 365 \times 10 = 7,300$.
2. Without the habit: $1 - (0.9999)^{7300} \approx 1 - e^{-0.73} = 0.518$.
3. With it: $p = 0.55 \times 0.0001 = 0.000055$; $1 - (0.999945)^{7300} \approx 1 - e^{-0.40} = 0.331$.

Answer: About 52% without the habit and 33% with it. A risk too small to notice on any one trip becomes the main event over ten years, and a steady habit changes it substantially.

Example 3. A micromort is a one-in-a-million chance of a fatal accident. If an activity carries 1 micromort each time (illustrative), how many times can it be done before the cumulative chance reaches one percent (0.01)?

Show solution

1. Solve $1 - (1 - 10^{-6})^n = 0.01$.
2. Approximately $n \times 10^{-6} \approx 0.01$, so $n \approx 10,000$. (More exactly, $n = \ln(0.99) / \ln(1 - 10^{-6}) \approx 10,050$.)

Answer: About 10,000 times: roughly once a day for 27 years.

PRACTICE

1. Easy Who built the first scientific life table, from which city's records, and in what year was it published?

Show answer

Edmond Halley, from the records of Breslau (now Wrocław) kept by Caspar Neumann; published in 1693.

2. Easy In Halley's table, 661 of 1,000 were living at age 10 and 598 at age 20. What fraction of 10-year-olds reached 20?

Show answer

$598/661 \approx 0.905$, about 90%.

3. **Easy** A safety measure reduces a risk by 45%. By what number do you multiply the original risk?

Show answer

0.55.

4. **Medium** Using Halley's table (age 40: 445; age 70: 142), find the chance a 40-year-old would reach 70.

Show answer

$142/445 \approx 0.319$.

5. **Medium** A player has a 1-in-500 chance of an ankle injury in any practice without proper warm-up (illustrative). Over 120 practices, what is the chance of at least one injury?

Show answer

$1 - (499/500)^{120} = 1 - 0.998^{120} \approx 1 - 0.787 = 0.213$.

6. **Medium** In problem 5, a warm-up routine cuts the per-practice risk in half. What is the new chance of at least one injury over 120 practices?

Show answer

$1 - (0.999)^{120} \approx 1 - 0.887 = 0.113$.

7. **Medium** Explain what a base rate is and why Halley needed a whole city's records rather than his own acquaintances' ages at death.

Show answer

A base rate is how often something happens in a whole population. A few families give a small, unrepresentative sample; a city over several years gives a large sample whose proportions are steady (the law of large numbers).

8. **Medium** A family member says, "I don't need a seat belt; I'm only going two miles." Answer using this lesson's mathematics.

Show answer

The risk on one short trip is small, but the habit is repeated thousands of times, and the cumulative risk $1 - (1 - p)^n$ becomes large. A seat belt reduces the risk on every trip by about 45%, so the habit removes a large share of the lifetime risk.

9. **Hard** Use $1 - e^{-np}$ to estimate the chance of at least one event if $p = 0.0002$ and $n = 5,000$.

Show answer

$$np = 1; 1 - e^{-1} \approx 0.632.$$

10. **Hard** Halley's table shows 1,000 at age 1 and 598 at age 20. If a person aged 1 in Breslau had a 59.8% chance of reaching 20, what was the chance of not reaching it? What does this tell you about life in 1690?

Show answer

40.2%. Childhood was dangerous: about two children in five did not reach adulthood, which is why careful records of risk were so valuable.

11. **Challenge** Show that for small p , $(1 - p)^n \approx e^{-np}$ by using $\ln(1 - p) \approx -p$, and check it for $p = 0.01$, $n = 100$.

Show answer

$$(1 - p)^n = e^{n \ln(1-p)} \approx e^{-np}. \text{ Check: } 0.99^{100} = 0.366 \text{ and } e^{-1} = 0.368: \text{ very close.}$$

COMMON MISTAKES

- Multiplying np and calling it a probability even when it is large (it can exceed 1); use $1 - (1 - p)^n$.
- Judging a risk by one occasion instead of by how often it is repeated.
- Confusing a relative reduction (45% less risk) with an absolute one (45 percentage points).

Make it a game • Risk Ladder. Write eight everyday risks on cards, such as riding in a car with and without a seat belt, and ask your parent to rank them from most to least risky in sixty seconds. Rank them yourself too. Check the order against the numbers in your lesson, including micromorts and relative risk reduction. Whoever is closest wins. Mention Halley's table from Breslau in 1693.

Scripture. "A prudent man foreseeth the evil, and hideth himself: but the simple pass on, and are punished." (Proverbs 22:3, KJV)

Prudence looks ahead at the numbers and takes simple, repeated precautions.

6.2 JUDGE THE DECISION, NOT THE RESULT

OBJECTIVES

- Separate the quality of a decision from the luck of its outcome, and explain hindsight bias.
- Analyze the final Seattle pass in Super Bowl XLIX (2015) with a simple probability model of the plays and the clock.
- Keep a decision journal that records probabilities before the result is known, and review it fairly afterward.

Warm-up. Write down three decisions you made this week (in a game, in schoolwork or at home). Beside each, mark whether it turned out well or badly. Then, separately, mark whether it was a good decision given what you knew at the time. Are the two columns the same?

WATCH

Why Psychology Tells Us What We Already Know

SciShow Psych · 5:50

https://www.youtube.com/watch?v=csqOL_KT4Go

WHEN TO WATCH

After the story paragraph. It explains hindsight bias: why, once we know how something turned out, it seems as if it was obvious all along.

BACKUP VIDEO

Worst play call in Super Bowl history?

NFL · 1:31

<https://www.youtube.com/watch?v=CZSqjfYaX4M>

With the story paragraph, to see the play and hear how it was described in the moment.

KEY TERMS

decision quality

How good a choice was, given the information, time and options available when it was made.

outcome

What actually happened, which depends on the decision and on luck.

hindsight bias

The tendency, after learning a result, to believe it was predictable all along.

outcome bias

Judging a decision by its result rather than by its reasoning.

decision journal

A written record of a decision, its options, the estimated probabilities and the reasons, made before the result is known.

CORE IDEAS

The story. On February 1, 2015, in Glendale, Arizona, the Seattle Seahawks trailed the New England Patriots 28–24 in Super Bowl XLIX. With 26 seconds left, Seattle had second-and-goal at the Patriots’ 1-yard line and one timeout. Seattle had one of the best running backs in the league, Marshawn Lynch, and nearly everyone watching expected a run. Instead quarterback Russell Wilson threw a quick pass, and an undrafted rookie cornerback, Malcolm Butler, stepped in front of the receiver and intercepted it. New England won. Within minutes, commentators were calling it the worst play call in Super Bowl history.

How they figured it out. Coach Pete Carroll explained his reasoning: with only one timeout, a pass on second down that fell incomplete would stop the clock and leave time for two more plays, so Seattle would get three chances to score instead of two. Interceptions from the 1-yard line were very rare. Several analysts who studied the call afterward concluded that it was defensible, perhaps even the better choice, and that the outcome was unlucky. Carroll later called it “the worst result of a call ever,” a careful choice of words. Whether or not you agree with the call, the lesson is that the result alone cannot tell you.

The mathematics. A decision is a choice among probability distributions; the outcome is one draw from the chosen distribution. A good decision can lose, and a bad one can win. To judge a decision, estimate the probabilities as they were *before* the play, compute the chance of success of each option, and compare. Then, separately, ask whether the estimates themselves were reasonable. Over many decisions, good choices produce good results more often; in any single case, luck may decide.

On the field. Great coaches review game film by asking “Was it the right call?” before “Did it work?” A **decision journal** makes this possible: before a big choice, write the options, your probabilities and your reasons. Afterward, sort each decision into one of four boxes: good decision and good outcome (deserved success), good decision and bad outcome (bad luck), bad decision and good outcome (dumb luck), bad decision and bad outcome (deserved failure). Only the first column teaches you to repeat; only the third warns you about a habit that luck is hiding.

Think it through. “A man’s heart deviseth his way: but the LORD directeth his steps” (Proverbs 16:9). We are responsible for the quality of our choices, not for controlling every outcome. That frees a player to decide well and accept results with humility, and it keeps him from boasting when luck goes his way.

PICTURE STUDY



The Fog Warning, Winslow Homer, 1885 • Oil on canvas • Museum of Fine Arts, Boston

Image: [Wikimedia Commons](#), public domain

Winslow Homer painted the fishermen of the North Atlantic, who rowed out from their schooner in small boats to fish. This man has made a good catch, but he has seen the fog bank coming. He must judge, from the distance, the wind and the fog, whether he can row back to the ship before he is lost. He cannot know the outcome; he can only decide well.

Look closely

- Find the schooner on the horizon. How far away is it?
- Find the fog bank. Which way is it moving, and how fast do you think?
- Find the fish in the boat. What has he already gained, and what could he lose?
- Look at his face and shoulders. Is he panicking or calculating?

Talk about it

1. What information does the fisherman have, and what does he not know?
2. If he rows hard and reaches the ship, was it a good decision? If he is caught in the fog, was it a bad one? What else would you need to know?
3. How would a fisherman's "decision journal" over many trips help him decide better next time?

PROJECT

A Decision Journal for Two Weeks

You will keep a real decision journal for two weeks of games (family games, sports or both), recording your probabilities before each decision and reviewing them fairly afterward.

You will need

- A notebook set aside as your decision journal
- A family game you play often, or a sport you play
- Calculator

Steps

1. Set up each journal entry with these headings: date, situation, options, my probability of success for each, my reasons, decision, result, review.
2. Over two weeks, record at least 20 decisions before you know the result. Each must include a probability estimate.
3. After each result, write it down without changing anything you wrote before.
4. At the end, sort all 20 decisions into the four boxes of the decision–outcome grid.
5. Group your decisions by your estimated probability (for example 0–40%, 40–70%, 70–100%) and compare your average estimate with your actual success rate in each group.
6. Compute the standard deviation from luck for one group and decide whether your estimates were too high, too low or about right.
7. Choose one decision with a bad outcome that you still believe was good, and explain why in a paragraph.
8. Choose one decision with a good outcome that you now believe was poor, and explain what you will do differently.

Tips

- Write quickly in the moment. A rough estimate written before the result is worth more than a careful one written after.
- Be honest in the review. The journal is for learning, not for looking good.

Check your work

- ☐ I recorded at least 20 decisions with probabilities before the results.
- ☐ I sorted them into the four boxes and checked my calibration by groups.
- ☐ I wrote one paragraph defending a good decision with a bad outcome and one admitting a lucky poor decision.

