COURSES · MATHEMATICS · GRADES 9–10

Geometry

First edition | 13 units | 73 lessons | About 11 hours of video

FIRST EDITION

Course guide

This course is the geometry of the plane and of solids, built the way Euclid built it: from a few definitions, postulates and common notions to a long chain of proved theorems. You begin with logic and the nature of proof, then learn to construct figures with compass and straightedge, prove triangles congruent and similar, study right triangles and take a first look at trigonometry, and work through circles, area and volume, transformations and coordinates. Each new result rests on what came before, so by the end of the year you will have built a whole body of knowledge yourself, one step and one reason at a time.

Every theorem in the course is either proved or clearly marked as a postulate we agree to accept. You will write two-column, paragraph, flow and indirect proofs, and you will meet Euclid's own propositions by number in the Elements. Constructions with compass and straightedge appear throughout the year, not only in one chapter. Short history notes follow the story from Thales and Pythagoras to Euclid, Archimedes and Descartes. Steady daily habits, such as reciting definitions and postulates, marking every given on the figure and stating a reason for every step, make the reasoning clear and lasting.

> Most videos are by Eddie Woo, a veteran classroom teacher in Australia. Math Antics and Math with Mr. J fill in. Eddie Woo says “interval” for segment, “gradient” for slope, “co-interior angles” for same-side interior angles, “vertically opposite angles” for vertical angles, “RHS” for HL, “equiangular” for AA and “trapezium” for trapezoid. Each lesson notes where a video’s words differ from the course.

Start with Lesson 1.1 →

Who it is for

For students in grades 9–10 (ages 14–16) who have finished Algebra I (equations, radicals, graphing lines and quadratics); it leads to Algebra II and then to Trigonometry or Precalculus.

How to use this course

1. The course fills one school year: 32 weeks, 5 days a week (160 school days), about 45 minutes a day, with 73 lessons in 13 units.
2. Daily rhythm: begin with the practice habit (recite the definitions, postulates and theorems you are learning); read the core ideas with pencil, compass and straightedge in hand; study the two worked examples; do the practice set; check the answers, compare your proofs with the model proofs (any valid proof is accepted), and correct every mistake.
3. Two-day lessons: on day 1, read the teaching, study the examples and do the first half of the practice; on day 2, review and do the second half.
4. Keep a theorem notebook: write every definition, postulate and theorem in order, numbered, with a small labeled figure. It becomes your own copy of the Elements.
5. Proofs: a model proof is one valid proof among many. A parent can check a student's proof step by step by pointing to each line and asking, "What is the reason for this line?"
6. Each unit ends with a review day and a unit test. Use the mastery checklist on the review day.
7. The midterm comes after Unit 6 (a review day and an exam day); the final comes after Unit 13 (two review days and the exam).
8. Calculators: work by hand through Unit 8; use a scientific calculator from Unit 9 on (trigonometry, and π and decimal approximations in Units 10–12).
9. Students who need more time may stretch the course; understanding every proof matters more than finishing on a set date.

Scope and sequence

UNIT 1 · WEEKS 1–3

Foundations: Logic and the Axiomatic Method

Students learn how Euclid built geometry from first principles and master the logic of conditional statements, valid arguments and algebraic proof.

1.1 Euclid's Elements and the Axiomatic Method

1.2 Inductive and Deductive Reasoning

1.3 Conditional Statements

1.4 The Laws of Logic

1.5 Algebraic Proof and the Two-Column Format

UNIT 2 · WEEKS 3–5

Points, Lines, Planes and Angles

Students measure and construct segments and angles, prove the Vertical Angles Theorem and write their first geometric proofs.

2.1 Points, Lines and Planes

2.2 Measuring and Constructing Segments

2.3 Measuring Angles

2.4 Angle Pairs and the Vertical Angles Theorem

2.5 Basic Constructions

2.6 Proving Statements About Segments and Angles

UNIT 3 · WEEKS 5–7

Parallel and Perpendicular Lines

Students meet the parallel postulate, prove the angle relationships formed by parallel lines and a transversal, and prove lines parallel.

3.1 Lines, Transversals and Angle Pairs

3.2 The Parallel Postulate and Parallel Lines

3.3 Proving Lines Parallel

3.4 Perpendicular Lines and Distance

3.5 Slopes of Parallel and Perpendicular Lines

UNIT 4 · WEEKS 7–10

Triangles and Congruence

Students prove the triangle angle sum, establish the congruence tests SSS, SAS, ASA, AAS and HL, and use congruent triangles to prove properties of figures.

4.1 Triangle Angle Sum and Exterior Angles

4.2 Congruent Figures

4.3 Proving Triangles Congruent: SSS and SAS

4.4 Proving Triangles Congruent: ASA and AAS

4.5 Using Congruent Triangles (CPCTC)

4.6 Isosceles and Equilateral Triangles

4.7 Right Triangles: HL and Proofs with Two Pairs of Triangles

UNIT 5 · WEEKS 10–13

Relationships Within Triangles

Students study bisectors, medians, altitudes and midsegments, learn indirect proof and prove the inequalities that govern the sides and angles of a triangle.

5.1 Perpendicular Bisectors and Angle Bisectors

5.2 Circumcenter and Incenter

5.3 Medians, Altitudes and the Centroid

5.4 The Triangle Midsegment Theorem

5.5 Indirect Proof and Inequalities in One Triangle

5.6 The Triangle Inequality and the Hinge Theorem

UNIT 6 · WEEKS 13–15

Quadrilaterals and Polygons

Students find the angle sums of polygons and prove the properties of parallelograms, rectangles, rhombuses, squares, trapezoids and kites.

6.1 Angles of Polygons

6.2 Properties of Parallelograms

6.3 Proving That a Quadrilateral Is a Parallelogram

6.4 Rectangles, Rhombuses and Squares

6.5 Trapezoids and Kites

UNIT 7 · WEEKS 15–17

Transformations

Students translate, reflect, rotate and dilate figures, study symmetry, and see congruence defined by rigid motions.

7.1 Translations and Vectors

7.2 Reflections and Symmetry

7.3 Rotations and Rotational Symmetry

7.4 Compositions and Congruence

7.5 Dilations

UNIT 8 · WEEKS 17–19

Similarity

Students work with proportions and the geometric mean, prove triangles similar by AA, SSS and SAS, and use proportions in triangles for indirect measurement.

8.1 Ratios, Proportions and the Geometric Mean

8.2 Similar Polygons

8.3 Proving Triangles Similar: AA

8.4 Proving Triangles Similar: SSS and SAS

8.5 Proportions in Triangles

UNIT 9 · WEEKS 19–22

Right Triangles and Trigonometry

Students prove the Pythagorean theorem several ways, master special right triangles and use the tangent, sine and cosine ratios to solve right triangles.

9.1 The Pythagorean Theorem and Its Proofs

9.2 The Converse of the Pythagorean Theorem

9.3 Similar Right Triangles and the Geometric Mean

9.4 Special Right Triangles

9.5 The Tangent Ratio

9.6 The Sine and Cosine Ratios

9.7 Angles of Elevation and Depression

UNIT 10 · WEEKS 22–25

Circles

Students prove the theorems about chords, tangents, inscribed angles and segment lengths in circles and write equations of circles.

10.1 Circles, Arcs and Central Angles

10.2 Chords

10.3 Tangents

10.4 Inscribed Angles

10.5 Angles Formed by Chords, Secants and Tangents

10.6 Segment Lengths in Circles

10.7 Circles in the Coordinate Plane

UNIT 11 · WEEKS 25–28

Area and Perimeter

Students derive the area formulas for polygons and circles, find arc lengths and sector areas, and compare the areas of similar figures.

11.1 Areas of Rectangles, Parallelograms and Triangles

11.2 Areas of Trapezoids, Rhombuses and Kites

11.3 Areas of Regular Polygons

11.4 Circumference and Arc Length

11.5 Areas of Circles and Sectors

11.6 Areas of Similar Figures and Composite Regions

UNIT 12 · WEEKS 28–30

Surface Area and Volume

Students study solids, nets and cross sections and find the surface areas and volumes of prisms, cylinders, pyramids, cones and spheres.

12.1 Solids, Nets and Cross Sections

12.2 Surface Area of Prisms and Cylinders

12.3 Surface Area of Pyramids and Cones

12.4 Volume of Prisms and Cylinders

12.5 Volume of Pyramids and Cones

12.6 Spheres and Similar Solids

UNIT 13 · WEEKS 31–32

Coordinate Geometry and Proof

Students combine distance, midpoint and slope to classify figures and write coordinate proofs, then review the course from Euclid to Descartes.

13.1 Coordinate Tools: Distance, Midpoint, Slope and Partitions

13.2 Classifying Figures with Coordinates

13.3 Coordinate Proofs

Weekly pacing calendar

WEEK	PLAN
1	• Day 1: Lesson 1.1 Euclid's Elements and the Axiomatic Method (day 1 of 2) • Day 2: Lesson 1.1 Euclid's Elements and the Axiomatic Method (day 2 of 2) • Day 3: Lesson 1.2 Inductive and Deductive Reasoning • Day 4: Lesson 1.3 Conditional Statements (day 1 of 2) • Day 5: Lesson 1.3 Conditional Statements (day 2 of 2) *Families on a 4-day week can combine the two days of a two-day lesson when the student is ready, or stretch the course to about 40 weeks.*
2	• Day 1: Lesson 1.4 The Laws of Logic (day 1 of 2) • Day 2: Lesson 1.4 The Laws of Logic (day 2 of 2) • Day 3: Lesson 1.5 Algebraic Proof and the Two-Column Format (day 1 of 2) • Day 4: Lesson 1.5 Algebraic Proof and the Two-Column Format (day 2 of 2) • Day 5: Unit 1 review (mastery checklist and missed problems)
3	• Day 1: Unit 1 test • Day 2: Lesson 2.1 Points, Lines and Planes • Day 3: Lesson 2.2 Measuring and Constructing Segments (day 1 of 2) • Day 4: Lesson 2.2 Measuring and Constructing Segments (day 2 of 2) • Day 5: Lesson 2.3 Measuring Angles
4	• Day 1: Lesson 2.4 Angle Pairs and the Vertical Angles Theorem (day 1 of 2) • Day 2: Lesson 2.4 Angle Pairs and the Vertical Angles Theorem (day 2 of 2) • Day 3: Lesson 2.5 Basic Constructions (day 1 of 2) • Day 4: Lesson 2.5 Basic Constructions (day 2 of 2)

WEEK	PLAN
	• Day 5: Lesson 2.6 Proving Statements About Segments and Angles (day 1 of 2)
5	• Day 1: Lesson 2.6 Proving Statements About Segments and Angles (day 2 of 2) • Day 2: Unit 2 review (mastery checklist and missed problems) • Day 3: Unit 2 test • Day 4: Lesson 3.1 Lines, Transversals and Angle Pairs • Day 5: Lesson 3.2 The Parallel Postulate and Parallel Lines (day 1 of 2)
6	• Day 1: Lesson 3.2 The Parallel Postulate and Parallel Lines (day 2 of 2) • Day 2: Lesson 3.3 Proving Lines Parallel (day 1 of 2) • Day 3: Lesson 3.3 Proving Lines Parallel (day 2 of 2) • Day 4: Lesson 3.4 Perpendicular Lines and Distance (day 1 of 2) • Day 5: Lesson 3.4 Perpendicular Lines and Distance (day 2 of 2)
7	• Day 1: Lesson 3.5 Slopes of Parallel and Perpendicular Lines • Day 2: Unit 3 review (mastery checklist and missed problems) • Day 3: Unit 3 test • Day 4: Lesson 4.1 Triangle Angle Sum and Exterior Angles (day 1 of 2) • Day 5: Lesson 4.1 Triangle Angle Sum and Exterior Angles (day 2 of 2)
8	• Day 1: Lesson 4.2 Congruent Figures • Day 2: Lesson 4.3 Proving Triangles Congruent: SSS and SAS (day 1 of 2) • Day 3: Lesson 4.3 Proving Triangles Congruent: SSS and SAS (day 2 of 2) • Day 4: Lesson 4.4 Proving Triangles Congruent: ASA and AAS (day 1 of 2) • Day 5: Lesson 4.4 Proving Triangles Congruent: ASA and AAS (day 2 of 2)
9	• Day 1: Lesson 4.5 Using Congruent Triangles (CPCTC) (day 1 of 2) • Day 2: Lesson 4.5 Using Congruent Triangles (CPCTC) (day 2 of 2) • Day 3: Lesson 4.6 Isosceles and Equilateral Triangles (day 1 of 2) • Day 4: Lesson 4.6 Isosceles and Equilateral Triangles (day 2 of 2) • Day 5: Lesson 4.7 Right Triangles: HL and Proofs with Two Pairs of Triangles (day 1 of 2)
10	• Day 1: Lesson 4.7 Right Triangles: HL and Proofs with Two Pairs of Triangles (day 2 of 2) • Day 2: Unit 4 review (mastery checklist and missed problems) • Day 3: Unit 4 test • Day 4: Lesson 5.1 Perpendicular Bisectors and Angle Bisectors (day 1 of 2) • Day 5: Lesson 5.1 Perpendicular Bisectors and Angle Bisectors (day 2 of 2)

WEEK	PLAN
11	• Day 1: Lesson 5.2 Circumcenter and Incenter (day 1 of 2) • Day 2: Lesson 5.2 Circumcenter and Incenter (day 2 of 2) • Day 3: Lesson 5.3 Medians, Altitudes and the Centroid (day 1 of 2) • Day 4: Lesson 5.3 Medians, Altitudes and the Centroid (day 2 of 2) • Day 5: Lesson 5.4 The Triangle Midsegment Theorem
12	• Day 1: Lesson 5.5 Indirect Proof and Inequalities in One Triangle (day 1 of 2) • Day 2: Lesson 5.5 Indirect Proof and Inequalities in One Triangle (day 2 of 2) • Day 3: Lesson 5.6 The Triangle Inequality and the Hinge Theorem (day 1 of 2) • Day 4: Lesson 5.6 The Triangle Inequality and the Hinge Theorem (day 2 of 2) • Day 5: Unit 5 review (mastery checklist and missed problems)
13	• Day 1: Unit 5 test • Day 2: Lesson 6.1 Angles of Polygons • Day 3: Lesson 6.2 Properties of Parallelograms (day 1 of 2) • Day 4: Lesson 6.2 Properties of Parallelograms (day 2 of 2) • Day 5: Lesson 6.3 Proving That a Quadrilateral Is a Parallelogram (day 1 of 2)
14	• Day 1: Lesson 6.3 Proving That a Quadrilateral Is a Parallelogram (day 2 of 2) • Day 2: Lesson 6.4 Rectangles, Rhombuses and Squares (day 1 of 2) • Day 3: Lesson 6.4 Rectangles, Rhombuses and Squares (day 2 of 2) • Day 4: Lesson 6.5 Trapezoids and Kites (day 1 of 2) • Day 5: Lesson 6.5 Trapezoids and Kites (day 2 of 2)
15	• Day 1: Unit 6 review (mastery checklist and missed problems) • Day 2: Unit 6 test • Day 3: Midterm review (Units 1–6) • Day 4: Midterm exam (Units 1–6) • Day 5: Lesson 7.1 Translations and Vectors *Midterm week. Use any spare time to correct missed problems from Units 1–6.*
16	• Day 1: Lesson 7.2 Reflections and Symmetry • Day 2: Lesson 7.3 Rotations and Rotational Symmetry (day 1 of 2) • Day 3: Lesson 7.3 Rotations and Rotational Symmetry (day 2 of 2) • Day 4: Lesson 7.4 Compositions and Congruence (day 1 of 2) • Day 5: Lesson 7.4 Compositions and Congruence (day 2 of 2)

WEEK	PLAN
17	• Day 1: Lesson 7.5 Dilations (day 1 of 2) • Day 2: Lesson 7.5 Dilations (day 2 of 2) • Day 3: Unit 7 review (mastery checklist and missed problems) • Day 4: Unit 7 test • Day 5: Lesson 8.1 Ratios, Proportions and the Geometric Mean
18	• Day 1: Lesson 8.2 Similar Polygons • Day 2: Lesson 8.3 Proving Triangles Similar: AA (day 1 of 2) • Day 3: Lesson 8.3 Proving Triangles Similar: AA (day 2 of 2) • Day 4: Lesson 8.4 Proving Triangles Similar: SSS and SAS (day 1 of 2) • Day 5: Lesson 8.4 Proving Triangles Similar: SSS and SAS (day 2 of 2)
19	• Day 1: Lesson 8.5 Proportions in Triangles (day 1 of 2) • Day 2: Lesson 8.5 Proportions in Triangles (day 2 of 2) • Day 3: Unit 8 review (mastery checklist and missed problems) • Day 4: Unit 8 test • Day 5: Lesson 9.1 The Pythagorean Theorem and Its Proofs (day 1 of 2)
20	• Day 1: Lesson 9.1 The Pythagorean Theorem and Its Proofs (day 2 of 2) • Day 2: Lesson 9.2 The Converse of the Pythagorean Theorem • Day 3: Lesson 9.3 Similar Right Triangles and the Geometric Mean (day 1 of 2) • Day 4: Lesson 9.3 Similar Right Triangles and the Geometric Mean (day 2 of 2) • Day 5: Lesson 9.4 Special Right Triangles (day 1 of 2)
21	• Day 1: Lesson 9.4 Special Right Triangles (day 2 of 2) • Day 2: Lesson 9.5 The Tangent Ratio (day 1 of 2) • Day 3: Lesson 9.5 The Tangent Ratio (day 2 of 2) • Day 4: Lesson 9.6 The Sine and Cosine Ratios (day 1 of 2) • Day 5: Lesson 9.6 The Sine and Cosine Ratios (day 2 of 2)
22	• Day 1: Lesson 9.7 Angles of Elevation and Depression (day 1 of 2) • Day 2: Lesson 9.7 Angles of Elevation and Depression (day 2 of 2) • Day 3: Unit 9 review (mastery checklist and missed problems) • Day 4: Unit 9 test • Day 5: Lesson 10.1 Circles, Arcs and Central Angles
23	• Day 1: Lesson 10.2 Chords (day 1 of 2) • Day 2: Lesson 10.2 Chords (day 2 of 2)

WEEK	PLAN
	• Day 3: Lesson 10.3 Tangents (day 1 of 2) • Day 4: Lesson 10.3 Tangents (day 2 of 2) • Day 5: Lesson 10.4 Inscribed Angles (day 1 of 2)
24	• Day 1: Lesson 10.4 Inscribed Angles (day 2 of 2) • Day 2: Lesson 10.5 Angles Formed by Chords, Secants and Tangents (day 1 of 2) • Day 3: Lesson 10.5 Angles Formed by Chords, Secants and Tangents (day 2 of 2) • Day 4: Lesson 10.6 Segment Lengths in Circles (day 1 of 2) • Day 5: Lesson 10.6 Segment Lengths in Circles (day 2 of 2)
25	• Day 1: Lesson 10.7 Circles in the Coordinate Plane (day 1 of 2) • Day 2: Lesson 10.7 Circles in the Coordinate Plane (day 2 of 2) • Day 3: Unit 10 review (mastery checklist and missed problems) • Day 4: Unit 10 test • Day 5: Lesson 11.1 Areas of Rectangles, Parallelograms and Triangles
26	• Day 1: Lesson 11.2 Areas of Trapezoids, Rhombuses and Kites (day 1 of 2) • Day 2: Lesson 11.2 Areas of Trapezoids, Rhombuses and Kites (day 2 of 2) • Day 3: Lesson 11.3 Areas of Regular Polygons (day 1 of 2) • Day 4: Lesson 11.3 Areas of Regular Polygons (day 2 of 2) • Day 5: Lesson 11.4 Circumference and Arc Length (day 1 of 2)
27	• Day 1: Lesson 11.4 Circumference and Arc Length (day 2 of 2) • Day 2: Lesson 11.5 Areas of Circles and Sectors (day 1 of 2) • Day 3: Lesson 11.5 Areas of Circles and Sectors (day 2 of 2) • Day 4: Lesson 11.6 Areas of Similar Figures and Composite Regions (day 1 of 2) • Day 5: Lesson 11.6 Areas of Similar Figures and Composite Regions (day 2 of 2)
28	• Day 1: Unit 11 review (mastery checklist and missed problems) • Day 2: Unit 11 test • Day 3: Lesson 12.1 Solids, Nets and Cross Sections • Day 4: Lesson 12.2 Surface Area of Prisms and Cylinders (day 1 of 2) • Day 5: Lesson 12.2 Surface Area of Prisms and Cylinders (day 2 of 2)
29	• Day 1: Lesson 12.3 Surface Area of Pyramids and Cones (day 1 of 2) • Day 2: Lesson 12.3 Surface Area of Pyramids and Cones (day 2 of 2) • Day 3: Lesson 12.4 Volume of Prisms and Cylinders (day 1 of 2) • Day 4: Lesson 12.4 Volume of Prisms and Cylinders (day 2 of 2)

WEEK	PLAN
	• Day 5: Lesson 12.5 Volume of Pyramids and Cones (day 1 of 2)
30	• Day 1: Lesson 12.5 Volume of Pyramids and Cones (day 2 of 2) • Day 2: Lesson 12.6 Spheres and Similar Solids (day 1 of 2) • Day 3: Lesson 12.6 Spheres and Similar Solids (day 2 of 2) • Day 4: Unit 12 review (mastery checklist and missed problems) • Day 5: Unit 12 test
31	• Day 1: Lesson 13.1 Coordinate Tools: Distance, Midpoint, Slope and Partitions • Day 2: Lesson 13.2 Classifying Figures with Coordinates (day 1 of 2) • Day 3: Lesson 13.2 Classifying Figures with Coordinates (day 2 of 2) • Day 4: Lesson 13.3 Coordinate Proofs (day 1 of 2) • Day 5: Lesson 13.3 Coordinate Proofs (day 2 of 2)
32	• Day 1: Unit 13 review (mastery checklist and missed problems) • Day 2: Unit 13 test • Day 3: Final review (Units 1–6) • Day 4: Final review (Units 7–13) • Day 5: Final exam (Units 1–13) *Final exam week.*

Materials

A good compass

Choose one that holds its width: a simple metal compass with a screw or a locking arm. Cheap compasses that slip make constructions frustrating.

A straightedge

An unmarked ruler, or the back edge of a ruler, for constructions.

A clear protractor

For measuring and drawing angles and for checking answers.

A ruler with inches and centimeters

For measuring and for drawing figures to scale.

Sharp pencils and an eraser

Constructions need a fine, sharp point.

Unlined paper and graph paper

Unlined paper for constructions; graph paper for coordinate work in Units 3, 7, 10 and 13.

A notebook or binder and a separate theorem notebook

Daily work goes in the notebook; every definition, postulate and theorem, numbered and with its figure, goes in the theorem notebook.

Index cards

For definitions, postulates and theorems to recite each day.

A scientific calculator

Needed from Unit 9 on (sine, cosine, tangent and their inverses, π).

Optional: colored pencils

For marking givens, congruent parts and overlapping triangles on diagrams.

Optional: a copy of Euclid's Elements

Book I is enough to start, for example the Heath translation or a modern edition; many are free online.

Optional: patty paper or tracing paper

For folding and tracing in the transformations unit.

Optional: string and stakes

For outdoor surveying activities, such as laying out a right angle with a 3-4-5 triangle.

Grading

Grades come mostly from unit tests, with daily practice, the midterm and the final making up the rest. The goal is mastery: every mistake is corrected, and a proof is judged by whether each step truly follows.

PART	%
Daily practice sets	**20**
Unit tests	**50**
Midterm exam	**15**
Final exam	**15**

Scale: A 90–100, B 80–89, C 70–79, D 60–69, below 60 redo

- Grade daily practice for completion and corrections: full credit when every problem is attempted and every mistake has been corrected.
- Grading a proof: give full credit for any valid proof, even if it differs from the model proof. Give partial credit for correct statements with missing or wrong reasons. A proof that assumes what it must prove earns little credit.
- Retakes: a student who scores below 80 on a unit test may study the missed material, correct the test, and take a second version; record the higher score, up to 90.
- Mastery over speed: it is better to take an extra week on a unit than to move on without understanding congruence or similarity, because later units depend on them.

A note to parents

You do not need to be a geometry expert to teach this course. Every answer, every model proof and every test key is provided, and each lesson explains the ideas step by step. Your part is to keep a steady daily rhythm, look over the work, and ask good questions.

When your student is stuck on a proof, resist the urge to give the next line. Ask instead: "What are you given?" "What must you prove?" "Did you mark the givens on the figure?" "What is your reason for that step?" These questions are often enough to get a student moving again. Expect daily practice, insist that every mistake be corrected, and listen as your student recites a few definitions and theorems each day; that small habit pays off all year.

Why study geometry this way? For twenty-three centuries Euclid's Elements was the standard training in clear reasoning. Geometry serves carpentry, building, surveying, design, navigation, science and engineering, but its greatest gift is a habit of mind: to reason honestly from first principles, to say exactly what you mean, and to accept a conclusion only when you can give a reason for every step.

UNIT 1

Foundations: Logic and the Axiomatic Method

Geometry is the first course in which you prove, rather than just compute. This unit lays the foundation. You will meet Euclid's *Elements*, the book that taught the world how to build a whole science from a few definitions, postulates and common notions, and you will learn the difference between noticing a pattern (inductive reasoning) and proving a conclusion (deductive reasoning).

Next you will study the logic that every proof uses: if-then statements and their converses, inverses and contrapositives, biconditionals and good definitions, and the Laws of Detachment and Syllogism, along with the common errors that look like logic but are not. The unit ends with your first two-column proofs, in which each step of solving an equation is justified by a property of equality, the modern form of Euclid's common notions.

Parents: no new algebra is required here. The goal is careful, honest reasoning: every statement needs a reason. Ask your student to explain each step aloud; if they can say *why*, they are ready for Unit 2.

PREREQUISITES

- Solve linear equations, including equations with parentheses and variables on both sides (Algebra I).
- Recognize even, odd and prime numbers and test divisibility.
- Find the perimeter of a triangle or rectangle and the area of a rectangle.
- Know the everyday meaning of point, line, circle, radius, right angle, square and rectangle.
- Use a compass to draw a circle with a given center and radius.

MASTERY CHECKLIST

- ☐ I can explain the difference between undefined terms, definitions, postulates and theorems.
- ☐ I can state Euclid's five postulates and five common notions in modern words.
- ☐ I can construct an equilateral triangle on a segment (Euclid I.1) and explain why it works.
- ☐ I can make a conjecture from a pattern and test it with a counterexample.
- ☐ I can explain why mathematics requires deductive proof rather than examples.
- ☐ I can write the converse, inverse and contrapositive of a conditional and decide whether each is true.
- ☐ I can write a biconditional and test whether a definition is reversible.
- ☐ I can use the Law of Detachment and the Law of Syllogism, and recognize affirming the conclusion and denying the hypothesis.
- ☐ I can use a Venn diagram to decide whether an argument is valid.
- ☐ I can name the property of equality that justifies each step in solving an equation.
- ☐ I can write a two-column proof of an algebraic statement, with a reason for every step.

LESSONS IN THIS UNIT

1.1	**Euclid's Elements and the Axiomatic Method**	2 sessions of 45 minutes
1.2	**Inductive and Deductive Reasoning**	45 minutes
1.3	**Conditional Statements**	2 sessions of 45 minutes
1.4	**The Laws of Logic**	2 sessions of 45 minutes
1.5	**Algebraic Proof and the Two-Column Format**	2 sessions of 45 minutes

Unit 1 test

You have 45–60 minutes. No calculator is needed or allowed. Show your work on every problem. In every proof, give a reason for each statement; the last statement must be exactly what you are asked to prove. Give exact answers with units. For Problem 16 you need a compass and a straightedge (an unmarked ruler); leave all construction marks visible.

1. **5 pts**
   Classify each as an undefined term, a definition, a postulate or a theorem: (a) point; (b) “A circle can be drawn with any center and any radius.”

2. **5 pts**
   A carpenter stretches a chalk line along the straight edge of a board and snaps it so that the line continues straight past the end of the board. Which of Euclid's postulates describes this action? Give its number and state it.

3. **6 pts**
   A master mason lays out an equilateral triangle for a church window by Euclid's Proposition I.1, swinging two arcs with a cord as a compass. The outline of the triangle uses exactly 36 inches of lead came. What radius did he use for each arc? Explain.

4. **6 pts**

   Samuel builds a row of square pens for his sheep, each new pen sharing a side with the one before. One pen takes 4 fence panels, two pens take 7, and three pens take 10. Make a conjecture about the number of panels for n pens, and use it to find the number of panels needed for 15 pens.

5. **6 pts**

   Conjecture: “For every whole number n, $2^n > n^2$.” Find the smallest whole number that is a counterexample, and show why it is one.

6. **5 pts**

   “Every rafter cut from this pattern board is 14 feet long. Henry's rafter was cut from this pattern board. So Henry's rafter is 14 feet long.” Is this inductive or deductive reasoning? Explain briefly.

7. **8 pts**

   Consider the true conditional “If $x > 5$, then $x > 3$.” Give the truth value of (a) its converse, (b) its inverse and (c) its contrapositive. Write each statement, and give a counterexample for any that is false.

8. **6 pts**

   Grace proposes the definition: “An angle is obtuse if and only if its measure is greater than 90°.” Is this a good definition? Explain by testing both conditionals.

9. **5 pts**

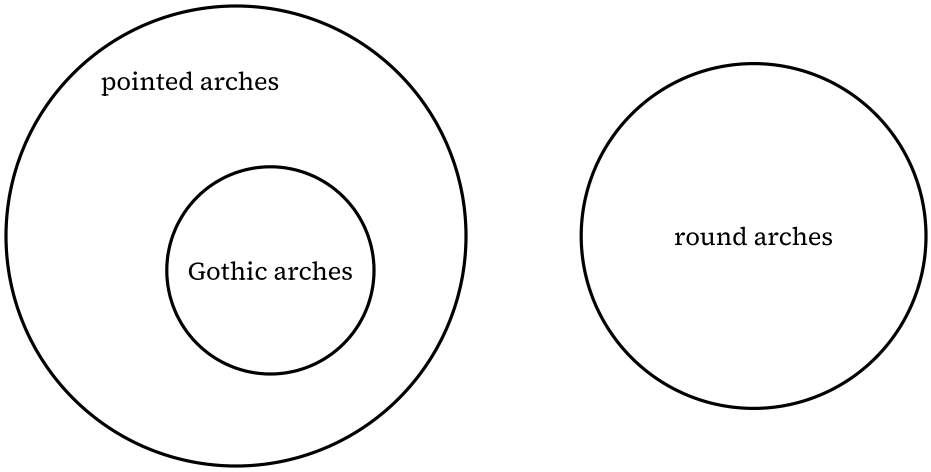
   “If the lighthouse lamp is lit, then ships can see the rocks. If ships can see the rocks, then they steer clear of them.” Tonight the lighthouse lamp is lit. Do the ships steer clear of the rocks? Name the laws you use.

10. **5 pts**

    “If a ship is a schooner, then it has at least two masts. The ship in the harbor has two masts. So it is a schooner.” Is the argument valid? If not, name the error.

11. **6 pts**

"Every Gothic arch is a pointed arch. No round arch is a pointed arch." Using the Venn diagram, decide whether a Gothic arch can be a round arch, and explain.

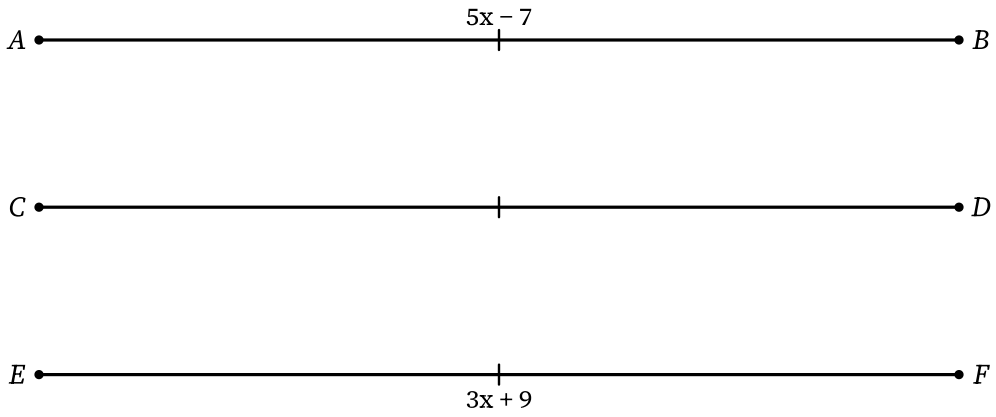


12. **8 pts**

Write a two-column proof. Given: $7x - 3 = 4x + 18$. Prove: $x = 7$.

13. **8 pts**

In the figure, $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, with $AB = 5x - 7$ and $EF = 3x + 9$. Write a two-column proof that $x = 8$.



14. **8 pts**

The Miller family is pouring a rectangular barn floor. Its length is 4 feet less than three times its width, and the forms around its edge total 152 feet. Find the width and the length, naming the property of equality (or the Distributive Property) used in each step.

15. **5 pts**

Explain why every system of mathematics must begin with some postulates that are accepted without proof.

16. **8 pts**

Copy segment $\overline{AB}$ from the figure. (a) With compass and straightedge, construct an equilateral triangle ABC on it by Euclid's Proposition I.1, leaving your arcs visible. (b) Write a two-column proof that $\triangle ABC$ is equilateral.



ANSWER KEY

1. (a) undefined term; (b) postulate (Euclid's Postulate 3).
2. Postulate 2: a segment can be extended continuously in a straight line.
3. 12 inches. In I.1 each arc's radius equals the given segment, and all three sides are equal to it, so each side is $36 \div 3 = 12$ inches.
4. Each new pen adds 3 panels, so n pens take $3n + 1$ panels. For 15 pens: $3(15) + 1 = 46$ panels.
5. $n = 2$: $2^n = 2^2 = 4$ and $n^2 = 2^2 = 4$, and $4 > 4$ is false. (For $n = 0$, $1 > 0$, and for $n = 1$, $2 > 1$, so those are not counterexamples.)
6. Deductive: the conclusion follows by logic (the Law of Detachment) from an accepted general statement and a given fact, not from observed examples.
7. (a) Converse: If $x > 3$, then $x > 5$. False ($x = 4$). (b) Inverse: If $x \leq 5$, then $x \leq 3$. False ($x = 4$). (c) Contrapositive: If $x \leq 3$, then $x \leq 5$. True.
8. No. “If an angle is obtuse, then its measure is greater than $90°$” is true, but “If an angle's measure is greater than $90°$, then it is obtuse” is false: a straight angle measures $180°$ and is not obtuse. A correct definition: an angle is obtuse if and only if its measure is greater than $90°$ and less than $180°$.
9. Yes. The Law of Syllogism gives “If the lamp is lit, then ships steer clear of the rocks,” and the Law of Detachment applies it to tonight.
10. Not valid: it affirms the conclusion. A ship with two masts could be another kind of vessel, such as a brig.
11. No. The Gothic-arch circle lies inside the pointed-arch circle, and the round-arch circle does not meet the pointed-arch circle, so nothing can be in both the Gothic and round circles.
12. See the model proof.

TWO-COLUMN PROOF

Given: $7x - 3 = 4x + 18$

Prove: $x = 7$

	STATEMENTS	REASONS
1	$7x - 3 = 4x + 18$	Given
2	$3x - 3 = 18$	Subtraction Property of Equality
3	$3x = 21$	Addition Property of Equality

	STATEMENTS	REASONS
4	$x = 7$	Division Property of Equality

∎

13. See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$; $\overline{CD} \cong \overline{EF}$; $AB = 5x - 7$, $EF = 3x + 9$

Prove: $x = 8$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{EF}$	Given
2	$\overline{AB} \cong \overline{EF}$	Transitive Property of Congruence
3	$AB = EF$	Definition of congruent segments
4	$AB = 5x - 7$, $EF = 3x + 9$	Given
5	$5x - 7 = 3x + 9$	Substitution Property of Equality
6	$2x - 7 = 9$	Subtraction Property of Equality
7	$2x = 16$	Addition Property of Equality
8	$x = 8$	Division Property of Equality

∎

14. Let the width be w; the length is $3w - 4$. Perimeter: $2w + 2(3w - 4) = 152$. Distributive: $2w + 6w - 8 = 152$; Simplify: $8w - 8 = 152$; Addition: $8w = 160$; Division: $w = 20$. Width 20 feet, length $3(20) - 4 = 56$ feet (check: $2(20) + 2(56) = 152$).

15. A proof justifies each statement by earlier statements. If every starting statement also needed a proof, the chain of reasons would go back forever or circle around to use what it was trying to prove. So mathematicians agree on a few clear starting statements (postulates) and prove everything else from them.

16. (a) Draw the circle with center A through B and the circle with center B through A; label a crossing point C; draw $\overline{AC}$ and $\overline{BC}$. (b) See the model proof.

TWO-COLUMN PROOF

Given: C lies on the circle with center A through B; C lies on the circle with center B through A

Prove: $\triangle ABC$ is equilateral

	STATEMENTS	REASONS
1	C lies on the circle with center A through B and on the circle with center B through A	Given (construction)
2	$AC = AB$	Definition of a circle (radii of the circle with center A)
3	$BC = AB$	Definition of a circle (radii of the circle with center B)
4	$AC = BC$	Common Notion 1 (things equal to the same thing are equal)
5	$\triangle ABC$ is equilateral	Definition of an equilateral triangle (all three sides equal)

∎

1.1 Euclid's Elements and the Axiomatic Method

OBJECTIVES

- Explain what a proof is and why mathematics is built on a small number of first principles.
- Distinguish undefined terms, definitions, postulates and theorems.
- State Euclid's five postulates and five common notions in modern words.
- Name the postulate, definition or common notion that justifies a simple step of reasoning.

WATCH

The Global/Historical Geometry Project (1 of 2: Euclid to Lobachevsky)

Eddie Woo · 10:25

https://www.youtube.com/watch?v=HqwQOUUokSE

> **WHEN TO WATCH**
> After the Core ideas: how geometry grew from Euclid's Elements, including the long story of the parallel postulate.

Note: Neither video lists Euclid's five postulates and five common notions; teach those from the Core ideas.

BACKUP VIDEO

Deductive Geometry (1 of 3: Why is it part of our syllabus?)

Eddie Woo · 4:04

https://www.youtube.com/watch?v=RXN0R5Trc7Q

Before Example 1: why a geometry course is really a course in proof.

KEY TERMS

undefined term

A basic word, such as point, line or plane, that is described and pictured but not defined, because every definition must use other words, and we have to start somewhere.

definition

A precise statement of the meaning of a new term, using undefined terms and terms already defined. Example: a circle is the set of all points in a plane at a given distance (the radius) from a given point (the center).

postulate (axiom)

A statement accepted as true without proof and used as a starting point for reasoning.

common notion

Euclid's name for a general truth about quantities that is accepted without proof, such as “things equal to the same thing are equal to each other.”

theorem

A statement that has been proved from undefined terms, definitions, postulates and earlier theorems. Euclid called his theorems and constructions propositions.

proof

A chain of statements, each one justified by given information, a definition, a postulate, a common notion or a theorem already proved, that ends with the statement to be proved.

CORE IDEAS

Imagine a curious younger brother who answers everything you say with “Why?” If you keep giving reasons, one of two things happens: you go on forever, or you go around in a circle and end up using the very thing you were trying to explain. Mathematics avoids both traps by agreeing at the start on a few first principles. Everything else is then proved from them. This plan is called the **axiomatic method**, and Euclid's *Elements* is its most famous example.

The first principles come in three kinds. **Undefined terms** (point, line, plane) are described but not defined: a point has position but no size, a line is straight, has no thickness and goes on without end in both directions, and a plane is a flat surface with no thickness that goes on without end. Euclid tried to describe them too (“A point is that which has no part”; “A line is breadthless length”), but these sentences

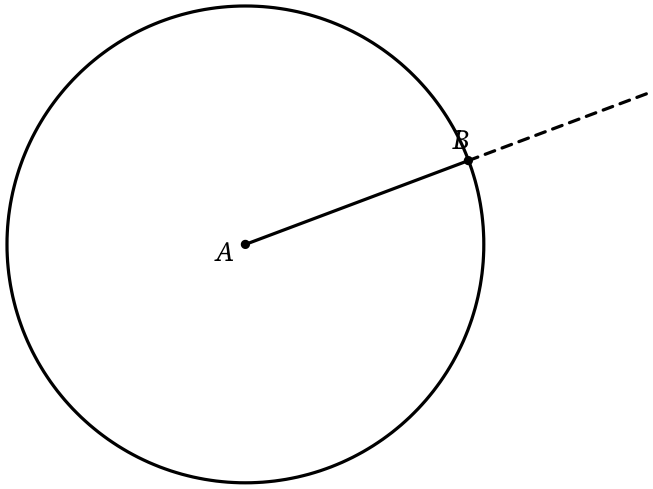
picture the idea rather than define it. **Definitions** then name new things using only words already understood. **Postulates** are statements accepted without proof.

Euclid's five postulates, in modern words: (1) a straight segment can be drawn from any point to any other point; (2) a segment can be extended as far as you like in a straight line; (3) a circle can be drawn with any center and any radius; (4) all right angles are congruent (equal); (5) if a line crosses two other lines so that the two interior angles on one side add up to less than 180°, then those two lines, extended far enough, meet on that side. Notice that Postulates 1 and 2 describe what a **straightedge** can do and Postulate 3 describes what a **compass** can do. Postulate 4 makes the right angle a fixed standard for measuring every other angle, and Postulate 5, the famous parallel postulate, is the subject of Unit 3.

Euclid also listed five **common notions**, truths about any kind of quantity (lengths, angles, areas, numbers): (1) things equal to the same thing are equal to each other; (2) if equals are added to equals, the wholes are equal; (3) if equals are subtracted from equals, the remainders are equal; (4) things that coincide (fit exactly on each other) are equal; (5) the whole is greater than the part. You already use these every day in algebra; in Lesson 1.5 they become the properties of equality.

A **theorem** is a statement that has been proved, and a **proof** is a chain of statements in which every link has a reason: something given, a definition, a postulate, a common notion or a theorem proved earlier. That is why order matters so much in Euclid. Proposition I.1 may use only the definitions, postulates and common notions; Proposition I.2 may also use I.1; and so on. Each new result rests on solid ground, and nothing is used before it is earned.

Why not simply trust a careful drawing? Because drawings can mislead: two segments can look equal and differ by a hair, and a drawing shows only one case, while a theorem must be true in every case. A figure helps you see what to prove and plan how to prove it, but it never counts as a reason. In this course every step of a proof gets a stated reason.



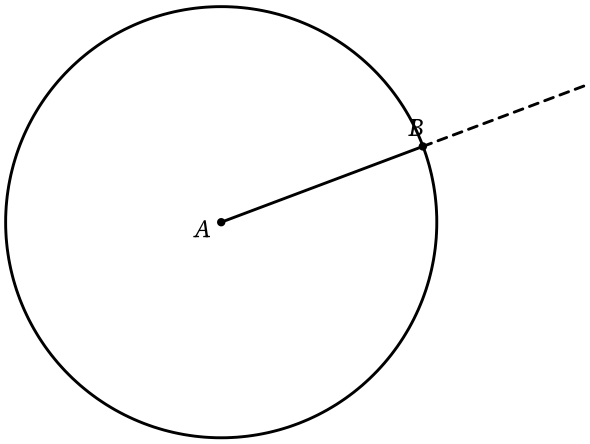
Post. 1 draws $\overline{AB}$; Post. 2 extends it (dashed); Post. 3 draws the circle with center A through B.

POSTULATES AND THEOREMS

POSTULATE **Euclid's Postulates 1–3 (Straightedge and Compass)**

Euclid Post. 1–3

(1) A straight segment can be drawn joining any two points. (2) Any segment can be extended continuously in a straight line. (3) A circle can be drawn with any center and any radius.



Post. 1 draws $\overline{AB}$; Post. 2 extends it (dashed); Post. 3 draws the circle with center A through B.

These are exactly the moves a straightedge (Post. 1 and 2) and a compass (Post. 3) allow, which is why constructions in this course use only those two tools.

POSTULATE **Euclid's Postulate 4 (Right Angles)** Euclid Post. 4

All right angles are congruent.

In Lesson 2.6 we prove this from the modern Protractor Postulate, so in our course it becomes a theorem.

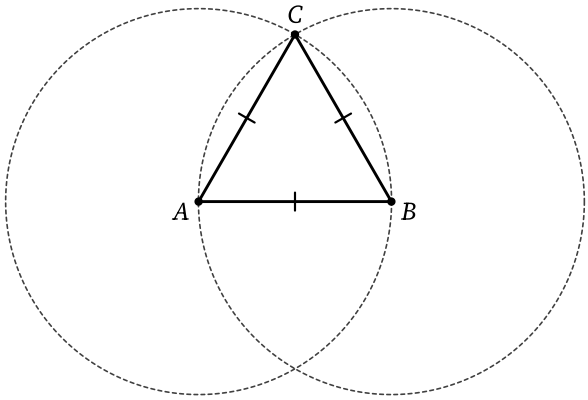
POSTULATE **Euclid's Postulate 5 (The Parallel Postulate)** Euclid Post. 5

If a line crossing two lines makes the two interior angles on the same side add up to less than $180°$, then the two lines, if extended far enough, meet on that side.

Euclid's words: "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side." We study it in Unit 3.

CONSTRUCTIONS

CONSTRUCTION **Construct an equilateral triangle on a given segment** Euclid I.1



1. Start with the given segment $\overline{AB}$.
2. Put the compass point on A and open it to B. Draw the circle with center A through B (Postulate 3).
3. Keeping the same idea, put the compass point on B, open it to A, and draw the circle with center B through A (Postulate 3).
4. Label C one of the two points where the circles cross.
5. With the straightedge, draw $\overline{AC}$ and $\overline{BC}$ (Postulate 1). Triangle ABC is equilateral.

Why it works. $AC = AB$ because both are radii of the circle with center A (definition of a circle), and $BC = AB$ because both are radii of the circle with center B. Things equal to the same thing

are equal to each other (Common Notion 1), so $AC = BC$ as well, and all three sides are equal. Euclid took for granted that the two circles really do cross; David Hilbert later supplied the missing assumption (see Lesson 2.5).

From history. Euclid taught mathematics in Alexandria, Egypt, about 300 BC. His *Elements*, in 13 books, gathered the geometry and number theory of the Greeks into one carefully ordered chain of proofs, and it served as the standard geometry textbook for more than two thousand years. Abraham Lincoln's 1860 campaign autobiography says that, after serving in Congress, he "studied and nearly mastered the six books of Euclid"; a later account reports that he did it to learn what it really means to *demonstrate* something. In 1899 the German mathematician David Hilbert published a modern set of axioms that filled the small gaps in Euclid's foundations.

Euclid connection. Book I of the *Elements* opens with 23 definitions, 5 postulates and 5 common notions, and then proves 48 propositions, ending with the Pythagorean theorem (I.47) and its converse (I.48). Proposition I.1, the equilateral triangle, uses nothing but the definitions of a circle and of an equilateral triangle, Postulates 1 and 3 and Common Notion 1, together with one unstated assumption: that the two circles really cross (Lesson 2.5).

Practice habit. Recite Euclid's five postulates and five common notions aloud until you can say them without looking. When you read any argument, ask of each step: "Which definition, postulate or common notion makes this true?"

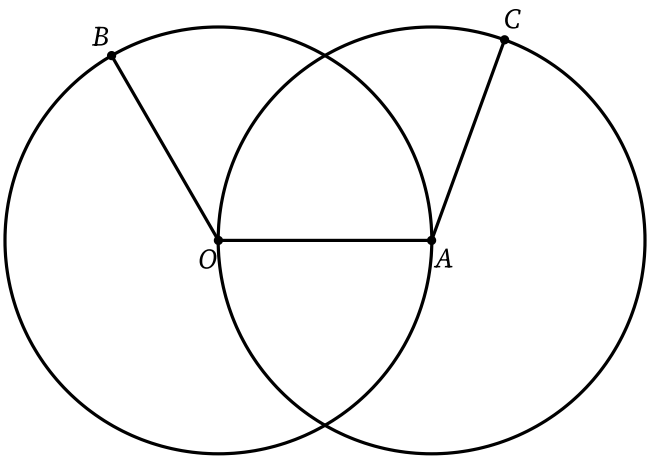
Example 1. Grace measures and finds that the porch rail is exactly as long as the kitchen table. Her brother finds that the garden bench is also exactly as long as the kitchen table. Without measuring the rail against the bench, what can they conclude, and which of Euclid's common notions justifies it?

Show solution

1. Name the lengths: rail $=$ table and bench $=$ table.
2. The rail and the bench are each equal to the same thing, the table.
3. Common Notion 1: things equal to the same thing are equal to each other. So rail $=$ bench.

Answer: The porch rail and the garden bench are the same length, by Common Notion 1.

Example 2. In the figure, the circle with center O passes through A and B, and the circle with center A passes through O and C. Prove that $OB = AC$.



Show solution

1. Look for a segment that belongs to both circles: $\overline{OA}$ is a radius of each one.
2. Use the definition of a circle twice to compare OB and AC with OA.
3. Finish with Common Notion 1.

PARAGRAPH PROOF

Given: The circle with center O passes through A and B; The circle with center A passes through O and C

Prove: $OB = AC$

Since A and B lie on the circle with center O, the segments $\overline{OA}$ and $\overline{OB}$ are radii of that circle, so $OB = OA$ by the definition of a circle (all its points are the same distance from the center).

Since O and C lie on the circle with center A, the segments $\overline{AO}$ and $\overline{AC}$ are radii of that circle, so $AC = AO$, again by the definition of a circle. The length AO is the same as OA.

Thus OB and AC are both equal to the same length OA. Things equal to the same thing are equal to each other (Common Notion 1), so $OB = AC$.

∎

Answer: See the proof: both OB and AC equal OA, so $OB = AC$ by Common Notion 1.

PRACTICE

1. Easy Name the three undefined terms of geometry.

   Show answer

   Point, line and plane.

2. Easy Is the following statement an undefined term, a definition, a postulate or a theorem? “A circle is the set of all points in a plane at a given distance from a given point.”

   Show answer

   A definition (it gives the meaning of the word circle).

3. Easy Samuel ties a 6-foot cord to a stake, pulls it tight, and walks all the way around the stake to mark the edge of a round flower bed. Which of Euclid's five postulates describes what he is doing?

   Show answer

   Postulate 3: a circle can be drawn with any center and any radius.

4. Easy Henry has two boards, each 12 feet long, and cuts a 4-foot piece off each one. (a) How long is each remaining piece? (b) Which common notion guarantees that the two remaining pieces are equal, even before you measure them?

   Show answer

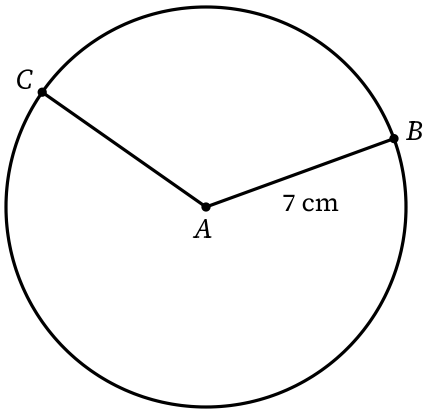
   (a) 8 feet each. (b) Common Notion 3: if equals are subtracted from equals, the remainders are equal.

5. Medium You are told that $AB = CD$, $CD = EF$ and $EF = 9$ inches. Find AB, and name the common notion you use.

   Show answer

   $AB = 9$ inches. Since $AB = CD$ and $CD = EF$, Common Notion 1 gives $AB = EF$; then $AB = 9$ inches, again because things equal to the same thing are equal.

6. **Medium** In the figure, B and C lie on the circle with center A, and $AB = 7$ cm. Find AC and give the reason.



Show answer

$AC = 7$ cm, because $\overline{AB}$ and $\overline{AC}$ are both radii of the same circle (definition of a circle).

7. **Medium** A stained-glass maker draws a 9-inch segment and uses Euclid's Proposition I.1 to build an equilateral triangle on it. How many inches of lead came (the metal strip around the glass) does she need to go once around the triangle?

Show answer

Each side is 9 inches, so she needs $3 \times 9 = 27$ inches.

8. **Medium** Ruth cuts a 10-foot board into a 4-foot piece and a 6-foot piece. Which of Euclid's common notions says that the 4-foot piece is shorter than the whole board?

Show answer

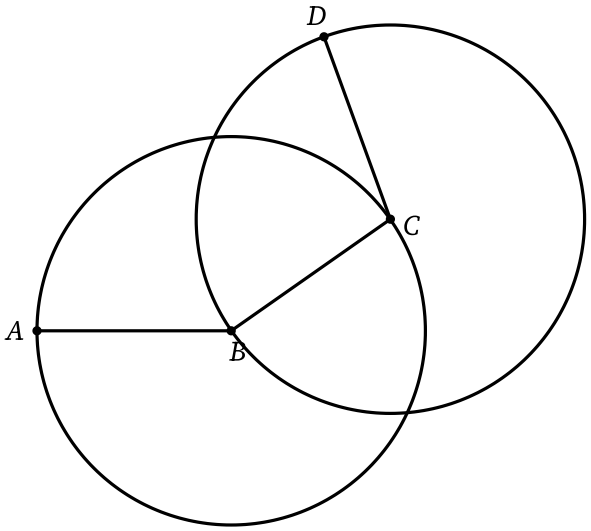
Common Notion 5: the whole is greater than the part.

9. **Medium** Explain why Euclid's Postulates 1, 2 and 3 can be called the “rules of the compass and straightedge.” Why do you think the straightedge has no markings for measuring?

Show answer

Postulates 1 and 2 say we may draw a straight segment between two points and extend it, which is exactly what a straightedge does; Postulate 3 says we may draw a circle with any center and radius, which is exactly what a compass does. The straightedge has no markings because none of the postulates allows measuring: Euclid's constructions get equal lengths from circles (all radii are equal), not from numbers on a ruler, so they are exact rather than approximate.

10. Hard In the figure, the circle with center B passes through A and C, and the circle with center C passes through B and D. Write a paragraph proof that $AB = CD$.



Show answer

See the model proof: $AB = BC$ and $BC = CD$, so $AB = CD$ by Common Notion 1.

PARAGRAPH PROOF

Given: The circle with center B passes through A and C; The circle with center C passes through B and D

Prove: $AB = CD$

Because A and C lie on the circle with center B, the segments $\overline{BA}$ and $\overline{BC}$ are radii of that circle, so $AB = BC$ by the definition of a circle.

Because B and D lie on the circle with center C, the segments $\overline{CB}$ and $\overline{CD}$ are radii of that circle, so $CD = CB$ by the definition of a circle; and CB is the same length as BC.

So AB and CD are each equal to the same length BC, and by Common Notion 1 (things equal to the same thing are equal to each other), $AB = CD$.

∎

11. Hard Explain why it is impossible to define every word in geometry. What do mathematicians do about it?

Show answer

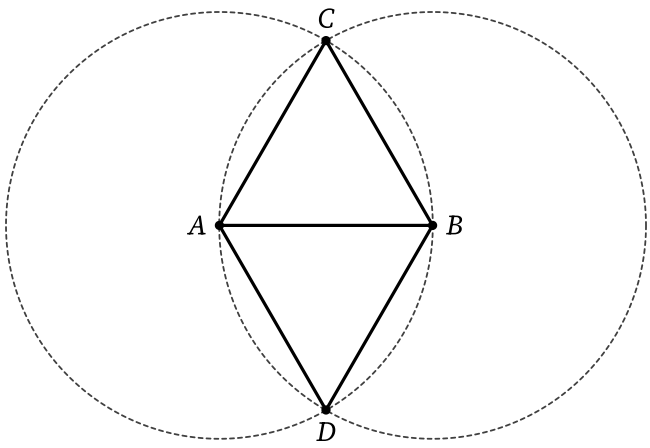
Every definition uses other words. If those words must also be defined, and their definitions use still other words, the process either goes on forever or eventually loops back and uses the word being defined, which explains nothing. So mathematicians choose a few undefined terms (point, line, plane), describe them clearly, and build every other definition from them.

12. Hard (a) Is Euclid's statement “All right angles are equal to one another” a postulate or a theorem in the *Elements*? (b) Is Euclid's Proposition I.5, “In isosceles triangles the angles at the base are equal to one another,” a postulate or a theorem? Explain the difference.

Show answer

(a) A postulate (Postulate 4): Euclid accepts it without proof. (b) A theorem: Euclid proves it from the definitions, postulates, common notions and earlier propositions I.1–I.4.

13. **Challenge** In the figure, the circle with center A passes through B, and the circle with center B passes through A. The two circles cross at C and D. Write a paragraph proof that $AC = BC = AD = BD$, so that $ACBD$ has four equal sides.



Show answer

See the model proof: all four segments equal AB, so they equal each other by Common Notion 1.

PARAGRAPH PROOF

Given: The circle with center A passes through B, C and D; The circle with center B passes through A, C and D
Prove: $AC = BC = AD = BD$
The points B, C and D all lie on the circle with center A, so $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are radii of that circle. By the definition of a circle, $AC = AB$ and $AD = AB$.

The points A, C and D all lie on the circle with center B, so $\overline{BA}$, $\overline{BC}$ and $\overline{BD}$ are radii of that circle. By the definition of a circle, $BC = AB$ and $BD = AB$.

Thus each of AC, BC, AD and BD is equal to the same length AB. Things equal to the same thing are equal to each other (Common Notion 1), so $AC = BC = AD = BD$.

∎

COMMON MISTAKES

- Using a drawing as a reason (“they look equal”). A figure suggests what might be true; only a definition, postulate, common notion or proved theorem can justify a

step.

- Mixing up postulates and theorems. A postulate is accepted without proof; a theorem must be proved. Ask: “Did anyone prove this, or did we agree to it at the start?”
- Thinking an undefined term is vague or meaningless. Point, line and plane are perfectly clear ideas; they are simply the starting words that every other definition is built from.
- Forgetting that every radius of a circle has the same length. Whenever a point lies on a circle, draw the radius to it and write down the equal length.

> **Make it a game · Sort the Statements.** Write ten statements on slips of paper, such as 'A point has no part,' 'Things equal to the same thing are equal to each other,' and 'The base angles of an isosceles triangle are equal.' Sort them into undefined term, definition, postulate or common notion, and theorem. Then defend each pile aloud. Redo the sort tomorrow and beat your score.

1.2 Inductive and Deductive Reasoning

OBJECTIVES

- Use inductive reasoning to find a pattern and make a conjecture.
- Disprove a conjecture with a counterexample.
- Distinguish inductive reasoning from deductive reasoning.
- Write a short deductive argument about whole numbers.

WATCH

Formal Geometry Proofs (1 of 3: What does it mean to "prove" something?)

Eddie Woo · 5:20

https://www.youtube.com/watch?v=IZ93Ah1dTnw

> **WHEN TO WATCH**
> Before the Core ideas: how proof in mathematics differs from evidence in history and science (deductive versus inductive reasoning).

Note: He does not use the words 'inductive' and 'conjecture'; the course defines them.

BACKUP VIDEO

Counter-Examples of Integers

Eddie Woo · 3:35

https://www.youtube.com/watch?v=eVShvGSGANM

Before Example 2: using a counterexample to disprove a claim about whole numbers.

KEY TERMS

inductive reasoning

Reasoning that looks at several examples, notices a pattern and guesses that the pattern always continues.

conjecture

An unproved statement based on observation; an educated guess that may be true or false.

counterexample

A single example for which the hypothesis of a statement is true but its conclusion is false. One counterexample shows the statement is false.

deductive reasoning

Reasoning that reaches a conclusion by logic from statements already accepted (given facts, definitions, postulates and proved theorems). If the starting statements are true and the logic is valid, the conclusion must be true.

even and odd integers

An integer is even if it equals $2k$ for some integer k, and odd if it equals $2k + 1$ for some integer k.

CORE IDEAS

Much of mathematics is discovered by **inductive reasoning**: you try examples, notice a pattern and make a **conjecture**. Add the first few odd numbers: $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$. The sums are $1, 4, 9, 16$, the perfect squares, so you might conjecture that the sum of the first n odd numbers is n^2. Inductive reasoning is how mathematicians get ideas.

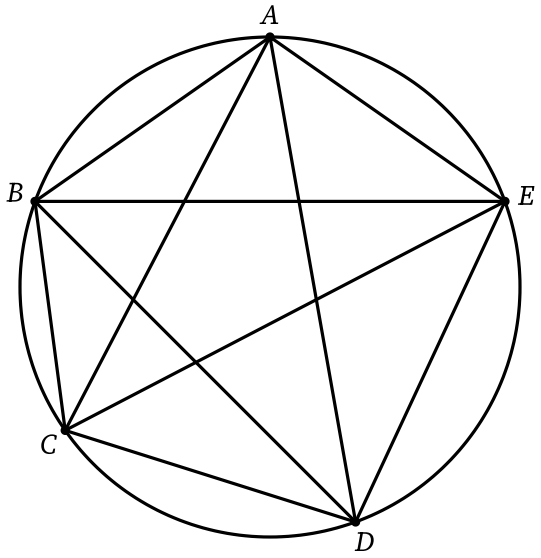
But a pattern is not a proof. Place points on a circle, join every pair of points with a segment, and count the regions inside the circle (placing the points so that no three segments pass through one point). With 1, 2, 3, 4 and 5 points you get $1, 2, 4, 8, 16$ regions. Surely 6 points give 32? Count carefully and you find only **31**. The doubling pattern was a coincidence for the first five cases. No number of agreeing examples can prove a statement about infinitely many cases.

A **counterexample** settles the question the other way: one case in which the hypothesis is true and the conclusion is false shows a statement is false. The conjecture "every prime number is odd" falls to the single counterexample 2, which is prime and even. When you test a conjecture, try unusual cases on purpose: 0, 1, 2, negative numbers, fractions, very large numbers, or a lopsided figure.

Deductive reasoning works the other way around. It starts from statements already accepted (definitions, postulates, theorems already proved and given facts) and uses logic to reach a conclusion that *must* be true whenever those statements are true. For example, using the definition of an odd number as $2k + 1$, we can prove that the sum

of two odd numbers is always even, for every pair of odd numbers at once, without checking a single example.

Why does mathematics insist on deduction? Because a builder, a navigator or a scientist who relies on a mathematical fact must be able to trust it in every case, including cases nobody has tried. Inductive reasoning suggests what might be true; deductive reasoning proves it. A good mathematician uses both: induction to discover, deduction to demonstrate.



Five points on a circle, every pair joined: 16 regions.

POSTULATES AND THEOREMS

THEOREM **Sum of Two Odd Integers** Euclid IX.22

The sum of any two odd integers is even.

PARAGRAPH PROOF

Given: a and b are odd integers

Prove: $a + b$ is even

By the definition of an odd integer, there are integers m and n with $a = 2m + 1$ and $b = 2n + 1$.

Then $a + b = (2m + 1) + (2n + 1) = 2m + 2n + 2 = 2(m + n + 1)$. Since m, n and 1 are integers, $m + n + 1$ is an integer.

So $a + b$ equals 2 times an integer, and by the definition of an even integer, $a + b$ is even.

∎

Euclid IX.22 says more generally that the sum of an even number of odd numbers is even.

From history. Egyptian and Babylonian scribes knew many useful rules for measuring fields and building, found largely by experience and handed down as recipes. The Greeks asked a new question: *why* is the rule true? According to Proclus, Thales of Miletus (about 600 BC) was the first to demonstrate geometric facts, such as that a diameter cuts a circle into two equal parts. From Thales to Euclid, Greek geometry became a science of proof.

Euclid connection. The *Elements* never proves anything by checking examples: every proposition is deduced from the definitions, postulates, common notions and earlier propositions, and many end with the words translated "which was to be demonstrated" (in Latin *quod erat demonstrandum*, Q.E.D.). Books VII–IX treat whole numbers, including propositions about even and odd numbers such as IX.22.

Practice habit. Before you believe a conjecture, hunt for a counterexample: test 0, 1, 2, a negative number and a large number. Before you claim a counterexample, check that it really makes the hypothesis true and the conclusion false.

Example 1. The Miller family stacks firewood logs in a triangle: 1 log in the top row, 2 in the next row, 3 in the next, and so on, one more log in each row. Stacks with 1, 2, 3 and 4 rows hold 1, 3, 6 and 10 logs. Make a conjecture for the number of logs in a stack with n rows, check it, and use it to predict how many logs are in a 12-row stack.

Show solution

1. Look at the totals $1, 3, 6, 10$. Doubling them gives $2, 6, 12, 20$, which are $1 \cdot 2$, $2 \cdot 3$, $3 \cdot 4$, $4 \cdot 5$.
2. Conjecture: a stack with n rows holds $\dfrac{n(n+1)}{2}$ logs.
3. Check a new case by adding: 5 rows give $1 + 2 + 3 + 4 + 5 = 15$, and $\dfrac{5 \times 6}{2} = 15$. The conjecture survives the test.
4. Predict: $\dfrac{12 \times 13}{2} = 78$. (Adding $1 + 2 + \cdots + 12$ directly also gives 78.)

Answer: Conjecture: $\dfrac{n(n+1)}{2}$ logs; a 12-row stack holds 78 logs.

Example 2. Ruth adds sets of three consecutive integers: $4 + 5 + 6 = 15$, $10 + 11 + 12 = 33$, $-2 + (-1) + 0 = -3$. Each sum is a multiple of 3. Make a conjecture, then prove it deductively.

Show solution

1. Inductive step: the examples suggest the conjecture "the sum of any three consecutive integers is divisible by 3."
2. To prove it for every case, name the integers with a variable: n, $n + 1$, $n + 2$.
3. Add, simplify, and factor out 3.

PARAGRAPH PROOF

Given: n, $n + 1$ and $n + 2$ are three consecutive integers
Prove: Their sum is divisible by 3
Any three consecutive integers can be written n, $n + 1$ and $n + 2$, where n is the smallest, because each consecutive integer is 1 more than the one before.

Their sum is $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$.

Since $n + 1$ is an integer, the sum is 3 times an integer, so it is divisible by 3. This holds for every integer n, so the conjecture is proved.

∎

Answer: Conjecture: the sum of three consecutive integers is divisible by 3.
Proof: $n + (n + 1) + (n + 2) = 3(n + 1)$.

PRACTICE

1. Easy Find the next number in the pattern $3, 7, 11, 15, \ldots$

   Show answer

   19 (add 4 each time).

2. Easy Find the next number in the pattern $2, 6, 18, 54, \ldots$

   Show answer

   162 (multiply by 3 each time).

3. Easy Find a counterexample to the conjecture "Every prime number is odd."

   Show answer

   2 is prime and even.

4. **Easy** Henry has watched the sun rise in the east every morning he can remember, so he concludes that it will rise in the east tomorrow. Is this inductive or deductive reasoning?

   Show answer

   Inductive: he generalizes from many observed cases.

5. **Medium** "Every square has four right angles. The window pane $ABCD$ is a square. Therefore pane $ABCD$ has four right angles." Is this inductive or deductive reasoning?

   Show answer

   Deductive: the conclusion follows by logic from an accepted general statement and a given fact.

6. **Medium** Find the smallest positive counterexample to the conjecture "If a whole number is divisible by 4, then it is divisible by 8."

   Show answer

   4: it is divisible by 4 but not by 8.

7. **Medium** For points on a circle with every pair joined (as in the lesson figure), 1, 2, 3, 4 and 5 points give 1, 2, 4, 8 and 16 regions. What number of regions does the doubling pattern predict for 6 points? What does the true count of 31 regions teach?

   Show answer

   The pattern predicts 32 regions. Because the true count is 31, the doubling pattern was only a conjecture, and it is false: agreeing examples do not prove a pattern continues.

8. **Medium** The front row of a church balcony has 12 seats, and each row behind it has 2 more seats than the row in front. How many seats are in the 15th row?

   Show answer

   Row n has $12 + 2(n-1)$ seats, so row 15 has $12 + 2 \times 14 = 40$ seats.

9. **Medium** The expression $n^2 + n + 41$ gives a prime number for every whole number n from 0 to 39. Evaluate it for $n = 40$ and show that the conjecture "$n^2 + n + 41$ is always prime" is false.

Show answer

$40^2 + 40 + 41 = 1600 + 40 + 41 = 1681 = 41 \times 41$, which is not prime. So $n = 40$ is a counterexample.

10. **Hard** Explain why a thousand examples that agree with a conjecture cannot prove it, while one counterexample can disprove it.

Show answer

A conjecture claims something for every case, and there are usually infinitely many cases; checking some of them says nothing certain about the ones not checked (the $n^2 + n + 41$ pattern works 40 times and then fails). A counterexample is a case where the hypothesis holds and the conclusion fails, so the claim "for every case" is plainly false.

11. **Hard** Write a paragraph proof that the product of any two odd integers is odd.

Show answer

See the model proof: $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$.

PARAGRAPH PROOF

Given: a and b are odd integers

Prove: ab is odd

By the definition of an odd integer, $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n.

Multiplying, $ab = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$.

Because $2mn + m + n$ is an integer, ab is 1 more than 2 times an integer, so ab is odd by the definition of an odd integer.

∎

12. Hard Write a paragraph proof that the sum of an even integer and an odd integer is odd.

Show answer

See the model proof: $2m + (2n + 1) = 2(m + n) + 1$.

PARAGRAPH PROOF

Given: a is an even integer and b is an odd integer

Prove: $a + b$ is odd

By the definitions of even and odd integers, $a = 2m$ and $b = 2n + 1$ for some integers m and n.

Then $a + b = 2m + 2n + 1 = 2(m + n) + 1$.

Since $m + n$ is an integer, $a + b$ is 1 more than 2 times an integer, so $a + b$ is odd by the definition of an odd integer.

∎

13. Challenge A diagonal of a polygon joins two vertices that are not next to each other. A triangle has 0 diagonals, a quadrilateral 2, a pentagon 5 and a hexagon 9. The conjecture is that a polygon with n sides has $\dfrac{n(n-3)}{2}$ diagonals. Give a deductive reason why this formula is right, then use it: a rose window has 12 points equally spaced around a circle, forming a 12-sided polygon, and the glazier joins every pair of points that are not neighbors with a straight strip of lead. How many strips does he need?

Show answer

Reason: each of the n vertices can be joined to $n - 3$ others (not to itself and not to its two neighbors), so counting from every vertex gives $n(n-3)$; but each diagonal has two ends, so it was counted twice, and the number of diagonals is $\dfrac{n(n-3)}{2}$. For $n = 12$: $\dfrac{12 \times 9}{2} = 54$ strips.

COMMON MISTAKES

- Believing a pattern after a few cases. Treat every pattern as a conjecture until it is proved, and look for a counterexample first.
- Offering a “counterexample” that does not satisfy the hypothesis. To disprove “if a number is divisible by 4, then it is divisible by 8,” you need a number that IS

divisible by 4 and is NOT divisible by 8, such as 4 or 12; the number 7 proves nothing.

- Proving a statement about all integers with a single example such as $3 + 5 = 8$. Use variables ($2m + 1$, $2n + 1$) so the argument covers every case at once.
- Calling an argument deductive because it sounds confident. Ask whether the conclusion is forced by accepted statements, or only suggested by observations.

> **Make it a game · Counterexample Court.** Parent reads a bold claim: 'All prime numbers are odd,' 'The sum of any two odd numbers is even,' 'n squared plus n plus 41 is always prime.' You have 60 seconds to hunt for a counterexample. Find one and the claim is thrown out. Find none and you must say whether you have proof or only a pattern. Swap roles and write three claims for your parent.

1.3 Conditional Statements

OBJECTIVES

- Identify the hypothesis and conclusion of a conditional statement and rewrite statements in if-then form.
- Write the converse, inverse and contrapositive of a conditional and decide whether each is true, using counterexamples.
- Explain why a conditional and its contrapositive are logically equivalent.
- Write biconditional statements and test whether a definition is reversible.

WATCH

Implications & Contrapositives (1 of 2: How do they relate?)

Eddie Woo · 11:01

https://www.youtube.com/watch?v=L49rrLkhvKU

> **WHEN TO WATCH**
> Before Example 1: if-then statements and the contrapositive.

Note: Eddie Woo teaches in Australia: he says 'implication' for a conditional statement. Biconditionals are taught from the Core ideas.

BACKUP VIDEO

How to Write the Conditional, Converse, Inverse, and Contrapositive

Mario's Math Tutoring · 6:06

https://www.youtube.com/watch?v=lmm9Xfg_nOU

Before the converse, inverse and contrapositive examples: writing all four forms of one statement and deciding which are true.

KEY TERMS

conditional statement

A statement of the form "if p, then q," written $p \to q$. The part after "if" (p) is the **hypothesis**; the part after "then" (q) is the **conclusion**.

negation

The opposite of a statement, written $\sim p$ and read "not p." The negation of "the angle is acute" is "the angle is not acute."

converse

The statement formed by switching the hypothesis and conclusion of a conditional: $q \to p$.

inverse

The statement formed by negating both the hypothesis and conclusion of a conditional: $\sim p \to \sim q$.

contrapositive

The statement formed by switching and negating the hypothesis and conclusion: $\sim q \to \sim p$.

logically equivalent statements

Statements that are always both true or both false.

biconditional

A statement of the form "p if and only if q," written $p \leftrightarrow q$. It means both $p \to q$ and $q \to p$, and it is true exactly when both of those conditionals are true.

CORE IDEAS

Geometry is full of if-then statements. A **conditional statement** has a hypothesis (what is assumed) and a conclusion (what then follows): "If two angles are right angles, then they are congruent." Many statements hide their if-then form. Euclid's Postulate 4, "All right angles are equal," is the conditional just given, and "All squares are rectangles" means "If a figure is a square, then it is a rectangle." Rewriting a statement in if-then form is the first step to using it in a proof.

When is a conditional true? Think of it as a promise: "If it rains on Saturday, then Father will drive us to the library." The promise is broken in only one situation: it rains and he does not drive. So a conditional is **false only when the hypothesis is true and the conclusion is false**, and one such case, a **counterexample**, is enough to show it is false. If it does not rain, the promise has not been broken, whatever happens, so a conditional with a false hypothesis counts as true.

From any conditional $p \to q$ we can form three related statements. Take "If an animal is a robin, then it is a bird" (true). The **converse** switches the parts: "If an animal is a bird, then it is a robin" (false: a sparrow). The **inverse** negates both parts: "If an animal is not a robin, then it is not a bird" (false: a sparrow again). The **contrapositive** switches and negates: "If an animal is not a bird, then it is not a robin" (true).

Why is the contrapositive always true when the original is? Suppose $p \to q$ is true and we are in a case where q is false. Could p be true there? No, because then the true conditional would force q to be true. So p must be false, which is exactly what $\sim q \to \sim p$ says. A conditional and its contrapositive are **logically equivalent**: they stand or fall together. In the same way, the inverse is the contrapositive of the converse, so the converse and inverse are equivalent to each other. But a conditional and its converse are **not** equivalent: "If a figure is a square, then it is a rectangle" is true, while its converse is false.

When a conditional and its converse are both true, we can join them in a **biconditional** using "if and only if": "An angle is a right angle if and only if its measure is $90°$." Every good definition is a biconditional: it must work in both directions, so that every object with the stated property gets the name and every object with the name has the property. A definition that fails in one direction lets the wrong objects in, or shuts the right ones out.

Because a converse does not follow automatically, Euclid proves converses separately. Proposition I.5 shows that in an isosceles triangle the base angles are equal; I.6 proves the converse. I.47 is the Pythagorean theorem; I.48 proves its converse. Whenever you want to use a converse in this course, first ask: has it been proved?

POSTULATES AND THEOREMS

THEOREM **Contrapositive Equivalence**

A conditional statement and its contrapositive are logically equivalent: they are both true or both false.

PARAGRAPH PROOF

Prove: $p \to q$ and $\sim q \to \sim p$ are both true or both false

First suppose $p \to q$ is true. Take any case in which $\sim q$ holds, that is, q is false. If p were true in this case, then, since $p \to q$ is true, q would be true, which it is not. So p is false, that is, $\sim p$ holds. Thus whenever $\sim q$ holds, $\sim p$ holds, so $\sim q \to \sim p$ is true.

Now suppose $\sim q \to \sim p$ is true. Its contrapositive is $\sim(\sim p) \to \sim(\sim q)$, which is $p \to q$. By the argument in the first paragraph (applied to the conditional $\sim q \to \sim p$), its contrapositive $p \to q$ is true.

So if either statement is true, the other is true. Hence if either is false, the other cannot be true, so it is false too. The two statements are logically equivalent.

∎

COROLLARY **Converse–Inverse Equivalence**

The converse and the inverse of a conditional are logically equivalent.

PARAGRAPH PROOF

Prove: $q \to p$ and $\sim p \to \sim q$ are both true or both false

The converse of $p \to q$ is $q \to p$. Switching and negating its parts gives its contrapositive, $\sim p \to \sim q$, which is exactly the inverse of $p \to q$.

By the Contrapositive Equivalence Theorem, a conditional and its contrapositive are logically equivalent, so the converse and the inverse are logically equivalent.

∎

KEY FORMULAS

$$\text{Conditional: } p \to q$$

$$\text{Converse: } q \to p$$

$$\text{Inverse: } \sim p \to \sim q$$

$$\text{Contrapositive: } \sim q \to \sim p$$

$$\text{Biconditional: } p \leftrightarrow q \text{ means } (p \to q) \text{ and } (q \to p)$$

Euclid connection. Each proposition of the *Elements* is a conditional: it states what is given (the hypothesis) and what follows (the conclusion). Euclid proves many converses as separate propositions, for example I.5 and its converse I.6, I.27 and its converse I.29, and I.47 and its converse I.48, because a converse has to be earned on its own.

Practice habit. Every time you meet a new conditional, write its contrapositive beside it (it comes free), and check whether its converse is true before you ever use it.

WORKED EXAMPLES

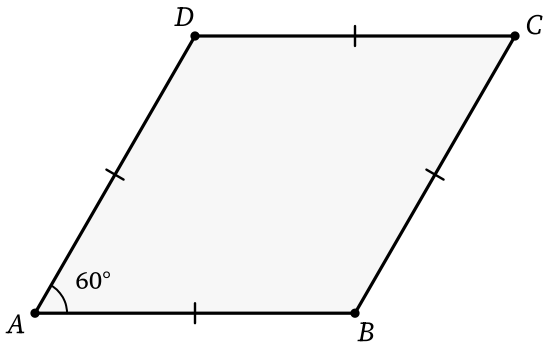
Example 1. Consider the true statement "If an angle measures 30°, then it is acute." Write its converse, inverse and contrapositive, and decide whether each is true. Give a counterexample for any that is false.

Show solution

1. Hypothesis p: an angle measures 30°. Conclusion q: it is acute.
2. Converse ($q \to p$): If an angle is acute, then it measures 30°. False: a 50° angle is acute but does not measure 30°.
3. Inverse ($\sim p \to \sim q$): If an angle does not measure 30°, then it is not acute. False: the same 50° angle does not measure 30°, yet it is acute.
4. Contrapositive ($\sim q \to \sim p$): If an angle is not acute, then it does not measure 30°. True, because it is equivalent to the true original statement.

Answer: Converse: false (counterexample 50°). Inverse: false (counterexample 50°). Contrapositive: true.

Example 2. Ruth proposes this definition: "A square is a quadrilateral with four congruent sides." Write it as a biconditional, split the biconditional into two conditionals, and decide whether it is a good definition. The figure shows quadrilateral $ABCD$ with four congruent sides and $m\angle DAB = 60°$.



Show solution

1. Biconditional: A quadrilateral is a square if and only if it has four congruent sides.
2. First conditional: If a quadrilateral is a square, then it has four congruent sides. True.
3. Second conditional: If a quadrilateral has four congruent sides, then it is a square. Look for a counterexample: $ABCD$ in the figure has four congruent sides, but its $60°$ angle is not a right angle, so it is not a square (it is a rhombus).
4. Since one direction is false, the biconditional is false, and the definition is not reversible. A good definition: a square is a quadrilateral with four congruent sides and four right angles.

Answer: No, it is not a good definition: the conditional "if a quadrilateral has four congruent sides, then it is a square" is false, as the rhombus $ABCD$ shows.

PRACTICE

1. Easy Identify the hypothesis and the conclusion: "If two angles are right angles, then they are congruent."

   Show answer

   Hypothesis: two angles are right angles. Conclusion: they are congruent.

2. **Easy** Rewrite in if-then form: “All squares are rectangles.”

   Show answer

   If a figure is a square, then it is a rectangle.

3. **Easy** Is this conditional true or false? “If a whole number is divisible by 10, then it is divisible by 5.”

   Show answer

   True: a number divisible by 10 is $10k = 5(2k)$, so it is divisible by 5.

4. **Easy** Write the converse of the statement in Problem 3. Is the converse true or false? If false, give a counterexample.

   Show answer

   Converse: If a whole number is divisible by 5, then it is divisible by 10. False: 15 is divisible by 5 but not by 10.

5. **Medium** Conditional: “If $x = 4$, then $x^2 = 16$.” Write its converse and decide whether the converse is true. If it is false, give a counterexample.

   Show answer

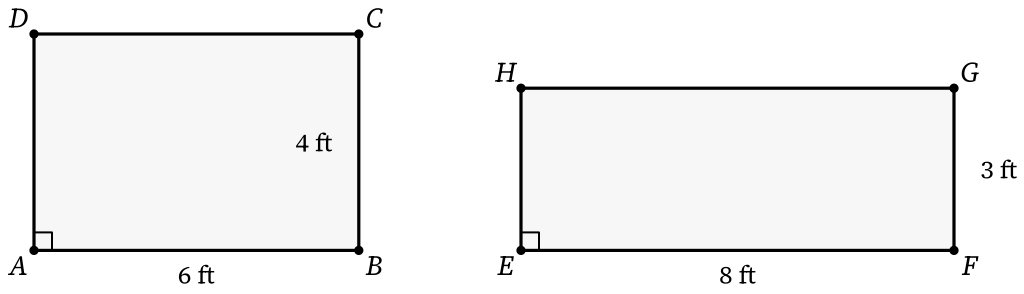
   Converse: If $x^2 = 16$, then $x = 4$. False: $x = -4$ gives $x^2 = 16$ but $x \neq 4$.

6. **Medium** Original statement: “If the church bell rings, then the service begins.” Is the statement “If the service does not begin, then the church bell does not ring” its converse, inverse or contrapositive?

   Show answer

   The contrapositive (hypothesis and conclusion switched and both negated).

7. **Medium** Henry claims: “If a rectangular garden has an area of 24 square feet, then its perimeter is 20 feet.” The figure shows two rectangular gardens. Is Henry's statement true or false? Explain using the figure.



Show answer

False. Both gardens have area $6 \times 4 = 24$ and $8 \times 3 = 24$ square feet, but the second has perimeter $2(8 + 3) = 22$ feet, not 20 feet. It is a counterexample.

8. **Medium** The statement “If a figure is a square, then it is a rectangle” is true. Without testing any examples, is its contrapositive true or false? Write the contrapositive and explain.

Show answer

True. Contrapositive: If a figure is not a rectangle, then it is not a square. A conditional and its contrapositive are logically equivalent, so it is true because the original is.

9. **Medium** Write the biconditional “An angle is a straight angle if and only if its measure is $180°$” as two conditional statements.

Show answer

If an angle is a straight angle, then its measure is $180°$. If an angle's measure is $180°$, then it is a straight angle.

10. **Hard** The statement “If two integers are both even, then their sum is even” is true. Can it be written as a true biconditional? Explain.

Show answer

No. The converse, “If the sum of two integers is even, then both integers are even,” is false: $3 + 5 = 8$ is even, but 3 and 5 are odd. A biconditional needs both directions true.

11. Hard Mother tells Samuel, “If you finish splitting the kindling, then you may go fishing.” Samuel does not finish, and he does not go fishing. Explain why Mother's statement still counts as true. Then describe the only situation that would make it false.

Show answer

The statement promises something only when the hypothesis (finishing the kindling) is true. Since Samuel did not finish, the promise was never tested, so it was not broken, and a conditional that has not been broken is counted true. It would be false only if Samuel finished the kindling and was still not allowed to go fishing (hypothesis true, conclusion false).

12. Hard The statement “If a quadrilateral is a square, then it has four right angles” is true. Which one of its converse, inverse and contrapositive must also be true? Check the other two with an example.

Show answer

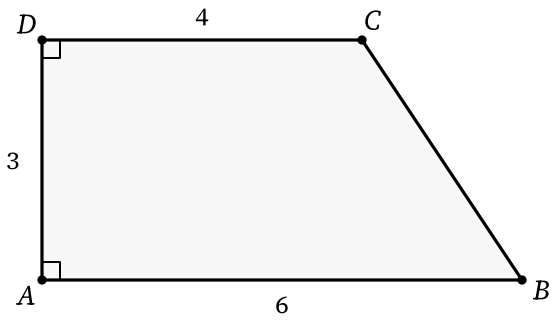
The contrapositive (“If a quadrilateral does not have four right angles, then it is not a square”) must be true. The converse (“If a quadrilateral has four right angles, then it is a square”) and the inverse (“If a quadrilateral is not a square, then it does not have four right angles”) are both false: a 4-by-6 rectangle is a counterexample to each.

13. Challenge Grace says, “The converse of a statement and its inverse always have the same truth value.” Explain why she is right, using the statement “If a number is divisible by 6, then it is even” as an illustration.

Show answer

The inverse $\sim p \rightarrow \sim q$ is the contrapositive of the converse $q \rightarrow p$ (switch and negate $q \rightarrow p$), and a conditional is always logically equivalent to its contrapositive. Illustration: converse “If a number is even, then it is divisible by 6” is false (4); inverse “If a number is not divisible by 6, then it is not even” is also false (4 again). Both fail for the same counterexample.

14. **Challenge** Samuel proposes: “A rectangle is a quadrilateral with two right angles.” Write the proposed definition as a biconditional and decide whether it is a good definition. Use quadrilateral $ABCD$ in the figure, which has right angles at A and D, $AB = 6$, $DC = 4$ and $AD = 3$.



Show answer

Biconditional: A quadrilateral is a rectangle if and only if it has two right angles. Not a good definition: $ABCD$ has two right angles, but it is not a rectangle, since its opposite sides $\overline{AB}$ and $\overline{DC}$ have different lengths (6 and 4), while opposite sides of a rectangle are equal. So “if a quadrilateral has two right angles, then it is a rectangle” is false.

COMMON MISTAKES

- Mixing up the converse and the inverse. Converse = switch; inverse = negate; contrapositive = switch AND negate. Say the three words with their actions until they are automatic.
- Assuming the converse of a true statement is true. “If it is a square, then it is a rectangle” is true; its converse is not. Always test a converse before using it.
- Giving a counterexample that does not satisfy the hypothesis. A counterexample to $p \to q$ must make p true and q false.
- Accepting a definition that works in only one direction. Test both conditionals of the biconditional: every good definition is reversible.

> **Make it a game · Flip It Four Ways.** Parent states an if-then sentence: 'If a shape is a square, then it is a rectangle.' You have 20 seconds to give the converse, the inverse and the contrapositive, and call each true or false, backing every false one with a counterexample. One point per correct form. Then write three conditionals of your own and try to stump your parent.

1.4 The Laws of Logic

OBJECTIVES

- Apply the Law of Detachment and the Law of Syllogism to draw valid conclusions.
- Recognize the invalid forms of affirming the conclusion and denying the hypothesis.
- Use Venn diagrams to test whether an argument is valid.
- Chain several given conditionals into a valid conclusion.

WATCH

Foundations of Proof (1 of 2: Statements, implications, negations)

Eddie Woo · 14:20

https://www.youtube.com/watch?v=hFULZsBQiqQ

> **WHEN TO WATCH**
> After the Core ideas: statements, implications and negations, the raw material of the laws of logic.

Note: He does not name the Law of Detachment or the Law of Syllogism; the course does.

BACKUP VIDEO

Implications & Contrapositives (2 of 2: Further examples)

Eddie Woo · 7:44

https://www.youtube.com/watch?v=HqnEi2ybKjk

Before the practice set: more on implications and their contrapositives.

KEY TERMS

argument

A set of statements called premises followed by a statement called the conclusion, which is claimed to follow from them.

valid argument

An argument in which the conclusion must be true whenever all the premises are true. Validity depends on the form of the argument, not on whether the premises are actually true.

Law of Detachment

If $p \to q$ is true and p is true, then q is true. (Also called *modus ponens*.)

Law of Syllogism

If $p \to q$ and $q \to r$ are both true, then $p \to r$ is true.

affirming the conclusion

The invalid argument "$p \to q$ is true and q is true, so p is true." It treats the converse as if it were true.

denying the hypothesis

The invalid argument "$p \to q$ is true and p is false, so q is false." It treats the inverse as if it were true.

Venn diagram

A drawing that shows sets as circles; "all A are B" is shown by drawing the circle for A inside the circle for B.

CORE IDEAS

An **argument** gives some statements, the premises, and claims that a conclusion follows from them. An argument is **valid** when the conclusion *must* be true whenever the premises are true. Validity is about the shape of the reasoning. "If an animal is a fish, then it can fly. A trout is a fish. So a trout can fly" is valid in form, but its first premise is false, so its conclusion cannot be trusted. A proof needs both: true premises and valid steps.

The **Law of Detachment** says: if $p \to q$ is true and p is true, then q is true. This is simply what a true conditional means; once the hypothesis is satisfied, you may "detach" the conclusion and use it. "If two angles are right angles, then they are congruent. $\angle A$ and $\angle B$ are right angles. So $\angle A \cong \angle B$." Almost every step of a geometry proof is a detachment: a definition, postulate or theorem is applied to a case that meets its hypothesis.

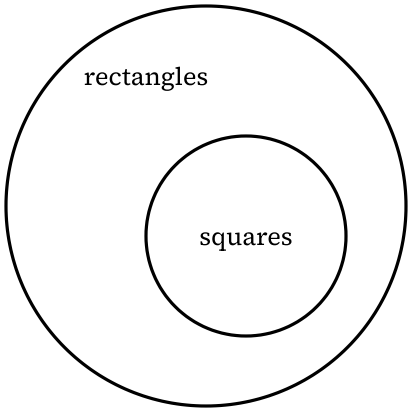
The **Law of Syllogism** chains conditionals: if $p \to q$ and $q \to r$ are true, then $p \to r$ is true. Why? Whenever p is true, detachment gives q; then detachment again gives r. A long proof is just a long chain of this kind, each link fastened to the next, so that the given information is joined to the conclusion.

Two tempting forms of argument are **invalid**. *Affirming the conclusion*: "If an animal is a robin, then it has feathers. My pet has feathers. So it is a robin." (It might be a parakeet.) *Denying the hypothesis*: "If an animal is a robin, then it has feathers. My pet

is not a robin. So it has no feathers." (Again, a parakeet.) The first quietly uses the converse and the second the inverse, and neither is guaranteed. By contrast, *denying the conclusion* is valid: "My pet has no feathers, so it is not a robin," because that is the contrapositive at work.

A **Venn diagram** makes these laws visible. "If p, then q" is the same as "every p is a q," so draw the p circle inside the q circle. Anything inside the small circle is inside the large one (detachment). Something inside the large circle may lie outside the small one (so affirming the conclusion fails). Something outside the small circle may still be inside the large one (so denying the hypothesis fails). And something outside the large circle cannot be inside the small one (so denying the conclusion is valid).

To draw a conclusion from several statements, first rewrite each one in if-then form. Then line them up so that the conclusion of one is the hypothesis of the next, using a contrapositive if a link does not match. Finally apply the Law of Syllogism to the chain, and the Law of Detachment if you know a particular case satisfies the first hypothesis.



POSTULATES AND THEOREMS

POSTULATE **Law of Detachment**

If $p \to q$ is a true conditional and p is true, then q is true.

Accepted as a basic rule of logic: it is the meaning of a true conditional, which is false only when its hypothesis is true and its conclusion false.

THEOREM **Law of Syllogism**

If $p \to q$ and $q \to r$ are true conditionals, then $p \to r$ is true.

PARAGRAPH PROOF

Given: $p \to q$ is true; $q \to r$ is true

Prove: $p \to r$ is true

To show $p \to r$ is true, we must show that in every case in which p is true, r is also true. So take any case in which p is true.

Since $p \to q$ is true and p is true, q is true by the Law of Detachment. Since $q \to r$ is true and q is true, r is true by the Law of Detachment again.

So every case with p true has r true, and $p \to r$ is true.

∎

THEOREM **Law of the Contrapositive (Denying the Conclusion)**

If $p \to q$ is true and q is false, then p is false.

PARAGRAPH PROOF

Given: $p \to q$ is true; q is false

Prove: p is false

Since $p \to q$ is true, its contrapositive $\sim q \to \sim p$ is true by the Contrapositive Equivalence Theorem (Lesson 1.3).

Since q is false, $\sim q$ is true. By the Law of Detachment applied to $\sim q \to \sim p$, $\sim p$ is true, which means p is false.

∎

KEY FORMULAS

Law of Detachment: $p \to q$ and p give q

Law of Syllogism: $p \to q$ and $q \to r$ give $p \to r$

Law of the Contrapositive: $p \to q$ and $\sim q$ give $\sim p$

From history. The Greek philosopher Aristotle (384–322 BC) wrote the first systematic study of valid reasoning, and the word *syllogism* comes from his work. The circle diagrams in this lesson are named for the English logician John Venn, who described them in 1880. Charles Dodgson, a mathematics lecturer at Christ Church, Oxford, better known as Lewis Carroll, wrote playful logic puzzles for young people and also a book, *Euclid and His Modern Rivals* (1879), defending Euclid as the best textbook for learning geometry.

Euclid connection. Every proposition of the *Elements* is a chain of these laws. In I.1, for example, Euclid applies the definition of a circle (detachment) to get $AC = AB$ and $BC = AB$, then applies Common Notion 1 to conclude $AC = BC$, linking each step to the next until the proposition is demonstrated.

Practice habit. Rewrite every premise in if-then form before you reason with it, and recite the two valid laws (Detachment and Syllogism) and the two invalid forms (affirming the conclusion, denying the hypothesis) until you can spot them instantly.

WORKED EXAMPLES

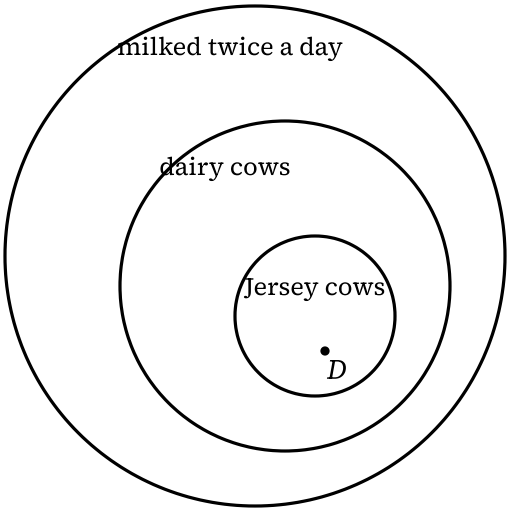
Example 1. Decide whether each argument is valid. Name the law used or the error made. (a) "If an angle measures 90°, then it is a right angle. $\angle K$ measures 90°. So $\angle K$ is a right angle." (b) "If Grace practices the hymn every day, then she plays it well on Sunday. Grace played it well on Sunday. So she practiced it every day."