SCOREBOOK

1. Draw the decision–outcome grid and place three plays from a game you watched into its boxes.
2. Draw the probability tree for worked example 1, with the pass-first branch and the run-run branch.
3. “What I would decide and why”: write your own call for Seattle’s second-and-goal, with your probabilities, before checking the worked example.

KEY FORMULAS

$$P(\text{score in } k \text{ tries}) = 1 - (1 - q)^k$$

$$P(\text{pass first}) = t + i_c [1 - (1 - r)^2]$$

From history. Malcolm Butler had been undrafted and was playing in his first season. He had studied the formation Seattle used in practice that week, which helped him jump the route: a reminder that good preparation is a decision too.

WORKED EXAMPLES

Example 1. Use illustrative numbers. From the 1-yard line, a run scores 55% of the time; otherwise the clock keeps running. A pass scores 45%, falls incomplete 53% (stopping the clock) and is intercepted 2%. With 26 seconds and one timeout, assume Seattle could run only two plays if it ran first (run, timeout, run), but three if it passed first and the pass fell incomplete (pass, run, timeout, run). Compare the chance of scoring.

Show solution

1. Run, run: $1 - (0.45)^2 = 1 - 0.2025 = 0.7975$.
2. Pass first: touchdown 0.45; incomplete 0.53, then two runs:
 $0.53 \times 0.7975 = 0.4227$; interception 0.02, then 0.
3. Total for passing first: $0.45 + 0.4227 = 0.873$.

Answer: With these numbers, passing first scores about 87% of the time and running twice about 80%. Even with a 2% interception risk, the pass gives more chances. The interception was a bad outcome; on these numbers it was not a bad decision.

Example 2. In worked example 1, how large would the interception rate have to be (taking it from the incompletions, so touchdown stays 45%) before running twice was better?

Show solution

1. Let the interception rate be i ; incompletions are $0.55 - i$.
2. Pass first: $0.45 + (0.55 - i)(0.7975)$. Set equal to 0.7975.
3. $(0.55 - i)(0.7975) = 0.3475$, so $0.55 - i = 0.4357$ and $i \approx 0.114$.

Answer: The interception rate would have to exceed about 11%, far higher than from the 1-yard line. That is a sensitivity check: the conclusion survives large errors in the estimate.

Example 3. A coach's decision journal shows 40 fourth-down attempts he chose to make, each with his before-the-play estimate of success averaging 60%. They succeeded 21 times. Is that evidence that his estimates were too high?

Show solution

1. Expected successes: $40 \times 0.60 = 24$.
2. Spread from luck: $\sqrt{40 \times 0.6 \times 0.4} = \sqrt{9.6} \approx 3.1$.
3. 21 is $3/3.1 \approx 1$ standard deviation below 24.

Answer: No: a shortfall of one standard deviation is ordinary luck. The journal should keep growing before he concludes that his estimates are too optimistic.

PRACTICE

1. **Easy** In one sentence, what is the difference between a decision and an outcome?

Show answer

A decision is a choice made with the information available; an outcome is what happened, which also depends on luck.

2. **Easy** What was the score, the down and distance, and the time left when Seattle threw its final pass in Super Bowl XLIX?

Show answer

Seattle trailed 28–24, second-and-goal at the 1-yard line, with 26 seconds left and one timeout.

3. **Easy** Name the four boxes of the decision–outcome grid.

Show answer

Good decision–good outcome, good decision–bad outcome, bad decision–good outcome, bad decision–bad outcome.

4. **Medium** A run from the 1 scores 60% of the time. What is the chance of scoring with two runs? With three?

Show answer

Two: $1 - 0.4^2 = 0.84$; three: $1 - 0.4^3 = 0.936$.

5. **Medium** Explain hindsight bias with one sports example.

Show answer

After a result is known, people believe it was predictable. Example: after the interception, many said “everyone knew they should run,” although before the play the pass had good reasons and a small interception risk.

6. **Medium** A player takes a high-percentage shot that rims out and loses the game. His coach criticizes the shot. Which box of the grid is this, and what should the coach say instead?

Show answer

Good decision, bad outcome. The coach should praise the decision and note that the shot will go in more often than not.

7. **Medium** A player throws a wild, low-percentage pass that happens to be caught for a touchdown. Which box is this, and why is it dangerous to praise it?

Show answer

Bad decision, good outcome (dumb luck). Praising it encourages a habit that will fail most of the time.

8. **Medium** List four things a good decision-journal entry should contain.

Show answer

The options, the estimated probability and value of each, the reasons, and the decision, all written before the result; afterward, the result and a review.

9. **Hard** In worked example 1, suppose a run scored only 45% and a pass 45% (with 53% incomplete and 2% intercepted). Compare running twice with passing first.

Show answer

Run, run: $1 - 0.55^2 = 0.6975$. Pass first:
 $0.45 + 0.53(0.6975) = 0.45 + 0.3697 = 0.820$. Passing first is still better, because it buys a third play.

10. **Hard** A coach made 50 “go for it” decisions with an average estimated success of 50% and succeeded 22 times. Compute the expected number and the standard deviation, and decide whether the shortfall is surprising.

Show answer

Expected 25; $\sqrt{50 \times 0.5 \times 0.5} \approx 3.54$. 22 is $3/3.54 \approx 0.85$ standard deviations below: not surprising.

11. **Challenge** Write a general formula for the pass-first plan in worked example 1 in terms of the touchdown rate t , incompleteness rate c and run success rate r , and find the condition under which passing first beats running twice.

Show answer

Pass first: $t + c[1 - (1 - r)^2]$. Run twice: $1 - (1 - r)^2$. Passing first is better when $t > (1 - c)[1 - (1 - r)^2]$. With $t = 0.45$, $c = 0.53$, $r = 0.55$:
 $0.45 > 0.47 \times 0.7975 = 0.375$, true.

COMMON MISTAKES

- Judging a decision only by its result.
- Changing your memory of what you expected after you know what happened (hindsight bias). A journal prevents this.
- Concluding that a strategy is wrong from a handful of failures, without comparing with the spread expected from luck.

Make it a game • Good Call, Bad Luck. Draw a four-box grid: good decision with good outcome, good decision with bad outcome, bad decision with good outcome, and bad decision with bad outcome. Your parent tells four short stories, and you place each in a box. Argue your reasons. Then add a hindsight-bias story of your own and let your parent place it on the grid.

Scripture. “A man’s heart deviseth his way: but the LORD directeth his steps.” (Proverbs 16:9, KJV)

We are responsible for planning and deciding well; the outcome is not ours to command.

6.3 TRAPS IN OUR THINKING

OBJECTIVES

- Explain loss aversion and prospect theory (Kahneman and Tversky, 1979) and compute how a loss-averse person values a 50–50 choice.
- Recognize the sunk-cost trap, overconfidence and base-rate neglect, each with a sports example.
- Use Bayes's rule to correct base-rate neglect in a scouting problem.

Warm-up. Answer quickly, without calculating: (1) In a game you play for points, would you accept a coin flip that adds 100 points to your score on heads and subtracts 100 on tails? (2) Would you accept one that adds 150 on heads and subtracts 100 on tails? Write yes or no for each, and keep your answers.

WATCH

Prospect Theory: An Overview

Ashley Hodgson · 13:14

<https://www.youtube.com/watch?v=Sl-qgzF70IM>

WHEN TO WATCH

After the story paragraph. An economist explains the three parts of prospect theory with a hand-drawn value curve.

BACKUP VIDEO

What Is Thinking on the Margin, and How Does It Solve the Sunk Cost Fallacy?

Marginal Revolution University · 6:18

<https://www.youtube.com/watch?v=B9MPklfyRHI>

Before worked example 3, for the sunk-cost trap and the habit of thinking about the next decision rather than the last one.

KEY TERMS

loss aversion

The tendency to feel a loss more strongly than a gain of the same size, often about twice as strongly.

prospect theory

Kahneman and Tversky's 1979 description of how people actually value risky choices: relative to a reference point, with losses weighted more than gains.

sunk cost

A cost already paid that cannot be recovered and should not affect the next decision.

overconfidence

Believing your estimates are more accurate than they are.

base-rate neglect

Ignoring how common something is overall when judging a particular case.

CORE IDEAS

The story. In 1979 two Israeli psychologists, Daniel Kahneman and Amos Tversky, published “Prospect Theory: An Analysis of Decision under Risk” in the journal *Econometrica*. Economists had long assumed that people weigh risky choices by their expected value or expected utility. Kahneman and Tversky showed, with simple choices given to hundreds of people, that real people judge outcomes as gains or losses from a reference point, and that losses loom larger than gains. Tversky died in 1996; Kahneman received the Nobel Prize in economics in 2002 for their work together.

How they figured it out. Their method was to offer simple choices and record the answers. Most people refuse a 50–50 chance to win or lose the same amount, and many refuse even when the possible gain is clearly larger than the possible loss. Later researchers found the same patterns among professionals. Devin Pope and Maurice Schweitzer studied more than 2.5 million putts on the PGA Tour and found that professionals make putts for par more often than birdie putts of the same length, by about two percentage points: they try harder to avoid a bogey (a “loss” relative to par) than to gain a birdie. Barry Staw and Ha Hoang found in 1995 that

NBA teams gave more playing time to players they had drafted early, even when their performance did not justify it: a sunk-cost trap.

The mathematics. In a simple version of prospect theory, a gain of x feels like x , but a loss of x feels like $-\lambda x$, with λ about 2. A 50–50 chance of +150 or –100 points has an expected value of +25 but feels like $0.5(150) - 0.5(2)(100) = -25$, so most people refuse it. **Base-rate neglect** is corrected with Bayes’s rule (Lesson 1.4): $P(A | B) = \frac{P(B | A)P(A)}{P(B)}$.

On the field. Each trap has a sports form. Loss aversion: protecting a lead so anxiously that you stop playing well (Lesson 4.5). Sunk cost: keeping a failing plan or an expensive player because of what has already been invested. Overconfidence: a coach who is “sure” of his fourth-down estimates without ever checking them (Lesson 6.2). Base-rate neglect: believing a scout’s “can’t-miss” label without asking how many “can’t-miss” prospects actually succeed. The cure is the habit of this course: count, record, compare, decide.

Think it through. These traps are not foolishness; they are the ordinary tendencies of human minds, including expert ones. “The simple believeth every word: but the prudent man looketh well to his going” (Proverbs 14:15). Knowing the traps does not make you immune, but it lets you build habits, like a decision journal or a checklist, that catch them.