Show solution

1. In (a), let p: an angle measures 90°, and q: it is a right angle. The premises are $p \to q$ and p (for $\angle K$), and the conclusion is q. This is the Law of Detachment, so (a) is valid.
2. In (b), let p: Grace practices every day, and q: she plays well on Sunday. The premises are $p \to q$ and q, and the conclusion is p. This is affirming the conclusion.
3. Check (b) for a way the premises could be true and the conclusion false: perhaps Grace already knew the hymn well and practiced only twice. So (b) is invalid.

Answer: (a) Valid (Law of Detachment). (b) Not valid (affirming the conclusion).

Example 2. On the Miller farm: "Every Jersey cow is a dairy cow. Every dairy cow is milked twice a day. Daisy is a Jersey cow." (a) Is Daisy milked twice a day? (b) Bess is milked twice a day. Can you conclude that Bess is a Jersey cow? Use the Venn diagram, in which point D stands for Daisy.



Show solution

1. Rewrite: If a cow is a Jersey, then it is a dairy cow ($p \to q$). If a cow is a dairy cow, then it is milked twice a day ($q \to r$).
2. Law of Syllogism: If a cow is a Jersey, then it is milked twice a day ($p \to r$). In the diagram, the Jersey circle lies inside the dairy circle, which lies inside the milked-twice circle.
3. (a) Daisy is a Jersey (p), so by the Law of Detachment she is milked twice a day (r). Point D lies inside all three circles.
4. (b) Knowing Bess is milked twice a day (r) and concluding she is a Jersey (p) would be affirming the conclusion. Bess could be anywhere in the largest circle, perhaps a dairy cow of another breed. No conclusion follows.

Answer: (a) Yes, by the Law of Syllogism and the Law of Detachment. (b) No; that would be affirming the conclusion.

PRACTICE

1. Easy "If the rain gauge shows more than 2 inches, then the creek floods. On Monday the rain gauge shows 3 inches. So on Monday the creek floods." Which law of logic is used?

   Show answer

   The Law of Detachment.

2. Easy Use the Law of Syllogism to write the conclusion: "If a figure is a square, then it is a rectangle. If a figure is a rectangle, then it is a parallelogram."

   Show answer

   If a figure is a square, then it is a parallelogram.

3. Easy Is this argument valid? "If a whole number ends in 0, then it is divisible by 10. The number 340 ends in 0. So 340 is divisible by 10."

   Show answer

   Yes, it is valid by the Law of Detachment.

4. Easy Is this argument valid? "If an animal is a robin, then it has feathers. Ruth's pet has feathers. So Ruth's pet is a robin."

   Show answer

   No. It affirms the conclusion; Ruth's pet could be a canary.

5. Medium Name the error: "If Henry practices the piano each day, then he improves. Henry did not practice each day this week. So he did not improve."

   Show answer

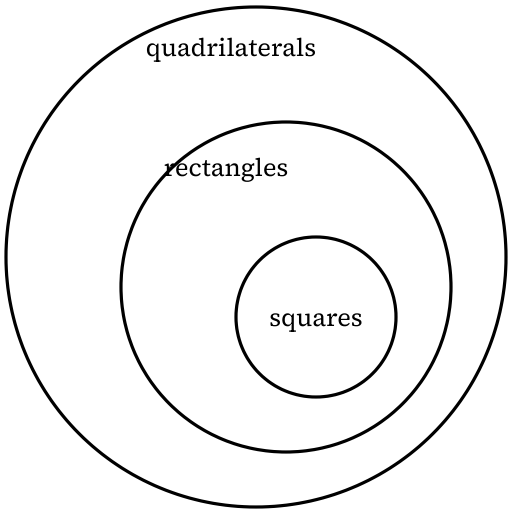
   Denying the hypothesis. Henry may still have improved, for example by practicing on some days.

6. Medium You know: "If $x + 5 = 12$, then $x = 7$" and "If $x = 7$, then $3x = 21$." You are also told that $x + 5 = 12$. What is the value of $3x$, and which laws did you use?

   Show answer

   $3x = 21$. The Law of Syllogism gives "if $x + 5 = 12$, then $3x = 21$," and the Law of Detachment applies it.

7. **Medium** The Venn diagram shows that all squares are rectangles and all rectangles are quadrilaterals. Figure $WXYZ$ is a rectangle. (a) Must $WXYZ$ be a quadrilateral? (b) Must $WXYZ$ be a square?



Show answer

(a) Yes: the rectangle circle lies inside the quadrilateral circle (Law of Detachment). (b) No: a point inside the rectangle circle may lie outside the square circle; concluding "square" would be affirming the conclusion of "if square, then rectangle."

8. **Medium** "If the barn door is locked, then the lantern inside is off." Tonight the lantern inside is on. Is the barn door locked? Name the law you use.

Show answer

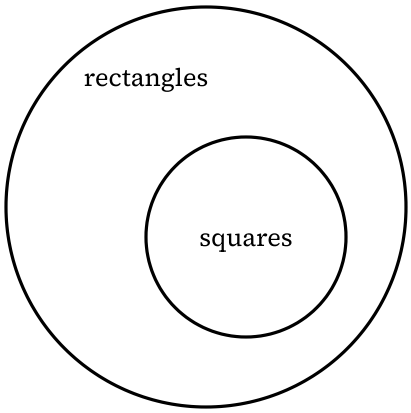
No. The lantern is not off, so by the Law of the Contrapositive (denying the conclusion), the door is not locked.

9. **Medium** Given: “If the creek floods, then the county road closes.” “If the school bus is late, then Grace misses the first class.” “If the county road closes, then the school bus is late.” This morning the creek floods. Does Grace miss the first class? Show the chain.

Show answer

Yes. Order the statements: creek floods $\rightarrow$ road closes $\rightarrow$ bus is late $\rightarrow$ Grace misses the first class. The Law of Syllogism gives “if the creek floods, then Grace misses the first class,” and the Law of Detachment applies it to this morning.

10. **Hard** Use the Venn diagram in the lesson (squares inside rectangles) to explain why “If a figure is a square, then it is a rectangle. $ABCD$ is a rectangle. So $ABCD$ is a square” is not valid.



Show answer

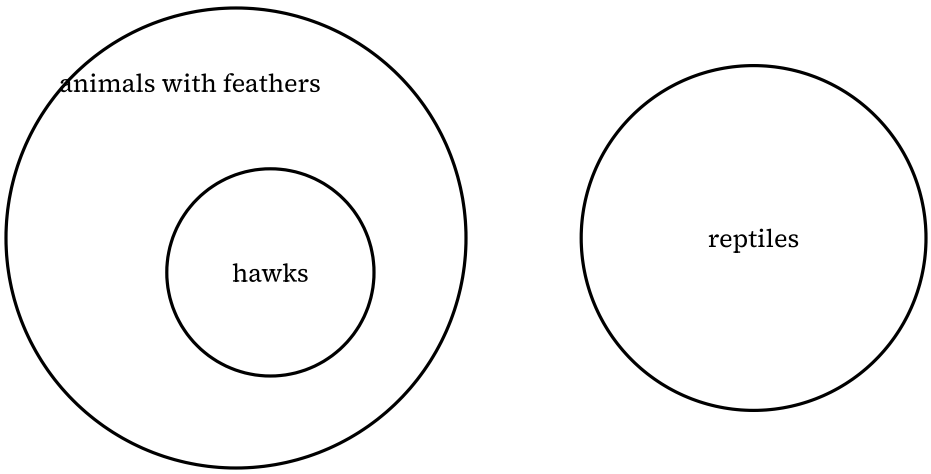
The premise “if square, then rectangle” puts the square circle inside the rectangle circle. Knowing $ABCD$ is a rectangle places it somewhere inside the larger circle, and it could be in the part outside the square circle, like a 4-by-6 rectangle. The premises can be true while the conclusion is false, so the argument (affirming the conclusion) is invalid.

11. Hard "If a whole number is divisible by 6, then it is divisible by 3. If a whole number is divisible by 3, then it is odd. So if a whole number is divisible by 6, then it is odd." (a) Is the argument valid? (b) Is its conclusion true?

Show answer

(a) Yes: it has the form of the Law of Syllogism. (b) No: 12 is divisible by 6 and is even. The second premise is false (6 is divisible by 3 and even), which is why a valid argument reached a false conclusion.

12. Hard "All hawks have feathers. No reptile has feathers." Using the Venn diagram, decide whether a hawk can be a reptile, and explain.



Show answer

No. The hawk circle lies inside the feathered circle, and the reptile circle does not touch the feathered circle, so no point can be in both the hawk circle and the reptile circle. In words: if an animal is a hawk, it has feathers; if it has feathers, it is not a reptile (contrapositive of "if a reptile, then no feathers"); so by the Law of Syllogism, if it is a hawk, it is not a reptile.

13. **Challenge** "All of the Millers' kittens are playful. No playful animal sleeps all day. Patch is one of the Millers' kittens." Does Patch sleep all day? Write the chain of reasoning, naming each law.

Show answer

No. If an animal is one of the Millers' kittens, then it is playful. If an animal is playful, then it does not sleep all day. By the Law of Syllogism, if an animal is one of the Millers' kittens, then it does not sleep all day. Patch is one of the Millers' kittens, so by the Law of Detachment, Patch does not sleep all day.

14. **Challenge** "If the oven light is on, then the bread is baking. If the timer has rung, then the bread is not baking." The timer has rung. Is the oven light on? Explain each step.

Show answer

No. The timer has rung, so by the Law of Detachment the bread is not baking. Since "if the oven light is on, then the bread is baking" is true and the bread is not baking, the Law of the Contrapositive gives: the oven light is not on.

COMMON MISTAKES

- Affirming the conclusion: from "if p, then q" and q, concluding p. Picture the Venn diagram: being in the big circle does not put you in the small one.
- Denying the hypothesis: from "if p, then q" and not p, concluding not q. Being outside the small circle does not put you outside the big one.
- Chaining statements whose links do not match. In the Law of Syllogism the conclusion of the first conditional must be exactly the hypothesis of the second; if not, try a contrapositive.
- Thinking a valid argument always has a true conclusion. Validity guarantees a true conclusion only when every premise is true, so check the premises too.

> **Make it a game · Valid or Fake.** Parent reads six short arguments. Three use the Law of Detachment or Syllogism; three commit a fallacy, such as 'If it rains, the street is wet. The street is wet. So it rained.' Call each one valid or fake, name the law or the error, and give a counterexample for every fake. Then write three arguments of your own to catch your parent.

1.5 Algebraic Proof and the Two-Column Format

OBJECTIVES

- Name the properties of equality and use them as reasons for the steps in solving an equation.
- Write two-column proofs of algebraic statements.
- Connect Euclid's common notions to the modern properties of equality.
- Use the reflexive, symmetric and transitive properties of congruence.

WATCH

Formal Geometry Proofs (2 of 3: Investigating angles in triangles)

Eddie Woo · 7:26

https://www.youtube.com/watch?v=VuMUwuIu6lA

> **WHEN TO WATCH**
> Before Example 1: writing each step of an argument with its reason.

Note: Eddie Woo teaches in Australia: he writes a statement and its reason line by line rather than in a two-column table, and does not name the properties of equality. The course gives those names.

BACKUP VIDEO

Formal Geometry Proofs (3 of 3: Completing the geometric proof)

Eddie Woo · 5:54

https://www.youtube.com/watch?v=9TUPX5zOZVI

Before the practice set: a second worked proof with a reason on every line.

KEY TERMS

properties of equality

Accepted rules for working with equal numbers: addition, subtraction, multiplication, division, substitution, reflexive, symmetric and transitive properties (plus the distributive property of numbers).

Substitution Property of Equality

If $a = b$, then a may be replaced by b in any expression or equation.

Reflexive, Symmetric and Transitive Properties

Reflexive: $a = a$. Symmetric: if $a = b$, then $b = a$. Transitive: if $a = b$ and $b = c$, then $a = c$.

two-column proof

A proof written as numbered statements in the left column with a reason for each in the right column, starting from the given information and ending with the statement to be proved.

congruent segments and angles

Segments are congruent ($\cong$) if they have equal lengths; angles are congruent if they have equal measures. ($\overline{AB} \cong \overline{CD}$ means $AB = CD$; $\angle A \cong \angle B$ means $m\angle A = m\angle B$.) These definitions are studied further in Lessons 2.2 and 2.3.

CORE IDEAS

You have solved equations for years. Solving an equation is really a small deductive proof: from the given equation, each new equation follows by a rule. In geometry we write those rules down as **reasons**, so that every step can be checked and nobody has to trust a guess.

The rules are the **properties of equality**. If $a = b$, then: $a + c = b + c$ (Addition); $a - c = b - c$ (Subtraction); $ac = bc$ (Multiplication); $\frac{a}{c} = \frac{b}{c}$ when $c \neq 0$ (Division); and b may replace a anywhere (Substitution). Also $a = a$ (Reflexive); if $a = b$ then $b = a$ (Symmetric); if $a = b$ and $b = c$ then $a = c$ (Transitive). And $a(b + c) = ab + ac$ (Distributive). Why can we trust them? An equation is a balance: a and b name the same number, so doing the same thing to that number on both sides cannot make the two sides different.

These properties have ancient ancestors in Euclid's common notions. Common Notion 1, “things equal to the same thing are equal to each other,” is the Transitive Property (with the Symmetric Property built in). Common Notion 2, “if equals are added to equals, the wholes are equal,” is the Addition Property, and Common Notion 3, “if equals are subtracted from equals, the remainders are equal,” is the

Subtraction Property. Common Notion 4, “things that coincide are equal,” is the idea behind congruence and the Reflexive Property: a figure fits exactly on itself.

A **two-column proof** sets this out clearly. Above it you write the **Given** (what you may assume) and the **Prove** (what you must reach). Then each row has one statement on the left and its reason on the right. The first rows are usually the given facts, with the reason “Given.” The last statement is exactly the Prove statement. A reason is never “because I can see it” or “algebra”; it is a named property, definition, postulate or theorem.

Be careful to choose the right name. If you subtract 7 from both sides, the reason is the Subtraction Property of Equality, not the Addition Property. If you replace a quantity by an equal one, it is Substitution. If you collect like terms on one side, such as $6x - 8 + 5$ becoming $6x - 3$, write “Simplify”: that step does not change either side, it only rewrites it.

In geometry we compare figures, not just numbers. Two segments are **congruent** when their lengths are equal, and two angles are congruent when their measures are equal. Use $=$ between numbers ($AB = CD$, $m\angle A = m\angle B$) and $\cong$ between figures ($\overline{AB} \cong \overline{CD}$, $\angle A \cong \angle B$). Because congruence is defined by equal measures, it inherits three properties of equality: congruence is reflexive, symmetric and transitive. You will use these in every unit that follows.

POSTULATES AND THEOREMS

POSTULATE **Properties of Equality**

For all real numbers a, b and c: if $a = b$, then $a + c = b + c$, $a - c = b - c$, $ac = bc$, and $\frac{a}{c} = \frac{b}{c}$ when $c \neq 0$; if $a = b$, then b may replace a in any expression; $a = a$; if $a = b$, then $b = a$; if $a = b$ and $b = c$, then $a = c$; and $a(b + c) = ab + ac$.

Accepted from algebra as starting points. Euclid's Common Notions 1, 2 and 3 are the ancient forms of the Transitive, Addition and Subtraction Properties.

THEOREM **Properties of Segment Congruence**

Segment congruence is reflexive ($\overline{AB} \cong \overline{AB}$), symmetric (if $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$) and transitive (if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$).

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$; $\overline{CD} \cong \overline{EF}$

Prove: $\overline{AB} \cong \overline{EF}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}, \overline{CD} \cong \overline{EF}$	Given
2	$AB = CD, CD = EF$	Definition of congruent segments
3	$AB = EF$	Transitive Property of Equality
4	$\overline{AB} \cong \overline{EF}$	Definition of congruent segments

∎

The proof shown is for the transitive part. The reflexive part follows the same way from $AB = AB$ (Reflexive Property of Equality), and the symmetric part is Practice Problem 12.

THEOREM **Properties of Angle Congruence**

Angle congruence is reflexive ($\angle A \cong \angle A$), symmetric (if $\angle A \cong \angle B$, then $\angle B \cong \angle A$) and transitive (if $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$).

TWO-COLUMN PROOF

Given: $\angle A \cong \angle B$

Prove: $\angle B \cong \angle A$

	STATEMENTS	REASONS
1	$\angle A \cong \angle B$	Given
2	$m\angle A = m\angle B$	Definition of congruent angles
3	$m\angle B = m\angle A$	Symmetric Property of Equality
4	$\angle B \cong \angle A$	Definition of congruent angles

∎

The proof shown is for the symmetric part; the reflexive and transitive parts are proved the same way, using the Reflexive and Transitive Properties of Equality.

KEY FORMULAS

Addition and Subtraction: if $a = b,$ then $a + c = b + c$ and $a - c = b - c$

Multiplication and Division: if $a = b,$ then $ac = bc$ and, if $c \neq 0,$ $\dfrac{a}{c} = \dfrac{b}{c}$

Substitution: if $a = b,$ then b may replace a in any expression or equation

Reflexive: $a = a$ Symmetric: if $a = b,$ then $b = a$ Transitive: if $a = b$ and $b = c,$ then $a = c$

Distributive: $a(b + c) = ab + ac$

From history. The word *algebra* comes from the title of a book written in Baghdad in the ninth century by the mathematician al-Khwarizmi. *Al-jabr*, "restoring," named the step of adding the same quantity to both sides of an equation to remove a subtracted term, the Addition Property of Equality in action. Its companion *al-muqabala*, "balancing," named the step of cancelling equal terms from both sides.

Euclid connection. Euclid's common notions are the ancestors of the properties of equality: C.N. 1 is the Transitive Property, C.N. 2 the Addition Property, C.N. 3 the Subtraction Property, and C.N. 4 (things that coincide are equal) lies behind congruence. Euclid applies them to lengths, angles and areas exactly as we apply them to numbers.

Practice habit. Recite the eight properties of equality (addition, subtraction, multiplication, division, substitution, reflexive, symmetric, transitive) and the distributive property until you can name each one instantly. In every proof, write a reason beside every statement, and make your last statement match the Prove line exactly.

WORKED EXAMPLES

Example 1. Write a two-column proof. Given: $5(x - 3) = 2x + 9$. Prove: $x = 8$.

Show solution

1. Plan: clear the parentheses, collect the x-terms on one side, collect the numbers on the other, then divide.
2. Name the property for each move: Distributive, then Subtraction (take $2x$ from both sides), then Addition (add 15 to both sides), then Division (by 3).
3. Check: $5(8 - 3) = 25$ and $2(8) + 9 = 25$.

TWO-COLUMN PROOF

Given: $5(x - 3) = 2x + 9$

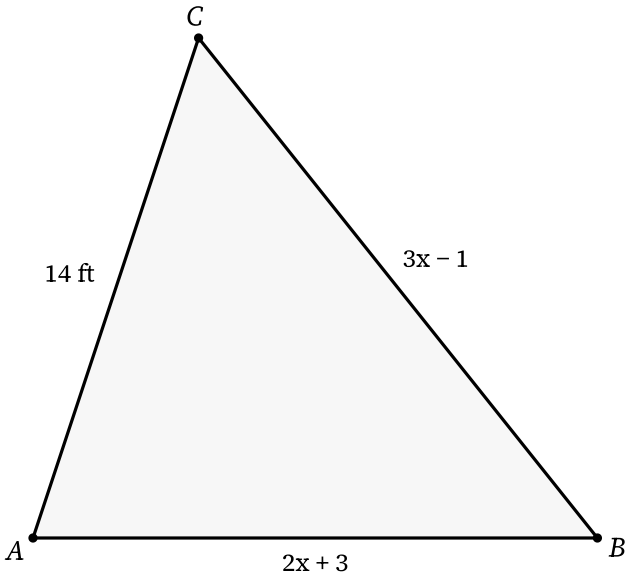
Prove: $x = 8$

	STATEMENTS	REASONS
1	$5(x - 3) = 2x + 9$	Given
2	$5x - 15 = 2x + 9$	Distributive Property
3	$3x - 15 = 9$	Subtraction Property of Equality
4	$3x = 24$	Addition Property of Equality
5	$x = 8$	Division Property of Equality

∎

Answer: $x = 8$

Example 2. Grace is edging a triangular herb bed ABC with 46 feet of edging, exactly enough to go around it once. In the figure, $AB = 2x + 3$ feet, $BC = 3x - 1$ feet and $AC = 14$ feet. Write a two-column proof that $x = 6$, and then find AB and BC.



Show solution

1. The edging goes once around, so the perimeter is 46 feet: $AB + BC + AC = 46$.
2. Substitute the three expressions, simplify, and solve, naming a property for each step.
3. Then substitute $x = 6$: $AB = 2(6) + 3 = 15$ feet and $BC = 3(6) - 1 = 17$ feet. Check: $15 + 17 + 14 = 46$.

TWO-COLUMN PROOF

Given: The perimeter of $\triangle ABC$ is 46; $AB = 2x + 3$, $BC = 3x - 1$, $AC = 14$

Prove: $x = 6$

	STATEMENTS	REASONS
1	The perimeter of $\triangle ABC$ is 46; $AB = 2x + 3$, $BC = 3x - 1$, $AC = 14$	Given
2	$AB + BC + AC = 46$	Definition of perimeter
3	$(2x + 3) + (3x - 1) + 14 = 46$	Substitution Property of Equality

	STATEMENTS	REASONS
4	$5x + 16 = 46$	Simplify
5	$5x = 30$	Subtraction Property of Equality
6	$x = 6$	Division Property of Equality

∎

Answer: $x = 6$, so $AB = 15$ feet and $BC = 17$ feet.

PRACTICE

1. **Easy** Name the property: "If $AB = 7$, then $7 = AB$."

   Show answer

   Symmetric Property of Equality.

2. **Easy** Name the property: "If $x - 4 = 10$, then $x = 14$."

   Show answer

   Addition Property of Equality (4 was added to both sides).

3. **Easy** Name the property: "$m\angle PQR = m\angle PQR$."

   Show answer

   Reflexive Property of Equality.

4. **Easy** Name the property: "If $3(x + 2) = 18$, then $3x + 6 = 18$."

   Show answer

   Distributive Property.

5. **Medium** Which of Euclid's common notions is the ancestor of the Transitive Property of Equality? State it in Euclid's words.

   Show answer

   Common Notion 1: "Things which are equal to the same thing are also equal to one another."

6. **Medium** Write a two-column proof. Given: $4x - 9 = 23$. Prove: $x = 8$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $4x - 9 = 23$

Prove: $x = 8$

	STATEMENTS	REASONS
1	$4x - 9 = 23$	Given
2	$4x = 32$	Addition Property of Equality
3	$x = 8$	Division Property of Equality

∎

7. **Medium** Write a two-column proof. Given: $\frac{x}{3} + 5 = 11$. Prove: $x = 18$.

Show answer

See the model proof.

TWO-COLUMN PROOF

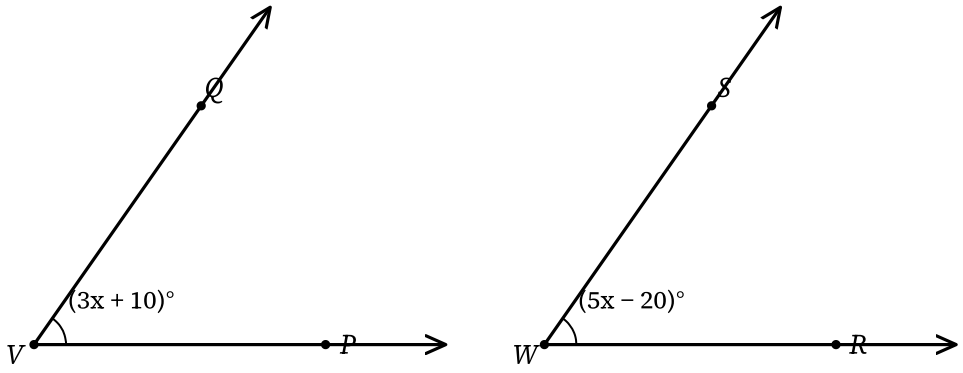
Given: $\frac{x}{3} + 5 = 11$

Prove: $x = 18$

	STATEMENTS	REASONS
1	$\frac{x}{3} + 5 = 11$	Given
2	$\frac{x}{3} = 6$	Subtraction Property of Equality
3	$x = 18$	Multiplication Property of Equality

∎

8. **Medium** In the figure, $\angle PVQ \cong \angle RWS$, $m\angle PVQ = (3x + 10)^\circ$ and $m\angle RWS = (5x - 20)^\circ$. Find $m\angle PVQ$, naming the property or definition for each step.



Show answer

$m\angle PVQ = m\angle RWS$ (definition of congruent angles), so $3x + 10 = 5x - 20$ (Substitution); $10 = 2x - 20$ (Subtraction); $30 = 2x$ (Addition); $15 = x$ (Division). Then $m\angle PVQ = 3(15) + 10 = 55^\circ$ (Substitution).

9. **Medium** Henry is fencing a rectangular garden. Its length is 3 feet more than twice its width, and the fence around it is 54 feet long. Find the width and the length, naming the property used in each step of solving.

Show answer

Let the width be w; the length is $2w + 3$. Perimeter: $2w + 2(2w + 3) = 54$. Distributive: $2w + 4w + 6 = 54$; Simplify: $6w + 6 = 54$; Subtraction: $6w = 48$; Division: $w = 8$. Width 8 feet, length $2(8) + 3 = 19$ feet (check: $2(8) + 2(19) = 54$).

10. Hard Write a two-column proof. Given: $2(3x - 4) + 5 = 4x + 11$. Prove: $x = 7$.

Show answer

See the model proof.

TWO-COLUMN PROOF

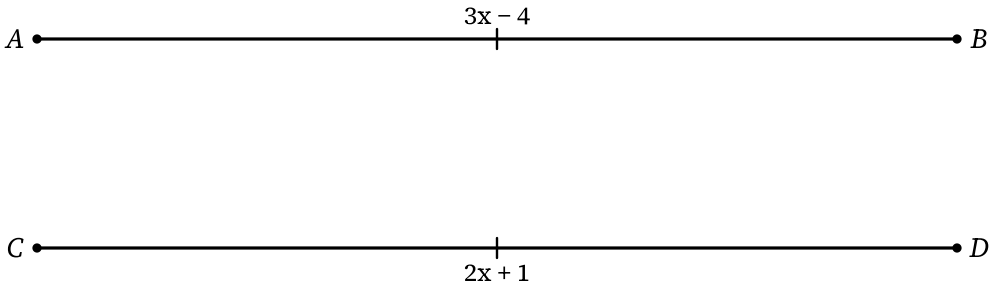
Given: $2(3x - 4) + 5 = 4x + 11$

Prove: $x = 7$

	STATEMENTS	REASONS
1	$2(3x - 4) + 5 = 4x + 11$	Given
2	$6x - 8 + 5 = 4x + 11$	Distributive Property
3	$6x - 3 = 4x + 11$	Simplify
4	$2x - 3 = 11$	Subtraction Property of Equality
5	$2x = 14$	Addition Property of Equality
6	$x = 7$	Division Property of Equality

∎

11. Hard In the figure, $\overline{AB} \cong \overline{CD}$, $AB = 3x - 4$ and $CD = 2x + 1$. Write a two-column proof that $x = 5$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$; $AB = 3x - 4$, $CD = 2x + 1$

Prove: $x = 5$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$	Given
2	$AB = CD$	Definition of congruent segments
3	$AB = 3x - 4, CD = 2x + 1$	Given
4	$3x - 4 = 2x + 1$	Substitution Property of Equality
5	$x - 4 = 1$	Subtraction Property of Equality
6	$x = 5$	Addition Property of Equality

∎

12. Hard Prove the symmetric property of segment congruence. Given: $\overline{AB} \cong \overline{CD}$. Prove: $\overline{CD} \cong \overline{AB}$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{CD} \cong \overline{AB}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$	Given
2	$AB = CD$	Definition of congruent segments
3	$CD = AB$	Symmetric Property of Equality
4	$\overline{CD} \cong \overline{AB}$	Definition of congruent segments

∎

13. **Challenge** Given: $m\angle A = m\angle B$, $m\angle A = (4x - 10)^\circ$, $m\angle B = (2x + 14)^\circ$, and $\angle C \cong \angle A$. Write a two-column proof that $m\angle C = 38^\circ$.

Show answer

See the model proof: $x = 12$, so $m\angle A = 38^\circ$ and $m\angle C = 38^\circ$.

TWO-COLUMN PROOF

Given: $m\angle A = m\angle B$; $m\angle A = 4x - 10$, $m\angle B = 2x + 14$; $\angle C \cong \angle A$

Prove: $m\angle C = 38^\circ$

	STATEMENTS	REASONS
1	$m\angle A = m\angle B$	Given
2	$m\angle A = 4x - 10$, $m\angle B = 2x + 14$	Given
3	$4x - 10 = 2x + 14$	Substitution Property of Equality
4	$2x - 10 = 14$	Subtraction Property of Equality
5	$2x = 24$	Addition Property of Equality
6	$x = 12$	Division Property of Equality
7	$m\angle A = 4(12) - 10$	Substitution Property of Equality
8	$m\angle A = 38^\circ$	Simplify
9	$\angle C \cong \angle A$	Given
10	$m\angle C = m\angle A$	Definition of congruent angles
11	$m\angle C = 38^\circ$	Substitution Property of Equality

∎

14. **Challenge** Samuel shows his sister this "proof" that $2 = 1$. Let $a = b$, with $b \neq 0$. Then $a^2 = ab$ (multiply both sides by a); $a^2 - b^2 = ab - b^2$ (subtract b^2); $(a + b)(a - b) = b(a - b)$ (factor); $a + b = b$ (divide both sides by $a - b$); $b + b = b$ (substitute $a = b$); $2b = b$; $2 = 1$ (divide by b). Find the step that breaks a property of equality, and explain why that property has its restriction.

Show answer

The step "divide both sides by $a - b$" is not allowed: since $a = b$, $a - b = 0$, and the Division Property of Equality applies only when the divisor is not 0. The restriction exists because $0 \cdot 5 = 0 \cdot 7$ is true while $5 = 7$ is false: multiplying by 0 makes unequal numbers look equal, so dividing by 0 cannot undo it. (The last step, dividing by b, is allowed because $b \neq 0$.)

COMMON MISTAKES

- Naming the opposite operation. If you subtracted 9 from both sides, the reason is the Subtraction Property of Equality, even though you did it to "undo" an addition.
- Confusing Substitution with Transitive. Use Transitive for the exact pattern "$a = b$ and $b = c$, so $a = c$"; use Substitution when you replace an expression inside a larger equation by an equal one.
- Writing $\overline{AB} = \overline{CD}$ or $AB \cong CD$. Lengths and measures are numbers and are equal ($=$); segments and angles are figures and are congruent ($\cong$).
- Stopping one step short. The last statement must be exactly the Prove statement; if the Prove line is $\overline{CD} \cong \overline{AB}$, do not stop at $CD = AB$.

> **Make it a game · Name That Reason.** Solve 3(x - 2) + 5 = 20 on paper, writing each step on its own strip. Cut the strips apart and shuffle them. Rebuild the solution in order, then say the property of equality that justifies each step. Time yourself, record the result, and try again with a new equation to beat your best.

UNIT 1

Foundations: Logic and the Axiomatic Method

Geometry is the first course in which you prove, rather than just compute. This unit lays the foundation. You will meet Euclid's *Elements*, the book that taught the world how to build a whole science from a few definitions, postulates and common notions, and you will learn the difference between noticing a pattern (inductive reasoning) and proving a conclusion (deductive reasoning).

Next you will study the logic that every proof uses: if-then statements and their converses, inverses and contrapositives, biconditionals and good definitions, and the Laws of Detachment and Syllogism, along with the common errors that look like logic but are not. The unit ends with your first two-column proofs, in which each step of solving an equation is justified by a property of equality, the modern form of Euclid's common notions.

Parents: no new algebra is required here. The goal is careful, honest reasoning: every statement needs a reason. Ask your student to explain each step aloud; if they can say *why*, they are ready for Unit 2.

PREREQUISITES

- Solve linear equations, including equations with parentheses and variables on both sides (Algebra I).
- Recognize even, odd and prime numbers and test divisibility.
- Find the perimeter of a triangle or rectangle and the area of a rectangle.
- Know the everyday meaning of point, line, circle, radius, right angle, square and rectangle.
- Use a compass to draw a circle with a given center and radius.

MASTERY CHECKLIST

- ☐ I can explain the difference between undefined terms, definitions, postulates and theorems.
- ☐ I can state Euclid's five postulates and five common notions in modern words.
- ☐ I can construct an equilateral triangle on a segment (Euclid I.1) and explain why it works.
- ☐ I can make a conjecture from a pattern and test it with a counterexample.
- ☐ I can explain why mathematics requires deductive proof rather than examples.
- ☐ I can write the converse, inverse and contrapositive of a conditional and decide whether each is true.
- ☐ I can write a biconditional and test whether a definition is reversible.
- ☐ I can use the Law of Detachment and the Law of Syllogism, and recognize affirming the conclusion and denying the hypothesis.
- ☐ I can use a Venn diagram to decide whether an argument is valid.
- ☐ I can name the property of equality that justifies each step in solving an equation.
- ☐ I can write a two-column proof of an algebraic statement, with a reason for every step.

LESSONS IN THIS UNIT

1.1	**Euclid's Elements and the Axiomatic Method**	2 sessions of 45 minutes
1.2	**Inductive and Deductive Reasoning**	45 minutes
1.3	**Conditional Statements**	2 sessions of 45 minutes
1.4	**The Laws of Logic**	2 sessions of 45 minutes
1.5	**Algebraic Proof and the Two-Column Format**	2 sessions of 45 minutes

Unit 1 Test

You have 45–60 minutes. No calculator is needed or allowed. Show your work on every problem. In every proof, give a reason for each statement; the last statement must be exactly what you are asked to prove. Give exact answers with units. For Problem 16 you need a compass and a straightedge (an unmarked ruler); leave all construction marks visible.

1. **5 pts**
   Classify each as an undefined term, a definition, a postulate or a theorem: (a) point; (b) “A circle can be drawn with any center and any radius.”

2. **5 pts**
   A carpenter stretches a chalk line along the straight edge of a board and snaps it so that the line continues straight past the end of the board. Which of Euclid's postulates describes this action? Give its number and state it.

3. **6 pts**
   A master mason lays out an equilateral triangle for a church window by Euclid's Proposition I.1, swinging two arcs with a cord as a compass. The outline of the triangle uses exactly 36 inches of lead came. What radius did he use for each arc? Explain.

4. **6 pts**

   Samuel builds a row of square pens for his sheep, each new pen sharing a side with the one before. One pen takes 4 fence panels, two pens take 7, and three pens take 10. Make a conjecture about the number of panels for n pens, and use it to find the number of panels needed for 15 pens.

5. **6 pts**

   Conjecture: “For every whole number n, $2^n > n^2$.” Find the smallest whole number that is a counterexample, and show why it is one.

6. **5 pts**

   “Every rafter cut from this pattern board is 14 feet long. Henry's rafter was cut from this pattern board. So Henry's rafter is 14 feet long.” Is this inductive or deductive reasoning? Explain briefly.

7. **8 pts**

   Consider the true conditional “If $x > 5$, then $x > 3$.” Give the truth value of (a) its converse, (b) its inverse and (c) its contrapositive. Write each statement, and give a counterexample for any that is false.

8. **6 pts**

   Grace proposes the definition: “An angle is obtuse if and only if its measure is greater than 90°.” Is this a good definition? Explain by testing both conditionals.

9. **5 pts**

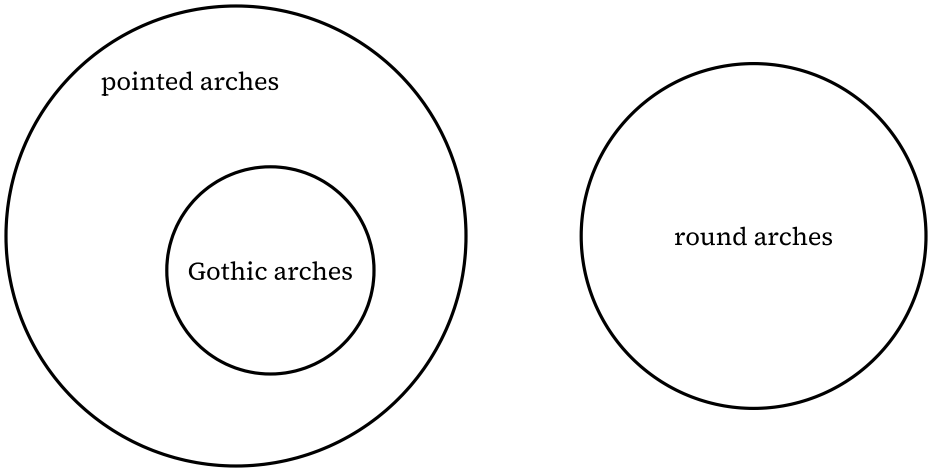
   “If the lighthouse lamp is lit, then ships can see the rocks. If ships can see the rocks, then they steer clear of them.” Tonight the lighthouse lamp is lit. Do the ships steer clear of the rocks? Name the laws you use.

10. **5 pts**

    “If a ship is a schooner, then it has at least two masts. The ship in the harbor has two masts. So it is a schooner.” Is the argument valid? If not, name the error.

11. **6 pts**

"Every Gothic arch is a pointed arch. No round arch is a pointed arch." Using the Venn diagram, decide whether a Gothic arch can be a round arch, and explain.

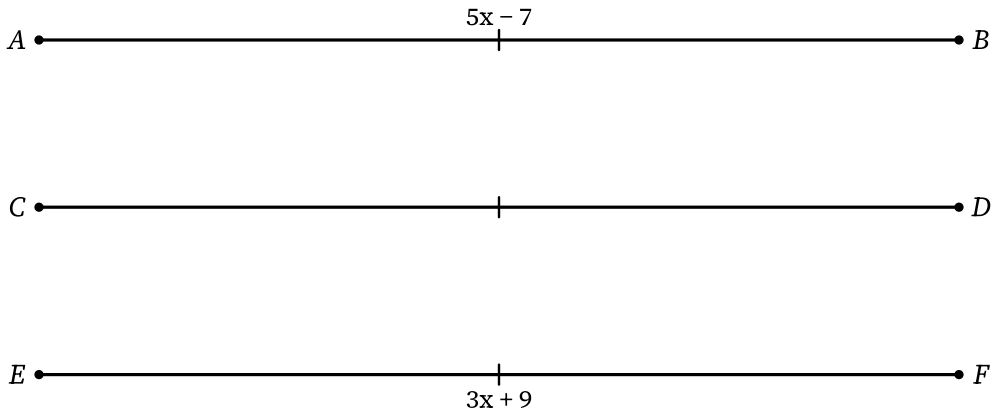


12. **8 pts**

Write a two-column proof. Given: $7x - 3 = 4x + 18$. Prove: $x = 7$.

13. **8 pts**

In the figure, $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, with $AB = 5x - 7$ and $EF = 3x + 9$. Write a two-column proof that $x = 8$.



14. **8 pts**

The Miller family is pouring a rectangular barn floor. Its length is 4 feet less than three times its width, and the forms around its edge total 152 feet. Find the width and the length, naming the property of equality (or the Distributive Property) used in each step.

15. **5 pts**

Explain why every system of mathematics must begin with some postulates that are accepted without proof.

16. **8 pts**

Copy segment $\overline{AB}$ from the figure. (a) With compass and straightedge, construct an equilateral triangle ABC on it by Euclid's Proposition I.1, leaving your arcs visible. (b) Write a two-column proof that $\triangle ABC$ is equilateral.



ANSWER KEY

1. (a) undefined term; (b) postulate (Euclid's Postulate 3).
2. Postulate 2: a segment can be extended continuously in a straight line.
3. 12 inches. In I.1 each arc's radius equals the given segment, and all three sides are equal to it, so each side is $36 \div 3 = 12$ inches.
4. Each new pen adds 3 panels, so n pens take $3n + 1$ panels. For 15 pens: $3(15) + 1 = 46$ panels.
5. $n = 2$: $2^n = 2^2 = 4$ and $n^2 = 2^2 = 4$, and $4 > 4$ is false. (For $n = 0$, $1 > 0$, and for $n = 1$, $2 > 1$, so those are not counterexamples.)
6. Deductive: the conclusion follows by logic (the Law of Detachment) from an accepted general statement and a given fact, not from observed examples.
7. (a) Converse: If $x > 3$, then $x > 5$. False ($x = 4$). (b) Inverse: If $x \leq 5$, then $x \leq 3$. False ($x = 4$). (c) Contrapositive: If $x \leq 3$, then $x \leq 5$. True.
8. No. “If an angle is obtuse, then its measure is greater than $90°$” is true, but “If an angle's measure is greater than $90°$, then it is obtuse” is false: a straight angle measures $180°$ and is not obtuse. A correct definition: an angle is obtuse if and only if its measure is greater than $90°$ and less than $180°$.
9. Yes. The Law of Syllogism gives “If the lamp is lit, then ships steer clear of the rocks,” and the Law of Detachment applies it to tonight.
10. Not valid: it affirms the conclusion. A ship with two masts could be another kind of vessel, such as a brig.
11. No. The Gothic-arch circle lies inside the pointed-arch circle, and the round-arch circle does not meet the pointed-arch circle, so nothing can be in both the Gothic and round circles.
12. See the model proof.

    TWO-COLUMN PROOF

    Given: $7x - 3 = 4x + 18$

    Prove: $x = 7$

	STATEMENTS	REASONS
1	$7x - 3 = 4x + 18$	Given
2	$3x - 3 = 18$	Subtraction Property of Equality
3	$3x = 21$	Addition Property of Equality

	STATEMENTS	REASONS
4	$x = 7$	Division Property of Equality

∎

13. See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$; $\overline{CD} \cong \overline{EF}$; $AB = 5x - 7$, $EF = 3x + 9$

Prove: $x = 8$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{EF}$	Given
2	$\overline{AB} \cong \overline{EF}$	Transitive Property of Congruence
3	$AB = EF$	Definition of congruent segments
4	$AB = 5x - 7$, $EF = 3x + 9$	Given
5	$5x - 7 = 3x + 9$	Substitution Property of Equality
6	$2x - 7 = 9$	Subtraction Property of Equality
7	$2x = 16$	Addition Property of Equality
8	$x = 8$	Division Property of Equality

∎

14. Let the width be w; the length is $3w - 4$. Perimeter: $2w + 2(3w - 4) = 152$. Distributive: $2w + 6w - 8 = 152$; Simplify: $8w - 8 = 152$; Addition: $8w = 160$; Division: $w = 20$. Width 20 feet, length $3(20) - 4 = 56$ feet (check: $2(20) + 2(56) = 152$).

15. A proof justifies each statement by earlier statements. If every starting statement also needed a proof, the chain of reasons would go back forever or circle around to use what it was trying to prove. So mathematicians agree on a few clear starting statements (postulates) and prove everything else from them.

16. (a) Draw the circle with center A through B and the circle with center B through A; label a crossing point C; draw $\overline{AC}$ and $\overline{BC}$. (b) See the model proof.

TWO-COLUMN PROOF

Given: C lies on the circle with center A through B; C lies on the circle with center B through A

Prove: $\triangle ABC$ is equilateral

	STATEMENTS	REASONS
1	C lies on the circle with center A through B and on the circle with center B through A	Given (construction)
2	$AC = AB$	Definition of a circle (radii of the circle with center A)
3	$BC = AB$	Definition of a circle (radii of the circle with center B)
4	$AC = BC$	Common Notion 1 (things equal to the same thing are equal)
5	$\triangle ABC$ is equilateral	Definition of an equilateral triangle (all three sides equal)

∎

UNIT 2

Points, Lines, Planes and Angles

In Unit 1 you learned what a proof is. In this unit you start building geometry itself, the way Euclid did: from undefined terms (point, line, plane), a few postulates, and careful definitions. You will measure segments and angles, learn the Segment and Angle Addition Postulates, and meet the angle pairs (complementary, supplementary, linear pairs, vertical angles) that appear in nearly every later proof.

You will also pick up a compass and straightedge. With Euclid's first propositions you will construct an equilateral triangle (I.1), copy segments and angles, bisect segments and angles, and draw perpendiculars, the same tools cathedral master masons used to lay out arches and windows. Finally you will write your own two-column and paragraph proofs about segments and angles, including Euclid's I.15, the Vertical Angles Theorem.

Parents: this unit is where proofs start to feel like real geometry. Encourage your student to say every definition and postulate aloud until it is automatic, to draw large marked figures, and to give a reason for every step. A compass, an unmarked straightedge (a ruler used without its marks is fine), a protractor and a sharp pencil are needed.

PREREQUISITES

- Solve linear equations, including with variables on both sides (Algebra I).
- Use the distance and midpoint formulas in the coordinate plane (Algebra I).
- Use the Pythagorean theorem to find a side of a right triangle (Algebra I).
- Write and analyze conditional statements (Lesson 1.3).
- Justify steps with the properties of equality and congruence in a two-column proof (Lesson 1.5).

MASTERY CHECKLIST

- ☐ I can name points, lines, planes, segments and rays and tell whether points are collinear or coplanar.
- ☐ I can state the postulates about points, lines and planes and use them to explain simple facts.
- ☐ I can use the Ruler Postulate, the Segment Addition Postulate and the definition of midpoint to find lengths, including with algebra and coordinates.
- ☐ I can use the Protractor Postulate, the Angle Addition Postulate and the definition of angle bisector to find angle measures, and work with degrees and minutes.
- ☐ I can identify complementary and supplementary angles, linear pairs and vertical angles and use them to write and solve equations.
- ☐ I can prove the Linear Pair Theorem, the Vertical Angles Theorem (Euclid I.15) and the Congruent Supplements and Complements Theorems.
- ☐ I can construct an equilateral triangle (I.1) and prove it is equilateral, and copy a segment and an angle.
- ☐ I can bisect a segment and an angle and construct perpendiculars through a point on a line and from a point not on a line, and explain why each construction works.

- ☐ I can write the Given and Prove for a statement, draw and mark its diagram, and plan a proof by working backward.
- ☐ I can write two-column and paragraph proofs about segments and angles with a correct reason for every step.

LESSONS IN THIS UNIT

2.1	**Points, Lines and Planes**	45 minutes
2.2	**Measuring and Constructing Segments**	2 sessions of 45 minutes
2.3	**Measuring Angles**	45 minutes
2.4	**Angle Pairs and the Vertical Angles Theorem**	2 sessions of 45 minutes
2.5	**Basic Constructions**	2 sessions of 45 minutes
2.6	**Proving Statements About Segments and Angles**	2 sessions of 45 minutes

Unit 2 test

You have 50 minutes. No calculator is allowed; all arithmetic can be done by hand. You will need a compass, a straightedge and a pencil for Problem 11. Show your work for every problem. In every proof, give a reason for each step: a given, a definition, a postulate, a property or a theorem we have proved; nothing may be assumed from the look of a figure except the order of points and which side of a line a point is on. Give exact answers unless a problem asks you to round, and include units.

1. **5 pts**

   True or false: three collinear points determine exactly one plane. Explain.

2. **5 pts**

   A quilter marks 6 points around the edge of a round quilt block, no three in a line, and sews a straight seam between every pair of points. How many seams does she sew?

3. **6 pts**

   Find the distance between $J(-3, -2)$ and $K(9, 3)$, and the coordinates of the midpoint of $\overline{JK}$.

4. **6 pts**

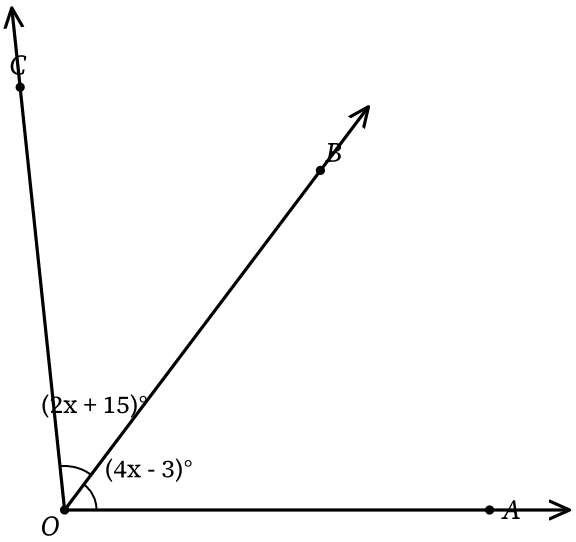
   Point R is between Q and S. $QR = 3x - 4$, $RS = 2x + 7$ and $QS = 43$. Find x and RS.

5. **6 pts**

   A straight highway has mile markers. One town is at mile 37.6 and the next town is at mile 92.4. The state builds a rest stop exactly halfway between them. At what mile marker is the rest stop, and how far is it from each town?

6. **6 pts**

   In the figure, B is in the interior of $\angle AOC$, $m\angle AOB = (4x - 3)^\circ$, $m\angle BOC = (2x + 15)^\circ$ and $m\angle AOC = 96^\circ$. Find x and $m\angle AOB$.



7. **6 pts**

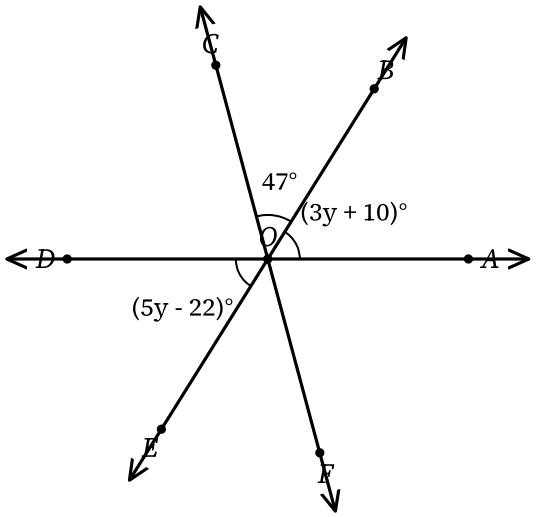
   A surveyor measures the angle between two property lines as $103°24'$ and wants to bisect it to set a drainage line. What angle should the drainage line make with each property line? Give your answer in degrees and minutes.

8. **6 pts**

   The supplement of an angle is $30°$ more than twice its complement. Find the angle.

9. **7 pts**

   In the figure, lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ meet at O. $m\angle AOB = (3y + 10)°$, $m\angle DOE = (5y - 22)°$ and $m\angle BOC = 47°$. Find y and $m\angle COD$.



10. **6 pts**

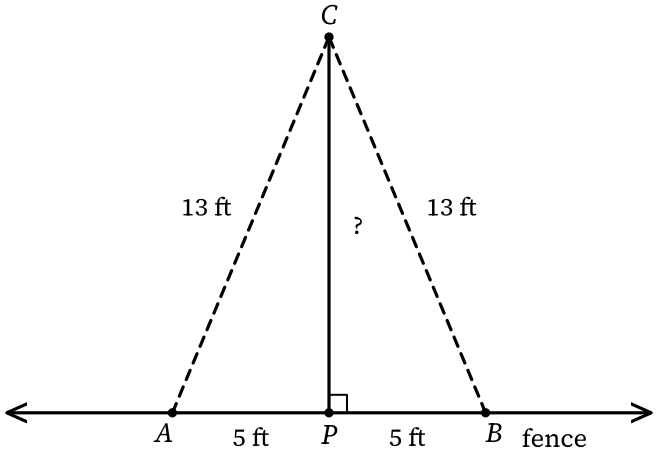
    Explain in two or three sentences why vertical angles must be congruent. Do not just name the theorem; give the reason behind it.

11. **10 pts**

    Draw a segment $\overline{AB}$ about 8 cm long. Using only a compass and straightedge, construct the perpendicular bisector of $\overline{AB}$, and then use it to construct an angle of $45°$. Leave all arcs showing and list your steps.

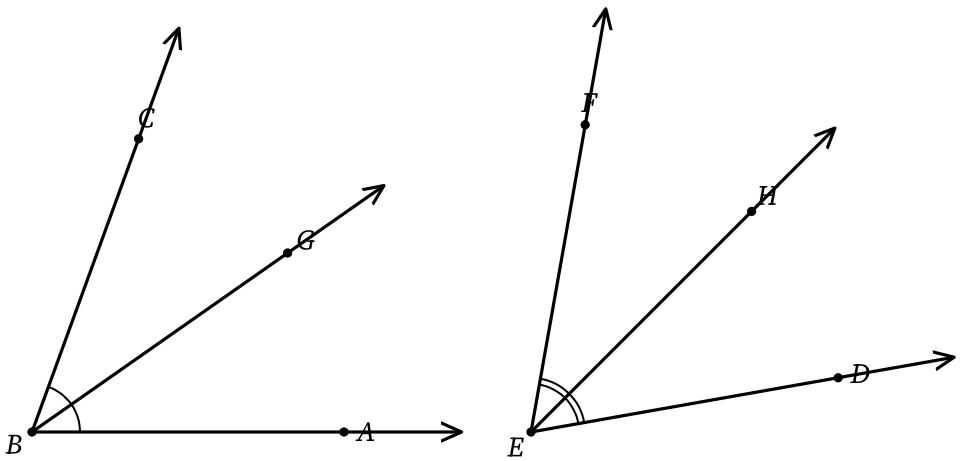
12. **8 pts**

Samuel is laying out a garden path perpendicular to a straight fence at gate post P. He marks points A and B on the fence, each 5 feet from P on opposite sides. Then, holding a 13-foot cord first at A and then at B, he finds the point C that is 13 feet from both, as in the figure. By the construction, $\overleftrightarrow{PC}$ is perpendicular to the fence. How far is C from the gate post P?



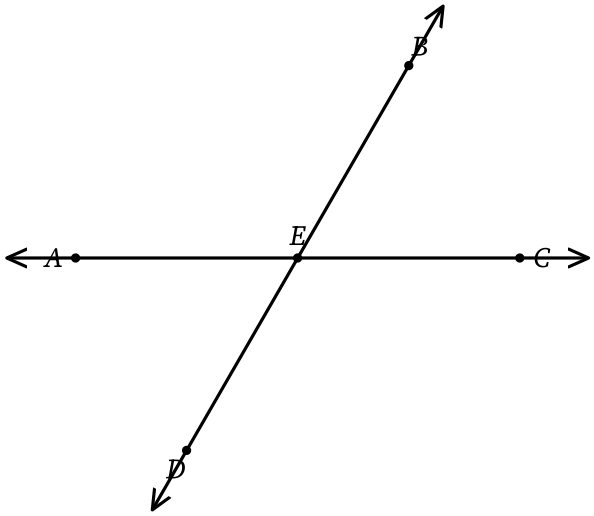
13. **12 pts**

In the figure, $\angle ABC \cong \angle DEF$, $\overrightarrow{BG}$ bisects $\angle ABC$ and $\overrightarrow{EH}$ bisects $\angle DEF$. Write a two-column proof that $\angle ABG \cong \angle DEH$.



14. **11 pts**

In the figure, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E, and $m\angle AEB = 2 \cdot m\angle BEC$. Write a paragraph proof that $m\angle AED = 60°$.



ANSWER KEY

1. False. Three collinear points lie on one line, and infinitely many planes contain that line (one through each point not on it). The Plane Postulate gives exactly one plane only for noncollinear points.
2. Each of the 6 points pairs with 5 others: $6 \times 5 = 30$, and each seam is counted twice, so $30 \div 2 = 15$ seams.
3. $JK = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; midpoint $\left(\dfrac{-3+9}{2}, \dfrac{-2+3}{2}\right) = (3, 0.5)$
4. By the Segment Addition Postulate $(3x - 4) + (2x + 7) = 43$, so $5x + 3 = 43$ and $x = 8$. Then $RS = 2(8) + 7 = 23$.
5. Mile $\dfrac{37.6 + 92.4}{2} = 65$; it is $65 - 37.6 = 27.4$ miles from each town.
6. By the Angle Addition Postulate $(4x - 3) + (2x + 15) = 96$, so $6x + 12 = 96$ and $x = 14$. Then $m\angle AOB = 4(14) - 3 = 53°$.
7. $103°24' \div 2$: $103° = 102°60'$, so $103°24' = 102°84'$, and half is $51°42'$ (which is $51.7°$).
8. $180 - x = 2(90 - x) + 30$, so $180 - x = 210 - 2x$ and $x = 30°$. Check: supplement $150°$, complement $60°$, and $2(60) + 30 = 150$. ✓
9. $\angle AOB$ and $\angle DOE$ are vertical angles, so $3y + 10 = 5y - 22$ and $y = 16$; then $m\angle AOB = 58°$. Since $\overrightarrow{OA}$ and $\overrightarrow{OD}$ are opposite rays, $m\angle COD = 180 - 58 - 47 = 75°$.
10. Each of two vertical angles forms a linear pair with the same angle between them, so each one equals $180°$ minus the measure of that same angle. Equals subtracted from equals leave equals (Common Notion 3), so the vertical angles have equal measures and are congruent.
11. With the compass opened to more than half of AB, draw arcs from A and from B above and below the segment, crossing at P and Q. Draw $\overleftrightarrow{PQ}$; it meets $\overline{AB}$ at the midpoint M and $m\angle PMB = 90°$. Bisect $\angle PMB$: an arc from M crosses $\overrightarrow{MP}$ at X and $\overrightarrow{MB}$ at Y; equal arcs from X and Y cross at Z. Draw $\overrightarrow{MZ}$. Then $m\angle ZMB = \dfrac{1}{2}(90°) = 45°$.
12. $\triangle APC$ has a right angle at P, with $AP = 5$ ft and $AC = 13$ ft. By the Pythagorean theorem $PC = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$ feet.
13. See the model proof.

TWO-COLUMN PROOF

Given: $\angle ABC \cong \angle DEF$; $\overrightarrow{BG}$ bisects $\angle ABC$; $\overrightarrow{EH}$ bisects $\angle DEF$

Prove: $\angle ABG \cong \angle DEH$

	STATEMENTS	REASONS
1	$\angle ABC \cong \angle DEF$	Given
2	$m\angle ABC = m\angle DEF$	Definition of congruent angles
3	$\overrightarrow{BG}$ bisects $\angle ABC$; $\overrightarrow{EH}$ bisects $\angle DEF$	Given
4	$m\angle ABG = \frac{1}{2}m\angle ABC$	Angle Bisector Halving Theorem
5	$m\angle DEH = \frac{1}{2}m\angle DEF$	Angle Bisector Halving Theorem
6	$\frac{1}{2}m\angle ABC = \frac{1}{2}m\angle DEF$	Multiplication Property of Equality
7	$m\angle ABG = m\angle DEH$	Substitution Property of Equality (rows 4 and 5 into row 6)
8	$\angle ABG \cong \angle DEH$	Definition of congruent angles

∎

14. See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E; $m\angle AEB = 2 \cdot m\angle BEC$

Prove: $m\angle AED = 60°$

Since the lines intersect at E, $\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays, so $\angle AEB$ and $\angle BEC$ form a linear pair. By the Linear Pair Theorem, $m\angle AEB + m\angle BEC = 180°$.

Substituting $2 \cdot m\angle BEC$ for $m\angle AEB$ gives $3 \cdot m\angle BEC = 180°$, so $m\angle BEC = 60°$ by the Division Property of Equality.

Also $\overrightarrow{EB}$ and $\overrightarrow{ED}$ are opposite rays, so $\angle AED$ and $\angle CEB$ are vertical angles. By the Vertical Angles Theorem they are congruent, so $m\angle AED = m\angle BEC = 60°$.

∎

2.1 Points, Lines and Planes

OBJECTIVES

- Name points, lines, planes, segments and rays with correct notation.
- Identify collinear points, coplanar points and opposite rays.
- State the basic postulates about points, lines and planes and use them to explain simple facts.
- Sketch and describe the intersections of lines and planes.

WATCH

An Intro to Points, Lines, Line Segments, and Rays | Geometry | Math with Mr. J

Math with Mr. J · 3:50

https://www.youtube.com/watch?v=lnV59_ELs1o

> **WHEN TO WATCH**
> Before the Core ideas: points, lines, segments and rays.

Note: The postulates are taught from the Core ideas.

BACKUP VIDEO

Geometry Lesson 1 - Points, Lines, and Planes

Mario's Math Tutoring · 10:32

https://www.youtube.com/watch?v=MD1Ob370TIA

Before the postulates: naming planes, collinear and coplanar points, and where lines and planes intersect.

KEY TERMS

undefined terms

The words point, line and plane. They are not defined; they are described, and their behavior is fixed by the postulates. Every other geometric word is defined from them.

collinear points

Points that lie on one line. Points that do not all lie on one line are noncollinear.

coplanar points

Points that lie in one plane. Points that do not all lie in one plane are noncoplanar.

segment

Segment $\overline{AB}$ is made of the points A and B (its endpoints) and all points of $\overleftrightarrow{AB}$ between A and B.

ray

Ray $\overrightarrow{AB}$ is made of the endpoint A, the point B, all points between them, and all points X of $\overleftrightarrow{AB}$ such that B is between A and X. The endpoint is always named first.

opposite rays

If B is between A and C, then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays: they share the endpoint B and together make up the whole line.

intersection

The intersection of two figures is the set of all points that lie in both of them.

CORE IDEAS

Every definition uses other words. If we tried to define every word, we would either go around in circles (a dictionary defines a line as a straight path and a path as a line) or go on forever. So geometry, like Euclid's Elements, starts with a few **undefined terms**: point, line and plane. A point marks a location and has no size. A line is straight, has no thickness, and goes on without end in both directions. A plane is a flat surface with no thickness that goes on without end. We do not define these words; instead the postulates tell us how they behave.

Names matter because proofs must be exact. A point is named by a capital letter, such as A. A line is named by two of its points, $\overleftrightarrow{AB}$, or by a lowercase letter such as line ℓ. A plane is named by a capital script-style letter such as plane M, or by three of its points that are not on one line, such as plane ABC. Segment $\overline{AB}$ has two endpoints, and $\overline{AB}$ and $\overline{BA}$ are the same segment. Ray $\overrightarrow{AB}$ starts at A and goes through B forever, so $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are different rays: the endpoint is always written first.

Points on one line are **collinear**; points in one plane are **coplanar**. Two points are always collinear, because there is a line through any two points. Three points may or

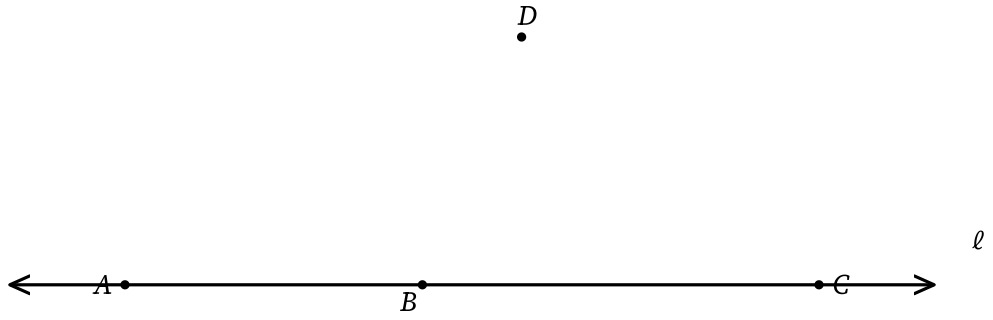
may not be collinear. When B is between A and C on a line, the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ point in opposite directions from B; they are **opposite rays**.

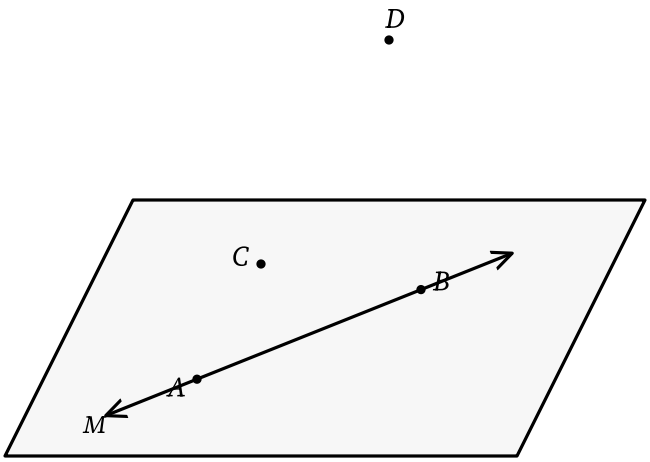
Postulates are the starting rules we accept without proof. **Line Postulate:** through any two points there is exactly one line, and every line contains at least two points. **Plane Postulates:** through any three noncollinear points there is exactly one plane, and if two points lie in a plane, the whole line through them lies in that plane. **Intersection Postulates:** if two distinct lines intersect, they meet in exactly one point; if two distinct planes intersect, they meet in a line. Each of these matches what we see in a straightedge, a tight string or a flat floor, and each is used again and again in later proofs.

Why does a three-legged stool never wobble on a flat floor, while a four-legged chair sometimes does? The three tips of the legs are three noncollinear points, and by the Plane Postulate exactly one plane passes through them, so all three tips always touch. Four points need not be coplanar, so if one leg is a little short, only three tips touch at a time. Postulates are not just rules on paper; they describe the world we build in.

Once we have postulates, we can prove new facts called theorems. For example, from the postulates we can prove that two lines that cross lie in exactly one plane (see the theorem below). This is our first taste of building geometry step by step from first principles, exactly as Euclid did.

To sketch planes, draw them as parallelograms, and draw hidden lines dashed. To show two planes meeting, draw the line where they cross. These pictures help you think, but remember that the postulates, not the picture, decide what is true.

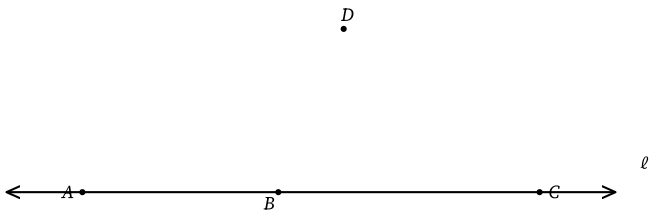




POSTULATES AND THEOREMS

POSTULATE **Line Postulate** Euclid Post. 1

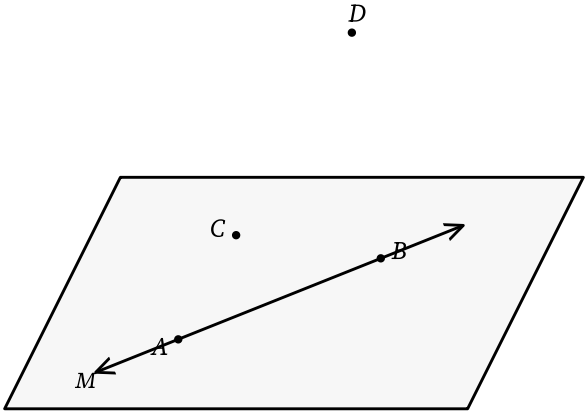
Through any two points there is exactly one line, and every line contains at least two points.



Euclid's Postulate 1 says a straight line can be drawn from any point to any point; he also treats this line as the only one.

POSTULATE **Plane Postulates**

Through any three noncollinear points there is exactly one plane, and every plane contains at least three noncollinear points. If two points lie in a plane, then the line containing them lies in that plane.



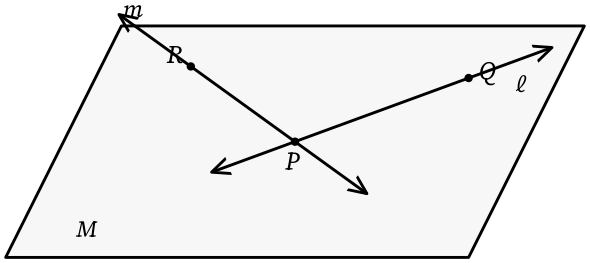
POSTULATE **Intersection Postulates**

If two distinct lines intersect, then they intersect in exactly one point. If two distinct planes intersect, then their intersection is a line.

The first part can also be explained from the Line Postulate: two different lines cannot share two points, because only one line passes through two points (see Practice Problem 11).

THEOREM **Intersecting Lines Determine a Plane**

If two distinct lines intersect, then exactly one plane contains both of them.



PARAGRAPH PROOF

Given: Lines ℓ and m are distinct and intersect at point P

Prove: Exactly one plane contains ℓ and m

By the Line Postulate, each line contains at least two points, so we can choose a point Q on ℓ other than P and a point R on m other than P.

The points P, Q and R are noncollinear. Indeed, the only line through P and Q is ℓ (Line Postulate). If R were on ℓ, then ℓ and m would share the two different points P and R, but by the Intersection Postulate two distinct lines meet in only one point. So R is not on ℓ.

By the Plane Postulate there is exactly one plane, call it M, through P, Q and R. Since P and Q lie in M, the whole line ℓ lies in M; since P and R lie in M, the whole line m lies in M (Plane Postulates). So M contains both lines.

Finally, any plane that contains both lines contains the points P, Q and R, and only one plane contains those three noncollinear points. Therefore exactly one plane contains ℓ and m.

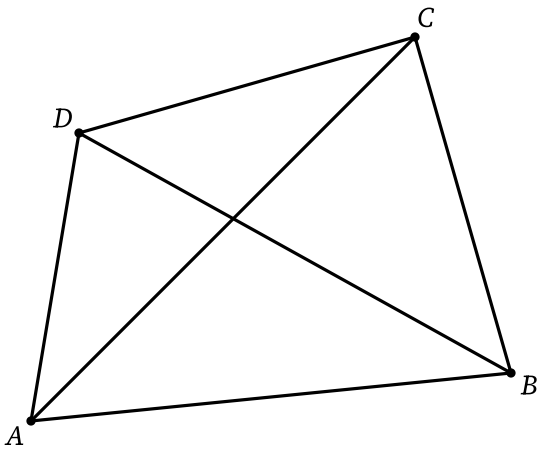
∎

From history. Euclid began the Elements with definitions such as "A point is that which has no part" and "A line is breadthless length." These are helpful descriptions, but they use words like part and length that are themselves undefined. In 1899 David Hilbert, in his Foundations of Geometry, took point, line and plane as undefined terms and let the axioms alone say how they behave. He is said to have remarked that one should be able to say "tables, chairs and beer mugs" in place of points, lines and planes, and the proofs would still work.

Euclid connection. Euclid's Book I opens with 23 definitions, beginning "A point is that which has no part," and his Postulate 1 lets us "draw a straight line from any point to any point," which is the ancestor of our Line Postulate. Modern geometry keeps Euclid's plan of starting from first principles but leaves point, line and plane undefined.

Practice habit. Recite the Line Postulate, the Plane Postulates and the Intersection Postulates until you can say them without looking. When you sketch, draw large figures, draw planes as parallelograms, and dash any line that would be hidden.

Example 1. In the figure, the four points A, B, C and D lie in one plane, and no three of them are collinear. How many different lines pass through two of these points?



Show solution

1. By the Line Postulate, each pair of points determines exactly one line, and since no three points are collinear, different pairs give different lines.
2. List the pairs in order so none is missed or counted twice: AB, AC, AD, BC, BD, CD.
3. Check by counting another way: each of the 4 points pairs with 3 others, giving $4 \times 3 = 12$, but this counts every pair twice (AB and BA), so there are $12 \div 2 = 6$ lines.

Answer: 6 lines

Example 2. A surveyor sets up a tripod on rough, rocky ground, and it stands firmly. At home, a four-legged table wobbles on an uneven porch. Use the postulates of this lesson to explain the difference.

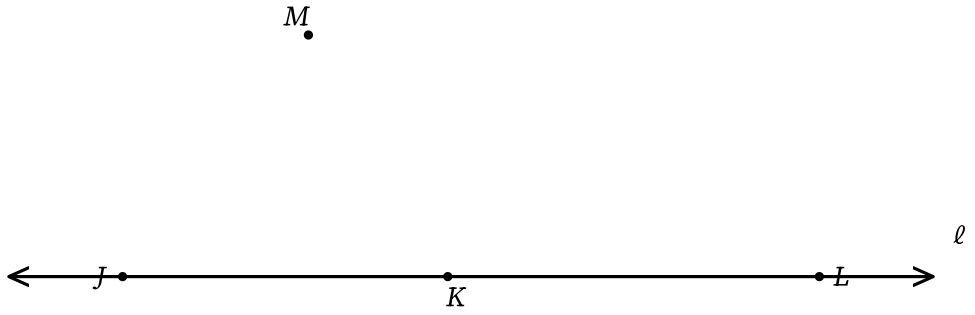
Show solution

1. Model the tips of the legs as points and the ground as a surface on which the tips rest.
2. The tripod's three tips are noncollinear (they are not in a line). By the Plane Postulate, exactly one plane passes through three noncollinear points, so there is always a flat position touching all three tips. The tripod can always settle so all three tips touch the ground at once.
3. Four points need not be coplanar. If the porch is uneven or one leg is slightly short, the four tips do not lie in one plane, so only three of them can touch at a time. The table rocks from one set of three legs to another.

Answer: Three noncollinear points always lie in exactly one plane, so all three legs of the tripod touch; four points need not be coplanar, so the table can rock on three legs at a time.

PRACTICE

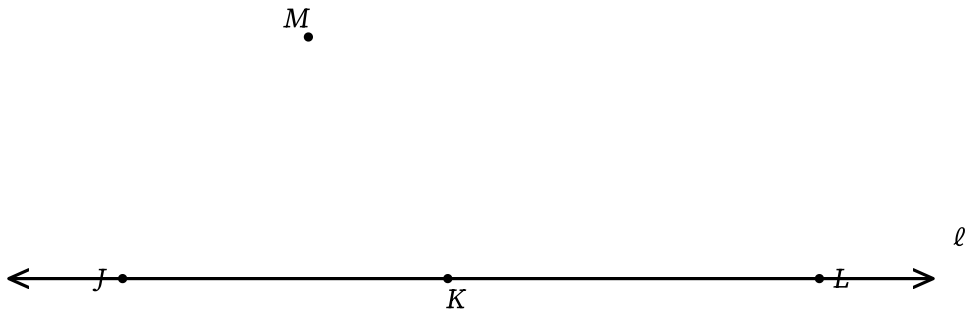
1. Easy In the figure, points J, K and L lie on line ℓ, and point M does not. Are J, K and M collinear?



Show answer

No. The only line through J and K is ℓ, and M is not on ℓ.

2. **Easy** In the figure for Problem 1, K is between J and L. Are $\overrightarrow{KJ}$ and $\overrightarrow{KL}$ opposite rays?



Show answer

Yes. They share the endpoint K, and K is between J and L, so they point in opposite directions along ℓ.

3. **Easy** Two distinct planes intersect. Is their intersection a point, a line or a plane?

Show answer

A line (Intersection Postulate).

4. **Easy** True or false: any three points are coplanar. Explain.

Show answer

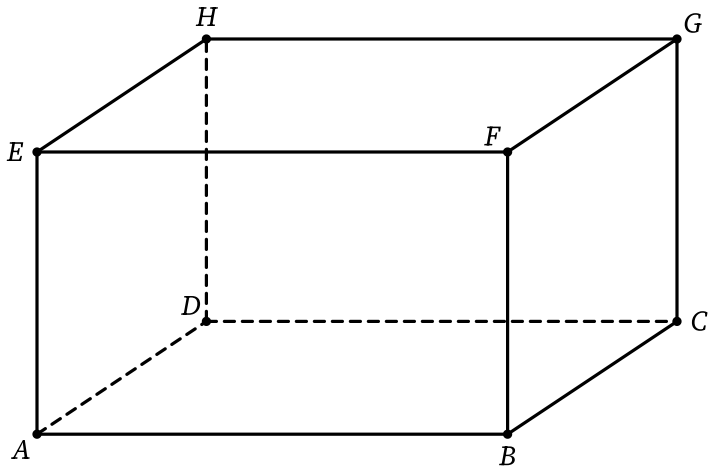
True. If the three points are noncollinear, a plane passes through them (Plane Postulate). If they are collinear, they lie on one line, and any plane containing that line (for example the plane through the line and some point off it) contains all three.

5. **Medium** True or false: any four points are coplanar. If false, give a counterexample from a classroom.

Show answer

False. For example, three corners of the floor and one corner of the ceiling are not in one plane.

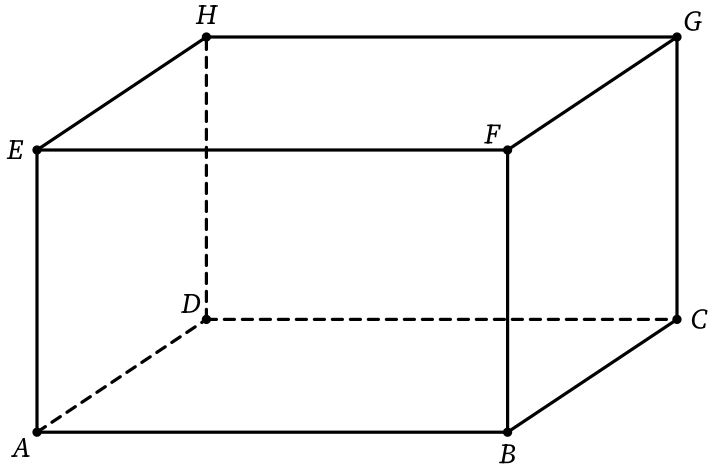
6. **Medium** The figure shows a box. Plane ABF is the front face and plane BCG is the right face. Name the intersection of these two planes.



Show answer

Line BF (written $\overleftrightarrow{BF}$).

7. **Medium** In the box of Problem 6, are the points A, B, C and G coplanar?



Show answer

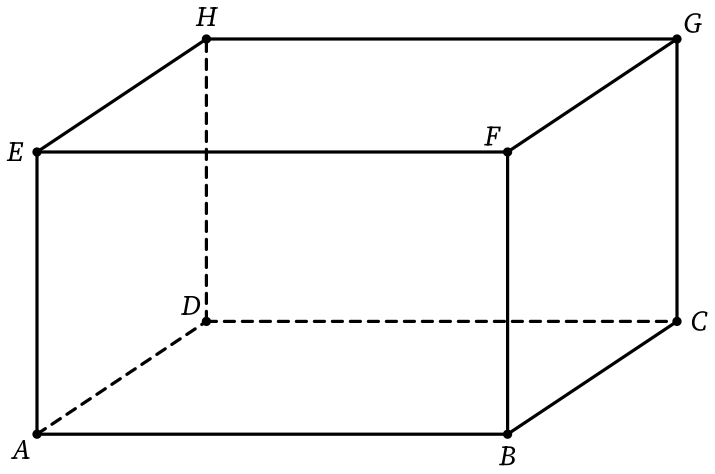
No. The only plane through A, B and C is the plane of the bottom face, and G is on the top face above C, not in that plane.

8. Medium Sketch plane M and a line ℓ that meets plane M in exactly one point P. Then add a second line m that lies in plane M and passes through P.

Show answer

Draw a parallelogram for M; draw ℓ crossing it at P, dashing the part of ℓ hidden below the plane; draw m inside the parallelogram through P.

9. Hard In the box of Problem 6, are the points A, B, G and H coplanar? Explain by describing the surface.



Show answer

Yes. Picture a flat ramp that rests on the front bottom edge $\overline{AB}$ and leans back to the top back edge $\overline{HG}$; it is a slanted plane cutting through the box, and it contains all four points.

10. Hard Grace sets 5 garden stakes in the ground, no three of them in a straight line, and stretches a straight string between every pair of stakes. How many strings does she need?

Show answer

Each of the 5 stakes pairs with 4 others: $5 \times 4 = 20$, and each string is counted twice, so $20 \div 2 = 10$ strings.

11. Hard Explain in a paragraph why two distinct lines cannot meet in two different points, using only the Line Postulate.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Lines ℓ and m are distinct

Prove: ℓ and m do not share two different points

Suppose ℓ and m both contained two different points P and Q.

By the Line Postulate there is exactly one line through P and Q. Both ℓ and m pass through P and Q, so ℓ and m would be the same line.

But ℓ and m are distinct, so this cannot happen. Therefore two distinct lines share at most one point.

∎

12. Hard Prove: If point P is not on line ℓ, then exactly one plane contains both ℓ and P.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Point P is not on line ℓ

Prove: Exactly one plane contains ℓ and P

By the Line Postulate, ℓ contains at least two points; call two of them A and B. The only line through A and B is ℓ, and P is not on ℓ, so A, B and P are noncollinear.

By the Plane Postulate there is exactly one plane M through A, B and P. Because A and B lie in M, the line ℓ through them lies in M (Plane Postulates). So M contains both ℓ and P.

Any plane containing ℓ and P contains the three noncollinear points A, B and P, and only one plane does. Therefore exactly one plane contains ℓ and P.

∎

13. **Challenge** Five points in space are placed so that no four of them are coplanar. How many different planes pass through three of the points? Explain why every set of three of these points determines its own plane.

Show answer

10 planes. No three of the points are collinear (if three were on a line, that line and any fourth point would lie in one plane, making four coplanar points). So each set of three points determines exactly one plane, and two different sets cannot give the same plane, because that plane would contain at least four of the points. The number of sets of three is $\dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$.

COMMON MISTAKES

- Writing a ray with the wrong endpoint first: $\overrightarrow{AB}$ starts at A, $\overrightarrow{BA}$ starts at B. Always put the endpoint first, and draw the arrow over the letters pointing right.
- Thinking a plane has edges because we draw it as a parallelogram. A plane goes on forever; the parallelogram shows only a piece of it.
- Mixing up collinear and coplanar. Any three points are coplanar, but three points are collinear only if they lie on one line. Ask yourself: one line, or one flat surface?
- Assuming four points are coplanar because they look flat in a drawing. Check whether one of them could be above or below the plane of the other three.

> **Make it a game · Notation Scavenger Hunt.** Walk the house with sticky notes labeled with letters. In 3 minutes, mark three collinear points along a shelf edge, four coplanar points not all on one line, and a pair of skew edges. Then write each answer with correct notation: line AB, segment CD, ray EF. Parent checks every symbol and counts your score.

2.2 Measuring and Constructing Segments

OBJECTIVES

- Use the Ruler Postulate and the Segment Addition Postulate to find lengths, including with algebra.
- Identify congruent segments, midpoints and segment bisectors, and prove simple facts about them.
- Find distances and midpoints on a number line and in the coordinate plane.
- Copy a segment with compass and straightedge, as in Euclid I.2 and I.3.

WATCH

Finding the Midpoint (1 of 3: Horizontal & Vertical Intervals)

Eddie Woo · 9:29

https://www.youtube.com/watch?v=yudeOCF3dy0

> **WHEN TO WATCH**
> Before the midpoint examples: finding the midpoint of a segment.

Note: Eddie Woo teaches in Australia: he says 'interval' for segment. The Segment Addition Postulate and copying a segment are taught from the Core ideas.

BACKUP VIDEO

Math Antics - Measuring Distance

mathantics · 10:58

https://www.youtube.com/watch?v=dNcJ4-JVN5M

Before Example 1: measuring a length with a ruler, the idea behind the Ruler Postulate.

KEY TERMS

Ruler Postulate

The points of a line can be matched one to one with the real numbers so that the distance between two points is the absolute value of the difference of their numbers (coordinates): $AB = |a - b|$.

between

Point B is between A and C when A, B and C are different collinear points and B lies on $\overline{AC}$.

Segment Addition Postulate

If B is between A and C, then $AB + BC = AC$.

congruent segments

Segments that have the same length. $\overline{AB} \cong \overline{CD}$ means exactly $AB = CD$.

midpoint

The point M on $\overline{AB}$ with $AM = MB$. It divides the segment into two congruent segments.

segment bisector

A point, ray, line, segment or plane that intersects a segment at its midpoint.

CORE IDEAS

A segment has a length, written without a bar: $\overline{AB}$ is the segment (a set of points) and AB is its length (a number). The **Ruler Postulate** is the promise that we can lay a ruler along any line: each point gets a number called its coordinate, and the distance between two points is the absolute value of the difference, $AB = |a - b|$. Absolute value is used because a distance is never negative, no matter which point you start from.

The **Segment Addition Postulate** says that the parts add up to the whole: if B is between A and C, then $AB + BC = AC$. Euclid uses this idea constantly without stating it as a rule of its own; his Common Notion 5, "the whole is greater than the part," is a close relative, since $AC = AB + BC$ is greater than either part. Notice the condition: B must be between A and C. If B is off the segment, the lengths do not add this way.

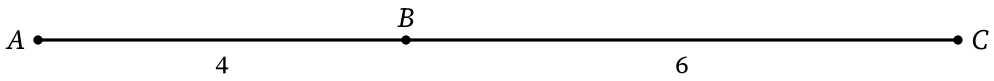
Two segments are **congruent** when they have the same length. We write $\overline{AB} \cong \overline{CD}$ for the segments and $AB = CD$ for the lengths; the two statements say the same thing, and moving between them is justified by the definition of congruent segments. On a diagram, congruent segments are shown with matching tick marks.

The **midpoint** of a segment cuts it into two congruent halves. Any line, ray or segment through the midpoint **bisects** the segment. Because the two halves are equal and add up to the whole, each half is exactly half the segment; this is proved below as the Midpoint Theorem, and it is a good model of how a proof uses a definition, a postulate and algebra together.

On a number line, the midpoint of the points with coordinates a and b is their average, $\frac{a+b}{2}$, because it is the same distance from each end. In the coordinate

plane you already know from Algebra I the distance formula (the Pythagorean theorem applied to the horizontal and vertical change) and the midpoint formula (average the x-coordinates and average the y-coordinates).

Euclid did not use rulers with numbers. He compared lengths by moving them with a compass. To **copy a segment**, open the compass to the length of the given segment and, keeping the same width, mark that length on a new ray. Euclid shows how to do this in I.2 ("to place at a given point a straight line equal to a given straight line") and I.3 ("given two unequal straight lines, to cut off from the greater a straight line equal to the less").



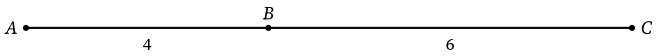
AC = 4 + 6 = 10

POSTULATES AND THEOREMS

POSTULATE **Ruler Postulate**

The points of a line can be put in one-to-one correspondence with the real numbers so that the distance between any two points A and B with coordinates a and b is $AB = |a - b|$.

POSTULATE **Segment Addition Postulate**

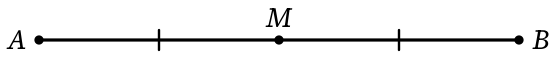


AC = 4 + 6 = 10

If B is between A and C, then $AB + BC = AC$.

Euclid never states this as a separate rule, but he uses it throughout. His Common Notion 5, "the whole is greater than the part," follows from it, since $AC = AB + BC > AB$. (His Common Notion 2, "if equals are added to equals, the wholes are equal," is our Addition Property of Equality.)

THEOREM **Midpoint Theorem**



If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$ and $MB = \frac{1}{2}AB$.

TWO-COLUMN PROOF

Given: M is the midpoint of $\overline{AB}$

Prove: $AM = \frac{1}{2}AB$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$	Given
2	$AM = MB$	Definition of midpoint
3	$AM + MB = AB$	Segment Addition Postulate
4	$AM + AM = AB$	Substitution Property of Equality
5	$2 \cdot AM = AB$	Distributive Property (combining like terms)
6	$AM = \frac{1}{2}AB$	Division Property of Equality

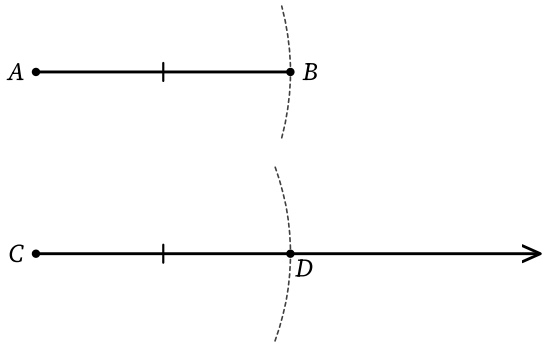
∎

Since $MB = AM$, the same steps give $MB = \frac{1}{2}AB$.

CONSTRUCTIONS

CONSTRUCTION **Copy a segment** Euclid I.2, I.3

1. Given $\overline{AB}$ and a point C. With the straightedge, draw a ray from C longer than $\overline{AB}$.
2. Place the compass point on A and open the compass until the pencil point is on B.
3. Without changing the compass width, place the compass point on C and draw an arc that crosses the ray. Label the crossing point D.
4. Then $\overline{CD} \cong \overline{AB}$.



Why it works. Every point of the arc is the same distance from C,

namely the compass width, which was set equal to AB. So $CD = AB$, and by the definition of congruent segments $\overline{CD} \cong \overline{AB}$. Euclid's Postulate 3 allows us to draw a circle with any center and any radius; his Proposition I.2 shows that a length can be carried to any point even if the compass closes whenever it is lifted, and I.3 cuts off the copied length along a longer line, exactly as in step 3.

KEY FORMULAS

$$AB = |a - b| \quad \text{(number line)}$$

$$M = \frac{a + b}{2} \quad \text{(midpoint on a number line)}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

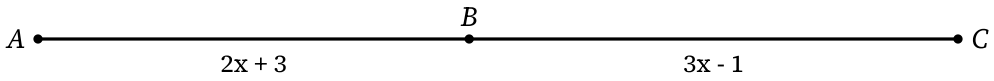
From history. Euclid's Postulate 3 allows us "to describe a circle with any center and distance." Many readers, beginning with ancient commentators, noticed that this does not say the compass can be lifted and set down somewhere else while keeping its opening. That is why Euclid proves Proposition I.2: using only circles drawn about given centers, he shows how to place a segment equal to a given one at any point. So a compass that "collapses" when lifted can do everything a modern compass can do.

Euclid connection. Euclid's I.2 places a copy of a given segment at a given point, and I.3 cuts off from a longer segment a part equal to a shorter one; together they are our construction for copying a segment. His Common Notion 4, "things which coincide with one another are equal to one another," is the idea behind congruent segments.

Practice habit. Recite the Ruler Postulate, the Segment Addition Postulate and the definition of midpoint until they are automatic. Before using $AB + BC = AC$, check that B really is between A and C, and mark the given congruent segments on your diagram with tick marks.

WORKED EXAMPLES

Example 1. Henry is building a straight fence 32 feet long from post A to post C, with a gate post B between them. The distance AB is $(2x + 3)$ feet and BC is $(3x - 1)$ feet. Find x and the length BC.

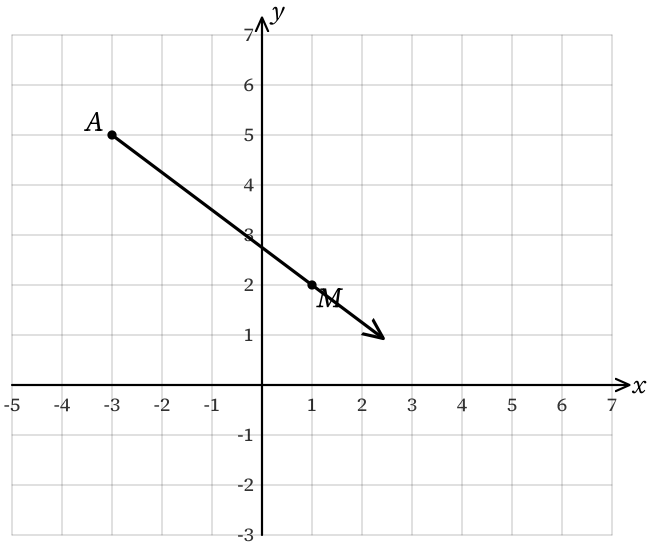


Show solution

1. Because B is between A and C, the Segment Addition Postulate gives $AB + BC = AC$.
2. Substitute: $(2x + 3) + (3x - 1) = 32$, so $5x + 2 = 32$.
3. Subtract 2: $5x = 30$. Divide by 5: $x = 6$.
4. Then $BC = 3(6) - 1 = 17$ feet. Check: $AB = 2(6) + 3 = 15$ feet and $15 + 17 = 32$. ✓

Answer: $x = 6$ and $BC = 17$ feet

Example 2. In the coordinate plane, $M(1, 2)$ is the midpoint of $\overline{AB}$, and A is the point $(-3, 5)$. Find the coordinates of B and the length AB.



Show solution

1. Let $B = (x, y)$. By the midpoint formula, $\dfrac{-3 + x}{2} = 1$ and $\dfrac{5 + y}{2} = 2$.
2. Solve: $-3 + x = 2$, so $x = 5$; and $5 + y = 4$, so $y = -1$. Thus $B = (5, -1)$.
3. Check with the idea of the midpoint: from A to M we move 4 right and 3 down; moving the same again from M reaches $(5, -1)$. ✓
4. Distance formula: $AB = \sqrt{(5 - (-3))^2 + (-1 - 5)^2} = \sqrt{8^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$.

Answer: $B = (5, -1)$ and $AB = 10$

PRACTICE

1. Easy A straight country road has mile markers. The Miller farm is at mile 14.5 and the church is at mile 22. How far apart are they along the road?

   Show answer

   $|22 - 14.5| = 7.5$ miles

2. **Easy** On a number line, A has coordinate -4 and B has coordinate 7. Find the coordinate of the midpoint of $\overline{AB}$.

   Show answer

   $$\frac{-4 + 7}{2} = 1.5$$

3. **Easy** Point B is between A and C, with $AB = 9$ and $BC = 14$. Find AC.

   Show answer

   $AC = 9 + 14 = 23$ (Segment Addition Postulate)

4. **Easy** M is the midpoint of $\overline{PQ}$ and $PM = 7.5$ cm. Find PQ.

   Show answer

   $PQ = 2 \times 7.5 = 15$ cm

5. **Medium** Find the distance between $A(1, -2)$ and $B(7, 6)$.

   Show answer

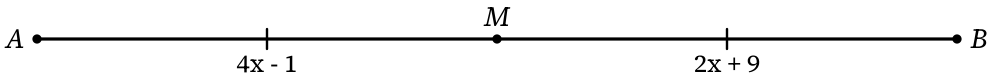
   $$AB = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

6. **Medium** Find the midpoint of the segment with endpoints $(-3, 4)$ and $(5, -8)$.

   Show answer

   $$\left(\frac{-3 + 5}{2}, \frac{4 + (-8)}{2}\right) = (1, -2)$$

7. **Medium** In the figure, M is the midpoint of $\overline{AB}$, $AM = 4x - 1$ and $MB = 2x + 9$. Find x and AB.

   Show answer

   $4x - 1 = 2x + 9$ (definition of midpoint), so $2x = 10$ and $x = 5$. Then $AM = MB = 19$ and $AB = 38$.

8. **Medium** A straight trail runs 5.1 miles from a trailhead T to a lake L and crosses a bridge B on the way. The bridge is twice as far from the lake as it is from the trailhead. How far is the bridge from the lake?

Show answer

Let $TB = x$; then $BL = 2x$ and $x + 2x = 5.1$, so $x = 1.7$. The bridge is $2(1.7) = 3.4$ miles from the lake.

9. **Medium** Explain why the Segment Addition Postulate requires B to be between A and C. Use the example $AB = 5$ and $BC = 3$ to show what can go wrong otherwise.

Show answer

If B is between A and C, then $AC = 5 + 3 = 8$. But if A, B, C are collinear with C between A and B, then $AC + CB = AB$, so $AC = 5 - 3 = 2$; and if the three points are not collinear, AC is some other number. So $AB + BC = AC$ is guaranteed only when B is between A and C.

10. **Hard** $M(2, -1)$ is the midpoint of $\overline{AB}$, and $A = (-4, 3)$. Find the coordinates of B and the length AB in simplest radical form.

Show answer

$B = (2 \cdot 2 - (-4),\ 2(-1) - 3) = (8, -5)$; $AB = \sqrt{12^2 + (-8)^2} = \sqrt{208} = 4\sqrt{13}$

11. **Hard** In the figure, points A, B, C and D are collinear in that order. Given: $AC = BD$. Prove: $AB = CD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

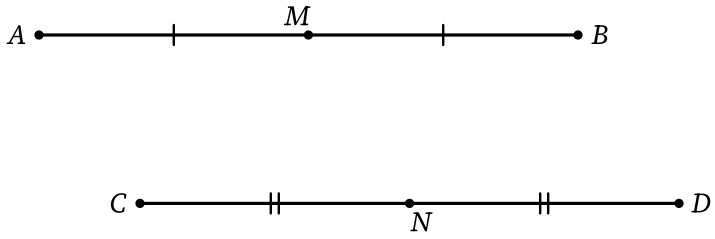
Given: A, B, C, D are collinear in that order; $AC = BD$

Prove: $AB = CD$

	STATEMENTS	REASONS
1	$AC = BD$	Given
2	$AB + BC = AC$	Segment Addition Postulate
3	$BC + CD = BD$	Segment Addition Postulate
4	$AB + BC = BD$	Substitution Property of Equality (row 2 into row 1)
5	$AB + BC = BC + CD$	Substitution Property of Equality (row 3 into row 4)
6	$AB = CD$	Subtraction Property of Equality

∎

12. Hard In the figure, M is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{CD}$, and $AB = CD$. Prove: $AM = CN$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: M is the midpoint of $\overline{AB}$; N is the midpoint of $\overline{CD}$; $AB = CD$

Prove: $AM = CN$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$; N is the midpoint of $\overline{CD}$	Given
2	$AM = \frac{1}{2}AB$	Midpoint Theorem
3	$CN = \frac{1}{2}CD$	Midpoint Theorem
4	$AB = CD$	Given
5	$\frac{1}{2}AB = \frac{1}{2}CD$	Multiplication Property of Equality
6	$AM = CN$	Substitution Property of Equality (rows 2 and 3 into row 5)

∎

13. Hard On a farm map drawn on a coordinate grid, each unit stands for 10 yards. The barn is at $(-2, 1)$ and the well is at $(10, 6)$. The family puts a water trough exactly halfway between them. How far is the trough from the barn, in yards?

Show answer

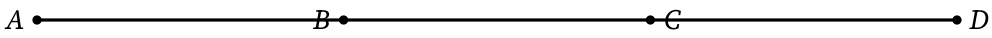
Trough at the midpoint $(4, 3.5)$. Distance from the barn: $\sqrt{6^2 + 2.5^2} = \sqrt{42.25} = 6.5$ units, which is 65 yards. (Check: barn to well is $\sqrt{12^2 + 5^2} = 13$ units, and half of that is 6.5.)

14. Challenge You are given two segments with lengths a and b, where $a > b$. Using only a compass and straightedge, describe how to construct a segment of length $2a - b$.

Show answer

Draw a ray from a point P. Copy length a onto the ray from P to get Q, then copy a again from Q (on the far side of Q from P) to get R, so $PR = 2a$. Now set the compass to b, put its point on R, and mark the point S on $\overline{RP}$ with $RS = b$. Since S is between P and R, $PS + SR = PR$, so $PS = 2a - b$.

15. **Challenge** Points A, B, C and D are collinear in that order. B is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. Prove that $AD = 3 \cdot AB$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: A, B, C, D are collinear in that order; B is the midpoint of $\overline{AC}$; C is the midpoint of $\overline{BD}$

Prove: $AD = 3 \cdot AB$

Because B is the midpoint of $\overline{AC}$, $AB = BC$ by the definition of midpoint. Because C is the midpoint of $\overline{BD}$, $BC = CD$. By the Transitive Property of Equality, $CD = AB$.

Since C is between B and D, the Segment Addition Postulate gives $BD = BC + CD$. Substituting AB for both BC and CD gives $BD = AB + AB = 2 \cdot AB$.

Since B is between A and D, the Segment Addition Postulate gives $AD = AB + BD$. Substituting, $AD = AB + 2 \cdot AB = 3 \cdot AB$.

∎

COMMON MISTAKES

- Writing $\overline{AB} = 7$. A segment is a set of points; its length is a number. Write $AB = 7$, and write $\overline{AB} \cong \overline{CD}$ for congruent segments.
- Using $AB + BC = AC$ when B is not between A and C. Check the order of the points on the line first.
- Finding the midpoint when the problem asks for an endpoint. If M is the midpoint and A is one endpoint, the other endpoint is $B = 2M - A$ in each coordinate; check that M is halfway between your answer and A.
- Changing the compass width while copying a segment. Set it once on the given segment and do not touch the hinge until the arc is drawn.

Make it a game · Split the Segment. Draw a segment on a paper strip and mark the midpoint by eye. Fold the strip to find the true midpoint and measure your error in millimeters; the smaller error wins a point. Then solve a parent-made problem: AB = 2x + 1, BC = 3x - 4, AC = 22. Find x and prove the Segment Addition Postulate holds.

2.3 Measuring Angles

OBJECTIVES

- Name angles, their vertices and sides, and classify angles as acute, right, obtuse or straight.
- Use the Protractor Postulate and the Angle Addition Postulate to find angle measures, including with algebra.
- Identify congruent angles and angle bisectors and prove simple facts about them.
- Convert between decimal degrees and degrees and minutes, as surveyors and astronomers do.

WATCH

Math Antics - Angles & Degrees

mathantics · 9:17

https://www.youtube.com/watch?v=_n3KZR1DSEo

> **WHEN TO WATCH**
> Before the Core ideas: what an angle is and how degrees measure it.

Note: Reading a protractor and converting minutes back to decimal degrees are taught from the Core ideas.

BACKUP VIDEO

Convert from Degrees to Degrees, Minutes, Seconds

Mario's Math Tutoring · 2:40

https://www.youtube.com/watch?v=kbN93Q82IB4

Before the degrees-and-minutes examples: turning a decimal degree into degrees, minutes and seconds.

KEY TERMS

angle

A figure made of two rays with a common endpoint. The rays are the sides and the common endpoint is the vertex. $\angle ABC$ has vertex B (always the middle letter) and sides $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

interior of an angle

The region between the sides of an angle (less than a straight angle): the points on the same side of $\overleftrightarrow{BA}$ as C and on the same side of $\overleftrightarrow{BC}$ as A.

acute, right, obtuse and straight angles

An acute angle measures between $0°$ and $90°$; a right angle measures exactly $90°$; an obtuse angle measures between $90°$ and $180°$; a straight angle is formed by opposite rays and measures $180°$.

congruent angles

Angles that have the same measure. $\angle A \cong \angle B$ means exactly $m\angle A = m\angle B$.

angle bisector

A ray in the interior of an angle, starting at the vertex, that divides the angle into two congruent angles.

minute (of angle)

One sixtieth of a degree: $1° = 60'$. A second is one sixtieth of a minute: $1' = 60''$.

CORE IDEAS

An **angle** is two rays with a common endpoint, the **vertex**. We name an angle by three points with the vertex in the middle, $\angle ABC$, or by its vertex alone, $\angle B$, when no other angle there could be meant. Its measure is written $m\angle ABC$. The sides of an angle are rays, so they go on forever: how long they are drawn in a picture has nothing to do with the size of the angle. The size is the amount of turning from one side to the other.

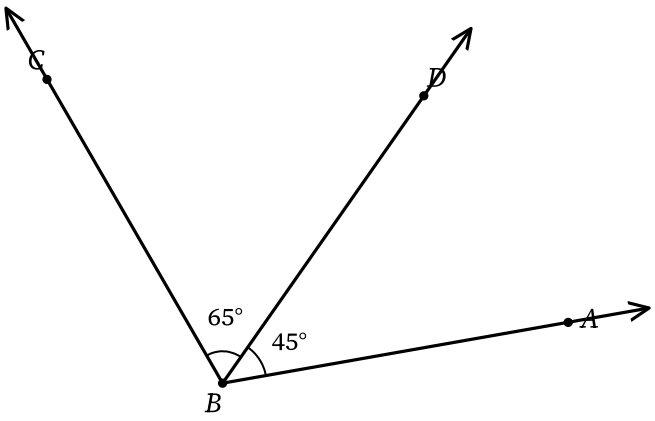
The **Protractor Postulate** is the angle version of the Ruler Postulate. Put the center of a protractor at the vertex: every ray from the vertex on one side of the base line gets a number from 0 to 180, and the measure of an angle is the absolute value of the difference of the numbers on its two sides. That is why you may read an angle from either scale of a protractor, as long as you subtract.

We classify angles by comparing them with a right angle: **acute** (less than $90°$), **right** ($90°$, shown with a small square), **obtuse** (between $90°$ and $180°$) and **straight** ($180°$, made by opposite rays). Euclid defined a right angle without numbers: "When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right." He then called an angle obtuse if it is greater than a right angle and acute if it is less.

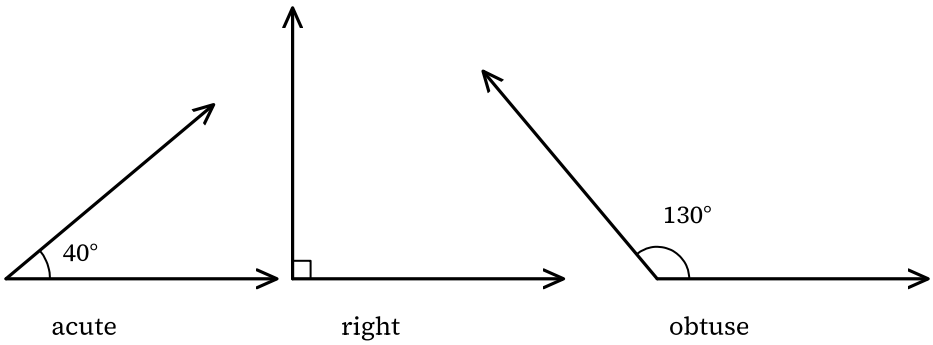
The **Angle Addition Postulate** says that if point D is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$. The two smaller angles fill the larger one exactly, with no gap and no overlap, so their measures add. Like the Segment Addition Postulate, it has a condition: D must be in the interior.

Angles with equal measures are **congruent**, shown with matching arcs. A ray that splits an angle into two congruent angles is its **bisector**. Since the two halves are equal and add up to the whole angle, each half is exactly half the angle; this is proved below, step by step, just as we proved the Midpoint Theorem for segments.

Surveyors and astronomers need more precision than whole degrees, so each degree is divided into 60 **minutes** ($60'$) and each minute into 60 seconds ($60''$). To change 25.75° to degrees and minutes, multiply the decimal part by 60: $0.75 \times 60 = 45$, so $25.75^\circ = 25^\circ 45'$. To subtract angles in degrees and minutes, borrow $1^\circ = 60'$ when you need more minutes, just as you borrow an hour as 60 minutes when subtracting times.



$m\angle ABC = 45^\circ + 65^\circ = 110^\circ$



POSTULATES AND THEOREMS

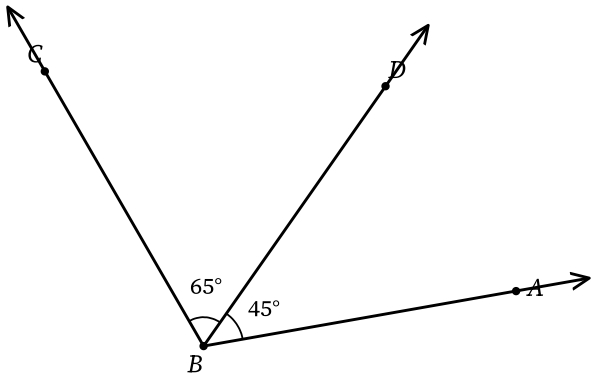
POSTULATE **Protractor Postulate**

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be opposite rays. The rays from O on one side of $\overleftrightarrow{AB}$, together with $\overrightarrow{OA}$ and $\overrightarrow{OB}$, can be paired one to one with the real numbers from 0 to 180 so that $\overrightarrow{OA}$

is paired with 0 *and* $\overrightarrow{OB}$ *with* 180. *The measure of the angle formed by two of these rays is the absolute value of the difference of their numbers.*

POSTULATE **Angle Addition Postulate**

If point D *is in the interior of* $\angle ABC$, *then* $m\angle ABD + m\angle DBC = m\angle ABC$.

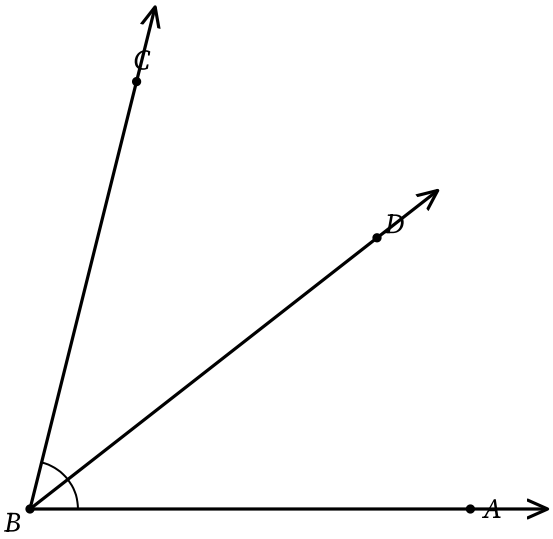


$m\angle ABC = 45° + 65° = 110°$

Euclid uses this idea for angles without stating it as a rule; his Common Notion 5, "the whole is greater than the part," is a close relative, since $m\angle ABC = m\angle ABD + m\angle DBC$ *is greater than either part.*

THEOREM **Angle Bisector Halving Theorem**

If $\overrightarrow{BD}$ *bisects* $\angle ABC$, *then* $m\angle ABD = m\angle DBC = \frac{1}{2}m\angle ABC$.



TWO-COLUMN PROOF

Given: $\overrightarrow{BD}$ bisects $\angle ABC$

Prove: $m\angle ABD = \frac{1}{2}m\angle ABC$

	STATEMENTS	REASONS
1	$\overrightarrow{BD}$ bisects $\angle ABC$	Given
2	$\angle ABD \cong \angle DBC$	Definition of angle bisector
3	$m\angle ABD = m\angle DBC$	Definition of congruent angles
4	$m\angle ABD + m\angle DBC = m\angle ABC$	Angle Addition Postulate (D is in the interior of $\angle ABC$ by the definition of angle bisector)
5	$m\angle ABD + m\angle ABD = m\angle ABC$	Substitution Property of Equality
6	$2 \cdot m\angle ABD = m\angle ABC$	Distributive Property (combining like terms)
7	$m\angle ABD = \frac{1}{2}m\angle ABC$	Division Property of Equality

∎

Since $m\angle DBC = m\angle ABD$, it also follows that $m\angle DBC = \frac{1}{2}m\angle ABC$.

KEY FORMULAS

$$m\angle ABD + m\angle DBC = m\angle ABC$$

$$1^\circ = 60' \qquad 1' = 60''$$

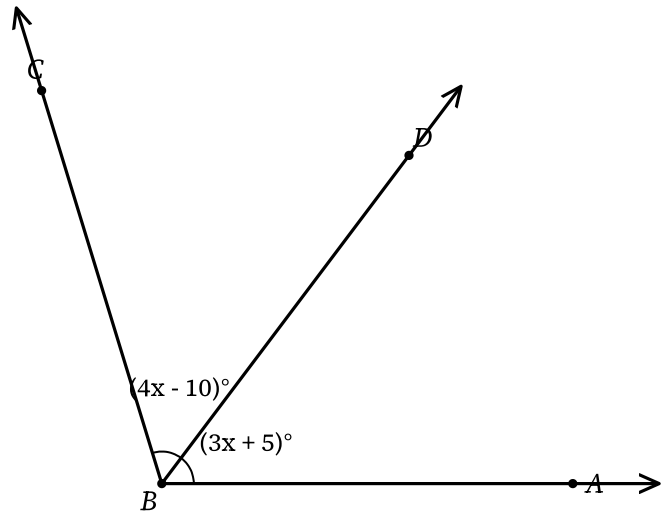
From history. The custom of dividing a circle into 360 degrees, and each degree into 60 minutes, goes back to the astronomers of ancient Babylon, who counted in sixties. Greek astronomers took it over, and Ptolemy's great astronomy book used it throughout. Ship's navigators measuring a star's height with a sextant and land surveyors measuring property corners with a transit still record angles in degrees, minutes and seconds today.

Euclid connection. Euclid's Book I, Definition 8 calls a plane angle "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line," so for Euclid a straight angle was not an angle at all. His Definitions 10 to 12 define right, obtuse and acute angles by comparison with a right angle, without numbers; our Protractor Postulate adds the numbers.

Practice habit. Recite the Protractor Postulate, the Angle Addition Postulate and the definition of angle bisector until they are automatic. Before computing, estimate each angle from the figure (is it acute or obtuse? about how large?) and use the estimate to check your answer.

WORKED EXAMPLES

Example 1. In the figure, D is in the interior of $\angle ABC$, $m\angle ABD = (3x + 5)^\circ$, $m\angle DBC = (4x - 10)^\circ$ and $m\angle ABC = 107^\circ$. Find x and $m\angle DBC$.

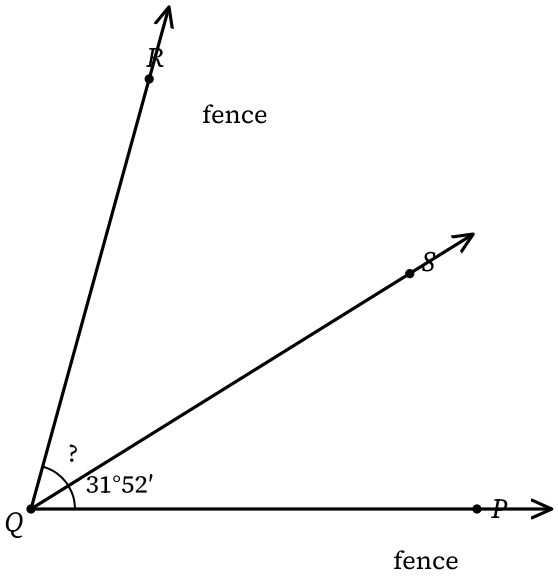


Show solution

1. Because D is in the interior of $\angle ABC$, the Angle Addition Postulate gives $m\angle ABD + m\angle DBC = m\angle ABC$.
2. Substitute: $(3x + 5) + (4x - 10) = 107$, so $7x - 5 = 107$.
3. Add 5: $7x = 112$. Divide by 7: $x = 16$.
4. Then $m\angle DBC = 4(16) - 10 = 54^\circ$. Check: $m\angle ABD = 3(16) + 5 = 53^\circ$ and $53 + 54 = 107$. ✓

Answer: $x = 16$ and $m\angle DBC = 54^\circ$

Example 2. A surveyor sets up a transit at corner Q of a field and measures $m\angle PQR = 74^\circ 38'$ between two fence lines. A stake S lies inside the angle, and $m\angle PQS = 31^\circ 52'$. Find $m\angle SQR$ in degrees and minutes.



Show solution

1. Since S is in the interior of $\angle PQR$, the Angle Addition Postulate gives $m\angle PQS + m\angle SQR = m\angle PQR$, so $m\angle SQR = 74^\circ 38' - 31^\circ 52'$.
2. We cannot take $52'$ from $38'$, so borrow one degree: $74^\circ 38' = 73^\circ 98'$.
3. Subtract: $73^\circ 98' - 31^\circ 52' = 42^\circ 46'$.
4. Check by adding: $31^\circ 52' + 42^\circ 46' = 73^\circ 98' = 74^\circ 38'$. ✓ In decimal degrees this is $42 + \dfrac{46}{60} \approx 42.77^\circ$.

Answer: $m\angle SQR = 42^\circ 46'$

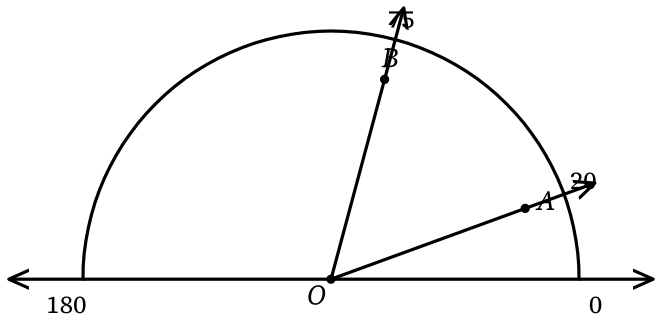
PRACTICE

1. **Easy** Classify an angle that measures 127° as acute, right, obtuse or straight.

   Show answer

   Obtuse, since $90 < 127 < 180$.

2. **Easy** In the figure, a protractor is centered at O with its base along the line. Ray $\overrightarrow{OA}$ passes through the mark 20 and ray $\overrightarrow{OB}$ passes through the mark 75. Find $m\angle AOB$.



Show answer

$|75 - 20| = 55^\circ$ (Protractor Postulate)

3. **Easy** Samuel is making a picture frame. Two pieces meet at a right-angled corner, and he cuts each end so that the seam bisects the 90° corner. What angle does the seam make with each side of the frame?

Show answer

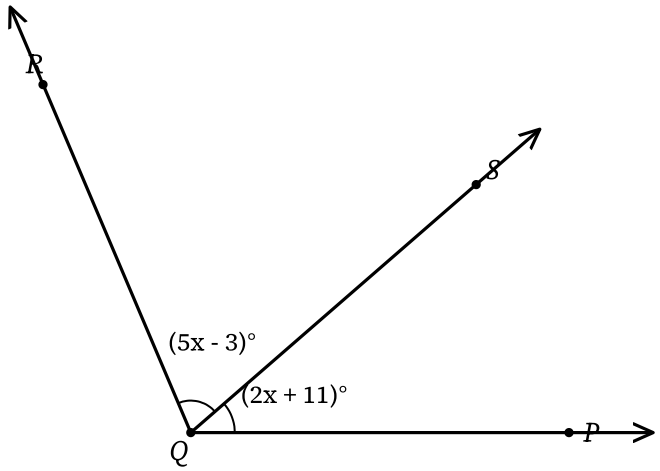
$\frac{1}{2} \times 90^\circ = 45^\circ$

4. **Easy** Write 25.75° in degrees and minutes.

Show answer

$0.75 \times 60 = 45$, so $25.75^\circ = 25^\circ 45'$.

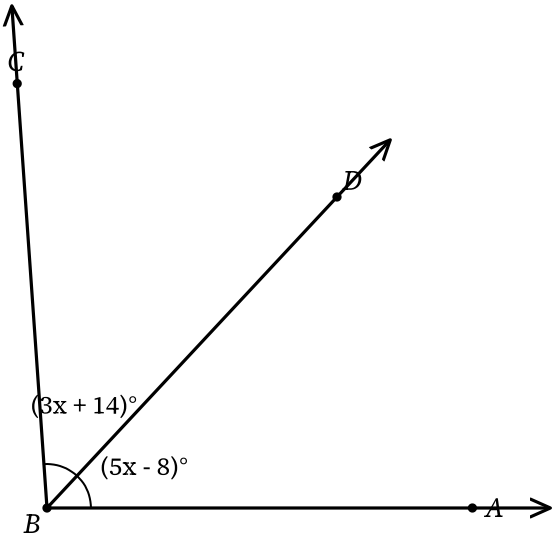
5. **Medium** In the figure, S is in the interior of $\angle PQR$, $m\angle PQS = (2x + 11)^\circ$, $m\angle SQR = (5x - 3)^\circ$ and $m\angle PQR = 113^\circ$. Find x and $m\angle SQR$.



Show answer

$(2x + 11) + (5x - 3) = 113$, so $7x + 8 = 113$ and $x = 15$. Then $m\angle SQR = 5(15) - 3 = 72^\circ$.

6. **Medium** In the figure, $\overrightarrow{BD}$ bisects $\angle ABC$, $m\angle ABD = (5x - 8)^\circ$ and $m\angle DBC = (3x + 14)^\circ$. Find $m\angle ABC$.



Show answer

$5x - 8 = 3x + 14$, so $x = 11$; each half is 47° and $m\angle ABC = 94^\circ$.

7. **Medium** What is the measure of the smaller angle between the hour hand and the minute hand of a clock at 2:30?

   Show answer

   The minute hand points at the 6, 180° from the 12. The hour hand is halfway between 2 and 3, at $2.5 \times 30^\circ = 75^\circ$ from the 12. The angle is $180 - 75 = 105^\circ$.

8. **Medium** A ship's navigator measures the angle of a star above the horizon as $31^\circ 48'$ at 9 p.m. and as $45^\circ 27'$ two hours later. By how much did the angle increase? Give your answer in degrees and minutes.

   Show answer

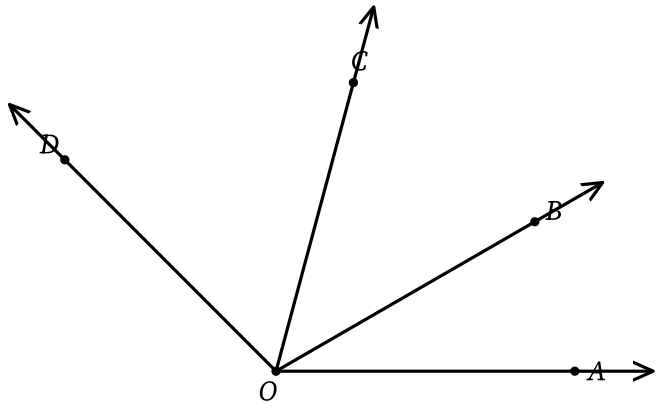
   $45^\circ 27' - 31^\circ 48' = 44^\circ 87' - 31^\circ 48' = 13^\circ 39'$ (which is 13.65°)

9. **Medium** Grace draws $\angle ABC$ with long sides and $\angle DEF$ with short sides, and both measure 95°. Ruth says $\angle ABC$ is bigger because its sides are longer. Is Ruth right? Explain why.

   Show answer

   No. The two angles are congruent. The sides of an angle are rays, which go on forever, so the length drawn means nothing. The measure of an angle is the amount of turning between its sides, and both angles measure 95°, so they are congruent by definition.

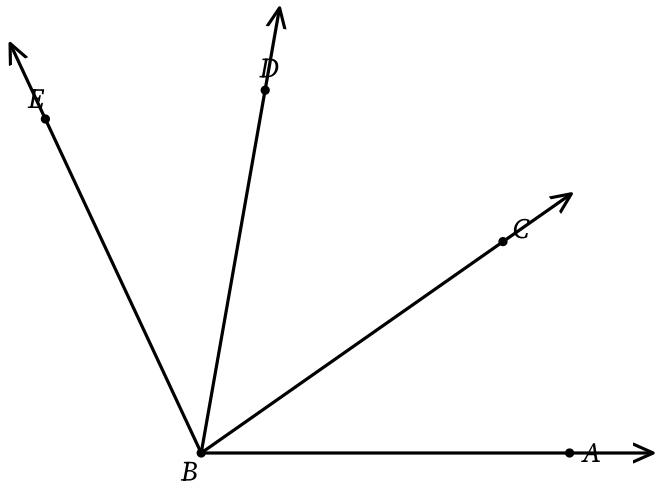
10. Hard In the figure, rays $\overrightarrow{OB}$ and $\overrightarrow{OC}$ lie in the interior of $\angle AOD$, and $m\angle AOB : m\angle BOC : m\angle COD = 2 : 3 : 4$. If $m\angle AOD = 135^\circ$, find $m\angle BOD$.



Show answer

Let the angles be $2k$, $3k$ and $4k$. By angle addition $9k = 135$, so $k = 15$: the angles are 30°, 45° and 60°. Then $m\angle BOD = 45 + 60 = 105^\circ$.

11. Hard In the figure, rays $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$ and $\overrightarrow{BE}$ are in that order, so C is in the interior of $\angle ABD$ and D is in the interior of $\angle CBE$. Given: $m\angle ABD = m\angle CBE$. Prove: $m\angle ABC = m\angle DBE$.



Show answer

See the model proof.

TWO-COLUMN PROOF

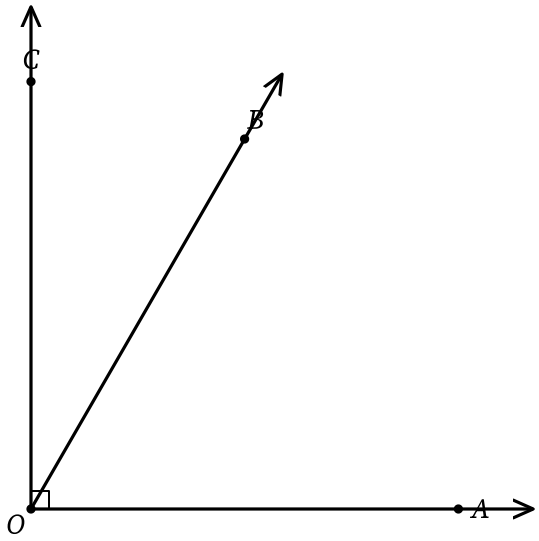
Given: C is in the interior of $\angle ABD$; D is in the interior of $\angle CBE$; $m\angle ABD = m\angle CBE$

Prove: $m\angle ABC = m\angle DBE$

	STATEMENTS	REASONS
1	$m\angle ABD = m\angle CBE$	Given
2	$m\angle ABC + m\angle CBD = m\angle ABD$	Angle Addition Postulate
3	$m\angle CBD + m\angle DBE = m\angle CBE$	Angle Addition Postulate
4	$m\angle ABC + m\angle CBD = m\angle CBD + m\angle DBE$	Substitution Property of Equality (rows 2 and 3 into row 1)
5	$m\angle ABC = m\angle DBE$	Subtraction Property of Equality

∎

12. Hard In the figure, $\angle AOC$ is a right angle and B is in the interior of $\angle AOC$. Given: $m\angle AOB = 2 \cdot m\angle BOC$. Prove: $m\angle BOC = 30°$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\angle AOC$ is a right angle; B is in the interior of $\angle AOC$; $m\angle AOB = 2 \cdot m\angle BOC$

Prove: $m\angle BOC = 30°$

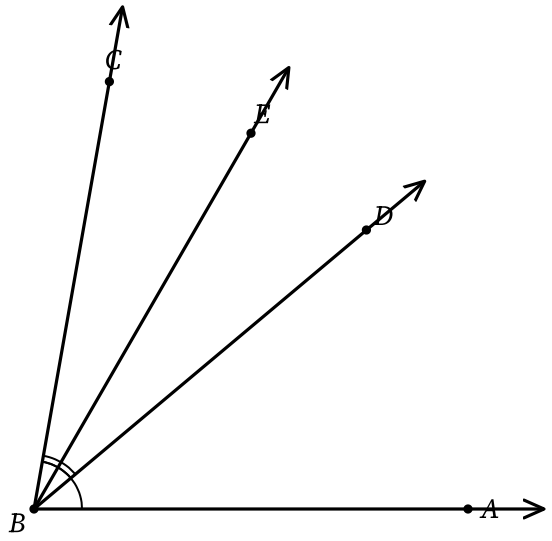
Since $\angle AOC$ is a right angle, $m\angle AOC = 90°$ by the definition of a right angle.

Since B is in the interior of $\angle AOC$, the Angle Addition Postulate gives $m\angle AOB + m\angle BOC = m\angle AOC = 90°$.

Substituting $2 \cdot m\angle BOC$ for $m\angle AOB$ gives $2 \cdot m\angle BOC + m\angle BOC = 90°$, so $3 \cdot m\angle BOC = 90°$, and by the Division Property of Equality $m\angle BOC = 30°$.

∎

13. Hard In the figure, $\overrightarrow{BD}$ bisects $\angle ABC$ and $\overrightarrow{BE}$ bisects $\angle DBC$. Prove: $m\angle ABC = 4 \cdot m\angle EBC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overrightarrow{BD}$ bisects $\angle ABC$; $\overrightarrow{BE}$ bisects $\angle DBC$

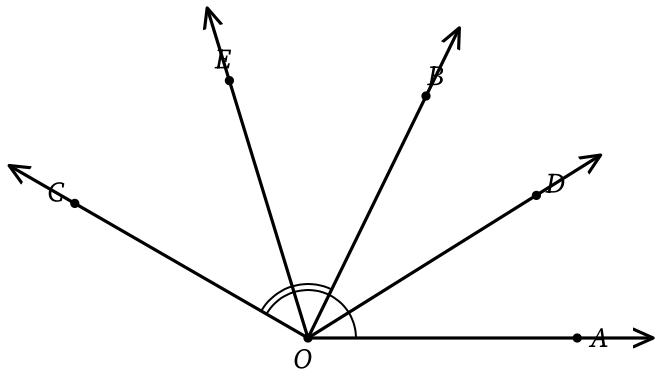
Prove: $m\angle ABC = 4 \cdot m\angle EBC$

	STATEMENTS	REASONS
1	$\overrightarrow{BD}$ bisects $\angle ABC$	Given
2	$m\angle DBC = \frac{1}{2}m\angle ABC$	Angle Bisector Halving Theorem
3	$\overrightarrow{BE}$ bisects $\angle DBC$	Given
4	$m\angle EBC = \frac{1}{2}m\angle DBC$	Angle Bisector Halving Theorem
5	$m\angle EBC = \frac{1}{2} \cdot \frac{1}{2}m\angle ABC$	Substitution Property of Equality (row 2 into row 4)
6	$m\angle EBC = \frac{1}{4}m\angle ABC$	Simplify (multiply the fractions)
7	$4 \cdot m\angle EBC = m\angle ABC$	Multiplication Property of Equality

	STATEMENTS	REASONS
8	$m\angle ABC = 4 \cdot m\angle EBC$	Symmetric Property of Equality

∎

14. **Challenge** In the figure, B is in the interior of $\angle AOC$, $m\angle AOB = 64°$ and $m\angle BOC = 86°$. Ray $\overrightarrow{OD}$ bisects $\angle AOB$ and ray $\overrightarrow{OE}$ bisects $\angle BOC$. Find $m\angle DOE$, and explain why the answer is always half of $m\angle AOC$, whatever the two angles are.



Show answer

$m\angle DOB = 32°$ and $m\angle BOE = 43°$, so $m\angle DOE = 32 + 43 = 75°$, which is half of $m\angle AOC = 150°$. In general $m\angle DOE = \frac{1}{2}m\angle AOB + \frac{1}{2}m\angle BOC = \frac{1}{2}(m\angle AOB + m\angle BOC) = \frac{1}{2}m\angle AOC$ by the Angle Bisector Halving Theorem, the Angle Addition Postulate and the Distributive Property.

COMMON MISTAKES

- Putting the vertex in the wrong place in the name. In $\angle ABC$ the vertex is B; $\angle BAC$ is a different angle with vertex A.
- Reading the wrong scale on a protractor. Estimate first: if the angle is clearly acute, an answer over $90°$ means you read the other scale.
- Treating minutes as hundredths: $25°45'$ is $25.75°$, not $25.45°$, because a degree has 60 minutes, not 100.
- Adding angle measures when the ray is not in the interior. Check that D really lies between the sides of $\angle ABC$ before writing $m\angle ABD + m\angle DBC = m\angle ABC$.

Make it a game · Human Protractor. Stand facing your parent. Parent calls an angle: 45 degrees, 135, 90, 180. Show it with your two arms from one shoulder, and name it acute, right, obtuse or straight. Check with a real protractor; within 5 degrees earns a point. Then try bisecting your own angle with one hand, and say its measure.

2.4 Angle Pairs and the Vertical Angles Theorem

OBJECTIVES

- Identify complementary, supplementary and adjacent angles, linear pairs and vertical angles.
- Prove the Linear Pair Theorem, the Vertical Angles Theorem and the Congruent Supplements and Congruent Complements Theorems.
- Use angle-pair relationships and algebra to find unknown angle measures.
- Write two-column and paragraph proofs about angle pairs.

WATCH

Complementary and Supplementary Angles | How to Find Missing Angles | Math with Mr. J

Math with Mr. J · 9:32

https://www.youtube.com/watch?v=E_ulN2FjJac

> **WHEN TO WATCH**
> Before Example 1: complementary and supplementary angles and finding a missing angle.

Note: Both videos find angles with numbers; the proofs of the Linear Pair and Vertical Angles Theorems are taught from the Core ideas.

BACKUP VIDEO

What are Vertical Angles? | Math with Mr. J

Math with Mr. J · 2:19

https://www.youtube.com/watch?v=DJRTIGwJlbU

Before the proofs: what vertical angles are.

KEY TERMS

complementary angles

Two angles whose measures add to 90°. Each is the complement of the other.

supplementary angles

Two angles whose measures add to 180°. Each is the supplement of the other.

adjacent angles

Two angles in a plane that share a vertex and a side but have no interior points in common, such as $\angle ABD$ and $\angle DBC$ when D is in the interior of $\angle ABC$.

linear pair

Two adjacent angles whose noncommon sides are opposite rays. If B is between A and C and D is not on $\overleftrightarrow{AC}$, then $\angle ABD$ and $\angle DBC$ form a linear pair.

vertical angles

Two angles whose sides form two pairs of opposite rays. When two lines cross, they form two pairs of vertical angles.

CORE IDEAS

Many proofs in geometry use pairs of angles. Two angles are **complementary** if their measures add to $90°$ and **supplementary** if they add to $180°$. These words describe only the sum of the measures; the two angles do not have to touch. The complement of a $34°$ angle is $56°$ and its supplement is $146°$.

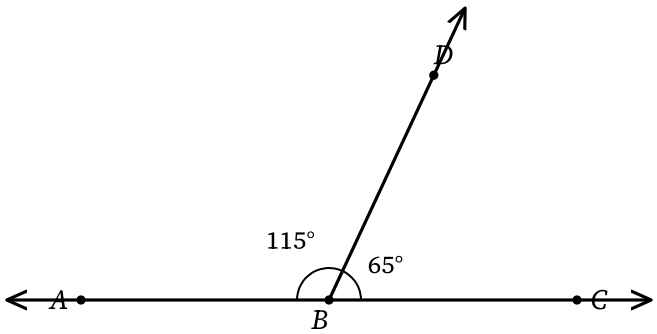
Adjacent angles sit side by side: they share a vertex and a side and do not overlap. When the two outer sides of adjacent angles are opposite rays, the angles form a **linear pair**; together they make a straight angle. Euclid proves in I.13 that "if a straight line set up on a straight line makes angles, it will make either two right angles or angles equal to two right angles." In our language: the angles of a linear pair are supplementary. We prove it below from the Protractor Postulate.

When two lines cross, they make four angles. Two angles across from each other are **vertical angles**; their sides are two pairs of opposite rays. The **Vertical Angles Theorem** says they are always congruent. Why? Each of the two vertical angles forms a linear pair with the same angle between them, so each equals $180°$ minus that same angle. Equals subtracted from equals leave equals (Euclid's Common Notion 3).

That same reasoning works more generally. If two angles are supplements of the same angle (or of congruent angles), they must be congruent: this is the **Congruent Supplements Theorem**. The same is true for complements: the **Congruent Complements Theorem**. These theorems let you prove two angles congruent without knowing any measures at all.

With algebra, angle pairs become equations. Vertical angles give an equation of the form expression $=$ expression; a linear pair or two supplementary angles give expression $+$ expression $= 180$; complementary angles give a sum of 90. Always say which relationship you are using; it is the reason for your equation.

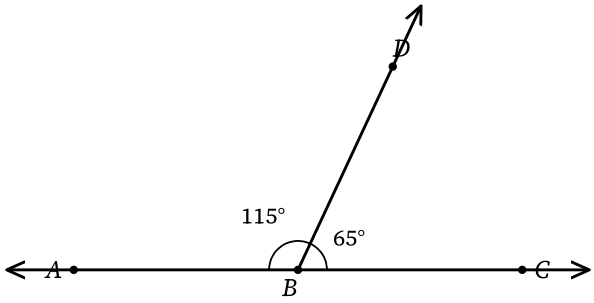
A warning: two adjacent angles need not be supplementary, and two supplementary angles need not be adjacent. Read the definitions carefully and check the converse before you use it.



POSTULATES AND THEOREMS

THEOREM **Linear Pair Theorem** Euclid I.13

If two angles form a linear pair, then they are supplementary.



PARAGRAPH PROOF

Given: $\angle ABD$ and $\angle DBC$ form a linear pair

Prove: $m\angle ABD + m\angle DBC = 180°$

Since the angles form a linear pair, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays and D is not on $\overleftrightarrow{AC}$.

By the Protractor Postulate, the rays from B on the side of $\overleftrightarrow{AC}$ that contains D can be numbered from 0 to 180 with $\overrightarrow{BA}$ numbered 0 and $\overrightarrow{BC}$ numbered 180. Let d be the number of $\overrightarrow{BD}$, where $0 < d < 180$.

Then $m\angle ABD = |d - 0| = d$ and $m\angle DBC = |180 - d| = 180 - d$. Adding, $m\angle ABD + m\angle DBC = d + (180 - d) = 180°$, so the angles are supplementary by definition.

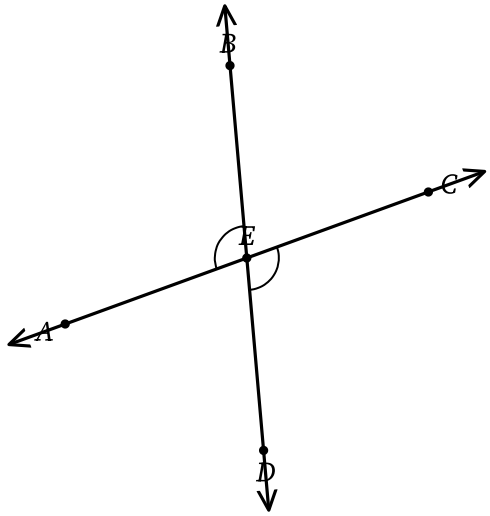
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Euclid proves this as Proposition I.13 by comparing the angles with right angles. Some textbooks accept it as a postulate.

THEOREM **Vertical Angles**

Theorem Euclid I.15

Vertical angles are congruent.



TWO-COLUMN PROOF

Given: Lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E

Prove: $\angle AEB \cong \angle CED$

	STATEMENTS	REASONS
1	$\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E	Given
2	$\angle AEB$ and $\angle BEC$ form a linear pair	Definition of linear pair ($\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays)
3	$m\angle AEB + m\angle BEC = 180°$	Linear Pair Theorem
4	$\angle BEC$ and $\angle CED$ form a linear pair	Definition of linear pair ($\overrightarrow{EB}$ and $\overrightarrow{ED}$ are opposite rays)
5	$m\angle BEC + m\angle CED = 180°$	Linear Pair Theorem
6	$m\angle AEB + m\angle BEC = m\angle BEC + m\angle CED$	Transitive Property of Equality

	STATEMENTS	REASONS
7	$m\angle AEB = m\angle CED$	Subtraction Property of Equality
8	$\angle AEB \cong \angle CED$	Definition of congruent angles

∎

The same steps, using the linear pairs $\angle BEC, \angle CED$ and $\angle CED, \angle DEA$, show $\angle BEC \cong \angle DEA$.

THEOREM **Congruent Supplements Theorem**

If two angles are supplementary to the same angle (or to congruent angles), then they are congruent.

PARAGRAPH PROOF

Given: $\angle 1$ and $\angle 2$ are supplementary; $\angle 3$ and $\angle 4$ are supplementary; $\angle 2 \cong \angle 4$
Prove: $\angle 1 \cong \angle 3$
By the definition of supplementary angles, $m\angle 1 + m\angle 2 = 180°$ and $m\angle 3 + m\angle 4 = 180°$. By the Subtraction Property of Equality, $m\angle 1 = 180° - m\angle 2$ and $m\angle 3 = 180° - m\angle 4$.

Since $\angle 2 \cong \angle 4$, $m\angle 2 = m\angle 4$ by the definition of congruent angles. Substituting, $m\angle 3 = 180° - m\angle 2$, so $m\angle 1 = m\angle 3$ by the Transitive Property of Equality, and $\angle 1 \cong \angle 3$ by the definition of congruent angles.

If both angles are supplementary to the same angle, take $\angle 4$ to be $\angle 2$ itself; the same proof applies.

∎

THEOREM **Congruent Complements Theorem**

If two angles are complementary to the same angle (or to congruent angles), then they are congruent.

TWO-COLUMN PROOF

Given: $\angle 1$ and $\angle 2$ are complementary; $\angle 3$ and $\angle 4$ are complementary; $\angle 2 \cong \angle 4$
Prove: $\angle 1 \cong \angle 3$

	STATEMENTS	REASONS
1	$\angle 1$ and $\angle 2$ are complementary; $\angle 3$ and $\angle 4$ are complementary	Given
2	$m\angle 1 + m\angle 2 = 90^\circ$	Definition of complementary angles
3	$m\angle 3 + m\angle 4 = 90^\circ$	Definition of complementary angles
4	$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	Transitive Property of Equality
5	$\angle 2 \cong \angle 4$	Given
6	$m\angle 2 = m\angle 4$	Definition of congruent angles
7	$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$	Substitution Property of Equality
8	$m\angle 1 = m\angle 3$	Subtraction Property of Equality
9	$\angle 1 \cong \angle 3$	Definition of congruent angles

∎

If both angles are complementary to the same angle, take $\angle 4$ to be $\angle 2$.

KEY FORMULAS

$$\text{complementary: } m\angle 1 + m\angle 2 = 90^\circ$$

$$\text{supplementary: } m\angle 1 + m\angle 2 = 180^\circ$$

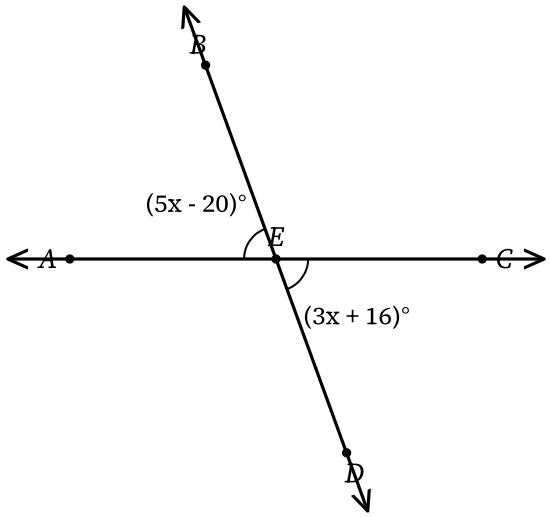
From history. According to Proclus, who wrote a commentary on Book I of the Elements in the fifth century AD, the historian Eudemus said that the Vertical Angles Theorem was first discovered by Thales of Miletus, one of the earliest Greek geometers, but that it was thought worthy of a scientific proof only by Euclid. Thales is remembered as one of the first to ask not just that a fact is true but why it is true.

Euclid connection. Euclid's I.15 reads: "If two straight lines cut one another, they make the vertical angles equal to one another." His proof is the one given here: each vertical angle together with the same adjacent angle makes two right angles (I.13), and when the common angle is taken away, the remainders are equal (Common Notion 3).

Practice habit. Recite the definitions of complementary, supplementary, linear pair and vertical angles and the four theorems of this lesson until they are automatic. Before writing an equation, name the relationship you are using (vertical angles, linear pair, complementary) as its reason.

WORKED EXAMPLES

Example 1. Two straight roads cross at an intersection E, as in the figure. One angle between them measures $(5x - 20)^\circ$ and the angle vertical to it measures $(3x + 16)^\circ$. Find x and $m\angle BEC$.

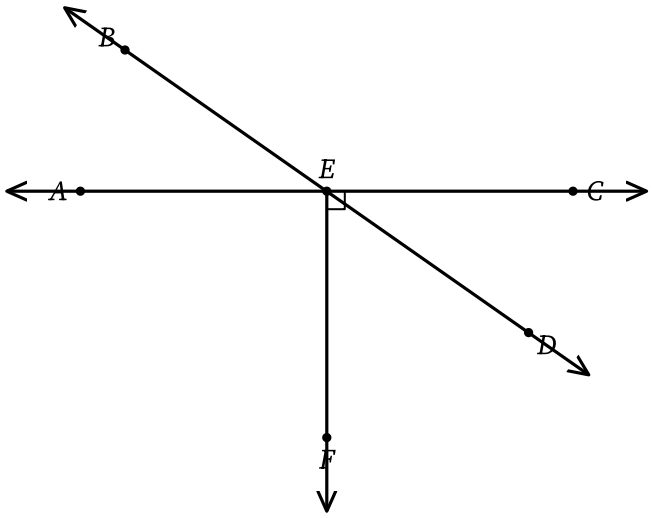


Show solution

1. $\angle AEB$ and $\angle CED$ are vertical angles, so they are congruent (Vertical Angles Theorem): $5x - 20 = 3x + 16$.
2. Subtract $3x$ and add 20: $2x = 36$, so $x = 18$. Then $m\angle AEB = 5(18) - 20 = 70^\circ$ (and $3(18) + 16 = 70^\circ$ ✓).
3. $\angle AEB$ and $\angle BEC$ form a linear pair, so they are supplementary (Linear Pair Theorem): $m\angle BEC = 180 - 70 = 110^\circ$.

Answer: $x = 18$ and $m\angle BEC = 110^\circ$

Example 2. In the figure, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E, $\angle CEF$ is a right angle, and D is in the interior of $\angle CEF$. Prove that $\angle AEB$ and $\angle DEF$ are complementary.



Show solution

1. Work backward: to show the angles are complementary we need $m\angle AEB + m\angle DEF = 90°$.
2. The right angle $\angle CEF$ is split by $\overrightarrow{ED}$, so $m\angle CED + m\angle DEF = 90°$.
3. $\angle AEB$ and $\angle CED$ are vertical angles, so we can replace $m\angle CED$ by $m\angle AEB$.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E; $\angle CEF$ is a right angle; D is in the interior of $\angle CEF$

Prove: $\angle AEB$ and $\angle DEF$ are complementary

	STATEMENTS	REASONS
1	$\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E	Given
2	$\angle AEB \cong \angle CED$	Vertical Angles Theorem
3	$m\angle AEB = m\angle CED$	Definition of congruent angles
4	$\angle CEF$ is a right angle	Given
5	$m\angle CEF = 90°$	Definition of right angle

	STATEMENTS	REASONS
6	$m\angle CED + m\angle DEF = m\angle CEF$	Angle Addition Postulate (D is in the interior of $\angle CEF$)
7	$m\angle CED + m\angle DEF = 90^\circ$	Substitution Property of Equality
8	$m\angle AEB + m\angle DEF = 90^\circ$	Substitution Property of Equality (row 3 into row 7)
9	$\angle AEB$ and $\angle DEF$ are complementary	Definition of complementary angles

∎

Answer: See the proof.

PRACTICE

1. Easy Find the complement of a 34° angle.

   Show answer

   $90 - 34 = 56^\circ$

2. Easy Find the supplement of a 117° angle.

   Show answer

   $180 - 117 = 63^\circ$

3. Easy The two arms of a pair of scissors are straight and cross at the pivot like two lines. The blades are opened to an angle of 38°. What is the angle between one blade and the handle of the other arm?

   Show answer

   The blade and that handle form a linear pair with the 38° angle, so the angle is $180 - 38 = 142^\circ$.

4. Easy True or false: two adjacent angles always form a linear pair.

   Show answer

   False. Adjacent angles form a linear pair only if their noncommon sides are opposite rays; for example, two adjacent 30° angles do not.

5. **Medium** Two complementary angles measure $(2x + 8)^\circ$ and $(3x - 3)^\circ$. Find x and the measure of the larger angle.

Show answer

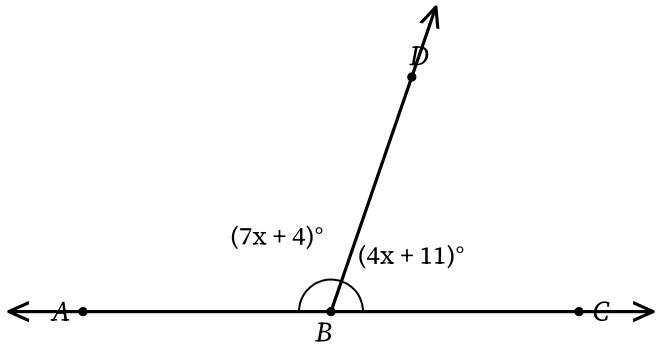
$(2x + 8) + (3x - 3) = 90$, so $5x + 5 = 90$ and $x = 17$. The angles are 42° and 48°; the larger is 48°.

6. **Medium** The supplement of an angle is four times its complement. Find the angle.

Show answer

$180 - x = 4(90 - x)$, so $180 - x = 360 - 4x$, $3x = 180$ and $x = 60^\circ$. Check: supplement 120°, complement 30°, and $120 = 4 \times 30$. ✓

7. **Medium** In the figure, B is between A and C, $m\angle ABD = (7x + 4)^\circ$ and $m\angle DBC = (4x + 11)^\circ$. Find x and $m\angle ABD$.



Show answer

The angles form a linear pair, so $(7x + 4) + (4x + 11) = 180$; $11x + 15 = 180$ and $x = 15$. Then $m\angle ABD = 109^\circ$.

8. **Medium** A straight county road crosses a straight railroad track. One of the four angles at the crossing measures 63°. What is the measure of each obtuse angle at the crossing?

Show answer

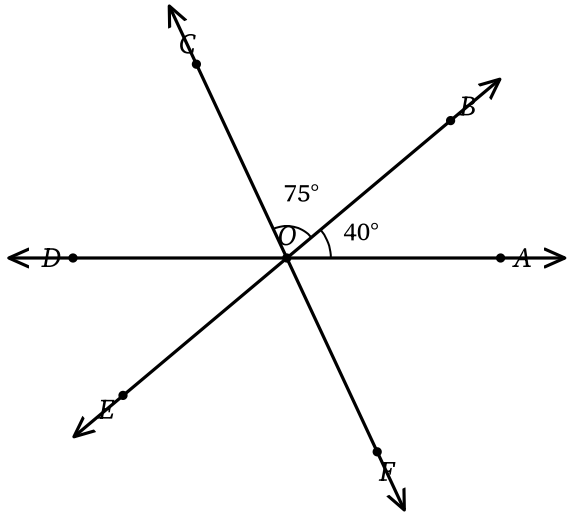
The angle next to the 63° angle forms a linear pair with it: $180 - 63 = 117^\circ$. The other obtuse angle is vertical to it, so it is also 117°.

9. **Medium** Ruth says: "If two angles are supplementary, they must form a linear pair." Is she right? Explain why or why not.

Show answer

No. The converse of the Linear Pair Theorem is false. Two angles of 70° and 110° drawn in different places are supplementary but are not adjacent, so they are not a linear pair.

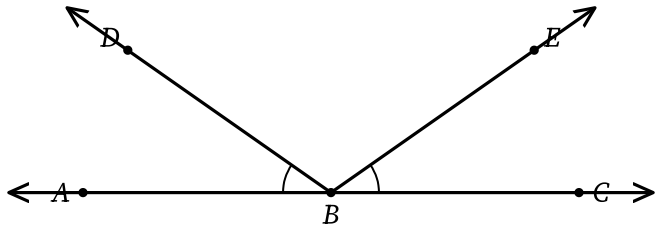
10. **Hard** In the figure, three lines meet at O: $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$. Given $m\angle AOB = 40^\circ$ and $m\angle BOC = 75^\circ$, find $m\angle COD$ and $m\angle EOF$.



Show answer

$\overrightarrow{OA}$ and $\overrightarrow{OD}$ are opposite rays, so $m\angle COD = 180 - 40 - 75 = 65^\circ$ (Angle Addition and Linear Pair). $\angle EOF$ is vertical to $\angle BOC$, so $m\angle EOF = 75^\circ$.

11. Hard In the figure, B is between A and C, and D and E are on the same side of $\overleftrightarrow{AC}$. Given: $\angle ABD \cong \angle CBE$. Prove: $\angle ABE \cong \angle CBD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: B is between A and C; $\angle ABD \cong \angle CBE$

Prove: $\angle ABE \cong \angle CBD$

	STATEMENTS	REASONS
1	$\angle ABD \cong \angle CBE$	Given
2	$\angle ABE$ and $\angle EBC$ form a linear pair	Definition of linear pair ($\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays)
3	$\angle ABE$ and $\angle EBC$ are supplementary	Linear Pair Theorem
4	$\angle CBD$ and $\angle DBA$ form a linear pair	Definition of linear pair
5	$\angle CBD$ and $\angle DBA$ are supplementary	Linear Pair Theorem
6	$\angle ABE \cong \angle CBD$	Congruent Supplements Theorem (supplements of the congruent angles $\angle CBE$ and $\angle ABD$)

∎

12. Hard Given: $\angle PQR$ and $\angle RQS$ form a linear pair, and $m\angle PQR = 3 \cdot m\angle RQS$. Prove: $m\angle RQS = 45°$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\angle PQR$ and $\angle RQS$ form a linear pair; $m\angle PQR = 3 \cdot m\angle RQS$

Prove: $m\angle RQS = 45°$

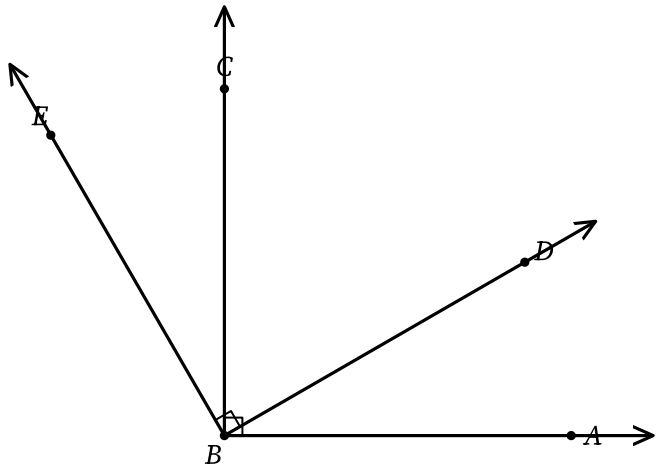
Because the angles form a linear pair, they are supplementary by the Linear Pair Theorem, so $m\angle PQR + m\angle RQS = 180°$.

Substituting $3 \cdot m\angle RQS$ for $m\angle PQR$ gives $3 \cdot m\angle RQS + m\angle RQS = 180°$, so $4 \cdot m\angle RQS = 180°$.

By the Division Property of Equality, $m\angle RQS = 45°$.

∎

13. Hard A carpenter lays two steel squares on a board so that their corners meet at B, as in the figure. $\angle ABC$ and $\angle DBE$ are right angles, D is in the interior of $\angle ABC$, and C is in the interior of $\angle DBE$. Prove: $\angle ABD \cong \angle CBE$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\angle ABC$ and $\angle DBE$ are right angles; D is in the interior of $\angle ABC$; C is in the interior of $\angle DBE$

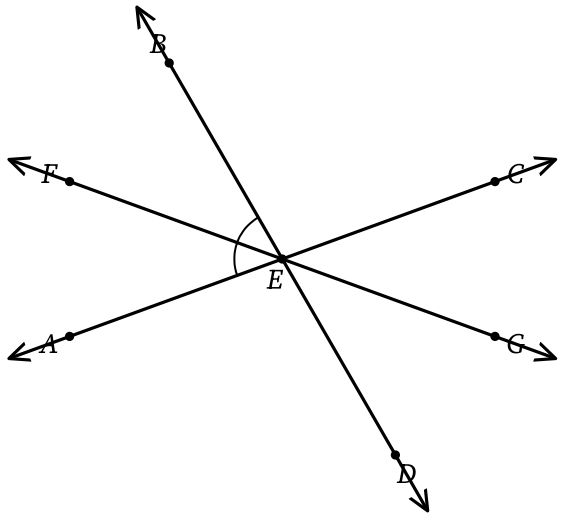
Prove: $\angle ABD \cong \angle CBE$

	STATEMENTS	REASONS
1	$\angle ABC$ and $\angle DBE$ are right angles	Given
2	$m\angle ABC = 90°$	Definition of right angle
3	$m\angle DBE = 90°$	Definition of right angle
4	$m\angle ABD + m\angle DBC = m\angle ABC$	Angle Addition Postulate
5	$m\angle DBC + m\angle CBE = m\angle DBE$	Angle Addition Postulate
6	$m\angle ABD + m\angle DBC = 90°$	Substitution Property of Equality
7	$m\angle DBC + m\angle CBE = 90°$	Substitution Property of Equality
8	$\angle ABD$ and $\angle DBC$ are complementary	Definition of complementary angles

	STATEMENTS	REASONS
9	$\angle CBE$ and $\angle DBC$ are complementary	Definition of complementary angles
10	$\angle ABD \cong \angle CBE$	Congruent Complements Theorem

∎

14. **Challenge** In the figure, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E. Ray $\overrightarrow{EF}$ bisects $\angle AEB$, and $\overrightarrow{EG}$ is the ray opposite $\overrightarrow{EF}$. Prove that $\overrightarrow{EG}$ bisects $\angle CED$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E; $\overrightarrow{EF}$ bisects $\angle AEB$; $\overrightarrow{EG}$ and $\overrightarrow{EF}$ are opposite rays

Prove: $\overrightarrow{EG}$ bisects $\angle CED$

	STATEMENTS	REASONS
1	$\overrightarrow{EF}$ bisects $\angle AEB$	Given
2	$\angle AEF \cong \angle FEB$	Definition of angle bisector
3	$\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays; $\overrightarrow{EB}$ and $\overrightarrow{ED}$ are opposite rays; $\overrightarrow{EF}$ and $\overrightarrow{EG}$ are opposite rays	Given (the lines meet at E; $\overrightarrow{EG}$ is opposite $\overrightarrow{EF}$)
4	$\angle AEF$ and $\angle CEG$ are vertical angles	Definition of vertical angles
5	$\angle AEF \cong \angle CEG$	Vertical Angles Theorem
6	$\angle FEB$ and $\angle GED$ are vertical angles	Definition of vertical angles
7	$\angle FEB \cong \angle GED$	Vertical Angles Theorem

	STATEMENTS	REASONS
8	$\angle CEG \cong \angle FEB$	Symmetric and Transitive Properties of Congruence (rows 5 and 2)
9	$\angle CEG \cong \angle GED$	Transitive Property of Congruence (rows 8 and 7)
10	G is in the interior of $\angle CED$	As shown in the figure (G lies between the sides of $\angle CED$)
11	$\overrightarrow{EG}$ bisects $\angle CED$	Definition of angle bisector

∎

COMMON MISTAKES

- Mixing up complementary ($90°$) and supplementary ($180°$). One memory aid: C comes before S in the alphabet, and 90 comes before 180.
- Assuming every pair of angles that looks like a linear pair or vertical pair is one. Vertical angles must be formed by two intersecting lines, so check that both pairs of sides really are opposite rays.
- Setting vertical angles supplementary or a linear pair equal. Vertical angles are equal; a linear pair adds to $180°$. Write the reason next to the equation before solving.
- Using the converse of a theorem without checking it. Supplementary angles need not be a linear pair, and congruent angles need not be vertical angles.

> **Make it a game · Angle Pair Charades.** Write complementary, supplementary, adjacent, linear pair and vertical angles on five slips. Draw one and act it out silently, using your arms or two rulers as the rays. Your parent guesses the name; then you state the theorem and a numeric example, such as vertical angles of 2x + 10 and 3x - 20. Swap turns.

2.5 Basic Constructions

OBJECTIVES

- Explain what a compass-and-straightedge construction is and why Euclid allows only these two tools.
- Construct an equilateral triangle on a segment (I.1) and prove that it is equilateral.
- Copy an angle (I.23), bisect an angle (I.9), bisect a segment with its perpendicular bisector (I.10), and construct perpendiculars through a point on a line (I.11) and from a point not on a line (I.12).
- Explain why each construction works, previewing the congruent-triangle proofs of Unit 4.

WATCH

How to Construct a Perpendicular Bisector and WHY it Works!

Mario's Math Tutoring · 4:30

https://www.youtube.com/watch?v=0fBkrYwcORk

> **WHEN TO WATCH**
> With the perpendicular-bisector construction: the compass steps, then why it works (congruent triangles, previewing Unit 4).

Note: The equilateral triangle, copying an angle and the perpendiculars through a point on and off a line are taught from the construction steps and figures in the lesson.

BACKUP VIDEO

How to Construct an Angle Bisector Using a Compass and WHY it Works!

Mario's Math Tutoring · 3:51

https://www.youtube.com/watch?v=Wz6t1ghuiPY

With the angle-bisector construction: the compass steps, then why it works (SSS congruence).

KEY TERMS

construction

A drawing made with only an unmarked straightedge (to draw lines through two points) and a compass (to draw circles with a given center through a given point), following steps that can be justified by postulates and theorems.

circle

The set of all points in a plane at a given distance (the radius) from a given point (the center). Every point of a circle is the same distance from the center.

equilateral triangle

A triangle whose three sides are all congruent.

perpendicular lines

Two lines that intersect to form a right angle. We write $\ell \perp m$; a right-angle mark shows it on a figure.

perpendicular bisector

A line (or ray or segment) that is perpendicular to a segment at its midpoint.

CORE IDEAS

A **construction** is a drawing made with only two tools: a straightedge with no marks, and a compass. These are exactly the tools of Euclid's first three postulates: draw a straight line from any point to any point (Post. 1), extend a straight line (Post. 2), and draw a circle with any center and radius (Post. 3). Why so strict? Because a construction is a kind of proof. A ruler or protractor gives only an approximate measurement, but a construction gives an exact result whose correctness we can prove from the postulates.

Every construction rests on one fact about circles: every point on a circle is the same distance from the center. So when you swing an arc with the compass, every point on that arc is exactly one compass-width from the point where the compass needle sits. Most constructions find a point that is the right distance from two given points by crossing two arcs.

Euclid's very first proposition, I.1, constructs an equilateral triangle on a given segment $\overline{AB}$: draw the circle with center A through B, the circle with center B

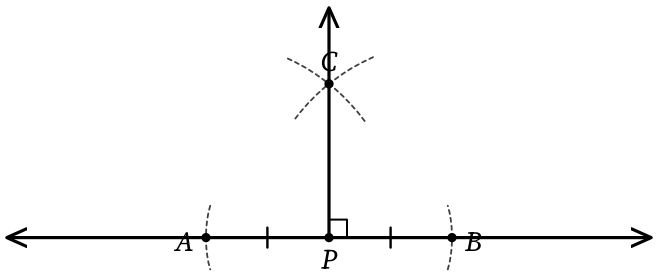
through A, and join a point C where they cross to A and to B. The proof is short and beautiful: $AC = AB$ and $BC = BA$ because they are radii, and "things equal to the same thing are equal to each other" (Common Notion 1). But there is a gap: Euclid never proved the two circles actually meet! It is clear from a drawing, but a figure cannot prove anything. Later mathematicians, David Hilbert among them in his *Foundations of Geometry* (1899), filled such gaps with axioms of continuity from which the meeting of the circles can be proved. In this course we simply state the fact as the Circle Intersection Postulate.

With the same idea of crossing arcs we can bisect a segment. Swing arcs of the same radius (more than half of AB) from A and from B; they cross at two points P and Q. Each of these points is the same distance from A as from B, and the line $\overleftrightarrow{PQ}$ turns out to be the **perpendicular bisector** of $\overline{AB}$ (Euclid I.10 and I.11). To draw a perpendicular through a point P on a line, first mark two points A and B on the line at equal distances from P, then build the perpendicular bisector of $\overline{AB}$: it passes through P (see the figure). To drop a perpendicular from a point not on a line (I.12), use the point itself as the first compass center (Worked Example 2).

To bisect an angle (I.9), mark equal distances along its two sides, then cross equal arcs from those marks; the ray from the vertex through the crossing point is the bisector. To copy an angle (I.23), copy its "triangle of arcs": the same radius along the sides and the same chord between them (Worked Example 1).

Why do the angle bisector, perpendicular bisector and angle-copying constructions really work? Each one builds two triangles whose sides are equal, compass-width by compass-width. In Unit 4 you will prove that triangles with three pairs of congruent sides are congruent (SSS, Lesson 4.3), and in Lesson 4.5 you will use that to prove each construction correct. For now, you can check each one with a protractor and ruler, while remembering that measuring checks a construction but does not prove it.

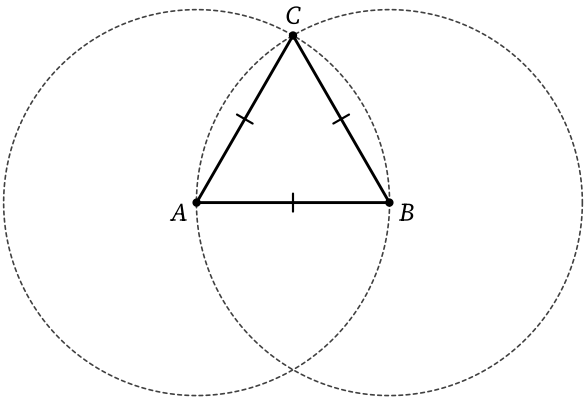
Constructions were not only exercises for students. The master masons who built the great cathedrals laid out walls, arches and windows with compass, cord and straightedge, and the pointed arch of many Gothic windows is drawn exactly like Euclid's I.1.



POSTULATES AND THEOREMS

POSTULATE **Circle Intersection Postulate**

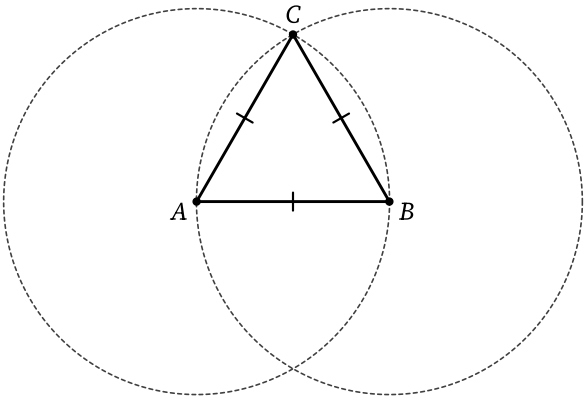
If two circles have radii r and s and their centers are d apart, and each of r, s and d is less than the sum of the other two, then the circles meet in exactly two points, one on each side of the line through the centers.



Euclid used this fact in I.1 without stating it; modern axiom systems such as Hilbert's include continuity axioms that guarantee it.

THEOREM **Equilateral Triangle Construction** Euclid I.1

If C is a point where the circle with center A through B meets the circle with center B through A, then $\triangle ABC$ is equilateral.



PARAGRAPH PROOF

Given: Circle with center A through B and circle with center B through A meet at C

Prove: $\triangle ABC$ is equilateral

The two circles meet by the Circle Intersection Postulate: both radii equal AB, and $AB < AB + AB$, so each of the three distances is less than the sum of the other two.

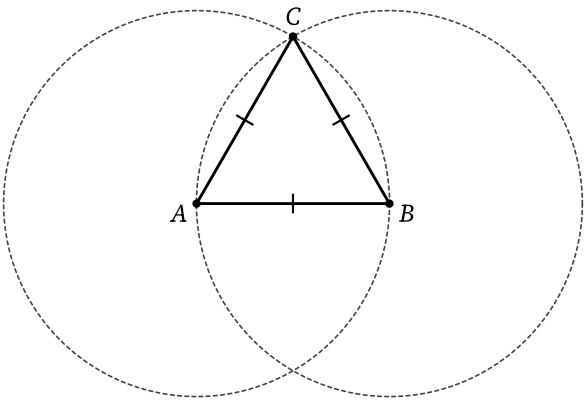
Since C lies on the circle with center A through B, $AC = AB$, because every point of a circle is the same distance from the center (definition of circle). Since C lies on the circle with center B through A, $BC = BA$ for the same reason.

Thus AC and BC are each equal to AB, so $AC = BC$ by the Transitive Property of Equality (Euclid's Common Notion 1: things equal to the same thing are equal to each other). All three sides of $\triangle ABC$ are equal, so $\triangle ABC$ is equilateral by definition.

∎

CONSTRUCTIONS

CONSTRUCTION **Construct an equilateral triangle on a segment** Euclid I.1

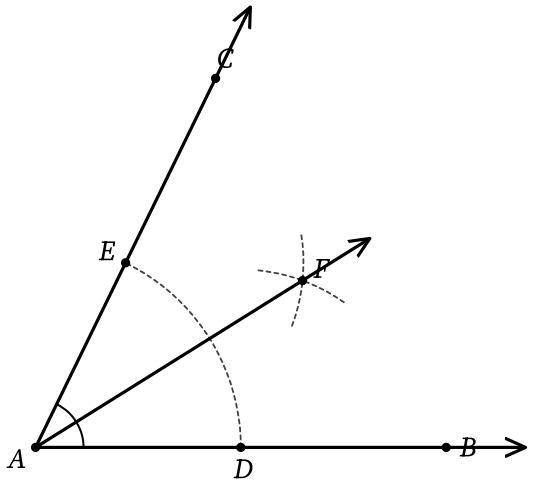


1. Given $\overline{AB}$. Place the compass point on A and open it to B. Draw the circle (or a large arc above the segment) with center A through B.
2. Without changing the width, place the compass point on B and draw the circle (or arc) with center B through A.
3. Label a point where the two circles cross C.
4. With the straightedge draw $\overline{AC}$ and $\overline{BC}$. Then $\triangle ABC$ is equilateral.

Why it works. $AC = AB$ because both are radii of the circle centered at A, and $BC = AB$ because both are radii of the circle centered at B; things equal to the same thing are equal to each other, so all three sides are equal (proved above).

CONSTRUCTION **Bisect an angle** Euclid I.9

1. Given $\angle BAC$. Place the compass point on the vertex A and draw an arc crossing both sides; label the crossings D (on $\overrightarrow{AB}$) and E (on $\overrightarrow{AC}$).
2. Place the compass point on D and draw an arc in the interior of the angle. Keeping the same width, place the compass point on E and draw an arc crossing the first one. Label the crossing F.
3. Draw $\overrightarrow{AF}$. It bisects $\angle BAC$.

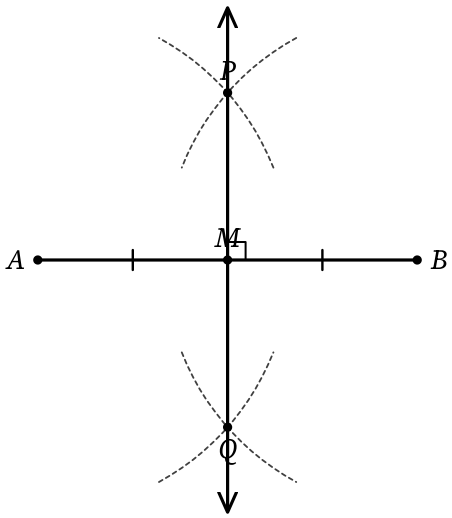


Why it works. $AD = AE$ (same arc from A) and $DF = EF$ (same compass width), and $\overline{AF}$ is shared, so triangles ADF and AEF have three pairs of equal sides. In Lesson 4.5 you will prove with SSS that the triangles are congruent, so $\angle DAF \cong \angle EAF$. Euclid's I.9

uses the same idea with an equilateral triangle built on $\overline{DE}$.

CONSTRUCTION **Bisect a segment: the perpendicular bisector** Euclid I.10

1. Given $\overline{AB}$. Open the compass to more than half of AB.
2. With the compass point on A, draw arcs above and below the segment. Keeping the same width, put the compass point on B and draw arcs crossing the first two. Label the crossings P (above) and Q (below).
3. Draw $\overleftrightarrow{PQ}$. It meets $\overline{AB}$ at its midpoint M, and $\overleftrightarrow{PQ} \perp \overline{AB}$.



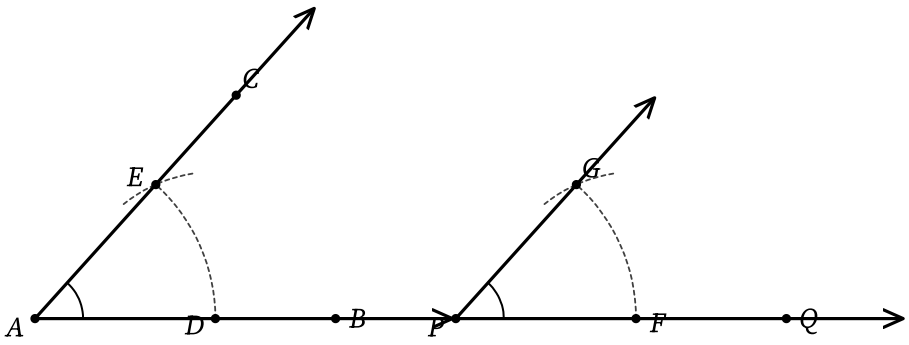
Why it works. By the construction $PA = PB$ and $QA = QB$ (equal compass widths), so P and Q are each equally far from A and B. In Lesson 4.5 you will prove with congruent triangles that $AM = MB$ and that the angles at M are right angles; in Lesson 5.1 you will see that the perpendicular bisector is exactly the set of points equally far from A and B. Euclid's I.10 bisects the segment, and I.11 gives the perpendicular.

From history. The master masons who designed medieval cathedrals had no calculators and often little arithmetic, but they knew their geometry. They laid out walls, arches and windows with compass, cord and straightedge, often drawing full-size patterns on a plaster "tracing floor." The pointed equilateral arch of many Gothic windows is Euclid's I.1: two arcs, each centered at one end of the span and passing through the other end, meet at the top. Rose windows were divided into equal parts by stepping a compass around a circle.

Euclid connection. Euclid's Book I mixes "problems," which construct something and end "which was to be done," with "theorems," which prove something and end "which was to be demonstrated." The constructions of this lesson are his problems I.1 (equilateral triangle), I.9 (bisect an angle), I.10 (bisect a segment), I.11 and I.12 (perpendiculars) and I.23 (copy an angle), all done with only the tools of Postulates 1 to 3.

Practice habit. Keep the compass width fixed when a step says "without changing the width," make arcs long enough to cross, and label every new point as soon as you make it. Recite Euclid's Postulates 1 to 3 and the definition of a circle, because every construction step is justified by one of them.

Example 1. Given $\angle BAC$ and a ray $\overrightarrow{PQ}$, construct an angle with vertex P and side $\overrightarrow{PQ}$ that is congruent to $\angle BAC$ (Euclid I.23).

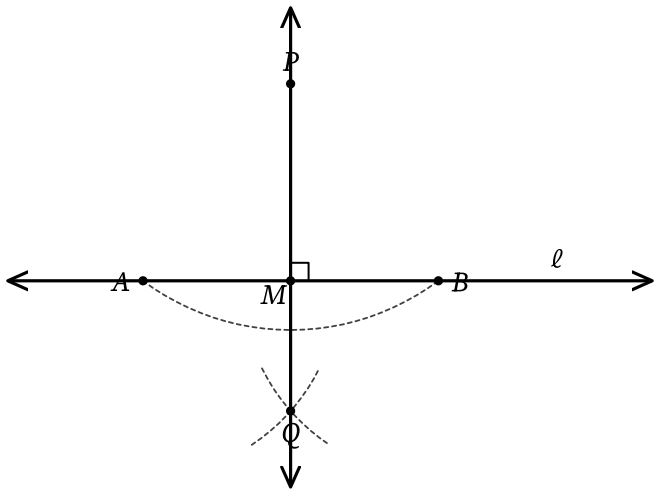


Show solution

1. With the compass point on A, draw an arc crossing both sides of $\angle BAC$; label the crossings D on $\overrightarrow{AB}$ and E on $\overrightarrow{AC}$.
2. Without changing the width, put the compass point on P and draw a long arc crossing $\overrightarrow{PQ}$; label the crossing F.
3. Set the compass to the distance DE (needle on D, pencil on E). With the needle on F, draw an arc crossing the long arc; label the crossing G.
4. Draw $\overrightarrow{PG}$. Then $\angle FPG \cong \angle BAC$. Why: $PF = AD$, $PG = AE$ and $FG = DE$, so triangles PFG and ADE have three pairs of equal sides; SSS (Lesson 4.3) will prove them congruent, so their angles at P and A are congruent. Euclid's I.23 builds the copy in just this way, using his I.22 to make a triangle from three given segments.

Answer: $\angle FPG \cong \angle BAC$, as constructed.

Example 2. Given line ℓ and a point P not on ℓ, construct the line through P perpendicular to ℓ (Euclid I.12).



Show solution

1. Put the compass point on P and open it wide enough to reach past ℓ. Draw an arc crossing ℓ at two points; label them A and B.
2. Open the compass to more than half of AB. With the needle on A and then on B, using the same width both times, draw two arcs on the side of ℓ opposite P that cross at a point Q.
3. Draw $\overleftrightarrow{PQ}$, and label the point where it meets ℓ as M. Then $\overleftrightarrow{PQ} \perp \ell$.
4. Why: $PA = PB$ (one arc from P) and $QA = QB$ (equal arcs from A and B), so both P and Q are equally far from A and B. Exactly as in the perpendicular bisector construction, $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$, which you will prove in Lesson 4.5. Drawing the arcs for Q on the far side of ℓ keeps Q from landing too close to P, which would make the line hard to draw accurately.

Answer: $\overleftrightarrow{PQ}$ is the perpendicular from P to ℓ, meeting ℓ at M.

PRACTICE

1. Easy Which of Euclid's postulates allows you to draw a circle with a given center through a given point?

   Show answer

   Postulate 3: to describe a circle with any center and distance.

2. Easy Draw a segment about 5 cm long and label it $\overline{AB}$. Construct an equilateral triangle on $\overline{AB}$, following Euclid I.1.

   Show answer

   Draw circles (or arcs) with center A through B and with center B through A; join one crossing point C to A and B. Check with a ruler that all three sides are about 5 cm.

3. Easy Draw a segment exactly 6 cm long and construct its perpendicular bisector. How long is each of the two halves?

   Show answer

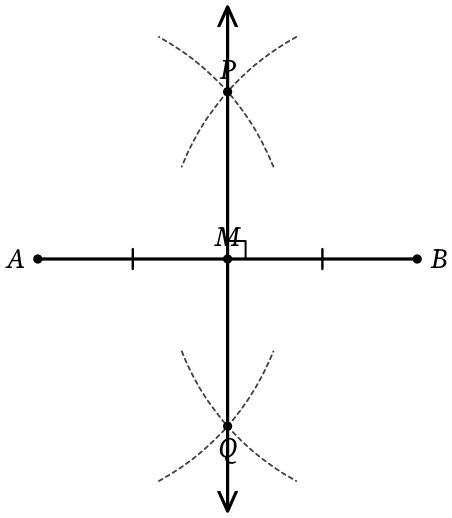
   The perpendicular bisector passes through the midpoint, so each half is 3 cm. (Check with your ruler.)

4. Easy Draw a 70° angle with a protractor and bisect it by construction. What should each of the two new angles measure?

   Show answer

   $70 \div 2 = 35^\circ$ each (check with your protractor).

5. **Medium** In the perpendicular bisector construction (see the figure), circles of the same radius r are drawn about A and B, and they cross at P and Q. Prove that $PA = PB$ and $QA = QB$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: The circle with center A and radius r and the circle with center B and radius r meet at P and Q

Prove: $PA = PB$ and $QA = QB$

P lies on the circle with center A and radius r, so $PA = r$ by the definition of a circle. P also lies on the circle with center B and radius r, so $PB = r$.

Since PA and PB both equal r, $PA = PB$ by the Transitive Property of Equality. The same reasoning for Q, which also lies on both circles, gives $QA = r = QB$.

∎

6. **Medium** Each angle of an equilateral triangle measures $60°$ (you will prove this in Unit 4). Use this fact to describe how to construct an angle of $30°$ with compass and straightedge.

Show answer

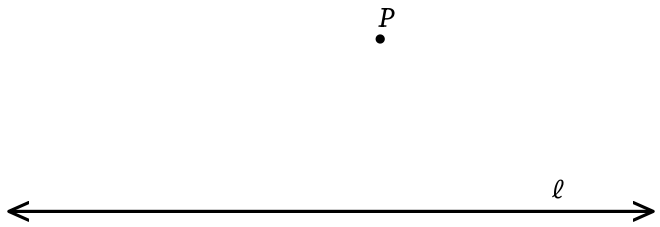
Construct an equilateral triangle ABC on a segment $\overline{AB}$ (I.1), so $m\angle BAC = 60°$. Then bisect $\angle BAC$ (I.9); each half measures $30°$.

7. **Medium** Draw an obtuse angle $\angle XYZ$ and a ray $\overrightarrow{RS}$. Construct an angle with vertex R and side $\overrightarrow{RS}$ congruent to $\angle XYZ$. Then check your copy with a protractor.

Show answer

Follow Worked Example 1: one arc from Y crossing both sides, the same arc from R, then the chord length between the first two crossings, marked from the crossing on $\overrightarrow{RS}$. The protractor readings of the two angles should agree.

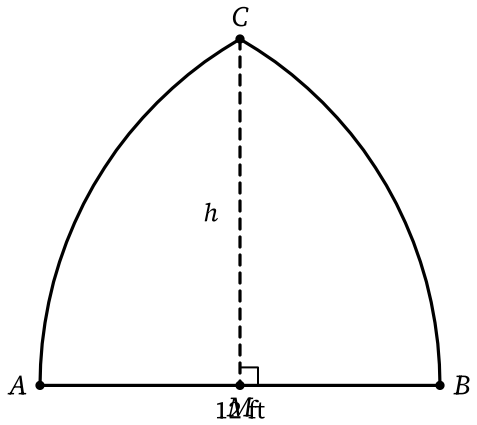
8. **Medium** In the figure, point P is not on line ℓ. Trace the figure and construct the perpendicular from P to ℓ. List your steps.



Show answer

Arc from P crossing ℓ at A and B; equal arcs from A and B crossing at Q on the other side of ℓ; draw $\overleftrightarrow{PQ}$, which is perpendicular to ℓ (Euclid I.12).

9. **Medium** A master mason draws an equilateral arch over a window opening $\overline{AB}$ that is 12 feet wide. He swings an arc centered at A through B and an arc centered at B through A; they meet at the peak C. The peak is directly above the midpoint M of the opening. How high is the peak above the opening? Give the exact answer and a decimal to the nearest tenth of a foot.



Show answer

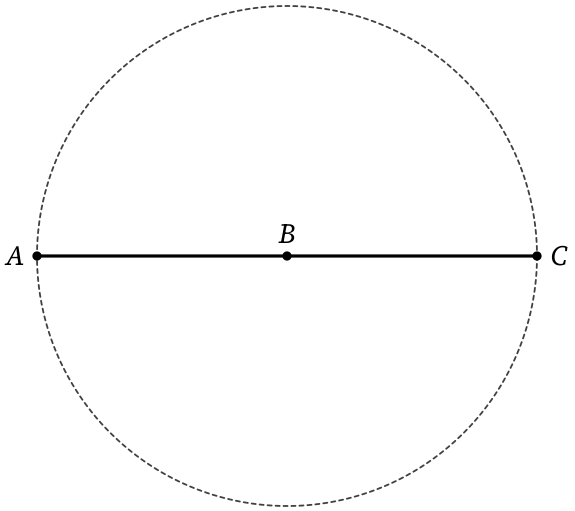
$AC = 12$ ft (a radius) and $AM = 6$ ft, and $\triangle AMC$ has a right angle at M. By the Pythagorean theorem, $CM = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3} \approx 10.4$ feet.

10. **Medium** Explain why the compass must be opened to more than half of AB when you construct the perpendicular bisector of $\overline{AB}$.

Show answer

The arcs from A and B must cross. If the radius r is less than half of AB, then $r + r < AB$, so the two circles are too far apart to meet (the Circle Intersection Postulate needs each distance less than the sum of the other two). If r is exactly half, the circles touch only at the midpoint, giving one point instead of two, so no line can be drawn.

11. Hard To double a segment $\overline{AB}$, Henry extends it past B with his straightedge (Euclid's Postulate 2) and draws the circle with center B through A. The circle crosses the extension at C, so B is between A and C. Prove that $AC = 2 \cdot AB$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: C lies on the circle with center B through A; B is between A and C

Prove: $AC = 2 \cdot AB$

Since A and C both lie on the circle with center B, $BC = BA$ by the definition of a circle.

Since B is between A and C, the Segment Addition Postulate gives $AC = AB + BC$.

Substituting AB for BC gives $AC = AB + AB = 2 \cdot AB$.

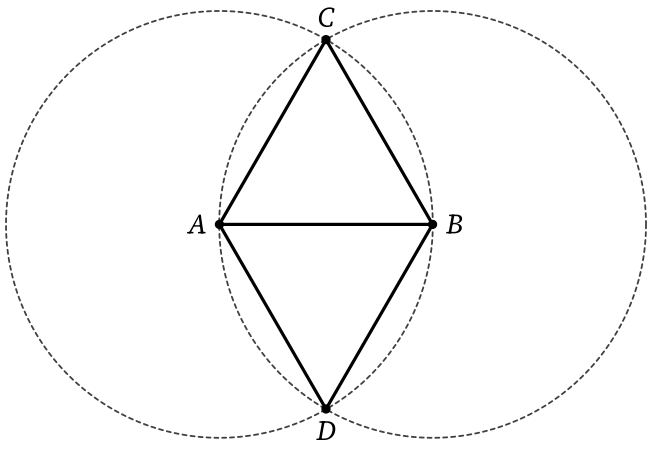
∎

12. Hard Samuel wants to mark the edge of a 10-inch board into 4 equal parts without measuring, using only a compass and straightedge. Describe how, and give the length of each part.

Show answer

Construct the perpendicular bisector of the edge to find its midpoint; then construct the perpendicular bisector of each half to find their midpoints. The three marks divide the edge into 4 equal parts, each $10 \div 4 = 2.5$ inches.

13. Hard In the figure, the circle with center A through B and the circle with center B through A meet at two points, C above $\overline{AB}$ and D below it. Prove that $AC = AD = BC = BD$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: The circle with center A through B and the circle with center B through A meet at C and D

Prove: $AC = AD = BC = BD$

C and D lie on the circle with center A through B, so $AC = AB$ and $AD = AB$ by the definition of a circle.

C and D also lie on the circle with center B through A, so $BC = BA$ and $BD = BA$, again by the definition of a circle.

All four lengths AC, AD, BC and BD equal AB, so they are equal to each other by the Transitive Property of Equality (Common Notion 1).

∎

14. **Challenge** Using only the constructions of this lesson and the fact that each angle of an equilateral triangle is $60°$, describe how to construct an angle of $75°$. Justify the measure.

Show answer

On a line through A with point B, construct equilateral triangle ABC, so $m\angle BAC = 60°$. Construct the perpendicular to $\overleftrightarrow{AB}$ at A (I.11) on the same side as C, giving ray $\overrightarrow{AD}$ with $m\angle BAD = 90°$. By angle addition $m\angle CAD = 90 - 60 = 30°$. Bisect $\angle CAD$ with $\overrightarrow{AE}$, so $m\angle CAE = 15°$. Then $m\angle BAE = 60 + 15 = 75°$ (Angle Addition Postulate).

COMMON MISTAKES

- Letting the compass slip or changing its width between two arcs that must be equal. Set the width once and do not touch the hinge until both arcs are drawn.
- Drawing arcs too short to cross. Make each arc long, and when bisecting a segment open the compass well past halfway.
- Measuring with a ruler or protractor and calling it a construction. Measuring is a good check, but a construction uses only the compass and an unmarked straightedge.
- In the perpendicular from a point (I.12), drawing the second pair of arcs on the same side of the line as P and too near P. Cross them on the far side so the line through P and Q can be drawn accurately.

> **Make it a game · String Compass Race.** Using a loop of string, a pencil and a book edge as straightedge, construct an equilateral triangle on a segment, as Euclid does in Proposition I.1. Time yourself. Then explain aloud why all three sides are equal, naming each radius. Go again and beat your time while keeping the construction accurate.

2.6 Proving Statements About Segments and Angles

OBJECTIVES

- Write the given, the prove statement and a marked diagram for a geometric statement.
- Plan a proof by working backward from the prove statement.
- Write two-column and paragraph proofs using the Segment and Angle Addition Postulates, the definitions of midpoint and bisector, and the angle-pair theorems.
- Prove that all right angles are congruent and that congruent angles forming a linear pair are right angles.

WATCH

Revolution Angles and Corresponding Angles (How to use mathematical reasoning to justify results)

Eddie Woo · 6:00

https://www.youtube.com/watch?v=4XdYb48bY0c

> **WHEN TO WATCH**
> Before Example 1: giving a reason for every statement about angles.

Note: Eddie Woo teaches in Australia: he writes statements and reasons line by line rather than in a two-column table.

BACKUP VIDEO

Using Quadrilateral Properties (1 of 2: Prologue about Proof)

Eddie Woo · 4:15

https://www.youtube.com/watch?v=lf5PGP2SvOk

Before the practice set: what a proof is and why properties are used as reasons.

KEY TERMS

given

The information you are told to accept at the start of a proof; usually the hypothesis of the conditional being proved.

prove statement

The conclusion the proof must reach; usually the conclusion of the conditional. It is the last statement of the proof.

two-column proof

A proof written as numbered statements in the left column with a reason for each in the right column.

paragraph proof

A proof written in sentences, in which each statement is still supported by a stated reason.

working backward

Planning a proof by asking what you would need to know to reach the prove statement, then what you would need to know to get that, until you reach the givens.

CORE IDEAS

A theorem is usually a conditional, "if (hypothesis), then (conclusion)." To prove it, first translate it: the hypothesis becomes the **Given**, the conclusion becomes the **Prove** statement, and you draw a diagram that shows exactly the given information and nothing more. Mark the givens on the diagram with tick marks, arcs and right-angle squares. For example, "If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$" becomes Given: M is the midpoint of $\overline{AB}$; Prove: $AM = \frac{1}{2}AB$.

Your tools for reasons are now a growing list: the properties of equality and congruence (Lesson 1.5), definitions (midpoint, angle bisector, congruent segments and angles, right angle, complementary, supplementary, linear pair, vertical angles), postulates (Segment Addition, Angle Addition, Protractor), and theorems already proved (Midpoint Theorem, Angle Bisector Halving Theorem, Linear Pair Theorem, Vertical Angles Theorem, Congruent Supplements and Complements). A reason must be one of these. "It looks that way" is never a reason: a figure may show which point is between which, but it cannot prove that two lengths or angles are equal.

How do you find a proof? **Work backward.** Ask: what is the last step? To prove $AC = BD$, I need to write AC and BD as sums (Segment Addition) and show the sums are equal. To prove two angles congruent, perhaps they are vertical angles, or supplements of congruent angles, or halves of congruent angles. Then work forward from the givens to meet your plan in the middle.

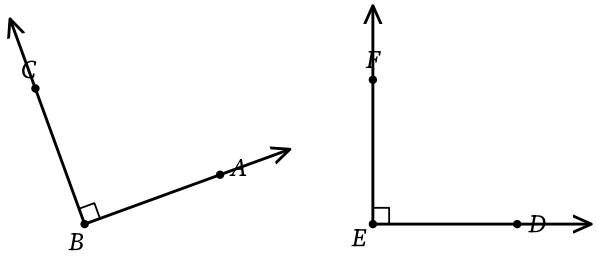
Euclid's Postulate 4 says "all right angles are equal to one another." Euclid had to assume it because he had no numbers for angles. We have the Protractor Postulate, which gives every right angle the measure $90°$, so for us it becomes a theorem with a two-line proof: the **Right Angle Congruence Theorem**. This is a good example of how a different set of starting assumptions can turn a postulate into a theorem.