PICTURE STUDY



The Parable of the Blind Leading the Blind, Pieter Bruegel the Elder, 1568 · Tempera on canvas · Museo di Capodimonte, Naples

Image: [Wikimedia Commons](#), public domain

Bruegel illustrated the words of Jesus: “if the blind lead the blind, both shall fall into the ditch” (Matthew 15:14). Each man trusts the one in front, and they all fall travels down the line. It is a painting about following without seeing, which is exactly what the traps in this lesson make us do.

Look closely

- Follow the line of men from right to left. Where does the trouble begin, and how far has it travelled?
- Look at the faces. How does the expression change from the back of the line to the front?
- Find the church in the background. Why might Bruegel have placed it there, standing still while the men fall?
- Look at how each man holds the one ahead. What would it take for one of them to stop?

Talk about it

1. Which trap in this lesson is most like following a leader you cannot see: overconfidence, sunk cost, loss aversion or base-rate neglect? Explain.

2. How do records, base rates and decision journals give a player “sight” he would not otherwise have?
3. When have you followed a crowd’s opinion about a game or sport without checking the numbers?

PROJECT

Testing the Traps

You will run three short experiments on willing family members and friends (with points only, never money), record their choices, and compare them with prospect theory and Bayes's rule.

You will need

- A coin
- Prepared question cards (written by you from the steps below)
- Scorebook and calculator
- At least six willing participants

Steps

1. **Loss aversion.** Ask each participant whether he would accept a coin flip in a game played for points: heads +100, tails -100. Then +150/-100, +200/-100, +250/-100 and +300/-100. Record the smallest gain each accepts.
2. Estimate each person's λ as (smallest accepted gain) $\div$ 100, and find the group average.
3. **Base rates.** Read participants the scouting problem from worked example 2 and ask for a quick estimate of the chance a "can't-miss" prospect becomes a star. Record the answers and compare with 15.5%.
4. **Overconfidence.** Ask ten factual questions with number answers (for example, the year a famous game was played) and ask each participant for a range he is 90% sure contains the answer. Count how many ranges contain the true answer.
5. Tabulate all results.
6. Compare: is the group average λ near 2? Did most people overestimate the scouting chance? Were the 90% ranges right 90% of the time?
7. Write a report on what you found, and one habit that would protect a coach from each trap.

Tips

- Ask everyone the same questions in the same order, and do not hint at the “right” answer.
- Thank your participants and explain the results afterward; most people enjoy learning about their own thinking.

Check your work

- ☐ I tested at least six people on all three experiments and recorded their answers.
- ☐ I estimated λ for each person and the group.
- ☐ My report names one protective habit for each trap.

SCOREBOOK

1. Draw the prospect-theory value curve: a line through the origin, steeper on the loss side than the gain side.
2. Make a two-row tree for the scouting problem and circle the branches with the label.
3. Return to your warm-up answers. Which trap did they show, if any?

KEY FORMULAS

$$V(x) = \begin{cases} x & x \geq 0 \\ \lambda x & x < 0 \end{cases}, \quad \lambda \approx 2$$

$$P(A | B) = \frac{P(B | A) P(A)}{P(B | A)P(A) + P(B | \text{not } A)P(\text{not } A)}$$

From history. Kahneman and Tversky first met at a university in Jerusalem in 1969 and worked together for more than two decades. Their friendship and partnership is one of the most famous in modern social science.

WORKED EXAMPLES

Example 1. With $\lambda = 2$, compute the expected value and the felt value of (a) a 50–50 chance of +100 or –100 points, and (b) a 50–50 chance of +250 or –100 points.

Show solution

1. (a) EV: $0.5(100) + 0.5(-100) = 0$; felt: $0.5(100) + 0.5(-200) = -50$.
2. (b) EV: $0.5(250) + 0.5(-100) = 75$; felt: $0.5(250) + 0.5(-200) = 25$.

Answer: (a) EV 0, felt –50: refused. (b) EV +75, felt +25: accepted. With $\lambda = 2$, a 50–50 choice is accepted only if the gain is more than twice the loss.

Example 2. A scout labels prospects “can’t miss.” Only 2% of prospects become stars. The scout gives the label to 90% of future stars, but also to 10% of prospects who do not become stars. If a prospect has the label, what is the chance he becomes a star?

Show solution

1. Stars with the label: $0.02 \times 0.90 = 0.018$.
2. Non-stars with the label: $0.98 \times 0.10 = 0.098$.
3. $P(\text{star} \mid \text{label}) = 0.018 / (0.018 + 0.098) = 0.018 / 0.116 = 0.155$.

Answer: About 15.5%. Because stars are rare, most “can’t-miss” prospects miss, even with a good scout.

Example 3. A team has spent three weeks learning a new offensive system. In scrimmages it has averaged 14 points per game, while its old system averaged 21. The coach says, “We can’t switch back now, after all that work.” Analyze.

Show solution

1. The three weeks are a sunk cost: they are spent whether the team switches or not.
2. The decision should compare the future: the expected points of each system from now on, including any expected improvement in the new one.
3. If the new system is not expected to pass 21 points per game soon enough to matter, the team should switch back.

Answer: The past effort should not decide the question; only the expected future results should.

PRACTICE

1. **Easy** Who published prospect theory, and in what year?

Show answer

Daniel Kahneman and Amos Tversky, in 1979.

2. **Easy** Define a sunk cost and give a sports example.

Show answer

A cost already paid that cannot be recovered. Example: practice time already spent on a play that does not work; it should not keep the play in the playbook.

3. **Easy** What did Pope and Schweitzer find about par putts and birdie putts?

Show answer

Professionals made par putts more often than birdie putts of the same length, by about two percentage points, consistent with loss aversion.

4. **Medium** With $\lambda = 2$, find the felt value of a 50–50 chance of +180 or –100 points. Would a loss-averse player accept it?

Show answer

$0.5(180) - 0.5(200) = -10$. He would refuse, though the expected value is +40.

5. **Medium** With $\lambda = 2$, what is the smallest gain G that makes a 50–50 chance of $+G$ or -50 points feel acceptable?

Show answer

$$0.5G - 0.5(100) \geq 0 \Rightarrow G \geq 100.$$

6. **Medium** A player says he is 90% sure of 20 predictions. 13 come true. What is his actual rate, and what trap does this show?

Show answer

$$13/20 = 65\%, \text{ well below } 90\%: \text{ overconfidence.}$$

7. **Medium** Explain why Staw and Hoang's NBA finding is an example of the sunk-cost trap.

Show answer

Teams gave more playing time to high draft picks because of what they had invested in choosing them, rather than because of how well the players were playing.

8. **Medium** One percent of players who try out for a team have a rare skill. A test identifies 95% of those who have it, but also wrongly flags 5% of those who do not. If a player is flagged, what is the chance he has the skill?

Show answer

$$0.0095 / (0.0095 + 0.0495) = 0.0095 / 0.059 \approx 0.161, \text{ about } 16\%.$$

9. **Hard** In problem 8, how would the answer change if 20% of the players trying out had the skill?

Show answer

$$0.19 / (0.19 + 0.04) = 0.19 / 0.23 \approx 0.826, \text{ about } 83\%. \text{ The base rate matters enormously.}$$

10. **Hard** A golfer has a 10-foot putt. Explain how loss aversion might make him play it differently if it is for par rather than for birdie, and why that is not the percentage play.

Show answer

Facing a bogey (a loss), he may concentrate harder or putt more firmly; facing a birdie (a gain), he may be more cautious and leave it short. But every stroke counts the same in the total score, so each putt should be played to minimize total strokes, regardless of par.

11. **Challenge** For a 50–50 choice of $+G$ or $-L$, show that a loss-averse player accepts only if $G > \lambda L$, while an expected-value player accepts if $G > L$. If $\lambda = 2.25$ and $L = 40$, how many good choices with $40 < G \leq 90$ does the loss-averse player refuse?

Show answer

Felt value $0.5G - 0.5\lambda L > 0 \iff G > \lambda L$; expected value $0.5G - 0.5L > 0 \iff G > L$. With $\lambda L = 90$, he refuses every choice with $40 < G \leq 90$, all of which have positive expected value.

COMMON MISTAKES

- Thinking loss aversion is always wrong. Avoiding ruin is wise (Lesson 2.5); the trap is refusing good bets of small size.
- Letting past effort decide a future choice.
- Trusting a test or a label without asking how common the thing is in the first place.

Make it a game • Name That Trap. Your parent reads five situations quickly: refusing to leave a bad movie because the ticket cost money, being too sure about a quiz, ignoring base rates, hating a loss more than loving an equal win. Name the trap in five seconds. Then compute how a loss-averse person values a 50-50 chance to win 10 or lose 10 pretend chips.

6.4 TIME AND CHANCE

OBJECTIVES

- Explain how the Bible speaks of chance and providence (Proverbs 16:33, Ecclesiastes 9:11, Acts 1:26, James 4:13-15), and how probability describes our knowledge rather than limiting God.
- Compute how often the better team or runner loses a single contest and a series, and the spread of wins over a season.
- Tell the stories of Pascal's faith and Eric Liddell's choice in 1924, and explain the difference between calculating odds and being ruled by them.

Warm-up. A “better” team wins 60% of its games against a rival. Roll a die for each game: 1–4 means the better team wins (about 67%, close enough for a warm-up), 5–6 means the rival wins. Play five best-of-seven series. How often did the better team lose a series?

WATCH

Eric Liddell: From Olympian to Martyred Missionary | Cody Crouch on TBN

Cody Crouch TV · 11:13

<https://www.youtube.com/watch?v=rZdGZLe-N2U>

WHEN TO WATCH

After the story paragraph. It tells Eric Liddell's life from the 1924 Olympics to his missionary work and death in China.

BACKUP VIDEO

The Flying Scotsman takes gold and sets a new record - Eric Liddell - Paris 1924 Olympic Games

Olympic Games · 1:12

<https://www.youtube.com/watch?v=IYNUxdolacA>

After the video, to see original film of Liddell's 400-meter victory in Paris.

KEY TERMS

providence

God's care and government of all things, including events that appear to us as chance.

chance (in probability)

A measure of our uncertainty about an outcome, given what we know.

upset

A win by the side with the lower chance of winning.

series

A set of games in which the first team to win a majority wins, such as a best-of-seven.

conviction

A settled belief about what is right that a person will act on even when it costs him.

CORE IDEAS

The story. In the summer of 1654 Blaise Pascal was corresponding with Pierre de Fermat about the problem of points (Lesson 1.1), the founding work of probability. On the night of November 23 of that same year he had an experience of God so powerful that he wrote it on a slip of parchment, beginning with the word “FIRE,” and sewed it into his coat, where it was found after his death. His later notes, published as the *Pensées*, include his famous “wager”: since we must live as if God exists or as if He does not, a reasonable person should weigh what is at stake. Pascal did not think faith was merely a calculation, but he saw no conflict between careful reasoning about chance and trust in God.