Two-column and paragraph proofs contain the same logic. A two-column proof makes every reason easy to check; a paragraph proof reads more naturally and is how mathematicians usually write. In both, every statement needs a reason, and the steps must go in an order where each one follows from earlier ones. Practice both styles.

POSTULATES AND THEOREMS

THEOREM **Right Angle Congruence Theorem** Euclid Post. 4

All right angles are congruent.



TWO-COLUMN PROOF

Given: $\angle ABC$ and $\angle DEF$ are right angles

Prove: $\angle ABC \cong \angle DEF$

	STATEMENTS	REASONS
1	$\angle ABC$ and $\angle DEF$ are right angles	Given
2	$m\angle ABC = 90°$	Definition of right angle
3	$m\angle DEF = 90°$	Definition of right angle
4	$m\angle ABC = m\angle DEF$	Transitive Property of Equality

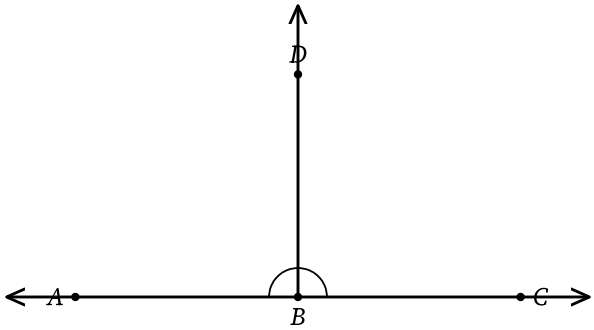
	STATEMENTS	REASONS
5	$\angle ABC \cong \angle DEF$	Definition of congruent angles

∎

Euclid took this as his Postulate 4; with the Protractor Postulate we can prove it.

THEOREM **Congruent Linear Pair Theorem**

If two angles are congruent and form a linear pair, then each is a right angle.



PARAGRAPH PROOF

Given: $\angle ABD$ and $\angle DBC$ form a linear pair; $\angle ABD \cong \angle DBC$

Prove: $\angle ABD$ and $\angle DBC$ are right angles

Because $\angle ABD$ and $\angle DBC$ form a linear pair, they are supplementary by the Linear Pair Theorem, so $m\angle ABD + m\angle DBC = 180°$.

Because the angles are congruent, $m\angle ABD = m\angle DBC$ by the definition of congruent angles. Substituting gives $m\angle ABD + m\angle ABD = 180°$, so $2 \cdot m\angle ABD = 180°$ and $m\angle ABD = 90°$ by the Division Property of Equality.

Then $m\angle DBC = m\angle ABD = 90°$ as well, so both angles are right angles by the definition of right angle.

∎

This is exactly Euclid's Definition 10: when a line standing on a line makes the adjacent angles equal, each is right. In Lesson 3.4 it becomes a test for perpendicular lines.

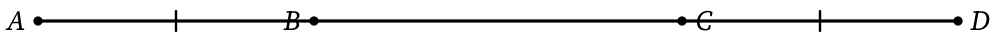
From history. Euclid's Postulate 4, "all right angles are equal to one another," puzzled readers for centuries, because it seems like something that ought to be proved. Ancient commentators, Proclus among them, discussed whether it should be proved rather than assumed. In modern axiom systems such as Hilbert's (1899), it is indeed a theorem, proved from other axioms, and in our course it follows at once from the Protractor Postulate.

Euclid connection. Euclid's Postulate 4 states "all right angles are equal to one another"; in this course it is the Right Angle Congruence Theorem. His proofs in Book I are written as paragraphs, each step justified by a definition, postulate, common notion or earlier proposition, which is the model for our paragraph proofs.

Practice habit. Before you write a single step, copy the Given and Prove, draw a large figure, and mark every given on it. Then plan by working backward from the Prove statement, and give a reason for every step; recite the theorems of Lessons 2.2 to 2.6 until you can name the right one quickly.

WORKED EXAMPLES

Example 1. In the figure, points A, B, C and D are collinear in that order. Given: $\overline{AB} \cong \overline{CD}$. Prove: $\overline{AC} \cong \overline{BD}$.



Show solution

1. Work backward: to show $\overline{AC} \cong \overline{BD}$, show $AC = BD$.
2. By Segment Addition, $AC = AB + BC$ and $BD = BC + CD$. Both sums contain BC.
3. The given says $AB = CD$; adding BC to both sides makes the two sums equal.

TWO-COLUMN PROOF

Given: A, B, C, D are collinear in that order; $\overline{AB} \cong \overline{CD}$

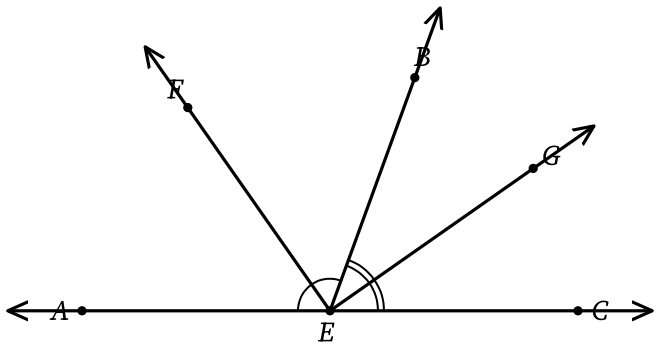
Prove: $\overline{AC} \cong \overline{BD}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$	Given
2	$AB = CD$	Definition of congruent segments
3	$AB + BC = CD + BC$	Addition Property of Equality
4	$AB + BC = AC$	Segment Addition Postulate
5	$BC + CD = BD$	Segment Addition Postulate
6	$AC = BD$	Substitution Property of Equality (rows 4 and 5 into row 3, using $CD + BC = BC + CD$)
7	$\overline{AC} \cong \overline{BD}$	Definition of congruent segments

∎

Answer: See the proof.

Example 2. In the figure, E is between A and C, and B is not on $\overleftrightarrow{AC}$. Ray $\overrightarrow{EF}$ bisects $\angle AEB$ and ray $\overrightarrow{EG}$ bisects $\angle BEC$. Prove that $\angle FEG$ is a right angle. (The bisectors of a linear pair meet at a right angle.)



Show solution

1. Work backward: to show $\angle FEG$ is right, show $m\angle FEG = 90°$.
2. $\angle FEG$ is split by $\overrightarrow{EB}$ into $\angle FEB$ and $\angle BEG$, which are halves of $\angle AEB$ and $\angle BEC$.
3. The whole angles $\angle AEB$ and $\angle BEC$ form a linear pair, so they add to $180°$; their halves add to $90°$.

PARAGRAPH PROOF

Given: E is between A and C; B is not on $\overleftrightarrow{AC}$; $\overrightarrow{EF}$ bisects $\angle AEB$; $\overrightarrow{EG}$ bisects $\angle BEC$
Prove: $\angle FEG$ is a right angle

Since E is between A and C, $\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays, so $\angle AEB$ and $\angle BEC$ form a linear pair. By the Linear Pair Theorem, $m\angle AEB + m\angle BEC = 180°$.

Since $\overrightarrow{EF}$ bisects $\angle AEB$, the Angle Bisector Halving Theorem gives $m\angle FEB = \frac{1}{2}m\angle AEB$. Likewise, since $\overrightarrow{EG}$ bisects $\angle BEC$, $m\angle BEG = \frac{1}{2}m\angle BEC$.

As the figure shows, B is in the interior of $\angle FEG$, so by the Angle Addition Postulate $m\angle FEG = m\angle FEB + m\angle BEG$. Substituting, $m\angle FEG = \frac{1}{2}m\angle AEB + \frac{1}{2}m\angle BEC = \frac{1}{2}(m\angle AEB + m\angle BEC)$ by the Distributive Property, and this equals $\frac{1}{2}(180°) = 90°$ by substitution.

Therefore $\angle FEG$ is a right angle by the definition of right angle.

∎

Answer: See the proof.

PRACTICE

1. Easy Give the reason: If M is the midpoint of $\overline{AB}$, then $AM = MB$.

   Show answer

   Definition of midpoint

2. Easy Give the reason: If $\angle P$ and $\angle Q$ are right angles, then $\angle P \cong \angle Q$.

   Show answer

   Right Angle Congruence Theorem

3. Easy Give the reason: If D is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.

   Show answer

   Angle Addition Postulate

4. Easy Give the reason: If $AB = CD$ and $CD = EF$, then $AB = EF$.

   Show answer

   Transitive Property of Equality

5. Medium In a proof, Grace wrote the statement "$AB = BC$" with the reason "they look the same length in the diagram." Explain why this reason is not acceptable and what kind of reason she needs.

   Show answer

   A figure can suggest but never prove: two segments can look equal and differ slightly, and a figure might not be drawn to scale. She needs a given, a definition (such as midpoint), a postulate or a theorem already proved that guarantees $AB = BC$.

6. Medium $\angle ABD$ and $\angle DBC$ form a linear pair and are congruent, and $m\angle ABD = (5x + 15)^\circ$. Find x.

   Show answer

   By the Congruent Linear Pair Theorem each angle is 90°, so $5x + 15 = 90$ and $x = 15$.

7. **Medium** Henry nails a board across a straight fence rail. The board makes two angles with the rail that form a linear pair, and he measures them and finds that they are equal. What is the measure of each angle, and which theorem tells you so?

Show answer

Each is $90°$, by the Congruent Linear Pair Theorem.

8. **Medium** Points A, B, C and D are collinear in that order. Given: B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$. Prove: $\overline{AB} \cong \overline{CD}$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: B is the midpoint of $\overline{AC}$; C is the midpoint of $\overline{BD}$

Prove: $\overline{AB} \cong \overline{CD}$

	STATEMENTS	REASONS
1	B is the midpoint of $\overline{AC}$	Given
2	$AB = BC$	Definition of midpoint
3	C is the midpoint of $\overline{BD}$	Given
4	$BC = CD$	Definition of midpoint
5	$AB = CD$	Transitive Property of Equality
6	$\overline{AB} \cong \overline{CD}$	Definition of congruent segments

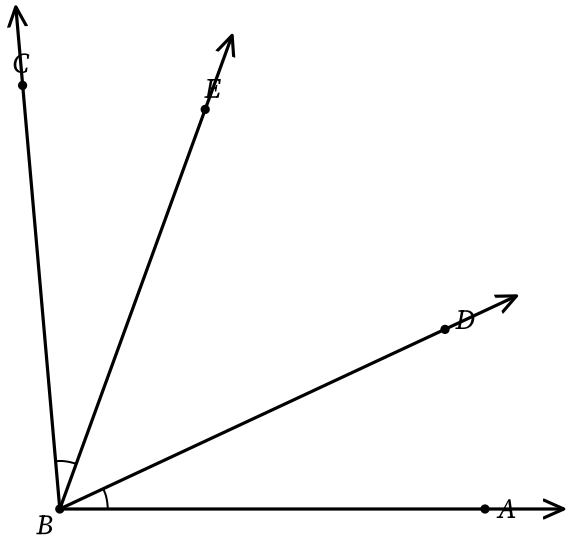
∎

9. **Medium** In a rose window, two straight lead lines cross at the center. One of the four angles they form is three times as large as an angle next to it. Find the measures of these two angles.

Show answer

Adjacent angles at the crossing form a linear pair, so $x + 3x = 180$, $x = 45$. The angles are $45°$ and $135°$.

10. Hard In the figure, rays $\overrightarrow{BA}$, $\overrightarrow{BD}$, $\overrightarrow{BE}$ and $\overrightarrow{BC}$ are in that order. Given: $\angle ABD \cong \angle EBC$. Prove: $\angle ABE \cong \angle DBC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

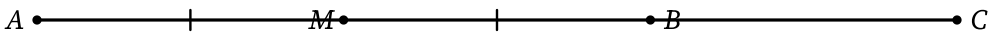
Given: D is in the interior of $\angle ABE$; E is in the interior of $\angle DBC$; $\angle ABD \cong \angle EBC$

Prove: $\angle ABE \cong \angle DBC$

	STATEMENTS	REASONS
1	$\angle ABD \cong \angle EBC$	Given
2	$m\angle ABD = m\angle EBC$	Definition of congruent angles
3	$m\angle ABD + m\angle DBE = m\angle EBC + m\angle DBE$	Addition Property of Equality
4	$m\angle ABD + m\angle DBE = m\angle ABE$	Angle Addition Postulate
5	$m\angle DBE + m\angle EBC = m\angle DBC$	Angle Addition Postulate
6	$m\angle ABE = m\angle DBC$	Substitution Property of Equality (rows 4 and 5 into row 3)
7	$\angle ABE \cong \angle DBC$	Definition of congruent angles

∎

11. Hard In the figure, points A, M, B and C are collinear in that order. Given: M is the midpoint of $\overline{AB}$ and $AB = 2 \cdot BC$. Prove: $\overline{MB} \cong \overline{BC}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

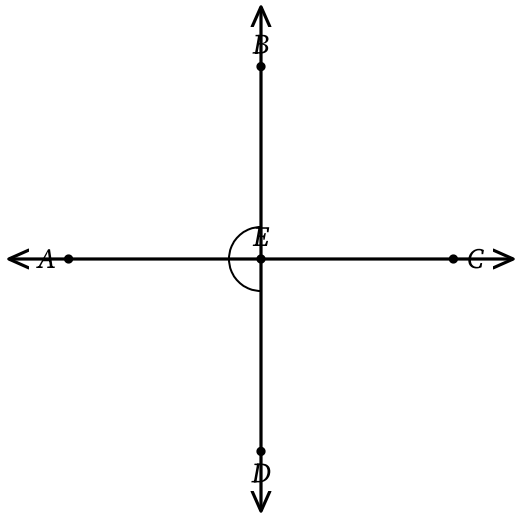
Given: M is the midpoint of $\overline{AB}$; $AB = 2 \cdot BC$

Prove: $\overline{MB} \cong \overline{BC}$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$	Given
2	$AM = MB$	Definition of midpoint
3	$AM + MB = AB$	Segment Addition Postulate
4	$MB + MB = AB$	Substitution Property of Equality
5	$2 \cdot MB = AB$	Distributive Property (combining like terms)
6	$AB = 2 \cdot BC$	Given
7	$2 \cdot MB = 2 \cdot BC$	Transitive Property of Equality
8	$MB = BC$	Division Property of Equality
9	$\overline{MB} \cong \overline{BC}$	Definition of congruent segments

∎

12. Hard In the figure, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E. Given: $\angle AEB \cong \angle AED$. Prove: $\angle BEC$ is a right angle.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E; $\angle AEB \cong \angle AED$

Prove: $\angle BEC$ is a right angle

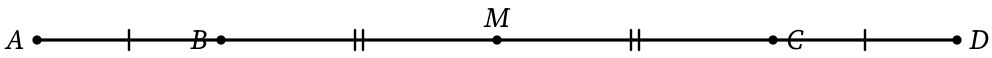
Since the lines intersect at E, $\overrightarrow{EB}$ and $\overrightarrow{ED}$ are opposite rays, so $\angle AEB$ and $\angle AED$ form a linear pair. They are also congruent (given), so by the Congruent Linear Pair Theorem each is a right angle; in particular $m\angle AEB = 90°$.

Also $\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays, so $\angle AEB$ and $\angle BEC$ form a linear pair, and by the Linear Pair Theorem $m\angle AEB + m\angle BEC = 180°$.

Substituting, $90° + m\angle BEC = 180°$, so $m\angle BEC = 90°$ by the Subtraction Property of Equality. Therefore $\angle BEC$ is a right angle by the definition of right angle.

∎

13. **Challenge** Henry is centering a window in a wall. Along the straight bottom edge, the wall runs from A to D and the window from B to C, with A, B, C, D in that order. He makes $AB = CD$ and marks M, the midpoint of $\overline{BC}$. In the figure, M is between B and C. Prove that M is also the midpoint of $\overline{AD}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: A, B, M, C, D are collinear in that order; $AB = CD$; M is the midpoint of $\overline{BC}$

Prove: M is the midpoint of $\overline{AD}$

	STATEMENTS	REASONS
1	$AB = CD$	Given
2	M is the midpoint of $\overline{BC}$	Given
3	$BM = MC$	Definition of midpoint
4	$AB + BM = CD + MC$	Addition Property of Equality (adding the equal quantities of rows 1 and 3)
5	$AB + BM = AM$	Segment Addition Postulate (B is between A and M)
6	$MC + CD = MD$	Segment Addition Postulate (C is between M and D)
7	$AM = MD$	Substitution Property of Equality (rows 5 and 6 into row 4, using $CD + MC = MC + CD$)
8	M is between A and D	Given (order of the points)
9	M is the midpoint of $\overline{AD}$	Definition of midpoint

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COMMON MISTAKES

- Using the prove statement (or something that depends on it) as a reason. That is circular; the conclusion may appear only as the last line.
- Writing a reason that is true but does not justify the step, such as "Segment Addition Postulate" for $AB = CD$. Ask: does this reason produce exactly this statement from earlier lines?
- Mixing up congruence and equality: write $\overline{AB} \cong \overline{CD}$ for segments and $AB = CD$ for their lengths, and switch between them with the definition of congruent segments (or angles).
- Reading equal lengths or angles from the diagram. Only betweenness and which side of a line a point lies on may be taken from a figure; equalities must come from givens, definitions, postulates or theorems.

> **Make it a game · Prove It Backward.** Parent gives a statement: 'M is the midpoint of AB and AM = 5. Prove AB = 10.' In 60 seconds write the given, the prove statement and a marked diagram. Then work backward on sticky notes: 'To show AB = 10, I need AM + MB = 10.' Parent plays the annoying judge who keeps asking 'Why?' until each link has a reason.

UNIT 2

Points, Lines, Planes and Angles

In Unit 1 you learned what a proof is. In this unit you start building geometry itself, the way Euclid did: from undefined terms (point, line, plane), a few postulates, and careful definitions. You will measure segments and angles, learn the Segment and Angle Addition Postulates, and meet the angle pairs (complementary, supplementary, linear pairs, vertical angles) that appear in nearly every later proof.

You will also pick up a compass and straightedge. With Euclid's first propositions you will construct an equilateral triangle (I.1), copy segments and angles, bisect segments and angles, and draw perpendiculars, the same tools cathedral master masons used to lay out arches and windows. Finally you will write your own two-column and paragraph proofs about segments and angles, including Euclid's I.15, the Vertical Angles Theorem.

Parents: this unit is where proofs start to feel like real geometry. Encourage your student to say every definition and postulate aloud until it is automatic, to draw large marked figures, and to give a reason for every step. A compass, an unmarked straightedge (a ruler used without its marks is fine), a protractor and a sharp pencil are needed.

PREREQUISITES

- Solve linear equations, including with variables on both sides (Algebra I).
- Use the distance and midpoint formulas in the coordinate plane (Algebra I).
- Use the Pythagorean theorem to find a side of a right triangle (Algebra I).
- Write and analyze conditional statements (Lesson 1.3).
- Justify steps with the properties of equality and congruence in a two-column proof (Lesson 1.5).

MASTERY CHECKLIST

- ☐ I can name points, lines, planes, segments and rays and tell whether points are collinear or coplanar.
- ☐ I can state the postulates about points, lines and planes and use them to explain simple facts.
- ☐ I can use the Ruler Postulate, the Segment Addition Postulate and the definition of midpoint to find lengths, including with algebra and coordinates.
- ☐ I can use the Protractor Postulate, the Angle Addition Postulate and the definition of angle bisector to find angle measures, and work with degrees and minutes.
- ☐ I can identify complementary and supplementary angles, linear pairs and vertical angles and use them to write and solve equations.
- ☐ I can prove the Linear Pair Theorem, the Vertical Angles Theorem (Euclid I.15) and the Congruent Supplements and Complements Theorems.
- ☐ I can construct an equilateral triangle (I.1) and prove it is equilateral, and copy a segment and an angle.
- ☐ I can bisect a segment and an angle and construct perpendiculars through a point on a line and from a point not on a line, and explain why each construction works.

- ☐ I can write the Given and Prove for a statement, draw and mark its diagram, and plan a proof by working backward.
- ☐ I can write two-column and paragraph proofs about segments and angles with a correct reason for every step.

LESSONS IN THIS UNIT

2.1	**Points, Lines and Planes**	45 minutes
2.2	**Measuring and Constructing Segments**	2 sessions of 45 minutes
2.3	**Measuring Angles**	45 minutes
2.4	**Angle Pairs and the Vertical Angles Theorem**	2 sessions of 45 minutes
2.5	**Basic Constructions**	2 sessions of 45 minutes
2.6	**Proving Statements About Segments and Angles**	2 sessions of 45 minutes

Unit 2 test

You have 50 minutes. No calculator is allowed; all arithmetic can be done by hand. You will need a compass, a straightedge and a pencil for Problem 11. Show your work for every problem. In every proof, give a reason for each step: a given, a definition, a postulate, a property or a theorem we have proved; nothing may be assumed from the look of a figure except the order of points and which side of a line a point is on. Give exact answers unless a problem asks you to round, and include units.

1. **5 pts**

   True or false: three collinear points determine exactly one plane. Explain.

2. **5 pts**

   A quilter marks 6 points around the edge of a round quilt block, no three in a line, and sews a straight seam between every pair of points. How many seams does she sew?

3. **6 pts**

   Find the distance between $J(-3, -2)$ and $K(9, 3)$, and the coordinates of the midpoint of $\overline{JK}$.

4. **6 pts**

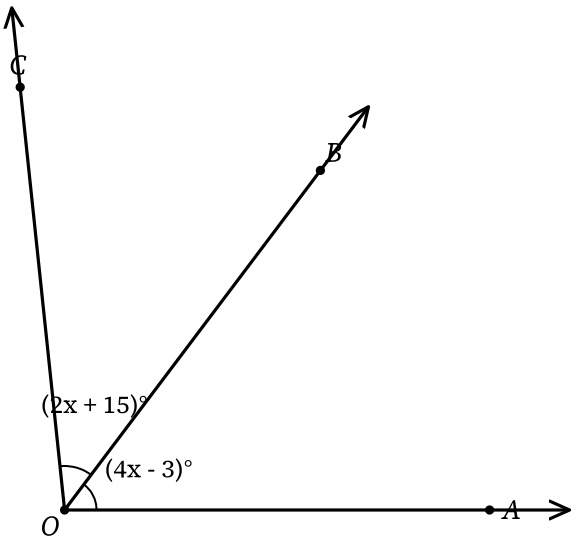
   Point R is between Q and S. $QR = 3x - 4$, $RS = 2x + 7$ and $QS = 43$. Find x and RS.

5. **6 pts**

   A straight highway has mile markers. One town is at mile 37.6 and the next town is at mile 92.4. The state builds a rest stop exactly halfway between them. At what mile marker is the rest stop, and how far is it from each town?

6. **6 pts**

   In the figure, B is in the interior of $\angle AOC$, $m\angle AOB = (4x - 3)^\circ$, $m\angle BOC = (2x + 15)^\circ$ and $m\angle AOC = 96^\circ$. Find x and $m\angle AOB$.



7. **6 pts**

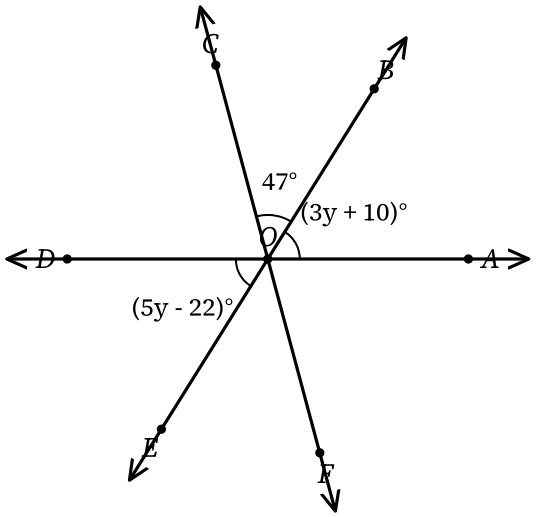
   A surveyor measures the angle between two property lines as $103°24'$ and wants to bisect it to set a drainage line. What angle should the drainage line make with each property line? Give your answer in degrees and minutes.

8. **6 pts**

   The supplement of an angle is $30°$ more than twice its complement. Find the angle.

9. **7 pts**

   In the figure, lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ meet at O. $m\angle AOB = (3y + 10)°$, $m\angle DOE = (5y - 22)°$ and $m\angle BOC = 47°$. Find y and $m\angle COD$.



10. **6 pts**

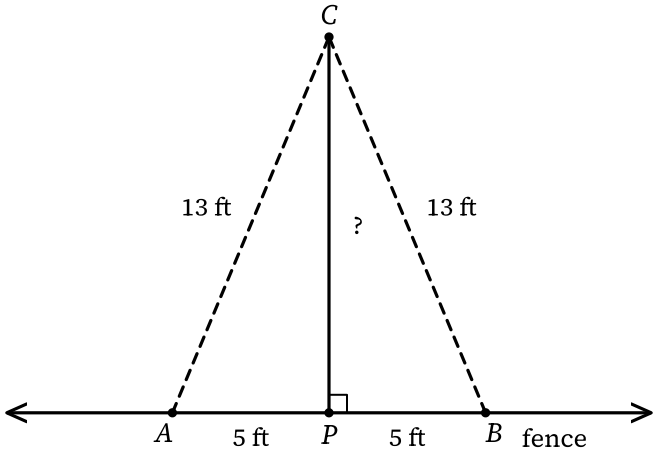
    Explain in two or three sentences why vertical angles must be congruent. Do not just name the theorem; give the reason behind it.

11. **10 pts**

    Draw a segment $\overline{AB}$ about 8 cm long. Using only a compass and straightedge, construct the perpendicular bisector of $\overline{AB}$, and then use it to construct an angle of $45°$. Leave all arcs showing and list your steps.

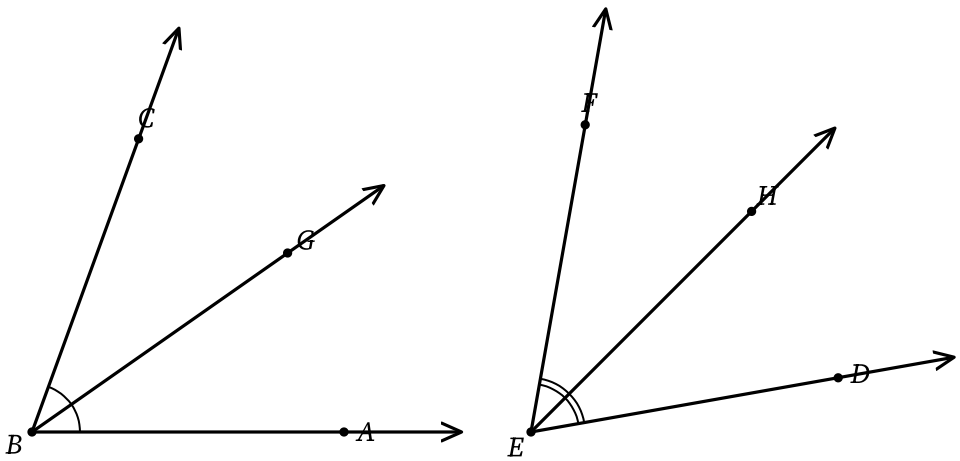
12. **8 pts**

Samuel is laying out a garden path perpendicular to a straight fence at gate post P. He marks points A and B on the fence, each 5 feet from P on opposite sides. Then, holding a 13-foot cord first at A and then at B, he finds the point C that is 13 feet from both, as in the figure. By the construction, $\overleftrightarrow{PC}$ is perpendicular to the fence. How far is C from the gate post P?



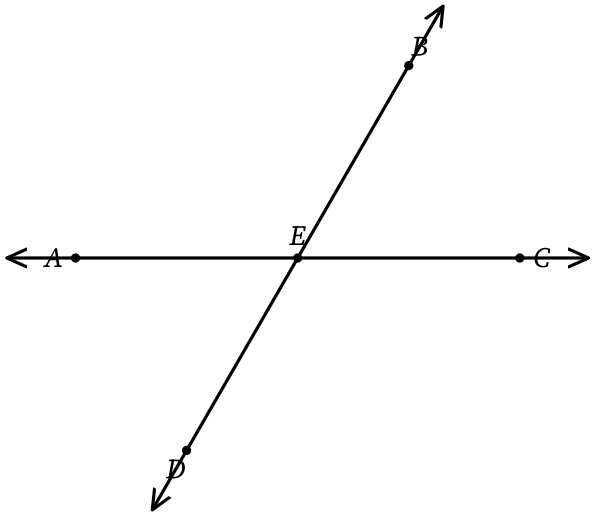
13. **12 pts**

In the figure, $\angle ABC \cong \angle DEF$, $\overrightarrow{BG}$ bisects $\angle ABC$ and $\overrightarrow{EH}$ bisects $\angle DEF$. Write a two-column proof that $\angle ABG \cong \angle DEH$.



14. **11 pts**

In the figure, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E, and $m\angle AEB = 2 \cdot m\angle BEC$. Write a paragraph proof that $m\angle AED = 60°$.



ANSWER KEY

1. False. Three collinear points lie on one line, and infinitely many planes contain that line (one through each point not on it). The Plane Postulate gives exactly one plane only for noncollinear points.
2. Each of the 6 points pairs with 5 others: $6 \times 5 = 30$, and each seam is counted twice, so $30 \div 2 = 15$ seams.
3. $JK = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; midpoint $\left(\dfrac{-3+9}{2}, \dfrac{-2+3}{2}\right) = (3, 0.5)$
4. By the Segment Addition Postulate $(3x - 4) + (2x + 7) = 43$, so $5x + 3 = 43$ and $x = 8$. Then $RS = 2(8) + 7 = 23$.
5. Mile $\dfrac{37.6 + 92.4}{2} = 65$; it is $65 - 37.6 = 27.4$ miles from each town.
6. By the Angle Addition Postulate $(4x - 3) + (2x + 15) = 96$, so $6x + 12 = 96$ and $x = 14$. Then $m\angle AOB = 4(14) - 3 = 53°$.
7. $103°24' \div 2$: $103° = 102°60'$, so $103°24' = 102°84'$, and half is $51°42'$ (which is $51.7°$).
8. $180 - x = 2(90 - x) + 30$, so $180 - x = 210 - 2x$ and $x = 30°$. Check: supplement $150°$, complement $60°$, and $2(60) + 30 = 150$. ✓
9. $\angle AOB$ and $\angle DOE$ are vertical angles, so $3y + 10 = 5y - 22$ and $y = 16$; then $m\angle AOB = 58°$. Since $\overrightarrow{OA}$ and $\overrightarrow{OD}$ are opposite rays, $m\angle COD = 180 - 58 - 47 = 75°$.
10. Each of two vertical angles forms a linear pair with the same angle between them, so each one equals $180°$ minus the measure of that same angle. Equals subtracted from equals leave equals (Common Notion 3), so the vertical angles have equal measures and are congruent.
11. With the compass opened to more than half of AB, draw arcs from A and from B above and below the segment, crossing at P and Q. Draw $\overleftrightarrow{PQ}$; it meets $\overline{AB}$ at the midpoint M and $m\angle PMB = 90°$. Bisect $\angle PMB$: an arc from M crosses $\overrightarrow{MP}$ at X and $\overrightarrow{MB}$ at Y; equal arcs from X and Y cross at Z. Draw $\overrightarrow{MZ}$. Then $m\angle ZMB = \dfrac{1}{2}(90°) = 45°$.
12. $\triangle APC$ has a right angle at P, with $AP = 5$ ft and $AC = 13$ ft. By the Pythagorean theorem $PC = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$ feet.
13. See the model proof.

TWO-COLUMN PROOF

Given: $\angle ABC \cong \angle DEF$; $\overrightarrow{BG}$ bisects $\angle ABC$; $\overrightarrow{EH}$ bisects $\angle DEF$

Prove: $\angle ABG \cong \angle DEH$

	STATEMENTS	REASONS
1	$\angle ABC \cong \angle DEF$	Given
2	$m\angle ABC = m\angle DEF$	Definition of congruent angles
3	$\overrightarrow{BG}$ bisects $\angle ABC$; $\overrightarrow{EH}$ bisects $\angle DEF$	Given
4	$m\angle ABG = \frac{1}{2}m\angle ABC$	Angle Bisector Halving Theorem
5	$m\angle DEH = \frac{1}{2}m\angle DEF$	Angle Bisector Halving Theorem
6	$\frac{1}{2}m\angle ABC = \frac{1}{2}m\angle DEF$	Multiplication Property of Equality
7	$m\angle ABG = m\angle DEH$	Substitution Property of Equality (rows 4 and 5 into row 6)
8	$\angle ABG \cong \angle DEH$	Definition of congruent angles

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14. See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E; $m\angle AEB = 2 \cdot m\angle BEC$

Prove: $m\angle AED = 60°$

Since the lines intersect at E, $\overrightarrow{EA}$ and $\overrightarrow{EC}$ are opposite rays, so $\angle AEB$ and $\angle BEC$ form a linear pair. By the Linear Pair Theorem, $m\angle AEB + m\angle BEC = 180°$.

Substituting $2 \cdot m\angle BEC$ for $m\angle AEB$ gives $3 \cdot m\angle BEC = 180°$, so $m\angle BEC = 60°$ by the Division Property of Equality.

Also $\overrightarrow{EB}$ and $\overrightarrow{ED}$ are opposite rays, so $\angle AED$ and $\angle CEB$ are vertical angles. By the Vertical Angles Theorem they are congruent, so $m\angle AED = m\angle BEC = 60°$.

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UNIT 3

Parallel and Perpendicular Lines

This unit is about the most famous assumption in mathematics: Euclid's fifth postulate, the parallel postulate. You will learn the names of the angle pairs a transversal makes with two lines, take the parallel postulate in a convenient modern form (the Corresponding Angles Postulate), and prove from it the angle theorems of Euclid I.29. Then you will turn them around and prove lines parallel (I.27, I.28, I.30).

Perpendicular lines come next: why one right angle gives four, why the perpendicular is the shortest path from a point to a line, and how to construct a parallel through a point with compass and straightedge (I.31). The unit ends in the coordinate plane, where you prove the Algebra I slope rules for parallel and perpendicular lines and use them to find distances.

For parents: the unit mixes two-column and paragraph proofs with practical problems from carpentry, surveying and farm life. The history of the parallel postulate (Proclus, Saccheri, Gauss, Bolyai and Lobachevsky) shows students why a careful foundation matters. No calculator is needed.

PREREQUISITES

- Name and use vertical angles, linear pairs, and the Congruent Supplements and Congruent Complements Theorems (Lesson 2.4).
- Write two-column and paragraph proofs with the Angle Addition Postulate and the definition of angle bisector (Lessons 1.5 and 2.6).
- Form the converse and contrapositive of a conditional, and know that a converse needs its own proof (Lesson 1.3).
- Copy an angle and construct perpendiculars with compass and straightedge (Lesson 2.5).
- Find slopes, write equations of lines, and use the distance formula and the Pythagorean Theorem numerically (Algebra I).

MASTERY CHECKLIST

- ☐ I can tell parallel, intersecting and skew lines apart and recognize parallel planes.
- ☐ I can name corresponding, alternate interior, alternate exterior and same-side interior angles in a figure.
- ☐ I can state Euclid's fifth postulate, Playfair's Axiom and the Corresponding Angles Postulate, and explain why the parallel postulate is assumed rather than proved.
- ☐ I can prove the Alternate Interior, Alternate Exterior and Same-Side Interior Angles Theorems and use them to find angle measures.
- ☐ I can use the converses to prove lines parallel, including Euclid I.30 and two lines perpendicular to the same line.
- ☐ I can prove that perpendicular lines form four right angles and use the Perpendicular Transversal Theorem.
- ☐ I can explain why the perpendicular is the shortest segment from a point to a line and find distances to lines and between parallel lines.
- ☐ I can construct the parallel to a line through a given point (Euclid I.31) and the perpendicular from a point to a line (I.12), and explain why each works.

- ☐ I can explain why parallel lines have equal slopes and perpendicular slopes multiply to -1, and write equations of parallel and perpendicular lines.
- ☐ I can find the distance from a point to a line in the coordinate plane by using a perpendicular.

LESSONS IN THIS UNIT

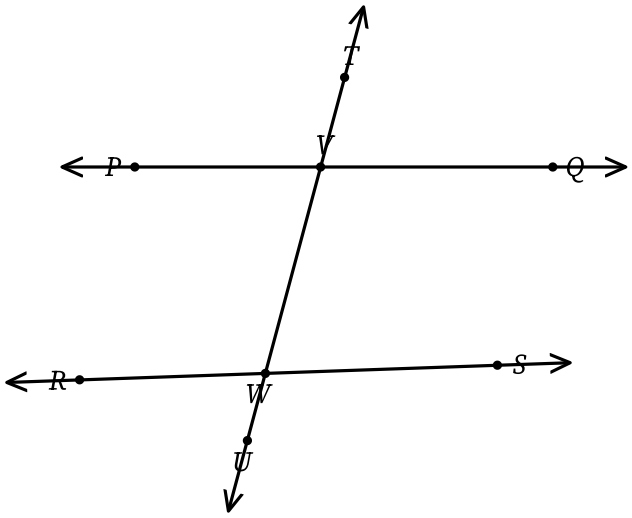
3.1	**Lines, Transversals and Angle Pairs**	45 minutes
3.2	**The Parallel Postulate and Parallel Lines**	2 sessions of 45 minutes
3.3	**Proving Lines Parallel**	2 sessions of 45 minutes
3.4	**Perpendicular Lines and Distance**	2 sessions of 45 minutes
3.5	**Slopes of Parallel and Perpendicular Lines**	45 minutes

Unit 3 test

You have 50 minutes. No calculator is needed or allowed. Show your work for every computation. In every proof, give a reason for each step; do not assume anything from the look of a figure except the order of points and which side of a line a point is on. Give exact answers (simplest radical form where needed). You will need a compass and straightedge for Problem 15.

1. **5 pts**

   In the figure, $\overleftrightarrow{UT}$ is a transversal of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$, crossing them at V and W. Classify the pair $\angle TVP$ and $\angle SWU$ as corresponding, alternate interior, alternate exterior or same-side interior angles.



2. **4 pts**

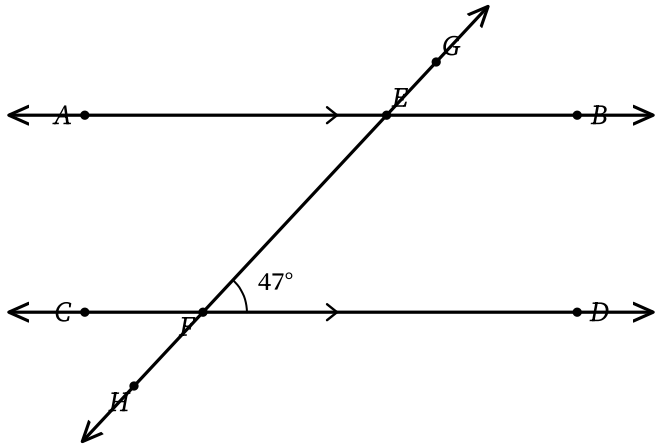
   In a rectangular classroom, consider the line where the front wall meets the floor and the line where the back wall meets the ceiling. Are these lines parallel, intersecting or skew?

3. **5 pts**

   A transversal crosses two lines. On one side of the transversal the interior angles measure 93° and 85°. According to Euclid's fifth postulate, do the lines meet? If so, on which side?

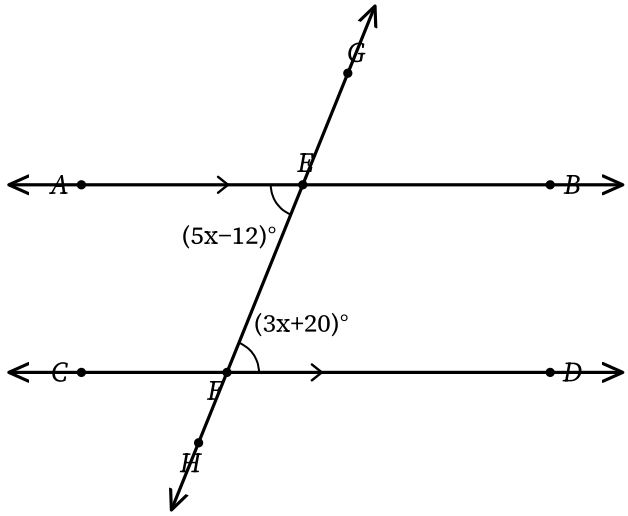
4. **6 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $m\angle DFE = 47°$. Find $m\angle BEF$ and name the theorem you used.



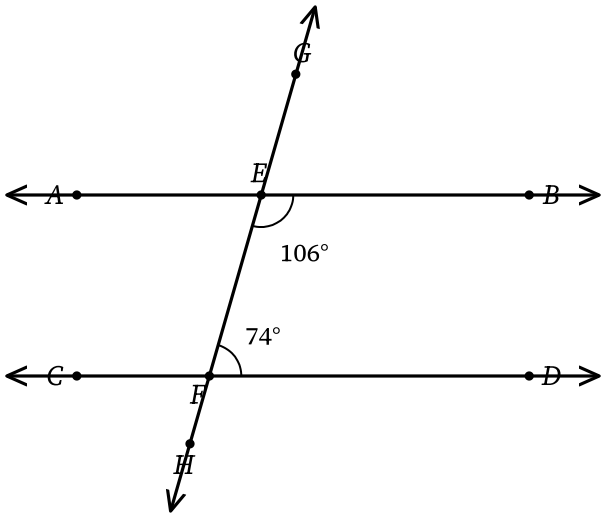
5. **7 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle AEF = (5x - 12)°$ and $m\angle DFE = (3x + 20)°$. Find x and $m\angle AEF$.



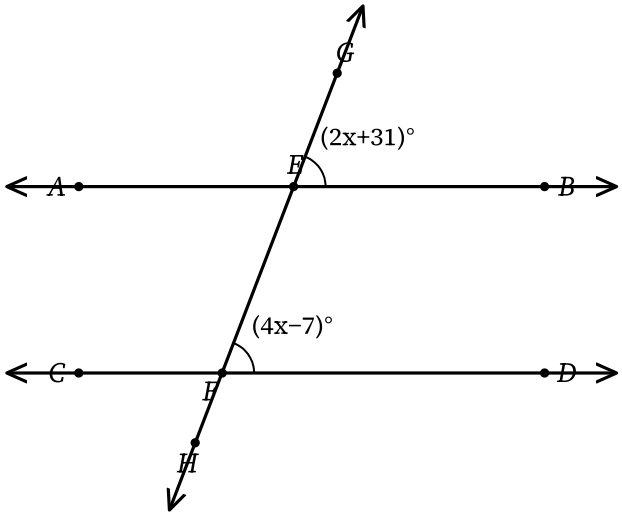
6. **6 pts**

A straight diagonal brace $\overleftrightarrow{GH}$ crosses two straight fence rails, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, at E and F. Ruth measures $m\angle BEF = 106°$ and $m\angle DFE = 74°$. Are the rails parallel? Name the theorem that decides it.



7. **6 pts**

In the figure, $m\angle BEG = (2x + 31)°$ and $m\angle DFE = (4x - 7)°$. What value of x makes $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?



8. **5 pts**

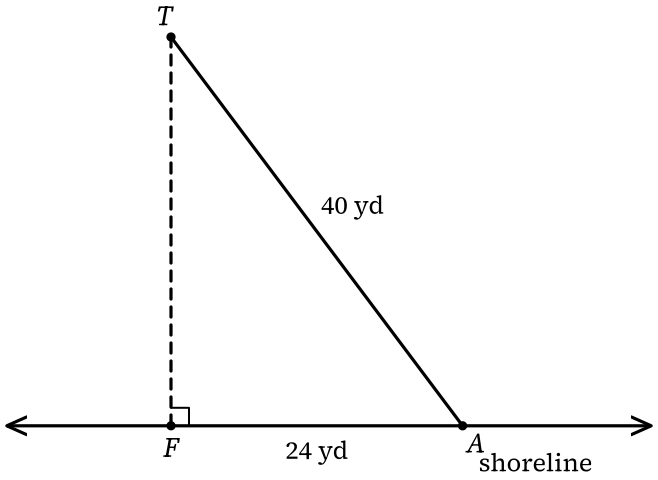
   Find the slope of a line perpendicular to $5x + 2y = 8$.

9. **7 pts**

   Write an equation of the line through $(-2, 3)$ that is parallel to the line through $(1, 4)$ and $(5, -4)$.

10. **7 pts**

    A lifeguard tower T stands near a straight shoreline. Point A on the shoreline is 40 yd from the tower, and the point F of the shoreline nearest the tower is 24 yd from A. How far is the tower from the shoreline?

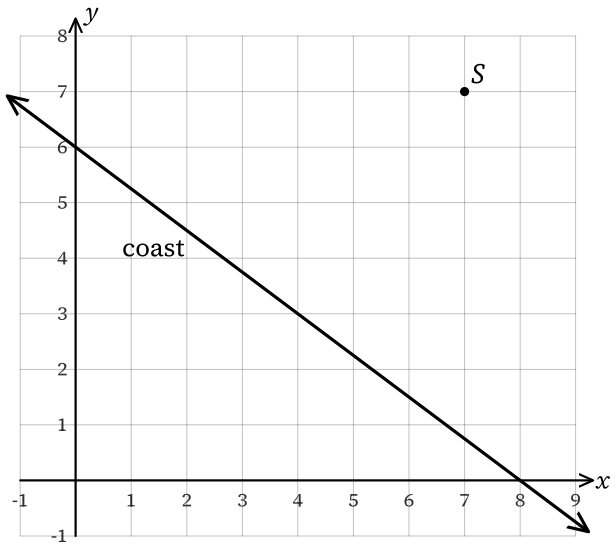


11. **6 pts**

    Henry has just shown that a pair of corresponding angles is congruent. His next step reads: "Statement: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Reason: Corresponding Angles Postulate." Explain what is wrong with his reason and give the correct one.

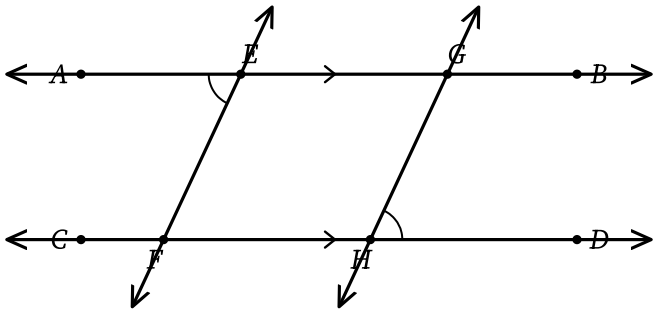
12. **8 pts**

On a chart where 1 unit is 1 mile, a straight coastline follows the line $y = -\frac{3}{4}x + 6$, and a ship is at $S(7, 7)$. Find the point of the coastline nearest the ship, and the ship's distance from the coastline.



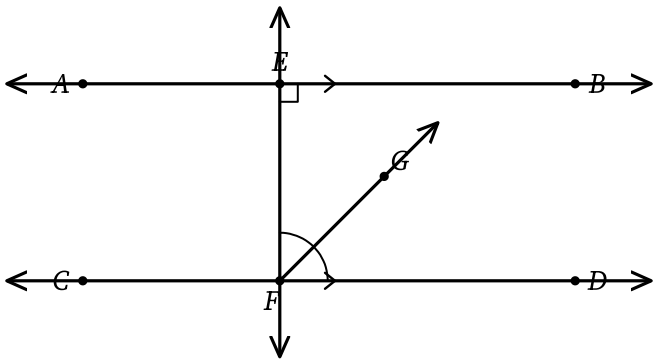
13. **10 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ cross both, with E and G on $\overleftrightarrow{AB}$ and F and H on $\overleftrightarrow{CD}$, and $\angle AEF \cong \angle GHD$. Prove that $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$.



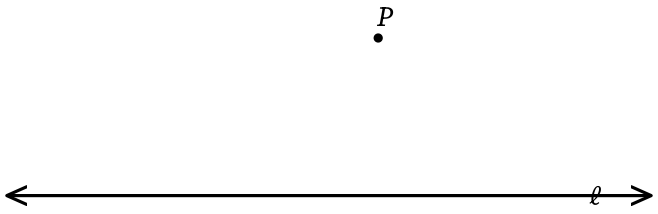
14. **9 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$ with E on $\overleftrightarrow{AB}$ and F on $\overleftrightarrow{CD}$, and $\overrightarrow{FG}$ bisects $\angle EFD$. Write a paragraph proof that $m\angle GFD = 45°$.



15. **9 pts**

Copy the figure: line ℓ and point P not on it. With compass and straightedge, construct the line through P parallel to ℓ. List your steps and name the postulate that guarantees your line is parallel to ℓ.



ANSWER KEY

1. Alternate exterior angles: both are outside the two lines, on opposite sides of the transversal.
2. Parallel. Both run straight across the room from side to side, so they point in the same direction, lie in one (slanted) plane, and never meet.
3. Yes. $93^\circ + 85^\circ = 178^\circ < 180^\circ$, so the lines meet on the side of the transversal where these two angles are.
4. 133°, by the Same-Side Interior Angles Theorem ($180^\circ - 47^\circ$).
5. Alternate interior angles are congruent: $5x - 12 = 3x + 20$, so $2x = 32$ and $x = 16$. Then $m\angle AEF = 5(16) - 12 = 68^\circ$.
6. Yes. $\angle BEF$ and $\angle DFE$ are same-side interior angles and $106^\circ + 74^\circ = 180^\circ$, so the rails are parallel by the Converse of the Same-Side Interior Angles Theorem.
7. $x = 19$. The angles are corresponding, so the lines are parallel when $2x + 31 = 4x - 7$, that is, $38 = 2x$ (each angle is then 69°).
8. $5x + 2y = 8$ is $y = -\dfrac{5}{2}x + 4$, so the perpendicular slope is $\dfrac{2}{5}$.
9. The slope is $\dfrac{-4 - 4}{5 - 1} = -2$, so $y - 3 = -2(x + 2)$ and $y = -2x - 1$.
10. 32 yd. The distance is TF, with $\overline{TF}$ perpendicular to the shore, and $TF^2 + 24^2 = 40^2$, so $TF^2 = 1024$ and $TF = 32$ yd.
11. The Corresponding Angles Postulate goes the other way: it starts with parallel lines and concludes that corresponding angles are congruent. Henry is going from congruent angles to parallel lines, so he needs the converse. The correct reason is the Converse of the Corresponding Angles Postulate.
12. The perpendicular through S has slope $\dfrac{4}{3}$: $y = \dfrac{4}{3}x - \dfrac{7}{3}$. Solve with the coastline: $-\dfrac{3}{4}x + 6 = \dfrac{4}{3}x - \dfrac{7}{3}$; times 12, $-9x + 72 = 16x - 28$, so $x = 4$ and $y = 3$. The nearest point is $(4, 3)$, and the distance is $\sqrt{3^2 + 4^2} = 5$ miles.
13. See the model proof.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\angle AEF \cong \angle GHD$
Prove: $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given

	STATEMENTS	REASONS
2	$\angle AEF \cong \angle EFD$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{EF}$)
3	$\angle AEF \cong \angle GHD$	Given
4	$\angle EFD \cong \angle GHD$	Symmetric and Transitive Properties of Congruence
5	$\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$	Converse of the Corresponding Angles Postulate ($\angle EFD$ and $\angle GHD$ are corresponding angles for transversal $\overleftrightarrow{CD}$)

∎

14. See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$; $\overrightarrow{FG}$ bisects $\angle EFD$

Prove: $m\angle GFD = 45°$

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$, the Perpendicular Transversal Theorem gives $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$. Perpendicular lines form four right angles, so $\angle EFD$ is a right angle and $m\angle EFD = 90°$.

Because $\overrightarrow{FG}$ bisects $\angle EFD$, $m\angle EFG = m\angle GFD$ by the definition of angle bisector, and $m\angle EFG + m\angle GFD = m\angle EFD = 90°$ by the Angle Addition Postulate.

Substituting gives $2 \cdot m\angle GFD = 90°$, so $m\angle GFD = 45°$ by the Division Property of Equality.

∎

15. Draw a line through P crossing ℓ at Q. With the compass at Q, draw an arc crossing ℓ at R and $\overrightarrow{QP}$ at S. With the same width and the compass at P, draw an arc crossing $\overrightarrow{QP}$ beyond P at T. Set the compass to RS and, from T, mark U on that arc on the same side as R. Draw $\overleftrightarrow{PU}$. Then $\angle UPT \cong \angle RQP$ are congruent corresponding angles, so $\overleftrightarrow{PU} \parallel \ell$ by the Converse of the Corresponding Angles Postulate (Euclid I.31).

3.1 Lines, Transversals and Angle Pairs

OBJECTIVES

- Distinguish parallel, intersecting and skew lines, and recognize parallel planes.
- Identify a transversal and the eight angles it forms with two lines.
- Classify pairs of angles as corresponding, alternate interior, alternate exterior or same-side interior.
- Use vertical angles and linear pairs to find the angles at one intersection.

WATCH

Parallel Lines & Transversals

Eddie Woo · 9:02

https://www.youtube.com/watch?v=UCigSz7yQpc

> **WHEN TO WATCH**
> Before Example 1: a transversal and the angle pairs it makes.

Note: Eddie Woo teaches in Australia: he says 'co-interior angles' for same-side interior angles. Skew lines and parallel planes are taught from the Core ideas.

BACKUP VIDEO

Angles on Parallel Lines (2 of 2: Naming the relationships)

Eddie Woo · 8:05

https://www.youtube.com/watch?v=2Ei_-6Lmm8E

Before the practice set: naming corresponding, alternate and co-interior angles.

KEY TERMS

parallel lines

Lines that lie in the same plane and do not intersect, however far they are extended. We write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

skew lines

Lines that do not intersect and do not lie in the same plane.

transversal

A line that intersects two or more coplanar lines at different points.

corresponding angles

Two angles formed by a transversal and two lines that sit in the same position at the two intersection points: on the same side of the transversal, and both above or both below their own lines.

alternate interior angles

Two angles that lie between the two lines, on opposite sides of the transversal, at different intersection points.

alternate exterior angles

Two angles that lie outside the two lines, on opposite sides of the transversal, at different intersection points.

same-side interior angles

Two angles that lie between the two lines and on the same side of the transversal, at different intersection points. They are also called consecutive interior angles.

CORE IDEAS

Two different lines in one plane either meet in exactly one point (Lesson 2.1) or never meet at all. Lines in a plane that never meet are **parallel**. Notice that the definition says *in the same plane*. In space, two lines can miss each other without running side by side: in your room, the edge where the front wall meets the ceiling and the edge where a side wall meets the floor never touch, yet they point in different directions. Such lines are **skew**. So in space there are three possibilities for two lines: intersecting, parallel or skew.

Planes can be parallel too: **parallel planes** are planes that do not intersect, like the floor and the ceiling of a room, or the front and back faces of a box. In the box in the second figure, the front face $ABCD$ and the back face $EFGH$ lie in parallel planes, $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$ are parallel, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$ are skew.

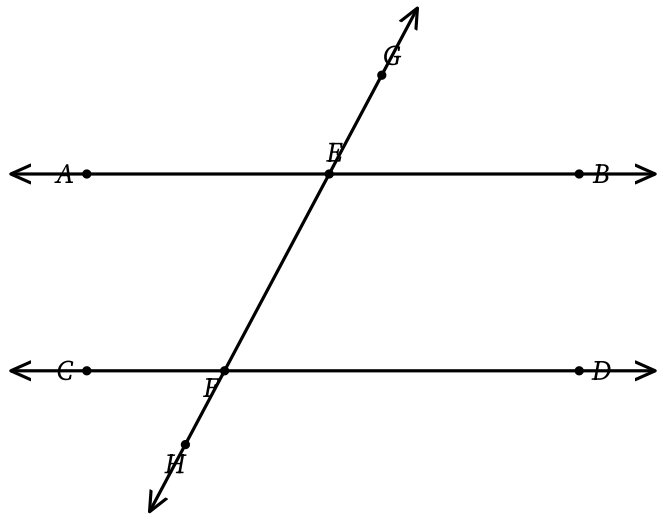
A **transversal** is a line that crosses two (or more) coplanar lines at different points. In the first figure, $\overleftrightarrow{GH}$ is a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, crossing them at E and F. It

forms eight angles, four at E and four at F. The four angles between the two lines (such as $\angle AEF$ and $\angle DFE$) are **interior** angles; the four outside them (such as $\angle BEG$ and $\angle CFH$) are **exterior** angles.

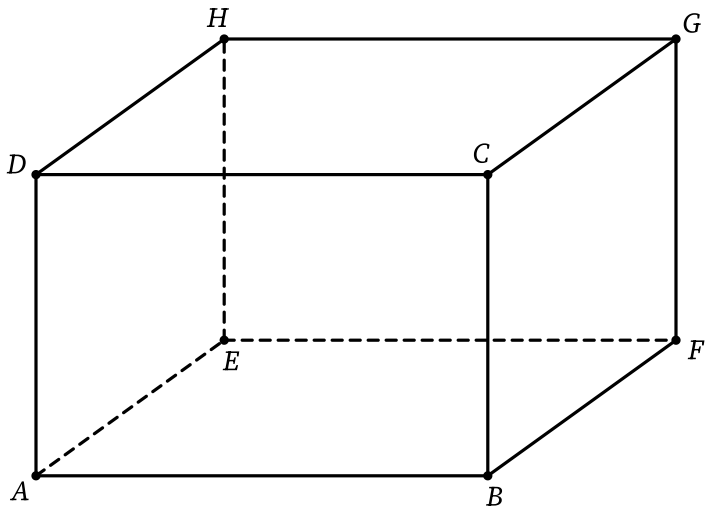
We name pairs of these angles, one angle at each intersection, by their positions. **Corresponding** angles are in matching positions: $\angle BEG$ and $\angle DFE$ are both above their lines and to the right of the transversal. **Alternate interior** angles are between the lines on opposite sides of the transversal: $\angle AEF$ and $\angle DFE$. **Alternate exterior** angles are outside the lines on opposite sides: $\angle AEG$ and $\angle DFH$. **Same-side interior** angles are between the lines on the same side: $\angle BEF$ and $\angle DFE$.

Why learn these names? Because they are exactly the pairs that tell us whether two lines are parallel. In Lesson 3.2 you will see that when the lines are parallel, corresponding, alternate interior and alternate exterior angles are congruent and same-side interior angles add to 180°; in Lesson 3.3 you will prove the reverse. Notice, though, that the names describe *positions only*. Two lines that are not parallel still have corresponding angles; those angles simply are not congruent.

A memory aid (not a reason in a proof): corresponding angles sit in an "F" shape, alternate interior angles in a "Z" shape, and same-side interior angles in a "C" shape. The F, Z or C may be backward or upside down. At a single intersection you already know more from Lesson 2.4: the angles across from each other are vertical angles and are congruent, and neighbors along a line form linear pairs and add to 180°. So one angle at E tells you all four angles at E, but by itself it tells you nothing about the angles at F.



Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.



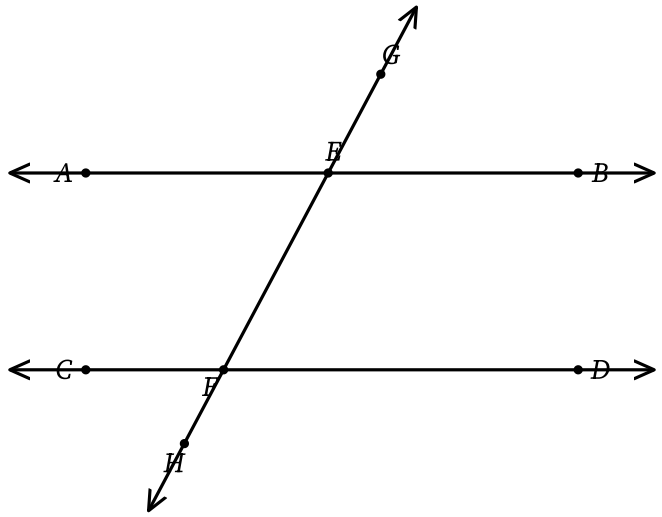
A box: front face $ABCD$, back face $EFGH$. Hidden edges are dashed.

> **From history.** The word *parallel* comes from the Greek *parallēlos*, meaning "beside one another." Euclid treats lines and planes in space in Book XI of the Elements, where he also defines parallel planes as planes that do not meet. Surveyors laying out roads and builders setting the beams of a floor have always needed to know when lines run side by side and when they will eventually cross.

Euclid connection. Euclid defines parallels in Book I, Definition 23: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." In Propositions I.27–I.29 he speaks of "a straight line falling on two straight lines" (our transversal) and of the "alternate angles," the "exterior angle" and "the interior and opposite angle," and the "interior angles on the same side": exactly the pairs named in this lesson.

Practice habit. Recite the definitions of transversal, corresponding, alternate interior, alternate exterior and same-side interior angles until they are automatic. Before naming a pair, trace the transversal with your finger and ask two questions: are the angles between the lines or outside them, and are they on the same side of the transversal or on opposite sides?

Example 1. In the figure, $\overleftrightarrow{GH}$ is a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, crossing them at E and F. Classify each pair of angles: (a) $\angle BEG$ and $\angle DFE$; (b) $\angle AEF$ and $\angle DFE$; (c) $\angle AEG$ and $\angle DFH$; (d) $\angle BEF$ and $\angle DFE$.



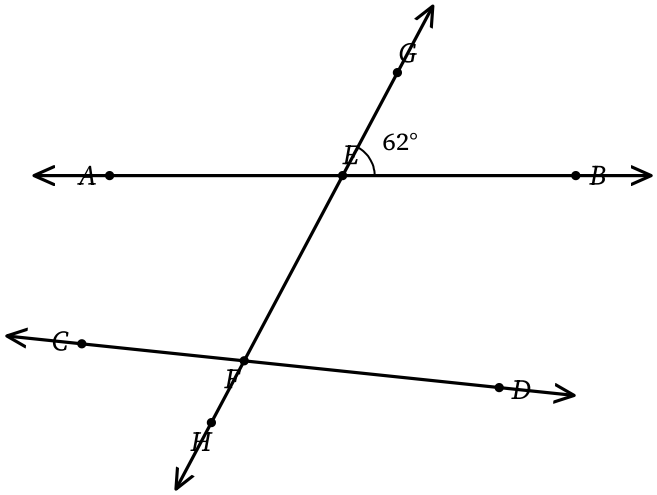
Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Show solution

1. For each angle, find its vertex (E or F), then ask whether it is between the lines (interior) or outside them (exterior), and which side of the transversal it is on.
2. (a) $\angle BEG$ has sides $\overrightarrow{EB}$ (right) and $\overrightarrow{EG}$ (up): above $\overleftrightarrow{AB}$, right of the transversal. $\angle DFE$ has sides $\overrightarrow{FD}$ (right) and $\overrightarrow{FE}$ (up): above $\overleftrightarrow{CD}$, right of the transversal. Same position at each crossing: corresponding angles.
3. (b) $\angle AEF$ is between the lines on the left; $\angle DFE$ is between the lines on the right. Interior, opposite sides: alternate interior angles.
4. (c) $\angle AEG$ is above both lines on the left; $\angle DFH$ is below both lines on the right. Exterior, opposite sides: alternate exterior angles.
5. (d) $\angle BEF$ and $\angle DFE$ are both between the lines on the right side: same-side interior angles.

Answer: (a) corresponding; (b) alternate interior; (c) alternate exterior; (d) same-side interior

Example 2. In the figure, $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F, and $m\angle BEG = 62°$. We are not told whether $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. Find $m\angle AEG$, $m\angle AEF$ and $m\angle BEF$. Then decide: can $m\angle DFE$ be found from this information?



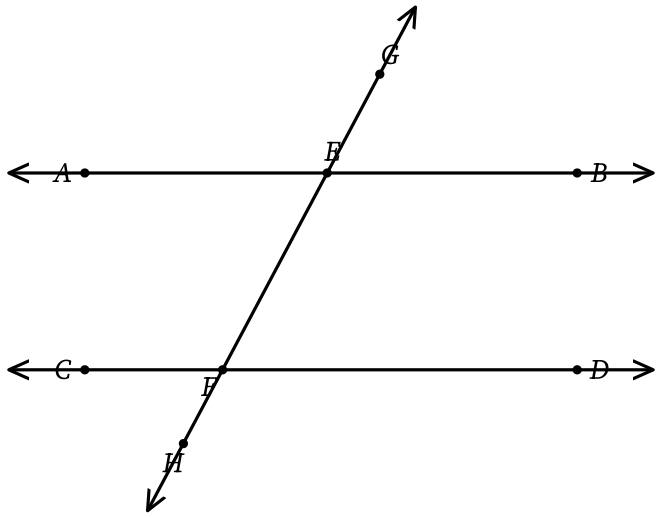
Show solution

1. $\angle BEG$ and $\angle AEG$ form a linear pair ($\overrightarrow{EA}$ and $\overrightarrow{EB}$ are opposite rays), so they are supplementary: $m\angle AEG = 180° - 62° = 118°$.
2. $\angle AEF$ and $\angle BEG$ are vertical angles ($\overrightarrow{EA}$ is opposite $\overrightarrow{EB}$, and $\overrightarrow{EF}$ is opposite $\overrightarrow{EG}$), so by the Vertical Angles Theorem $m\angle AEF = 62°$.
3. $\angle BEF$ and $\angle AEG$ are vertical angles, so $m\angle BEF = 118°$.
4. The angles at F depend on the direction of $\overleftrightarrow{CD}$, which could be tilted any amount. Nothing we know connects the angles at E to the angles at F unless the lines are known to be parallel (Lesson 3.2). In this figure the lines are in fact not parallel. So $m\angle DFE$ cannot be found.

Answer: $m\angle AEG = 118°$, $m\angle AEF = 62°$, $m\angle BEF = 118°$; no, $m\angle DFE$ cannot be found without knowing the lines are parallel.

PRACTICE

1. **Easy** In the figure, classify the pair $\angle AEG$ and $\angle CFE$.

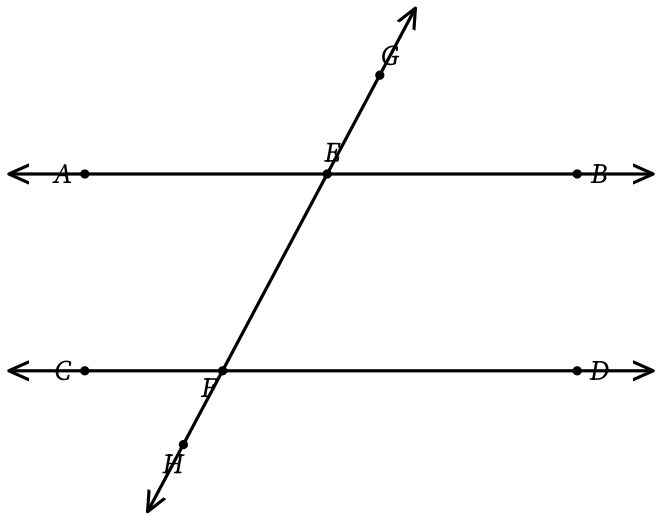


Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Show answer

Corresponding angles (both above their lines, left of the transversal).

2. **Easy** In the figure, classify the pair $\angle BEF$ and $\angle CFE$.

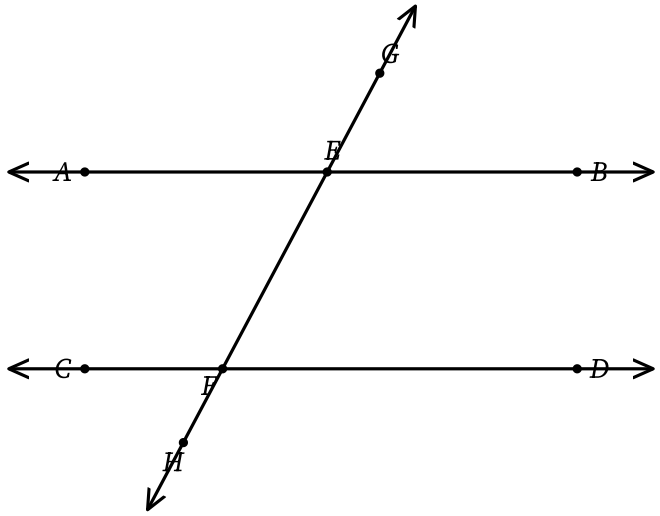


Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Show answer

Alternate interior angles (between the lines, on opposite sides of the transversal).

3. **Easy** In the figure, classify the pair $\angle BEG$ and $\angle CFH$.

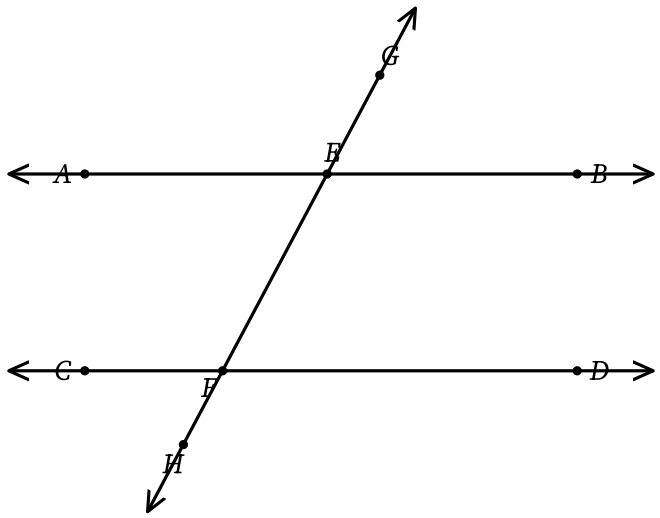


Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Show answer

Alternate exterior angles (outside the lines, on opposite sides of the transversal).

4. **Easy** In the figure, classify the pair $\angle AEF$ and $\angle CFE$.

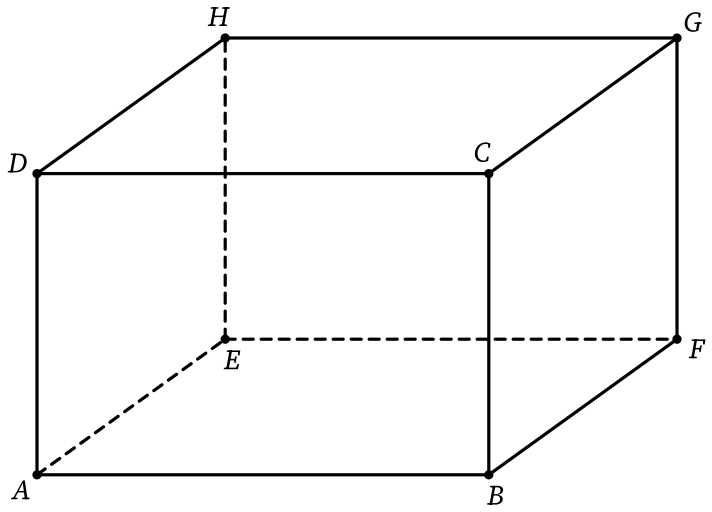


Transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Show answer

Same-side interior angles (between the lines, both on the left of the transversal).

5. **Medium** In the box, front face $ABCD$ is directly in front of back face $EFGH$ (E is behind A, F behind B, G behind C, H behind D). Name every edge whose line is parallel to $\overleftrightarrow{AB}$.

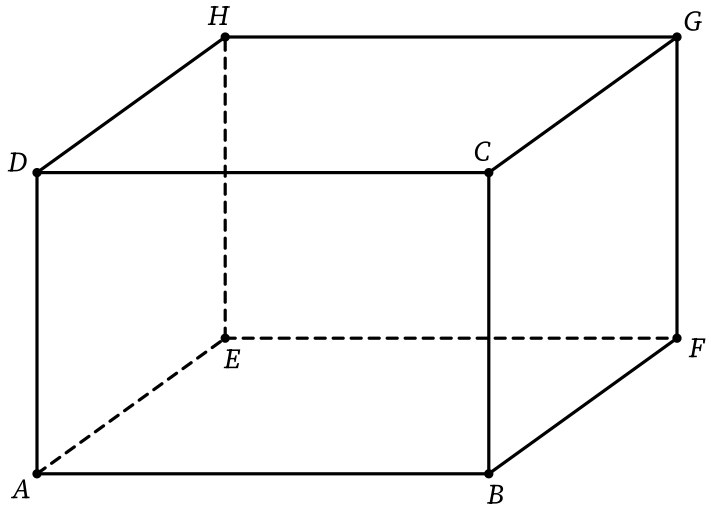


A box: front face $ABCD$, back face $EFGH$. Hidden edges are dashed.

Show answer

$\overline{CD}$, $\overline{EF}$ and $\overline{GH}$

6. **Medium** In the same box, name every edge whose line is skew to $\overleftrightarrow{AB}$.

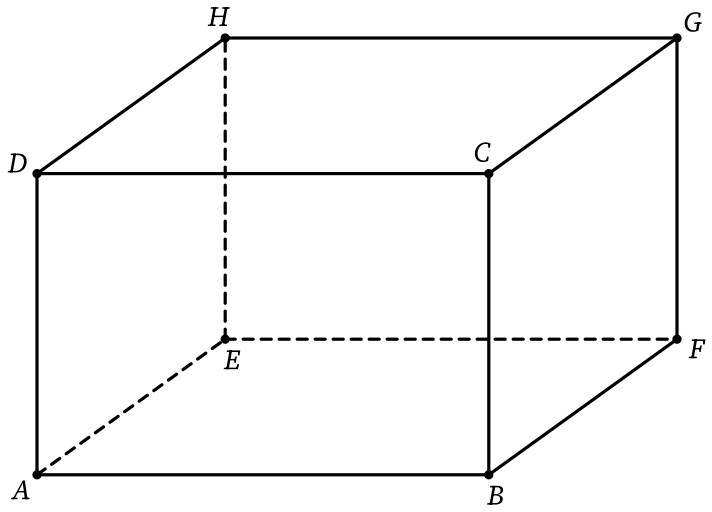


A box: front face $ABCD$, back face $EFGH$. Hidden edges are dashed.

Show answer

$\overline{CG}$, $\overline{DH}$, $\overline{EH}$ and $\overline{FG}$ (they neither meet $\overleftrightarrow{AB}$ nor run in its direction). The other edges either meet $\overleftrightarrow{AB}$ ($\overline{AD}$, $\overline{AE}$, $\overline{BC}$, $\overline{BF}$) or are parallel to it.

7. **Medium** In the same box, are $\overleftrightarrow{DC}$ and $\overleftrightarrow{BF}$ parallel, intersecting or skew?

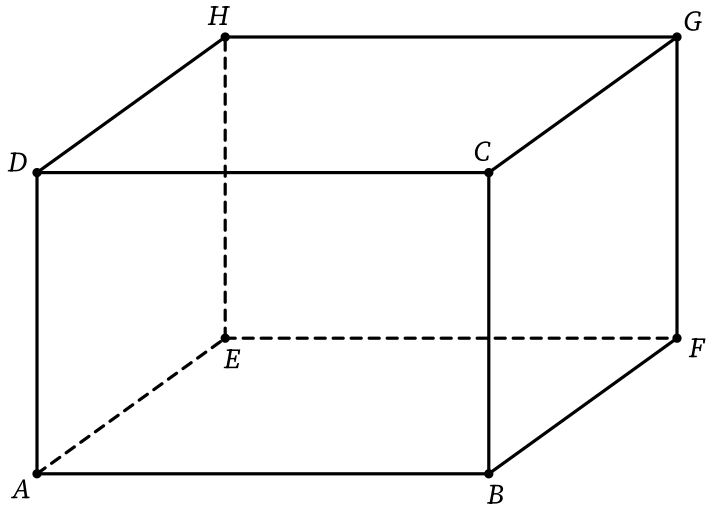


A box: front face $ABCD$, back face $EFGH$. Hidden edges are dashed.

Show answer

Skew: $\overleftrightarrow{DC}$ runs across the top front edge and $\overleftrightarrow{BF}$ runs front to back along the bottom right; they do not meet and are not in one plane.

8. **Medium** In the same box, name the plane parallel to plane ABC and the plane parallel to plane BCG.



A box: front face $ABCD$, back face $EFGH$. Hidden edges are dashed.

Show answer

Plane EFG (the back face) is parallel to plane ABC; plane ADH (the left face) is parallel to plane BCG.

9. Medium Grace studies the edges of her rectangular bedroom. Consider the edge where the front wall meets the ceiling and the edge where the left wall meets the floor. Are these lines parallel, intersecting or skew?

Show answer

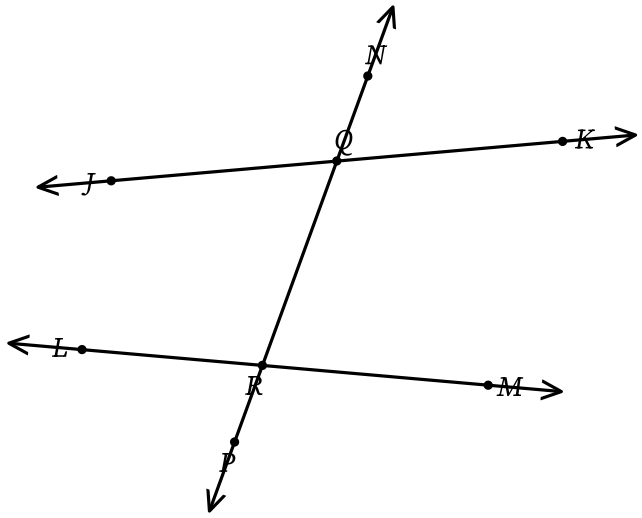
Skew. One runs side to side along the top of the front wall, the other runs front to back along the bottom of the left wall. They never meet and they point in different directions, so they cannot lie in one plane.

10. Medium True or false: "Two lines that do not intersect are parallel." Explain.

Show answer

False. In space, two lines that do not intersect may be skew, like the edges in Problem 9. The statement is true only for lines known to lie in the same plane, which is why the definition of parallel lines says "in the same plane."

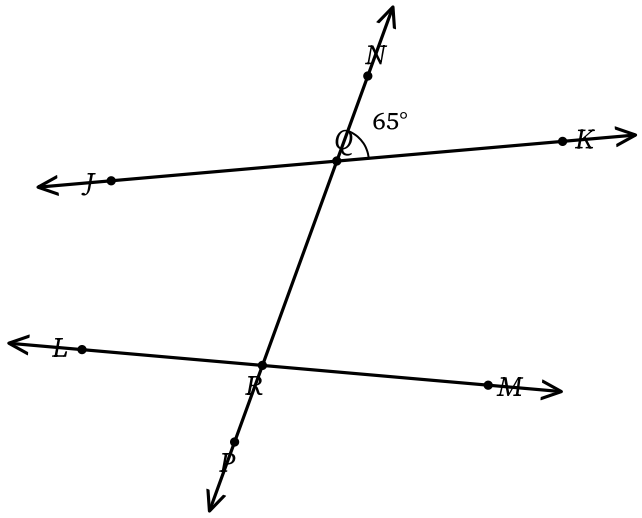
11. Hard In the figure, $\overleftrightarrow{PN}$ is a transversal of $\overleftrightarrow{JK}$ and $\overleftrightarrow{LM}$, crossing them at Q and R. Name the angle that is alternate interior to $\angle JQR$, and the angle that corresponds to $\angle JQN$.



Show answer

$\angle QRM$ is alternate interior to $\angle JQR$; $\angle LRQ$ corresponds to $\angle JQN$.

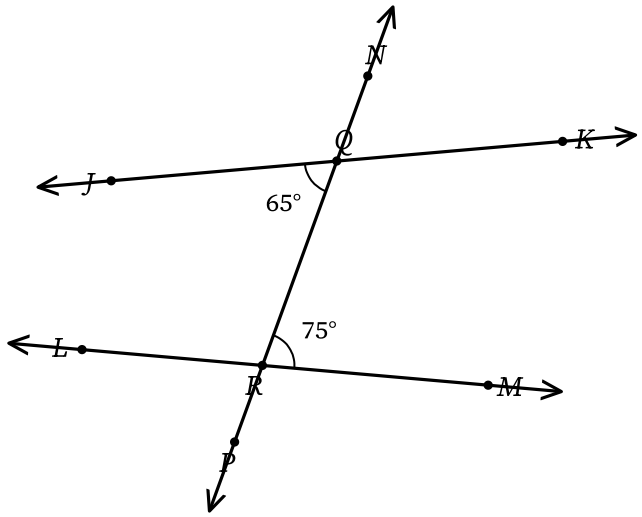
12. **Hard** In the figure, $m\angle KQN = 65°$. Find $m\angle JQR$, $m\angle JQN$ and $m\angle KQR$, giving a reason for each.



Show answer

$m\angle JQR = 65°$ (vertical to $\angle KQN$); $m\angle JQN = 115°$ (linear pair with $\angle KQN$); $m\angle KQR = 115°$ (vertical to $\angle JQN$).

13. **Hard** In the figure, $m\angle JQR = 65°$ and $m\angle QRM = 75°$. Samuel says these cannot be alternate interior angles because they are not equal. Are they alternate interior angles? Explain why or why not.



Show answer

Yes. Both angles lie between $\overleftrightarrow{JK}$ and $\overleftrightarrow{LM}$, on opposite sides of the transversal, at different intersection points. The name depends only on position. Their measures differ because $\overleftrightarrow{JK}$ and $\overleftrightarrow{LM}$ are not parallel.

14. **Challenge** A transversal crosses three coplanar lines at three different points. Taking the three lines two at a time, how many pairs of corresponding angles are formed in all? Explain your count.

Show answer

12. Each pair of lines, with the transversal, gives 4 pairs of corresponding angles (one for each of the four positions). There are 3 ways to choose two of the three lines, so $3 \times 4 = 12$.

COMMON MISTAKES

- Calling two angles at the same intersection "alternate interior" or "corresponding." These names always pair one angle at each of the two intersection points; angles at one point are vertical angles or linear pairs.
- Assuming that lines that do not meet must be parallel. In space they may be skew; ask first whether the lines lie in one plane.

- Assuming corresponding angles are congruent just because they are corresponding. The names tell position only; congruence needs parallel lines (Lesson 3.2).
- Using the wrong line as the transversal in a figure with many lines. The transversal is the line that contains the vertices of both angles in the pair.

> **Make it a game · Angle Pair Touch.** Draw two parallel lines cut by a transversal on a sheet and number the eight angles. Parent calls a pair type and one angle, such as 'corresponding, angle 3.' Touch the partner angle within 5 seconds. Then hold two pencils in the air to show skew lines versus parallel lines. Go ten rounds and chase a perfect score.

3.2 The Parallel Postulate and Parallel Lines

OBJECTIVES

- State Euclid's fifth postulate, Playfair's Axiom and the Corresponding Angles Postulate, and explain why the parallel postulate must be assumed rather than proved.
- Prove the Alternate Interior Angles, Alternate Exterior Angles and Same-Side Interior Angles Theorems from the Corresponding Angles Postulate.
- Use these theorems to find angle measures and solve equations when parallel lines are cut by a transversal.

WATCH

Patterns on Parallel Lines (1 of 2: Proving that Alternate Angles on Parallel Lines are equal)

Eddie Woo · 11:01

https://www.youtube.com/watch?v=nLP2w_thXfU

> **WHEN TO WATCH**
> After the postulates: a proof that alternate angles on parallel lines are equal.

Note: Eddie Woo teaches in Australia: he says 'alternate angles' for alternate interior angles and 'co-interior' for same-side interior angles.

BACKUP VIDEO

A Problem with the Parallel Postulate - Numberphile

Numberphile · 13:45

https://www.youtube.com/watch?v=H8Q0SoKT-A8

After the lesson, as enrichment: why the parallel postulate must be assumed and what happens without it (Numberphile, alternate list).

KEY TERMS

Euclid's fifth postulate (the parallel postulate)

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two lines, if extended far enough, meet on that side.

Playfair's Axiom

Through a point not on a given line there is exactly one line parallel to the given line.

Corresponding Angles Postulate

If two parallel lines are cut by a transversal, then each pair of corresponding angles is congruent.

non-Euclidean geometry

A consistent geometry in which the parallel postulate is replaced by a different assumption about parallels. It shows that the parallel postulate cannot be proved from Euclid's other postulates.

CORE IDEAS

Draw two parallel lines and a transversal carefully, and the corresponding angles always look equal. But a look is not a proof, and here something surprising happens: the postulates of Units 1 and 2 are not enough to prove it. We need a new starting assumption about parallel lines. Euclid saw this. His fifth postulate says: "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." In the first figure the interior angles on the right add to $75^\circ + 80^\circ = 155^\circ$, less than 180°, and the lines do meet on the right.

Euclid's wording is long, so in 1795 the Scottish mathematician John Playfair popularized a simpler statement, now called **Playfair's Axiom**: through a point not on a line there is exactly one parallel to that line (second figure). Together with Euclid's other postulates, each of these two statements can be proved from the other, so either one may serve as "the parallel postulate."

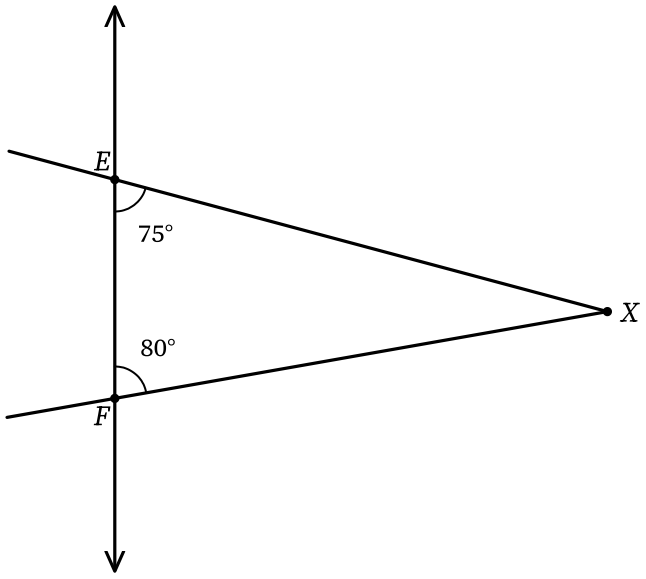
In this course we use a third form that is the most convenient for proofs, the **Corresponding Angles Postulate**: if two parallel lines are cut by a transversal, then corresponding angles are congruent. Like Euclid's fifth postulate, it is a genuine

starting assumption about the flat plane. Because the forms are equivalent, in this course we also accept Playfair's Axiom as a postulate and may cite it as a reason in proofs. Every result in this unit about angles and parallel lines rests on these assumptions. The Corresponding Angles Postulate is the modern form of the second part of Euclid's I.29 (\"the exterior angle equal to the interior and opposite angle\"), which Euclid derives from the alternate-angle part.

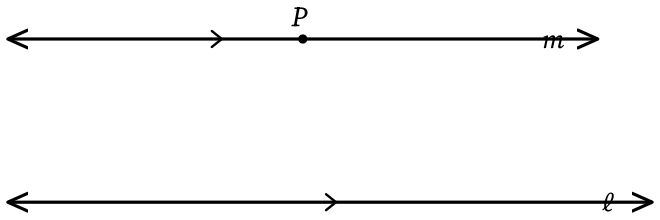
With this postulate and the tools of Lesson 2.4 (vertical angles and linear pairs) we can prove three more theorems. The idea is always the same: get from the angle you know to a corresponding angle, then move around one intersection using vertical angles or a linear pair. The **Alternate Interior Angles Theorem** and the **Alternate Exterior Angles Theorem** say those pairs are congruent. The **Same-Side Interior Angles Theorem** says same-side interior angles are supplementary, because each is the linear-pair partner of an angle congruent to the other.

Here is the big picture. When a transversal crosses two parallel lines at an angle of x°, all eight angles measure either x° or $(180 - x)^\circ$: four of each. So one angle measure tells you all eight. In algebra problems, set congruent pairs **equal** to each other, and set the **sum** of a same-side interior pair equal to 180.

All of this works only when the lines are known to be parallel. If the problem does not say so (in words or with arrowheads on the figure), none of these theorems may be used.



Euclid's fifth postulate: $75^\circ + 80^\circ < 180^\circ$, so the lines meet on that side, at X.

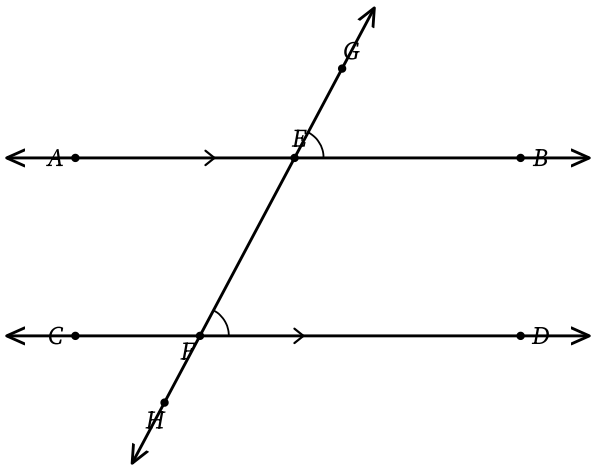


Playfair's Axiom: through P there is exactly one line m parallel to ℓ.

POSTULATES AND THEOREMS

POSTULATE **Corresponding Angles Postulate** Euclid I.29

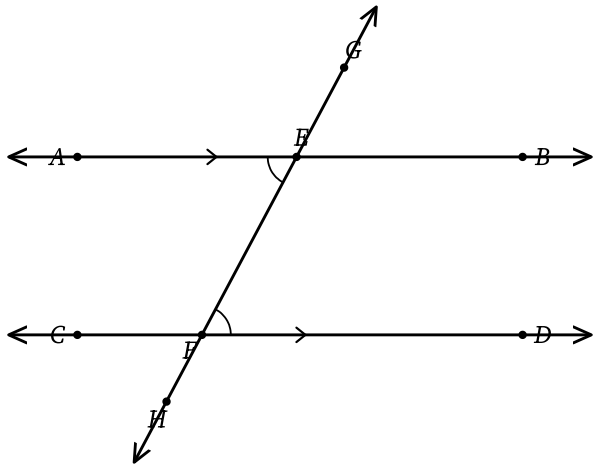
If two parallel lines are cut by a transversal, then each pair of corresponding angles is congruent.



Euclid proves this as part of I.29, using his fifth postulate. We take it as our form of the parallel postulate: given the other postulates it is equivalent to Euclid's fifth postulate and to Playfair's Axiom, so it cannot be proved from the postulates of Units 1 and 2.

THEOREM **Alternate Interior Angles Theorem** Euclid I.29

If two parallel lines are cut by a transversal, then each pair of alternate interior angles is congruent.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, cut by transversal $\overleftrightarrow{GH}$ at E and F

Prove: $\angle AEF \cong \angle DFE$

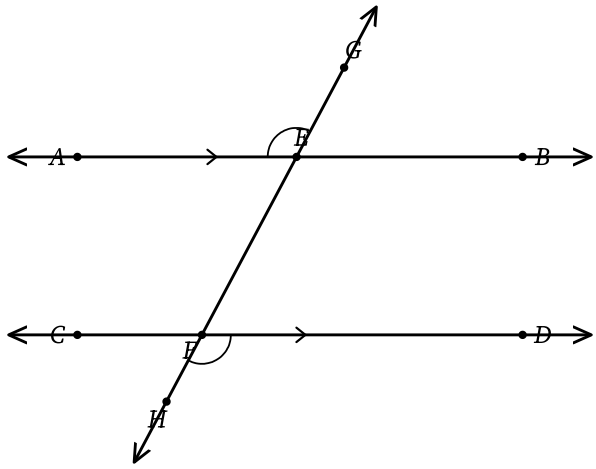
	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given
2	$\angle BEG \cong \angle DFE$	Corresponding Angles Postulate
3	$\angle AEF \cong \angle BEG$	Vertical Angles Theorem
4	$\angle AEF \cong \angle DFE$	Transitive Property of Congruence

∎

The same argument shows $\angle BEF \cong \angle CFE$, the other pair of alternate interior angles.

THEOREM **Alternate Exterior Angles Theorem** Euclid I.29

If two parallel lines are cut by a transversal, then each pair of alternate exterior angles is congruent.



PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, cut by transversal $\overleftrightarrow{GH}$ at E and F

Prove: $\angle AEG \cong \angle DFH$

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, the corresponding angles $\angle AEG$ and $\angle CFE$ (both above their lines, left of the transversal) are congruent by the Corresponding Angles Postulate.

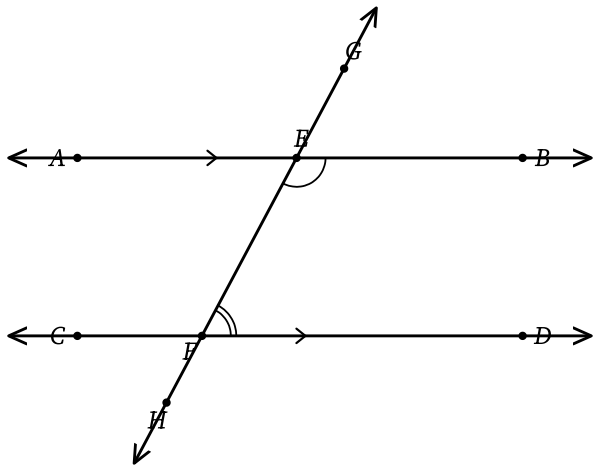
Angles $\angle CFE$ and $\angle DFH$ are vertical angles, because $\overrightarrow{FC}$ and $\overrightarrow{FD}$ are opposite rays and so are $\overrightarrow{FE}$ and $\overrightarrow{FH}$. So $\angle CFE \cong \angle DFH$ by the Vertical Angles Theorem.

By the Transitive Property of Congruence, $\angle AEG \cong \angle DFH$.

∎

THEOREM **Same-Side Interior Angles Theorem** Euclid I.29

If two parallel lines are cut by a transversal, then each pair of same-side interior angles is supplementary.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, cut by transversal $\overleftrightarrow{GH}$ at E and F

Prove: $\angle BEF$ and $\angle DFE$ are supplementary

	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given
2	$\angle BEG \cong \angle DFE$	Corresponding Angles Postulate
3	$m\angle BEG = m\angle DFE$	Definition of congruent angles
4	$\angle BEG$ and $\angle BEF$ are supplementary	Linear Pair Theorem ($\overrightarrow{EG}$ and $\overrightarrow{EF}$ are opposite rays)
5	$m\angle BEG + m\angle BEF = 180°$	Definition of supplementary angles
6	$m\angle DFE + m\angle BEF = 180°$	Substitution Property of Equality
7	$\angle BEF$ and $\angle DFE$ are supplementary	Definition of supplementary angles

∎

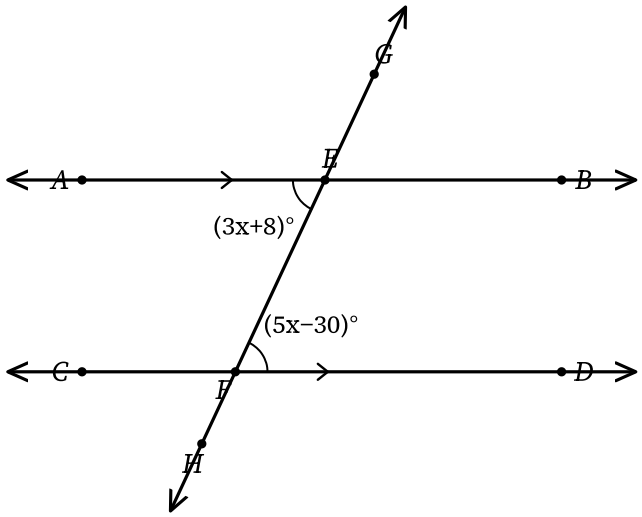
Euclid's wording: the interior angles on the same side are "equal to two right angles."

From history. For about two thousand years mathematicians felt that the fifth postulate was too complicated to be a starting assumption and tried to prove it from the other four; Proclus, in his commentary on Euclid, already argued that it ought to be proved. In 1733 the Italian priest and mathematician Girolamo Saccheri tried a proof by contradiction: he assumed the postulate false and hoped to reach an impossibility, but never reached a genuine contradiction. In the early 1800s Carl Friedrich Gauss, János Bolyai (published 1832) and Nikolai Lobachevsky (published 1829) independently developed a geometry in which the fifth postulate is false and found no contradiction in it. In 1868 Eugenio Beltrami built a model showing that this geometry is exactly as consistent as Euclid's. This settled the question: the parallel postulate cannot be proved from the others. It is a true starting assumption, and it is the right one for the flat plane we study.

Euclid connection. Proposition I.29 is the first in the Elements that uses the fifth postulate; Euclid proves the first 28 propositions without it. I.29 reads: "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles." Euclid proves the alternate-angle part by contradiction from Post. 5, then gets the other two parts with vertical angles (I.15) and linear pairs (I.13), just as we do starting from corresponding angles.

Practice habit. Recite the Corresponding Angles Postulate and the three theorems of this lesson until they are automatic, always beginning "If two parallel lines are cut by a transversal, then ...". Before using any of them, put your finger on the words or arrowheads that tell you the lines are parallel.

Example 1. Two straight, parallel streets, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, are crossed by a straight railroad line $\overleftrightarrow{GH}$ at E and F, as in the figure. A surveyor's notes give $m\angle AEF = (3x + 8)^\circ$ and $m\angle DFE = (5x - 30)^\circ$. Find x and $m\angle BEF$.

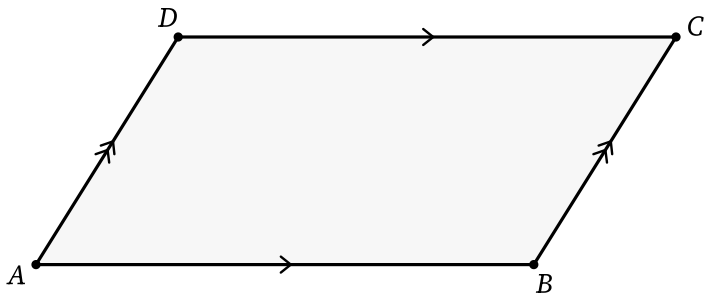


Show solution

1. $\angle AEF$ and $\angle DFE$ are between the streets on opposite sides of the railroad: alternate interior angles. The streets are parallel, so by the Alternate Interior Angles Theorem the angles are congruent.
2. Set the measures equal: $3x + 8 = 5x - 30$. Subtract $3x$ and add 30: $38 = 2x$, so $x = 19$.
3. Then $m\angle AEF = 3(19) + 8 = 65^\circ$. Check: $5(19) - 30 = 65^\circ$.
4. $\angle AEF$ and $\angle BEF$ form a linear pair, so $m\angle BEF = 180^\circ - 65^\circ = 115^\circ$.

Answer: $x = 19$ and $m\angle BEF = 115^\circ$

Example 2. In quadrilateral $ABCD$, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. Prove that $\angle DAB \cong \angle BCD$.



Show solution

1. Work backward: two angles are congruent if they are supplementary to the same angle (Congruent Supplements Theorem). Look for an angle that pairs with both $\angle DAB$ and $\angle BCD$.
2. $\angle ABC$ works. With parallels $\overline{AD}$ and $\overline{BC}$ and transversal $\overline{AB}$, $\angle DAB$ and $\angle ABC$ are same-side interior angles.
3. With parallels $\overline{AB}$ and $\overline{DC}$ and transversal $\overline{BC}$, $\angle ABC$ and $\angle BCD$ are same-side interior angles.

TWO-COLUMN PROOF

Given: $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \parallel \overline{BC}$

Prove: $\angle DAB \cong \angle BCD$

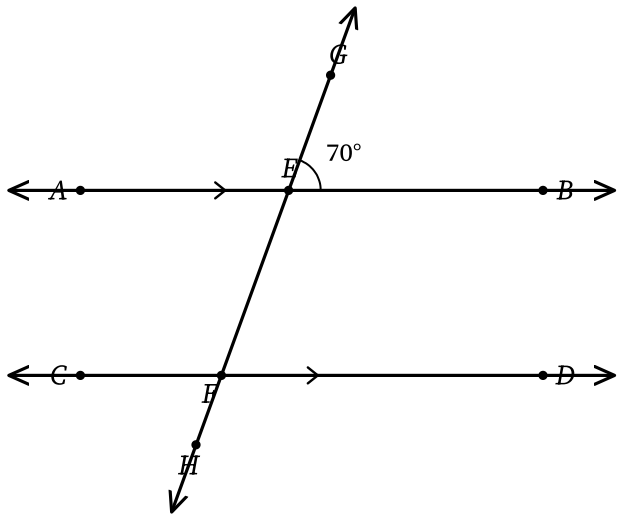
	STATEMENTS	REASONS
1	$\overline{AD} \parallel \overline{BC}$	Given
2	$\angle DAB$ and $\angle ABC$ are supplementary	Same-Side Interior Angles Theorem (transversal $\overline{AB}$)
3	$\overline{AB} \parallel \overline{DC}$	Given
4	$\angle ABC$ and $\angle BCD$ are supplementary	Same-Side Interior Angles Theorem (transversal $\overline{BC}$)
5	$\angle DAB \cong \angle BCD$	Congruent Supplements Theorem

∎

Answer: See the model proof. (This is part of Euclid I.34: opposite angles of a parallelogram are equal; see Lesson 6.2.)

PRACTICE

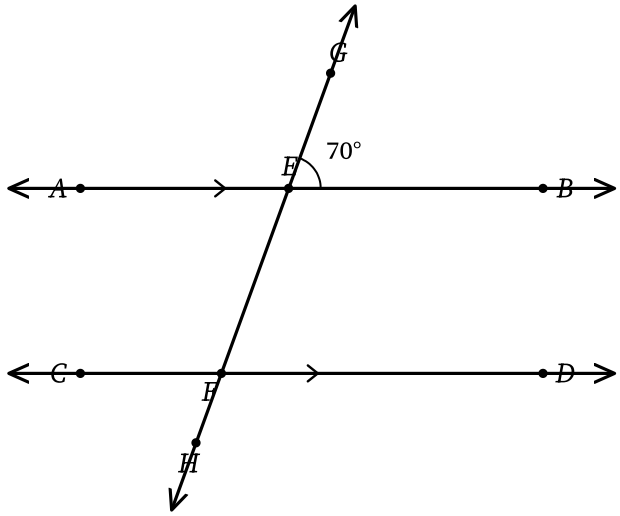
1. **Easy** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $m\angle BEG = 70°$. Find $m\angle DFE$ and name the postulate or theorem you used.



Show answer

$70°$, by the Corresponding Angles Postulate.

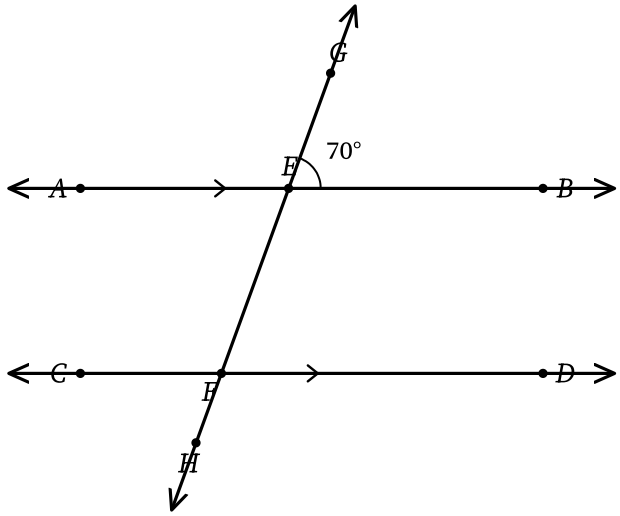
2. **Easy** In the same figure, find $m\angle CFE$.



Show answer

$110°$: $\angle CFE$ and $\angle DFE$ form a linear pair, and $m\angle DFE = 70°$.

3. **Easy** In the same figure, find $m\angle CFH$ and name the theorem that gives it directly from $m\angle BEG$.



Show answer

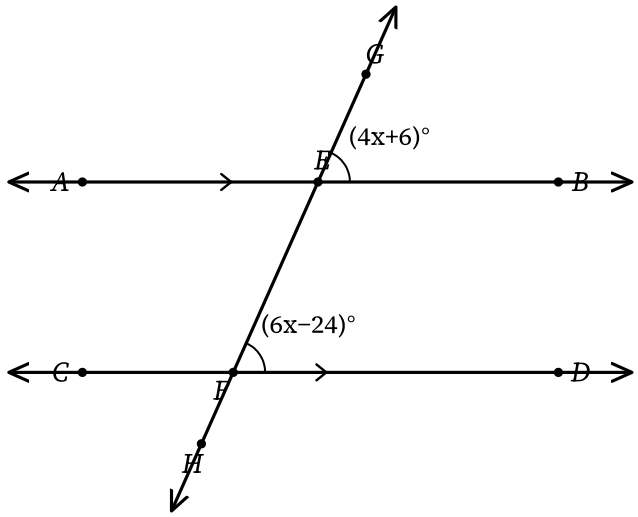
$70°$, by the Alternate Exterior Angles Theorem.

4. **Easy** Two straight rows of grapevines in the Miller vineyard are parallel, and a straight path crosses both rows. On one side of the path, the angle between the path and the first row, measured in the strip between the rows, is 128°. Find the angle between the path and the second row on the same side of the path, also measured in the strip between the rows.

Show answer

52°. The two angles are same-side interior angles of parallel lines, so they are supplementary: $180^\circ - 128^\circ = 52^\circ$.

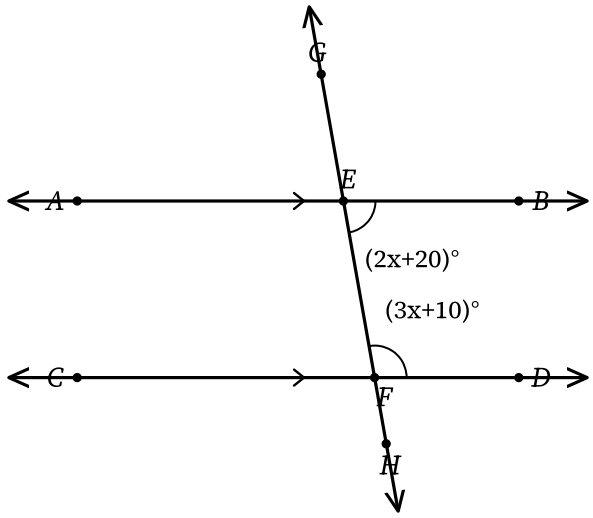
5. **Medium** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle BEG = (4x + 6)^\circ$ and $m\angle DFE = (6x - 24)^\circ$. Find x.



Show answer

$x = 15$. The angles are corresponding, so $4x + 6 = 6x - 24$, $30 = 2x$, $x = 15$ (each angle is 66°).

6. **Medium** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle BEF = (2x + 20)^\circ$ and $m\angle DFE = (3x + 10)^\circ$. Find x and the measures of both angles.



Show answer

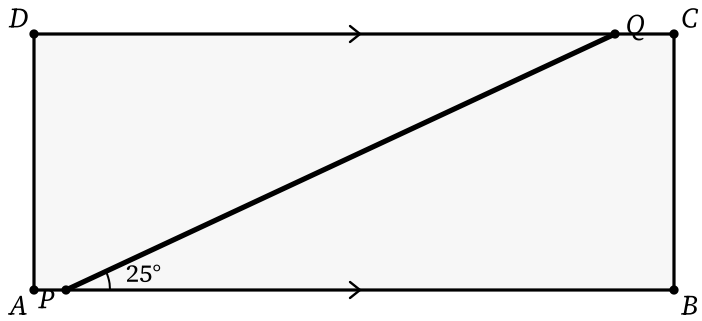
Same-side interior angles are supplementary: $(2x + 20) + (3x + 10) = 180$, so $5x = 150$ and $x = 30$. Then $m\angle BEF = 80^\circ$ and $m\angle DFE = 100^\circ$.

7. **Medium** The two long edges of a straight wooden plank are parallel. Henry saws straight across the plank at a slant. At one edge, the cut and the edge form a 58° angle on the plank's surface. At the other edge, what angle does the cut make with that edge, on the plank's surface and on the same side of the cut?

Show answer

122°. The two angles are same-side interior angles of the parallel edges with the cut as transversal, so they are supplementary: $180^\circ - 58^\circ = 122^\circ$.

8. **Medium** A farm gate has a straight top board $\overline{DC}$ parallel to its straight bottom board $\overline{AB}$. A diagonal brace $\overline{PQ}$ runs from P on the bottom board to Q on the top board, and $m\angle BPQ = 25^\circ$. Find $m\angle DQP$ and $m\angle CQP$.



Show answer

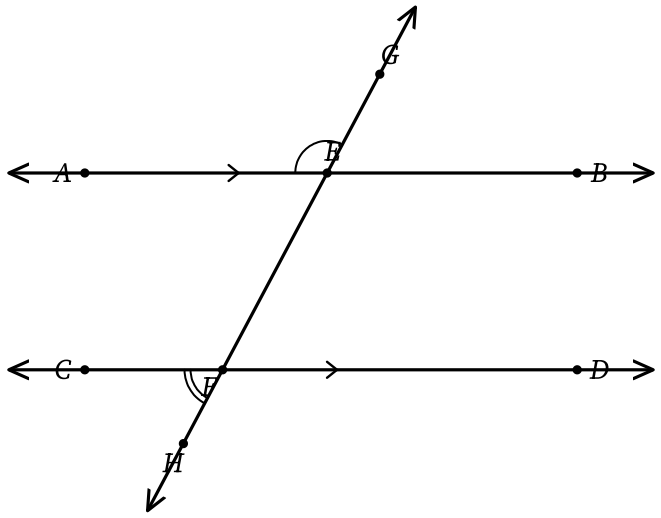
$m\angle DQP = 25^\circ$ (alternate interior angles with $\angle BPQ$); $m\angle CQP = 155^\circ$ (linear pair with $\angle DQP$).

9. **Medium** A transversal crosses two lines. On one side of the transversal the two interior angles measure 88° and 89°. According to Euclid's fifth postulate, must the two lines meet? If so, on which side of the transversal?

Show answer

Yes. $88^\circ + 89^\circ = 177^\circ$, which is less than two right angles (180°), so the lines meet on the side where these two angles are, even though they meet very far away.

10. **Medium** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Prove that the exterior angles $\angle AEG$ and $\angle CFH$, on the same side of the transversal, are supplementary.



Show answer

See the model proof.

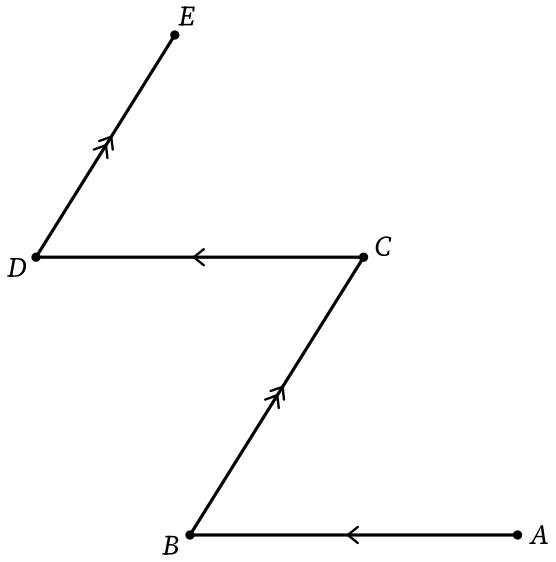
TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, cut by transversal $\overleftrightarrow{GH}$ at E and F

Prove: $\angle AEG$ and $\angle CFH$ are supplementary

	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given
2	$\angle AEG \cong \angle CFE$	Corresponding Angles Postulate
3	$m\angle AEG = m\angle CFE$	Definition of congruent angles
4	$\angle CFE$ and $\angle CFH$ are supplementary	Linear Pair Theorem ($\overrightarrow{FE}$ and $\overrightarrow{FH}$ are opposite rays)
5	$m\angle CFE + m\angle CFH = 180°$	Definition of supplementary angles
6	$m\angle AEG + m\angle CFH = 180°$	Substitution Property of Equality
7	$\angle AEG$ and $\angle CFH$ are supplementary	Definition of supplementary angles

11. **Hard** In the figure, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DE}$. Prove that $\angle ABC \cong \angle CDE$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \parallel \overline{CD}$; $\overline{BC} \parallel \overline{DE}$

Prove: $\angle ABC \cong \angle CDE$

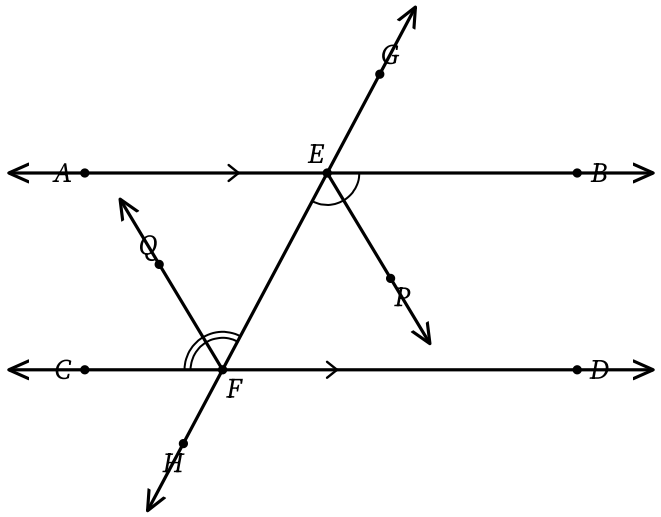
	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{CD}$	Given
2	$\angle ABC \cong \angle BCD$	Alternate Interior Angles Theorem (transversal $\overline{BC}$)
3	$\overline{BC} \parallel \overline{DE}$	Given
4	$\angle BCD \cong \angle CDE$	Alternate Interior Angles Theorem (transversal $\overline{CD}$)
5	$\angle ABC \cong \angle CDE$	Transitive Property of Congruence

12. Hard The Same-Side Interior Angles Theorem says same-side interior angles of parallel lines are *supplementary*, not *congruent*. Explain why they usually are not congruent, and describe the one situation in which they are congruent.

Show answer

Each same-side interior angle is the linear-pair partner of an angle congruent to the other, so the two add to 180°; usually one is acute and the other obtuse. They are congruent only when each is half of 180°, that is, 90°: when the transversal is perpendicular to the parallel lines.

13. Hard In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overrightarrow{EP}$ bisects $\angle BEF$ and $\overrightarrow{FQ}$ bisects $\angle CFE$. Prove that $\angle PEF \cong \angle QFE$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overrightarrow{EP}$ bisects $\angle BEF$; $\overrightarrow{FQ}$ bisects $\angle CFE$

Prove: $\angle PEF \cong \angle QFE$

Because $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, the alternate interior angles $\angle BEF$ and $\angle CFE$ are congruent by the Alternate Interior Angles Theorem, so $m\angle BEF = m\angle CFE$.

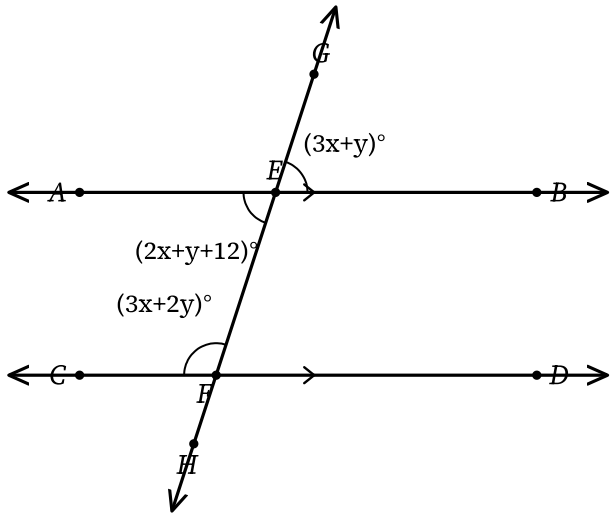
Since $\overrightarrow{EP}$ bisects $\angle BEF$, $m\angle BEP = m\angle PEF$ by the definition of angle bisector, and $m\angle BEP + m\angle PEF = m\angle BEF$ by the Angle Addition Postulate. Substituting gives $2 \cdot m\angle PEF = m\angle BEF$, so $m\angle PEF = \frac{1}{2} m\angle BEF$ by the Division Property of Equality.

In the same way, since $\overrightarrow{FQ}$ bisects $\angle CFE$, $m\angle QFE = \frac{1}{2} m\angle CFE$. Substituting $m\angle CFE = m\angle BEF$ gives $m\angle QFE = \frac{1}{2} m\angle BEF = m\angle PEF$.

Therefore $\angle PEF \cong \angle QFE$ by the definition of congruent angles.

∎

14. **Challenge** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle BEG = (3x + y)^\circ$, $m\angle AEF = (2x + y + 12)^\circ$ and $m\angle CFE = (3x + 2y)^\circ$. Find x and y.



Show answer

$\angle BEG$ and $\angle AEF$ are vertical angles, so $3x + y = 2x + y + 12$ and $x = 12$. $\angle BEG \cong \angle DFE$ (corresponding), and $\angle DFE$ and $\angle CFE$ form a linear pair, so $(3x + y) + (3x + 2y) = 180$, that is, $6x + 3y = 180$. With $x = 12$: $72 + 3y = 180$, so $y = 36$. Check: $3(12) + 36 = 72$, $2(12) + 36 + 12 = 72$, $3(12) + 2(36) = 108$, and $72 + 108 = 180$. So $x = 12$, $y = 36$.

COMMON MISTAKES

- Using these theorems when the lines are not known to be parallel. Look for the word "parallel" or arrowheads on the lines before writing any of them as a reason.
- Setting same-side interior angles equal instead of adding them to 180°. Check your answer: unless the transversal is perpendicular, one angle should be acute and the other obtuse.
- Naming the wrong pair, for example writing "Corresponding Angles Postulate" for alternate interior angles. Name the pair carefully, then the matching postulate or theorem.
- Stopping after finding x when the question asks for an angle. Substitute back, and check that both expressions give the same (or supplementary) measures.

Make it a game · Parallel Postulate Fibs. Write two true statements and one tall tale about the parallel postulate. Truths might be that Euclid's fifth postulate puzzled mathematicians for centuries, or that two lines drawn straight on a globe, as great circles, always meet. A tall tale might claim Playfair allows two parallels through a point. Your parent finds the fib and explains the correction.

3.3 Proving Lines Parallel

OBJECTIVES

- State and use the converses of the Corresponding Angles Postulate and of the Alternate Interior, Alternate Exterior and Same-Side Interior Angles Theorems.
- Prove that two lines parallel to the same line are parallel (Euclid I.30) and that two lines perpendicular to the same line are parallel.
- Write two-column and paragraph proofs that conclude two lines are parallel.
- Decide from angle measures whether two lines are parallel, and find values of a variable that make them parallel.

WATCH

Proving Parallel Lines (2 of 2: Using the properties of angles on parallel lines for proofs)

Eddie Woo · 10:15

https://www.youtube.com/watch?v=mUxapzNzMoU

> **WHEN TO WATCH**
> Before the proof examples: writing proofs with the angle facts for parallel lines.

Note: He states the converses informally; the course names each converse theorem and proves Euclid I.30.

BACKUP VIDEO

Parallel Lines & Angles: Converse Statements

Eddie Woo · 3:02

https://www.youtube.com/watch?v=85WFves6Vl8

Before the converse theorems: turning the parallel-line angle facts around.

KEY TERMS

converse

The statement formed by exchanging the hypothesis and conclusion of a conditional. The converse of a true statement may be false, so it needs its own proof or must be taken as a postulate.

auxiliary line

A line added to a figure to help with a proof or a calculation. Every auxiliary line must be justified: we must know that it exists and what it is (for example, by Playfair's Axiom, the one line through a point parallel to a given line).

Converse of the Corresponding Angles Postulate

If two lines are cut by a transversal so that a pair of corresponding angles is congruent, then the lines are parallel.

CORE IDEAS

In Lesson 3.2 we started with parallel lines and concluded facts about angles. Now we go the other way: we start with facts about angles and conclude that lines are parallel. Each new statement is the **converse** of an old one. Recall from Lesson 1.3 that a converse is a new claim: it is not automatically true, so each one needs its own justification.

We take the first one as a postulate: **if corresponding angles are congruent, the lines are parallel.** It matches Euclid's I.28. Euclid actually proves it without the fifth postulate, using the Exterior Angle Inequality (I.16, Lesson 4.1); we accept it as a postulate so that we can use it now.

From that postulate, vertical angles and linear pairs, the other converses follow, just as the theorems of Lesson 3.2 followed from the Corresponding Angles Postulate. Two lines cut by a transversal are parallel if any one of these is true: corresponding angles are congruent; alternate interior angles are congruent (Euclid I.27); alternate exterior angles are congruent (you will prove this in Practice Problem 5); or same-side interior angles are supplementary (I.28). Notice that one pair is enough.

Two useful results come next. First, **two lines parallel to the same line are parallel to each other** (Euclid I.30). The reason is Playfair's Axiom: if the two lines met at a point, there would be two different parallels to the third line through that one point, which is impossible. Second, **in a plane, two lines perpendicular to the same line are parallel**: the two right angles are corresponding angles, and all right angles are congruent. A carpenter uses this every time she sets studs square to the same floor plate: the studs come out parallel.

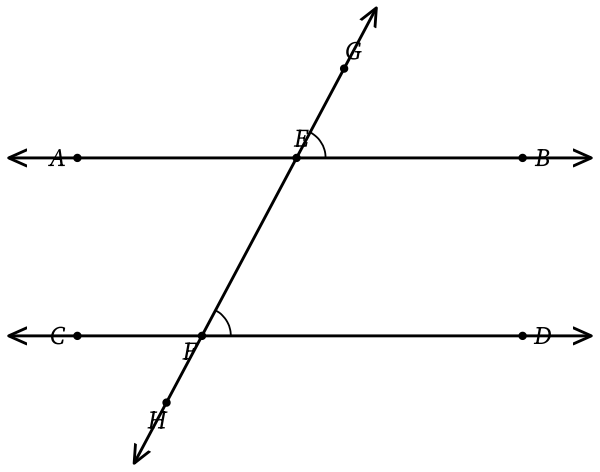
To prove two lines parallel, find a transversal and a pair of angles it forms with *those two lines*. In a figure with many lines this takes care: the pair $\angle ABD$ and $\angle CDB$, with transversal $\overleftrightarrow{BD}$, tells you about $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, the lines that contain the other sides of the two angles. Then show the pair is congruent (or supplementary, for same-side interior angles) and cite the matching converse.

The theorems of Lesson 3.2 also tell us when lines are **not** parallel. For example, if same-side interior angles add to something other than $180°$, the lines cannot be parallel, because if they were, the Same-Side Interior Angles Theorem would make the sum $180°$. (This is the contrapositive of that theorem, Lesson 1.3.) Surveyors use exactly this check.

POSTULATES AND THEOREMS

POSTULATE **Converse of the Corresponding Angles Postulate** Euclid I.28

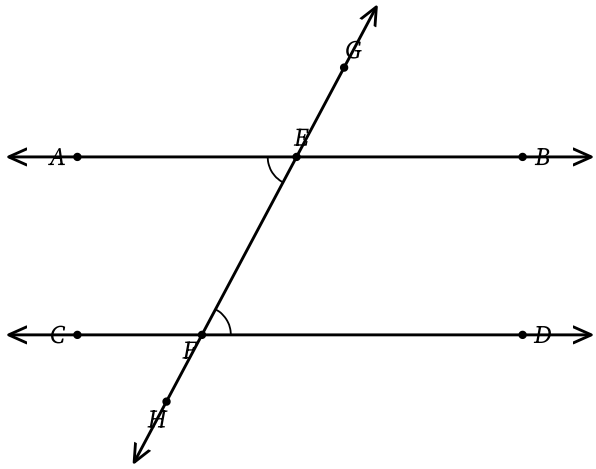
If two lines are cut by a transversal so that a pair of corresponding angles is congruent, then the lines are parallel.



Euclid proves the alternate-angle form (I.27) by contradiction, using the Exterior Angle Inequality (I.16), and derives this statement in I.28; none of it needs the fifth postulate. We take it as a postulate in this course.

THEOREM **Converse of the Alternate Interior Angles Theorem** Euclid I.27

If two lines are cut by a transversal so that a pair of alternate interior angles is congruent, then the lines are parallel.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{GH}$ cuts $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F; $\angle AEF \cong \angle DFE$

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

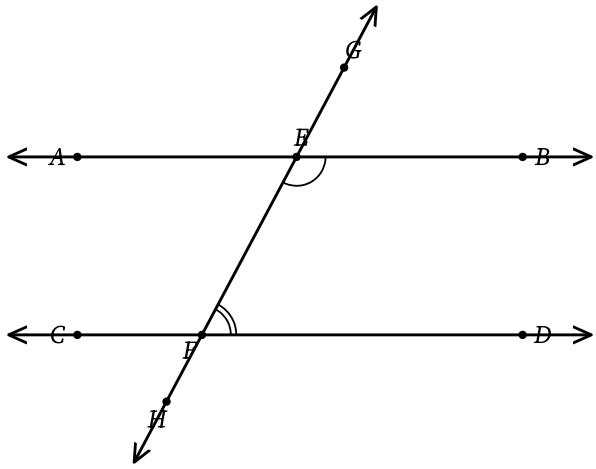
	STATEMENTS	REASONS
1	$\angle AEF \cong \angle DFE$	Given
2	$\angle BEG \cong \angle AEF$	Vertical Angles Theorem
3	$\angle BEG \cong \angle DFE$	Transitive Property of Congruence
4	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Converse of the Corresponding Angles Postulate ($\angle BEG$ and $\angle DFE$ are corresponding angles)

∎

Euclid I.27: "If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another."

THEOREM **Converse of the Same-Side Interior Angles Theorem** Euclid I.28

If two lines are cut by a transversal so that a pair of same-side interior angles is supplementary, then the lines are parallel.



PARAGRAPH PROOF

Given: $\overleftrightarrow{GH}$ cuts $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F; $\angle BEF$ and $\angle DFE$ are supplementary

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

We are given that $\angle BEF$ and $\angle DFE$ are supplementary. Also, $\angle BEF$ and $\angle BEG$ form a linear pair, because $\overrightarrow{EF}$ and $\overrightarrow{EG}$ are opposite rays; so they are supplementary by the Linear Pair Theorem.

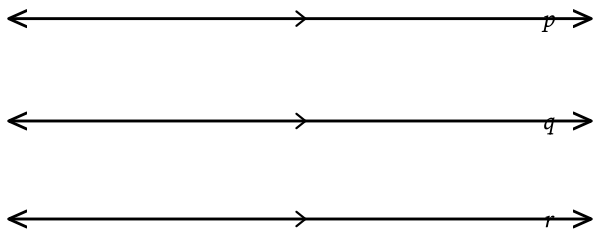
Now $\angle DFE$ and $\angle BEG$ are both supplementary to the same angle, $\angle BEF$. By the Congruent Supplements Theorem, $\angle DFE \cong \angle BEG$.

But $\angle BEG$ and $\angle DFE$ are corresponding angles. By the Converse of the Corresponding Angles Postulate, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

∎

THEOREM **Lines Parallel to the Same Line** Euclid I.30

In a plane, if two different lines are each parallel to a third line, then they are parallel to each other.



PARAGRAPH PROOF

Given: Lines p, q and r lie in one plane; $p \neq q$; $p \parallel r$ and $q \parallel r$

Prove: $p \parallel q$

Suppose p and q had a point X in common. Since X is on p and p does not meet r, the point X is not on r.

Then p and q would be two different lines through X, each parallel to r. Playfair's Axiom says there is exactly one line through X parallel to r. So p and q cannot have a point in common.

Lines in one plane that do not meet are parallel by definition, so $p \parallel q$.

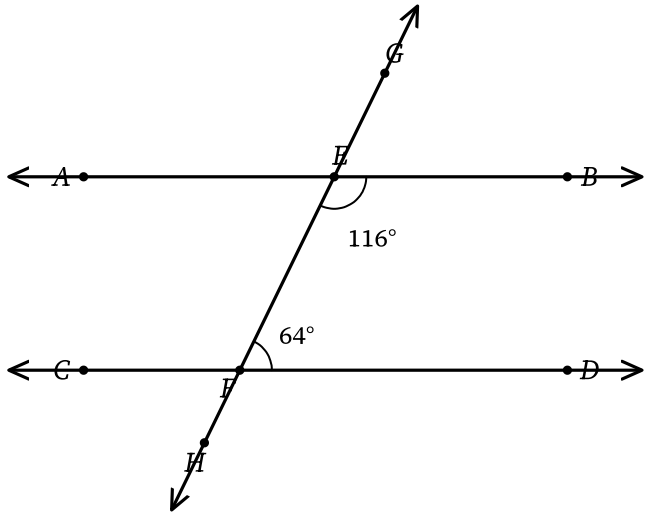
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Euclid's own proof of I.30 uses a transversal and I.29; this shorter proof uses Playfair's Axiom. Reasoning of the form "if they met, something impossible would follow" is called indirect proof; you will study it carefully in Lesson 5.5.

Euclid connection. Euclid proves the tests for parallel lines, I.27 and I.28, before he ever uses the fifth postulate; Post. 5 is needed only for the other direction, I.29 (Lesson 3.2). Then I.30 shows that "straight lines parallel to the same straight line are also parallel to one another," and I.31 (Lesson 3.4) constructs the parallel through a given point.

Practice habit. Before you write "the lines are parallel," say aloud which two lines and which transversal form your angle pair. Then check the direction of your reason: from angles to parallel lines you need a *converse* (this lesson), not a theorem from Lesson 3.2.

Example 1. Henry is building a set of shelves. A straight diagonal brace $\overleftrightarrow{GH}$ crosses the edge of the upper shelf, $\overleftrightarrow{AB}$, at E and the edge of the lower shelf, $\overleftrightarrow{CD}$, at F. He measures $m\angle BEF = 116°$ and $m\angle DFE = 64°$. Are the shelf edges parallel? Name the theorem that decides it.

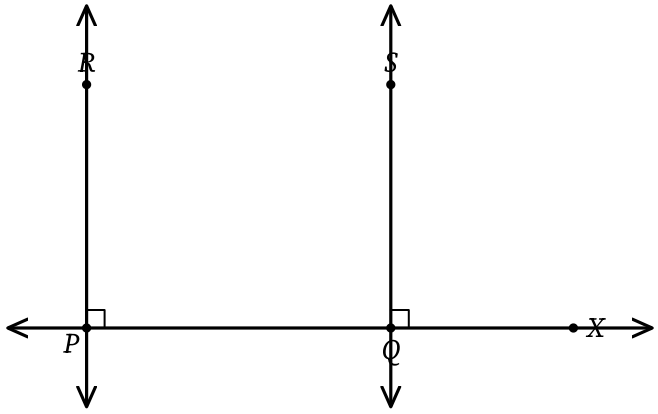


Show solution

1. Identify the pair: $\angle BEF$ and $\angle DFE$ are between the shelf edges, both to the right of the brace. They are same-side interior angles.
2. Add: $116° + 64° = 180°$, so the angles are supplementary.
3. By the Converse of the Same-Side Interior Angles Theorem, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Answer: Yes. The same-side interior angles are supplementary ($116° + 64° = 180°$), so the edges are parallel by the Converse of the Same-Side Interior Angles Theorem.

Example 2. Prove: In a plane, if two lines are perpendicular to the same line, then they are parallel. Use the figure, in which $\overleftrightarrow{PR} \perp \overleftrightarrow{PQ}$, $\overleftrightarrow{QS} \perp \overleftrightarrow{PQ}$, and X is a point of $\overleftrightarrow{PQ}$ beyond Q.



Show solution

1. Work backward: to prove $\overleftrightarrow{PR} \parallel \overleftrightarrow{QS}$, find a pair of angles that $\overleftrightarrow{PQ}$ (the transversal) forms with them.
2. $\angle RPQ$ and $\angle SQX$ are corresponding angles: both above $\overleftrightarrow{PQ}$, on the side toward X.
3. Both are right angles, and all right angles are congruent (Lesson 2.6). So the Converse of the Corresponding Angles Postulate finishes the proof.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{PR} \perp \overleftrightarrow{PQ}$; $\overleftrightarrow{QS} \perp \overleftrightarrow{PQ}$

Prove: $\overleftrightarrow{PR} \parallel \overleftrightarrow{QS}$

	STATEMENTS	REASONS
1	$\overleftrightarrow{PR} \perp \overleftrightarrow{PQ}$ and $\overleftrightarrow{QS} \perp \overleftrightarrow{PQ}$	Given
2	$\angle RPQ$ and $\angle SQX$ are right angles	Definition of perpendicular lines
3	$\angle RPQ \cong \angle SQX$	Right Angle Congruence Theorem (all right angles are congruent)

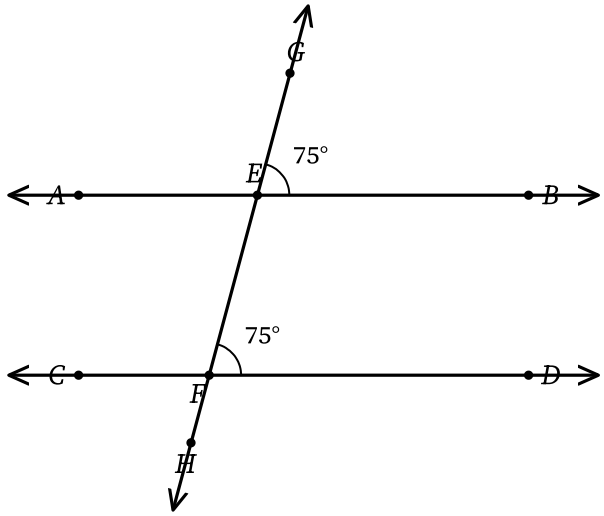
	STATEMENTS	REASONS
4	$\overleftrightarrow{PR} \parallel \overleftrightarrow{QS}$	Converse of the Corresponding Angles Postulate

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Answer: See the model proof. (Euclid gets the same result from I.28: the two interior angles on the same side are two right angles.)

PRACTICE

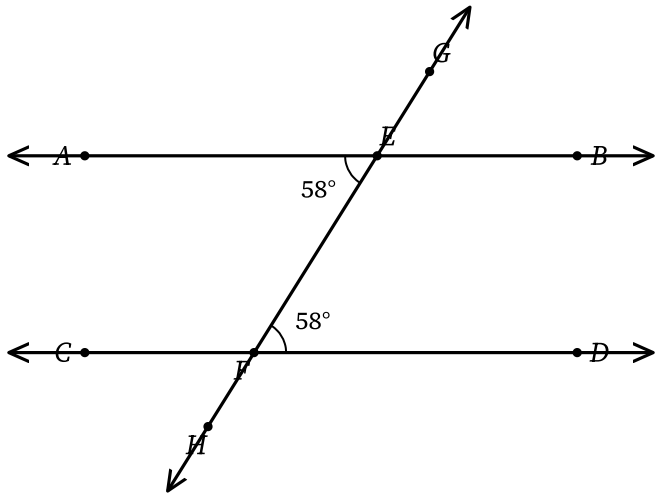
1. **Easy** In the figure, $m\angle BEG = 75°$ and $m\angle DFE = 75°$. Is $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$? Name the postulate or theorem.



Show answer

Yes, by the Converse of the Corresponding Angles Postulate.

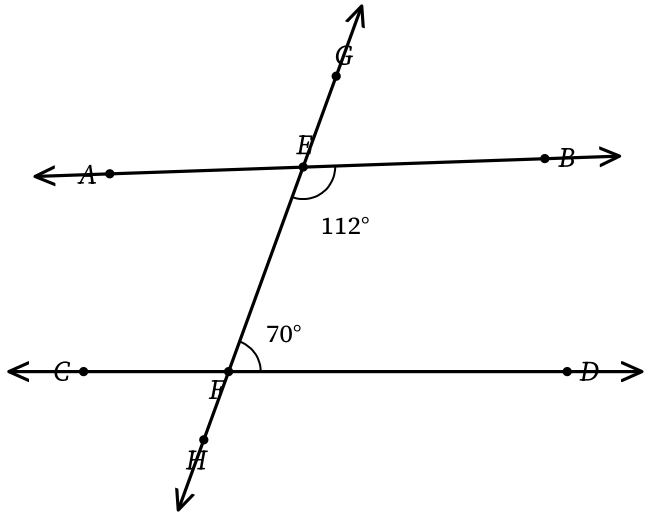
2. **Easy** In the figure, $m\angle AEF = 58^\circ$ and $m\angle DFE = 58^\circ$. Is $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$? Name the theorem.



Show answer

Yes, by the Converse of the Alternate Interior Angles Theorem.

3. **Easy** A surveyor checks whether two straight country roads, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, are parallel. A straight lane $\overleftrightarrow{GH}$ crosses them at E and F, and she measures $m\angle BEF = 112^\circ$ and $m\angle DFE = 70^\circ$. Are the roads parallel? Explain.



Show answer

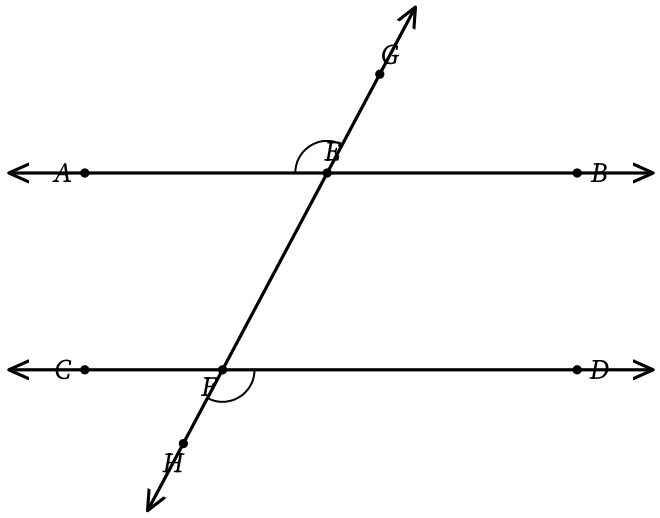
No. $\angle BEF$ and $\angle DFE$ are same-side interior angles, and $112^\circ + 70^\circ = 182^\circ \neq 180^\circ$. If the roads were parallel, the Same-Side Interior Angles Theorem would make the sum 180°.

4. **Easy** Grace fastens two shelves to a straight vertical upright so that the edge of each shelf is perpendicular to the upright. Must the two shelf edges be parallel? Name the theorem.

Show answer

Yes: in a plane, two lines perpendicular to the same line are parallel.

5. **Medium** Prove the Converse of the Alternate Exterior Angles Theorem: in the figure, if $\angle AEG \cong \angle DFH$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

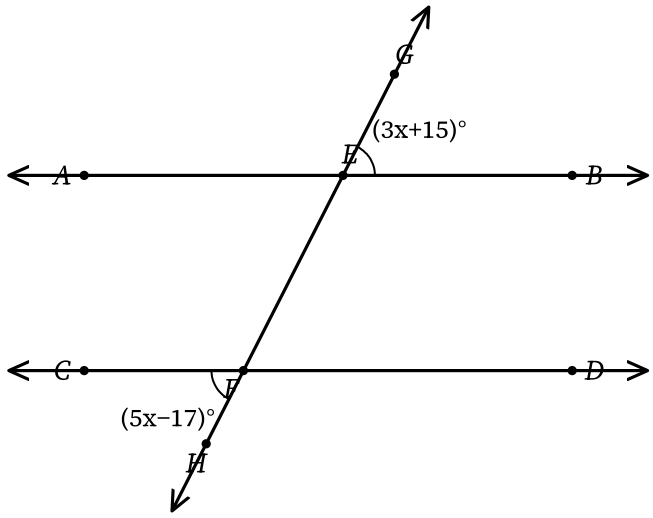
Given: $\overleftrightarrow{GH}$ cuts $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F; $\angle AEG \cong \angle DFH$

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

	STATEMENTS	REASONS
1	$\angle AEG \cong \angle DFH$	Given
2	$\angle DFH \cong \angle CFE$	Vertical Angles Theorem
3	$\angle AEG \cong \angle CFE$	Transitive Property of Congruence
4	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Converse of the Corresponding Angles Postulate ($\angle AEG$ and $\angle CFE$ are corresponding angles)

∎

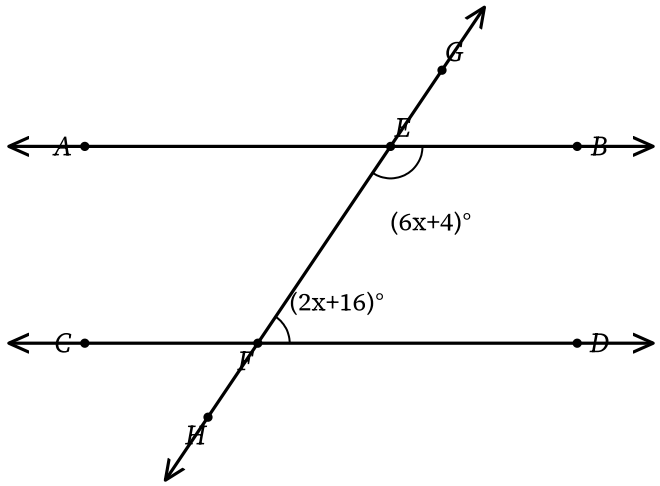
6. **Medium** In the figure, $m\angle BEG = (3x + 15)^\circ$ and $m\angle CFH = (5x - 17)^\circ$. What value of x makes $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?



Show answer

$x = 16$. The angles are alternate exterior angles, so the lines are parallel when $3x + 15 = 5x - 17$, that is, $32 = 2x$, $x = 16$ (each angle is then 63°).

7. **Medium** In the figure, $m\angle BEF = (6x + 4)^\circ$ and $m\angle DFE = (2x + 16)^\circ$. What value of x makes $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?



Show answer

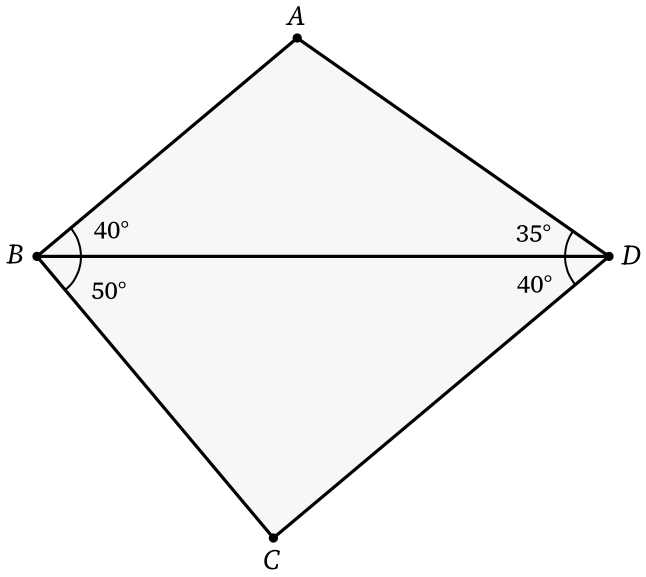
$x = 20$. Same-side interior angles must be supplementary: $(6x + 4) + (2x + 16) = 180$, so $8x = 160$ and $x = 20$ (the angles are 124° and 56°).

8. **Medium** Samuel plants two straight rows of beans, each parallel to the straight garden fence, and the two rows are different lines. Must the two rows be parallel to each other? Name the theorem.

Show answer

Yes: two lines parallel to the same line are parallel to each other (Euclid I.30).

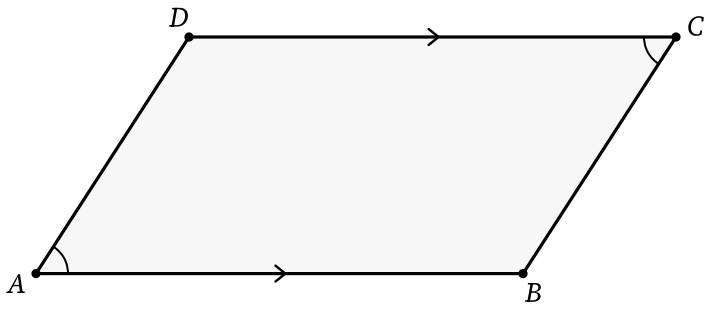
9. **Medium** In quadrilateral $ABCD$ with diagonal $\overline{BD}$, $m\angle ABD = 40°$, $m\angle CDB = 40°$, $m\angle ADB = 35°$ and $m\angle CBD = 50°$. (a) Is $\overline{AB} \parallel \overline{DC}$? (b) Is $\overline{AD} \parallel \overline{BC}$? Explain each answer.



Show answer

(a) Yes. With transversal $\overline{BD}$, $\angle ABD$ and $\angle CDB$ are alternate interior angles for $\overline{AB}$ and $\overline{DC}$, and they are congruent, so $\overline{AB} \parallel \overline{DC}$ (Converse of the Alternate Interior Angles Theorem). (b) No. $\angle ADB$ and $\angle CBD$ are the alternate interior angles for $\overline{AD}$ and $\overline{BC}$; if those segments were parallel, the angles would be congruent, but $35° \neq 50°$.

10. **Hard** In quadrilateral $ABCD$, $\overline{AB} \parallel \overline{DC}$ and $\angle DAB \cong \angle BCD$. Prove that $\overline{AD} \parallel \overline{BC}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

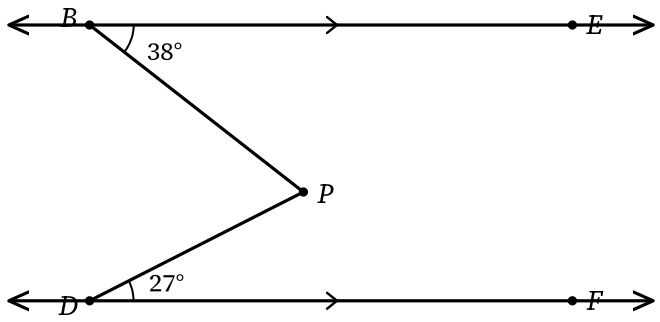
Given: $\overline{AB} \parallel \overline{DC}$; $\angle DAB \cong \angle BCD$

Prove: $\overline{AD} \parallel \overline{BC}$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$	Given
2	$\angle DAB$ and $\angle ADC$ are supplementary	Same-Side Interior Angles Theorem (transversal $\overline{AD}$)
3	$m\angle DAB + m\angle ADC = 180°$	Definition of supplementary angles
4	$\angle DAB \cong \angle BCD$	Given
5	$m\angle DAB = m\angle BCD$	Definition of congruent angles
6	$m\angle BCD + m\angle ADC = 180°$	Substitution Property of Equality
7	$\angle BCD$ and $\angle ADC$ are supplementary	Definition of supplementary angles
8	$\overline{AD} \parallel \overline{BC}$	Converse of the Same-Side Interior Angles Theorem (transversal $\overline{DC}$)

∎

11. Hard In the figure, $\overleftrightarrow{BE} \parallel \overleftrightarrow{DF}$, point P lies between the two lines, $m\angle EBP = 38^\circ$ and $m\angle FDP = 27^\circ$. Find $m\angle BPD$. (Hint: draw an auxiliary line through P parallel to $\overleftrightarrow{BE}$, and justify it.)



Show answer

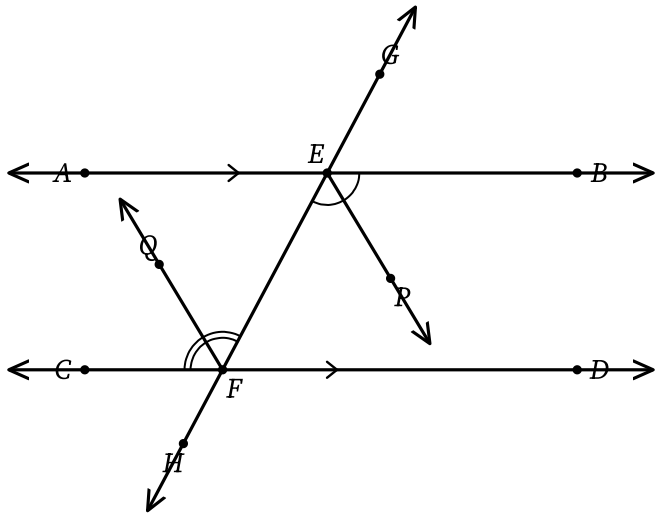
65°. By Playfair's Axiom there is exactly one line through P parallel to $\overleftrightarrow{BE}$; mark a point X on it to the left of P. This line is also parallel to $\overleftrightarrow{DF}$, since two lines parallel to the same line are parallel (I.30). With transversal $\overline{BP}$, $\angle BPX$ and $\angle EBP$ are alternate interior angles, so $m\angle BPX = 38^\circ$; with transversal $\overline{DP}$, $m\angle XPD = m\angle FDP = 27^\circ$. By the Angle Addition Postulate, $m\angle BPD = 38^\circ + 27^\circ = 65^\circ$.

12. Hard Ruth says, "If two lines are cut by a transversal and a pair of same-side interior angles are congruent, then the lines are parallel." Is she right? Explain, and give a counterexample if she is not.

Show answer

No. The converse requires the same-side interior angles to be *supplementary*. Counterexample: same-side interior angles of 80° and 80° are congruent, but they add to $160^\circ < 180^\circ$, so by Euclid's fifth postulate the lines meet on that side. Congruent same-side interior angles give parallel lines only when both are 90°.

13. **Challenge** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overrightarrow{EP}$ bisects $\angle BEF$ and $\overrightarrow{FQ}$ bisects $\angle CFE$. Prove that $\overleftrightarrow{EP} \parallel \overleftrightarrow{FQ}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overrightarrow{EP}$ bisects $\angle BEF$; $\overrightarrow{FQ}$ bisects $\angle CFE$

Prove: $\overleftrightarrow{EP} \parallel \overleftrightarrow{FQ}$

	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given
2	$\angle BEF \cong \angle CFE$	Alternate Interior Angles Theorem
3	$m\angle BEF = m\angle CFE$	Definition of congruent angles
4	$\overrightarrow{EP}$ bisects $\angle BEF$; $\overrightarrow{FQ}$ bisects $\angle CFE$	Given
5	$m\angle BEP = m\angle PEF$ and $m\angle CFQ = m\angle QFE$	Definition of angle bisector
6	$m\angle BEP + m\angle PEF = m\angle BEF$ and $m\angle CFQ + m\angle QFE = m\angle CFE$	Angle Addition Postulate
7	$2 \cdot m\angle PEF = m\angle BEF$ and $2 \cdot m\angle QFE = m\angle CFE$	Substitution Property of Equality

	STATEMENTS	REASONS
8	$2 \cdot m\angle PEF = 2 \cdot m\angle QFE$	Substitution Property of Equality
9	$m\angle PEF = m\angle QFE$	Division Property of Equality
10	$\angle PEF \cong \angle QFE$	Definition of congruent angles
11	$\overleftrightarrow{EP} \parallel \overleftrightarrow{FQ}$	Converse of the Alternate Interior Angles Theorem (transversal $\overleftrightarrow{EF}$)

∎

COMMON MISTAKES

- Writing a Lesson 3.2 theorem ("Alternate Interior Angles Theorem") as the reason for concluding that lines are parallel. That theorem starts with parallel lines; to conclude parallel lines you need its converse.
- Using an angle pair that belongs to the wrong two lines. Before citing a converse, name the transversal and check that the two angles' other sides lie on the two lines you want to prove parallel.
- Concluding "parallel" from same-side interior angles that are congruent. They must be supplementary.
- Deciding lines are parallel because they look parallel. Only the given information and the converses can show it; Problem 3's roads look nearly parallel but are not.

> **Make it a game · Converse Match-Up.** Make eight cards: four theorems about parallel lines, such as 'If lines are parallel, alternate interior angles are congruent,' and their four converses. Lay them face down, flip two at a time, and keep a pair only if one is the converse of the other. Say which direction of reasoning proves lines parallel.

3.4 Perpendicular Lines and Distance

OBJECTIVES

- Prove that perpendicular lines form four right angles and that a linear pair of congruent angles makes perpendicular lines.
- Prove and use the Perpendicular Transversal Theorem and the fact that the perpendicular segment is the shortest segment from a point to a line.
- Find the distance from a point to a line and between two parallel lines.
- Construct a parallel through a given point (Euclid I.31) and a perpendicular from a point to a line (I.12) with compass and straightedge.

WATCH

How to Construct Parallel Lines: Demonstration (Step-by-Step)

Eddie Woo · 9:46

https://www.youtube.com/watch?v=qcktB5gRVS8

> **WHEN TO WATCH**
> With the construction of a parallel through a point: follow the compass steps.

Note: The proof of the Shortest Distance Theorem is taught from the lesson.

BACKUP VIDEO

Constructing Perpendicular from Point to Line (Geometry)

Mario's Math Tutoring · 2:16

https://www.youtube.com/watch?v=CFefKshkSdI

With the perpendicular from a point not on a line: the compass steps, and why that segment gives the shortest distance.

KEY TERMS

perpendicular lines

Two lines that intersect to form right angles. We write $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$. Segments and rays are perpendicular when the lines containing them are.

foot of the perpendicular

The point F where the perpendicular from a point P to a line ℓ meets ℓ.

distance from a point to a line

The length of the perpendicular segment from the point to the line. (If the point is on the line, the distance is 0.)

distance between parallel lines

The length of a segment that is perpendicular to both parallel lines and has one endpoint on each.

CORE IDEAS

Two intersecting lines form four angles. If even one of them is a right angle, all four are: the angle across from it is a vertical angle, so it is $90°$, and the two neighbors each form a linear pair with it, so each is $180° - 90° = 90°$. That is why a carpenter needs to check only one corner where two boards cross.

Going the other way: if two lines form a linear pair of congruent angles, the lines are perpendicular. Two equal angles that together make $180°$ must each be $90°$. Fold a sheet of paper so that one edge lies exactly along itself, and the crease makes two equal angles along the edge; so the crease is perpendicular to the edge.

Perpendiculars and parallels work together. **Perpendicular Transversal Theorem:** a line perpendicular to one of two parallel lines is perpendicular to the other too, because the corresponding angle on the other line is also $90°$. And from Lesson 3.3, two lines perpendicular to the same line are parallel. A carpenter who sets every stud square to the floor plate gets parallel studs, and each stud is then square to the top plate as well, if the plates are parallel.

How far is a point from a line? There are many segments from point P to line ℓ, but the perpendicular one, $\overline{PF}$, is the **shortest**. The reason: for any other point Q on ℓ, triangle PFQ has a right angle at F, so by the Pythagorean Theorem $PQ^2 = PF^2 + FQ^2$, which is more than PF^2. So we define the **distance from a point to a line** as the length of the perpendicular segment. A farmer laying the shortest pipe from a well to a ditch, or a swimmer heading straight to shore, uses the perpendicular.

The **distance between two parallel lines** is the length of a segment perpendicular to both (by the Perpendicular Transversal Theorem, a segment perpendicular to one is perpendicular to the other). Every such segment has the same length; we will prove this in Unit 6, where these segments become opposite sides of a rectangle. That is why a road or a canal with parallel edges has one width.

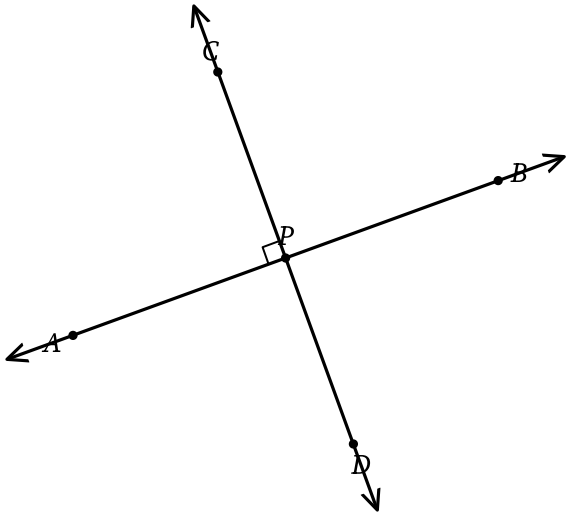
With compass and straightedge you can now draw a parallel through any point (Euclid I.31): draw any line through the point that crosses the given line, then copy the angle it makes as a corresponding angle. The Converse of the Corresponding Angles Postulate guarantees the new line is parallel, and Playfair's Axiom says it is the only one. To find the distance from a point to a line, construct the perpendicular from the point (I.12, first seen in Lesson 2.5) and measure it.

Builders check right angles with the numbers 3, 4, 5. Mark 3 ft along one board and 4 ft along the other from the corner; if the corner is square, the diagonal between the marks is exactly 5 ft, because $3^2 + 4^2 = 5^2$ and the converse of the Pythagorean Theorem (Algebra I; proved in Lesson 9.2) says the angle is right exactly when that equation holds. Surveyors lay out the corners of fields the same way, with longer tapes.

POSTULATES AND THEOREMS

THEOREM **Perpendicular Lines Form Four Right Angles**

If two lines intersect to form one right angle, then they form four right angles.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at P; $\angle APC$ is a right angle

Prove: $\angle CPB$, $\angle BPD$ and $\angle DPA$ are right angles

	STATEMENTS	REASONS
1	$\angle APC$ is a right angle	Given
2	$m\angle APC = 90^\circ$	Definition of right angle

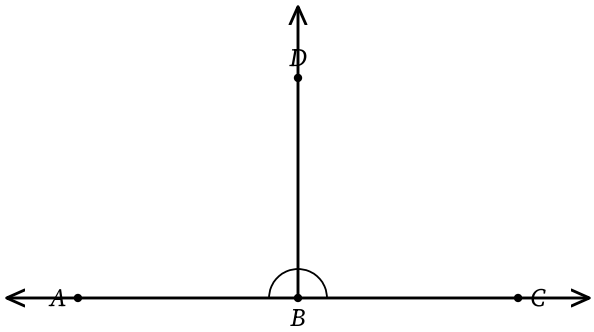
	STATEMENTS	REASONS
3	$\angle APC$ and $\angle CPB$ are supplementary	Linear Pair Theorem ($\overrightarrow{PA}$ and $\overrightarrow{PB}$ are opposite rays)
4	$m\angle APC + m\angle CPB = 180°$	Definition of supplementary angles
5	$90° + m\angle CPB = 180°$	Substitution Property of Equality
6	$m\angle CPB = 90°$	Subtraction Property of Equality
7	$\angle BPD \cong \angle APC$ and $\angle DPA \cong \angle CPB$	Vertical Angles Theorem
8	$m\angle BPD = 90°$ and $m\angle DPA = 90°$	Definition of congruent angles and Substitution Property of Equality
9	$\angle CPB$, $\angle BPD$ and $\angle DPA$ are right angles	Definition of right angle

∎

So to show two lines are perpendicular it is enough to show that one of the four angles is a right angle.

THEOREM **Congruent Linear Pair Theorem**

If two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.



PARAGRAPH PROOF

Given: $\angle ABD$ and $\angle DBC$ form a linear pair; $\angle ABD \cong \angle DBC$

Prove: $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$

Because $\angle ABD$ and $\angle DBC$ form a linear pair, they are supplementary by the Linear Pair Theorem, so $m\angle ABD + m\angle DBC = 180°$.

Because they are congruent, $m\angle ABD = m\angle DBC$. Substituting gives $2 \cdot m\angle ABD = 180°$, and by the Division Property of Equality, $m\angle ABD = 90°$.

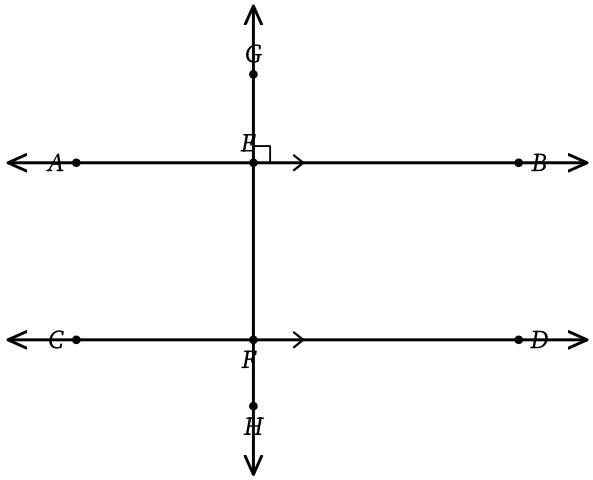
So $\angle ABD$ is a right angle, and the lines $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ that form it are perpendicular by the definition of perpendicular lines (and by the previous theorem all four angles are right).

∎

Euclid's definition of a right angle (Book I, Definition 10) is exactly this: when a line standing on a line makes the adjacent angles equal, each of the equal angles is right.

THEOREM **Perpendicular Transversal Theorem** Euclid I.29

In a plane, if a line is perpendicular to one of two parallel lines, then it is perpendicular to the other.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overleftrightarrow{GH} \perp \overleftrightarrow{AB}$ at E, and $\overleftrightarrow{GH}$ meets $\overleftrightarrow{CD}$ at F

Prove: $\overleftrightarrow{GH} \perp \overleftrightarrow{CD}$

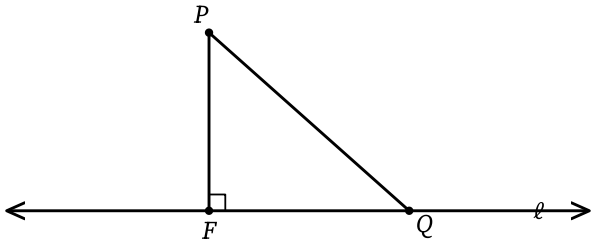
	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given
2	$\overleftrightarrow{GH} \perp \overleftrightarrow{AB}$	Given
3	$\angle BEG$ is a right angle	Definition of perpendicular lines
4	$\angle BEG \cong \angle DFE$	Corresponding Angles Postulate
5	$m\angle DFE = 90°$	Definition of congruent angles and definition of right angle
6	$\overleftrightarrow{GH} \perp \overleftrightarrow{CD}$	Definition of perpendicular lines ($\angle DFE$ is a right angle)

∎

A line that crosses one of two parallel lines must cross the other: otherwise it would be a second line through E parallel to $\overleftrightarrow{CD}$, which Playfair's Axiom forbids. That is why F exists.

THEOREM **Shortest Distance Theorem**

The perpendicular segment from a point to a line is shorter than any other segment from the point to the line.



PARAGRAPH PROOF

Given: Point P is not on line ℓ; $\overline{PF} \perp \ell$ with F on ℓ; Q is any point of ℓ other than F

Prove: $PF < PQ$

Since $\overline{PF} \perp \ell$, $\angle PFQ$ is a right angle, so $\triangle PFQ$ is a right triangle with hypotenuse $\overline{PQ}$. By the Pythagorean Theorem, $PQ^2 = PF^2 + FQ^2$.

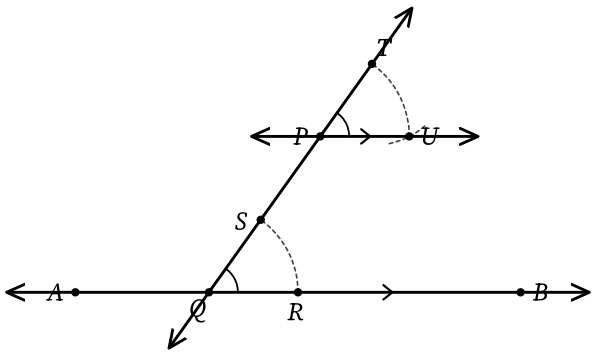
Because $Q \neq F$, the length FQ is positive, so $FQ^2 > 0$. Therefore $PQ^2 > PF^2$.

Both lengths are positive, and for positive numbers the larger square belongs to the larger number, so $PQ > PF$. Since Q was any point of ℓ other than F, $\overline{PF}$ is the shortest segment from P to ℓ.

∎

This proof uses the Pythagorean Theorem, which you know from Algebra I; it is proved in Lesson 9.1, and that proof does not depend on this theorem. In Lesson 5.5 you will see a second proof that compares the angles of $\triangle PFQ$.

CONSTRUCTION **Construct the line through a given point parallel to a given line** Euclid I.31

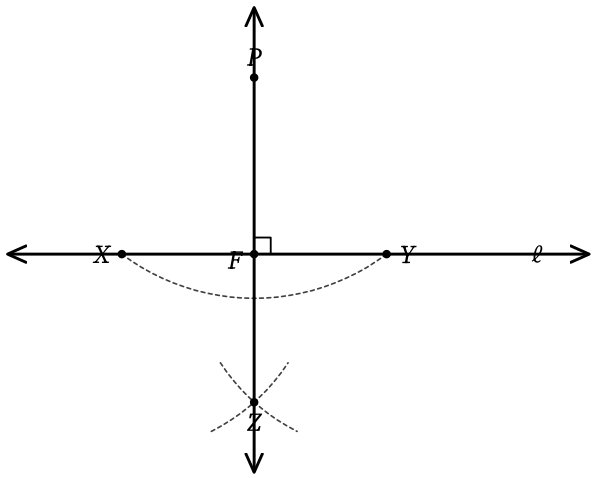


1. Let ℓ be the given line (through A and B) and P the point not on it. With the straightedge, draw a line through P that crosses ℓ at a point Q.
2. Put the compass point at Q and, with any convenient width, draw an arc that crosses ℓ at R (on the side toward B) and crosses $\overrightarrow{QP}$ at S.
3. Keeping the same compass width, put the compass point at P and draw an arc that crosses $\overrightarrow{QP}$ beyond P at T and continues toward the side where R is.
4. Set the compass to the distance RS. With the compass point at T, draw an arc that crosses the arc of Step 3 at U, on the same side of $\overleftrightarrow{QP}$ as R.
5. Draw $\overleftrightarrow{PU}$. It is the line through P parallel to ℓ.

Why it works. Steps 2–4 copy $\angle RQS$ to $\angle UPT$ (the angle copy of Euclid I.23, which is justified

by congruent triangles in Lesson 4.5). So $\angle UPT \cong \angle RQP$, and these are corresponding angles for $\overleftrightarrow{PU}$ and ℓ with transversal $\overleftrightarrow{QT}$. By the Converse of the Corresponding Angles Postulate, $\overleftrightarrow{PU} \parallel \ell$, and by Playfair's Axiom it is the only such line. Euclid's own I.31 copies the angle in the alternate position and uses I.27; the idea is the same.

CONSTRUCTION **Construct the perpendicular from a point to a line** Euclid I.12



1. Put the compass point at P and draw an arc that crosses line ℓ at two points, X and Y.
2. With the compass point at X and a width greater than half of XY, draw an arc on the side of ℓ opposite P.
3. With the same width and the compass point at Y, draw an arc that crosses the arc of Step 2 at Z.
4. Draw $\overleftrightarrow{PZ}$. It meets ℓ at F, and $\overline{PF} \perp \ell$. The length PF is the distance from P to ℓ.

Why it works. P is the same distance from X and Y (both lie on the first arc), and so is Z (Steps 2 and 3 use one width).

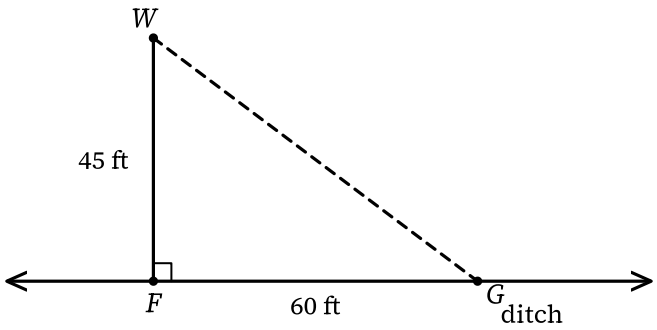
Points equidistant from X and Y lie on the perpendicular bisector of $\overline{XY}$, which you will prove with congruent triangles in Lessons 4.5 and 5.1. So $\overleftrightarrow{PZ}$ is that perpendicular bisector and is perpendicular to ℓ. This is Euclid I.12.

From history. Roman land surveyors used an instrument called a groma, a cross of two arms with plumb lines hanging from the ends, to sight lines at right angles when laying out roads, camps and fields. The medieval master masons who built the great cathedrals set out the long parallel walls of a nave and the square corners of towers with cords, stakes and large squares, using the same facts about perpendiculars and parallels proved in this lesson.

Euclid connection. Euclid constructs a perpendicular to a line at a point on it in I.11 and from a point off the line in I.12, and in I.31 he solves the problem "through a given point to draw a straight line parallel to a given straight line." His fourth postulate, that all right angles are equal to one another (proved as a theorem in Lesson 2.6), is what lets a carpenter's square give the same right angle everywhere.

Practice habit. Mark every right angle on your diagram with a small square before you begin, and say which theorem lets you add each new one (four right angles, Perpendicular Transversal, or a linear pair of congruent angles). When measuring a distance to a line, always measure along the perpendicular.

Example 1. Henry needs to run a water pipe in a straight line from the well W to a straight irrigation ditch. F is the point of the ditch with $\overline{WF}$ perpendicular to the ditch, $WF = 45$ ft, and G is another point of the ditch with $FG = 60$ ft. (a) How long would a straight pipe from W to G be? (b) How many feet of pipe does Henry save by laying it along $\overline{WF}$ instead?

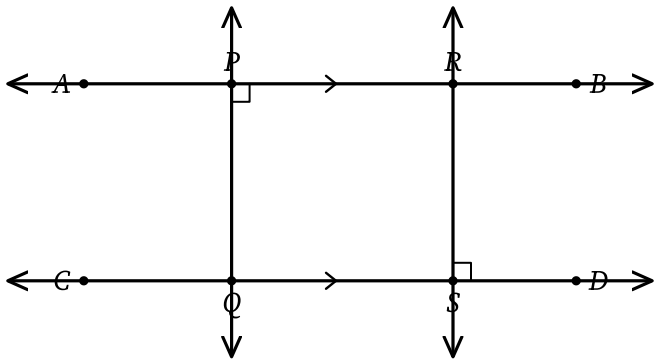


Show solution

1. $\angle WFG$ is a right angle, so $\triangle WFG$ is a right triangle with hypotenuse $\overline{WG}$.
2. Pythagorean Theorem: $WG^2 = 45^2 + 60^2 = 2025 + 3600 = 5625$, so $WG = \sqrt{5625} = 75$ ft.
3. The perpendicular is the shortest segment, as the Shortest Distance Theorem promises: $75 - 45 = 30$ ft saved.

Answer: (a) $WG = 75$ ft; (b) he saves 30 ft of pipe.

Example 2. In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Line $\overleftrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$, with P on $\overleftrightarrow{AB}$ and Q on $\overleftrightarrow{CD}$, and line $\overleftrightarrow{RS}$ is perpendicular to $\overleftrightarrow{CD}$, with R on $\overleftrightarrow{AB}$ and S on $\overleftrightarrow{CD}$. Prove that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.



Show solution

1. Work backward: two lines perpendicular to the same line are parallel (Lesson 3.3). Is there one line that both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are perpendicular to?
2. $\overleftrightarrow{RS} \perp \overleftrightarrow{CD}$ is given. $\overleftrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$, and the Perpendicular Transversal Theorem moves that to $\overleftrightarrow{CD}$.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$; $\overleftrightarrow{RS} \perp \overleftrightarrow{CD}$

Prove: $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$, the Perpendicular Transversal Theorem gives $\overleftrightarrow{PQ} \perp \overleftrightarrow{CD}$.

We are also given $\overleftrightarrow{RS} \perp \overleftrightarrow{CD}$. The lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are different, since they meet $\overleftrightarrow{AB}$ at the different points P and R.

So $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are two lines in a plane perpendicular to the same line $\overleftrightarrow{CD}$, and therefore $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$ (two lines perpendicular to the same line are parallel).

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Answer: See the model proof. All the perpendicular segments between two parallel lines are parallel to one another.

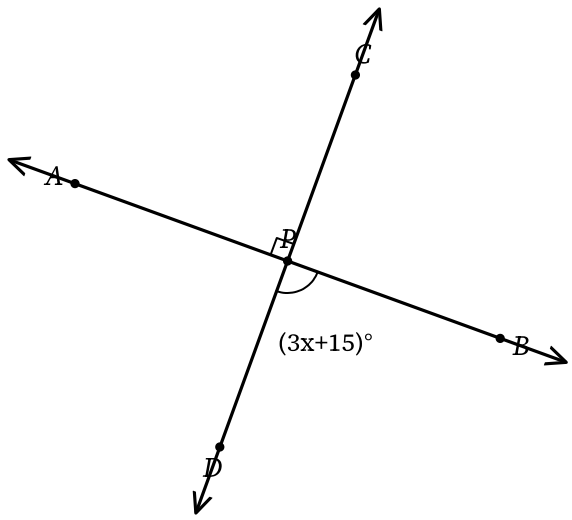
PRACTICE

1. **Easy** $\angle PQR$ and $\angle RQS$ form a linear pair, and $m\angle PQR = m\angle RQS$. Find $m\angle PQR$, and say what you can conclude about $\overleftrightarrow{QR}$ and $\overleftrightarrow{PS}$.

   Show answer

   $m\angle PQR = 90^\circ$ (two equal angles adding to 180°), so $\overleftrightarrow{QR} \perp \overleftrightarrow{PS}$.

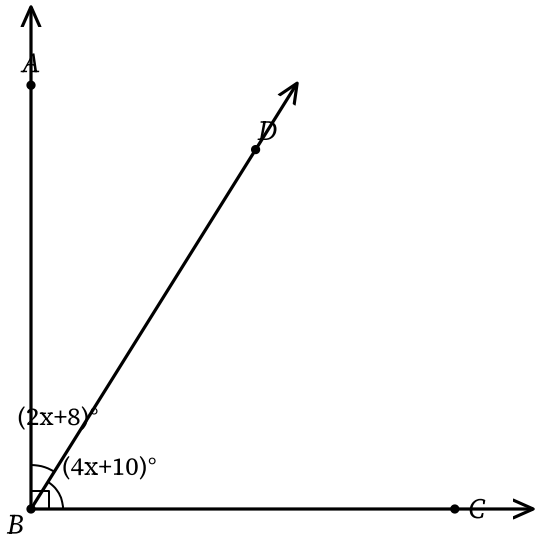
2. **Easy** In the figure, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ at P, and $m\angle BPD = (3x + 15)^\circ$. Find x.



   Show answer

   $x = 25$. Perpendicular lines form four right angles, so $3x + 15 = 90$ and $x = 25$.

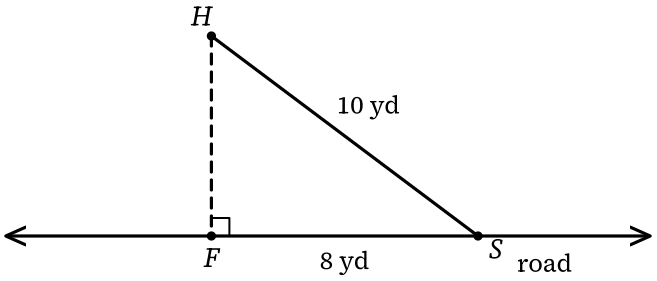
3. Easy In the figure, $\overrightarrow{BA} \perp \overrightarrow{BC}$, $m\angle ABD = (2x + 8)^\circ$ and $m\angle DBC = (4x + 10)^\circ$. Find x and $m\angle DBC$.



Show answer

$\angle ABC$ is a right angle, so by the Angle Addition Postulate $(2x + 8) + (4x + 10) = 90$, $6x = 72$, $x = 12$. Then $m\angle DBC = 4(12) + 10 = 58^\circ$.

4. Easy Ruth's house H stands beside a straight road. A stop sign S on the road is 10 yd from the house, and the point F of the road nearest the house (so $\overline{HF}$ is perpendicular to the road) is 8 yd from the sign. How far is the house from the road?



Show answer

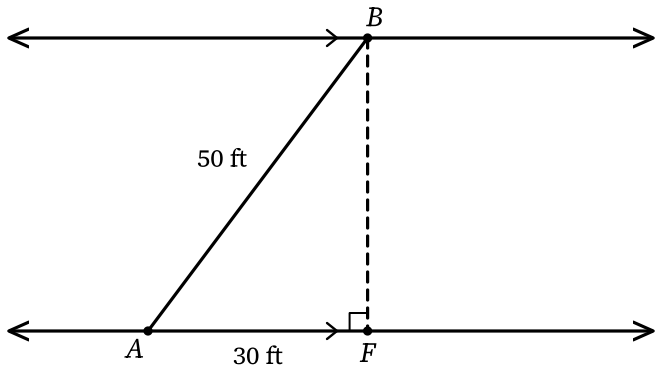
6 yd. The distance is HF, and $HF^2 + 8^2 = 10^2$, so $HF^2 = 36$ and $HF = 6$ yd.

5. **Medium** Grace is framing a wall and wants the stud to be perpendicular to the floor plate. From the corner she marks a point 6 ft along the plate and a point 8 ft up the stud. (a) If the corner is square, how long is the diagonal between the two marks? (b) She measures the diagonal as 9 ft 11 in. Is the stud perpendicular to the plate?

Show answer

(a) 10 ft, since $6^2 + 8^2 = 100 = 10^2$. (b) No. If the angle were right, the Pythagorean Theorem would make the diagonal exactly 10 ft; it is 1 inch short, so the angle is not a right angle.

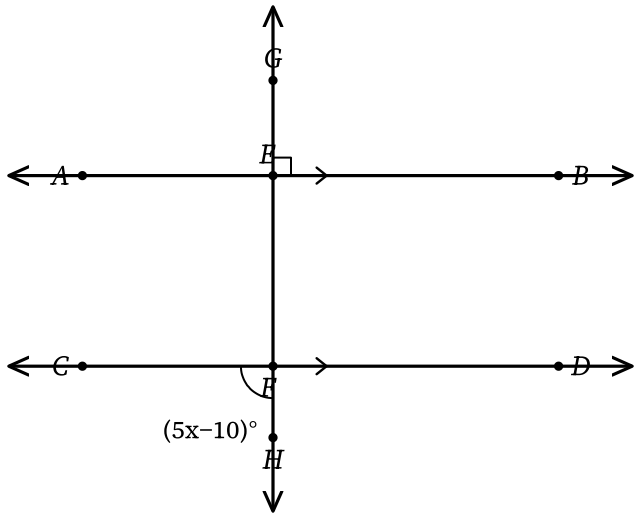
6. **Medium** Henry wants the width of a straight canal whose banks are parallel. From point A on one bank he stretches a 50-ft rope straight across to point B on the other bank. The foot F of the perpendicular from B to the first bank is 30 ft from A. How wide is the canal?



Show answer

40 ft. The width is the perpendicular distance BF, and $BF^2 + 30^2 = 50^2$, so $BF^2 = 1600$ and $BF = 40$ ft.

7. **Medium** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{GH} \perp \overleftrightarrow{AB}$, and $m\angle CFH = (5x - 10)^\circ$. Find x and name the theorem you used.



Show answer

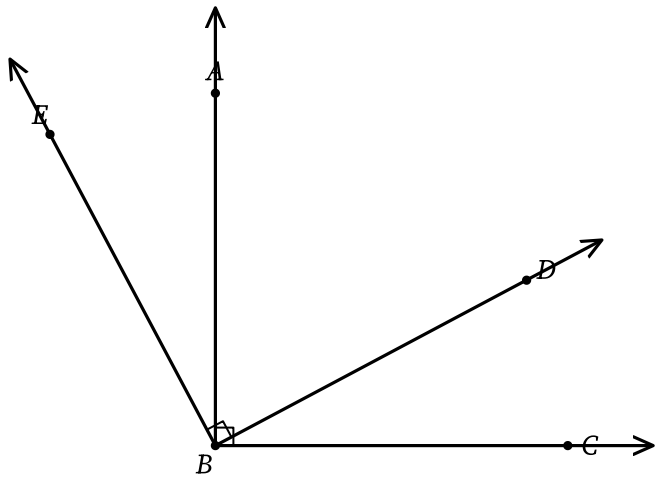
$x = 20$. By the Perpendicular Transversal Theorem $\overleftrightarrow{GH} \perp \overleftrightarrow{CD}$, so all four angles at F are right angles: $5x - 10 = 90$, $x = 20$.

8. **Medium** Samuel says the distance from his house to the straight county road is the length of his driveway, which meets the road at a slant. Explain why he is not right, and say how the distance should be measured.

Show answer

The distance from a point to a line is the length of the perpendicular segment, because that segment is shorter than every other segment to the line (Shortest Distance Theorem). A slanted driveway is the hypotenuse of a right triangle whose leg is the perpendicular, so it is longer than the true distance. He should measure along the line from the house that meets the road at a right angle.

9. **Medium** In the figure, $\overrightarrow{BA} \perp \overrightarrow{BC}$ and $\overrightarrow{BE} \perp \overrightarrow{BD}$. Prove that $\angle DBC \cong \angle ABE$.



Show answer

See the model proof.

TWO-COLUMN PROOF

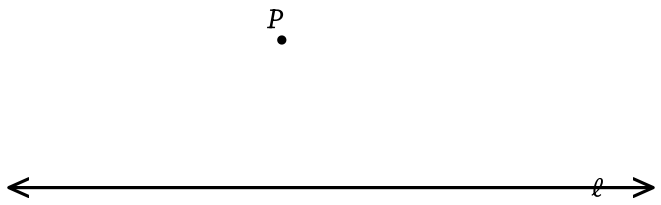
Given: $\overrightarrow{BA} \perp \overrightarrow{BC}$; $\overrightarrow{BE} \perp \overrightarrow{BD}$

Prove: $\angle DBC \cong \angle ABE$

	STATEMENTS	REASONS
1	$\overrightarrow{BA} \perp \overrightarrow{BC}$ and $\overrightarrow{BE} \perp \overrightarrow{BD}$	Given
2	$\angle ABC$ and $\angle EBD$ are right angles	Definition of perpendicular lines
3	$m\angle ABC = 90°$ and $m\angle EBD = 90°$	Definition of right angle
4	$m\angle DBC + m\angle ABD = m\angle ABC$	Angle Addition Postulate
5	$m\angle ABD + m\angle ABE = m\angle EBD$	Angle Addition Postulate
6	$m\angle DBC + m\angle ABD = 90°$ and $m\angle ABD + m\angle ABE = 90°$	Substitution Property of Equality
7	$\angle DBC$ and $\angle ABD$ are complementary; $\angle ABE$ and $\angle ABD$ are complementary	Definition of complementary angles
8	$\angle DBC \cong \angle ABE$	Congruent Complements Theorem

∎

10. Hard Copy the figure: line ℓ and point P not on it. Construct the line through P parallel to ℓ by copying an angle, and explain which postulate guarantees that your line is parallel to ℓ.



Show answer

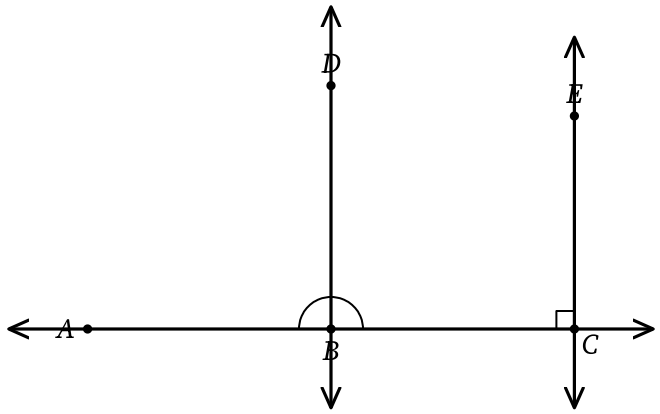
Draw a line through P crossing ℓ at Q. Copy the angle that $\overleftrightarrow{QP}$ makes with ℓ at Q to the point P, in the corresponding position (arc at Q, same arc at P, transfer the chord, draw the line through P and the arc crossing). The copied angle and the original are congruent corresponding angles, so the Converse of the Corresponding Angles Postulate makes the new line parallel to ℓ (Euclid I.31).

11. Hard Point P is 9 cm from line ℓ. Points A and B are two different points of ℓ, each 15 cm from P. How far apart are A and B? Explain.

Show answer

24 cm. Let F be the foot of the perpendicular from P, so $PF = 9$. In right triangle PFA, $FA^2 = 15^2 - 9^2 = 144$, so $FA = 12$; likewise $FB = 12$. On ℓ there are only two points 12 cm from F, one on each side of F, so A and B are on opposite sides and $AB = 12 + 12 = 24$ cm.

12. Hard In the figure, A, B and C are collinear with B between A and C, $\angle ABD \cong \angle CBD$, and $\overleftrightarrow{CE} \perp \overleftrightarrow{AC}$. Prove that $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

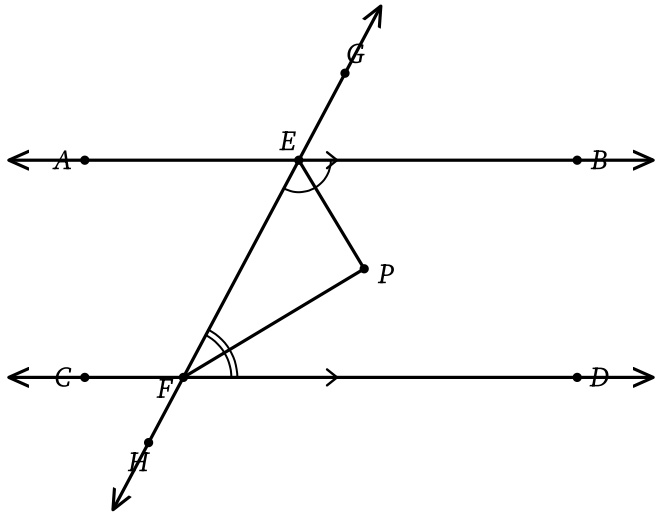
Given: B is between A and C on $\overleftrightarrow{AC}$; $\angle ABD \cong \angle CBD$; $\overleftrightarrow{CE} \perp \overleftrightarrow{AC}$

Prove: $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$

	STATEMENTS	REASONS
1	$\angle ABD \cong \angle CBD$	Given
2	$\angle ABD$ and $\angle CBD$ form a linear pair	Definition of linear pair ($\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays)
3	$\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$	Congruent Linear Pair Theorem
4	$\overleftrightarrow{CE} \perp \overleftrightarrow{AC}$	Given
5	$\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$	Two lines perpendicular to the same line are parallel

∎

13. **Challenge** In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and a transversal crosses them at E and F. $\overrightarrow{EP}$ bisects $\angle BEF$, $\overrightarrow{FP}$ bisects $\angle DFE$, and the bisectors meet at P between the lines. Prove that $\overline{EP} \perp \overline{FP}$. (Hint: draw the line through P parallel to $\overleftrightarrow{AB}$.)



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, cut by a transversal at E and F; $\overrightarrow{EP}$ bisects $\angle BEF$; $\overrightarrow{FP}$ bisects $\angle DFE$

Prove: $\overline{EP} \perp \overline{FP}$

By Playfair's Axiom there is exactly one line m through P parallel to $\overleftrightarrow{AB}$. Choose a point X on m on the side of P away from B and D (to the left in the figure), so that X lies in the interior of $\angle EPF$. Since $m \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$, and $m \neq \overleftrightarrow{CD}$ (P is not on $\overleftrightarrow{CD}$), we have $m \parallel \overleftrightarrow{CD}$ because two lines parallel to the same line are parallel.

With transversal $\overleftrightarrow{EP}$, the angles $\angle XPE$ and $\angle BEP$ are alternate interior angles of the parallel lines m and $\overleftrightarrow{AB}$, so $m\angle XPE = m\angle BEP$ by the Alternate Interior Angles Theorem. With transversal $\overleftrightarrow{FP}$, the angles $\angle XPF$ and $\angle DFP$ are alternate interior angles of the parallel lines m and $\overleftrightarrow{CD}$, so $m\angle XPF = m\angle DFP$.

By the Angle Addition Postulate, $m\angle EPF = m\angle XPE + m\angle XPF = m\angle BEP + m\angle DFP$.

Because $\overrightarrow{EP}$ bisects $\angle BEF$, $m\angle BEP = \frac{1}{2}m\angle BEF$; because $\overrightarrow{FP}$ bisects $\angle DFE$, $m\angle DFP = \frac{1}{2}m\angle DFE$. The angles $\angle BEF$ and $\angle DFE$ are same-side interior angles of the parallel lines, so $m\angle BEF + m\angle DFE = 180^\circ$ by the Same-Side Interior Angles Theorem.

Therefore $m\angle EPF = \frac{1}{2}(m\angle BEF + m\angle DFE) = \frac{1}{2}(180^\circ) = 90^\circ$, and $\overline{EP} \perp \overline{FP}$ by the definition of perpendicular lines.

∎

COMMON MISTAKES

- Measuring the distance from a point to a line along a slanted segment. The distance is always along the perpendicular; any other segment is longer.
- Thinking a right-angle mark at one corner says nothing about the other three. If one angle where two lines cross is right, all four are, by the first theorem of this lesson.
- Using the Perpendicular Transversal Theorem when the two lines are not known to be parallel. Check the given first.
- Changing the compass width in the middle of the parallel construction. Steps 2 and 3 must use the same width; only Step 4 resets it to RS.

> **Make it a game · Shortest Path Duel.** On notebook paper, place a dot above a ruler edge and draw five segments to the edge. Guess which is shortest, then measure to see. Next, guess the distance between two ruled lines in millimeters and measure along the perpendicular. Closest guess scores. Explain why the perpendicular always wins.

3.5 Slopes of Parallel and Perpendicular Lines

OBJECTIVES

- Explain why parallel lines have equal slopes and why perpendicular slopes multiply to -1.
- Write equations of lines parallel or perpendicular to a given line through a given point.
- Find the distance from a point to a line, and between two parallel lines, in the coordinate plane by using a perpendicular.
- Decide whether the sides of a figure on the coordinate plane are parallel or perpendicular.

WATCH

Perpendicular Lines (1 of 2: Using congruent triangles)

Eddie Woo · 6:54

https://www.youtube.com/watch?v=Ut4bl3m95xU

> **WHEN TO WATCH**
> After the Core ideas: a proof that perpendicular slopes multiply to −1, using a rotated congruent triangle.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Perpendicular Lines (2 of 2: Understanding the relationship)

Eddie Woo · 6:35

https://www.youtube.com/watch?v=JEKkjpGTtEA

Before Example 2: using perpendicular slopes in problems.