How they figured it out. A century ago, the Scottish runner Eric Liddell faced a different kind of decision. The heats of the 100 meters, his best event, were scheduled on a Sunday at the 1924 Paris Olympics, and Liddell, a devout Christian who kept the Lord's Day, would not run. He did not calculate his chances; he had already decided. He trained instead for the 400 meters, where he was not the favorite. Before the final, an American team masseur handed him a note drawn from 1 Samuel 2:30, where the LORD says, “them that honour me I will honour.” On July 11, 1924, Liddell won the 400 meters in a world-record 47.6 seconds. He also won

bronze in the 200. The next year he went to China as a missionary, and he died there in 1945 in a wartime internment camp.

The mathematics. Scripture is plain that the outcomes we call chance are not outside God's rule: "The lot is cast into the lap; but the whole disposing thereof is of the LORD" (Proverbs 16:33). When the apostles chose a successor to Judas, "they gave forth their lots; and the lot fell upon Matthias" (Acts 1:26). From where we stand, the lot was a 1-in-2 chance; from God's side, it was His choice. Probability measures our knowledge, not the limits of God's power. And it explains why "the race is not to the swift" (Ecclesiastes 9:11): a team that wins 60% of its games still loses 40% of them, and even a best-of-seven series gives the weaker team a 29% chance.

On the field. This course has taught you to do the percentages: take the high-percentage shot, avoid ruin, judge your decisions honestly. That is diligence, and Scripture commends it. But James warns against those who say "to morrow we will go into such a city ... and get gain," when "ye know not what shall be on the morrow," and tells us instead to say, "If the Lord will, we shall live, and do this, or that" (James 4:13-15). Liddell did both: he trained hard and ran to win, and he set the limits of his ambition by conviction, not by odds.

Think it through. Probability is a tool, not a master. Some choices are not decided by expected value at all, because they are matters of right and wrong. A wise person does the percentages wherever the percentages belong, and trusts God with the outcome.

PICTURE STUDY



Eric Liddell at the 1924 Olympic Games, Unknown photographer, published in *Le Miroir des sports*, 1924
· Photograph · Paris, France

Image: [Wikimedia Commons](#), public domain

This photograph appeared in the French sports magazine *Le Miroir des sports* after Liddell won the 400 meters. He was famous for an awkward running style, with his head thrown back, and he was not the favorite in the event. Nothing in the photograph shows the choice that brought him there: months earlier he had given up his best event rather than race on a Sunday.

Look closely

- Look at Liddell's face and posture. What do they suggest about him?
- Look at his simple kit. How does it compare with what Olympic runners wear today?
- Look at the background. What can you tell about the stadium and the crowd?

Talk about it

1. Liddell changed events and trained hard for the 400 meters. How does that show both conviction and diligence?
2. If he had lost the 400 meters, would his decision about the 100 meters have been wrong? Use Lesson 6.2.
3. What would you be unwilling to do in a game or sport, no matter how good the odds?

PROJECT

How Often Does the Swift Win?

You will measure, from real results, how often the favorite actually wins in a sport you follow, and compare it with the series mathematics of this lesson.

You will need

- Results from a free statistics site (Baseball-Reference, Basketball-Reference, Hockey-Reference) or a local league's published results
- Spreadsheet software
- Dice for the simulation

Steps

1. Choose a sport and a season. From the standings, mark which team in each of 30 games had the better record at the time (the “favorite”).
2. Count how often the favorite won those 30 games. Compute the percentage.
3. Choose at least 8 past playoff series and record whether the team with the better regular-season record won.
4. Compare: did favorites win series more often than single games? By how much?
5. Simulate 200 best-of-seven series in a spreadsheet with $p = 0.6$, and compare the favorite's series win rate with the exact 71.0%.
6. Repeat the simulation with $p = 0.55$ and $p = 0.65$.
7. Graph the chance of winning a series against the chance of winning one game for best-of-1, -3, -5 and -7.
8. Write a paragraph connecting your results to Ecclesiastes 9:11 and to the way a Christian athlete should think about winning and losing.

Tips

- Use the team's record before the game, not after, so you are not using information from the future.
- In a spreadsheet, a game won by the favorite is $= RAND() < 0.6$.

Check your work

- ☐ I recorded at least 30 real games and 8 real series.
- ☐ I simulated at least 200 series and compared them with the formula.
- ☐ My paragraph connects the numbers to Scripture thoughtfully and briefly.

SCOREBOOK

1. Make a table of series win chances for $p = 0.55, 0.6$ and 0.65 in best-of-1, -3, -5 and -7.
2. Write Proverbs 16:33 and James 4:15 in your scorebook, and beside them one sentence on how they shape the way you play.
3. “What I would decide and why”: describe a choice in sport where you would refuse a high-percentage play because it would be wrong.

KEY FORMULAS

$$P(\text{win best-of-7}) = \sum_{k=0}^3 \binom{3+k}{k} p^4 (1-p)^k$$

$$\sigma_{\text{wins}} = \sqrt{np(1-p)}$$

From history. In the Weihsien internment camp, Liddell taught science to the children, organized games and races for them, and was much loved. He died there of a brain tumor on February 21, 1945, a few months before the camp was liberated.

WORKED EXAMPLES

Example 1. A team wins each game against a rival with probability 0.6 , independently. Find its chance of winning a best-of-seven series.

Show solution

1. It must win 4 games before losing 4. It wins in exactly $4 + k$ games (k losses, the last game a win) in $\binom{3+k}{k}$ ways.
2. $k = 0 : 0.6^4 = 0.1296$; $k = 1 : 4(0.6^4)(0.4) = 0.2074$;
 $k = 2 : 10(0.6^4)(0.4^2) = 0.2074$; $k = 3 : 20(0.6^4)(0.4^3) = 0.1659$.
3. Sum: $0.1296 + 0.2074 + 0.2074 + 0.1659 = 0.710$.

Answer: About 71%. The weaker team still wins the series 29% of the time.

Example 2. A team that truly wins 60% of its games plays a 162-game season. Find the expected number of wins and the spread from luck.

Show solution

1. Expected wins: $162 \times 0.6 = 97.2$.
2. Spread: $\sqrt{162 \times 0.6 \times 0.4} = \sqrt{38.88} \approx 6.2$ wins.

Answer: About 97 wins, give or take about 6 from luck alone. Two teams of exactly equal skill can easily finish 10 games apart.

Example 3. In Acts 1, two men were put forward, Joseph called Barsabas and Matthias, and lots were cast. From the apostles' point of view, what was each man's chance? What does Proverbs 16:33 add?

Show solution

1. Two candidates, one lot: each had probability $\frac{1}{2}$.
2. Proverbs 16:33 says the whole disposing of the lot is of the LORD: the outcome was uncertain to the apostles but not to God.

Answer: $\frac{1}{2}$ each, from the human point of view. Probability describes what we know; it does not mean that no one is in charge.

PRACTICE

1. Easy Quote Ecclesiastes 9:11 from “the race” to “the strong,” and explain it in one sentence of probability.

Show answer

“The race is not to the swift, nor the battle to the strong.” The better side has a higher chance of winning, but not certainty.

2. Easy Why did Eric Liddell not run the 100 meters in Paris in 1924?

Show answer

The heats were held on a Sunday, and he would not race on the Lord's Day.

3. Easy What did Liddell win in the 400 meters, and in what time?

Show answer

The gold medal, in a world-record 47.6 seconds.

4. **Medium** A team wins each game with probability 0.6. Find its chance of winning a best-of-three series.

Show answer

$$0.6^2 + 2(0.6^2)(0.4) = 0.36 + 0.288 = 0.648.$$

5. **Medium** For the same team, the chance of winning a best-of-five series is 0.683. Compare single game, best-of-three, best-of-five and best-of-seven. What pattern do you see?

Show answer

0.600, 0.648, 0.683, 0.710: longer series favor the better team more, because luck has more chances to even out (the law of large numbers).

6. **Medium** Find the spread of wins from luck for a team that truly wins 50% of its games over an 82-game season.

Show answer

$$\sqrt{82 \times 0.5 \times 0.5} = \sqrt{20.5} \approx 4.5 \text{ wins.}$$

7. **Medium** What did Pascal experience on November 23, 1654, and why is the date significant for this course?

Show answer

A powerful experience of God, recorded in his “Memorial”; it came in the same year as his letters with Fermat that founded probability theory.

8. **Medium** Explain in your own words how a Christian can believe both that the lot is “of the LORD” and that a coin lands heads with probability $\frac{1}{2}$.

Show answer

Probability describes our knowledge and the long-run pattern we can observe; providence describes God’s rule over every event. The two are different descriptions, not rivals.

9. **Hard** Two runners: A finishes ahead of B in 70% of their races. In a three-race championship decided by most wins, what is B’s chance?

Show answer

$$B \text{ needs at least 2 of 3: } 0.3^3 + 3(0.3^2)(0.7) = 0.027 + 0.189 = 0.216.$$

10. **Hard** James 4:13-15 warns against confident plans for tomorrow. Explain how a decision journal (Lesson 6.2) fits the spirit of this passage.

Show answer

A journal records plans with honest probabilities and reviews them humbly, admitting uncertainty about tomorrow. It keeps diligence and humility together: we plan carefully but say “if the Lord will.”

11. **Challenge** Show that in a best-of-seven series between equal teams ($p = 0.5$), the chance the series goes the full seven games is $20/64 = 0.3125$.

Show answer

The series reaches game 7 exactly when the first six games split 3–3:
 $\binom{6}{3}(0.5)^6 = 20/64 = 0.3125$.

COMMON MISTAKES

- Thinking that “chance” means that nobody is in control. In Scripture, the lot is cast, but its disposing is of the LORD.
- Thinking that because outcomes are uncertain, preparation does not matter. It changes the odds even when it cannot guarantee the result.
- Using expected value for questions that are really about right and wrong.

Make it a game • Verse Match. Write each reference on one card, Proverbs 16:33, Ecclesiastes 9:11, Acts 1:26 and James 4:13-15, and each key idea on another. Shuffle them face down and match them. Read each verse aloud from your Bible. Then say in your own words how probability describes what we know while God remains the one who governs the outcome.

Scripture. “I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, nor yet favour to men of skill; but time and chance happeneth to them all.” (Ecclesiastes 9:11, KJV)

The better side has the better odds, not a guarantee. We prepare diligently and receive the outcome humbly.