KEY TERMS

slope

For a nonvertical line through (x_1, y_1) and (x_2, y_2), the ratio $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (rise over run). It is the same for any two points of the line. Horizontal lines have slope 0; vertical lines have no slope (undefined).

point-slope form

The equation $y - y_1 = m(x - x_1)$ of the line with slope m through the point (x_1, y_1).

opposite reciprocals

Two numbers whose product is -1, such as $\dfrac{2}{3}$ and $-\dfrac{3}{2}$. The slope of a line perpendicular to a line of slope $m \neq 0$ is $-\dfrac{1}{m}$.

CORE IDEAS

In Algebra I you learned two rules: parallel lines have equal slopes, and perpendicular lines have slopes that multiply to -1. In a geometry course we ask *why*. Both rules can be proved from what you already know, and the proofs are given below as theorems.

Parallel lines. Write two different nonvertical lines as $y = mx + b$ and $y = nx + c$. A point on both would have to satisfy $mx + b = nx + c$, that is, $(m - n)x = c - b$. If the slopes are equal, the left side is always 0 while the right side is not (different lines with equal slopes have different intercepts), so there is no common point: the lines are parallel. If the slopes are different, we can solve for x, so the lines meet. That is the whole reason: equal slopes, never meet; different slopes, must meet.

Perpendicular lines. Picture a slope triangle with run 3 and rise 2, slope $\dfrac{2}{3}$. Turn the whole picture a quarter turn: the run and the rise trade places and one of them changes sign, giving slope $-\dfrac{3}{2}$. The proof below makes this precise with the Pythagorean Theorem and its converse: the lines $y = m_1 x$ and $y = m_2 x$ meet at a right angle exactly when $m_1 m_2 = -1$. Vertical and horizontal lines are the special case: a vertical line has no slope, but it is perpendicular to every horizontal line.

Writing equations. To write a line parallel or perpendicular to a given line: (1) find the given line's slope, putting it in slope-intercept form if needed; (2) use the

same slope for a parallel line, or the opposite reciprocal for a perpendicular line; (3) use point-slope form with the given point; (4) check that the point satisfies your equation.

Distance from a point to a line. Lesson 3.4 proved that the shortest segment from a point to a line is the perpendicular one. So to find the distance from P to a line: write the equation of the perpendicular through P, solve the two equations together to find the foot F, and use the distance formula to find PF. For two parallel lines, pick any point on one line and find its distance to the other.

Checking figures. Compute the slope of every side. Sides with equal slopes are parallel; sides whose slopes multiply to -1 are perpendicular and meet at a right angle. You will use this often in Units 6 and 13 to classify quadrilaterals.

POSTULATES AND THEOREMS

THEOREM **Slopes of Parallel Lines**

Two different nonvertical lines in the coordinate plane are parallel if and only if they have the same slope. Any two different vertical lines are parallel.

PARAGRAPH PROOF

Given: Different nonvertical lines $y = mx + b$ and $y = nx + c$

Prove: The lines are parallel if and only if $m = n$

A point (x, y) lies on both lines exactly when $mx + b = nx + c$, that is, when $(m - n)x = c - b$.

Suppose $m = n$. Then $b \neq c$, since equal slopes and equal intercepts would make the two equations, and so the two lines, the same. The equation becomes $0 \cdot x = c - b$ with $c - b \neq 0$, which no x satisfies. So the lines have no point in common, and since they lie in one plane they are parallel.

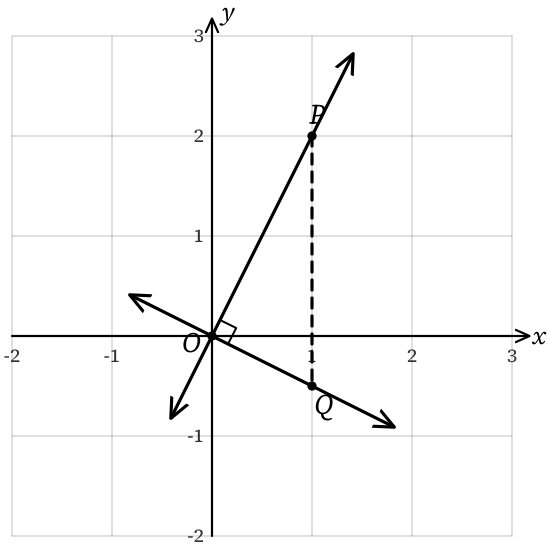
Suppose $m \neq n$. Then $x = \dfrac{c - b}{m - n}$ satisfies the equation, and this x with $y = mx + b$ gives a point on both lines. So the lines intersect and are not parallel.

Together these show that the lines are parallel exactly when $m = n$. Two different vertical lines $x = h$ and $x = k$ never meet, because no point has two different x-coordinates.

∎

THEOREM **Slopes of Perpendicular Lines**

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1. A vertical line and a horizontal line are perpendicular.



PARAGRAPH PROOF

Given: Two nonvertical lines with slopes m_1 and m_2

Prove: The lines are perpendicular if and only if $m_1 m_2 = -1$

First move to lines through the origin. The line $y = m_1 x$ has the same slope as the first line, so by the previous theorem it is parallel to it or is the same line; likewise $y = m_2 x$ for the second line. By the Perpendicular Transversal Theorem (Lesson 3.4), a line perpendicular to one of two parallel lines is perpendicular to the other. Using it twice shows that the original lines are perpendicular exactly when $y = m_1 x$ and $y = m_2 x$ are.

If $m_1 = m_2$, these are the same line, which is not perpendicular to itself, and $m_1 m_2 = m_1^2$ is not -1; so the statement holds. Now suppose $m_1 \neq m_2$. Take $O(0, 0)$, $P(1, m_1)$ on the first line and $Q(1, m_2)$ on the second; these are three noncollinear points.

By the distance formula, $OP^2 = 1 + m_1^2$, $OQ^2 = 1 + m_2^2$ and $PQ^2 = (m_1 - m_2)^2 = m_1^2 - 2m_1 m_2 + m_2^2$.

By the Pythagorean Theorem and its converse, $\angle POQ$ is a right angle exactly when $OP^2 + OQ^2 = PQ^2$, that is, when $2 + m_1^2 + m_2^2 = m_1^2 - 2m_1 m_2 + m_2^2$. Subtracting $m_1^2 + m_2^2$ from both sides gives $2 = -2m_1 m_2$, that is, $m_1 m_2 = -1$. So the lines are perpendicular exactly when $m_1 m_2 = -1$.

∎

This proof uses the Pythagorean Theorem and its converse from Algebra I; they are proved in Lessons 9.1 and 9.2 without using this theorem.

KEY FORMULAS

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = m(x - x_1)$$

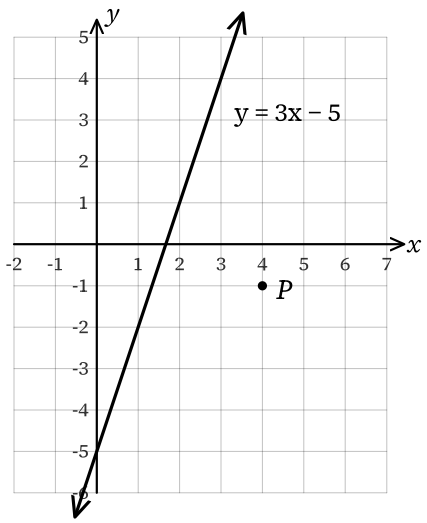
$$\text{parallel: } m_1 = m_2 \qquad \text{perpendicular: } m_1 \cdot m_2 = -1$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Euclid connection. Euclid had no coordinates or slopes; that joining of algebra and geometry came with Descartes in the 1600s (Lesson 10.7). But the slope rules rest on Euclid's ideas: the perpendicular-slope proof uses the Pythagorean Theorem (Euclid I.47) and its converse (I.48), and the distance from a point to a line is measured along the perpendicular of I.12.

Practice habit. Recite the two slope theorems until automatic: "parallel: equal slopes; perpendicular: opposite reciprocals, product -1." Always put an equation in $y = mx + b$ form before reading its slope, and check your final line by substituting the given point.

Example 1. Write equations of the lines through $P(4, -1)$ that are (a) parallel to $y = 3x - 5$ and (b) perpendicular to $y = 3x - 5$.

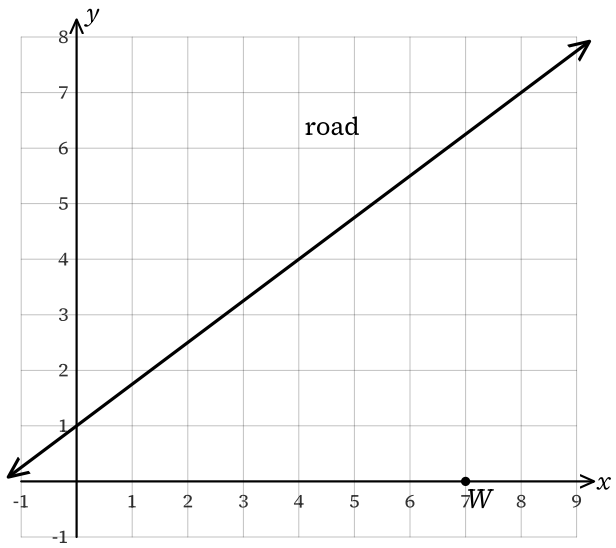


Show solution

1. The given line has slope 3.
2. (a) A parallel line has slope 3: $y - (-1) = 3(x - 4)$, so $y + 1 = 3x - 12$ and $y = 3x - 13$. Check: $3(4) - 13 = -1$.
3. (b) A perpendicular line has slope $-\frac{1}{3}$, since $3 \cdot \left(-\frac{1}{3}\right) = -1$: $y + 1 = -\frac{1}{3}(x - 4)$, so $y = -\frac{1}{3}x + \frac{4}{3} - 1 = -\frac{1}{3}x + \frac{1}{3}$. Check: $-\frac{4}{3} + \frac{1}{3} = -1$.

Answer: (a) $y = 3x - 13$; (b) $y = -\frac{1}{3}x + \frac{1}{3}$

Example 2. On the Miller farm map, 1 unit is 100 ft. A straight road follows the line $y = \frac{3}{4}x + 1$, and the well is at $W(7, 0)$. The family wants to lay the shortest possible pipe from the well to the road. Find the point F where the pipe should meet the road and the length of the pipe in feet.



Show solution

1. The shortest pipe is the perpendicular from W to the road (Shortest Distance Theorem, Lesson 3.4). The road has slope $\frac{3}{4}$, so the perpendicular has slope $-\frac{4}{3}$.
2. Perpendicular through $W(7, 0)$: $y = -\frac{4}{3}(x - 7) = -\frac{4}{3}x + \frac{28}{3}$.
3. Solve together: $\frac{3}{4}x + 1 = -\frac{4}{3}x + \frac{28}{3}$. Multiply by 12: $9x + 12 = -16x + 112$, so $25x = 100$ and $x = 4$. Then $y = \frac{3}{4}(4) + 1 = 4$, so $F = (4, 4)$.
4. Distance formula: $WF = \sqrt{(7-4)^2 + (0-4)^2} = \sqrt{9 + 16} = 5$ units, which is $5 \times 100 = 500$ ft.

Answer: $F = (4, 4)$; the pipe is 5 units, or 500 ft, long.

PRACTICE

1. Easy What is the slope of every line parallel to $y = -2x + 7$?

   Show answer

   -2

2. Easy What is the slope of every line perpendicular to $y = \frac{2}{3}x - 1$?

   Show answer

   $-\frac{3}{2}$, since $\frac{2}{3} \cdot \left(-\frac{3}{2}\right) = -1$.

3. Easy Are the lines $y = 4x + 1$ and $8x - 2y = 6$ parallel?

   Show answer

   Yes. The second line is $y = 4x - 3$: the same slope, 4, and a different intercept.

4. Easy Are the lines $y = -\frac{3}{4}x + 2$ and $4x - 3y = 9$ perpendicular?

   Show answer

   Yes. The second line is $y = \frac{4}{3}x - 3$, and $-\frac{3}{4} \cdot \frac{4}{3} = -1$.

5. Medium Write an equation of the line through $(2, 5)$ parallel to $y = -3x + 4$.

   Show answer

   $y - 5 = -3(x - 2)$, so $y = -3x + 11$.

6. Medium Write an equation of the line through $(4, 3)$ perpendicular to $y = 2x - 1$.

   Show answer

   The slope is $-\frac{1}{2}$: $y - 3 = -\frac{1}{2}(x - 4)$, so $y = -\frac{1}{2}x + 5$.

7. Medium Write an equation of the line through $(-2, 5)$ perpendicular to the line $x = 3$.

   Show answer

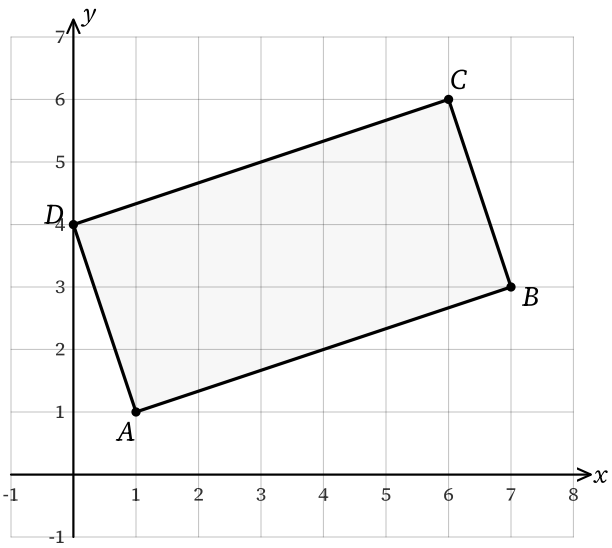
   $x = 3$ is vertical, so the perpendicular is horizontal: $y = 5$.

8. **Medium** The rule $m_1 m_2 = -1$ cannot be used for a vertical line and a horizontal line. Explain why, and explain why such lines are still perpendicular.

Show answer

A vertical line has no slope (its run is 0, and we cannot divide by 0), so there is no product to compute. But a vertical line $x = h$ and a horizontal line $y = k$ meet at (h, k), and the directions straight up and straight across make a right angle, just like the axes. So they are perpendicular even though the slope rule does not apply.

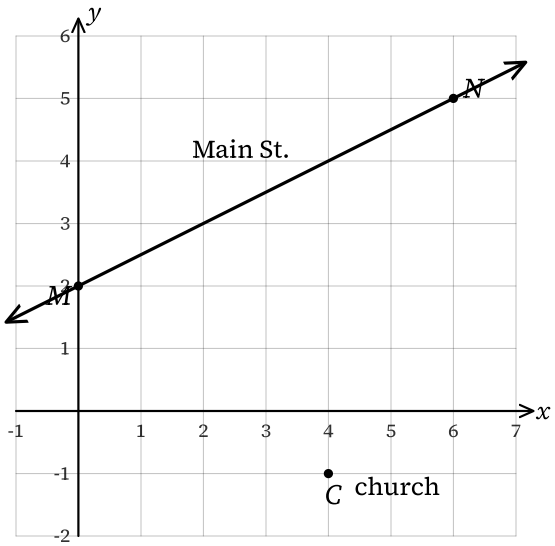
9. **Medium** Quadrilateral $ABCD$ has vertices $A(1, 1)$, $B(7, 3)$, $C(6, 6)$ and $D(0, 4)$. Find the slope of each side. Which sides are parallel? Is $\overline{AB} \perp \overline{BC}$?



Show answer

Slopes: AB: $\frac{1}{3}$; BC: -3; CD: $\frac{1}{3}$; DA: -3. So $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$. Yes, $\overline{AB} \perp \overline{BC}$, because $\frac{1}{3} \cdot (-3) = -1$ (in fact every pair of neighboring sides is perpendicular).

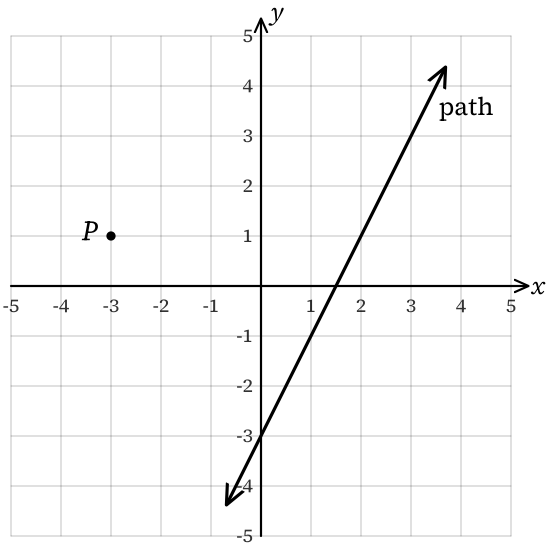
10. **Medium** On a town map, Main Street is the straight line through $M(0, 2)$ and $N(6, 5)$. A new street is to run from the church at $C(4, -1)$ and meet Main Street at a right angle. Write an equation of the new street's line.



Show answer

Main Street has slope $\frac{5-2}{6-0} = \frac{1}{2}$, so the new street has slope -2:
$y + 1 = -2(x - 4)$, so $y = -2x + 7$. (It meets Main Street at $(2, 3)$.)

11. Hard On a garden plan where 1 unit is 1 yard, a straight path follows $y = 2x - 3$ and a bench sits at $P(-3, 1)$. How far is the bench from the path? Give the exact answer in simplest radical form.



Show answer

The perpendicular through P has slope $-\frac{1}{2}$: $y = -\frac{1}{2}x - \frac{1}{2}$. Solve with $y = 2x - 3$: $2x - 3 = -\frac{1}{2}x - \frac{1}{2}$, so $\frac{5}{2}x = \frac{5}{2}$ and $x = 1$, $y = -1$. The foot is $(1, -1)$, and the distance is $\sqrt{(1+3)^2 + (-1-1)^2} = \sqrt{20} = 2\sqrt{5}$ yd.

12. Hard On a vineyard plan measured in meters, two straight rows of vines follow the parallel lines $y = \frac{3}{4}x + 2$ and $y = \frac{3}{4}x - 3$. How far apart are the rows?

Show answer

4 m. Take $(0, 2)$ on the first row. The perpendicular through it is $y = -\frac{4}{3}x + 2$. Solve with $y = \frac{3}{4}x - 3$: multiply by 12, $9x - 36 = -16x + 24$, so $25x = 60$, $x = 2.4$ and $y = -1.2$. The distance from $(0, 2)$ to $(2.4, -1.2)$ is $\sqrt{5.76 + 10.24} = \sqrt{16} = 4$ m.

13. Hard For what value of k is the line through $(1, k)$ and $(4, 2)$ perpendicular to the line $3x - 2y = 6$?

Show answer

$3x - 2y = 6$ is $y = \frac{3}{2}x - 3$, slope $\frac{3}{2}$, so we need slope $-\frac{2}{3}$: $\frac{2 - k}{4 - 1} = -\frac{2}{3}$, so $2 - k = -2$ and $k = 4$.

14. Challenge Points $A(0, 0)$, $B(4, 2)$ and $C(3, 4)$ are three vertices of quadrilateral $ABCD$, in which $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. Find the coordinates of D, and decide whether $\overline{AB} \perp \overline{BC}$.

Show answer

D is on the line through C parallel to $\overline{AB}$ (slope $\frac{1}{2}$): $y = \frac{1}{2}x + \frac{5}{2}$, and on the line through A parallel to $\overline{BC}$ (slope $\frac{4 - 2}{3 - 4} = -2$): $y = -2x$. Solving, $\frac{1}{2}x + \frac{5}{2} = -2x$, so $x = -1$ and $y = 2$: $D = (-1, 2)$. Yes, $\overline{AB} \perp \overline{BC}$, since $\frac{1}{2} \cdot (-2) = -1$.

COMMON MISTAKES

- Reading the slope from an equation not in slope-intercept form, for example taking 8 as the slope of $8x - 2y = 6$. Solve for y first: $y = 4x - 3$.
- Using the reciprocal without changing the sign (or changing the sign without taking the reciprocal). The perpendicular slope to $\frac{2}{3}$ is $-\frac{3}{2}$; check that the product is -1.
- Finding a distance from a point to a line by measuring to some convenient point of the line, such as its y-intercept. The distance must be measured to the foot of the perpendicular.
- Calling two lines with equal slopes parallel when they are actually the same line. Compare the intercepts too.

> **Make it a game · Slope Stair Walk.** Use floor tiles or paper squares. Walk a line with slope 2/3: three steps right, two steps up, repeating. Then your parent names a point and a slope, and you walk the perpendicular line by swapping and flipping signs. Say its slope aloud and write the point-slope equation for each. Ten correct walks beat your record.

UNIT 3

Parallel and Perpendicular Lines

This unit is about the most famous assumption in mathematics: Euclid's fifth postulate, the parallel postulate. You will learn the names of the angle pairs a transversal makes with two lines, take the parallel postulate in a convenient modern form (the Corresponding Angles Postulate), and prove from it the angle theorems of Euclid I.29. Then you will turn them around and prove lines parallel (I.27, I.28, I.30).

Perpendicular lines come next: why one right angle gives four, why the perpendicular is the shortest path from a point to a line, and how to construct a parallel through a point with compass and straightedge (I.31). The unit ends in the coordinate plane, where you prove the Algebra I slope rules for parallel and perpendicular lines and use them to find distances.

For parents: the unit mixes two-column and paragraph proofs with practical problems from carpentry, surveying and farm life. The history of the parallel postulate (Proclus, Saccheri, Gauss, Bolyai and Lobachevsky) shows students why a careful foundation matters. No calculator is needed.

PREREQUISITES

- Name and use vertical angles, linear pairs, and the Congruent Supplements and Congruent Complements Theorems (Lesson 2.4).
- Write two-column and paragraph proofs with the Angle Addition Postulate and the definition of angle bisector (Lessons 1.5 and 2.6).
- Form the converse and contrapositive of a conditional, and know that a converse needs its own proof (Lesson 1.3).
- Copy an angle and construct perpendiculars with compass and straightedge (Lesson 2.5).
- Find slopes, write equations of lines, and use the distance formula and the Pythagorean Theorem numerically (Algebra I).

MASTERY CHECKLIST

- ☐ I can tell parallel, intersecting and skew lines apart and recognize parallel planes.
- ☐ I can name corresponding, alternate interior, alternate exterior and same-side interior angles in a figure.
- ☐ I can state Euclid's fifth postulate, Playfair's Axiom and the Corresponding Angles Postulate, and explain why the parallel postulate is assumed rather than proved.
- ☐ I can prove the Alternate Interior, Alternate Exterior and Same-Side Interior Angles Theorems and use them to find angle measures.
- ☐ I can use the converses to prove lines parallel, including Euclid I.30 and two lines perpendicular to the same line.
- ☐ I can prove that perpendicular lines form four right angles and use the Perpendicular Transversal Theorem.
- ☐ I can explain why the perpendicular is the shortest segment from a point to a line and find distances to lines and between parallel lines.
- ☐ I can construct the parallel to a line through a given point (Euclid I.31) and the perpendicular from a point to a line (I.12), and explain why each works.

- ☐ I can explain why parallel lines have equal slopes and perpendicular slopes multiply to -1, and write equations of parallel and perpendicular lines.
- ☐ I can find the distance from a point to a line in the coordinate plane by using a perpendicular.

LESSONS IN THIS UNIT

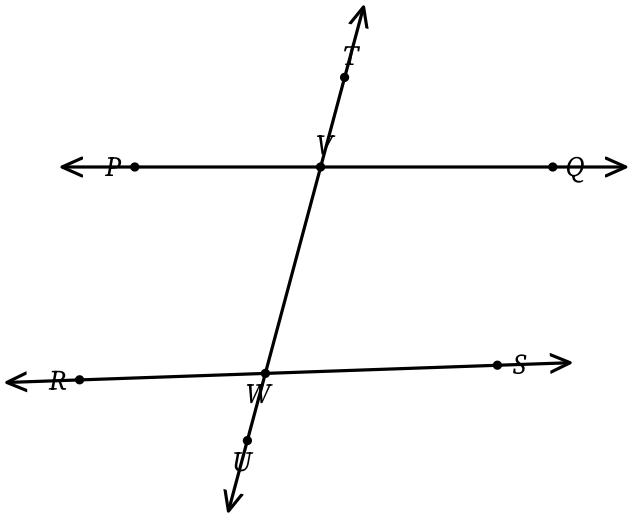
3.1	**Lines, Transversals and Angle Pairs**	45 minutes
3.2	**The Parallel Postulate and Parallel Lines**	2 sessions of 45 minutes
3.3	**Proving Lines Parallel**	2 sessions of 45 minutes
3.4	**Perpendicular Lines and Distance**	2 sessions of 45 minutes
3.5	**Slopes of Parallel and Perpendicular Lines**	45 minutes

Unit 3 test

You have 50 minutes. No calculator is needed or allowed. Show your work for every computation. In every proof, give a reason for each step; do not assume anything from the look of a figure except the order of points and which side of a line a point is on. Give exact answers (simplest radical form where needed). You will need a compass and straightedge for Problem 15.

1. **5 pts**

   In the figure, $\overleftrightarrow{UT}$ is a transversal of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$, crossing them at V and W. Classify the pair $\angle TVP$ and $\angle SWU$ as corresponding, alternate interior, alternate exterior or same-side interior angles.



2. **4 pts**

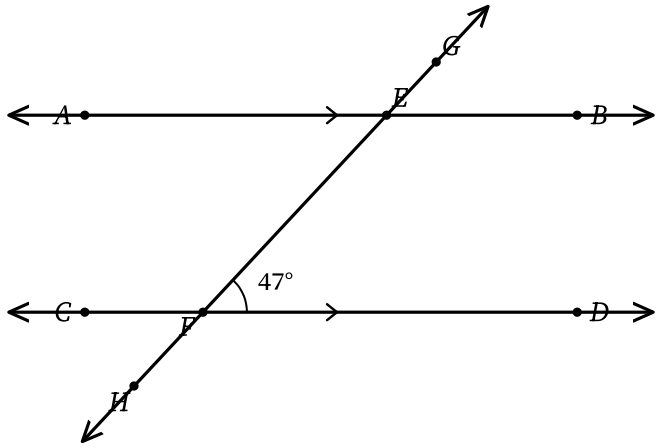
   In a rectangular classroom, consider the line where the front wall meets the floor and the line where the back wall meets the ceiling. Are these lines parallel, intersecting or skew?

3. **5 pts**

   A transversal crosses two lines. On one side of the transversal the interior angles measure 93° and 85°. According to Euclid's fifth postulate, do the lines meet? If so, on which side?

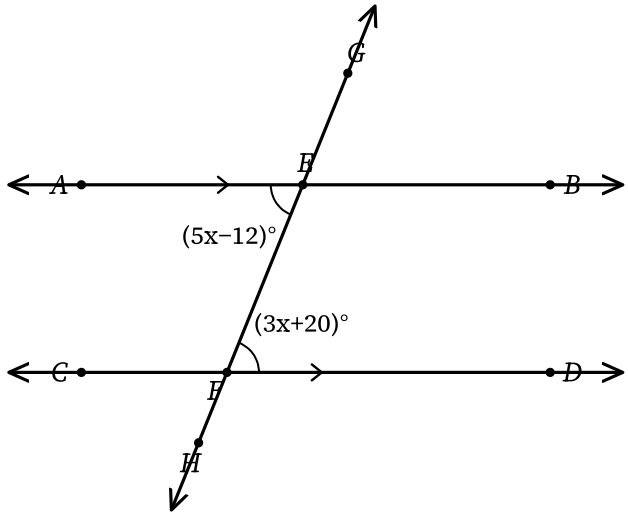
4. **6 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $m\angle DFE = 47°$. Find $m\angle BEF$ and name the theorem you used.



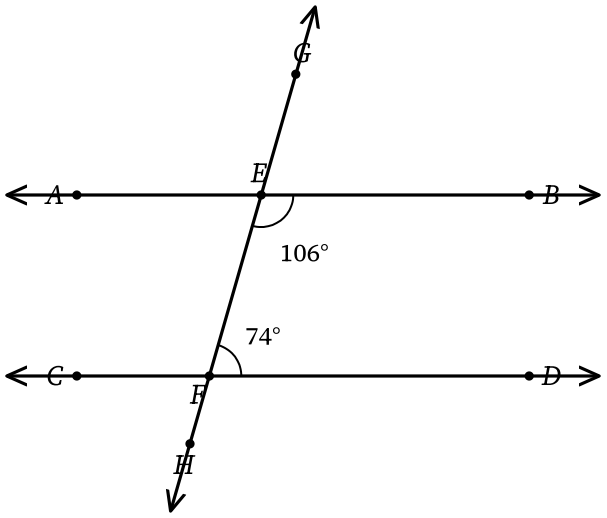
5. **7 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle AEF = (5x - 12)°$ and $m\angle DFE = (3x + 20)°$. Find x and $m\angle AEF$.



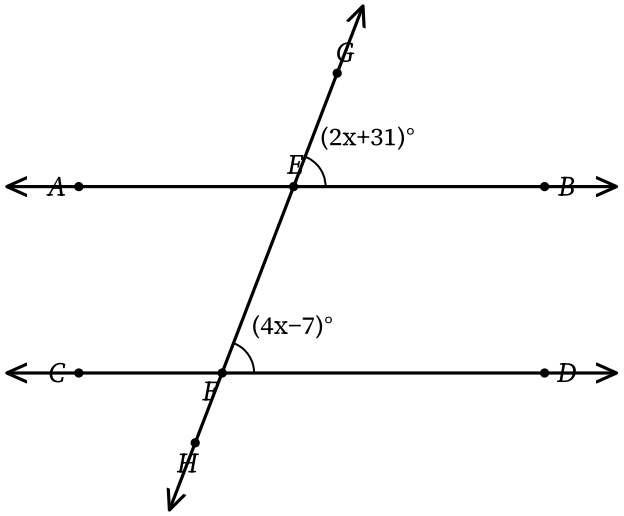
6. **6 pts**

A straight diagonal brace $\overleftrightarrow{GH}$ crosses two straight fence rails, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, at E and F. Ruth measures $m\angle BEF = 106°$ and $m\angle DFE = 74°$. Are the rails parallel? Name the theorem that decides it.



7. **6 pts**

In the figure, $m\angle BEG = (2x + 31)°$ and $m\angle DFE = (4x - 7)°$. What value of x makes $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?



8. **5 pts**

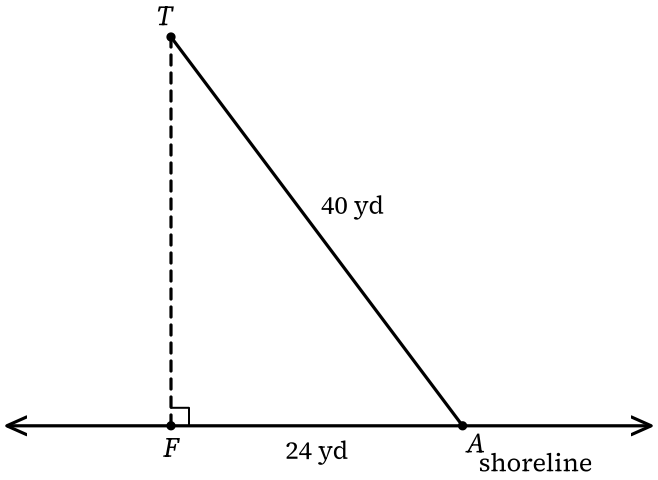
   Find the slope of a line perpendicular to $5x + 2y = 8$.

9. **7 pts**

   Write an equation of the line through $(-2, 3)$ that is parallel to the line through $(1, 4)$ and $(5, -4)$.

10. **7 pts**

    A lifeguard tower T stands near a straight shoreline. Point A on the shoreline is 40 yd from the tower, and the point F of the shoreline nearest the tower is 24 yd from A. How far is the tower from the shoreline?

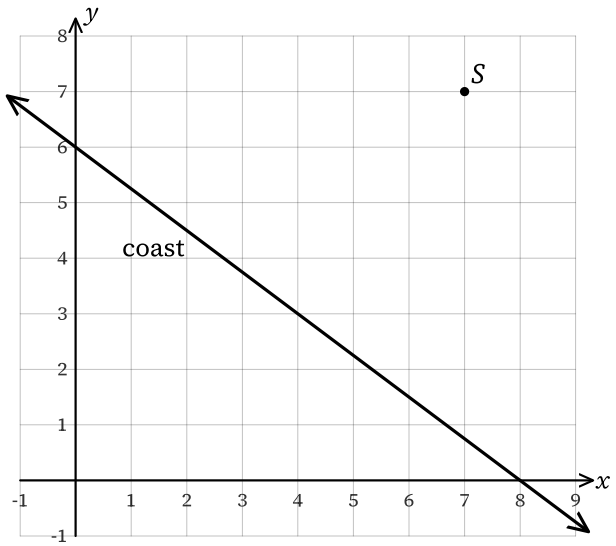


11. **6 pts**

    Henry has just shown that a pair of corresponding angles is congruent. His next step reads: "Statement: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Reason: Corresponding Angles Postulate." Explain what is wrong with his reason and give the correct one.

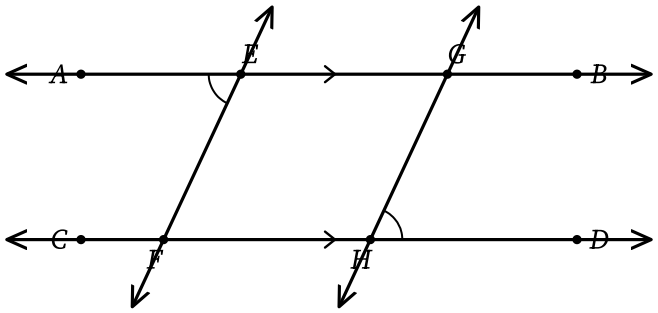
12. **8 pts**

On a chart where 1 unit is 1 mile, a straight coastline follows the line $y = -\frac{3}{4}x + 6$, and a ship is at $S(7, 7)$. Find the point of the coastline nearest the ship, and the ship's distance from the coastline.



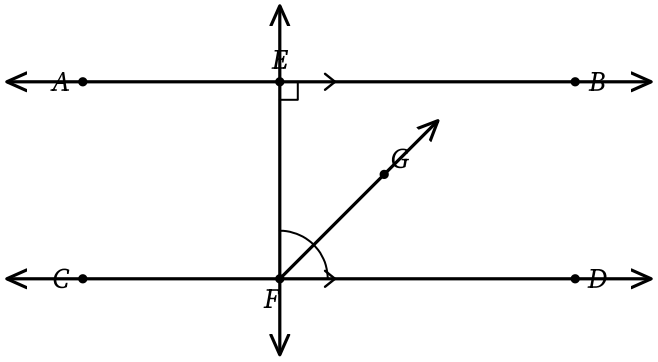
13. **10 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ cross both, with E and G on $\overleftrightarrow{AB}$ and F and H on $\overleftrightarrow{CD}$, and $\angle AEF \cong \angle GHD$. Prove that $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$.



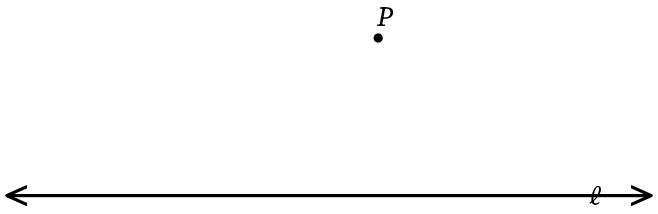
14. **9 pts**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$ with E on $\overleftrightarrow{AB}$ and F on $\overleftrightarrow{CD}$, and $\overrightarrow{FG}$ bisects $\angle EFD$. Write a paragraph proof that $m\angle GFD = 45°$.



15. **9 pts**

Copy the figure: line ℓ and point P not on it. With compass and straightedge, construct the line through P parallel to ℓ. List your steps and name the postulate that guarantees your line is parallel to ℓ.



ANSWER KEY

1. Alternate exterior angles: both are outside the two lines, on opposite sides of the transversal.
2. Parallel. Both run straight across the room from side to side, so they point in the same direction, lie in one (slanted) plane, and never meet.
3. Yes. $93^\circ + 85^\circ = 178^\circ < 180^\circ$, so the lines meet on the side of the transversal where these two angles are.
4. 133°, by the Same-Side Interior Angles Theorem ($180^\circ - 47^\circ$).
5. Alternate interior angles are congruent: $5x - 12 = 3x + 20$, so $2x = 32$ and $x = 16$. Then $m\angle AEF = 5(16) - 12 = 68^\circ$.
6. Yes. $\angle BEF$ and $\angle DFE$ are same-side interior angles and $106^\circ + 74^\circ = 180^\circ$, so the rails are parallel by the Converse of the Same-Side Interior Angles Theorem.
7. $x = 19$. The angles are corresponding, so the lines are parallel when $2x + 31 = 4x - 7$, that is, $38 = 2x$ (each angle is then 69°).
8. $5x + 2y = 8$ is $y = -\dfrac{5}{2}x + 4$, so the perpendicular slope is $\dfrac{2}{5}$.
9. The slope is $\dfrac{-4 - 4}{5 - 1} = -2$, so $y - 3 = -2(x + 2)$ and $y = -2x - 1$.
10. 32 yd. The distance is TF, with $\overline{TF}$ perpendicular to the shore, and $TF^2 + 24^2 = 40^2$, so $TF^2 = 1024$ and $TF = 32$ yd.
11. The Corresponding Angles Postulate goes the other way: it starts with parallel lines and concludes that corresponding angles are congruent. Henry is going from congruent angles to parallel lines, so he needs the converse. The correct reason is the Converse of the Corresponding Angles Postulate.
12. The perpendicular through S has slope $\dfrac{4}{3}$: $y = \dfrac{4}{3}x - \dfrac{7}{3}$. Solve with the coastline: $-\dfrac{3}{4}x + 6 = \dfrac{4}{3}x - \dfrac{7}{3}$; times 12, $-9x + 72 = 16x - 28$, so $x = 4$ and $y = 3$. The nearest point is $(4, 3)$, and the distance is $\sqrt{3^2 + 4^2} = 5$ miles.
13. See the model proof.

TWO-COLUMN PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\angle AEF \cong \angle GHD$
Prove: $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

	STATEMENTS	REASONS
1	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Given

	STATEMENTS	REASONS
2	$\angle AEF \cong \angle EFD$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{EF}$)
3	$\angle AEF \cong \angle GHD$	Given
4	$\angle EFD \cong \angle GHD$	Symmetric and Transitive Properties of Congruence
5	$\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$	Converse of the Corresponding Angles Postulate ($\angle EFD$ and $\angle GHD$ are corresponding angles for transversal $\overleftrightarrow{CD}$)

∎

14. See the model proof.

PARAGRAPH PROOF

Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$; $\overrightarrow{FG}$ bisects $\angle EFD$

Prove: $m\angle GFD = 45°$

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$, the Perpendicular Transversal Theorem gives $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$. Perpendicular lines form four right angles, so $\angle EFD$ is a right angle and $m\angle EFD = 90°$.

Because $\overrightarrow{FG}$ bisects $\angle EFD$, $m\angle EFG = m\angle GFD$ by the definition of angle bisector, and $m\angle EFG + m\angle GFD = m\angle EFD = 90°$ by the Angle Addition Postulate.

Substituting gives $2 \cdot m\angle GFD = 90°$, so $m\angle GFD = 45°$ by the Division Property of Equality.

∎

15. Draw a line through P crossing ℓ at Q. With the compass at Q, draw an arc crossing ℓ at R and $\overrightarrow{QP}$ at S. With the same width and the compass at P, draw an arc crossing $\overrightarrow{QP}$ beyond P at T. Set the compass to RS and, from T, mark U on that arc on the same side as R. Draw $\overleftrightarrow{PU}$. Then $\angle UPT \cong \angle RQP$ are congruent corresponding angles, so $\overleftrightarrow{PU} \parallel \ell$ by the Converse of the Corresponding Angles Postulate (Euclid I.31).

UNIT 4

Triangles and Congruence

This unit is the heart of Euclid's Book I. You will prove that the angles of every triangle add to 180° (Euclid I.32), learn exactly what it means for two triangles to be congruent, and discover that just three well-chosen parts (SSS, SAS, ASA, AAS, and HL for right triangles) are enough to guarantee all six. Along the way you will meet Euclid's famous 'bridge of asses' (I.5) about isosceles triangles.

Congruent triangles are the main tool of the rest of the course. With CPCTC you will finally prove that the compass constructions of Unit 2 really work, measure across rivers and ponds without crossing them, and write longer proofs in two-column, paragraph and flow form, sometimes using two pairs of triangles in a row.

Parents: the key skill is careful reasoning. Ask your student to say aloud which test they are using and to point to the three parts on the figure; a correct answer always names an included angle or included side in the right place.

PREREQUISITES

- Solve linear equations and simple systems of equations (Algebra I).
- Use the Vertical Angles Theorem, linear pairs and the Congruent Supplements and Complements Theorems (Lesson 2.4).
- Use the properties of parallel lines, especially the Alternate Interior Angles Theorem and its converse (Lessons 3.2 and 3.3).
- Bisect an angle, copy a segment and an angle, and construct a perpendicular bisector with compass and straightedge (Lesson 2.5).
- Write two-column proofs with the properties of equality and congruence (Lessons 1.5 and 2.6).
- Use the distance formula (Algebra I).

MASTERY CHECKLIST

- ☐ I can classify a triangle by its sides and by its angles.
- ☐ I can prove the Triangle Angle Sum Theorem with a parallel line and use the Exterior Angle Theorem.
- ☐ I can name corresponding parts from a congruence statement and use the Third Angles Theorem.
- ☐ I can prove triangles congruent using SSS, SAS, ASA, AAS and HL, and I can explain why SSA and AAA are not tests.
- ☐ I can construct a triangle from three given segments and copy a triangle with compass and straightedge.
- ☐ I can use CPCTC to prove segments and angles congruent, including in overlapping triangles.
- ☐ I can prove that the angle bisector and perpendicular bisector constructions work.
- ☐ I can measure an inaccessible distance by laying out congruent triangles.
- ☐ I can use the Isosceles Triangle Theorem and its converse, and facts about equilateral triangles, in proofs and with algebra.
- ☐ I can write two-column, paragraph and flow proofs, including proofs that use two pairs of congruent triangles.

- ☐ I can check whether two triangles in the coordinate plane are congruent using the distance formula.

LESSONS IN THIS UNIT

4.1	**Triangle Angle Sum and Exterior Angles**	2 sessions of 45 minutes
4.2	**Congruent Figures**	45 minutes
4.3	**Proving Triangles Congruent: SSS and SAS**	2 sessions of 45 minutes
4.4	**Proving Triangles Congruent: ASA and AAS**	2 sessions of 45 minutes
4.5	**Using Congruent Triangles (CPCTC)**	2 sessions of 45 minutes
4.6	**Isosceles and Equilateral Triangles**	2 sessions of 45 minutes
4.7	**Right Triangles: HL and Proofs with Two Pairs of Triangles**	2 sessions of 45 minutes

Unit 4 Test

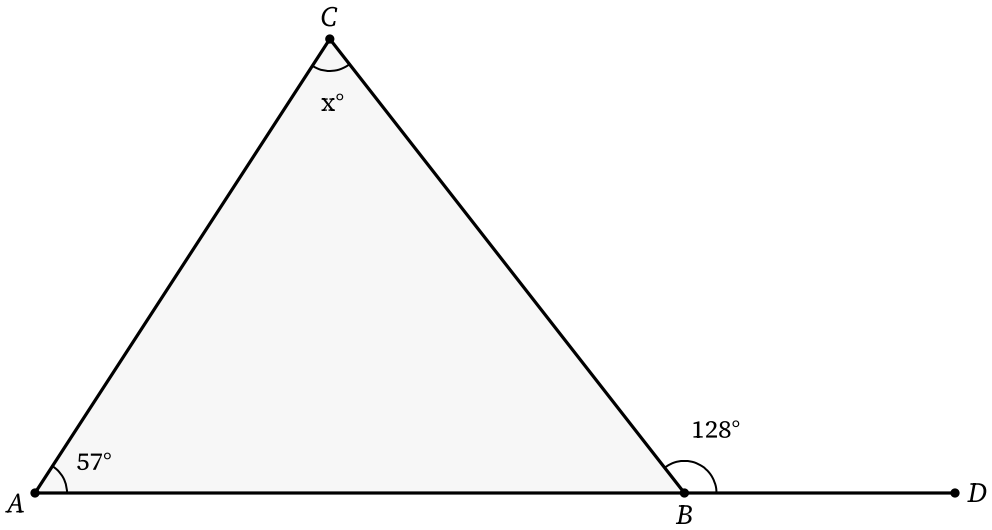
You have 60 minutes. No calculator is needed or allowed. Bring a compass, a straightedge and a sharp pencil for Problem 8. Show your work on every problem. In every proof, give a reason for every statement. Give exact answers, with units where the problem has units. Figures are drawn to scale, but use only the given information and the marks on a figure in your reasoning.

1. **5 pts**

   The angles of a triangle measure $(2x + 3)^\circ$, $(3x - 7)^\circ$ and $(4x + 4)^\circ$. Find x and the measure of the largest angle.

2. **5 pts**

The Miller family's triangular field ABC is shown in the figure. The fence along $\overline{AB}$ is extended past B to D. The corner at A measures $57°$, and the exterior angle $\angle CBD$ measures $128°$. Find $x = m\angle ACB$.



3. **5 pts**

Two interior angles of a triangle measure $(3x)°$ and $(2x)°$, and the exterior angle at the third vertex measures $110°$. Find x and the measure of the third interior angle.

4. **5 pts**

$\triangle MNP \cong \triangle RST$, with $MN = 4x - 3$, $RS = 2x + 9$ and $m\angle P = 52°$. Find x, MN and $m\angle T$.

5. **6 pts**

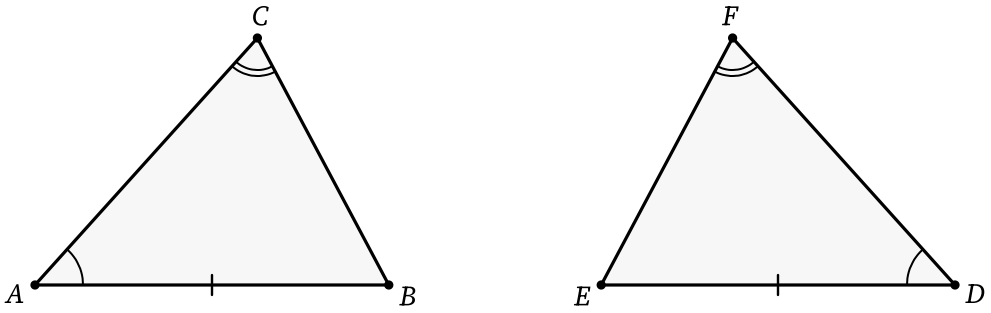
In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. (a) Must $\angle C \cong \angle F$? (b) Must the triangles be congruent? Explain both answers.

6. **5 pts**

In $\triangle PQR$ and $\triangle XYZ$, $\overline{PQ} \cong \overline{XY}$, $\overline{QR} \cong \overline{YZ}$ and $\angle P \cong \angle X$. Can you conclude that $\triangle PQR \cong \triangle XYZ$? If so, name the test; if not, write 'not enough information.'

7. **5 pts**

In the figure, $\angle A \cong \angle D$, $\angle C \cong \angle F$ and $\overline{AB} \cong \overline{DE}$. Which test proves $\triangle ABC \cong \triangle DEF$?

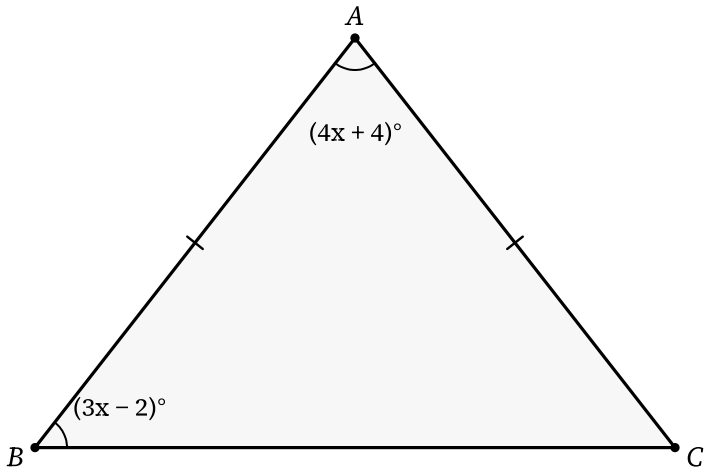


8. **8 pts**

Using only a compass and straightedge, construct $\triangle PQR$ with $PQ = 7$ cm, $QR = 5$ cm and $PR = 6$ cm (measure the three lengths with a ruler once, then use only the compass). List your steps, and explain why every student who does this construction correctly gets a triangle congruent to yours.

9. **7 pts**

In the figure, $\overline{AB} \cong \overline{AC}$, $m\angle A = (4x + 4)^\circ$ and $m\angle B = (3x - 2)^\circ$. Find x and $m\angle C$.

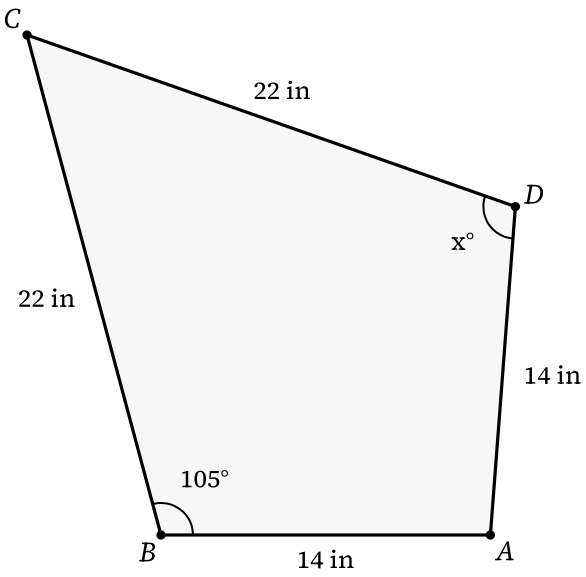


10. **6 pts**

A church window is topped by an equilateral arch with a span of 9 ft between the springing points A and B; each curve is drawn with radius AB from the opposite springing point, and the curves meet at the apex C. How far is the apex from each springing point, and what is $m\angle CAB$? Explain.

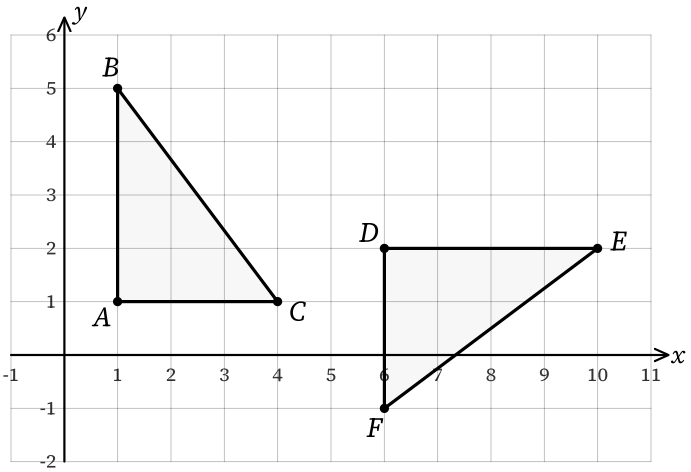
11. **8 pts**

A kite-shaped stained-glass pane $ABCD$ is shown in the figure, with $AB = AD = 14$ in and $CB = CD = 22$ in. The angle at B measures 105°. Find $x = m\angle ADC$, and justify your answer with a congruence test.



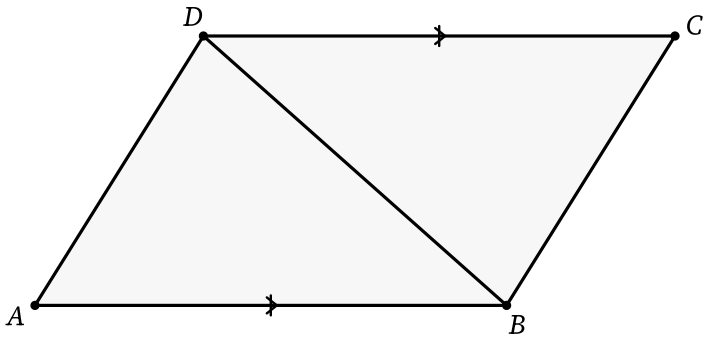
12. **7 pts**

In the figure, $A(1,1)$, $B(1,5)$, $C(4,1)$, $D(6,2)$, $E(10,2)$ and $F(6,-1)$. Explain why both triangles are right triangles, find the side lengths, and decide whether $\triangle ABC \cong \triangle DEF$.



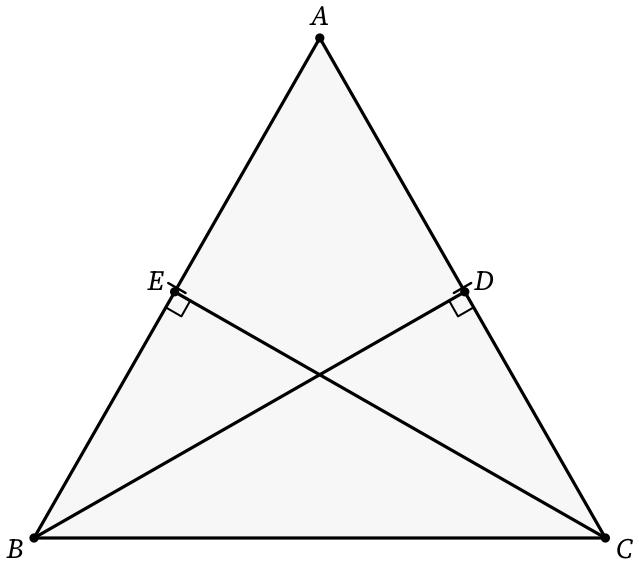
13. **10 pts**

In quadrilateral $ABCD$ in the figure, $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$, and diagonal $\overline{BD}$ is drawn. Write a two-column proof that $\overline{AD} \parallel \overline{BC}$.



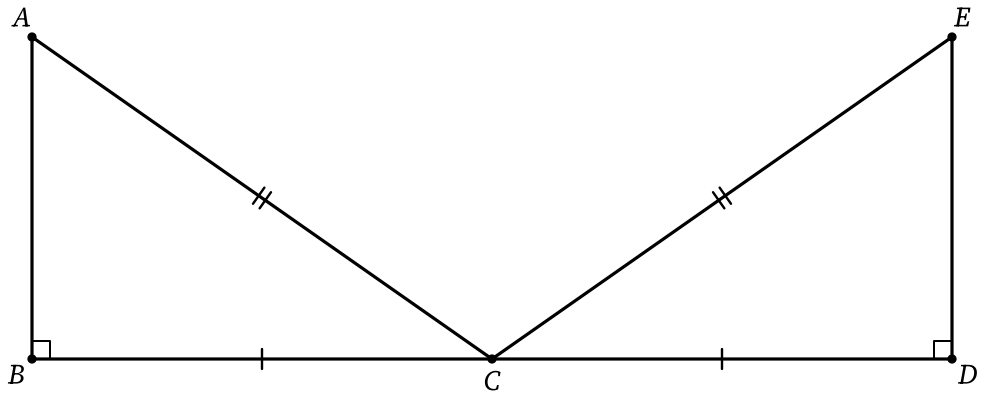
14. **10 pts**

In $\triangle ABC$ in the figure, $\overline{AB} \cong \overline{AC}$, $\overline{BD} \perp \overline{AC}$ at D, and $\overline{CE} \perp \overline{AB}$ at E. Write a flow proof that $\overline{BD} \cong \overline{CE}$.



15. **8 pts**

In the figure, $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{BD}$, C is the midpoint of $\overline{BD}$ and $\overline{AC} \cong \overline{EC}$. Write a paragraph proof that $\angle A \cong \angle E$.



ANSWER KEY

1. $9x = 180$, so $x = 20$; the angles are $43°$, $53°$ and $84°$, and the largest is $84°$.
2. By the Exterior Angle Theorem, $128 = 57 + x$, so $x = 71$.
3. Exterior Angle Theorem: $3x + 2x = 110$, so $x = 22$. The third interior angle is $180 - 110 = 70°$ (linear pair).
4. $4x - 3 = 2x + 9$, so $x = 6$ and $MN = 21$. $\angle T$ corresponds to $\angle P$, so $m\angle T = 52°$.
5. (a) Yes, by the Third Angles Theorem: each third angle is $180°$ minus the same sum. (b) No. Matching angles give the same shape but not the same size; for example, equilateral triangles with sides 2 in and 4 in have all angles $60°$ but are not congruent. (AAA is not a congruence test.)
6. Not enough information. The angle included between $\overline{PQ}$ and $\overline{QR}$ is $\angle Q$, not $\angle P$, so this is SSA, which is not a congruence test.
7. AAS ($\overline{AB}$ is opposite $\angle C$ and $\overline{DE}$ is opposite $\angle F$, so the side is not included).
8. Draw a ray from P and mark Q with $PQ = 7$ cm. With the compass set to 6 cm and centered at P, draw an arc; with the compass set to 5 cm and centered at Q, draw an arc crossing it at R. Draw $\overline{PR}$ and $\overline{QR}$. The arcs meet because each pair of lengths adds to more than the third ($5 + 6 > 7$, and so on). Any two triangles built this way have three pairs of congruent sides, so they are congruent by SSS (Euclid I.22 and I.8).
9. By the Isosceles Triangle Theorem $m\angle C = m\angle B$, so $(4x + 4) + 2(3x - 2) = 180$, $10x = 180$, $x = 18$. Then $m\angle C = 3(18) - 2 = 52°$.
10. C lies on both arcs of radius 9 ft, so $AC = BC = 9$ ft. With $AB = 9$ ft the triangle is equilateral, hence equiangular, so $m\angle CAB = 60°$.
11. Draw diagonal $\overline{AC}$. $\triangle ABC \cong \triangle ADC$ by SSS ($AB = AD$, $CB = CD$, $AC = AC$), so $\angle ADC \cong \angle ABC$ by CPCTC, and $x = 105$.
12. $\overline{AB}$ is vertical and $\overline{AC}$ is horizontal, so $\angle A$ is a right angle; likewise $\overline{DE}$ is horizontal and $\overline{DF}$ is vertical, so $\angle D$ is right. $AB = 4$, $AC = 3$, $BC = \sqrt{9 + 16} = 5$; $DE = 4$, $DF = 3$, $EF = \sqrt{16 + 9} = 5$. The hypotenuses ($BC = EF = 5$) and a pair of legs ($AB = DE = 4$) are congruent, so yes, $\triangle ABC \cong \triangle DEF$ by HL (SSS also works).
13. See the model proof. (This is Euclid I.33.)

    TWO-COLUMN PROOF

    Given: $\overline{AB} \parallel \overline{DC}$; $\overline{AB} \cong \overline{DC}$

    Prove: $\overline{AD} \parallel \overline{BC}$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$	Given
2	$\angle ABD \cong \angle CDB$	Alternate Interior Angles Theorem (transversal $\overline{BD}$)
3	$\overline{AB} \cong \overline{CD}$	Given
4	$\overline{BD} \cong \overline{DB}$	Reflexive Property of Congruence
5	$\triangle ABD \cong \triangle CDB$	SAS
6	$\angle ADB \cong \angle CBD$	CPCTC
7	$\overline{AD} \parallel \overline{BC}$	If alternate interior angles are congruent, the lines are parallel (Euclid I.27)

∎

14. See the model proof.

FLOW PROOF

Given: $\overline{AB} \cong \overline{AC}$; $\overline{BD} \perp \overline{AC}$ at D; $\overline{CE} \perp \overline{AB}$ at E

Prove: $\overline{BD} \cong \overline{CE}$

(1) $\overline{AB} \cong \overline{AC}$
Given

(2) $\angle ACB \cong \angle ABC$, that is, $\angle DCB \cong \angle EBC$
Isosceles Triangle Theorem (D is on $\overline{CA}$ and E is on $\overline{BA}$)
from (1)

(3) $\overline{BD} \perp \overline{AC}$ and $\overline{CE} \perp \overline{AB}$
Given

(4) $\angle BDC$ and $\angle CEB$ are right angles
Definition of perpendicular lines
from (3)

(5) $\angle BDC \cong \angle CEB$
All right angles are congruent
from (4)

(6) $\overline{BC} \cong \overline{CB}$
Reflexive Property of Congruence

(7) $\triangle BDC \cong \triangle CEB$
AAS
from (5), (2), (6)

(8) $\overline{BD} \cong \overline{CE}$
CPCTC
from (7)

∎

15. See the model proof.

PARAGRAPH PROOF

Given: $\overline{AB} \perp \overline{BD}$; $\overline{ED} \perp \overline{BD}$; C is the midpoint of $\overline{BD}$; $\overline{AC} \cong \overline{EC}$

Prove: $\angle A \cong \angle E$

Because $\overline{AB} \perp \overline{BD}$ and $\overline{ED} \perp \overline{BD}$, the angles $\angle ABC$ and $\angle EDC$ are right angles by the definition of perpendicular lines, so $\triangle ABC$ and $\triangle EDC$ are right triangles. Their hypotenuses $\overline{AC}$ and $\overline{EC}$ are congruent (given).

Since C is the midpoint of $\overline{BD}$, $\overline{BC} \cong \overline{DC}$ by the definition of midpoint; these are legs of the two right triangles. So $\triangle ABC \cong \triangle EDC$ by HL, and $\angle A \cong \angle E$ by CPCTC.

∎

4.1 Triangle Angle Sum and Exterior Angles

OBJECTIVES

- Classify triangles by their sides and by their angles.
- Prove the Triangle Angle Sum Theorem with an auxiliary parallel line, as Euclid does in I.32.
- Prove and use the Exterior Angle Theorem and the Exterior Angle Inequality.
- Use the angle sum and its corollaries to find unknown angles with algebra and to write proofs.

WATCH

Important Angle Sums

Eddie Woo · 9:40

https://www.youtube.com/watch?v=kWSSvXOF93I

> **WHEN TO WATCH**
> Before the Triangle Angle Sum Theorem: why the angles of a triangle add to 180°.

Note: The proof of the Exterior Angle Theorem and the Exterior Angle Inequality are taught from the Core ideas.

BACKUP VIDEO

Exterior Angle Theorem (Triangles)

Mario's Math Tutoring · 1:55

https://www.youtube.com/watch?v=vCz_j47uK2U

Before the Exterior Angle Theorem examples: an exterior angle equals the sum of the two remote interior angles.

KEY TERMS

Scalene, isosceles and equilateral triangles

A triangle is scalene if no two sides are congruent, isosceles if at least two sides are congruent, and equilateral if all three sides are congruent.

Acute, right, obtuse and equiangular triangles

A triangle is acute if all three angles are acute, right if it has a right angle, obtuse if it has an obtuse angle, and equiangular if all three angles are congruent.

Exterior angle

An angle formed by one side of a triangle and the extension of an adjacent side. It forms a linear pair with the interior angle at that vertex.

Remote interior angles

The two interior angles of a triangle that are not adjacent to a given exterior angle (Euclid calls them the 'interior and opposite' angles).

Auxiliary line

A line, ray or segment added to a figure to help with a proof. You must be able to justify that it exists (for example by a postulate or a construction).

Corollary

A theorem that follows quickly from another theorem that has just been proved.

CORE IDEAS

Triangles are sorted in two independent ways. By sides: scalene (no congruent sides), isosceles (at least two congruent sides) and equilateral (three congruent sides). By angles: acute (three acute angles), right (one right angle), obtuse (one obtuse angle). A triangle with three congruent angles is called equiangular. A full name uses both, such as 'an obtuse isosceles triangle'.

You have probably heard that the angles of a triangle add up to 180°. But why must that be true for every triangle, even one you have never drawn? Measuring a few triangles is inductive reasoning and proves nothing. The proof needs a new line that is not in the triangle: through one vertex, draw the line parallel to the opposite side. Such an added line is an auxiliary line, and we are allowed to draw it because the Parallel Postulate (Lesson 3.2) guarantees exactly one parallel through a point not on a line.

Once the parallel line is drawn, the three angles at the top vertex lie side by side along a straight line, so they add to $180°$. The two outer ones are alternate interior angles with the two base angles of the triangle, so they are congruent to them. In effect the parallel line carries the three angles of the triangle to one place, where you can see them fill a straight angle. Notice what the proof depends on: the Alternate Interior Angles Theorem, which depends on the Parallel Postulate. Without Euclid's fifth postulate, the angle sum of a triangle cannot be proved to be $180°$.

If you extend one side of a triangle past a vertex, you make an exterior angle. It forms a linear pair with the interior angle at that vertex, so the two add to $180°$. The other two angles of the triangle also add to $180°$ minus that same interior angle. Two things that equal the same thing are equal, so the exterior angle equals the sum of the two remote interior angles. That is the Exterior Angle Theorem, the first half of Euclid I.32.

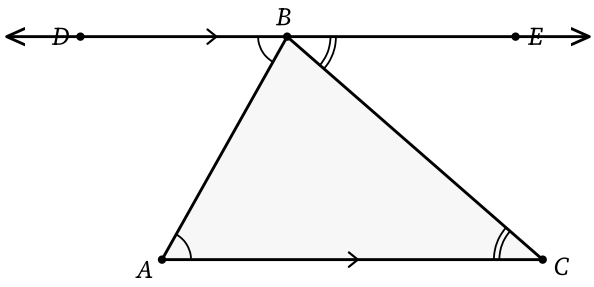
Because an exterior angle equals the sum of two positive angles, it is larger than either one. This is the Exterior Angle Inequality (Euclid I.16). Euclid proved it much earlier than I.32, before he ever used the parallel postulate, which makes it true even in the geometry of Bolyai and Lobachevsky (Lesson 3.2), where the angle sum of a triangle is less than $180°$. (On a sphere, where the angle sum is more than $180°$, it can fail.)

Quick consequences, called corollaries: a triangle can have at most one right or obtuse angle (two of them would already add to $180°$ or more, leaving nothing for the third angle); the two acute angles of a right triangle are complementary; and each angle of an equiangular triangle measures $180° \div 3 = 60°$.

POSTULATES AND THEOREMS

THEOREM **Triangle Angle Sum Theorem** Euclid I.32

The sum of the measures of the three angles of a triangle is $180°$.



TWO-COLUMN PROOF

Given: $\triangle ABC$

Prove: $m\angle BAC + m\angle ABC + m\angle BCA = 180°$

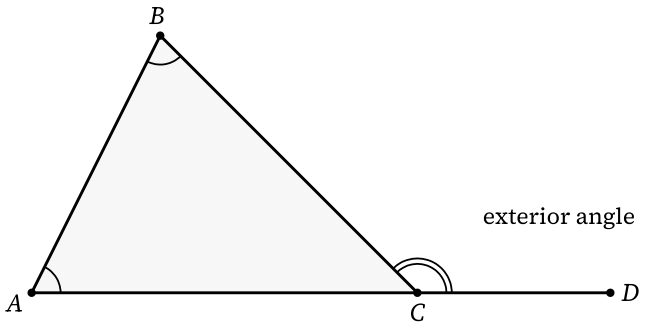
	STATEMENTS	REASONS
1	$\triangle ABC$	Given
2	Draw $\overleftrightarrow{DE}$ through B parallel to $\overleftrightarrow{AC}$, with D on the opposite side of $\overleftrightarrow{AB}$ from C and E on the opposite side of $\overleftrightarrow{BC}$ from A	Parallel Postulate (through a point not on a line there is exactly one parallel)
3	$m\angle DBA + m\angle ABE = 180°$	Linear pairs are supplementary (Euclid I.13)
4	$m\angle ABE = m\angle ABC + m\angle CBE$	Angle Addition Postulate
5	$m\angle DBA + m\angle ABC + m\angle CBE = 180°$	Substitution Property of Equality
6	$\angle DBA \cong \angle BAC$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{AB}$)
7	$\angle CBE \cong \angle BCA$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BC}$)
8	$m\angle DBA = m\angle BAC$ and $m\angle CBE = m\angle BCA$	Definition of congruent angles
9	$m\angle BAC + m\angle ABC + m\angle BCA = 180°$	Substitution Property of Equality

∎

Euclid's own proof extends side BC and draws the parallel through C instead of through B; the idea is the same.

THEOREM **Exterior Angle Theorem** Euclid I.32

The measure of an exterior angle of a triangle equals the sum of the measures of its two remote interior angles.



PARAGRAPH PROOF

Given: $\angle BCD$ is an exterior angle of $\triangle ABC$ (D lies on the ray opposite $\overrightarrow{CA}$)

Prove: $m\angle BCD = m\angle A + m\angle ABC$

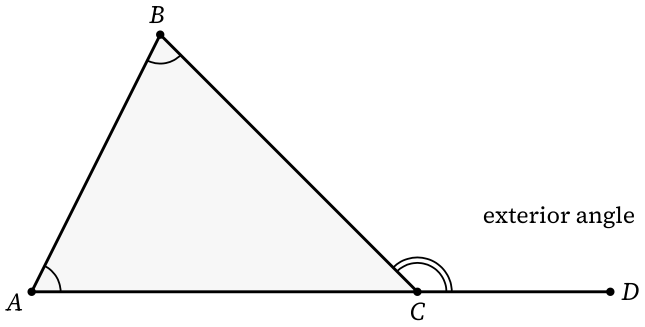
Because D lies on the ray opposite $\overrightarrow{CA}$, the angles $\angle BCD$ and $\angle BCA$ form a linear pair, so $m\angle BCD + m\angle BCA = 180°$ (linear pairs are supplementary). By the Triangle Angle Sum Theorem, $m\angle A + m\angle ABC + m\angle BCA = 180°$.

Both sums equal 180°, so by the Transitive Property of Equality $m\angle BCD + m\angle BCA = m\angle A + m\angle ABC + m\angle BCA$. Subtracting $m\angle BCA$ from both sides (Subtraction Property of Equality) gives $m\angle BCD = m\angle A + m\angle ABC$.

∎

THEOREM **Exterior Angle Inequality** Euclid I.16

The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.



PARAGRAPH PROOF

Given: $\angle BCD$ is an exterior angle of $\triangle ABC$

Prove: $m\angle BCD > m\angle A$ and $m\angle BCD > m\angle ABC$

By the Exterior Angle Theorem, $m\angle BCD = m\angle A + m\angle ABC$. Every angle of a triangle has a positive measure, so $m\angle ABC > 0$. Adding a positive number to $m\angle A$ makes the result larger, so $m\angle BCD > m\angle A$. In the same way, since $m\angle A > 0$, $m\angle BCD > m\angle ABC$.

∎

This quick proof uses I.32 and so depends on the Parallel Postulate. Euclid proved I.16 without it: he bisects $\overline{AC}$ at M, extends $\overline{BM}$ its own length to F, and shows by Side-Angle-Side that $\angle MCF \cong \angle A$, an angle that is only part of the exterior angle. You will be able to follow that proof after Lesson 4.3, and you will meet it again in Lesson 5.5.

COROLLARY **Corollaries to the Triangle Angle Sum Theorem**

(1) The acute angles of a right triangle are complementary. (2) Each angle of an equiangular triangle measures $60°$.

PARAGRAPH PROOF

Prove: Statements (1) and (2)
(1) In right $\triangle ABC$ with right angle C, the Triangle Angle Sum Theorem gives $m\angle A + m\angle B + 90^\circ = 180^\circ$, so $m\angle A + m\angle B = 90^\circ$ by the Subtraction Property of Equality. By definition, $\angle A$ and $\angle B$ are complementary.

(2) In an equiangular triangle, call the common measure x. By the Triangle Angle Sum Theorem $x + x + x = 180^\circ$, so $3x = 180^\circ$ and $x = 60^\circ$ by the Division Property of Equality.

∎

KEY FORMULAS

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

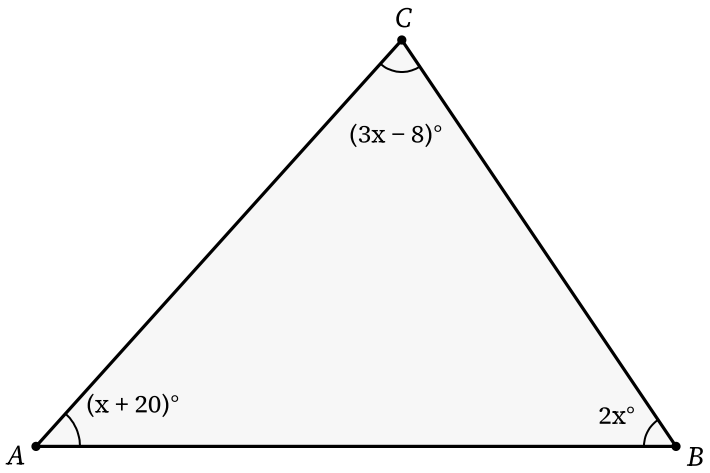
$$m(\text{exterior angle}) = m(\text{remote interior angle 1}) + m(\text{remote interior angle 2})$$

From history. According to Proclus, the historian Eudemus credited the Pythagoreans with discovering that every triangle has its interior angles equal to two right angles, and with proving it. Euclid puts the theorem in Book I as Proposition 32, and it is no accident that it comes right after the propositions on parallel lines (I.27 to I.31). Centuries later, mathematicians who tried to do without the fifth postulate found that the angle sum and the parallel postulate stand or fall together.

Euclid connection. Euclid I.32 says: 'In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.' Notice that Euclid says 'two right angles' where we say 180°. The weaker I.16, 'the exterior angle is greater than either of the interior and opposite angles,' comes earlier because it does not need the parallel postulate.

Practice habit. Recite the Triangle Angle Sum Theorem and the Exterior Angle Theorem word for word until they are automatic. When you see an exterior angle, first point to its two remote interior angles with your pencil, and estimate every angle from the figure to check that your answer is reasonable.

Example 1. In $\triangle ABC$, $m\angle A = (x + 20)^\circ$, $m\angle B = (2x)^\circ$ and $m\angle C = (3x - 8)^\circ$. Find x and the three angle measures, then classify the triangle by its angles.

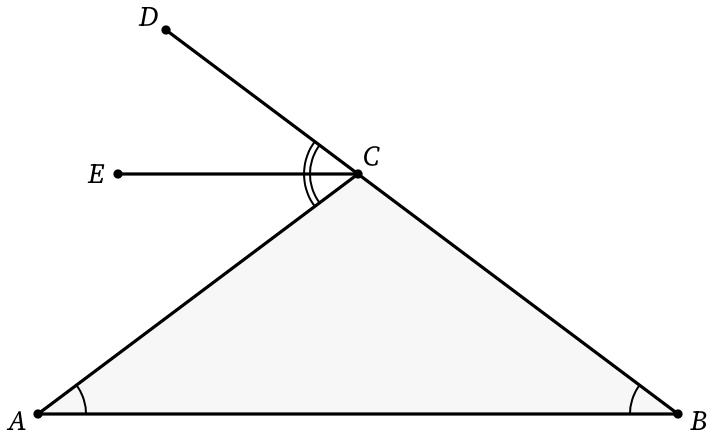


Show solution

1. By the Triangle Angle Sum Theorem, $(x + 20) + 2x + (3x - 8) = 180$.
2. Combine like terms: $6x + 12 = 180$, so $6x = 168$ and $x = 28$.
3. Substitute: $m\angle A = 28 + 20 = 48^\circ$, $m\angle B = 2(28) = 56^\circ$, $m\angle C = 3(28) - 8 = 76^\circ$.
4. Check: $48 + 56 + 76 = 180$. All three angles are less than 90°, so the triangle is acute.

Answer: $x = 28$; $m\angle A = 48^\circ$, $m\angle B = 56^\circ$, $m\angle C = 76^\circ$; an acute triangle

Example 2. In the figure, D lies on $\overrightarrow{BC}$ beyond C, so $\angle ACD$ is an exterior angle of $\triangle ABC$. $\overrightarrow{CE}$ bisects $\angle ACD$, and $m\angle CAB = m\angle ABC$. Prove that $\overleftrightarrow{CE} \parallel \overleftrightarrow{AB}$.



Show solution

1. Work backward: to prove lines parallel, look for congruent alternate interior angles. With transversal $\overleftrightarrow{AC}$, the pair is $\angle ACE$ and $\angle CAB$.
2. The Exterior Angle Theorem says $m\angle ACD = m\angle CAB + m\angle ABC$, and the two remote angles are equal, so the exterior angle is twice $m\angle CAB$.
3. The bisector cuts the exterior angle in half, so $m\angle ACE$ equals $m\angle CAB$. Then use the converse of the Alternate Interior Angles Theorem (Euclid I.27).

TWO-COLUMN PROOF

Given: $\angle ACD$ is an exterior angle of $\triangle ABC$; $\overrightarrow{CE}$ bisects $\angle ACD$; $m\angle CAB = m\angle ABC$

Prove: $\overleftrightarrow{CE} \parallel \overleftrightarrow{AB}$

	STATEMENTS	REASONS
1	$\angle ACD$ is an exterior angle of $\triangle ABC$	Given
2	$m\angle ACD = m\angle CAB + m\angle ABC$	Exterior Angle Theorem
3	$m\angle CAB = m\angle ABC$	Given
4	$m\angle ACD = 2 \cdot m\angle CAB$	Substitution Property of Equality
5	$\overrightarrow{CE}$ bisects $\angle ACD$	Given
6	$m\angle ACE = m\angle ECD$	Definition of angle bisector

	STATEMENTS	REASONS
7	$m\angle ACE + m\angle ECD = m\angle ACD$	Angle Addition Postulate
8	$2 \cdot m\angle ACE = m\angle ACD$	Substitution Property of Equality
9	$2 \cdot m\angle ACE = 2 \cdot m\angle CAB$	Transitive Property of Equality
10	$m\angle ACE = m\angle CAB$	Division Property of Equality
11	$\angle ACE \cong \angle CAB$	Definition of congruent angles
12	$\overleftrightarrow{CE} \parallel \overleftrightarrow{AB}$	If alternate interior angles are congruent, the lines are parallel (Euclid I.27)

∎

Answer: See the model proof.

PRACTICE

1. Easy Grace is edging a triangular garden bed. Two of its corners measure $47°$ and $68°$. What is the measure of the third corner?

   Show answer

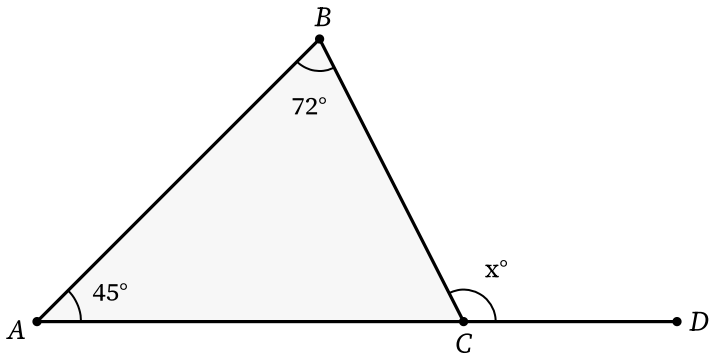
   $180 - 47 - 68 = 65$, so the third corner measures $65°$.

2. Easy In right $\triangle PQR$, $\angle Q$ is the right angle and $m\angle P = 38°$. Find $m\angle R$.

   Show answer

   The acute angles of a right triangle are complementary, so $m\angle R = 90 - 38 = 52°$.

3. **Easy** In the figure, $\angle BCD$ is an exterior angle of $\triangle ABC$, $m\angle A = 45^\circ$ and $m\angle ABC = 72^\circ$. Find x.



Show answer

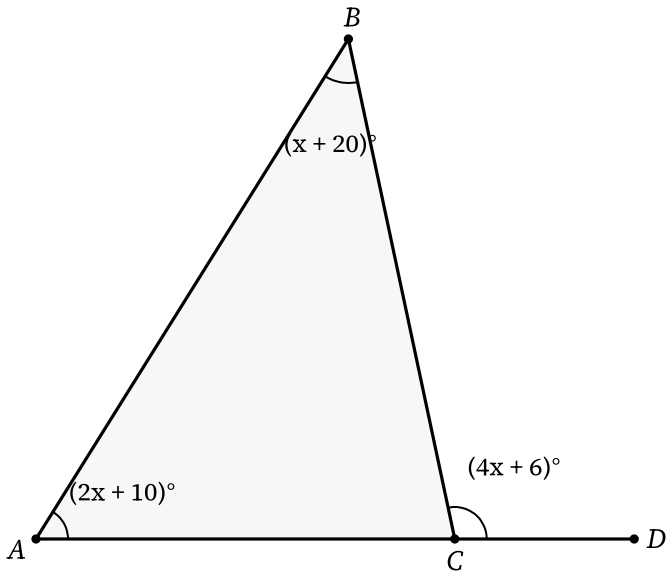
By the Exterior Angle Theorem, $x = 45 + 72 = 117$.

4. **Easy** Can a triangle have two right angles? Explain why or why not.

Show answer

No. Two right angles already add to 180°, so the third angle would have to measure 0°, which is impossible because every angle of a triangle has a positive measure. (In fact the two sides perpendicular to the third side would be parallel and could never meet.)

5. **Medium** In the figure, $\angle BCD$ is an exterior angle of $\triangle ABC$ with $m\angle BCD = (4x + 6)^\circ$, $m\angle A = (2x + 10)^\circ$ and $m\angle ABC = (x + 20)^\circ$. Find x and $m\angle BCD$.



Show answer

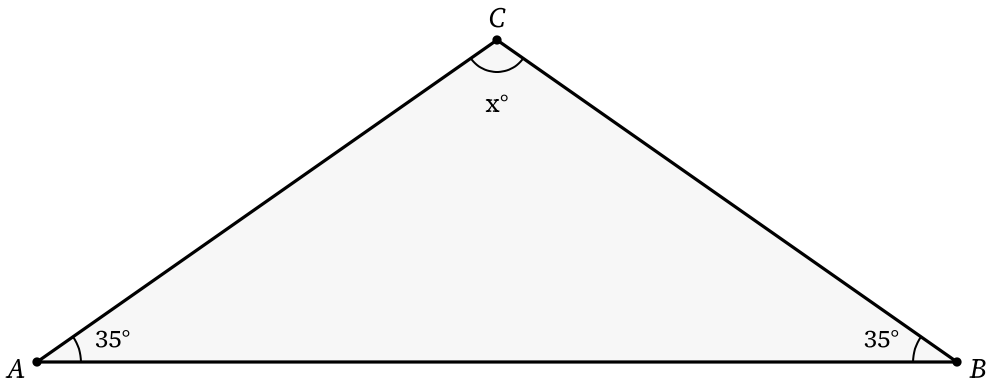
$4x + 6 = (2x + 10) + (x + 20)$, so $4x + 6 = 3x + 30$ and $x = 24$. Then $m\angle BCD = 4(24) + 6 = 102^\circ$.

6. **Medium** The angles of a triangle are in the ratio $2 : 3 : 4$. Find the measure of each angle and classify the triangle by its angles.

Show answer

Let the angles be $2k$, $3k$ and $4k$. Then $9k = 180$, so $k = 20$ and the angles are 40°, 60° and 80°. All are acute, so the triangle is acute.

7. **Medium** The Miller family is framing a shed roof. Each rafter makes a 35° angle with the horizontal ceiling joist, as in the figure. At what angle x do the two rafters meet at the peak?



ceiling joist

Show answer

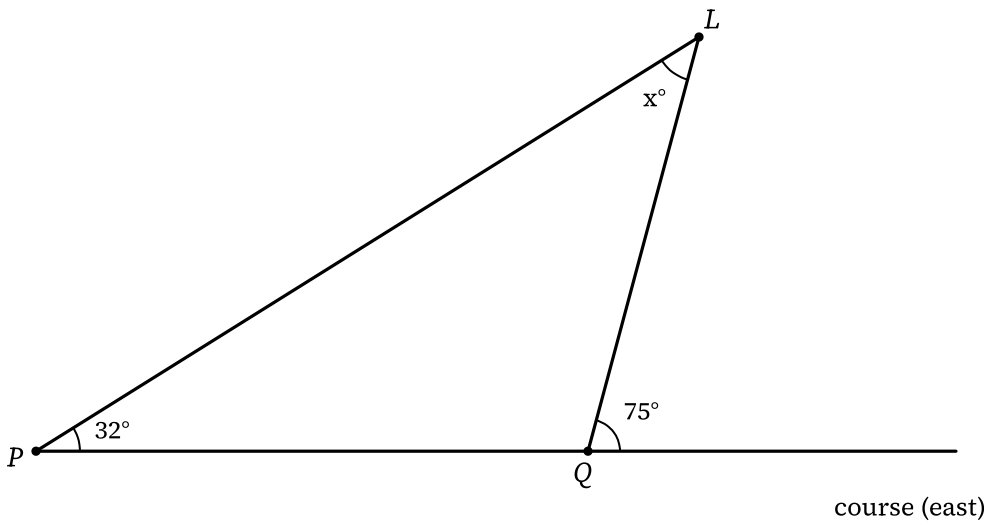
$x = 180 - 35 - 35 = 110$, so the rafters meet at 110°.

8. **Medium** The acute angles of a right triangle measure $(3x + 2)^\circ$ and $(4x + 4)^\circ$. Find x and the measure of each acute angle.

Show answer

The acute angles are complementary: $(3x + 2) + (4x + 4) = 90$, so $7x + 6 = 90$ and $x = 12$. The angles are $3(12) + 2 = 38^\circ$ and $4(12) + 4 = 52^\circ$.

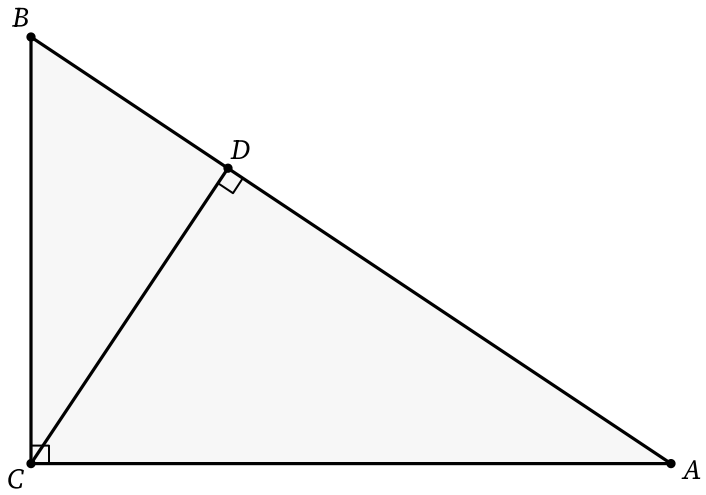
9. **Medium** A ship sails due east along a straight course past a lighthouse L. At point P the navigator measures the angle between the course and the line of sight to the lighthouse as 32°. Later, at point Q farther east, the angle between the course ahead and the line of sight is 75° (see the figure). Find $x = m\angle PLQ$, the angle at the lighthouse between the two sight lines.



Show answer

The 75° angle is an exterior angle of $\triangle PQL$ with remote interior angles 32° and x°. So $75 = 32 + x$ and $x = 43$. The angle at the lighthouse is 43°.

10. **Hard** In the figure, $\triangle ABC$ has a right angle at C, and $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$. Prove that $\angle ACD \cong \angle B$.



Show answer

See the model proof.

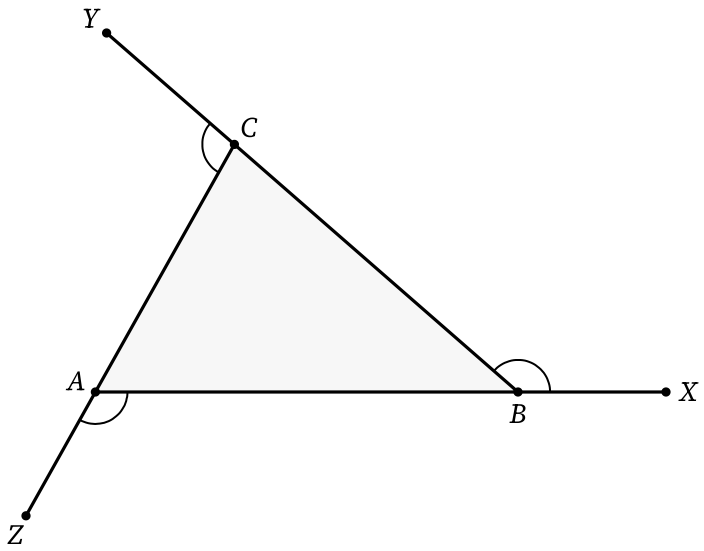
TWO-COLUMN PROOF

Given: $\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$

Prove: $\angle ACD \cong \angle B$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle	Given
2	$\overline{CD} \perp \overline{AB}$	Given
3	$\angle ADC$ is a right angle	Definition of perpendicular lines
4	$\angle A$ and $\angle ACD$ are complementary	The acute angles of a right triangle are complementary (in right $\triangle ADC$)
5	$\angle A$ and $\angle B$ are complementary	The acute angles of a right triangle are complementary (in right $\triangle ACB$)
6	$\angle ACD \cong \angle B$	Congruent Complements Theorem (complements of the same angle are congruent)

11. Hard In the figure, one exterior angle is drawn at each vertex of $\triangle ABC$: $\angle BAZ$, $\angle CBX$ and $\angle ACY$. Prove that $m\angle BAZ + m\angle CBX + m\angle ACY = 360°$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

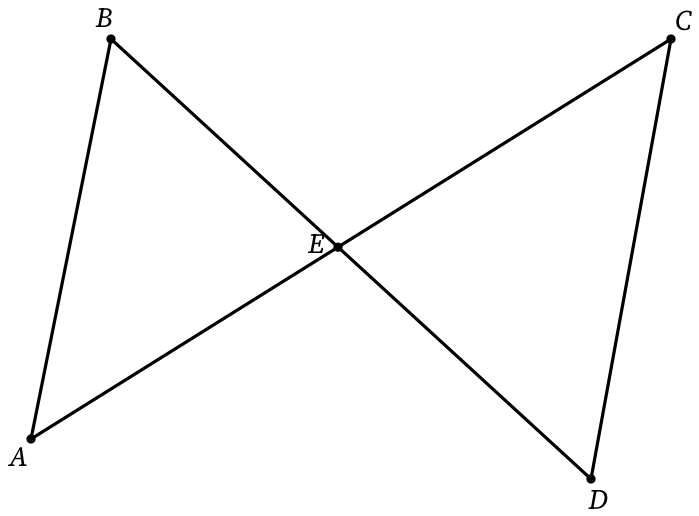
Given: $\angle BAZ$, $\angle CBX$ and $\angle ACY$ are exterior angles of $\triangle ABC$ at A, B and C

Prove: $m\angle BAZ + m\angle CBX + m\angle ACY = 360°$

Each exterior angle forms a linear pair with the interior angle at the same vertex, and linear pairs are supplementary. So $m\angle BAZ + m\angle BAC = 180°$, $m\angle CBX + m\angle ABC = 180°$ and $m\angle ACY + m\angle BCA = 180°$.

Adding the three equations (Addition Property of Equality) gives $(m\angle BAZ + m\angle CBX + m\angle ACY) + (m\angle BAC + m\angle ABC + m\angle BCA) = 540°$. By the Triangle Angle Sum Theorem the second group equals $180°$, so by substitution $(m\angle BAZ + m\angle CBX + m\angle ACY) + 180° = 540°$. Subtracting $180°$ from both sides gives $m\angle BAZ + m\angle CBX + m\angle ACY = 360°$.

12. **Hard** In the figure, $\overline{AC}$ and $\overline{BD}$ intersect at E. Prove that $m\angle BAE + m\angle ABE = m\angle DCE + m\angle CDE$.



Show answer

See the model proof.

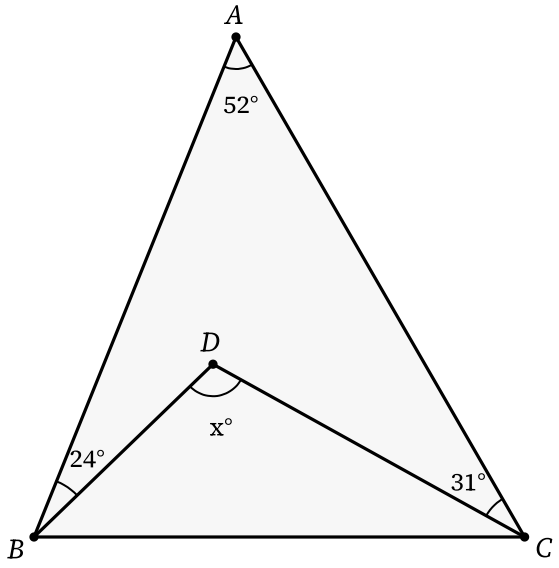
TWO-COLUMN PROOF

Given: $\overline{AC}$ and $\overline{BD}$ intersect at E

Prove: $m\angle BAE + m\angle ABE = m\angle DCE + m\angle CDE$

	STATEMENTS	REASONS
1	$\overline{AC}$ and $\overline{BD}$ intersect at E	Given
2	$\angle AEB \cong \angle CED$	Vertical Angles Theorem
3	$m\angle AEB = m\angle CED$	Definition of congruent angles
4	$m\angle BAE + m\angle ABE + m\angle AEB = 180°$	Triangle Angle Sum Theorem ($\triangle ABE$)
5	$m\angle DCE + m\angle CDE + m\angle CED = 180°$	Triangle Angle Sum Theorem ($\triangle CDE$)
6	$m\angle BAE + m\angle ABE + m\angle AEB = m\angle DCE + m\angle CDE + m\angle CED$	Transitive Property of Equality
7	$m\angle BAE + m\angle ABE + m\angle AEB = m\angle DCE + m\angle CDE + m\angle AEB$	Substitution Property of Equality
8	$m\angle BAE + m\angle ABE = m\angle DCE + m\angle CDE$	Subtraction Property of Equality

13. **Challenge** In the figure, point D lies inside $\triangle ABC$. $m\angle BAC = 52°$, $m\angle ABD = 24°$ and $m\angle ACD = 31°$. Find $x = m\angle BDC$ and explain your reasoning. (Hint: draw the auxiliary ray $\overrightarrow{AD}$ and extend it to meet $\overline{BC}$.)



Show answer

Extend $\overrightarrow{AD}$ to meet $\overline{BC}$ at P. In $\triangle ABD$, $\angle BDP$ is an exterior angle, so $m\angle BDP = m\angle BAD + 24°$. In $\triangle ACD$, $m\angle PDC = m\angle CAD + 31°$. Adding, $m\angle BDC = (m\angle BAD + m\angle CAD) + 24° + 31° = 52° + 55° = 107°$. So $x = 107$.

COMMON MISTAKES

- Adding the adjacent interior angle when using the Exterior Angle Theorem. The exterior angle equals the sum of the two angles NOT touching it; the angle next to it is its supplement.
- Using the angle sum on a figure that is not a triangle, or forgetting an angle. Before writing the equation, point to the three vertices of the triangle and list its three angles.
- Classifying a triangle by its sides from how it looks. Only given or proved information counts; a triangle that looks isosceles may not be.
- Drawing an auxiliary line and giving it two properties at once (for example 'draw the line through B parallel to $\overline{AC}$ and bisecting $\angle B$'). You may only add a line with one condition you can justify.

Make it a game · Tear and Prove. Tear off the three corners of a paper triangle and line them up along a straight edge to see 180 degrees. Then play the lightning round: parent names two angles, and you give the third, the exterior angle and the classification, such as acute scalene. Finish by explaining Euclid's proof using an auxiliary parallel line.

4.2 Congruent Figures

OBJECTIVES

- Identify corresponding sides and angles of congruent polygons and write correct congruence statements.
- Prove and apply the Third Angles Theorem.
- Use the reflexive, symmetric and transitive properties of triangle congruence.
- Prove two triangles congruent from the definition by showing all six pairs of corresponding parts congruent.

WATCH

Congruent Figures (1 of 2: Introduction)

Eddie Woo · 8:00

https://www.youtube.com/watch?v=3qxZYeVgvBk

> **WHEN TO WATCH**
> Before the Core ideas: what congruent means.

Note: The Third Angles Theorem and congruence statements are taught from the Core ideas.

BACKUP VIDEO

What are Congruent Figures? | An Intro to Congruent Figures | Math with Mr. J

Math with Mr. J · 8:20

https://www.youtube.com/watch?v=_-4aFZxaj_g

Before Example 1: matching corresponding sides and angles of congruent figures.

KEY TERMS

Congruent polygons

Two polygons are congruent if and only if their vertices can be matched so that every pair of corresponding sides is congruent and every pair of corresponding angles is congruent. Informally: same size and same shape.

Corresponding parts

Sides or angles that are matched with each other by the pairing of vertices in two polygons.

Congruence statement

A statement such as $\triangle ABC \cong \triangle DEF$, in which the order of the letters tells which vertices correspond: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$.

Third Angles Theorem

If two angles of one triangle are congruent to two angles of another triangle, then the third angles are also congruent.

CORE IDEAS

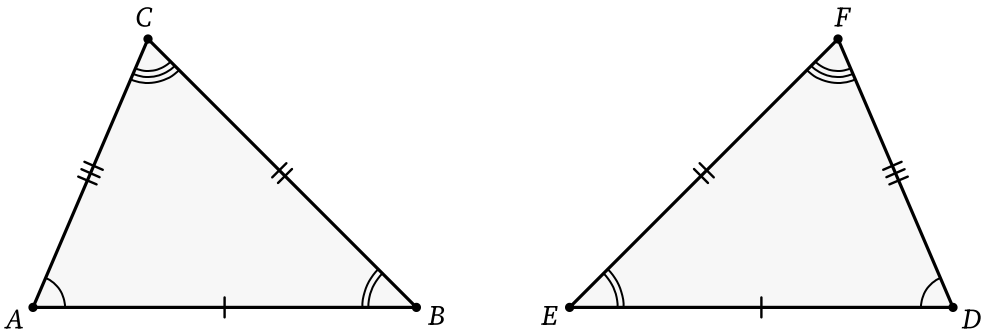
Two figures are congruent when they have exactly the same size and shape: one could be picked up and laid exactly on top of the other, perhaps after turning it or flipping it over. Euclid used just this idea when he said that things which coincide with one another are equal. In Unit 7 we will make 'picking up and laying down' precise with rigid motions; for now we use the definition in terms of parts.

Congruent polygons are defined by their parts: every side of one matches a congruent side of the other, and every angle matches a congruent angle. Because this is a definition, it works in both directions. If you know two triangles are congruent, you may conclude that every pair of corresponding parts is congruent; and if you show all six pairs of parts congruent, you may conclude the triangles are congruent.

Order matters in a congruence statement. Writing $\triangle ABC \cong \triangle DEF$ says A matches D, B matches E and C matches F. So $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$. The same two triangles could not be written $\triangle ABC \cong \triangle EFD$ unless they happened to be equilateral. A good habit is to read the statement letter by letter: first letters match, second letters match, third letters match.

The Third Angles Theorem follows from the angle sum. If two angles of one triangle match two angles of another, the third angle in each is 180° minus the same total, so the third angles match too. This is often the quickest way to supply the last pair of angles. Be careful: three pairs of congruent angles do NOT make triangles congruent. A small equilateral triangle and a large one have all angles 60°, yet they are not the same size.

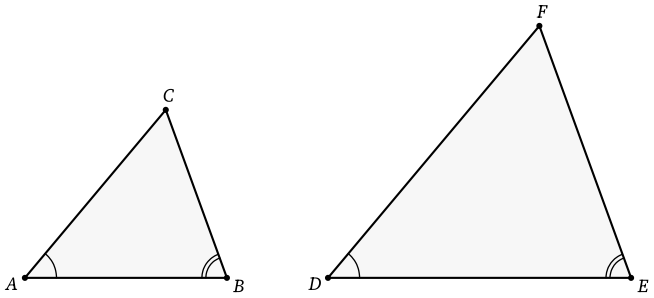
Triangle congruence behaves like equality. Every triangle is congruent to itself (reflexive); if $\triangle ABC \cong \triangle DEF$ then $\triangle DEF \cong \triangle ABC$ (symmetric); and two triangles congruent to the same triangle are congruent to each other (transitive). Each of these is true because the same property already holds for congruent segments and congruent angles (Lesson 1.5).



POSTULATES AND THEOREMS

THEOREM **Third Angles Theorem**

If two angles of one triangle are congruent to two angles of another triangle, then the third angles are also congruent.



TWO-COLUMN PROOF

Given: $\angle A \cong \angle D$; $\angle B \cong \angle E$

Prove: $\angle C \cong \angle F$

	STATEMENTS	REASONS
1	$\angle A \cong \angle D$	Given
2	$\angle B \cong \angle E$	Given
3	$m\angle A = m\angle D$ and $m\angle B = m\angle E$	Definition of congruent angles
4	$m\angle A + m\angle B + m\angle C = 180°$	Triangle Angle Sum Theorem
5	$m\angle D + m\angle E + m\angle F = 180°$	Triangle Angle Sum Theorem
6	$m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F$	Transitive Property of Equality
7	$m\angle A + m\angle B + m\angle C = m\angle A + m\angle B + m\angle F$	Substitution Property of Equality
8	$m\angle C = m\angle F$	Subtraction Property of Equality
9	$\angle C \cong \angle F$	Definition of congruent angles

∎

The two triangles need not be congruent; they only have the same angles.

THEOREM **Properties of Triangle Congruence**

Reflexive: $\triangle ABC \cong \triangle ABC$. *Symmetric: if* $\triangle ABC \cong \triangle DEF$, *then* $\triangle DEF \cong \triangle ABC$. *Transitive: if* $\triangle ABC \cong \triangle DEF$ *and* $\triangle DEF \cong \triangle JKL$, *then* $\triangle ABC \cong \triangle JKL$.

PARAGRAPH PROOF

Given: $\triangle ABC \cong \triangle DEF$; $\triangle DEF \cong \triangle JKL$

Prove: $\triangle ABC \cong \triangle JKL$ (the transitive property)

By the definition of congruent polygons, $\overline{AB} \cong \overline{DE}$ and $\overline{DE} \cong \overline{JK}$, so $\overline{AB} \cong \overline{JK}$ by the Transitive Property of Segment Congruence. In the same way $\overline{BC} \cong \overline{KL}$ and $\overline{CA} \cong \overline{LJ}$.

Also $\angle A \cong \angle D$ and $\angle D \cong \angle J$, so $\angle A \cong \angle J$ by the Transitive Property of Angle Congruence, and likewise $\angle B \cong \angle K$ and $\angle C \cong \angle L$. All six pairs of corresponding parts of $\triangle ABC$ and $\triangle JKL$ are congruent, so $\triangle ABC \cong \triangle JKL$ by the definition of congruent polygons.

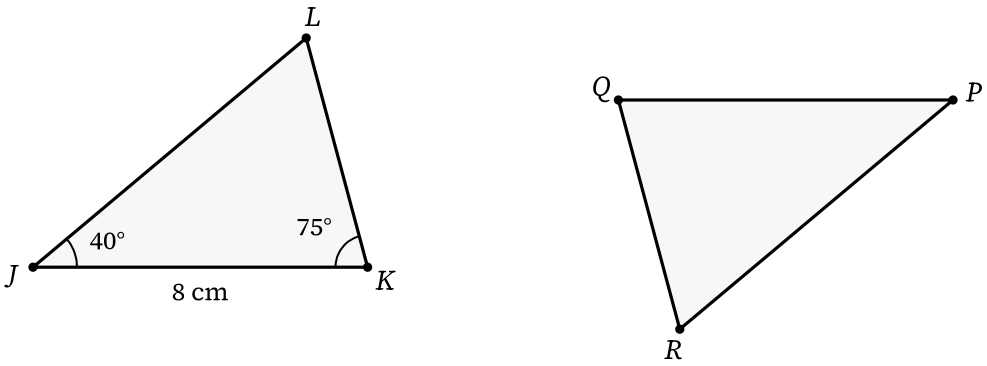
The reflexive property is proved the same way, since every segment and angle is congruent to itself. The symmetric property is left for you in the practice set.

∎

Euclid connection. Euclid never uses the word 'congruent.' His Common Notion 4 says 'Things which coincide with one another are equal to one another,' and when he proves triangles congruent he says they are 'equal' and that 'the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend.' That is his way of naming corresponding parts.

Practice habit. Before you read off any parts, write the vertex matching under the congruence statement ($A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$). Mark every pair of corresponding parts on your diagram with matching tick marks or arcs.

Example 1. In the figure, $\triangle JKL \cong \triangle PQR$. In $\triangle JKL$, $JK = 8$ cm, $m\angle J = 40°$ and $m\angle K = 75°$. Find PQ, $m\angle Q$ and $m\angle R$.

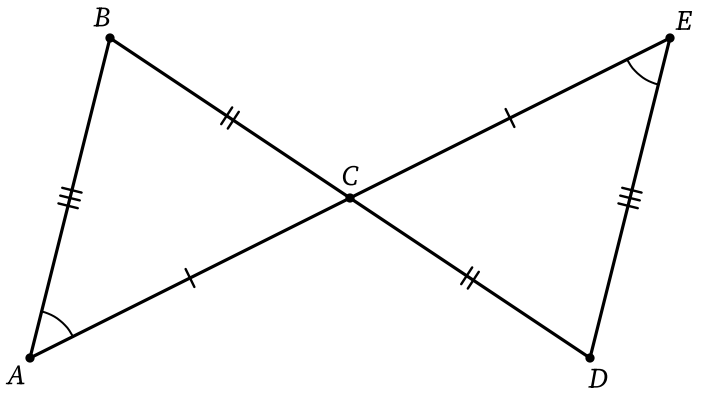


Show solution

1. Write the matching from the statement: $J \leftrightarrow P$, $K \leftrightarrow Q$, $L \leftrightarrow R$.
2. $\overline{PQ}$ corresponds to $\overline{JK}$, so $PQ = 8$ cm.
3. $\angle Q$ corresponds to $\angle K$, so $m\angle Q = 75°$.
4. $\angle R$ corresponds to $\angle L$. By the Triangle Angle Sum Theorem $m\angle L = 180 - 40 - 75 = 65°$, so $m\angle R = 65°$. Notice that the figure shows $\triangle PQR$ turned upside down; the statement, not the picture's position, tells you which parts match.

Answer: $PQ = 8$ cm, $m\angle Q = 75°$, $m\angle R = 65°$

Example 2. In the figure, C is the midpoint of both $\overline{AE}$ and $\overline{BD}$, $\overline{AB} \cong \overline{ED}$ and $\angle BAC \cong \angle DEC$. Prove that $\triangle ABC \cong \triangle EDC$ using the definition of congruent triangles.



Show solution

1. The definition needs all six pairs: three sides and three angles.
2. Sides: two pairs come from the midpoints, and the third is given.
3. Angles: one pair is given, the angles at C are vertical angles, and the Third Angles Theorem gives the last pair.

TWO-COLUMN PROOF

Given: C is the midpoint of $\overline{AE}$ and of $\overline{BD}$; $\overline{AB} \cong \overline{ED}$; $\angle BAC \cong \angle DEC$

Prove: $\triangle ABC \cong \triangle EDC$

	STATEMENTS	REASONS
1	C is the midpoint of $\overline{AE}$	Given
2	$\overline{AC} \cong \overline{EC}$	Definition of midpoint
3	C is the midpoint of $\overline{BD}$	Given
4	$\overline{BC} \cong \overline{DC}$	Definition of midpoint
5	$\overline{AB} \cong \overline{ED}$	Given
6	$\angle BAC \cong \angle DEC$	Given
7	$\angle ACB \cong \angle ECD$	Vertical Angles Theorem
8	$\angle ABC \cong \angle EDC$	Third Angles Theorem
9	$\triangle ABC \cong \triangle EDC$	Definition of congruent polygons (all corresponding

STATEMENTS	REASONS
	sides and angles congruent)

∎

Answer: See the model proof. (In Lessons 4.3 and 4.4 you will learn shortcuts that need only three of the six pairs.)

PRACTICE

1. Easy Given $\triangle PQR \cong \triangle XYZ$, name the angle of $\triangle XYZ$ that is congruent to $\angle Q$.

   Show answer

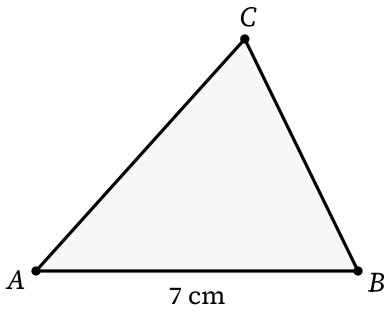
   $\angle Y$ (second letter matches second letter)

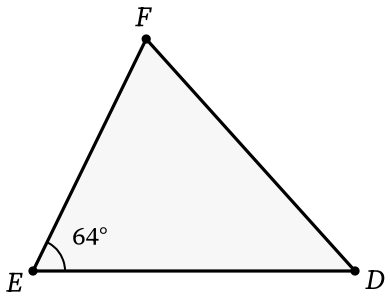
2. Easy Given $\triangle PQR \cong \triangle XYZ$, name the side of $\triangle XYZ$ that is congruent to $\overline{PR}$.

   Show answer

   $\overline{XZ}$ ($P \leftrightarrow X$ and $R \leftrightarrow Z$)

3. Easy In the figure, $\triangle ABC \cong \triangle DEF$. $AB = 7$ cm and $m\angle E = 64°$. Find DE and $m\angle B$.

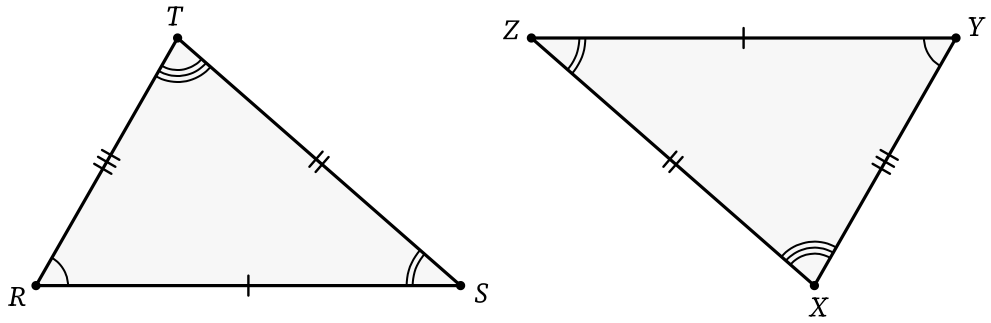




   Show answer

   $\overline{DE}$ corresponds to $\overline{AB}$, so $DE = 7$ cm; $\angle B$ corresponds to $\angle E$, so $m\angle B = 64°$.

4. **Easy** In the figure, the marks show which sides and angles of $\triangle RST$ and $\triangle XYZ$ are congruent. Write a correct congruence statement that begins with $\triangle RST$.



Show answer

$\triangle RST \cong \triangle YZX$ (the marks match R with Y, S with Z and T with X)

5. **Medium** $\triangle ABC \cong \triangle DEF$, $AB = 2x + 3$ and $DE = 5x - 9$. Find x and AB.

Show answer

Corresponding sides are congruent: $2x + 3 = 5x - 9$, so $12 = 3x$ and $x = 4$. Then $AB = 2(4) + 3 = 11$.

6. **Medium** Two triangular gable vents on a barn each have one angle of 48° and one angle of 71°. Samuel measures the third angle of the first vent as 61°. Without measuring, what is the third angle of the second vent? Name the theorem that guarantees it.

Show answer

61°, by the Third Angles Theorem (two angles of one triangle are congruent to two angles of the other).

7. **Medium** $\triangle ABC \cong \triangle DEF$, $m\angle C = (4y - 6)^\circ$ and $m\angle F = (3y + 10)^\circ$. Find y and $m\angle C$.

Show answer

$4y - 6 = 3y + 10$, so $y = 16$ and $m\angle C = 4(16) - 6 = 58^\circ$.

8. **Medium** Ruth is cutting congruent triangles for a quilt, with $\triangle RST \cong \triangle UVW$. She knows $RS = 5$ in, $ST = 7$ in and $UW = 9$ in. What is the perimeter of $\triangle RST$?

Show answer

$\overline{RT}$ corresponds to $\overline{UW}$, so $RT = 9$ in. The perimeter is $5 + 7 + 9 = 21$ in.

9. **Medium** If all three angles of one triangle are congruent to the three angles of another triangle, must the triangles be congruent? Explain why or why not with an example.

Show answer

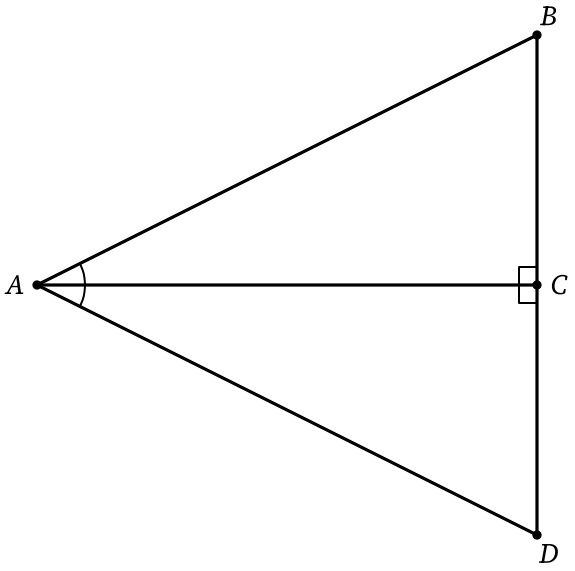
No. An equilateral triangle with sides 3 cm and one with sides 5 cm both have three 60° angles (Lesson 4.1), but their sides are not congruent, so they are not congruent. Matching angles fix the shape but not the size.

10. **Hard** $\triangle ABC \cong \triangle DEF$ with $AB = 3x - y$, $DE = 11$, $BC = x + y$ and $EF = 9$. Find x and y.

Show answer

$3x - y = 11$ and $x + y = 9$. Adding the equations gives $4x = 20$, so $x = 5$ and $y = 4$. Check: $3(5) - 4 = 11$ and $5 + 4 = 9$.

11. Hard In the figure, $\angle BAC \cong \angle DAC$, and $\angle ACB$ and $\angle ACD$ are right angles. Prove that $\angle B \cong \angle D$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\angle BAC \cong \angle DAC$; $\angle ACB$ and $\angle ACD$ are right angles

Prove: $\angle B \cong \angle D$

	STATEMENTS	REASONS
1	$\angle BAC \cong \angle DAC$	Given
2	$\angle ACB$ and $\angle ACD$ are right angles	Given
3	$\angle ACB \cong \angle ACD$	All right angles are congruent
4	$\angle B \cong \angle D$	Third Angles Theorem (in $\triangle ABC$ and $\triangle ADC$)

∎

12. Hard Prove the symmetric property of triangle congruence: if $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$. Write a paragraph proof.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\triangle ABC \cong \triangle DEF$

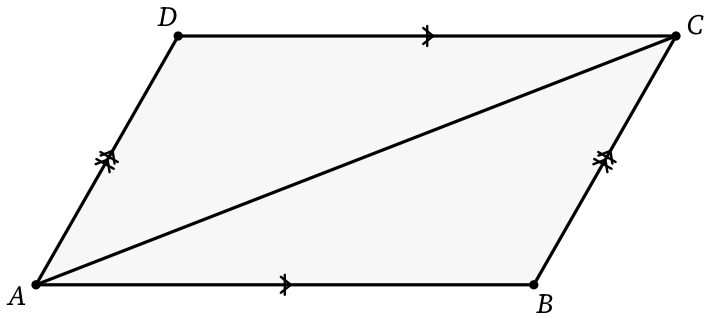
Prove: $\triangle DEF \cong \triangle ABC$

By the definition of congruent polygons, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.

By the Symmetric Properties of Segment and Angle Congruence, $\overline{DE} \cong \overline{AB}$, $\overline{EF} \cong \overline{BC}$, $\overline{FD} \cong \overline{CA}$, $\angle D \cong \angle A$, $\angle E \cong \angle B$ and $\angle F \cong \angle C$. These are all six pairs of corresponding parts under the matching $D \leftrightarrow A$, $E \leftrightarrow B$, $F \leftrightarrow C$, so $\triangle DEF \cong \triangle ABC$ by the definition of congruent polygons.

∎

13. Hard In quadrilateral $ABCD$ in the figure, $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$, $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$. Diagonal $\overline{AC}$ is drawn. Prove that $\triangle ABC \cong \triangle CDA$ using the definition of congruent triangles.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \parallel \overline{BC}$; $\overline{AB} \cong \overline{CD}$; $\overline{AD} \cong \overline{CB}$

Prove: $\triangle ABC \cong \triangle CDA$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$	Given
2	$\overline{BC} \cong \overline{DA}$	Given (and the Symmetric Property of Congruence)
3	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
4	$\overline{AB} \parallel \overline{DC}$	Given
5	$\angle BAC \cong \angle DCA$	Alternate Interior Angles Theorem (transversal $\overline{AC}$)
6	$\overline{AD} \parallel \overline{BC}$	Given
7	$\angle BCA \cong \angle DAC$	Alternate Interior Angles Theorem (transversal $\overline{AC}$)
8	$\angle B \cong \angle D$	Third Angles Theorem
9	$\triangle ABC \cong \triangle CDA$	Definition of congruent polygons (all corresponding sides and angles congruent)

14. **Challenge** Suppose $\triangle ABC \cong \triangle BCA$. (Read the statement carefully: it matches A with B, B with C and C with A.) What kind of triangle must $\triangle ABC$ be? Explain.

Show answer

Equilateral. The matching gives $\overline{AB} \cong \overline{BC}$ (first two letters) and $\overline{BC} \cong \overline{CA}$ (last two letters), so all three sides are congruent by the Transitive Property. (It is also equiangular: $\angle A \cong \angle B \cong \angle C$.)

COMMON MISTAKES

- Reading parts from the picture instead of the statement. If $\triangle ABC \cong \triangle FED$, then $\overline{AB}$ matches $\overline{FE}$, even if $\overline{DE}$ looks closer. Always write the vertex matching first.
- Writing a congruence statement in the wrong order, such as $\triangle RST \cong \triangle XYZ$ when the marks match R with Y. Find each vertex's partner by its marked angle before writing.
- Thinking that three pairs of congruent angles make congruent triangles. Equal angles give the same shape, not the same size.
- Using the Third Angles Theorem when only one pair of angles is known to be congruent. It needs two pairs.

> **Make it a game · Congruence Statement Scramble.** Fold paper twice and cut two congruent triangles. Label one ABC and the other with different letters, then turn or flip it. Parent rotates the second triangle; you write the correct congruence statement and list all six corresponding parts. Add a challenge: parent states two angle pairs, and you name the third pair and the theorem.

4.3 Proving Triangles Congruent: SSS and SAS

OBJECTIVES

- State the SSS and SAS congruence postulates and identify an included angle.
- Prove triangles congruent using SSS and SAS in two-column and paragraph proofs.
- Explain why SSA is not a congruence test.
- Construct a triangle from three given segments (Euclid I.22) and copy a triangle with compass and straightedge.

WATCH

Triangle Congruence (quick review)

Eddie Woo · 12:56

https://www.youtube.com/watch?v=7ynkddMyV94

> **WHEN TO WATCH**
> Before Example 1: the triangle congruence tests; focus on SSS and SAS.

Note: Eddie Woo teaches in Australia: he also covers AAS and 'RHS', which is the course's HL (Lesson 4.7). The SSA warning is in the course.

BACKUP VIDEO

Congruent Figures (2 of 2: Transformations)

Eddie Woo · 7:17

https://www.youtube.com/watch?v=N8ABWeeGWU4

After the lesson: congruence as moving one figure exactly onto another.

KEY TERMS

Included angle

The angle formed by two sides of a triangle; for sides $\overline{AB}$ and $\overline{AC}$ it is $\angle A$, the angle at their common endpoint.

Included side

The side of a triangle that lies between two angles; for $\angle A$ and $\angle B$ it is $\overline{AB}$.

Side-Side-Side (SSS) Postulate

If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

Side-Angle-Side (SAS) Postulate

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

Rigid

Unable to change shape without changing the length of a side. A triangle made of fixed-length bars is rigid; a four-sided frame is not.

CORE IDEAS

In Lesson 4.2 you proved triangles congruent by checking all six pairs of parts. That is a lot of work. The good news is that three well-chosen pairs are enough, because three well-chosen parts already fix the triangle completely.

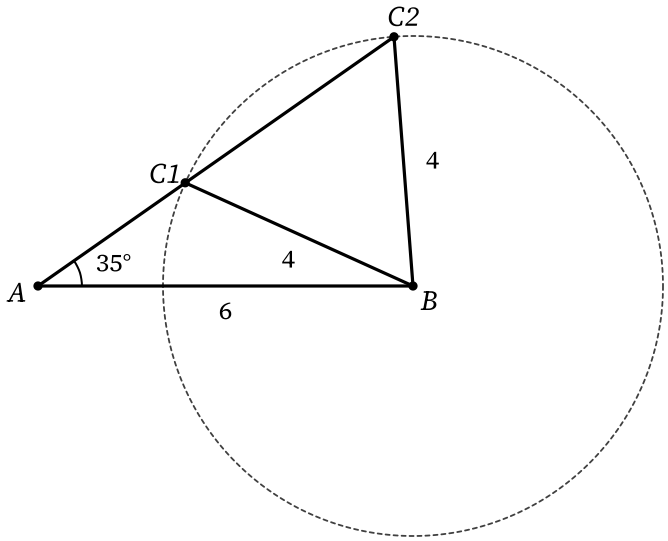
Why do three sides fix a triangle? Try it with a compass. Lay down one side. The third vertex must be at the length of the second side from one end, so it lies on a circle; and at the length of the third side from the other end, so it lies on a second circle. On one side of the base those two circles cross in exactly one point. So there is only one triangle (up to its position) with those three sides. That is the Side-Side-Side (SSS) Postulate, Euclid I.8, and it is why a triangle of three bolted boards is rigid while a rectangle of four boards can be pushed into a slanted shape.

Why do two sides and the angle between them fix a triangle? Lay down one side, turn through the given angle at one end, and measure off the second side. The third vertex is now pinned down, and the third side is simply the segment joining two fixed points. That is the Side-Angle-Side (SAS) Postulate, Euclid I.4. The angle must be the included angle, the one formed by the two given sides.

Why is Side-Side-Angle (SSA) not a test? If the angle is not between the two sides, the third vertex may land in two different places. In the figure, $\angle A = 35^\circ$, $AB = 6$ and the side opposite $\angle A$ is 4; a circle of radius 4 about B crosses the other ray of $\angle A$ twice, at C_1 and C_2. Triangles ABC_1 and ABC_2 have the same two sides and the

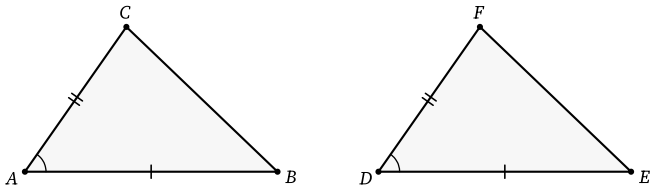
same non-included angle, yet they are clearly not congruent. One counterexample is enough to show SSA is not a valid test.

In a proof, a side shared by two triangles is congruent to itself by the Reflexive Property of Congruence, and vertical angles give a free pair of congruent angles. Look for these first. Then check that your three pairs are in the right pattern and in the right order before you write SSS or SAS.



POSTULATES AND THEOREMS

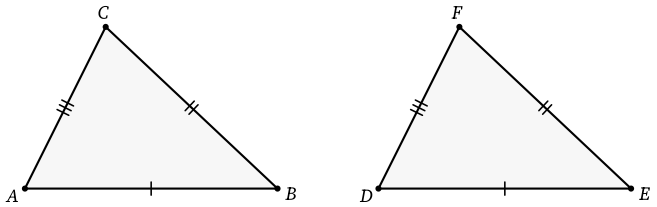
POSTULATE **Side-Angle-Side (SAS) Congruence Postulate** Euclid I.4



If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

Euclid proves I.4 by 'applying' one triangle to the other: he places the angle of one on the equal angle of the other and argues that everything must coincide. Modern writers such as Hilbert noticed that moving figures is not one of the postulates, so they take SAS itself as a postulate. In Unit 7 you will see how rigid motions justify it.

POSTULATE **Side-Side-Side (SSS) Congruence Postulate** Euclid I.8

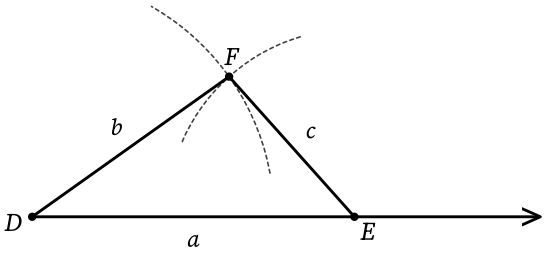


If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

Euclid proves I.8 from I.7 (two triangles on the same base with equal sides cannot have different apexes on the same side). We accept SSS as a postulate in this course; it can be proved from SAS, and Unit 7 shows how rigid motions explain it.

CONSTRUCTIONS

CONSTRUCTION **Construct a triangle with three given sides** Euclid I.22

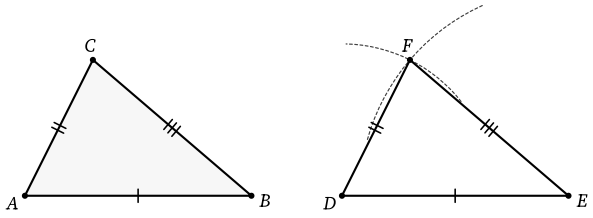


1. You are given three segments of lengths a, b and c. Draw a ray and mark its endpoint D.
2. Set your compass to length a. With the point at D, mark E on the ray, so $DE = a$.
3. Set your compass to length b. With the point at D, draw a long arc above the ray.
4. Set your compass to length c. With the point at E, draw an arc crossing the first arc. Call the crossing point F.
5. Draw $\overline{DF}$ and $\overline{EF}$. Then $\triangle DEF$ has sides a, b and c.

Why it works. $DF = b$ and $EF = c$ because every point on a compass arc is the compass width from its center. The two

arcs meet only when each pair of lengths adds to more than the third (the Triangle Inequality, Euclid I.20, proved in Lesson 5.6); Euclid states this condition at the start of I.22. Any two triangles built from the same three segments are congruent by SSS, so the result does not depend on where you start.

CONSTRUCTION **Copy a triangle**



1. Given $\triangle ABC$, draw a ray with endpoint D and copy $\overline{AB}$ onto it to get $\overline{DE}$ (Lesson 2.2).
2. Set the compass to length AC (point on A, pencil on C). Draw an arc centered at D.
3. Set the compass to length BC. Draw an arc centered at E crossing the first arc at F.
4. Draw $\overline{DF}$ and $\overline{EF}$.

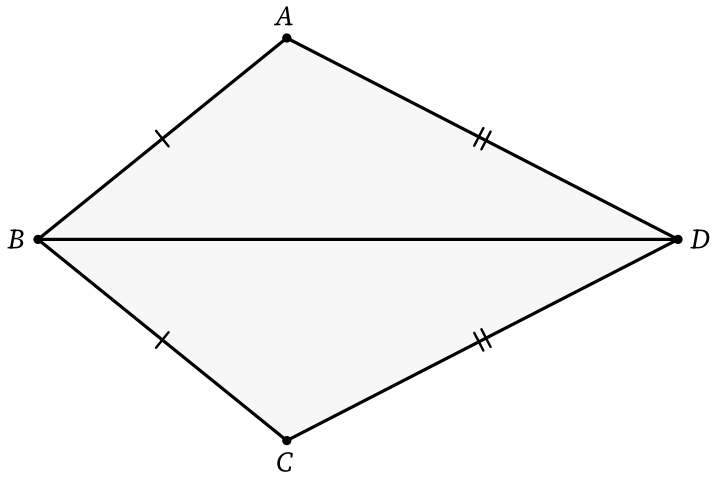
Why it works. By construction $DE = AB$, $DF = AC$ and $EF = BC$, so $\triangle DEF \cong \triangle ABC$ by SSS. This is also the heart of copying an angle (Euclid I.23): $\angle D$ is a copy of $\angle A$ because the triangles are congruent.

From history. Builders have long relied on the rigidity of the triangle. The roof trusses of timber barns, the braced frames of church roofs and the iron truss bridges of the 1800s are networks of triangles, because a triangle cannot change its shape unless one of its sides changes length, which is exactly what SSS says.

Euclid connection. Euclid I.4 is the first congruence theorem in the Elements: 'If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles.' Euclid I.8 is SSS, and I.22 is the construction of a triangle from three given straight lines.

Practice habit. Recite SSS and SAS until automatic, and say the word 'included' every time you use SAS. Before writing the last line, put your finger on the angle and check that it really is between the two sides you have listed.

Example 1. In the figure, $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. Prove that $\triangle ABD \cong \triangle CBD$.



Show solution

1. Mark the givens: two pairs of sides.
2. The triangles share $\overline{BD}$, which gives the third pair by the Reflexive Property.
3. Three pairs of sides: use SSS.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CB}$; $\overline{AD} \cong \overline{CD}$

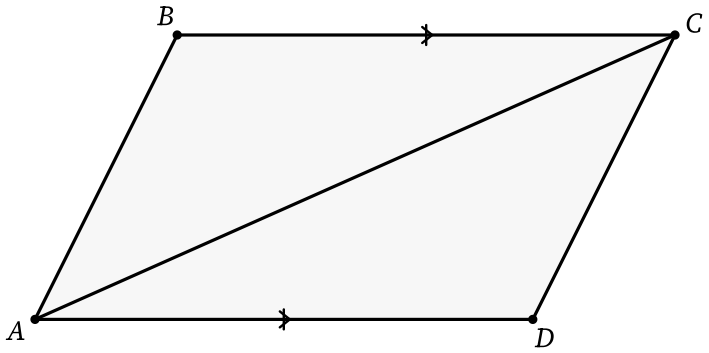
Prove: $\triangle ABD \cong \triangle CBD$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CB}$	Given
2	$\overline{AD} \cong \overline{CD}$	Given
3	$\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
4	$\triangle ABD \cong \triangle CBD$	SSS

∎

Answer: See the model proof.

Example 2. In quadrilateral $ABCD$ in the figure, $\overline{AD} \parallel \overline{BC}$ and $\overline{AD} \cong \overline{CB}$. Diagonal $\overline{AC}$ is drawn. Prove that $\triangle ADC \cong \triangle CBA$.



Show solution

1. One pair of sides is given and the diagonal $\overline{AC}$ is shared.
2. We need the angle between them in each triangle: $\angle DAC$ and $\angle BCA$. These are alternate interior angles for the parallel segments $\overline{AD}$ and $\overline{BC}$ with transversal $\overline{AC}$.
3. Side, included angle, side: use SAS.

TWO-COLUMN PROOF

Given: $\overline{AD} \parallel \overline{BC}$; $\overline{AD} \cong \overline{CB}$

Prove: $\triangle ADC \cong \triangle CBA$

	STATEMENTS	REASONS
1	$\overline{AD} \cong \overline{CB}$	Given
2	$\overline{AD} \parallel \overline{BC}$	Given
3	$\angle DAC \cong \angle BCA$	Alternate Interior Angles Theorem
4	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
5	$\triangle ADC \cong \triangle CBA$	SAS

∎

Answer: See the model proof.

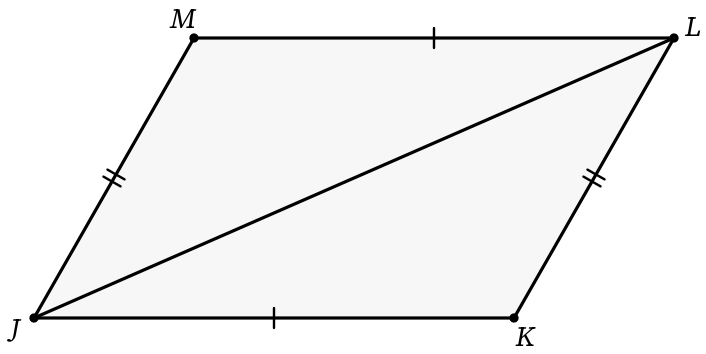
PRACTICE

1. **Easy** In $\triangle ABC$, name the angle included between sides $\overline{AB}$ and $\overline{BC}$.

   Show answer

   $\angle B$ (the common endpoint of the two sides)

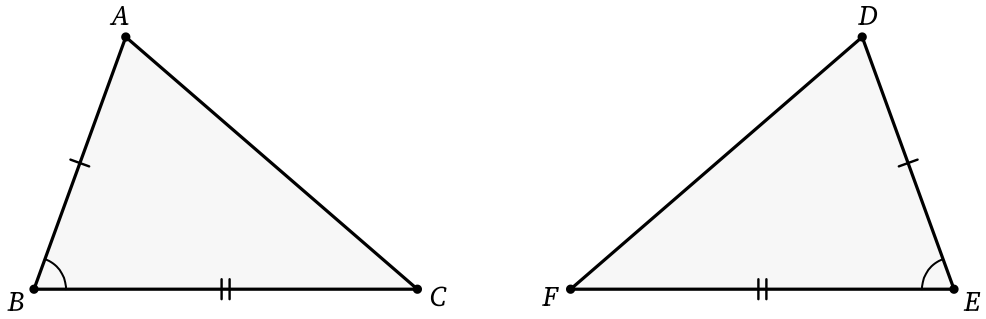
2. **Easy** In the figure, $\overline{JK} \cong \overline{LM}$ and $\overline{KL} \cong \overline{MJ}$. Which postulate, SSS or SAS, proves $\triangle JKL \cong \triangle LMJ$?



   Show answer

   SSS: the third pair is the shared side $\overline{JL} \cong \overline{LJ}$ (Reflexive Property).

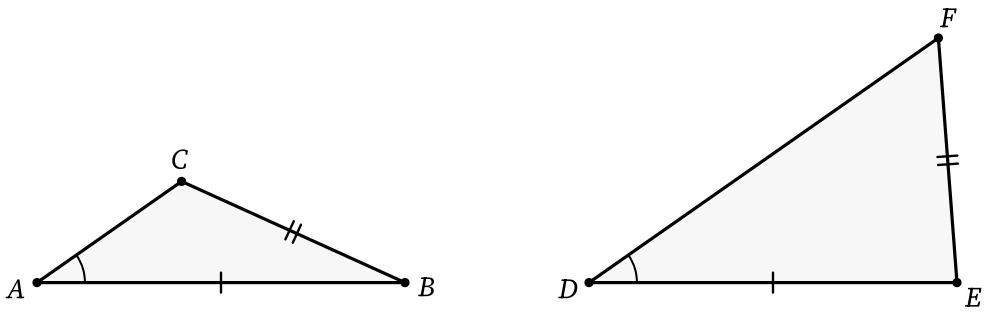
3. **Easy** In the figure, $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$. Which postulate proves $\triangle ABC \cong \triangle DEF$?



   Show answer

   SAS ($\angle B$ is included between $\overline{AB}$ and $\overline{BC}$, and $\angle E$ between $\overline{DE}$ and $\overline{EF}$).

4. **Easy** In the figure, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle A \cong \angle D$. Can you conclude that the triangles are congruent by SSS or SAS? If not, write 'not enough information' and explain.



Show answer

Not enough information. The congruent angle is not between the two congruent sides, so this is SSA, which is not a congruence test; the figure itself shows two such triangles that are not congruent.

5. **Medium** $\triangle ABC$ has $AB = 7$, $BC = 9$ and $CA = 12$. $\triangle DEF$ has $DE = 7$, $EF = 2x + 1$ and $FD = 12$. For what value of x are the triangles congruent by SSS?

Show answer

Need $EF = BC$: $2x + 1 = 9$, so $x = 4$.

6. **Medium** Henry builds a rectangular wooden garden gate, but after a month it sags into a slanted shape. His grandfather tells him to screw one diagonal board from corner to corner. Explain, using a congruence postulate, why the diagonal keeps the gate from changing shape.

Show answer

The diagonal divides the gate into two triangles whose sides are boards of fixed length. By SSS, a triangle with three fixed side lengths has only one possible shape, so its angles cannot change; therefore the corner angles of the gate stay fixed. A four-sided frame without the diagonal is not rigid: its sides can stay the same length while its angles change.

7. **Medium** Grace wants to follow Euclid I.22 to build a triangle from three sticks measuring 3 in, 4 in and 8 in. Will her compass arcs meet? Explain.

Show answer

No. With the 8-in stick as the base, the arcs of radius 3 in and 4 in centered at its ends reach only $3 + 4 = 7$ in along the base in total, which is less than 8 in, so the arcs never meet. Each pair of lengths must add to more than the third.

8. **Medium** In $\triangle ABC$ and $\triangle DEF$, you know $\overline{AB} \cong \overline{DE}$ and $\angle B \cong \angle E$. What other pair of congruent parts would let you prove $\triangle ABC \cong \triangle DEF$ by SAS?

Show answer

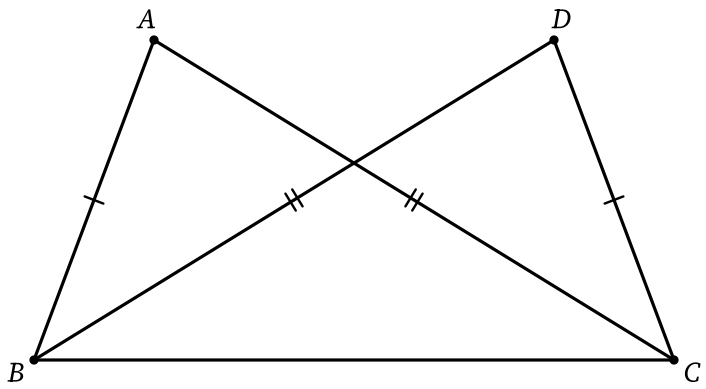
$\overline{BC} \cong \overline{EF}$ (the other sides that include $\angle B$ and $\angle E$)

9. **Medium** Samuel builds two roof trusses for a shed. Each has two rafters 8 ft long and a bottom chord 14 ft long. He measures the angle at the peak of the first truss as about 122°. Without measuring, what is the peak angle of the second truss? Explain.

Show answer

About 122°. The trusses have three pairs of congruent sides, so they are congruent by SSS, and corresponding angles of congruent triangles are congruent.

10. **Hard** In the figure, $\overline{AB} \cong \overline{DC}$ and $\overline{AC} \cong \overline{DB}$. Prove that $\triangle ABC \cong \triangle DCB$.



Show answer

See the model proof.

TWO-COLUMN PROOF

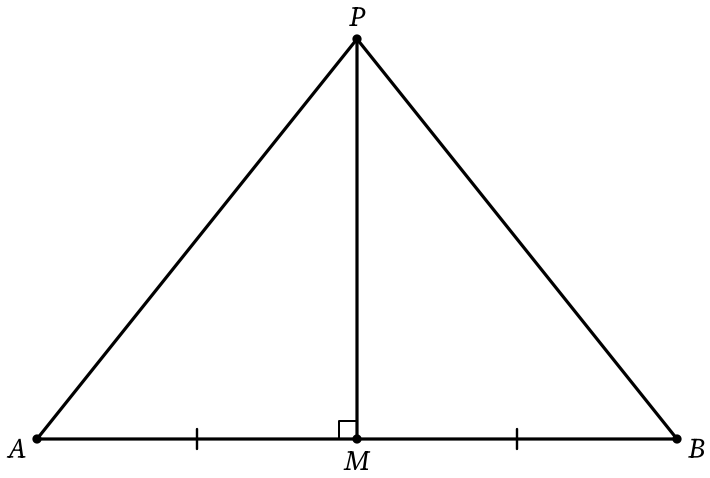
Given: $\overline{AB} \cong \overline{DC}$; $\overline{AC} \cong \overline{DB}$

Prove: $\triangle ABC \cong \triangle DCB$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{DC}$	Given
2	$\overline{AC} \cong \overline{DB}$	Given
3	$\overline{BC} \cong \overline{CB}$	Reflexive Property of Congruence
4	$\triangle ABC \cong \triangle DCB$	SSS

∎

11. Hard In the figure, M is the midpoint of $\overline{AB}$ and $\overline{PM} \perp \overline{AB}$. Write a paragraph proof that $\triangle PMA \cong \triangle PMB$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: M is the midpoint of $\overline{AB}$; $\overline{PM} \perp \overline{AB}$

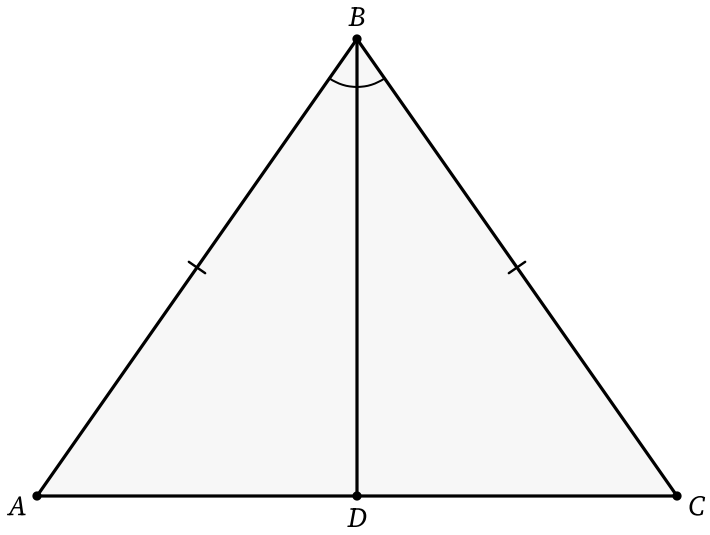
Prove: $\triangle PMA \cong \triangle PMB$

Since M is the midpoint of $\overline{AB}$, $\overline{AM} \cong \overline{BM}$ by the definition of midpoint. Since $\overline{PM} \perp \overline{AB}$, the angles $\angle PMA$ and $\angle PMB$ are right angles by the definition of perpendicular lines, and all right angles are congruent, so $\angle PMA \cong \angle PMB$.

Also $\overline{PM} \cong \overline{PM}$ by the Reflexive Property of Congruence. The angle at M is included between $\overline{PM}$ and $\overline{MA}$ in one triangle and between $\overline{PM}$ and $\overline{MB}$ in the other, so $\triangle PMA \cong \triangle PMB$ by SAS.

∎

12. Hard In the figure, $\overline{AB} \cong \overline{CB}$ and $\overrightarrow{BD}$ bisects $\angle ABC$, with D on $\overline{AC}$. Prove that $\triangle ABD \cong \triangle CBD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

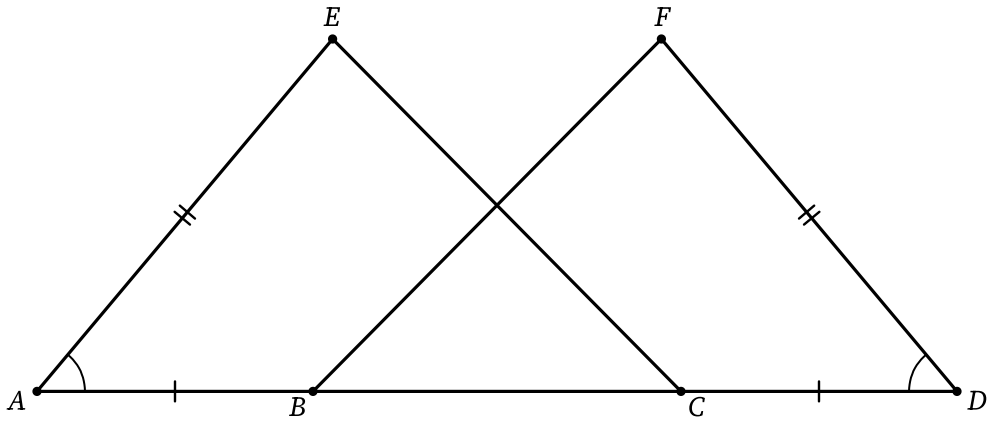
Given: $\overline{AB} \cong \overline{CB}$; $\overrightarrow{BD}$ bisects $\angle ABC$

Prove: $\triangle ABD \cong \triangle CBD$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CB}$	Given
2	$\overrightarrow{BD}$ bisects $\angle ABC$	Given
3	$\angle ABD \cong \angle CBD$	Definition of angle bisector
4	$\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
5	$\triangle ABD \cong \triangle CBD$	SAS

∎

13. **Challenge** In the figure, A, B, C and D lie on a line in that order, $\overline{AB} \cong \overline{CD}$, $\overline{AE} \cong \overline{DF}$ and $\angle EAB \cong \angle FDC$. Prove that $\triangle ACE \cong \triangle DBF$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: A, B, C, D are collinear in that order; $\overline{AB} \cong \overline{CD}$; $\overline{AE} \cong \overline{DF}$; $\angle EAB \cong \angle FDC$

Prove: $\triangle ACE \cong \triangle DBF$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$	Given
2	$AB = CD$	Definition of congruent segments
3	$BC = BC$	Reflexive Property of Equality
4	$AB + BC = CD + BC$	Addition Property of Equality
5	$AB + BC = AC$ and $CD + BC = DB$	Segment Addition Postulate
6	$AC = DB$	Substitution Property of Equality
7	$\overline{AC} \cong \overline{DB}$	Definition of congruent segments
8	$\overline{AE} \cong \overline{DF}$	Given
9	$\angle EAB \cong \angle FDC$	Given

	STATEMENTS	REASONS
10	$\angle EAC \cong \angle FDB$	Same angles as in the previous row ($\overrightarrow{AB} = \overrightarrow{AC}$ and $\overrightarrow{DC} = \overrightarrow{DB}$)
11	$\triangle ACE \cong \triangle DBF$	SAS

∎

14. **Challenge** In $\triangle ABC$, $m\angle A = 30°$, $AB = 10$ cm and $BC = 6$ cm. How many different (non-congruent) triangles fit this description? Explain using a compass picture. (A careful drawing shows that the distance from B to the other side of $\angle A$ is 5 cm; you will prove this in Unit 9.)

Show answer

Two. Draw $\angle A$ and mark B 10 cm from A on one side. The closest point to B on the other side is 5 cm away, and a circle of radius 6 cm about B reaches past it, crossing that side twice (both crossings are on the ray because $6 < 10$). Each crossing gives a triangle, so there are two non-congruent triangles: another example of why SSA is not a test.

COMMON MISTAKES

- Using SAS with an angle that is not included. Check that the angle's vertex is the shared endpoint of the two sides; otherwise it is SSA, which proves nothing.
- Forgetting the shared side. When two triangles share a side, write it in the proof with the reason 'Reflexive Property of Congruence'; it is often the missing third pair.
- Writing the congruence statement in an order that does not match the parts. If $\overline{AB}$ matches $\overline{DC}$, then A matches D and B matches C.
- Concluding congruence from the look of the figure. Tick marks you add yourself must come from the givens or from earlier steps.

> **Make it a game · SSS or SAS Verdict.** Parent reveals three measurements on cards, such as two sides and the angle between them. Call the verdict in 10 seconds: SSS, SAS or not enough. Then both of you draw the triangle from the same data with ruler and protractor, and lay one drawing over the other. Name the included angle each time. Ten rounds scores.

4.4 Proving Triangles Congruent: ASA and AAS

OBJECTIVES

- State and apply the ASA and AAS congruence tests (Euclid I.26).
- Explain why AAA and SSA are not congruence tests.
- Choose among SSS, SAS, ASA and AAS for a given set of congruent parts.
- Read and write flow proofs.

WATCH

Triangle Similiarity, Congruence & the Sine Rule

Eddie Woo · 6:33

https://www.youtube.com/watch?v=N4qrAuphV2w

> **WHEN TO WATCH**
> After the ASA and AAS theorems: why AAS works and why SSA is not a congruence test.

Note: Eddie Woo teaches in Australia: he uses the sine rule (Unit 9 and the Trigonometry course) to show why SSA fails; the picture is the important part. He says 'RHS' for HL and 'equiangular' for AA.

BACKUP VIDEO

Deductive Geometry (2 of 3: General Tips - Diagrams & Clarity)

Eddie Woo · 7:21

https://www.youtube.com/watch?v=pLMMyjpnKJc

Before writing your own proofs: how to make each reason clear.

KEY TERMS

Angle-Side-Angle (ASA)

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Angle-Angle-Side (AAS)

If two angles and a non-included side of one triangle are congruent to the corresponding two angles and side of another triangle, then the triangles are congruent.

Non-included side

A side of a triangle that is not between two given angles; it lies opposite one of them.

Flow proof

A proof written as boxes (statements) joined by arrows. Each box has its reason written under it, and arrows show which earlier statements it follows from.

CORE IDEAS

Why do two angles and the side between them fix a triangle? Lay down the side $\overline{AB}$. At A draw a ray making the first angle, and at B a ray making the second angle, both on the same side. The third vertex must lie on both rays, and two lines cross in at most one point. So the third vertex has only one possible position. That is Angle-Side-Angle (ASA).

Angle-Angle-Side (AAS) adds nothing new: if two angles of one triangle match two angles of another, the Third Angles Theorem (Lesson 4.2) says the third angles match too. Then any matching side lies between two matching angles, and ASA applies. Euclid proves both cases in one proposition, I.26.

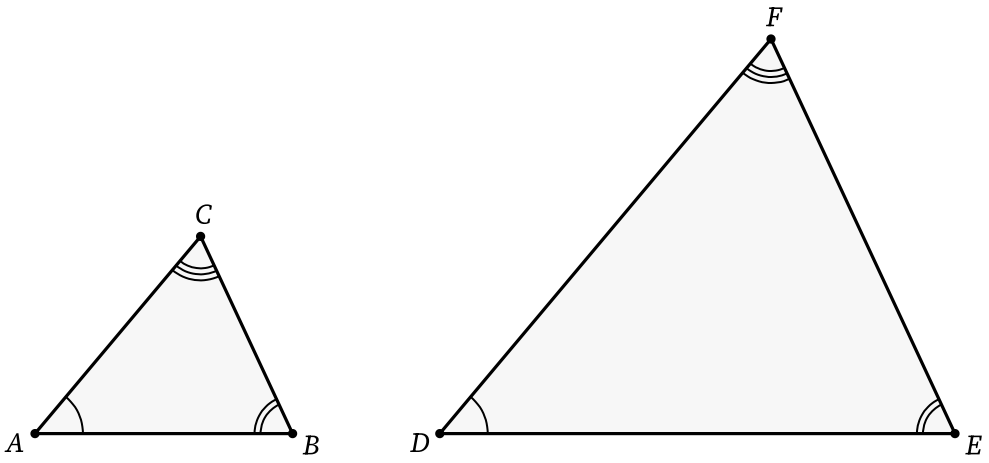
Be careful with the word 'corresponding' in AAS. The side must be opposite the same angle in both triangles. If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\overline{BC}$ (opposite $\angle A$) must be matched with $\overline{EF}$ (opposite $\angle D$), not with $\overline{DF}$.

Why is AAA not a test? Three angles fix the shape but not the size: a small triangle and a large triangle can have the same three angles. Together with SSA from Lesson 4.3, that leaves exactly four tests for any triangles: SSS, SAS, ASA and AAS (and a special right-triangle test in Lesson 4.7).

Choosing a test: mark everything you know, including shared sides (Reflexive Property), vertical angles, and angles from parallel lines. Then read the pattern around one triangle: three sides is SSS; two sides with the angle between them is SAS; two angles with the side between them is ASA; two angles and another side is AAS.

A flow proof shows the logic as a map. Each statement goes in a box with its reason underneath, and arrows run from the statements it depends on. Givens and facts like vertical angles that need nothing earlier start their own arrows. Flow proofs make it

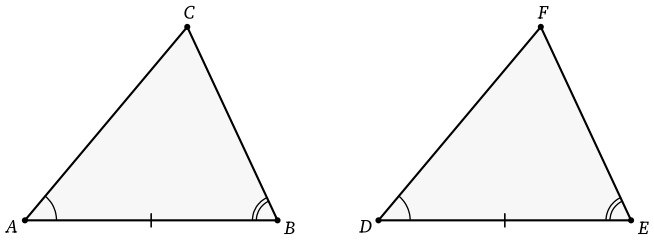
easy to see that every step is supported and that the final box uses exactly the pieces a test requires.



POSTULATES AND THEOREMS

THEOREM **Angle-Side-Angle (ASA) Congruence Theorem** Euclid I.26

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.



PARAGRAPH PROOF

Given: $\angle A \cong \angle D$; $\overline{AB} \cong \overline{DE}$; $\angle B \cong \angle E$

Prove: $\triangle ABC \cong \triangle DEF$

On ray $\overrightarrow{DF}$, mark the point F' with $DF' = AC$ (Ruler Postulate). Then $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$ (here $\angle EDF'$ is the same angle as $\angle EDF$, because F' is on ray $\overrightarrow{DF}$), and $\overline{AC} \cong \overline{DF'}$. So $\triangle ABC \cong \triangle DEF'$ by SAS, and therefore $\angle ABC \cong \angle DEF'$ by the definition of congruent triangles.

We were also given $\angle ABC \cong \angle DEF$, so $\angle DEF' \cong \angle DEF$ by the Transitive Property. The points F' and F lie on the same side of $\overleftrightarrow{DE}$, and by the Protractor Postulate only one ray from E on that side makes an angle of that measure with $\overrightarrow{ED}$. So $\overrightarrow{EF'}$ and $\overrightarrow{EF}$ are the same ray, and F' lies on line EF.

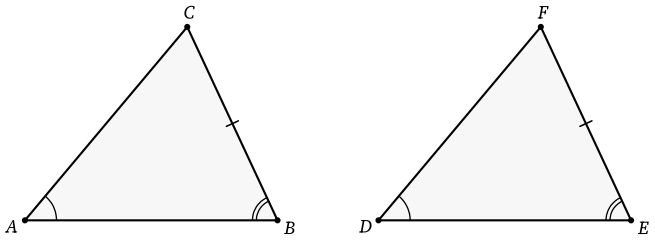
Now F' lies on both line DF and line EF. These are two different lines, and two different lines meet in at most one point; F is on both, so $F' = F$. Therefore $\triangle ABC \cong \triangle DEF$.

∎

Euclid's proof of I.26 also rests on I.4 (SAS): he supposes the sides are unequal, cuts off a part equal to the shorter side, and reaches the absurd result that a part equals the whole.

THEOREM **Angle-Angle-Side (AAS) Congruence Theorem** Euclid I.26

If two angles and a non-included side of one triangle are congruent to two angles and the corresponding non-included side of another triangle, then the triangles are congruent.



FLOW PROOF

Given: $\angle A \cong \angle D$; $\angle B \cong \angle E$; $\overline{BC} \cong \overline{EF}$

Prove: $\triangle ABC \cong \triangle DEF$

(1) $\angle A \cong \angle D$
Given

(2) $\angle B \cong \angle E$
Given

(3) $\overline{BC} \cong \overline{EF}$
Given

(4) $\angle C \cong \angle F$
Third Angles Theorem
from (1), (2)

(5) $\triangle ABC \cong \triangle DEF$
ASA (using $\angle B$, $\overline{BC}$, $\angle C$)
from (2), (3), (4)

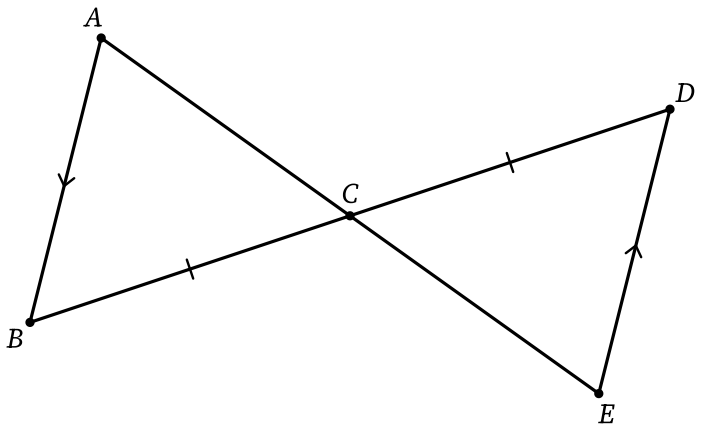
∎

From history. According to Proclus, the historian Eudemus credited Thales of Miletus with the theorem that two triangles with two angles and one side equal are equal, 'for he says that the method by which Thales showed how to find the distance of ships at sea necessarily involves this theorem.' We do not know Thales' exact method, but any such method works the same way: measure one side on land and two angles, and the ship's position is fixed.

Euclid connection. Euclid I.26: 'If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.' The 'side adjoining the equal angles' is ASA, and the side 'subtending' (opposite) one of them is AAS.

Practice habit. Recite the four tests (SSS, SAS, ASA, AAS) and the two non-tests (SSA, AAA). Mark every given on your diagram before starting, then trace the three marked parts around one triangle with your pencil to read the pattern in order.

Example 1. In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, $\overline{AB} \parallel \overline{ED}$ and C is the midpoint of $\overline{BD}$. Write a flow proof that $\triangle ABC \cong \triangle EDC$.



Show solution

1. Parallel lines give alternate interior angles with transversal $\overline{BD}$: $\angle ABC \cong \angle EDC$.
2. The midpoint gives $\overline{BC} \cong \overline{DC}$, and the angles at C are vertical angles.
3. The side $\overline{BC}$ lies between $\angle B$ and $\angle ACB$, so use ASA.

FLOW PROOF

Given: $\overline{AB} \parallel \overline{ED}$; C is the midpoint of $\overline{BD}$

Prove: $\triangle ABC \cong \triangle EDC$

(1) $\overline{AB} \parallel \overline{ED}$
Given

(2) C is the midpoint of $\overline{BD}$
Given

(3) $\angle ABC \cong \angle EDC$
Alternate Interior Angles Theorem
from (1)

(4) $\overline{BC} \cong \overline{DC}$
Definition of midpoint
from (2)

(5) $\angle ACB \cong \angle ECD$

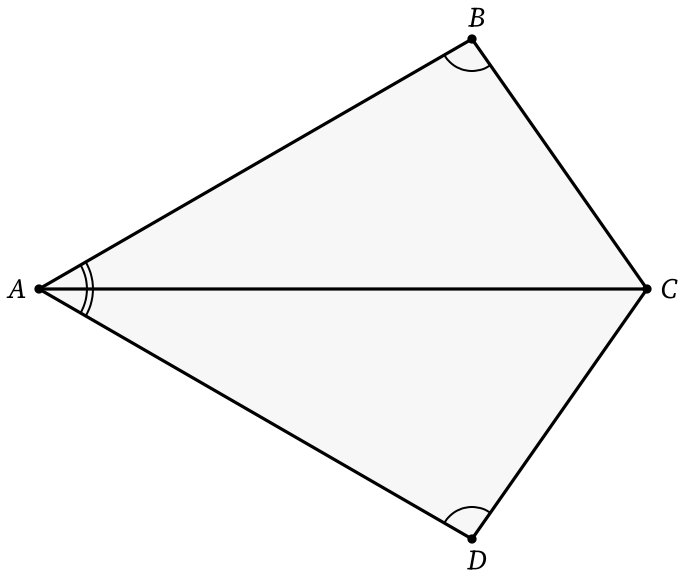
Vertical Angles Theorem

(6) $\triangle ABC \cong \triangle EDC$
ASA
from (3), (4), (5)

∎

Answer: See the model proof.

Example 2. In the figure, $\angle B \cong \angle D$ and $\overline{AC}$ bisects $\angle BAD$. Prove that $\triangle ABC \cong \triangle ADC$.



Show solution

1. The bisector gives a second pair of angles: $\angle BAC \cong \angle DAC$.
2. The shared side $\overline{AC}$ is the third pair. Is it between the two angles? It is between $\angle BAC$ and $\angle BCA$, not between $\angle B$ and $\angle BAC$; it lies opposite $\angle B$.
3. Two angles and a non-included side: use AAS.

TWO-COLUMN PROOF

Given: $\angle B \cong \angle D$; $\overline{AC}$ bisects $\angle BAD$

Prove: $\triangle ABC \cong \triangle ADC$

	STATEMENTS	REASONS
1	$\angle B \cong \angle D$	Given
2	$\overline{AC}$ bisects $\angle BAD$	Given
3	$\angle BAC \cong \angle DAC$	Definition of angle bisector
4	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
5	$\triangle ABC \cong \triangle ADC$	AAS

∎

Answer: See the model proof.

PRACTICE

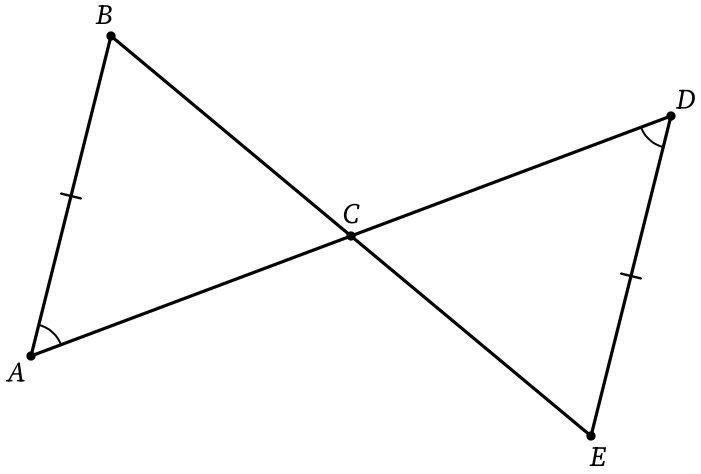
1. **Easy** In the figure, $\angle K \cong \angle R$, $\overline{KL} \cong \overline{RS}$ and $\angle L \cong \angle S$. Which test proves $\triangle KLM \cong \triangle RST$?



Show answer

ASA ($\overline{KL}$ is between $\angle K$ and $\angle L$)

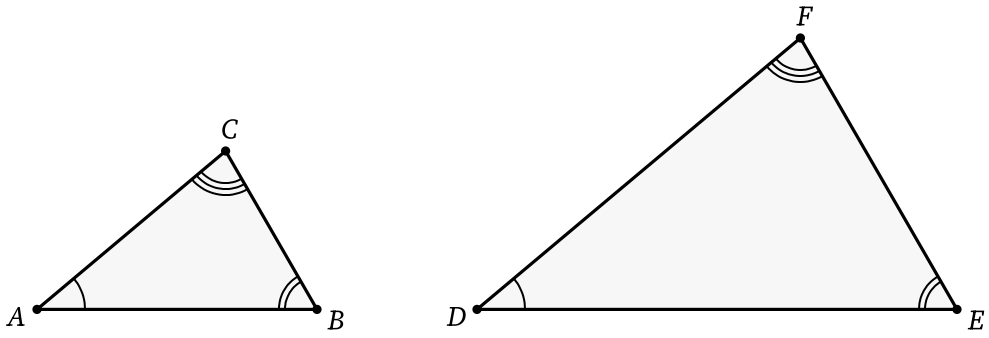
2. **Easy** In the figure, $\overline{AD}$ and $\overline{BE}$ intersect at C, $\angle A \cong \angle D$ and $\overline{AB} \cong \overline{DE}$. Which test proves $\triangle ABC \cong \triangle DEC$?



Show answer

AAS: $\angle ACB \cong \angle DCE$ (vertical angles) and $\angle A \cong \angle D$, with the non-included sides $\overline{AB}$ and $\overline{DE}$.

3. **Easy** In the figure, all three pairs of corresponding angles are marked congruent. Can you conclude $\triangle ABC \cong \triangle DEF$? If not, write 'not enough information.'



Show answer

Not enough information. AAA is not a congruence test; the triangles have the same shape but may have different sizes.

4. **Easy** In $\triangle PQR$, name the side included between $\angle P$ and $\angle Q$.

Show answer

$\overline{PQ}$

5. **Medium** In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$ and $\angle C \cong \angle F$. What pair of congruent sides would let you use ASA? What pair would let you use AAS? (Give one AAS answer.)

Show answer

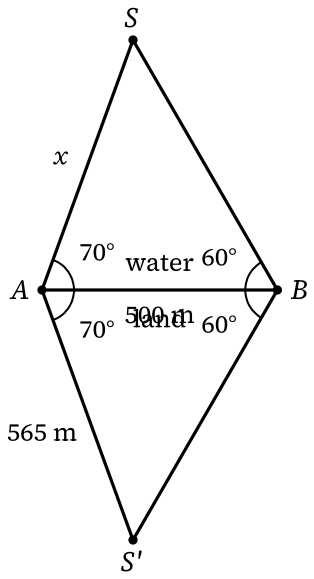
ASA: $\overline{AC} \cong \overline{DF}$. AAS: $\overline{AB} \cong \overline{DE}$ (or $\overline{BC} \cong \overline{EF}$).

6. **Medium** Ruth writes: '$\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\overline{AB} \cong \overline{DE}$, so $\triangle ABC \cong \triangle DEF$ by AAS.' The triangles are congruent, but which test should she have named, and why?

Show answer

ASA, because $\overline{AB}$ is the side included between $\angle A$ and $\angle B$.

7. **Medium** To find the distance from point A on a straight shore to a ship S, Henry stands at A and Grace at B, 500 m along the shore. Henry measures $m\angle SAB = 70^\circ$ and Grace measures $m\angle SBA = 60^\circ$. On the land side of the shore they lay out a point S' with $m\angle S'AB = 70^\circ$ and $m\angle S'BA = 60^\circ$, and they pace off $AS' = 565$ m (see the figure). How far is the ship from A? Name the test that justifies your answer.



Show answer

$\triangle SAB \cong \triangle S'AB$ by ASA (the shared side $\overline{AB}$ lies between the matching angles). Corresponding sides are congruent, so $AS = AS' = 565$ m.

8. **Medium** $\triangle ABC \cong \triangle DEF$ by AAS, with $m\angle B = (4x - 2)^\circ$ and $m\angle E = (3x + 9)^\circ$. Find x and $m\angle B$.

Show answer

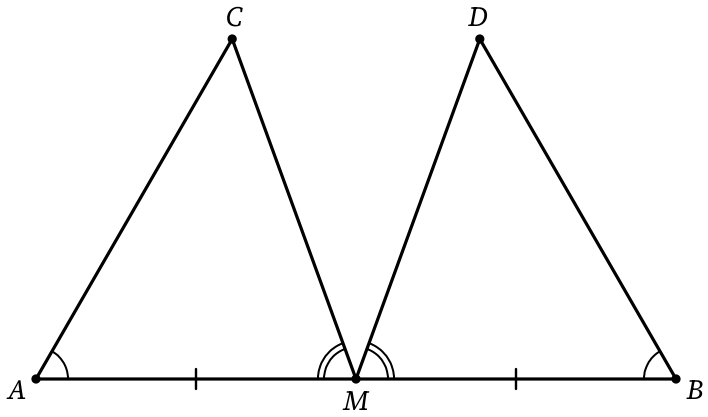
$4x - 2 = 3x + 9$, so $x = 11$ and $m\angle B = 4(11) - 2 = 42^\circ$.

9. **Medium** In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{DF}$. Can you conclude that $\triangle ABC \cong \triangle DEF$? Explain.

Show answer

No. $\overline{BC}$ is opposite $\angle A$, so it corresponds to $\overline{EF}$ (opposite $\angle D$), not to $\overline{DF}$. The given sides do not correspond, so neither ASA nor AAS applies.

10. Hard In the figure, M is the midpoint of $\overline{AB}$, $\angle A \cong \angle B$ and $\angle AMC \cong \angle BMD$. Prove that $\triangle AMC \cong \triangle BMD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

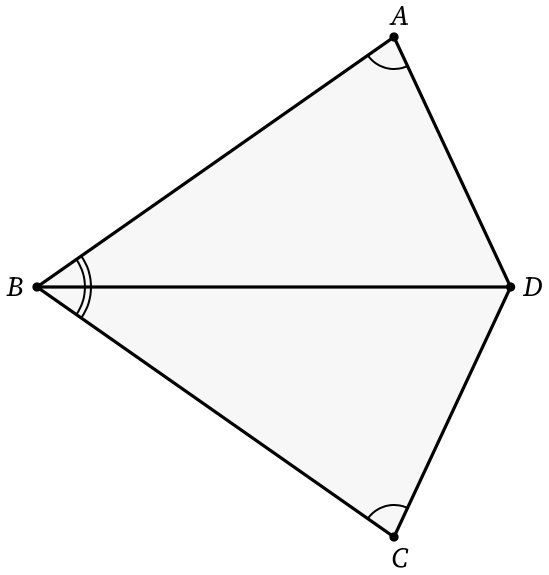
Given: M is the midpoint of $\overline{AB}$; $\angle A \cong \angle B$; $\angle AMC \cong \angle BMD$

Prove: $\triangle AMC \cong \triangle BMD$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$	Given
2	$\overline{AM} \cong \overline{BM}$	Definition of midpoint
3	$\angle A \cong \angle B$	Given
4	$\angle AMC \cong \angle BMD$	Given
5	$\triangle AMC \cong \triangle BMD$	ASA

∎

11. **Hard** In the figure, $\angle A \cong \angle C$ and $\overline{BD}$ bisects $\angle ABC$. Write a flow proof that $\triangle ABD \cong \triangle CBD$.



Show answer

See the model proof.

FLOW PROOF

Given: $\angle A \cong \angle C$; $\overline{BD}$ bisects $\angle ABC$

Prove: $\triangle ABD \cong \triangle CBD$

(1) $\angle A \cong \angle C$
Given

(2) $\overline{BD}$ bisects $\angle ABC$
Given

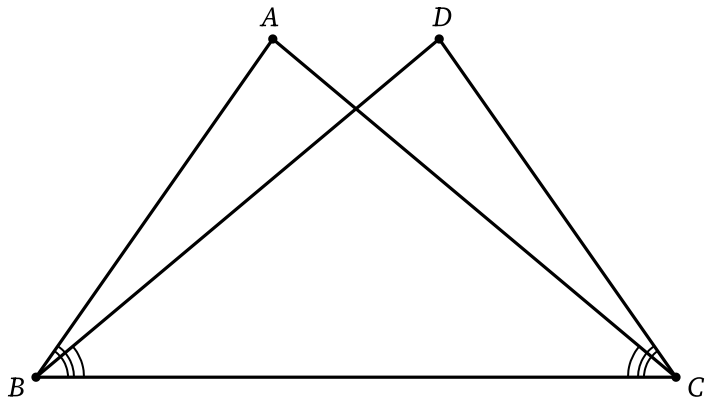
(3) $\angle ABD \cong \angle CBD$
Definition of angle bisector
from (2)

(4) $\overline{BD} \cong \overline{BD}$
Reflexive Property of Congruence

(5) $\triangle ABD \cong \triangle CBD$
AAS
from (1), (3), (4)

∎

12. Hard In the figure, $\angle ABC \cong \angle DCB$ and $\angle ACB \cong \angle DBC$. Write a paragraph proof that $\triangle ABC \cong \triangle DCB$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\angle ABC \cong \angle DCB$; $\angle ACB \cong \angle DBC$

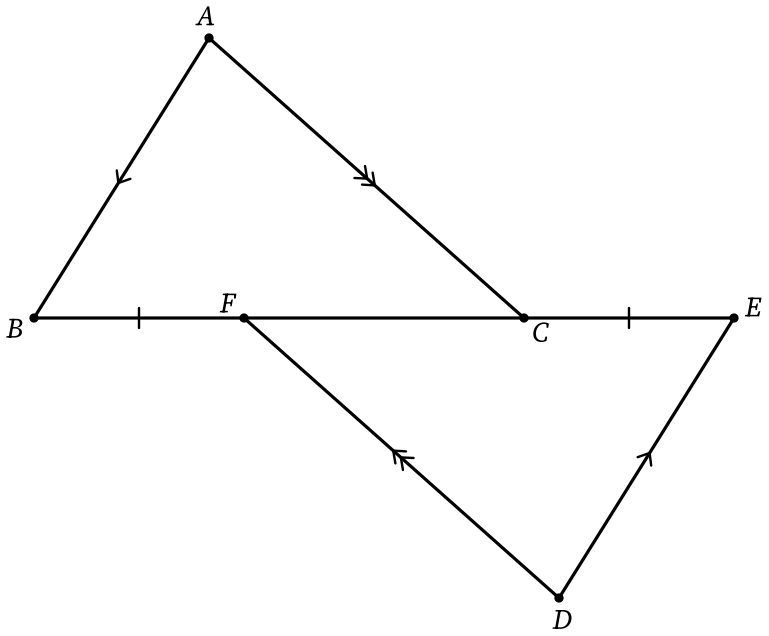
Prove: $\triangle ABC \cong \triangle DCB$

We are given $\angle ABC \cong \angle DCB$ and $\angle ACB \cong \angle DBC$. The two overlapping triangles share side $\overline{BC}$, and $\overline{BC} \cong \overline{CB}$ by the Reflexive Property of Congruence.

In $\triangle ABC$ the side $\overline{BC}$ is included between $\angle ABC$ and $\angle ACB$; in $\triangle DCB$ the side $\overline{CB}$ is included between the matching angles $\angle DCB$ and $\angle DBC$. Therefore $\triangle ABC \cong \triangle DCB$ by ASA.

∎

13. **Challenge** In the figure, B, F, C and E lie on a line in that order, $\overline{BF} \cong \overline{CE}$, $\overline{AB} \parallel \overline{DE}$ and $\overline{AC} \parallel \overline{DF}$, with A and D on opposite sides of the line. Prove that $\triangle ABC \cong \triangle DEF$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: B, F, C, E are collinear in that order; $\overline{BF} \cong \overline{CE}$; $\overline{AB} \parallel \overline{DE}$; $\overline{AC} \parallel \overline{DF}$

Prove: $\triangle ABC \cong \triangle DEF$

	STATEMENTS	REASONS
1	$\overline{BF} \cong \overline{CE}$	Given
2	$BF = CE$	Definition of congruent segments
3	$FC = FC$	Reflexive Property of Equality
4	$BF + FC = CE + FC$	Addition Property of Equality
5	$BF + FC = BC$ and $CE + FC = EF$	Segment Addition Postulate
6	$BC = EF$	Substitution Property of Equality
7	$\overline{BC} \cong \overline{EF}$	Definition of congruent segments

	STATEMENTS	REASONS
8	$\overline{AB} \parallel \overline{DE}$	Given
9	$\angle ABC \cong \angle DEF$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BE}$)
10	$\overline{AC} \parallel \overline{DF}$	Given
11	$\angle ACB \cong \angle DFE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BE}$)
12	$\triangle ABC \cong \triangle DEF$	ASA

∎

COMMON MISTAKES

- Mixing up ASA and AAS. Say where the side is: between the two angles (ASA) or opposite one of them (AAS). Both are valid, but the reason must match.
- Matching a side that does not correspond. In AAS the side must be opposite the same angle in both triangles; check by naming the angle opposite each side.
- Using AAA or SSA as if they were tests. Neither proves congruence.
- In a flow proof, drawing an arrow from a statement that the next box does not actually use. Each arrow should point from a reason's real ingredients.

> **Make it a game · Break the Test.** Prove two shortcuts fail by building counterexamples. In 90 seconds, draw two triangles with identical angles but different sizes to defeat AAA. Then swing a strip around a fixed side and angle to find two different triangles that defeat SSA. Parent then names conditions, and you answer ASA, AAS or neither.

4.5 Using Congruent Triangles (CPCTC)

OBJECTIVES

- Use CPCTC to prove segments and angles congruent after proving triangles congruent.
- Prove that the compass constructions for an angle bisector, a perpendicular bisector and a copied angle are correct.
- Separate overlapping triangles in a sketch and use two pairs of congruent triangles in one proof.
- Measure distances across ponds and rivers by laying out congruent triangles.

WATCH

Congruent Triangles in Circles (1 of 2: Equal chords)

Eddie Woo · 8:45

https://www.youtube.com/watch?v=P5tP9xY_f38

> **WHEN TO WATCH**
> After Example 1: proving triangles congruent and then using the matching parts (CPCTC).

Note: Both examples sit inside circles; only 'all radii are equal' is needed for the first. The backup also uses an angle fact from Unit 10.

BACKUP VIDEO

Equal Overlapping Circles Question (1 of 2: Identifying congruent triangles)

Eddie Woo · 12:27

https://www.youtube.com/watch?v=azPYGChtIRE

As a challenge after the practice set: a longer proof that ends with corresponding parts of congruent triangles.

KEY TERMS

CPCTC

Short for 'Corresponding Parts of Congruent Triangles are Congruent.' Once two triangles are proved congruent, each pair of corresponding sides and angles is congruent.

Indirect measurement

Finding a distance that cannot be measured directly by measuring an equal distance that can be reached, using congruent (or, later, similar) triangles.

Overlapping triangles

Triangles that share part of their interiors, and often a side or an angle, in a figure.

CORE IDEAS

The congruence tests (SSS, SAS, ASA, AAS) are usually not the goal of a proof; they are a tool. Most often you want to show that two segments or two angles are congruent. The strategy is: find two triangles that contain those parts, prove the triangles congruent, then conclude that the parts are congruent because corresponding parts of congruent triangles are congruent (CPCTC).

Why is CPCTC true? It is just the definition of congruent polygons from Lesson 4.2, read in the 'if congruent, then' direction. A congruence test tells you that three pairs of parts are enough to make the triangles congruent; the definition then gives you all six pairs. That is the whole point of the tests: three pairs in, six pairs out.

You may use CPCTC only after the triangles have been proved congruent, and only for parts that truly correspond. Write the congruence statement in the correct order and read the parts off it letter by letter.

Now we can finally prove that the constructions of Unit 2 work. In the angle bisector construction, equal compass settings give two triangles with three pairs of congruent sides, so by SSS and CPCTC the two halves of the angle are congruent. Euclid proves I.9 the same way, using I.8 (SSS). The perpendicular bisector construction needs two pairs of triangles, one after the other, as Euclid's I.10 does with I.4 (SAS).

When triangles overlap, sketch them separately, side by side, and copy the marks onto each. Shared sides and shared angles show up clearly that way. In a two-step proof, the first pair of congruent triangles supplies, through CPCTC, a part you need for the second pair.

POSTULATES AND THEOREMS

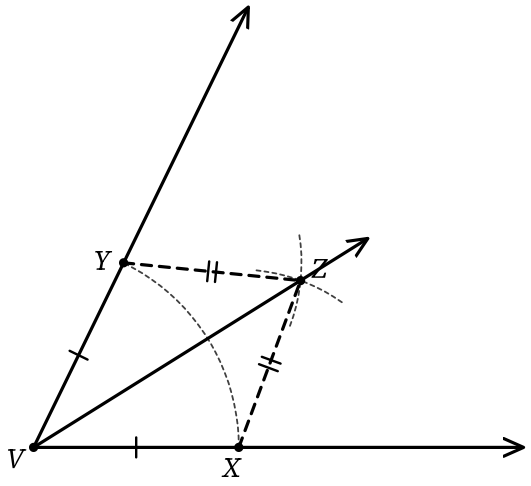
THEOREM **CPCTC**

If two triangles are congruent, then their corresponding sides are congruent and their corresponding angles are congruent.

This is the definition of congruent polygons (Lesson 4.2) used in the 'if congruent, then' direction, so it needs no separate proof.

THEOREM **The angle bisector construction is correct** Euclid I.9

If $\overline{VX} \cong \overline{VY}$ *and* $\overline{XZ} \cong \overline{YZ}$*, with* Z *in the interior of* $\angle XVY$*, then* $\overrightarrow{VZ}$ *bisects* $\angle XVY$*.*



FLOW PROOF

Given: $\overline{VX} \cong \overline{VY}$ (first compass setting); $\overline{XZ} \cong \overline{YZ}$ (second compass setting)

Prove: $\overrightarrow{VZ}$ bisects $\angle XVY$

(1) $\overline{VX} \cong \overline{VY}$
Given (same compass setting)

(2) $\overline{XZ} \cong \overline{YZ}$
Given (same compass setting)

(3) $\overline{VZ} \cong \overline{VZ}$
Reflexive Property of Congruence

(4) $\triangle VXZ \cong \triangle VYZ$
SSS
from (1), (2), (3)

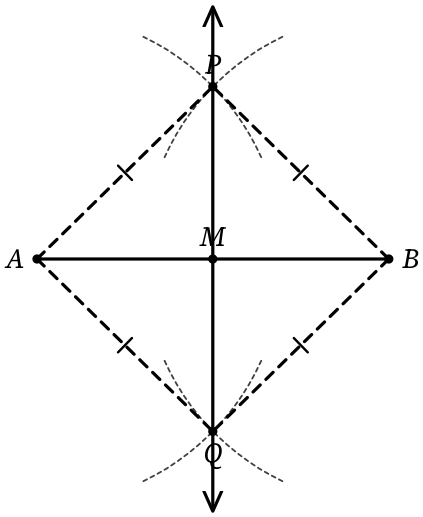
(5) $\angle XVZ \cong \angle YVZ$
CPCTC
from (4)

(6) $\overrightarrow{VZ}$ bisects $\angle XVY$
Definition of angle bisector
from (5)

∎

THEOREM **The perpendicular bisector construction is correct** Euclid I.10

If P and Q lie on opposite sides of $\overline{AB}$ with $\overline{AP} \cong \overline{BP} \cong \overline{AQ} \cong \overline{BQ}$, then $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.



TWO-COLUMN PROOF

Given: $\overline{AP} \cong \overline{BP}$ and $\overline{AQ} \cong \overline{BQ}$ (all four made with one compass setting); P and Q are on opposite sides of $\overline{AB}$; $\overline{PQ}$ meets $\overline{AB}$ at M

Prove: $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$

	STATEMENTS	REASONS
1	$\overline{AP} \cong \overline{BP}$ and $\overline{AQ} \cong \overline{BQ}$	Given (same compass setting)
2	$\overline{PQ} \cong \overline{PQ}$	Reflexive Property of Congruence
3	$\triangle APQ \cong \triangle BPQ$	SSS

	STATEMENTS	REASONS
4	$\angle APM \cong \angle BPM$	CPCTC ($\angle APQ$ and $\angle BPQ$ are the same angles as $\angle APM$ and $\angle BPM$, since M is on $\overline{PQ}$)
5	$\overline{PM} \cong \overline{PM}$	Reflexive Property of Congruence
6	$\triangle APM \cong \triangle BPM$	SAS
7	$\overline{AM} \cong \overline{BM}$	CPCTC
8	$\angle AMP \cong \angle BMP$	CPCTC
9	$\overleftrightarrow{PQ} \perp \overline{AB}$	If two lines form a linear pair of congruent angles, the lines are perpendicular
10	$\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector (M is the midpoint of $\overline{AB}$ by row 7, and $\overleftrightarrow{PQ} \perp \overline{AB}$ at M by row 9)

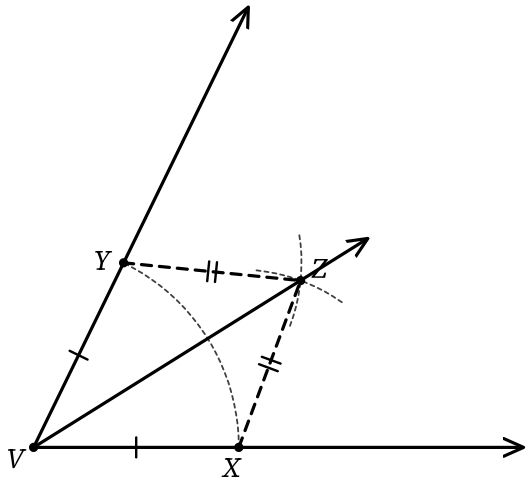
∎

So M is the midpoint of $\overline{AB}$ and $\overleftrightarrow{PQ}$ is perpendicular to it at M: the perpendicular bisector.

CONSTRUCTIONS

CONSTRUCTION **Bisect an angle** Euclid I.9

1. With the compass point on the vertex V, draw an arc crossing both sides of the angle at X and Y.
2. With the compass point on X, draw an arc in the interior of the angle. Keeping the same compass width, draw an arc from Y crossing the first one at Z.

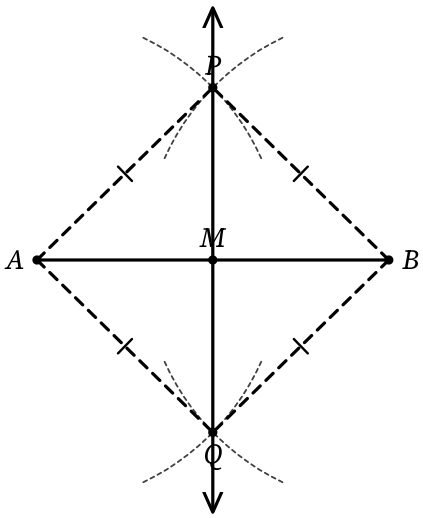


3. Draw $\overrightarrow{VZ}$. It bisects $\angle XVY$.

Why it works. Step 1 makes $VX = VY$ and step 2 makes $XZ = YZ$. So $\triangle VXZ \cong \triangle VYZ$ by SSS, and $\angle XVZ \cong \angle YVZ$ by CPCTC (see the theorem above).

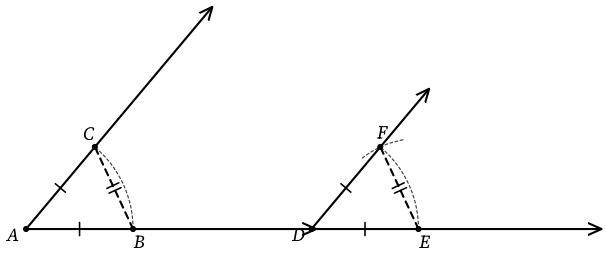
CONSTRUCTION **Construct the perpendicular bisector of a segment** Euclid I.10

1. Open the compass to more than half of AB. With the point on A, draw arcs above and below $\overline{AB}$.
2. With the same width and the point on B, draw arcs crossing the first two at P (above) and Q (below).
3. Draw $\overleftrightarrow{PQ}$. It crosses $\overline{AB}$ at its midpoint M and is perpendicular to it.



Why it works. All four distances AP, BP, AQ, BQ equal the compass width. $\triangle APQ \cong \triangle BPQ$ by SSS gives $\angle APM \cong \angle BPM$; then $\triangle APM \cong \triangle BPM$ by SAS gives $AM = BM$ and congruent angles at M that form a linear pair, so they are right angles.

CONSTRUCTION **Copy an angle** Euclid I.23



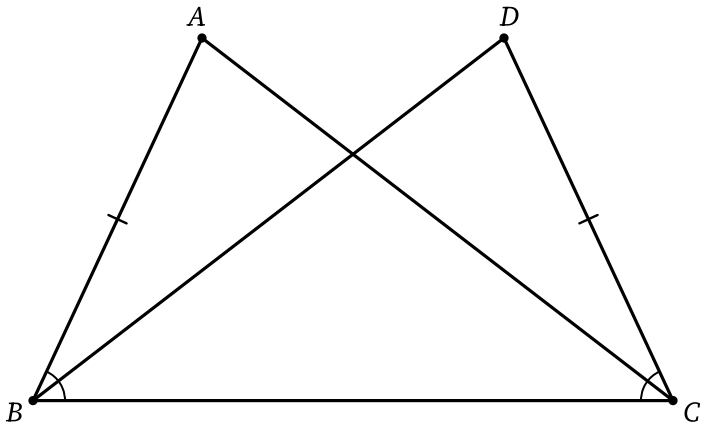
1. Given $\angle A$, draw a ray with endpoint D.
2. With the compass point on A, draw an arc crossing the sides of $\angle A$ at B and C. With the same width and the point on D, draw an arc crossing the ray at E.
3. Set the compass to the distance BC. With the point on E, draw an arc crossing the arc from step 2 at F.
4. Draw $\overrightarrow{DF}$. Then $\angle EDF \cong \angle BAC$.

Why it works. By construction $AB = DE$, $AC = DF$ and $BC = EF$, so $\triangle ABC \cong \triangle DEF$ by SSS, and $\angle BAC \cong \angle EDF$ by CPCTC.

Euclid connection. Euclid justifies his constructions exactly as we do here. In I.9 (to bisect a given rectilineal angle) he makes two equal sides and an equilateral triangle, then uses I.8 (SSS) to show the two halves of the angle are equal. In I.10 (to bisect a given finite straight line) he builds an equilateral triangle on the line, bisects its top angle, and uses I.4 (SAS) to show the two halves of the line are equal.

Practice habit. Plan every CPCTC proof by working backward: circle the two parts you must prove congruent, find the two triangles that contain them, and decide which test will prove those triangles congruent. When triangles overlap, redraw them separately before you start.

Example 1. In the figure, $\angle ABC \cong \angle DCB$ and $\overline{AB} \cong \overline{DC}$. Prove that $\overline{AC} \cong \overline{DB}$.



Show solution

1. Work backward: $\overline{AC}$ and $\overline{DB}$ are sides of the overlapping triangles $\triangle ABC$ and $\triangle DCB$. Sketch them separately.
2. Each has a marked side, a marked angle, and the shared side $\overline{BC}$. The angle is included, so use SAS.
3. Then $\overline{AC}$ and $\overline{DB}$ correspond, so CPCTC finishes the proof.

TWO-COLUMN PROOF

Given: $\angle ABC \cong \angle DCB$; $\overline{AB} \cong \overline{DC}$

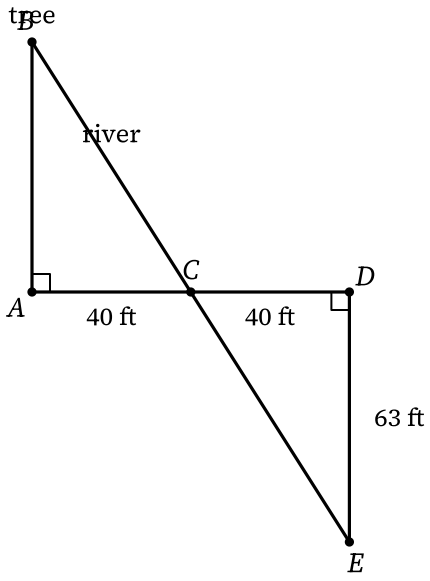
Prove: $\overline{AC} \cong \overline{DB}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{DC}$	Given
2	$\angle ABC \cong \angle DCB$	Given
3	$\overline{BC} \cong \overline{CB}$	Reflexive Property of Congruence
4	$\triangle ABC \cong \triangle DCB$	SAS
5	$\overline{AC} \cong \overline{DB}$	CPCTC

∎

Answer: See the model proof.

Example 2. Samuel wants the width of a river. From point A on his bank he sights a tree B directly across, so $\overline{AB}$ is perpendicular to the bank. He walks 40 ft along the bank to C and plants a stake, then walks 40 ft more to D. At D he turns at a right angle away from the river and walks until the stake C and the tree B line up, at point E. He measures $DE = 63$ ft (see the figure). How wide is the river, AB? Explain why the method works.