6.5 CAPSTONE: YOUR ODDS REPORT

OBJECTIVES

- Plan and carry out an original study of a decision in a sport or game: gather data, compute probabilities and expected values, and measure uncertainty.
- Compute a margin of error for a proportion and the sample size needed for a given precision, and test how sensitive a recommendation is to its assumptions.
- Present a clear, honest recommendation with a chart, stating both the percentage and the risk.

Warm-up. Take 20 shots with a soft ball at a basket or bucket from a marked spot and record makes. Then estimate: if you took 1,000 shots from there, what range would your make rate almost certainly fall in? Write your range before you learn the formula.

WATCH

Telling Stories with Data in 3 Steps (Quick Study).

Harvard Business Review • 4:47

https://www.youtube.com/watch?v=r5_34YnCmMY

WHEN TO WATCH

Before you begin writing your report. It shows how to turn a chart into a story with a setup, a conflict and a resolution.

BACKUP VIDEO

Avoid these 5 slide mistakes

storytelling with data • 9:10

<https://www.youtube.com/watch?v=CfBHKy2sebQ>

Before you make your slides or poster, to avoid common mistakes in presenting data.

KEY TERMS

margin of error

For a proportion $\hat{p}$ from n trials, about $2\sqrt{\hat{p}(1 - \hat{p})/n}$; the true rate is very likely within this distance.

sample size

The number of trials or observations in a study.

sensitivity analysis

Checking whether a recommendation still holds when the assumptions change within a reasonable range.

recommendation

A clear statement of what should be done, supported by the numbers and honest about uncertainty.

CORE IDEAS

The story. In 1858 Florence Nightingale, back from nursing British soldiers in the Crimean War, published a report on the health of the army. She had collected the numbers of soldiers who died each month in the military hospitals and sorted them by cause. Most had died not of wounds but of diseases spread by poor sanitation. To make the numbers impossible to ignore, she drew them as a new kind of chart, wedges of a circle for each month, with the preventable deaths shaded in blue. The charts helped persuade the government to reform army hospitals, and that same year she was elected to the Statistical Society of London. Her report was an odds report: data gathered, analyzed, drawn clearly and turned into a recommendation.

How they figured it out. Every lesson in this course has followed the same path: ask a clear question, count, compute, compare, decide. Romer asked whether teams should go for it on fourth down; Palacios-Huerta asked whether penalty takers play minimax; Halley asked how long the people of Breslau lived. A good report does the same and adds two things: an honest statement of uncertainty (how sure are we?) and a clear presentation that a coach, a teammate or a parent can understand.

The mathematics. If you make 60 of 100 shots, your true rate is probably not exactly 60%. The **margin of error** is about $2\sqrt{0.6 \times 0.4/100} \approx 0.098$, so your true rate is very likely between about 50% and 70%. To cut the margin of error to E , you need about $n = 4\hat{p}(1 - \hat{p})/E^2$ trials. A **sensitivity analysis** asks whether the recommendation changes anywhere inside that range. If it does not, you can recommend with confidence; if it does, say so and gather more data.

On the field. Your capstone project is your own odds report on a sport or game. Choose one decision that recurs (where to shoot from, when to go for it, which columns to choose in *Can't Stop*, when to accept a double, whether to serve wide or down the middle), gather at least 200 real observations, compute the expected value of each option, state the risk side, test the sensitivity, and present your recommendation in a five-minute talk with one clear chart. *Fabricando fit faber*: by making, one becomes a maker. By analyzing, you become an analyst.

The risk side. The commonest failures in an odds report are too little data, a chart that hides the point, and a conclusion stronger than the evidence. Say what you found, how sure you are, and what would change your mind. “Prove all things; hold fast that which is good” (1 Thessalonians 5:21).

PICTURE STUDY

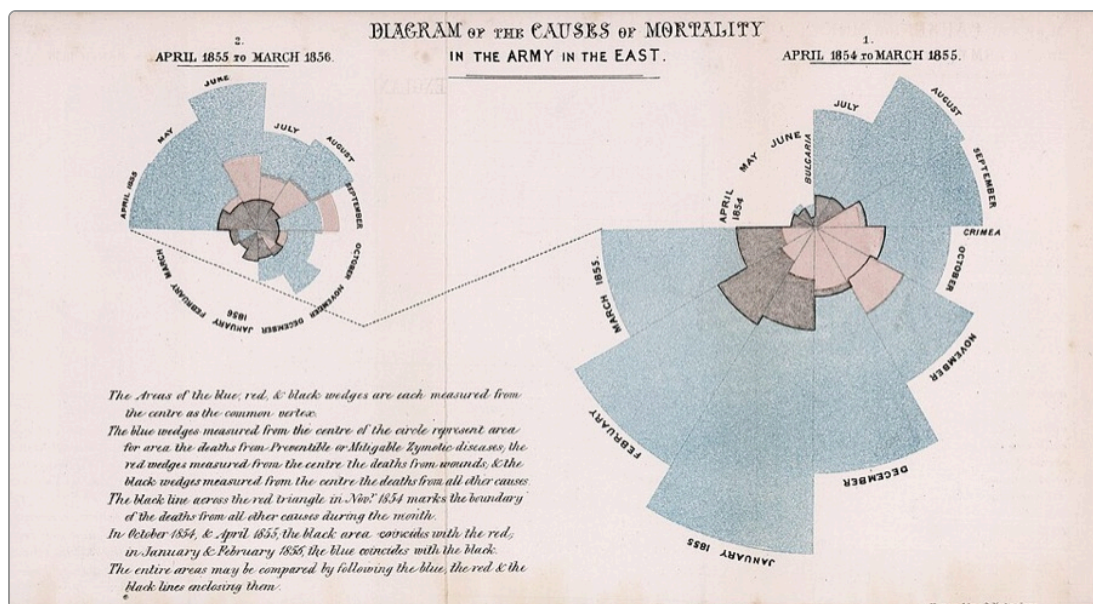


Diagram of the Causes of Mortality in the Army in the East, Florence Nightingale, 1858 · Color lithograph · From Notes on Matters Affecting the Health, Efficiency, and Hospital Administration of the British Army, London

Image: [Wikimedia Commons](#), public domain

Florence Nightingale drew these “rose” diagrams to show the monthly deaths in the British army hospitals during the Crimean War, two years of data side by side. The area of each wedge shows the number of deaths. The blue areas, deaths from diseases that better sanitation could prevent, dwarf the others. The right-hand diagram shows the year before sanitary improvements; the left, the year after.

Look closely

- Find the two circles. Each wedge is a month. Which months have the largest wedges?
- Find the blue, red and black areas. What does each color stand for?
- Compare the two diagrams. What changed after sanitary improvements were made?
- Read the text between the diagrams. Why did Nightingale explain how to read the chart?

Talk about it

1. Why might a picture persuade busy officials where a table of numbers would not?
2. What is the one message of this chart, in a single sentence?
3. What one chart will carry the main message of your own odds report?

Safety first. For any shooting, kicking or throwing experiment, warm up first, use a soft ball indoors, keep clear of windows and people, and have a parent supervise any trip to a court, field or alley.

PROJECT

Your Odds Report

This is the capstone of the course. You will choose a recurring decision in a sport or game, gather at least 200 real observations, compute the expected value and risk of each option, test the sensitivity of your conclusion, and present a recommendation in writing and in a five-minute talk.

You will need

- Your scorebook and project data from earlier lessons
- A soft ball and basket, dice, cards or a game board, depending on your topic
- Spreadsheet software for tables, charts and simulations
- Free statistics sites if you use real league data (with a parent's help)

Steps

1. **Choose a question.** One recurring decision with two or three options.
Examples: shoot from 3 m or 5 m in backyard basketball; go for it or kick on fourth down at a given spot in real NFL data; which columns to choose in *Can't Stop*; when to hold in Pig; where to aim a penalty kick against a particular keeper.
2. Write the question and your prediction in your scorebook before collecting any data.
3. **Plan the data.** Decide how many observations you need, using $n \approx 4\hat{p}(1 - \hat{p})/E^2$ for a target margin of error, and plan at least 200 in all.
4. **Gather the data.** Record every observation, dated, without discarding any.
5. **Compute.** For each option, find the success rate, its margin of error and the expected value (points per attempt, chance of winning, expected score).
6. **Weigh the risk.** For each option, state the worst realistic outcome and how often it happens.
7. **Test sensitivity.** Recompute your recommendation at the edges of your margins of error. Does it change?

8. **Make one chart** that carries the main message, with a title that states the finding.
9. **Write the report** (2–4 pages): question, method, data, results, risk, sensitivity, recommendation, and what would change your mind.
10. **Present** it in five minutes to your family or class, and answer questions.
11. Revise the report in light of the questions you were asked.

Tips

- Choose a question you can gather data on yourself. A modest, well-measured question beats a grand, unmeasured one.
- State your uncertainty plainly. “Probably better, by about 0.2 points per shot, give or take 0.2” is more honest and more useful than “definitely better.”
- Keep every score in points, never in money.

Check your work

- ☐ My report has at least 200 recorded observations and a margin of error for each option.
- ☐ I computed an expected value for each option, stated the risk side, and ran a sensitivity check.
- ☐ My chart has a title that states the finding, labelled axes and the sample size.
- ☐ I presented the report and revised it after questions.

SCOREBOOK

1. Sketch three possible charts for your report and choose the clearest.
2. Write your question, prediction and data plan on one page before collecting data.
3. “What I would decide and why”: your final recommendation in three sentences, including the percentage and the risk.

KEY FORMULAS

$$\text{margin of error} \approx 2\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

$$n \approx \frac{4\hat{p}(1 - \hat{p})}{E^2}$$

$$E[\text{option}] = \sum p_i x_i$$

From history. Nightingale was one of the first to use statistical graphics to change public policy. She corresponded with the leading statisticians of her day and argued that governments should collect and publish health statistics regularly, as nations now do.

WORKED EXAMPLES

Example 1. A student makes 60 of 100 free throws with a soft ball. Find the margin of error and the likely range of his true rate.

Show solution

1. $\hat{p} = 0.60$.

2. $2\sqrt{0.6 \times 0.4/100} = 2\sqrt{0.0024} = 2(0.049) = 0.098$.

3. Range: 0.60 ± 0.098 , about 50% to 70%.

Answer: His true rate is very likely between about 50% and 70%.

Example 2. How many shots does he need to bring the margin of error down to 3 percentage points, if his rate is about 60%?

Show solution

1. $n \approx 4(0.6)(0.4)/0.03^2 = 0.96/0.0009 \approx 1,067$.

Answer: About 1,070 shots. Precision is expensive: cutting the margin by a factor of about 3 takes about 10 times as many trials.

Example 3. A student compares two shots in backyard basketball: a 2-point shot made 60 of 100 times and a 3-point shot made 30 of 100 times. Compute the points per shot, the margins of error, and test whether the recommendation is sensitive to the uncertainty.

Show solution

1. Points per shot: 2-pointer $2(0.60) = 1.20$; 3-pointer $3(0.30) = 0.90$.
2. Margins: 2-pointer ± 0.098 in rate, so ± 0.196 points; 3-pointer $2\sqrt{0.3 \times 0.7/100} = 0.092$ in rate, so ± 0.275 points.
3. Worst case for the 2-pointer: $1.20 - 0.196 = 1.00$; best case for the 3-pointer: $0.90 + 0.275 = 1.18$. The ranges overlap.

Answer: The 2-pointer is better on the evidence (1.20 against 0.90 points per shot), but the ranges overlap, so the honest report says “probably” and recommends more shots to confirm.

PRACTICE

1. **Easy** What did Florence Nightingale’s 1858 charts show, and what did they help bring about?

Show answer

That most soldiers in the military hospitals died of preventable disease rather than wounds; they helped persuade the government to reform army sanitation.

2. **Easy** Find the margin of error for 50 makes in 100 attempts.

Show answer

$$2\sqrt{0.5 \times 0.5/100} = 2(0.05) = 0.10, 10 \text{ percentage points.}$$

3. **Easy** List the steps of an odds report in order.

Show answer

Question, data, computation (probabilities and expected values), uncertainty and sensitivity, risk side, recommendation, presentation.

4. **Medium** Find the margin of error for 30 makes in 100 attempts and for 300 in 1,000.

Show answer

$$100: 2\sqrt{0.21/100} \approx 0.092; 1,000: 2\sqrt{0.21/1000} \approx 0.029.$$

5. **Medium** How many trials are needed for a margin of error of 5 percentage points when the rate is near 50%?

Show answer

$$4(0.25)/0.0025 = 400.$$

6. **Medium** An extra point is made 94% of the time and a two-point try succeeds about half the time. Find the break-even two-point rate and decide whether a two-point rate measured as $50\% \pm 10\%$ settles the question.

Show answer

Break-even: $2p = 0.94 \Rightarrow p = 0.47$. The range 40% to 60% includes 0.47, so the data do not settle it; more data are needed.

7. **Medium** Explain what a sensitivity analysis is and why it makes a recommendation more trustworthy.

Show answer

It checks whether the recommendation stays the same when the assumptions change within reasonable limits. If it does, small errors in the data cannot overturn it.

8. **Medium** Give two features of a good chart for an odds report.

Show answer

One clear message stated in the title; labelled axes with units; only the colors needed to highlight the key comparison; the source and sample size shown.

9. **Hard** A Can't Stop player recorded 250 turns choosing middle columns and busted on 46 of them. Find the bust rate and its margin of error, and compare with the theoretical 8.0% for columns 6, 7, 8 on a single roll. Why might the turn bust rate be higher?

Show answer

$46/250 = 0.184$; margin $2\sqrt{0.184 \times 0.816/250} \approx 0.049$. A turn includes several rolls, so the chance of busting during a turn is higher than on a single roll.

10. **Hard** Two penalty-kick directions scored 41 of 50 (left) and 37 of 50 (right) in a backyard study. Compute each rate and margin of error. Can the student conclude that left is better?

Show answer

Left $82\% \pm 2\sqrt{0.82 \times 0.18/50} \approx 0.109$; right $74\% \pm 2\sqrt{0.74 \times 0.26/50} \approx 0.124$. The ranges overlap widely, so he cannot conclude that left is better; and if he always kicked left, the keeper would adjust (Lesson 4.3).

11. **Challenge** Show that the margin of error $2\sqrt{p(1-p)/n}$ is largest when $p = 0.5$, and explain why studies often plan their sample size with $p = 0.5$.

Show answer

$p(1-p) = p - p^2$ is a downward parabola with maximum where $1 - 2p = 0$, so $p = 0.5$. Planning with 0.5 gives the largest needed sample, so the study will be precise enough whatever the true rate turns out to be.

COMMON MISTAKES

- Collecting too little data and stating the conclusion too strongly.
- Throwing out inconvenient trials. Keep every observation, and explain any you cannot use.
- Making a chart with many colors and no clear point.
- Forgetting the risk side: an option with a slightly higher average and a much worse worst case may not be the better choice.

Make it a game • Peer Review Hot Seat. Pitch your Odds Report in ninety seconds. Your parent is a skeptical reviewer who asks five questions: What is your sample? What is your margin of error? What did you assume? What would change your answer? What remains uncertain? Answer each in under twenty seconds. Earn a point for each solid answer, and aim for five.

Scripture. “The thoughts of the diligent tend only to plenteousness; but of every one that is hasty only to want.” (Proverbs 21:5, KJV)

A careful report, built patiently from real data, is the diligent path; a hasty conclusion from a few trials is the other.

WISDOM, RISK AND PROVIDENCE

The last unit turns from the field and the game table to the player himself. You will read Halley's life table and learn why small risks repeated thousands of times matter more than dramatic rare ones; judge decisions by their quality rather than their results, using Super Bowl XLIX's final pass; meet the thinking traps that catch even experts; and consider how probability fits with providence, through Pascal's faith and Eric Liddell's choice in 1924. The course ends with your own odds report.

This unit asks for maturity. The mathematics is the same (probability, expected value, conditional probability, the law of large numbers), but now it is applied to habits, to judgment and to character. The goal of the whole course is a person who does the percentages diligently, keeps honest records, plays lawfully and graciously, and trusts God with the outcome.

PREREQUISITES

- Expected value, variance and risk of ruin (Unit 2)
- Conditional probability and Bayes's idea (Lesson 1.4)
- Regression to the mean and the spread from luck (Lesson 4.4)
- A spreadsheet with $= RAND()$ for simulations

MASTERY CHECKLIST

- ☐ I can compute survival chances from a life table and the cumulative risk of a repeated small risk.
- ☐ I can separate decision quality from outcome and keep an honest decision journal.
- ☐ I can explain loss aversion, sunk cost, overconfidence and base-rate neglect, and correct base-rate neglect with Bayes's rule.
- ☐ I can compute series and season probabilities and explain how Scripture speaks of chance and providence.
- ☐ I can produce an odds report with data, expected values, a margin of error, a sensitivity check and a clear chart.

OPTIONAL BOOKS AND GAMES FOR THIS UNIT

- **Pascal's *Pensées* (Project Gutenberg)**: Free. Pascal's notes on faith and reason in English translation, including the wager discussed in Lesson 6.4. Best read with a parent; some passages are difficult. Ages 15+.
- **Laplace, *A Philosophical Essay on Probabilities* (Project Gutenberg)**: Free. A classic 1814 essay that explains probability in words, with chapters on life tables, testimony and errors in judgment that fit Lessons 6.1 and 6.3. Ages 15+.
- **Retrosheet**: Free. Play-by-play baseball data going back more than a century, a rich source for a capstone odds report in Lesson 6.5. Ages 13+.

Not required. Every project in this unit works without them. School Launch has no ties to these authors, publishers or companies.

LESSONS IN THIS UNIT

6.1	Life Tables and Seat Belts	3 sessions of 50 minutes
6.2	Judge the Decision, Not the Result	3 sessions of 50 minutes
6.3	Traps in Our Thinking	3 sessions of 50 minutes
6.4	Time and Chance	3 sessions of 50 minutes
6.5	Capstone: Your Odds Report	3 sessions of 50 minutes

UNIT 6 TEST

Calculators are allowed. Show your work for every calculation: write the formula, substitute the numbers, and give the answer with a sentence explaining it. Answer the concept questions in complete sentences. The test has 15 problems worth a total of 100 points.

1. **6 pts**

From Halley's table (age 20: 598; age 40: 445), find the chance a 20-year-old would reach 40.

2. (7 pts)

A risk of 1 in 2,000 per occasion is repeated 1,000 times. Find the chance of at least one occurrence.

3. (6 pts)

A seat belt reduces the risk of fatal injury to front-seat occupants by about 45%. If a trip carried a risk p without a belt, what is it with one? Why does the habit matter even on short trips?

4. (6 pts)

Explain the difference between decision quality and outcome, using the four-box grid.

5. (8 pts)

From the 1-yard line, a run scores 55% and a pass scores 45%, falls incomplete 53% and is intercepted 2%. Passing first allows three plays; running first allows two. Compare the chance of scoring.

6. (6 pts)

What is hindsight bias? How does a decision journal guard against it?

7. (7 pts)

With $\lambda = 2$, find the expected value and the felt value of a 50–50 chance of +120 or –80 points.

8. (8 pts)

3% of prospects become starters. A tryout drill flags 80% of future starters and 15% of non-starters. If a prospect is flagged, what is the chance he becomes a starter?

9. (6 pts)

Define a sunk cost and explain why it should not affect the next decision, with one sports example.

10. (7 pts)

A team wins each game with probability 0.6. Find its chance of winning a best-of-three series.

11. (6 pts)

Find the spread of wins from luck for a 0.6 team over 162 games.

12. (6 pts)

Why did Eric Liddell switch from the 100 meters to the 400 meters in 1924, and what happened?

13. (7 pts)

A student makes 70 of 100 shots. Find the margin of error and the likely range of his true rate.

14. (6 pts)

How many attempts are needed for a margin of error of 4 percentage points when the rate is near 50%?

15. (8 pts)

Short essay (one paragraph): Using Proverbs 16:33 or Ecclesiastes 9:11 and one example from this course, explain how a Christian athlete can both play the percentages diligently and trust God with the outcome.

ANSWER KEY

1. $445/598 \approx 0.744$.
2. $1 - (0.9995)^{1000} \approx 1 - e^{-0.5} = 0.393$.
3. $0.55p$. The habit is repeated thousands of times, and the cumulative risk depends on every trip.
4. Decision quality is judged by the reasoning and information at the time; the outcome includes luck. The grid: good decision/good outcome, good decision/bad outcome, bad decision/good outcome, bad decision/bad outcome.
5. Run, run: $1 - 0.45^2 = 0.7975$. Pass first: $0.45 + 0.53(0.7975) = 0.873$. Passing first is better on these numbers.
6. The belief, after learning a result, that it was predictable all along. A journal records probabilities and reasons before the result, so they cannot be rewritten afterward.
7. EV: $0.5(120) - 0.5(80) = 20$; felt: $60 - 80 = -20$. A loss-averse person refuses a favorable choice.
8. $0.024/(0.024 + 0.1455) = 0.024/0.1695 \approx 0.142$, about 14%.
9. A cost already paid that cannot be recovered. It is the same whichever choice is made next, so only future costs and benefits matter. Example: continuing a failing game plan because of practice time already spent.
10. $0.36 + 2(0.36)(0.4) = 0.648$.
11. $\sqrt{162(0.6)(0.4)} \approx 6.2$ wins.
12. The 100-meter heats were on a Sunday, which he would not race on. He trained for the 400 meters and won the gold medal in a world-record 47.6 seconds.
13. $2\sqrt{0.7 \times 0.3/100} \approx 0.092$; about 61% to 79%.
14. $4(0.25)/0.0016 = 625$.
15. Answers vary. A strong essay explains that probability describes our knowledge and the long-run pattern while God governs every outcome; uses a specific example (taking the high-percentage shot, judging a decision by its quality, Liddell's conviction); and concludes that diligence and humility belong together.

PORTFOLIO PIECE: THE ODDS REPORT AND DECISION JOURNAL

Submit your capstone odds report from Lesson 6.5 (2–4 pages with at least 200 observations, expected values, margins of error, a sensitivity check and one clear chart), together with your two-week decision journal from Lesson 6.2 and a one-page reflection. The reflection should name one decision in your journal that turned out badly but was sound, one that turned out well but was not, and one way this course has changed how you play, practice or decide.

LOOK FOR	WHAT IT MEANS
Data and computation	At least 200 recorded observations; correct success rates, expected values and margins of error for each option.
Uncertainty and risk	A sensitivity check is done and reported honestly; the risk side of each option is stated.
Presentation	One clear chart with a title that states the finding; a well-organized report and a confident five-minute talk.
Decision journal and reflection	At least 20 journal entries with probabilities recorded before results; a thoughtful reflection that separates decision quality from outcome.
Character	Honest records with no discarded trials, points only, and a gracious, humble conclusion.

MIDTERM EXAM

Covers Units 1–3. Time: 100 minutes.

You have 100 minutes. A calculator is allowed; no notes. Show your work for every calculation: write the sample space, formula or table you used, substitute the numbers, and give probabilities as fractions, decimals or percents as asked; correct work earns partial credit even if the final number slips. Use the run-expectancy values given in the problems. Answer the concept questions in complete sentences. The exam has 26 problems worth a total of 100 points.

1. **3 pts** Lesson 1.1

A card is drawn from a standard 52-card deck. Find the probability that it is a king or a queen, as a fraction, a decimal and a percent.

2. **4 pts** Lesson 1.1

Two evenly matched players play a match to 4 wins for 80 points. The game stops with A leading 2–1. Divide the points fairly by Fermat's method.

3. **3 pts** Lesson 1.2

Two dice are rolled. Find the probability that the sum is 9.

4. **4 pts** Lesson 1.2

A Catan settlement touches hexes numbered 6, 8 and 10. How many pips does it have, how often does it collect per roll, and about how many cards should it collect in 72 rolls?

5. **4 pts** Lesson 1.2

A coach has 9 players. In how many ways can she choose 4 for a relay team (order not mattering)? In how many ways can she choose and order 4 runners?

6. **3 pts** Lesson 1.3

An 85% free-throw shooter takes 4 independent shots. Find the probability she makes all four.

7. **4 pts** Lesson 1.3

Find the probability of at least one 6 in 6 rolls of a die.

8. **4 pts** Lesson 1.3

A 40% three-point shooter has missed five in a row. A fan says she is “due”; another says she is “cold.” If her shots are independent, explain who is right and why streaks like this happen.

9. **3 pts** Lesson 1.4

A Monty Hall game has 5 doors and one car. You pick a door; the host, who knows where the car is, opens 3 goat doors, leaving your door and one other. What is the probability of winning if you switch?

10. **4 pts** Lesson 1.4

In practice a player took 40 layups and made 30, and 60 jump shots and made 24. Given that a shot was made, what is the probability it was a layup? Given that it was a layup, what is the probability it was made?

11. **4 pts** Lesson 1.4

A bag holds 4 dice: one weighted to show a 6 half the time, three fair. You draw one at random and roll a 6. What is the probability you drew the weighted die?

12. **4 pts** Lesson 1.5

A shooter makes 64 of 100 shots. Find the standard error of her shooting percentage and the range that likely contains her true rate (about 95%).

13. **3 pts** Lesson 2.1

A spinner scores 0, 2 or 5 points with probabilities 0.5, 0.3 and 0.2. Find its expected value, and the expected total over 40 spins.

14. **4 pts** Lesson 2.2

A player makes 57% of her two-point shots. What three-point percentage gives the same points per attempt? If she makes 36% of threes, which is better?

15. **4 pts** Lesson 2.2

A kicker makes 93% of extra points and a team converts 49% of two-point tries. Compare expected points and find the break-even two-point rate.

16. (4 pts) Lesson 2.3

Find the mean, variance and standard deviation of a two-point shot made 60% of the time.

17. (4 pts) Lesson 2.3

A team trails by 3 points with time for one shot. Its two-point shot goes in 60% of the time; its three goes in 33% of the time. Which shot has more expected points, which should it take, and why?

18. (4 pts) Lesson 2.4

A game costs 2 points to play. Two dice are rolled; if both show 6 the player receives 30 points. Find the expected net result per play and the house edge.

19. (4 pts) Lesson 2.4

A family's boat is worth 24,000 dollars. Suppose the chance of losing it in a year is 1 in 400, and insurance costs 90 dollars a year. Find the expected loss without insurance and the expected net cost of insuring. Why might insuring still be wise?

20. (4 pts) Lesson 2.5

A plan has a 4% chance of ruin each time it is used, independently. Find the chance of surviving 25 uses.

21. (3 pts) Lesson 2.5

In the St. Petersburg game the bank can pay at most 2^{12} coins. Find the expected value.

22. (4 pts) Lesson 3.1

A player has 150 hits in 450 at bats, with 60 walks, 5 hit-by-pitches and 5 sacrifice flies. Find his BA and OBP.

23. (4 pts) Lesson 3.2

Use the run-expectancy values: runner on 1st, 1 out = 0.51; runner on 2nd, 1 out = 0.66; bases empty, 2 outs = 0.10. Find the break-even success rate for stealing second with 1 out.

24. **4 pts** Lesson 3.3

A team's shots have xG 0.3, 0.3 and 0.1. Find its expected goals and the probability it scores at least once (assume independence).

25. **4 pts** Lesson 3.5

A tennis player gets 64% of first serves in and wins 73% of those points; she gets 91% of second serves in and wins 54% of those points. Find her probability of winning a service point and of double-faulting.

26. **6 pts** Lesson essay

Short essay. In about 150 words, explain what “the high-percentage shot” means, using at least three ideas from Units 1–3 (for example expected value, variance, the law of large numbers, base rates, break-even rates). Give one example from a sport and one from a family game, and name one situation in which the high-percentage choice is not the safest one.

ANSWER KEY

1. $\frac{8}{52} = \frac{2}{13} \approx 0.154 \approx 15.4\%$.
2. A needs 2, B needs 3: at most 4 more rounds, 16 sequences. B wins with at least 3 B's: $4 + 1 = 5$ sequences, so $P(B) = \frac{5}{16}$, $P(A) = \frac{11}{16}$. A gets 55 points, B gets 25.
3. Sum 9: (3,6), (4,5), (5,4), (6,3). $\frac{4}{36} = \frac{1}{9} \approx 11.1\%$.
4. Pips $5 + 5 + 3 = 13$; $\frac{13}{36} \approx 0.361$ per roll; $72 \times \frac{13}{36} = 26$ cards.
5. $\binom{9}{4} = 126$; $9 \cdot 8 \cdot 7 \cdot 6 = 3,024$.
6. $0.85^4 \approx 0.522$.
7. $1 - (5/6)^6 \approx 1 - 0.335 = 0.665$.
8. Neither: if the shots are independent her chance on the next shot is still 40%. Five misses in a row happen about $0.6^5 \approx 7.8\%$ of the time, so over a season such streaks are certain by chance alone. Believing she is "due" is the gambler's fallacy.
9. Your first pick is right with probability $\frac{1}{5}$; otherwise the car is behind the other closed door. Switching wins $\frac{4}{5}$.
10. $P(\text{layup} \mid \text{make}) = \frac{30}{54} \approx 55.6\%$; $P(\text{make} \mid \text{layup}) = \frac{30}{40} = 75\%$.
11. Weighted path $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$; fair paths $\frac{3}{4} \cdot \frac{1}{6} = \frac{1}{8}$. $P(W \mid 6) = \frac{1/8}{1/8 + 1/8} = \frac{1}{2}$.
12. $\sqrt{0.64 \times 0.36/100} = 0.048$; range about 0.544 to 0.736.
13. $0 + 0.6 + 1.0 = 1.6$ points per spin; $40 \times 1.6 = 64$ points.
14. Break-even $\frac{2}{3}(0.57) = 0.38$. Two 1.14; three 1.08. The two is better for her.
15. Kick 0.93; two-point try 0.98. Break-even $0.93/2 = 46.5\%$.
16. $\mu = 1.2$; $\text{Var} = 2^2(0.6)(0.4) = 0.96$; $\sigma \approx 0.98$.
17. Two: 1.20; three: 0.99. It should take the three, because only the three can tie the game (33% against 0%). When behind, the team needs variance, not average.
18. $\frac{1}{36}(30) - 2 = 0.833 - 2 \approx -1.17$ points per play; house edge $1.17/2 \approx 58\%$.
19. Expected loss $24,000/400 = 60$ dollars; net cost $90 - 60 = 30$ dollars a year. Insuring is wise if losing the boat would be a heavy blow: a small, sure cost removes the risk of a large loss.
20. $0.96^{25} \approx 0.360$, about 36%.
21. Tosses 1 to 12 contribute 1 each, and the capped tail contributes 1: $E = 13$ coins.
22. $BA = 150/450 \approx .333$; $OBP = \frac{215}{520} \approx .413$.

23. $\frac{0.51 - 0.10}{0.66 - 0.10} = \frac{0.41}{0.56} \approx 73\%$.

24. $xG\ 0.7$; $P(0) = 0.7 \times 0.7 \times 0.9 = 0.441$, so $P(\geq 1) \approx 0.559$.

25. $0.64(0.73) + 0.36(0.91)(0.54) = 0.4672 + 0.1769 \approx 0.644$; double fault
 $0.36 \times 0.09 \approx 0.032$.

26. Answers vary. A strong essay defines the high-percentage shot as the option with the highest expected value, uses at least three ideas correctly, gives two concrete examples (for example threes against long twos, Catan spots by pips, steals above break-even), and explains a case where the best choice adds variance (trailing late) or where the safe-looking choice lowers the chance of winning.

FINAL EXAM

Covers Units 1–6. Time: 120 minutes.

You have 120 minutes. A calculator is allowed; no notes. Show your work for every calculation: write the sample space, formula or table you used, substitute the numbers, and give probabilities as fractions, decimals or percents; correct work earns partial credit even if the final number slips. Use the values given in each problem. Answer the concept questions in complete sentences. About a third of the exam reviews Units 1–3 and the rest covers Units 4–6. The exam has 26 problems worth a total of 100 points.

1. **3 pts** Lesson 1.1

Two evenly matched players play a game to 3 wins for 64 points. The game is stopped with A leading 2–1. Divide the 64 points fairly by Pascal and Fermat's method.

2. **3 pts** Lesson 1.2

Two dice are rolled. Find the probability of a total of 7, and the probability of a total of 6 or 8 (the red numbers in *Catan*).

3. **4 pts** Lesson 1.4

Two dice are rolled, and you are told that at least one die shows a 6. What is the probability that the total is 10 or more?

4. **3 pts** Lesson 1.5

A shooter truly makes 40% of his shots. Find the spread from luck in his make rate over 25 shots and over 400 shots, and explain what the difference shows.

5. **4 pts** Lesson 2.2

A team makes 50% of its two-point shots. What three-point percentage gives the same expected points per shot? If the team makes 38% of its threes, which shot is the higher-percentage play?

6. **4 pts** Lesson 2.5

In a family game played for points, you may take a sure 40 points or a 50% chance at 100 points (otherwise 0). Find the expected value of each. Which should you choose if 40 points is exactly what you need to win the game?

7. **4 pts** Lesson 3.2

Use the run-expectancy values from Lesson 3.2: runner on 1st, 0 outs, 0.86; runner on 2nd, 0 outs, 1.10; bases empty, 1 out, 0.25. Find the break-even success rate for a stolen-base attempt.

8. **3 pts** Lesson 3.1

A hitter has 150 hits, 70 walks and 5 hit-by-pitches in 500 at-bats, with 5 sacrifice flies. Find his on-base percentage.

9. **3 pts** Lesson 3.3

Team A takes 12 shots each worth 0.08 expected goals; Team B takes 3 shots each worth 0.35. Which team created more expected goals?

10. **4 pts** Lesson 4.1

Fourth-and-2 at the opponent's 40. Success is worth $S = 2.8$, failure $F = -1.4$, and a punt $K = 0.6$. Find the break-even probability, and the expected value of going for it if you convert 55% of the time.

11. **3 pts** Lesson 4.1

State David Romer's main finding about NFL fourth-down decisions and one reason he suggested for it.

12. **4 pts** Lesson 4.2

Trailing by one goal with the goalie pulled, you score with probability 0.08 per minute and the opponent with probability 0.20. Find the chance you tie the game within 3 minutes, scoring before they do.

13. **5 pts** Lesson 4.3

A kicker scores 95% when the keeper guesses wrong, 55% when both go left and 75% when both go right. Let g be the probability the keeper dives left. Find the equilibrium g and the kicker's scoring rate.

14. **4 pts** Lesson 4.4

A hitter bats .330 over 60 at-bats; the league average is .250. Using $r = n/(n + k)$ with $k = 200$, predict his future average, and explain why the prediction is lower than .330.

15. **3 pts** Lesson 4.5

Runner on 2nd, 1 out: 0.66 expected runs. Runners on 1st and 2nd, 1 out: 0.88.
What does an intentional walk cost on average?

16. **4 pts** Lesson 5.1

(a) How many rolls of two dice hit a blot 6 pips away with no blocked points? (b)
In a 1-point tournament game you are doubled with a 22% chance of winning.
Accept or decline?

17. **4 pts** Lesson 5.2

After a win, a team wins its next game 70% of the time; after a loss, 50%. Find its
long-run winning percentage.

18. **4 pts** Lesson 5.3

A 1800 player draws with a 1600 player. Find the 1800 player's Elo expected score
and new rating with $K = 20$.

19. **4 pts** Lesson 5.4

A bridge declarer can make his contract either by a plan that needs a 5-card suit
to split 3–2 (67.8%) or by a plan that succeeds if at least one of two independent
finesses works. Which plan is better?

20. **4 pts** Lesson 5.5

In Pig, rolling a 1 loses the turn total and 2–6 add their value. Show that, to
maximize points per turn, you should roll again at a turn total of 17 and hold at
23.

21. **4 pts** Lesson 6.1

A risk of 1 in 5,000 per occasion is repeated 3,000 times. Find the chance of at
least one occurrence.

22. **3 pts** Lesson 6.2

A player takes a well-chosen, high-percentage shot that misses at the buzzer.
Using the decision–outcome grid, classify it and say how a good coach should
respond.

23. **4 pts** Lesson 6.3

5% of prospects become starters. A tryout drill flags 80% of future starters and 10% of non-starters. If a prospect is flagged, what is the chance he becomes a starter? Name the trap this problem guards against.

24. **4 pts** Lesson 6.4

A team wins each game with probability 0.6. Find its chance of winning a best-of-five series, and connect the answer to Ecclesiastes 9:11.

25. **4 pts** Lesson 6.5

A student makes 45 of 100 shots from a new spot. Find the margin of error and the likely range of his true rate, and the number of shots needed to bring the margin to 3 percentage points.

26. **7 pts** Lesson 6.4

Essay (two paragraphs): The owner of this course described it as “taking the highest-percentage shots on goal while allowing for the fewest risks.” Using at least three examples from different units (one from sport, one from a game, one from life), explain what a high-percentage decision is, why it must count both the expected gain and the risk, and how a Christian can play the percentages diligently while trusting God with the outcome.

ANSWER KEY

1. A needs 1 win, B needs 2. B wins only by taking the next two rounds:
 $(1/2)^2 = 1/4$. A gets $\frac{3}{4} \times 64 = 48$ points, B gets 16.
2. $P(7) = 6/36 = 1/6 \approx 16.7\%$; $P(6 \text{ or } 8) = 10/36 \approx 27.8\%$.
3. At least one 6: 11 outcomes. Of these, (6,4), (4,6), (6,5), (5,6), (6,6) total 10 or more: $5/11 \approx 0.455$.
4. $\sqrt{0.4 \times 0.6/25} \approx 0.098$ and $\sqrt{0.24/400} \approx 0.0245$. Small samples vary widely; large samples settle near the true rate (the law of large numbers).
5. Break-even: $3p = 2(0.5) \Rightarrow p = 33.3\%$. At 38%: $3(0.38) = 1.14$ points per shot against 1.00 for twos, so the three is the higher-expected-points shot.
6. Sure: 40. Chance: $0.5 \times 100 = 50$. The chance has the higher expected value, but if 40 points wins the game, take the sure 40: a certain win beats a 50% win. The goal, not the average, decides.
7. $p^* = \frac{0.86 - 0.25}{1.10 - 0.25} = \frac{0.61}{0.85} \approx 0.72$, about 72%.
8. $\frac{150 + 70 + 5}{500 + 70 + 5 + 5} = \frac{225}{580} \approx .388$.
9. A: $12 \times 0.08 = 0.96$; B: $3 \times 0.35 = 1.05$. B created more, with a quarter of the shots: shot quality beats shot volume.
10. $p^* = (0.6 + 1.4)/(2.8 + 1.4) = 2.0/4.2 \approx 0.476$.
 $E = 0.55(2.8) + 0.45(-1.4) = 1.54 - 0.63 = 0.91$, better than 0.6: go for it.
11. On the 1,068 fourth downs where going for it was better on average, teams kicked 959 times; teams were far too conservative. One explanation: coaches avoid choices that will be second-guessed (blame avoidance) and follow custom.
12. $0.08 + 0.72(0.08) + 0.72^2(0.08) = 0.08 + 0.0576 + 0.0415 \approx 0.179$, about 18%.
13. Kick left: $55g + 95(1 - g) = 95 - 40g$. Kick right: $95g + 75(1 - g) = 75 + 20g$.
 Equal: $20 = 60g$, $g = 1/3$. Scoring rate $95 - 40/3 \approx 81.7\%$.
14. $r = 60/260 \approx 0.231$; $0.250 + 0.231(0.080) \approx .268$. Part of a short hot start is luck, which does not repeat: regression to the mean.
15. $0.88 - 0.66 = 0.22$ runs.
16. (a) 17 of 36 (11 with a 6, plus 5-1, 1-5, 4-2, 2-4, 3-3, 2-2). (b) Accept:
 $4(0.22) - 2 = -1.12$; decline: -1. Decline, since 22% is below the 25% break-even.
17. $\pi_W(0.3) = (1 - \pi_W)(0.5) \Rightarrow 0.8\pi_W = 0.5 \Rightarrow \pi_W = 0.625$,

18. $E = 1/(1 + 10^{-0.5}) \approx 0.760$; new rating $1800 + 20(0.5 - 0.760) = 1794.8$.
19. Two finesses: at least one works with probability $1 - 0.5^2 = 0.75$. 75% beats 67.8%, so take the two-finesse plan.
20. Expected change $\frac{5}{6}(4) - \frac{1}{6}t = (20 - t)/6$. At 17: +0.5, roll. At 23: -0.5, hold.
21. $1 - (0.9998)^{3000} \approx 1 - e^{-0.6} \approx 0.451$,
22. Good decision, bad outcome. The coach should praise the choice, because over many games it will succeed more often than any alternative, and not let one result change a sound habit.
23. $\frac{0.05(0.8)}{0.05(0.8) + 0.95(0.1)} = \frac{0.04}{0.135} \approx 0.30$. Base-rate neglect.
24. $0.6^3 [1 + 3(0.4) + 6(0.4)^2] = 0.216(3.16) \approx 0.683$. The weaker team still wins about 32% of the time: “the race is not to the swift.”
25. $2\sqrt{0.45 \times 0.55}/100 \approx 0.0995$; about 35% to 55%. $n \approx 4(0.2475)/0.03^2 = 1100$ shots.
26. Answers vary. A strong essay defines a high-percentage decision as the option with the best expected result toward the real goal; gives three specific, correct examples (for instance fourth-down break-even rates, pulling the goalie or the three-point shot; accepting a backgammon double or holding at 20 in Pig; seat belts or a decision journal); explains that risk matters because a bad outcome can be ruinous or because the goal is winning, not averaging (increase variance when behind, reduce it when ahead); and connects diligence and humility with Scripture such as Proverbs 16:33, Ecclesiastes 9:11 or James 4:13-15.