Show solution

1. In $\triangle BAC$ and $\triangle EDC$: $\angle BAC$ and $\angle EDC$ are right angles, so they are congruent.
2. $\overline{AC} \cong \overline{DC}$, since both are 40 ft.
3. $\angle ACB \cong \angle DCE$ because they are vertical angles (B, C, E are in line, and A, C, D are in line).
4. So $\triangle BAC \cong \triangle EDC$ by ASA, and $AB = DE$ by CPCTC. The river is 63 ft wide.

Answer: $AB = 63$ ft

PRACTICE

1. Easy $\triangle ABC \cong \triangle XYZ$, $BC = 12$ cm and $m\angle A = 47°$. Find YZ and $m\angle X$.

   Show answer

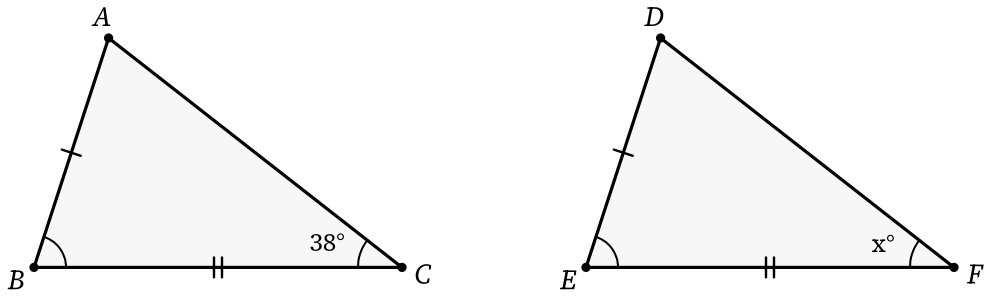
   By CPCTC, $YZ = BC = 12$ cm and $m\angle X = m\angle A = 47°$.

2. **Easy** Henry wants to prove $\overline{PQ} \cong \overline{RS}$, and his first step is 'CPCTC.' Explain why that cannot be the first step.

Show answer

CPCTC can be used only after two triangles have been proved congruent. Henry must first name two triangles that contain $\overline{PQ}$ and $\overline{RS}$ as corresponding sides and prove them congruent with SSS, SAS, ASA or AAS.

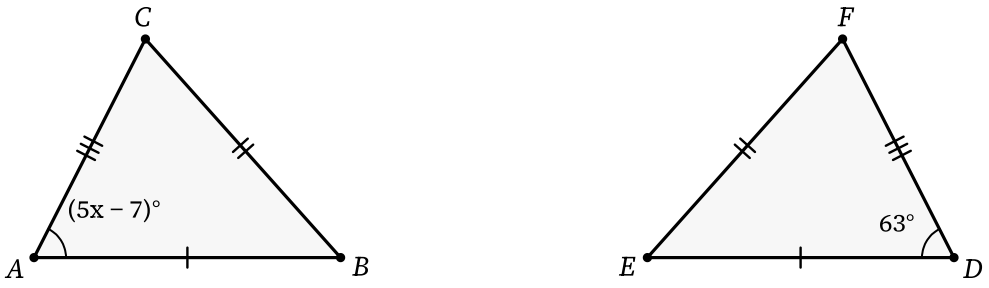
3. **Easy** In the figure, $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, $\overline{BC} \cong \overline{EF}$ and $m\angle C = 38°$. Find $x = m\angle F$.



Show answer

$\triangle ABC \cong \triangle DEF$ by SAS, so $\angle F \cong \angle C$ by CPCTC and $x = 38$.

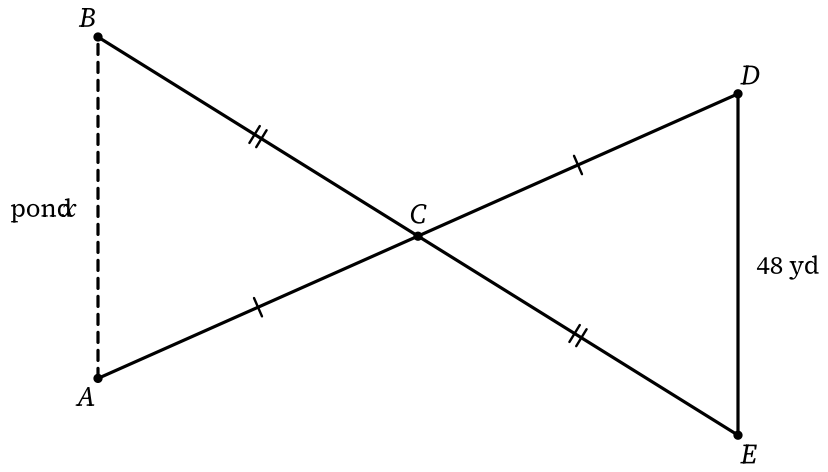
4. **Easy** In the figure, the three pairs of sides are marked congruent, $m\angle A = (5x - 7)°$ and $m\angle D = 63°$. Find x.



Show answer

$\triangle ABC \cong \triangle DEF$ by SSS, so $\angle A \cong \angle D$ by CPCTC: $5x - 7 = 63$, $5x = 70$, $x = 14$.

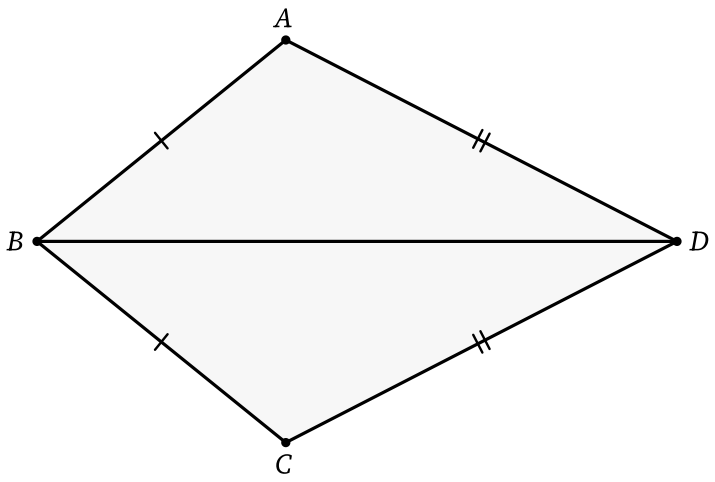
5. **Medium** Grace needs the distance AB across a pond. She chooses a point C from which she can walk straight to both A and B. She extends $\overline{AC}$ through C to D so that $CD = CA$, and $\overline{BC}$ through C to E so that $CE = CB$. She measures $DE = 48$ yd (see the figure). Find AB and name the test and the reason that justify it.



Show answer

$\angle ACB \cong \angle DCE$ (vertical angles), so $\triangle ACB \cong \triangle DCE$ by SAS, and $AB = DE = 48$ yd by CPCTC.

6. **Medium** In the figure, $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. Prove that $\angle BAD \cong \angle BCD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CB}$; $\overline{AD} \cong \overline{CD}$

Prove: $\angle BAD \cong \angle BCD$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CB}$	Given
2	$\overline{AD} \cong \overline{CD}$	Given
3	$\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
4	$\triangle ABD \cong \triangle CBD$	SSS
5	$\angle BAD \cong \angle BCD$	CPCTC

∎

7. **Medium** $\triangle ABD \cong \triangle CBD$, $AD = 3x + 2$ and $CD = 5x - 10$. Find x and AD.

Show answer

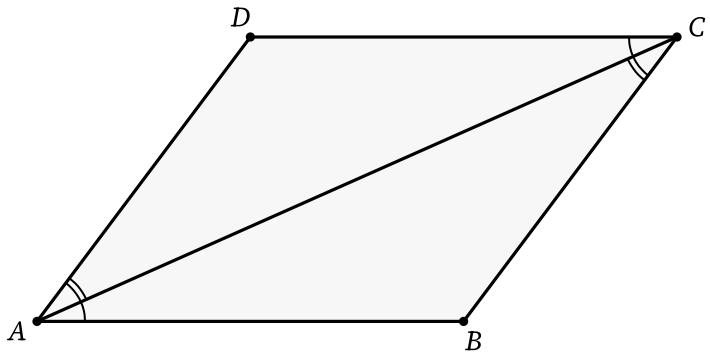
By CPCTC, $3x + 2 = 5x - 10$, so $12 = 2x$, $x = 6$ and $AD = 3(6) + 2 = 20$.

8. **Medium** A flagpole stands perpendicular to level ground at point M. Two guy wires run from the same point P on the pole to stakes A and B on opposite sides of the pole, each stake 9 ft from M. Wire $\overline{PA}$ is 15 ft long. How long is wire $\overline{PB}$? Explain.

Show answer

$\overline{MA} \cong \overline{MB}$ (both 9 ft), $\angle PMA \cong \angle PMB$ (both right angles), and $\overline{PM} \cong \overline{PM}$, so $\triangle PMA \cong \triangle PMB$ by SAS. By CPCTC, $PB = PA = 15$ ft.

9. **Hard** In quadrilateral $ABCD$ in the figure, diagonal $\overline{AC}$ is drawn, $\angle BAC \cong \angle DCA$ and $\angle BCA \cong \angle DAC$. Write a paragraph proof that $\overline{AB} \cong \overline{CD}$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\angle BAC \cong \angle DCA$; $\angle BCA \cong \angle DAC$

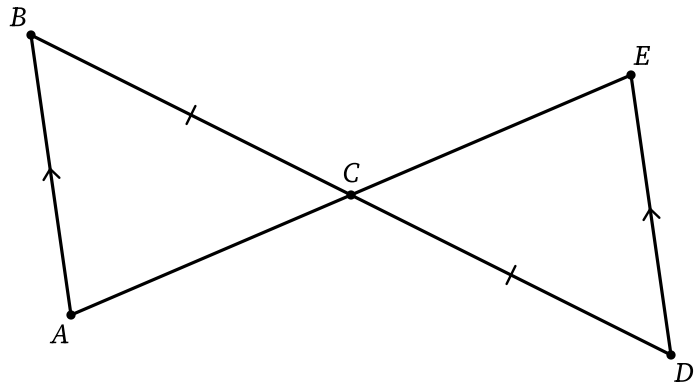
Prove: $\overline{AB} \cong \overline{CD}$

Consider $\triangle ABC$ and $\triangle CDA$. We are given $\angle BAC \cong \angle DCA$ and $\angle BCA \cong \angle DAC$, and $\overline{AC} \cong \overline{CA}$ by the Reflexive Property of Congruence. The side $\overline{AC}$ is included between the two given angles in each triangle, so $\triangle ABC \cong \triangle CDA$ by ASA.

In this correspondence $A \leftrightarrow C$ and $B \leftrightarrow D$, so $\overline{AB}$ corresponds to $\overline{CD}$. Therefore $\overline{AB} \cong \overline{CD}$ by CPCTC.

∎

10. Hard In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, $\overline{AB} \parallel \overline{DE}$ and C is the midpoint of $\overline{BD}$. Write a flow proof that C is also the midpoint of $\overline{AE}$.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{AE}$ and $\overline{BD}$ intersect at C; $\overline{AB} \parallel \overline{DE}$; C is the midpoint of $\overline{BD}$

Prove: C is the midpoint of $\overline{AE}$

(1) $\overline{AB} \parallel \overline{DE}$
Given

(2) C is the midpoint of $\overline{BD}$
Given

(3) $\angle ABC \cong \angle EDC$
Alternate Interior Angles Theorem
from (1)

(4) $\overline{BC} \cong \overline{DC}$
Definition of midpoint
from (2)

(5) $\angle ACB \cong \angle ECD$
Vertical Angles Theorem

(6) $\triangle ABC \cong \triangle EDC$
ASA
from (3), (4), (5)

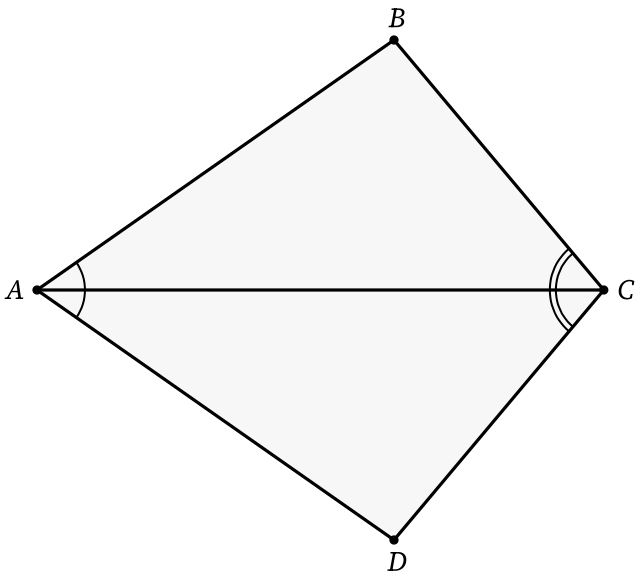
(7) $\overline{AC} \cong \overline{EC}$
CPCTC

from (6)

(8) C is the midpoint of $\overline{AE}$
Definition of midpoint (C lies on $\overline{AE}$)
from (7)

∎

11. **Hard** In the figure, $\overline{AC}$ bisects both $\angle BAD$ and $\angle BCD$. Prove that $\overline{AB} \cong \overline{AD}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

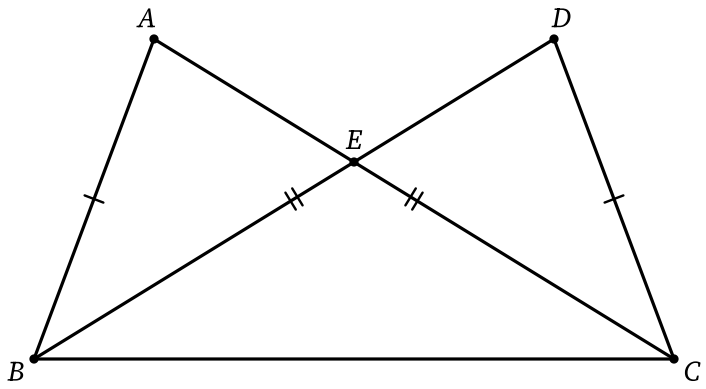
Given: $\overline{AC}$ bisects $\angle BAD$; $\overline{AC}$ bisects $\angle BCD$

Prove: $\overline{AB} \cong \overline{AD}$

	STATEMENTS	REASONS
1	$\overline{AC}$ bisects $\angle BAD$	Given
2	$\angle BAC \cong \angle DAC$	Definition of angle bisector
3	$\overline{AC}$ bisects $\angle BCD$	Given
4	$\angle BCA \cong \angle DCA$	Definition of angle bisector
5	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
6	$\triangle ABC \cong \triangle ADC$	ASA
7	$\overline{AB} \cong \overline{AD}$	CPCTC

∎

12. **Challenge** In the figure, $\overline{AB} \cong \overline{DC}$ and $\overline{AC} \cong \overline{DB}$, and $\overline{AC}$ and $\overline{DB}$ intersect at E. Prove that $\overline{AE} \cong \overline{DE}$. (You will need two pairs of congruent triangles.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{DC}$; $\overline{AC} \cong \overline{DB}$; $\overline{AC}$ and $\overline{DB}$ intersect at E

Prove: $\overline{AE} \cong \overline{DE}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{DC}$	Given
2	$\overline{AC} \cong \overline{DB}$	Given
3	$\overline{BC} \cong \overline{CB}$	Reflexive Property of Congruence
4	$\triangle ABC \cong \triangle DCB$	SSS
5	$\angle BAE \cong \angle CDE$	CPCTC ($\angle BAC \cong \angle CDB$, and these are the same angles because E lies on $\overline{AC}$ and on $\overline{DB}$)
6	$\angle AEB \cong \angle DEC$	Vertical Angles Theorem
7	$\triangle ABE \cong \triangle DCE$	AAS
8	$\overline{AE} \cong \overline{DE}$	CPCTC

∎

13. **Challenge** Euclid I.11 constructs a perpendicular to a line ℓ at a point P on it: with the compass point on P, mark A and B on ℓ with $PA = PB$; then, with a wider setting, draw arcs from A and B that meet at Q off the line. Prove that $\overleftrightarrow{QP} \perp \ell$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: A, P, B lie on ℓ with $\overline{PA} \cong \overline{PB}$; $\overline{QA} \cong \overline{QB}$ (same compass setting)
Prove: $\overleftrightarrow{QP} \perp \ell$
In $\triangle QPA$ and $\triangle QPB$, $\overline{PA} \cong \overline{PB}$ and $\overline{QA} \cong \overline{QB}$ by construction, and $\overline{QP} \cong \overline{QP}$ by the Reflexive Property. So $\triangle QPA \cong \triangle QPB$ by SSS, and $\angle QPA \cong \angle QPB$ by CPCTC.

Since A and B are on opposite rays from P, the angles $\angle QPA$ and $\angle QPB$ form a linear pair. Two lines that form a linear pair of congruent angles are perpendicular, so $\overleftrightarrow{QP} \perp \ell$.

∎

COMMON MISTAKES

- Using CPCTC before proving the triangles congruent. CPCTC is never the first reason; it always follows a line that says two triangles are congruent.
- Reading the wrong corresponding parts. Use the order of the letters in your congruence statement, not the picture, to decide which parts match.
- Losing track in overlapping figures. Redraw the two triangles separately and copy every mark onto each before you plan.
- Trying to prove triangles congruent with the very parts you are asked to prove. You cannot use the conclusion as a given.

> **Make it a game · CPCTC Story Chain.** Build a proof as a chain, one sentence each. Parent begins: 'I draw an arc from vertex A that meets both sides.' You add: 'Equal arcs from those points meet at D.' Continue until you reach the triangles, name the congruence, and finish with CPCTC proving the angle bisected. Any sentence without a reason breaks the chain.

4.6 Isosceles and Equilateral Triangles

OBJECTIVES

- Name the legs, base, base angles and vertex angle of an isosceles triangle.
- Prove the Isosceles Triangle Theorem (Euclid I.5) and its converse (Euclid I.6).
- Prove that a triangle is equilateral exactly when it is equiangular, and use these facts with algebra.
- Construct an equilateral (Gothic) arch and explain why its angles measure 60°.

WATCH

Equal Angles Opposite Equal Sides

Eddie Woo · 7:48

https://www.youtube.com/watch?v=0EWM9OZjFpk

> **WHEN TO WATCH**
> Before the Isosceles Triangle Theorem: discovering that the base angles are equal.

Note: Eddie Woo teaches in Australia: he says 'equal angles are opposite equal sides' for the Isosceles Triangle Theorem and 'pronumeral' for variable.

BACKUP VIDEO

Using Triangle Properties in Geometric Proofs

Eddie Woo · 8:13

https://www.youtube.com/watch?v=zfzEw7VjfaQ

Before Example 1: finding angles and sides with reasons.

KEY TERMS

Legs and base of an isosceles triangle

The two congruent sides are the legs; the third side is the base.

Vertex angle

The angle formed by the two legs of an isosceles triangle; it lies opposite the base.

Base angles

The two angles of an isosceles triangle that have the base as a side; each lies opposite a leg.

Equilateral arch

A pointed arch whose two curves are drawn from the two springing points with radius equal to the span, so the apex and the springing points form an equilateral triangle.

Isosceles Triangle Theorem

If two sides of a triangle are congruent, then the angles opposite them are congruent (Euclid I.5).

CORE IDEAS

An isosceles triangle has two congruent sides, its legs. The angle between the legs is the vertex angle, the third side is the base, and the two angles touching the base are the base angles. Notice that each base angle lies opposite a leg.

Why are the base angles congruent? Fold the triangle in your mind along the bisector of the vertex angle. To make that idea a proof, draw the bisector $\overline{AD}$ of the vertex angle. Now the two halves have two pairs of congruent sides (a leg each, and the shared bisector) and congruent included angles, so they are congruent by SAS, and the base angles are congruent by CPCTC. That is the Isosceles Triangle Theorem, Euclid I.5.

The converse is also true: if two angles of a triangle are congruent, the sides opposite them are congruent (Euclid I.6). Again draw the bisector of the third angle; this time the two halves have two pairs of congruent angles and the shared side, so they are congruent by AAS. Because both the theorem and its converse are true, we can say: two sides of a triangle are congruent if and only if the angles opposite them are congruent.

Apply both facts around an equilateral triangle. Every pair of sides is congruent, so every pair of angles is congruent; the triangle is equiangular, and by the angle sum each angle measures $60°$. Conversely, an equiangular triangle is equilateral.

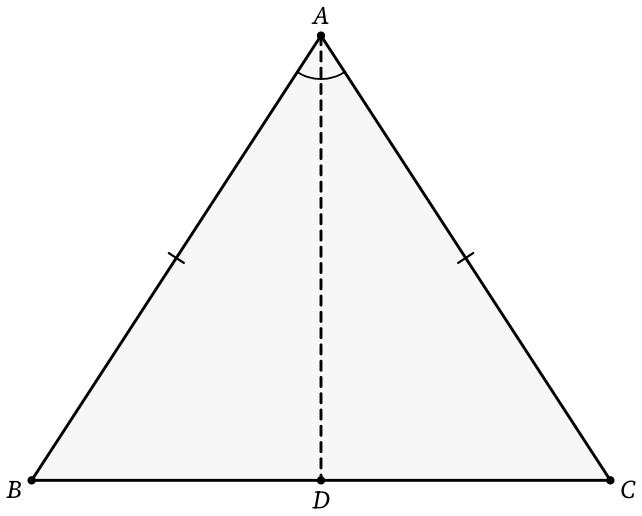
Gothic church builders drew the pointed 'equilateral arch' with nothing but a cord and two pins: one pin at each springing point of the arch, the cord as long as the span. Each curve is an arc of a circle centered at the opposite springing point, so the

apex is the same distance from both springing points as they are from each other. That is Euclid's very first proposition, I.1, standing in stone.

POSTULATES AND THEOREMS

THEOREM **Isosceles Triangle Theorem (Base Angles Theorem)** Euclid I.5

If two sides of a triangle are congruent, then the angles opposite those sides are congruent.



TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$

Prove: $\angle B \cong \angle C$

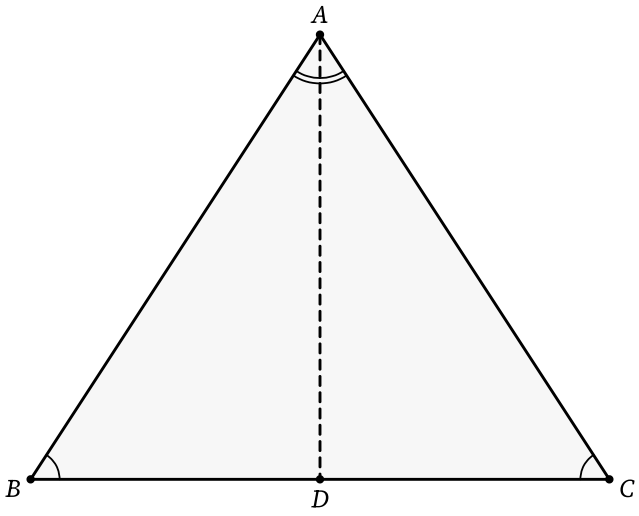
	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	Draw $\overrightarrow{AD}$, the bisector of $\angle BAC$, meeting $\overline{BC}$ at D	Every angle has a bisector (construction, Euclid I.9)
3	$\angle BAD \cong \angle CAD$	Definition of angle bisector
4	$\overline{AD} \cong \overline{AD}$	Reflexive Property of Congruence
5	$\triangle ABD \cong \triangle ACD$	SAS
6	$\angle B \cong \angle C$	CPCTC

∎

Euclid's own proof extends both legs below the base and compares two pairs of triangles, a harder argument that gave I.5 its nickname. According to Proclus, Pappus gave a very short proof: match $\triangle ABC$ with $\triangle ACB$ (the same triangle, read in mirror order); since $\overline{AB} \cong \overline{AC}$, $\angle A \cong \angle A$ and $\overline{AC} \cong \overline{AB}$, SAS gives $\triangle ABC \cong \triangle ACB$, so $\angle B \cong \angle C$ by CPCTC.

THEOREM **Converse of the Isosceles Triangle Theorem** Euclid I.6

If two angles of a triangle are congruent, then the sides opposite those angles are congruent.



TWO-COLUMN PROOF

Given: $\angle B \cong \angle C$

Prove: $\overline{AB} \cong \overline{AC}$

	STATEMENTS	REASONS
1	$\angle B \cong \angle C$	Given
2	Draw $\overrightarrow{AD}$, the bisector of $\angle BAC$, meeting $\overline{BC}$ at D	Every angle has a bisector (construction, Euclid I.9)
3	$\angle BAD \cong \angle CAD$	Definition of angle bisector
4	$\overline{AD} \cong \overline{AD}$	Reflexive Property of Congruence
5	$\triangle ABD \cong \triangle ACD$	AAS
6	$\overline{AB} \cong \overline{AC}$	CPCTC

∎

Euclid proves I.6 indirectly. Suppose $AB > AC$. Cut off $\overline{DB}$ on $\overline{BA}$ equal to $\overline{AC}$. Then $\triangle DBC$ and $\triangle ACB$ have $DB = AC$, a common side BC and the given equal angles at B and C, so by SAS they are congruent; but $\triangle DBC$ is only part of $\triangle ACB$, and 'the whole is greater than the part.' This contradiction shows AB cannot be greater than AC, and in the same way it cannot be less. You will write indirect proofs of your own in Lesson 5.5.

COROLLARY **Equilateral triangles are equiangular**

If a triangle is equilateral, then it is equiangular, and each angle measures $60°$.

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{BC} \cong \overline{CA}$

Prove: $\angle A \cong \angle B \cong \angle C$, each $60°$

Since $\overline{AB} \cong \overline{AC}$, the angles opposite them are congruent: $\angle C \cong \angle B$ (Isosceles Triangle Theorem). Since $\overline{BA} \cong \overline{BC}$, likewise $\angle C \cong \angle A$. By the Transitive Property all three angles are congruent, and by the Triangle Angle Sum Theorem each measures $180° \div 3 = 60°$.

∎

COROLLARY **Equiangular triangles are equilateral**

If a triangle is equiangular, then it is equilateral.

PARAGRAPH PROOF

Given: $\angle A \cong \angle B \cong \angle C$

Prove: $\overline{AB} \cong \overline{BC} \cong \overline{CA}$

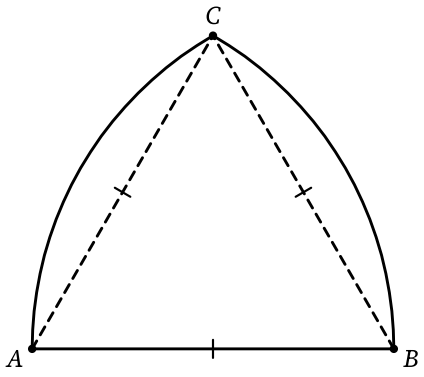
Since $\angle B \cong \angle C$, the sides opposite them are congruent: $\overline{AC} \cong \overline{AB}$ (Converse of the Isosceles Triangle Theorem). Since $\angle A \cong \angle C$, likewise $\overline{BC} \cong \overline{AB}$. By the Transitive Property all three sides are congruent.

∎

CONSTRUCTIONS

CONSTRUCTION **Draw an equilateral (Gothic) arch** Euclid I.1

1. Draw the span $\overline{AB}$ between the two springing points of the arch.
2. Set the compass (or a cord) to the length AB. With the point on A, draw an arc upward from B.
3. Keeping the same width, put the point on B and draw an



arc upward from A. The arcs meet at the apex C.

4. The two arcs from A to C and from B to C form the arch.

Why it works. C is on the circle about A with radius AB and on the circle about B with radius AB, so $AC = AB = BC$ and $\triangle ABC$ is equilateral (Euclid I.1). By the corollary above, each of its angles measures 60°, so the arch rises at a pleasing, fixed proportion no matter how wide it is.

KEY FORMULAS

$$\text{base angle} = \frac{180^\circ - \text{vertex angle}}{2}$$

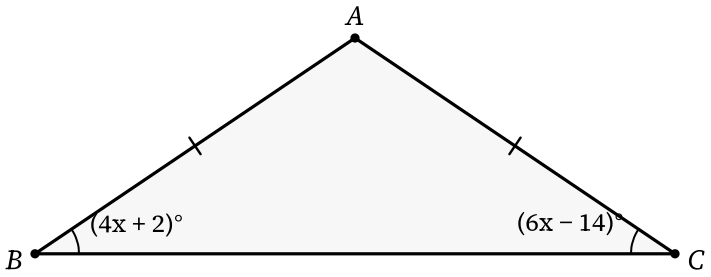
$$\text{vertex angle} = 180^\circ - 2(\text{base angle})$$

From history. Euclid I.5 came to be called the pons asinorum, the 'bridge of asses,' perhaps because its figure looks like a trestle bridge and because it was the first real test for a beginner: those who could not cross it went no further in geometry. Gothic master masons used the equilateral arch, along with other arches drawn from compass centers, for windows and vaults in many great churches.

Euclid connection. Euclid I.5 reads: 'In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.' I.6 is its converse, and Euclid's proof of it is one of the first indirect proofs in the Elements.

Practice habit. Recite the Isosceles Triangle Theorem and its converse until automatic, and always say which sides are opposite which angles. Before using either one, check that the two parts are in the SAME triangle; and remember the converse needs angles to prove sides, the theorem needs sides to prove angles.

Example 1. In the figure, $\overline{AB} \cong \overline{AC}$, $m\angle B = (4x + 2)^\circ$ and $m\angle C = (6x - 14)^\circ$. Find x and the measure of the vertex angle $\angle A$.

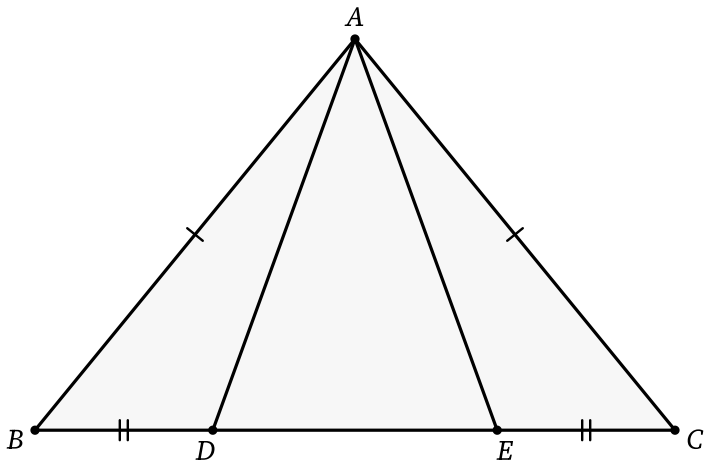


Show solution

1. $\angle B$ and $\angle C$ are opposite the congruent sides $\overline{AC}$ and $\overline{AB}$, so they are congruent (Isosceles Triangle Theorem): $4x + 2 = 6x - 14$.
2. Solve: $16 = 2x$, so $x = 8$. Each base angle measures $4(8) + 2 = 34^\circ$ (check: $6(8) - 14 = 34$).
3. By the Triangle Angle Sum Theorem, $m\angle A = 180 - 34 - 34 = 112^\circ$.

Answer: $x = 8$; $m\angle A = 112^\circ$

Example 2. In the figure, $\overline{AB} \cong \overline{AC}$, and D and E lie on $\overline{BC}$ with $\overline{BD} \cong \overline{CE}$. Write a paragraph proof that $\overline{AD} \cong \overline{AE}$.



Show solution

1. Work backward: $\overline{AD}$ and $\overline{AE}$ are sides of $\triangle ABD$ and $\triangle ACE$.
2. Those triangles have $\overline{AB} \cong \overline{AC}$ and $\overline{BD} \cong \overline{CE}$. The included angles are $\angle B$ and $\angle C$, which are congruent by the Isosceles Triangle Theorem.
3. SAS, then CPCTC.

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{AC}$; D and E lie on $\overline{BC}$ with $\overline{BD} \cong \overline{CE}$

Prove: $\overline{AD} \cong \overline{AE}$

Since $\overline{AB} \cong \overline{AC}$, the angles opposite these sides in $\triangle ABC$ are congruent by the Isosceles Triangle Theorem: $\angle ABC \cong \angle ACB$. Because D and E lie on $\overline{BC}$, these are the same angles as $\angle ABD$ and $\angle ACE$.

Now in $\triangle ABD$ and $\triangle ACE$ we have $\overline{AB} \cong \overline{AC}$ (given), $\angle ABD \cong \angle ACE$, and $\overline{BD} \cong \overline{CE}$ (given), with each angle included between the two sides. So $\triangle ABD \cong \triangle ACE$ by SAS, and $\overline{AD} \cong \overline{AE}$ by CPCTC.

∎

Answer: See the model proof.

PRACTICE

1. **Easy** The gable end of a barn is an isosceles triangle whose base angles each measure $40°$. What is the measure of the vertex angle at the peak?

   Show answer

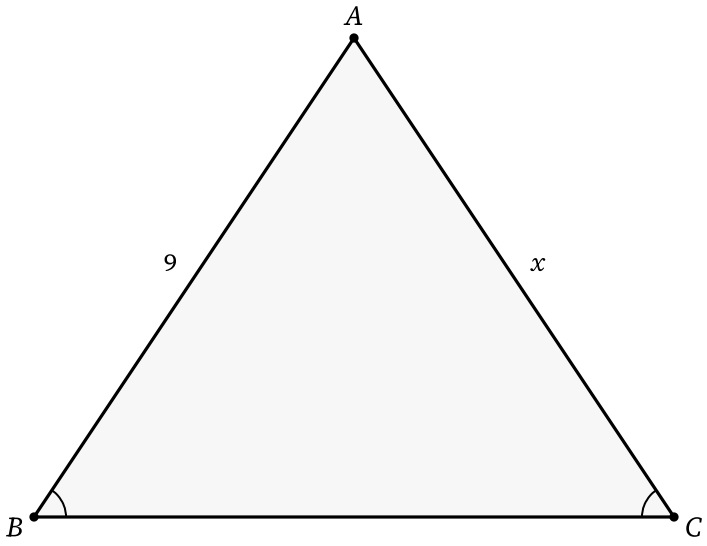
   $180 - 2(40) = 100°$

2. **Easy** The vertex angle of an isosceles triangle measures $50°$. Find the measure of each base angle.

   Show answer

   $(180 - 50) \div 2 = 65°$

3. **Easy** In the figure, $\angle B \cong \angle C$ and $AB = 9$. Find $x = AC$ and name the theorem you used.



   Show answer

   $x = 9$, by the Converse of the Isosceles Triangle Theorem (sides opposite congruent angles are congruent).

4. **Easy** Two sides of an equilateral triangle measure $3x - 2$ and $2x + 5$. Find x and the length of each side.

   Show answer

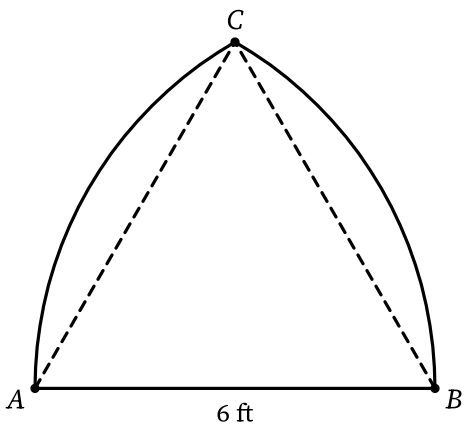
   $3x - 2 = 2x + 5$, so $x = 7$ and each side is $3(7) - 2 = 19$.

5. **Medium** In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$, $m\angle B = (2x + 5)^\circ$ and $m\angle A = (x + 20)^\circ$. Find x and $m\angle A$.

Show answer

$\angle C \cong \angle B$, so $2(2x + 5) + (x + 20) = 180$: $5x + 30 = 180$, $x = 30$, and $m\angle A = 30 + 20 = 50^\circ$. (Check: base angles 65° each; $65 + 65 + 50 = 180$.)

6. **Medium** A stonemason draws an equilateral arch over a window with a span AB of 6 ft, as in the figure: the arc through B and C is centered at A, and the arc through A and C is centered at B, both with radius 6 ft. Explain why $\triangle ABC$ is equilateral, and find $m\angle CAB$.



Show answer

C is on both arcs, so $AC = 6$ ft and $BC = 6$ ft, and $AB = 6$ ft; all three sides are congruent. An equilateral triangle is equiangular, so $m\angle CAB = 60^\circ$.

7. **Medium** Ruth opens a stepladder whose two legs are the same length. Each leg makes a 74° angle with the level floor. What angle do the legs make with each other at the top?

Show answer

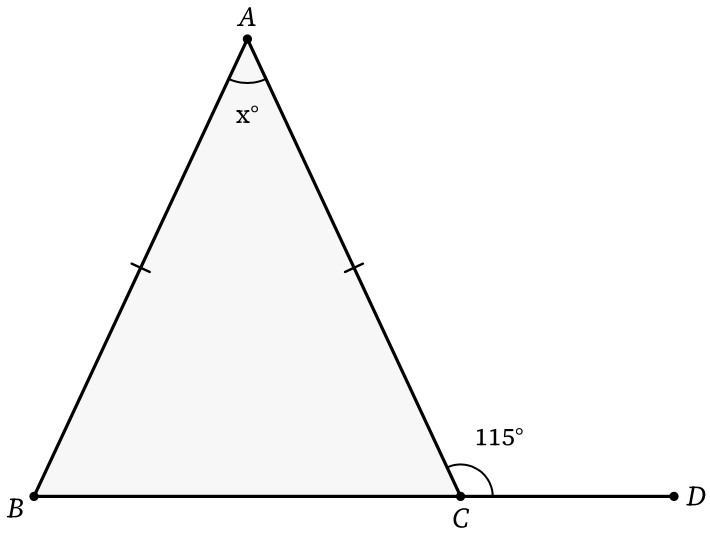
The legs and the floor form an isosceles triangle with base angles 74°, so the top angle is $180 - 2(74) = 32^\circ$.

8. **Medium** Can an isosceles triangle have a base angle of $90°$? Explain.

Show answer

No. The base angles are congruent, so both would be $90°$, and their sum alone would be $180°$, leaving $0°$ for the vertex angle. A triangle cannot have two right angles.

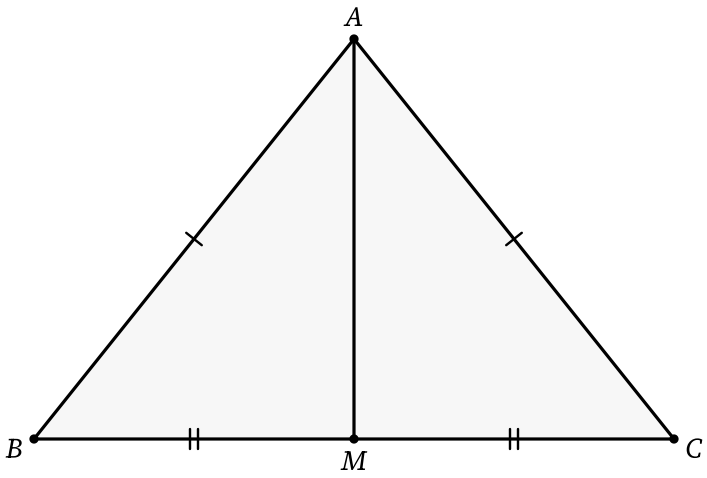
9. **Medium** In the figure, $\overline{AB} \cong \overline{AC}$, D lies on $\overrightarrow{BC}$ beyond C, and $m\angle ACD = 115°$. Find $x = m\angle BAC$.



Show answer

$m\angle ACB = 180 - 115 = 65°$ (linear pair). By the Isosceles Triangle Theorem $m\angle B = 65°$. Then $x = 180 - 65 - 65 = 50$.

10. Hard In the figure, $\overline{AB} \cong \overline{AC}$ and M is the midpoint of $\overline{BC}$. Prove that $\overline{AM} \perp \overline{BC}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

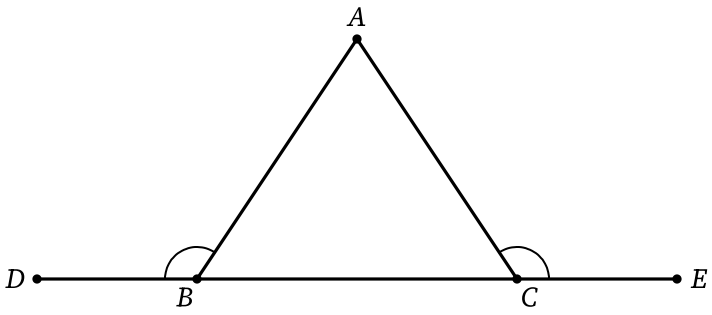
Given: $\overline{AB} \cong \overline{AC}$; M is the midpoint of $\overline{BC}$

Prove: $\overline{AM} \perp \overline{BC}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	M is the midpoint of $\overline{BC}$	Given
3	$\overline{BM} \cong \overline{CM}$	Definition of midpoint
4	$\overline{AM} \cong \overline{AM}$	Reflexive Property of Congruence
5	$\triangle ABM \cong \triangle ACM$	SSS
6	$\angle AMB \cong \angle AMC$	CPCTC
7	$\overline{AM} \perp \overline{BC}$	If two lines form a linear pair of congruent angles, the lines are perpendicular

∎

11. **Hard** In the figure, D, B, C, E lie on a line in that order and $\angle ABD \cong \angle ACE$. Write a flow proof that $\overline{AB} \cong \overline{AC}$.



Show answer

See the model proof.

FLOW PROOF

Given: D, B, C, E are collinear in that order; $\angle ABD \cong \angle ACE$

Prove: $\overline{AB} \cong \overline{AC}$

(1) $\angle ABD \cong \angle ACE$
Given

(2) $\angle ABD$ and $\angle ABC$ are supplementary
Linear pairs are supplementary

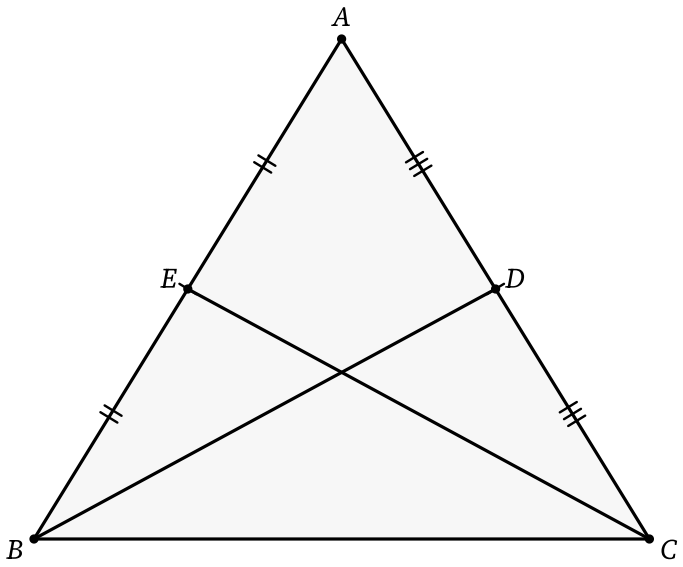
(3) $\angle ACE$ and $\angle ACB$ are supplementary
Linear pairs are supplementary

(4) $\angle ABC \cong \angle ACB$
Congruent Supplements Theorem
from (1), (2), (3)

(5) $\overline{AB} \cong \overline{AC}$
Converse of the Isosceles Triangle Theorem
from (4)

∎

12. **Hard** In the figure, $\overline{AB} \cong \overline{AC}$, E is the midpoint of $\overline{AB}$ and D is the midpoint of $\overline{AC}$. Prove that $\overline{BD} \cong \overline{CE}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$; E is the midpoint of $\overline{AB}$; D is the midpoint of $\overline{AC}$

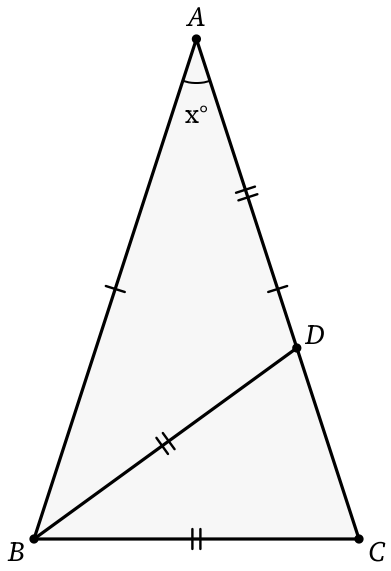
Prove: $\overline{BD} \cong \overline{CE}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$AB = AC$	Definition of congruent segments
3	E is the midpoint of $\overline{AB}$ and D is the midpoint of $\overline{AC}$	Given
4	$EB = \frac{1}{2}AB$ and $DC = \frac{1}{2}AC$	Definition of midpoint (with the Segment Addition Postulate)
5	$\frac{1}{2}AB = \frac{1}{2}AC$	Multiplication Property of Equality
6	$EB = DC$	Substitution Property of Equality

	STATEMENTS	REASONS
7	$\overline{EB} \cong \overline{DC}$	Definition of congruent segments
8	$\angle EBC \cong \angle DCB$	Isosceles Triangle Theorem ($\angle ABC \cong \angle ACB$; same angles, since E is on $\overline{BA}$ and D is on $\overline{CA}$)
9	$\overline{BC} \cong \overline{CB}$	Reflexive Property of Congruence
10	$\triangle EBC \cong \triangle DCB$	SAS
11	$\overline{BD} \cong \overline{CE}$	CPCTC

∎

13. **Challenge** In the figure, $\overline{AB} \cong \overline{AC}$, and D lies on $\overline{AC}$ so that $AD = BD = BC$. Find $x = m\angle A$.



Show answer

Since $AD = BD$, $m\angle ABD = x°$. $\angle BDC$ is an exterior angle of $\triangle ABD$, so $m\angle BDC = 2x°$. Since $BD = BC$, $m\angle BCD = m\angle BDC = 2x°$. Since $AB = AC$, $m\angle ABC = m\angle ACB = 2x°$. In $\triangle ABC$: $x + 2x + 2x = 180$, so $x = 36$.

COMMON MISTAKES

- Matching the wrong angles to the congruent sides. The congruent angles are OPPOSITE the congruent sides. If $\overline{AB} \cong \overline{BC}$, then $\angle C \cong \angle A$, not $\angle B$.
- Using the theorem when you need the converse (or the reverse). Ask: am I given sides and want angles (theorem), or given angles and want sides (converse)?
- Assuming the vertex angle is always at the top of the picture. Find it as the angle between the two congruent sides.
- Forgetting that 'isosceles' means at least two congruent sides, so an equilateral triangle is also isosceles; every fact about isosceles triangles applies to it.

> **Make it a game · Two Answers Race.** Draw an isosceles triangle and label its legs, base, vertex angle and base angles. Parent names one angle, such as 40 degrees; you must give both possible triangles, since 40 may be a base angle or the vertex angle. Say all three angles each time. Race ten rounds, then state Euclid's I.5 and I.6 aloud.

4.7 Right Triangles: HL and Proofs with Two Pairs of Triangles

OBJECTIVES

- Prove the Hypotenuse-Leg (HL) Congruence Theorem and use it in proofs.
- Recognize the right-triangle tests LL, HA and LA as cases of SAS, AAS and ASA.
- Write multi-step proofs that use two pairs of congruent triangles.
- Check whether two triangles in the coordinate plane are congruent using the distance formula.

WATCH

Congruent Triangles in Circles (2 of 2: Chord bisector)

Eddie Woo · 11:05

https://www.youtube.com/watch?v=GEAk5fmJe1s

> **WHEN TO WATCH**
> After the HL theorem: HL (he calls it RHS) used to prove that a perpendicular from the centre bisects a chord.

Note: Eddie Woo teaches in Australia: he says 'RHS' for HL. Both examples use circles; only 'all radii are equal' is needed for the primary.

BACKUP VIDEO

Tough Circle Geometry Question (1 of 3: Finding & using congruent triangles)

Eddie Woo · 11:53

https://www.youtube.com/watch?v=hRPkAFZW24E

As a challenge after the practice set: a multi-step proof built on congruent triangles.

KEY TERMS

Hypotenuse

The side of a right triangle opposite the right angle; it is the longest side.

Legs of a right triangle

The two sides that form the right angle.

Hypotenuse-Leg (HL) Theorem

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.

LL, HA and LA

Leg-Leg, Hypotenuse-Angle and Leg-Angle: shortcut names for SAS, AAS and ASA (or AAS) when the triangles are right triangles.

CORE IDEAS

In Lesson 4.3 you saw that Side-Side-Angle is not a congruence test, because a circle drawn about one vertex can cross the other side of the angle in two places. But there is one important exception: when the angle is a right angle. Then the two crossing points lie on opposite sides of the right angle's vertex, at equal distances from it, and the two triangles they make are mirror images of each other, so they are congruent. That is the Hypotenuse-Leg Theorem.

We prove HL using only what we already have. Put the two right triangles together along their congruent legs, so the right angles sit side by side and their other legs lie along one straight line. The two hypotenuses then form an isosceles triangle, so the angles at its base are congruent (Euclid I.5). Now the two original triangles have two pairs of congruent angles and a congruent side, and AAS finishes the proof. (If you prefer, the Pythagorean Theorem from Algebra I gives a quicker argument: the missing legs must be equal, so SSS applies.)

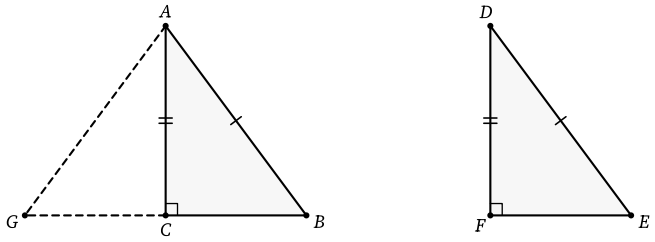
The other right-triangle tests are old tests with new names. Two legs (LL) are two sides with the right angle between them: SAS. A hypotenuse and an acute angle (HA) give two angles and a side opposite one of them: AAS. A leg and an acute angle (LA) is ASA if the angle touches the leg, and AAS if it is opposite the leg. In every case you need the word 'right' in your proof, since the right angle is one of the three parts.

Many proofs need two pairs of congruent triangles. The first pair is proved congruent only to supply, through CPCTC, a part that the second pair needs. Plan by working backward: decide which triangles contain the parts in the prove statement, list what those triangles still lack, and then look for another pair of triangles that can supply it.

In the coordinate plane you can check congruence with numbers. Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to find all three side lengths of each triangle. If the three lengths match in some order, the triangles are congruent by SSS.

THEOREM **Hypotenuse-Leg (HL) Congruence Theorem**

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.



TWO-COLUMN PROOF

Given: $\angle ACB$ and $\angle DFE$ are right angles; $\overline{AB} \cong \overline{DE}$; $\overline{AC} \cong \overline{DF}$

Prove: $\triangle ABC \cong \triangle DEF$

	STATEMENTS	REASONS
1	$\angle ACB$ and $\angle DFE$ are right angles	Given
2	$\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$	Given
3	On the ray opposite $\overrightarrow{CB}$, mark G with $CG = FE$	Ruler Postulate (copying a segment)
4	$\angle ACG$ is a right angle	$\angle ACG$ forms a linear pair with right $\angle ACB$, and linear pairs are supplementary
5	$\angle ACG \cong \angle DFE$	All right angles are congruent
6	$\triangle ACG \cong \triangle DFE$	SAS
7	$\overline{AG} \cong \overline{DE}$	CPCTC
8	$\overline{AG} \cong \overline{AB}$	Transitive and Symmetric Properties of Congruence
9	$\angle ABC \cong \angle AGC$	Isosceles Triangle Theorem (in $\triangle ABG$)
10	$\angle ACB \cong \angle ACG$	All right angles are congruent
11	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
12	$\triangle ACB \cong \triangle ACG$	AAS

	STATEMENTS	REASONS
13	$\triangle ABC \cong \triangle DEF$	Transitive Property of Triangle Congruence (rows 6 and 12 give $\triangle ACB \cong \triangle DFE$, the matching $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$)

∎

A shorter proof uses the Pythagorean Theorem from Algebra I (proved in Lesson 9.1): $BC^2 = AB^2 - AC^2 = DE^2 - DF^2 = EF^2$, so $BC = EF$ and the triangles are congruent by SSS.

COROLLARY **Leg-Leg (LL)**

If the two legs of one right triangle are congruent to the two legs of another right triangle, then the triangles are congruent.

PARAGRAPH PROOF

Prove: LL

The right angles are congruent because all right angles are congruent, and each right angle is included between the two legs. So the triangles are congruent by SAS.

∎

COROLLARY **Hypotenuse-Angle (HA)**

If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and the corresponding acute angle of another right triangle, then the triangles are congruent.

PARAGRAPH PROOF

Prove: HA

The right angles are congruent, the given acute angles are congruent, and the hypotenuse lies opposite the right angle in each triangle, so it is a non-included side. The triangles are congruent by AAS.

∎

COROLLARY **Leg-Angle (LA)**

If a leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.

PARAGRAPH PROOF

Prove: LA

The right angles are congruent. If the acute angle touches the given leg, that leg is included between the acute angle and the right angle, so ASA applies. If the acute angle is opposite the given leg, the leg is a non-included side, so AAS applies.

∎

KEY FORMULAS

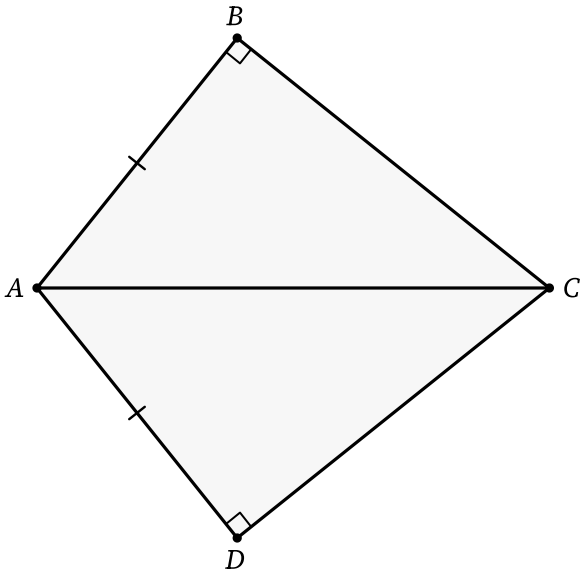
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Euclid connection. Euclid has no separate proposition for HL, but our joining proof is built entirely from his tools: I.4 (SAS), I.5 (the base angles of an isosceles triangle) and I.26 (AAS). It shows how a new result grows from earlier ones, which is the method of the whole Elements.

Practice habit. Before writing HL, write the line 'the triangles are right triangles' with its reason; HL may be used only for right triangles. In a two-pair proof, label your sketch with which triangles are 'pair 1' and 'pair 2' and write down what pair 1 must hand to pair 2.

WORKED EXAMPLES

Example 1. In the figure, $\angle ABC$ and $\angle ADC$ are right angles and $\overline{AB} \cong \overline{AD}$. Prove that $\triangle ABC \cong \triangle ADC$.



Show solution

1. Both triangles are right triangles, with right angles at B and D.
2. The shared side $\overline{AC}$ is opposite both right angles, so it is the hypotenuse of each.
3. A pair of legs is given. Hypotenuse and leg: use HL.

TWO-COLUMN PROOF

Given: $\angle ABC$ and $\angle ADC$ are right angles; $\overline{AB} \cong \overline{AD}$

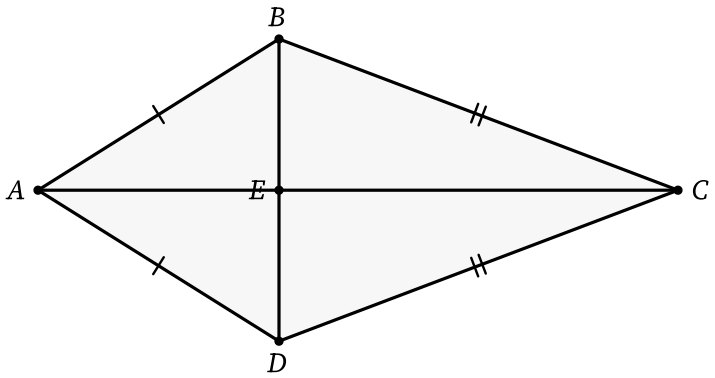
Prove: $\triangle ABC \cong \triangle ADC$

	STATEMENTS	REASONS
1	$\angle ABC$ and $\angle ADC$ are right angles	Given
2	$\triangle ABC$ and $\triangle ADC$ are right triangles	Definition of right triangle
3	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
4	$\overline{AB} \cong \overline{AD}$	Given
5	$\triangle ABC \cong \triangle ADC$	HL

∎

Answer: See the model proof.

Example 2. In kite $ABCD$ in the figure, $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$, and the diagonals $\overline{AC}$ and $\overline{BD}$ meet at E. Write a paragraph proof that $\overline{AC} \perp \overline{BD}$.



Show solution

1. Work backward: to show perpendicular, show that the linear pair $\angle AEB$ and $\angle AED$ are congruent. They are in $\triangle ABE$ and $\triangle ADE$ (pair 2).
2. Pair 2 has $\overline{AB} \cong \overline{AD}$ and the shared $\overline{AE}$, but it needs the included angles $\angle BAE \cong \angle DAE$.
3. Get those angles from pair 1, $\triangle ABC$ and $\triangle ADC$, which are congruent by SSS.

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{AD}$; $\overline{BC} \cong \overline{DC}$; $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\overline{AC} \perp \overline{BD}$

In $\triangle ABC$ and $\triangle ADC$, $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$ are given, and $\overline{AC} \cong \overline{AC}$ by the Reflexive Property. So $\triangle ABC \cong \triangle ADC$ by SSS, and $\angle BAC \cong \angle DAC$ by CPCTC. Since E lies on $\overline{AC}$, these are the angles $\angle BAE$ and $\angle DAE$.

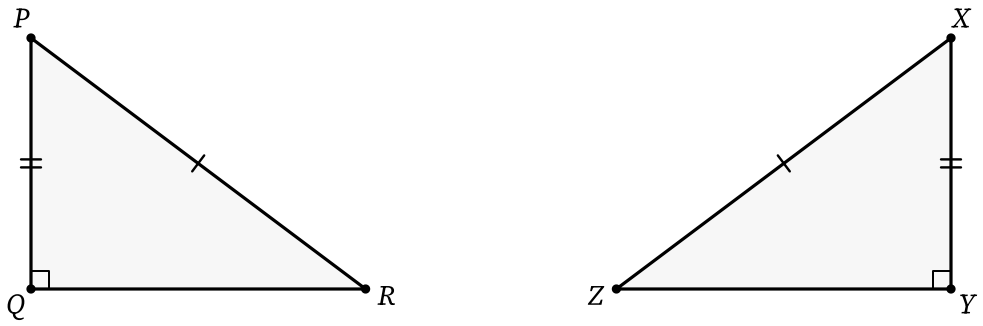
Now in $\triangle ABE$ and $\triangle ADE$, $\overline{AB} \cong \overline{AD}$, $\angle BAE \cong \angle DAE$, and $\overline{AE} \cong \overline{AE}$ (Reflexive Property), so $\triangle ABE \cong \triangle ADE$ by SAS. By CPCTC, $\angle AEB \cong \angle AED$. These two angles form a linear pair because E lies between B and D, and two lines that form a linear pair of congruent angles are perpendicular. Therefore $\overline{AC} \perp \overline{BD}$.

∎

Answer: See the model proof.

PRACTICE

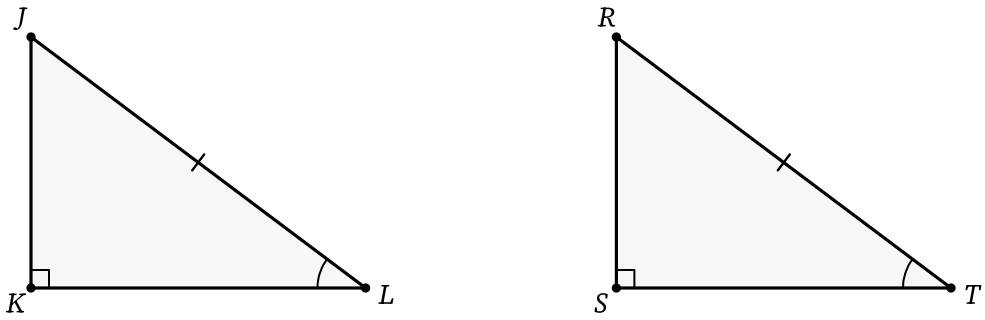
1. **Easy** In the figure, $\triangle PQR$ and $\triangle XYZ$ have right angles at Q and Y, $\overline{PR} \cong \overline{XZ}$ and $\overline{PQ} \cong \overline{XY}$. Which test proves $\triangle PQR \cong \triangle XYZ$?



Show answer

HL (hypotenuses $\overline{PR}$, $\overline{XZ}$ and legs $\overline{PQ}$, $\overline{XY}$)

2. **Easy** In the figure, $\triangle JKL$ and $\triangle RST$ have right angles at K and S, $\overline{JL} \cong \overline{RT}$ and $\angle L \cong \angle T$. Which right-triangle test (HL, LL, HA or LA) proves them congruent?



Show answer

HA (hypotenuse and an acute angle), a case of AAS

3. **Easy** Right triangles ABC and DEF have right angles at C and F, and $\overline{AB} \cong \overline{DE}$. What else must you know to prove them congruent by HL?

Show answer

One pair of corresponding legs: $\overline{AC} \cong \overline{DF}$ or $\overline{BC} \cong \overline{EF}$.

4. **Easy** Right triangles ABC and DEF have right angles at C and F. The hypotenuses are $AB = 3x - 1$ and $DE = 2x + 5$, and the legs are $AC = 8$ and $DF = y + 3$. Find x and y so that the triangles are congruent by HL.

   Show answer

   $3x - 1 = 2x + 5$ gives $x = 6$ (each hypotenuse is 17), and $y + 3 = 8$ gives $y = 5$.

5. **Medium** Two support cables, each 15 ft long, run from the top T of a vertical pole $\overline{TM}$ to stakes A and B on level ground. Stake A is 9 ft from the base M. How far is stake B from M? Explain with a congruence test.

   Show answer

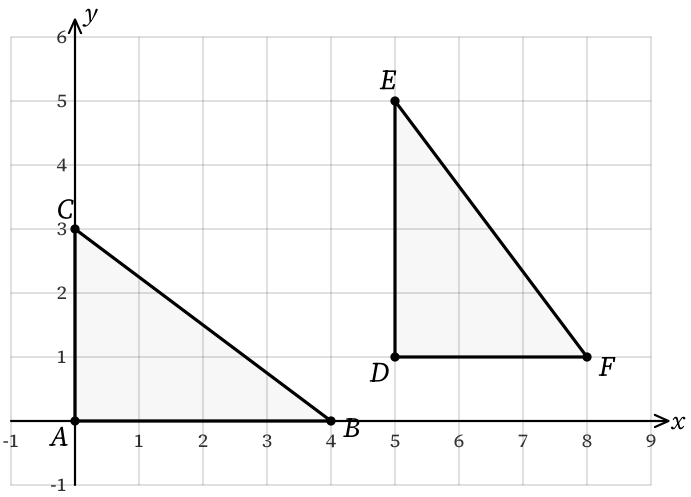
   $\triangle TMA$ and $\triangle TMB$ are right triangles (the pole is perpendicular to the ground) with congruent hypotenuses $\overline{TA} \cong \overline{TB}$ and the shared leg $\overline{TM}$. By HL they are congruent, so $MB = MA = 9$ ft by CPCTC.

6. **Medium** SSA is not a congruence test, yet HL (which is a kind of SSA) is. Explain what is different when the angle is a right angle.

   Show answer

   In SSA, the circle about one vertex can cross the other side of the angle at two points that give two different triangles. When the given angle is right, the side through it is perpendicular to the other leg's line, and the two crossing points lie on opposite sides of the right angle's vertex at equal distances; the two triangles are mirror images, so they are congruent. Also, the hypotenuse of a right triangle is longer than either leg, so only one triangle is possible on each side.

7. **Medium** In the figure, $A(0,0)$, $B(4,0)$, $C(0,3)$, $D(5,1)$, $E(5,5)$ and $F(8,1)$. Use the distance formula to decide whether $\triangle ABC \cong \triangle DEF$.



Show answer

Yes. $AB = 4$, $AC = 3$, $BC = \sqrt{16 + 9} = 5$; $DE = 4$, $DF = 3$, $EF = \sqrt{9 + 16} = 5$. All three pairs of corresponding sides are congruent, so $\triangle ABC \cong \triangle DEF$ by SSS.

8. **Medium** Henry cuts two right-triangle shelf brackets. Each has a hypotenuse of 20 in and one leg of 12 in. Are the brackets congruent? How long is the other leg of each?

Show answer

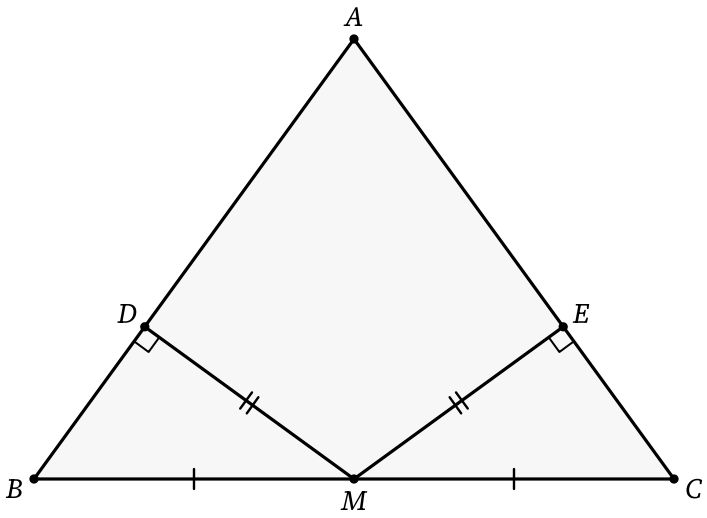
Yes, by HL. The other leg is $\sqrt{20^2 - 12^2} = \sqrt{256} = 16$ in on each bracket.

9. **Medium** Right triangles ABC and DEF have right angles at C and F, $\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$. Which right-triangle test proves them congruent, and which general test is it a case of?

Show answer

LA, a case of ASA: the leg $\overline{AC}$ is included between $\angle A$ and the right angle $\angle C$.

10. **Hard** In the figure, M is the midpoint of $\overline{BC}$, $\overline{MD} \perp \overline{AB}$ at D, $\overline{ME} \perp \overline{AC}$ at E, and $\overline{MD} \cong \overline{ME}$. Prove that $\angle B \cong \angle C$.



Show answer

See the model proof.

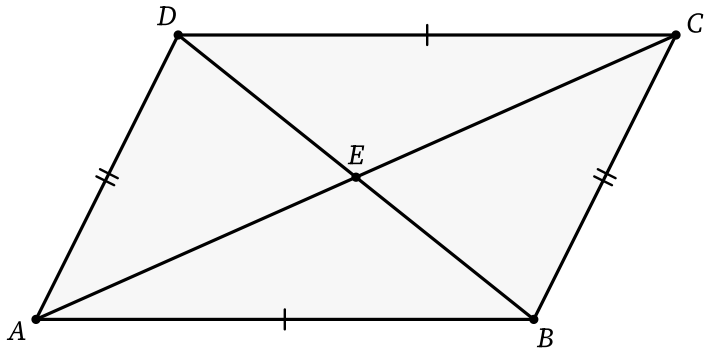
TWO-COLUMN PROOF

Given: M is the midpoint of $\overline{BC}$; $\overline{MD} \perp \overline{AB}$ and $\overline{ME} \perp \overline{AC}$; $\overline{MD} \cong \overline{ME}$

Prove: $\angle B \cong \angle C$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{BC}$	Given
2	$\overline{BM} \cong \overline{CM}$	Definition of midpoint
3	$\overline{MD} \perp \overline{AB}$ and $\overline{ME} \perp \overline{AC}$	Given
4	$\angle BDM$ and $\angle CEM$ are right angles	Definition of perpendicular lines
5	$\triangle BDM$ and $\triangle CEM$ are right triangles	Definition of right triangle
6	$\overline{MD} \cong \overline{ME}$	Given
7	$\triangle BDM \cong \triangle CEM$	HL (hypotenuses $\overline{BM}$, $\overline{CM}$; legs $\overline{MD}$, $\overline{ME}$)
8	$\angle B \cong \angle C$	CPCTC ($\angle DBM \cong \angle ECM$, which are $\angle B$ and $\angle C$ of $\triangle ABC$)

11. **Hard** In quadrilateral $ABCD$ in the figure, $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$, and the diagonals $\overline{AC}$ and $\overline{BD}$ meet at E. Write a flow proof that $\overline{AE} \cong \overline{CE}$.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{AB} \cong \overline{CD}$; $\overline{AD} \cong \overline{CB}$; $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\overline{AE} \cong \overline{CE}$

(1) $\overline{AB} \cong \overline{CD}$
Given

(2) $\overline{AD} \cong \overline{CB}$
Given

(3) $\overline{AC} \cong \overline{CA}$
Reflexive Property of Congruence

(4) $\triangle ABC \cong \triangle CDA$
SSS
from (1), (2), (3)

(5) $\angle BAE \cong \angle DCE$
CPCTC ($\angle BAC \cong \angle DCA$; E is on $\overline{AC}$)
from (4)

(6) $\overline{BD} \cong \overline{DB}$
Reflexive Property of Congruence

(7) $\triangle ABD \cong \triangle CDB$
SSS
from (1), (2), (6)

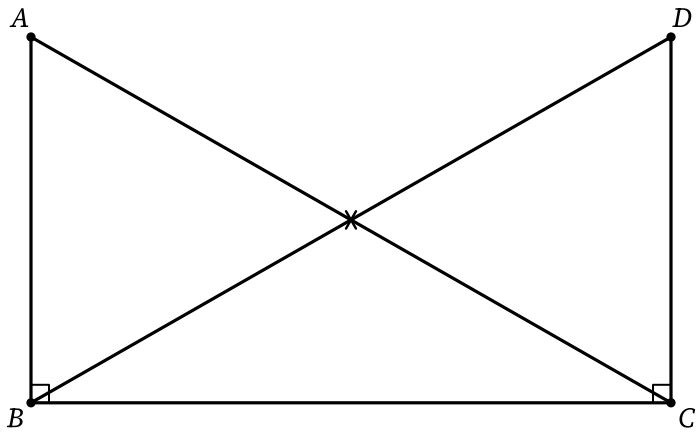
(8) $\angle ABE \cong \angle CDE$
CPCTC ($\angle ABD \cong \angle CDB$; E is on $\overline{BD}$)
from (7)

(9) $\triangle ABE \cong \triangle CDE$
ASA
from (5), (1), (8)

(10) $\overline{AE} \cong \overline{CE}$
CPCTC
from (9)

∎

12. Hard In the figure, $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$ and $\overline{AC} \cong \overline{DB}$. Write a paragraph proof that $\overline{AB} \cong \overline{DC}$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overline{AB} \perp \overline{BC}$; $\overline{DC} \perp \overline{BC}$; $\overline{AC} \cong \overline{DB}$

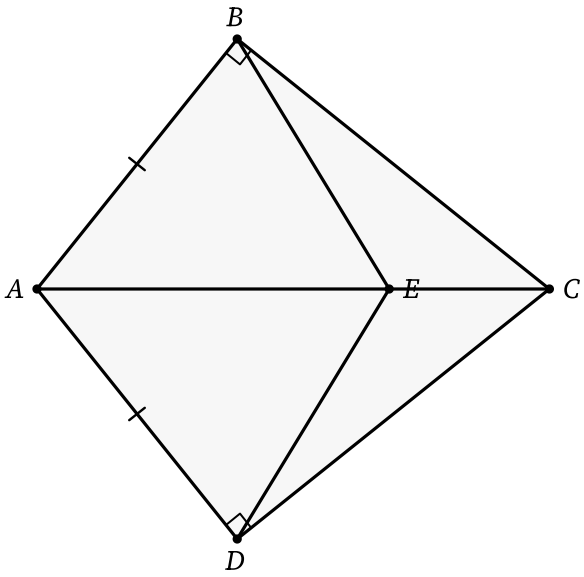
Prove: $\overline{AB} \cong \overline{DC}$

Because $\overline{AB} \perp \overline{BC}$ and $\overline{DC} \perp \overline{BC}$, the angles $\angle ABC$ and $\angle DCB$ are right angles by the definition of perpendicular lines, so $\triangle ABC$ and $\triangle DCB$ are right triangles.

Their hypotenuses $\overline{AC}$ and $\overline{DB}$ are congruent (given), and they share the leg $\overline{BC} \cong \overline{CB}$ (Reflexive Property). So $\triangle ABC \cong \triangle DCB$ by HL, and $\overline{AB} \cong \overline{DC}$ by CPCTC.

∎

13. **Challenge** In the figure, $\angle ABC$ and $\angle ADC$ are right angles, $\overline{AB} \cong \overline{AD}$, and E is a point on $\overline{AC}$. Prove that $\overline{BE} \cong \overline{DE}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\angle ABC$ and $\angle ADC$ are right angles; $\overline{AB} \cong \overline{AD}$; E lies on $\overline{AC}$
Prove: $\overline{BE} \cong \overline{DE}$

	STATEMENTS	REASONS
1	$\angle ABC$ and $\angle ADC$ are right angles	Given
2	$\triangle ABC$ and $\triangle ADC$ are right triangles	Definition of right triangle
3	$\overline{AB} \cong \overline{AD}$	Given
4	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
5	$\triangle ABC \cong \triangle ADC$	HL
6	$\overline{BC} \cong \overline{DC}$	CPCTC
7	$\angle BCE \cong \angle DCE$	CPCTC ($\angle BCA \cong \angle DCA$; E is on $\overline{CA}$)
8	$\overline{CE} \cong \overline{CE}$	Reflexive Property of Congruence

	STATEMENTS	REASONS
9	$\triangle BCE \cong \triangle DCE$	SAS
10	$\overline{BE} \cong \overline{DE}$	CPCTC

∎

COMMON MISTAKES

- Using HL on triangles that have not been shown to be right triangles. State the right angles, with a reason, first.
- Matching a leg with a hypotenuse. The hypotenuse is always opposite the right angle; find it before you pair sides.
- Jumping to the second pair of triangles before you have the part it needs. In a two-pair proof, the CPCTC step from pair 1 must come before the congruence step for pair 2.
- In coordinate checks, comparing only two sides. SSS needs all three pairs of side lengths.

> **Make it a game · Riddle Me a Test.** Parent reads riddles: 'I need the hypotenuse and one leg. Which test am I?' Answer HL, LL, HA or LA, and say which general test it secretly belongs to: SAS, AAS or ASA. Then write four riddles of your own and read them to your parent. Score a point for each correct answer and each good riddle.

UNIT 4

Triangles and Congruence

This unit is the heart of Euclid's Book I. You will prove that the angles of every triangle add to 180° (Euclid I.32), learn exactly what it means for two triangles to be congruent, and discover that just three well-chosen parts (SSS, SAS, ASA, AAS, and HL for right triangles) are enough to guarantee all six. Along the way you will meet Euclid's famous 'bridge of asses' (I.5) about isosceles triangles.

Congruent triangles are the main tool of the rest of the course. With CPCTC you will finally prove that the compass constructions of Unit 2 really work, measure across rivers and ponds without crossing them, and write longer proofs in two-column, paragraph and flow form, sometimes using two pairs of triangles in a row.

Parents: the key skill is careful reasoning. Ask your student to say aloud which test they are using and to point to the three parts on the figure; a correct answer always names an included angle or included side in the right place.

PREREQUISITES

- Solve linear equations and simple systems of equations (Algebra I).
- Use the Vertical Angles Theorem, linear pairs and the Congruent Supplements and Complements Theorems (Lesson 2.4).
- Use the properties of parallel lines, especially the Alternate Interior Angles Theorem and its converse (Lessons 3.2 and 3.3).
- Bisect an angle, copy a segment and an angle, and construct a perpendicular bisector with compass and straightedge (Lesson 2.5).
- Write two-column proofs with the properties of equality and congruence (Lessons 1.5 and 2.6).
- Use the distance formula (Algebra I).

MASTERY CHECKLIST

- ☐ I can classify a triangle by its sides and by its angles.
- ☐ I can prove the Triangle Angle Sum Theorem with a parallel line and use the Exterior Angle Theorem.
- ☐ I can name corresponding parts from a congruence statement and use the Third Angles Theorem.
- ☐ I can prove triangles congruent using SSS, SAS, ASA, AAS and HL, and I can explain why SSA and AAA are not tests.
- ☐ I can construct a triangle from three given segments and copy a triangle with compass and straightedge.
- ☐ I can use CPCTC to prove segments and angles congruent, including in overlapping triangles.
- ☐ I can prove that the angle bisector and perpendicular bisector constructions work.
- ☐ I can measure an inaccessible distance by laying out congruent triangles.
- ☐ I can use the Isosceles Triangle Theorem and its converse, and facts about equilateral triangles, in proofs and with algebra.
- ☐ I can write two-column, paragraph and flow proofs, including proofs that use two pairs of congruent triangles.

- ☐ I can check whether two triangles in the coordinate plane are congruent using the distance formula.

LESSONS IN THIS UNIT

4.1	**Triangle Angle Sum and Exterior Angles**	2 sessions of 45 minutes
4.2	**Congruent Figures**	45 minutes
4.3	**Proving Triangles Congruent: SSS and SAS**	2 sessions of 45 minutes
4.4	**Proving Triangles Congruent: ASA and AAS**	2 sessions of 45 minutes
4.5	**Using Congruent Triangles (CPCTC)**	2 sessions of 45 minutes
4.6	**Isosceles and Equilateral Triangles**	2 sessions of 45 minutes
4.7	**Right Triangles: HL and Proofs with Two Pairs of Triangles**	2 sessions of 45 minutes

Unit 4 Test

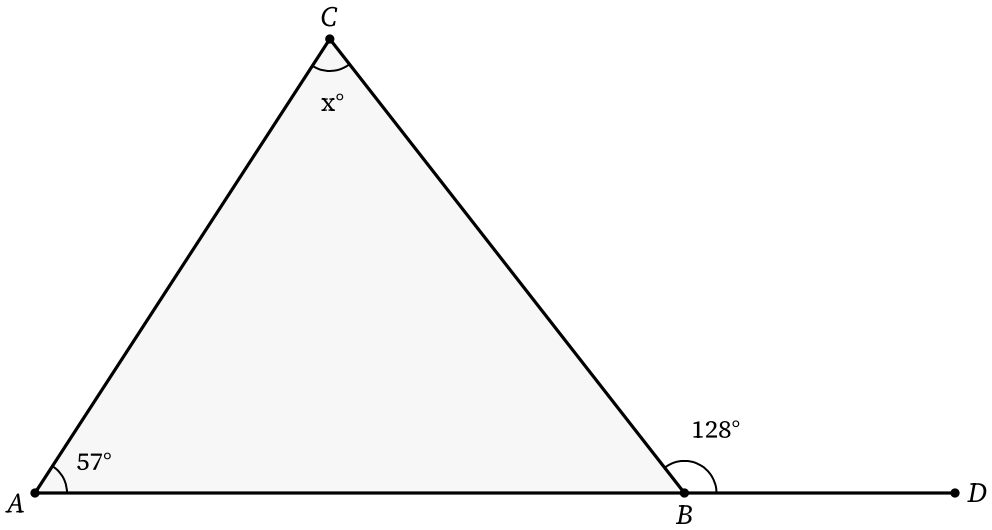
You have 60 minutes. No calculator is needed or allowed. Bring a compass, a straightedge and a sharp pencil for Problem 8. Show your work on every problem. In every proof, give a reason for every statement. Give exact answers, with units where the problem has units. Figures are drawn to scale, but use only the given information and the marks on a figure in your reasoning.

1. **5 pts**

   The angles of a triangle measure $(2x + 3)^\circ$, $(3x - 7)^\circ$ and $(4x + 4)^\circ$. Find x and the measure of the largest angle.

2. **5 pts**

The Miller family's triangular field ABC is shown in the figure. The fence along $\overline{AB}$ is extended past B to D. The corner at A measures $57°$, and the exterior angle $\angle CBD$ measures $128°$. Find $x = m\angle ACB$.



3. **5 pts**

Two interior angles of a triangle measure $(3x)°$ and $(2x)°$, and the exterior angle at the third vertex measures $110°$. Find x and the measure of the third interior angle.

4. **5 pts**

$\triangle MNP \cong \triangle RST$, with $MN = 4x - 3$, $RS = 2x + 9$ and $m\angle P = 52°$. Find x, MN and $m\angle T$.

5. **6 pts**

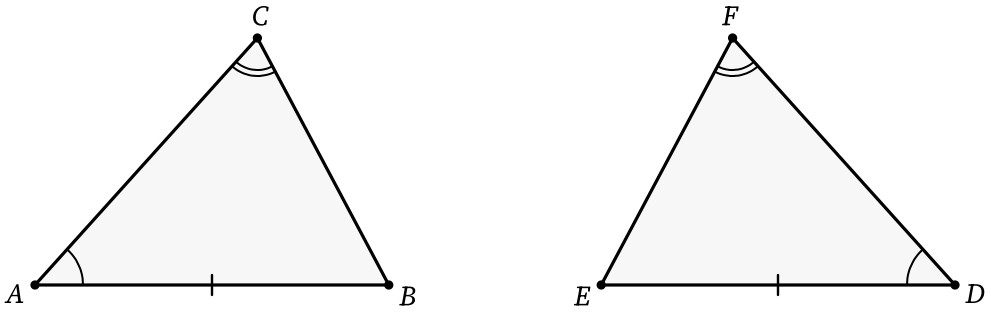
In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. (a) Must $\angle C \cong \angle F$? (b) Must the triangles be congruent? Explain both answers.

6. **5 pts**

In $\triangle PQR$ and $\triangle XYZ$, $\overline{PQ} \cong \overline{XY}$, $\overline{QR} \cong \overline{YZ}$ and $\angle P \cong \angle X$. Can you conclude that $\triangle PQR \cong \triangle XYZ$? If so, name the test; if not, write 'not enough information.'

7. **5 pts**

In the figure, $\angle A \cong \angle D$, $\angle C \cong \angle F$ and $\overline{AB} \cong \overline{DE}$. Which test proves $\triangle ABC \cong \triangle DEF$?

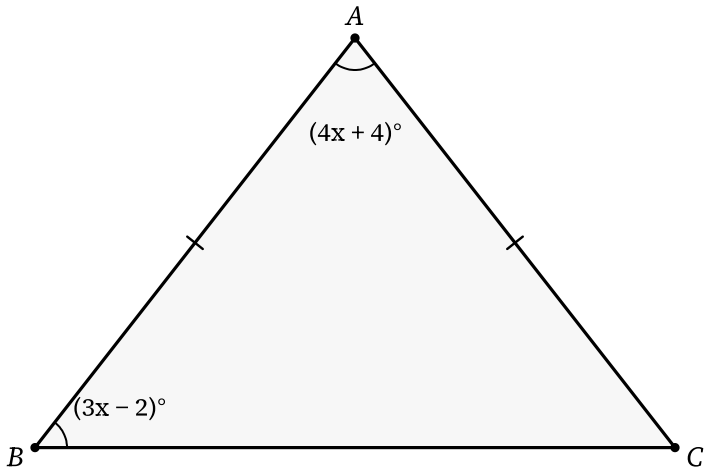


8. **8 pts**

Using only a compass and straightedge, construct $\triangle PQR$ with $PQ = 7$ cm, $QR = 5$ cm and $PR = 6$ cm (measure the three lengths with a ruler once, then use only the compass). List your steps, and explain why every student who does this construction correctly gets a triangle congruent to yours.

9. **7 pts**

In the figure, $\overline{AB} \cong \overline{AC}$, $m\angle A = (4x + 4)^\circ$ and $m\angle B = (3x - 2)^\circ$. Find x and $m\angle C$.

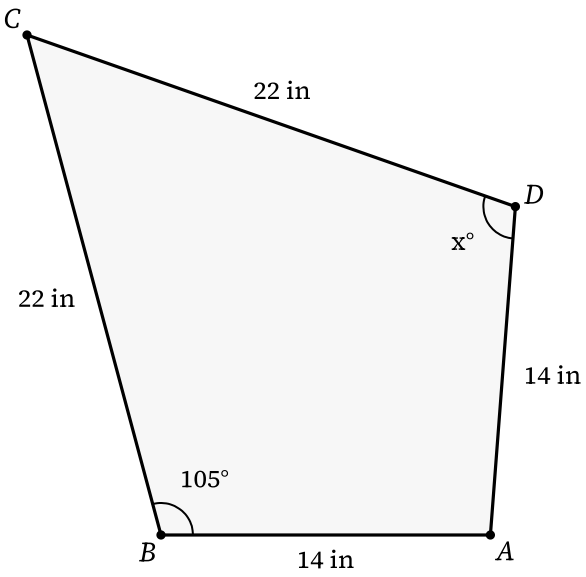


10. **6 pts**

A church window is topped by an equilateral arch with a span of 9 ft between the springing points A and B; each curve is drawn with radius AB from the opposite springing point, and the curves meet at the apex C. How far is the apex from each springing point, and what is $m\angle CAB$? Explain.

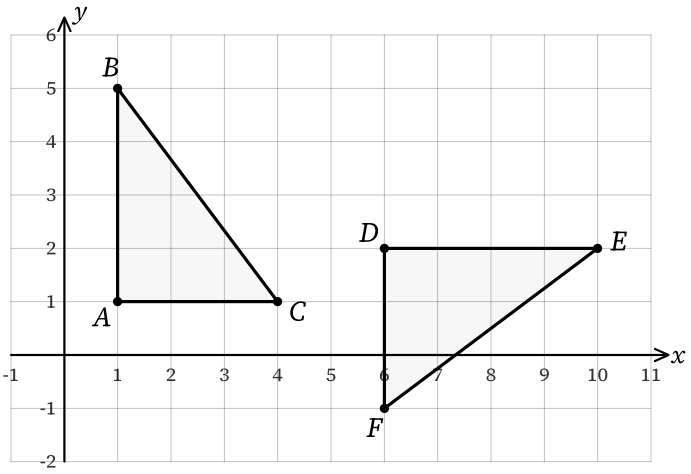
11. **8 pts**

A kite-shaped stained-glass pane $ABCD$ is shown in the figure, with $AB = AD = 14$ in and $CB = CD = 22$ in. The angle at B measures 105°. Find $x = m\angle ADC$, and justify your answer with a congruence test.



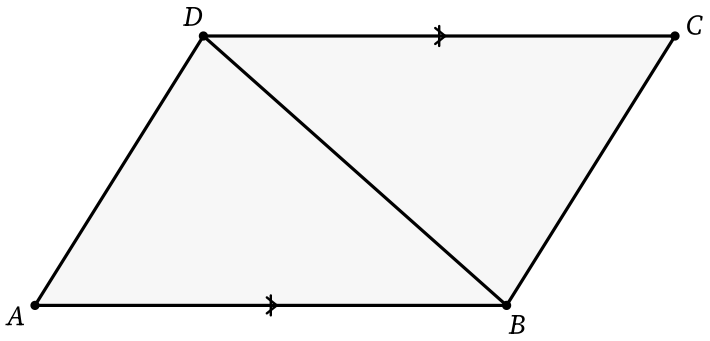
12. **7 pts**

In the figure, $A(1,1)$, $B(1,5)$, $C(4,1)$, $D(6,2)$, $E(10,2)$ and $F(6,-1)$. Explain why both triangles are right triangles, find the side lengths, and decide whether $\triangle ABC \cong \triangle DEF$.



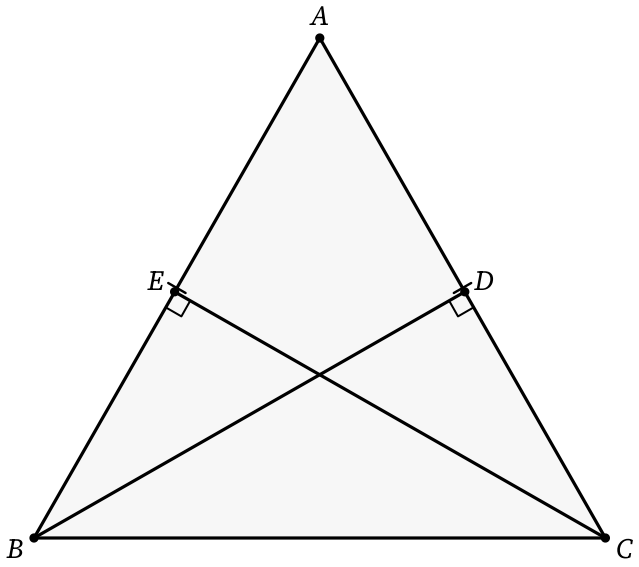
13. **10 pts**

In quadrilateral $ABCD$ in the figure, $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$, and diagonal $\overline{BD}$ is drawn. Write a two-column proof that $\overline{AD} \parallel \overline{BC}$.



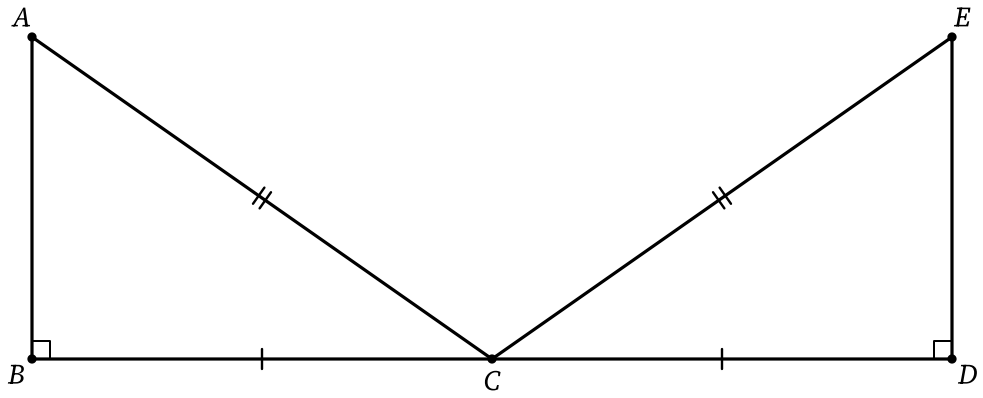
14. **10 pts**

In $\triangle ABC$ in the figure, $\overline{AB} \cong \overline{AC}$, $\overline{BD} \perp \overline{AC}$ at D, and $\overline{CE} \perp \overline{AB}$ at E. Write a flow proof that $\overline{BD} \cong \overline{CE}$.



15. **8 pts**

In the figure, $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{BD}$, C is the midpoint of $\overline{BD}$ and $\overline{AC} \cong \overline{EC}$. Write a paragraph proof that $\angle A \cong \angle E$.



ANSWER KEY

1. $9x = 180$, so $x = 20$; the angles are $43°$, $53°$ and $84°$, and the largest is $84°$.
2. By the Exterior Angle Theorem, $128 = 57 + x$, so $x = 71$.
3. Exterior Angle Theorem: $3x + 2x = 110$, so $x = 22$. The third interior angle is $180 - 110 = 70°$ (linear pair).
4. $4x - 3 = 2x + 9$, so $x = 6$ and $MN = 21$. $\angle T$ corresponds to $\angle P$, so $m\angle T = 52°$.
5. (a) Yes, by the Third Angles Theorem: each third angle is $180°$ minus the same sum. (b) No. Matching angles give the same shape but not the same size; for example, equilateral triangles with sides 2 in and 4 in have all angles $60°$ but are not congruent. (AAA is not a congruence test.)
6. Not enough information. The angle included between $\overline{PQ}$ and $\overline{QR}$ is $\angle Q$, not $\angle P$, so this is SSA, which is not a congruence test.
7. AAS ($\overline{AB}$ is opposite $\angle C$ and $\overline{DE}$ is opposite $\angle F$, so the side is not included).
8. Draw a ray from P and mark Q with $PQ = 7$ cm. With the compass set to 6 cm and centered at P, draw an arc; with the compass set to 5 cm and centered at Q, draw an arc crossing it at R. Draw $\overline{PR}$ and $\overline{QR}$. The arcs meet because each pair of lengths adds to more than the third ($5 + 6 > 7$, and so on). Any two triangles built this way have three pairs of congruent sides, so they are congruent by SSS (Euclid I.22 and I.8).
9. By the Isosceles Triangle Theorem $m\angle C = m\angle B$, so $(4x + 4) + 2(3x - 2) = 180$, $10x = 180$, $x = 18$. Then $m\angle C = 3(18) - 2 = 52°$.
10. C lies on both arcs of radius 9 ft, so $AC = BC = 9$ ft. With $AB = 9$ ft the triangle is equilateral, hence equiangular, so $m\angle CAB = 60°$.
11. Draw diagonal $\overline{AC}$. $\triangle ABC \cong \triangle ADC$ by SSS ($AB = AD$, $CB = CD$, $AC = AC$), so $\angle ADC \cong \angle ABC$ by CPCTC, and $x = 105$.
12. $\overline{AB}$ is vertical and $\overline{AC}$ is horizontal, so $\angle A$ is a right angle; likewise $\overline{DE}$ is horizontal and $\overline{DF}$ is vertical, so $\angle D$ is right. $AB = 4$, $AC = 3$, $BC = \sqrt{9 + 16} = 5$; $DE = 4$, $DF = 3$, $EF = \sqrt{16 + 9} = 5$. The hypotenuses ($BC = EF = 5$) and a pair of legs ($AB = DE = 4$) are congruent, so yes, $\triangle ABC \cong \triangle DEF$ by HL (SSS also works).
13. See the model proof. (This is Euclid I.33.)

    TWO-COLUMN PROOF

    Given: $\overline{AB} \parallel \overline{DC}$; $\overline{AB} \cong \overline{DC}$
    Prove: $\overline{AD} \parallel \overline{BC}$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$	Given
2	$\angle ABD \cong \angle CDB$	Alternate Interior Angles Theorem (transversal $\overline{BD}$)
3	$\overline{AB} \cong \overline{CD}$	Given
4	$\overline{BD} \cong \overline{DB}$	Reflexive Property of Congruence
5	$\triangle ABD \cong \triangle CDB$	SAS
6	$\angle ADB \cong \angle CBD$	CPCTC
7	$\overline{AD} \parallel \overline{BC}$	If alternate interior angles are congruent, the lines are parallel (Euclid I.27)

∎

14. See the model proof.

FLOW PROOF

Given: $\overline{AB} \cong \overline{AC}$; $\overline{BD} \perp \overline{AC}$ at D; $\overline{CE} \perp \overline{AB}$ at E

Prove: $\overline{BD} \cong \overline{CE}$

(1) $\overline{AB} \cong \overline{AC}$
Given

(2) $\angle ACB \cong \angle ABC$, that is, $\angle DCB \cong \angle EBC$
Isosceles Triangle Theorem (D is on $\overline{CA}$ and E is on $\overline{BA}$)
from (1)

(3) $\overline{BD} \perp \overline{AC}$ and $\overline{CE} \perp \overline{AB}$
Given

(4) $\angle BDC$ and $\angle CEB$ are right angles
Definition of perpendicular lines
from (3)

(5) $\angle BDC \cong \angle CEB$
All right angles are congruent
from (4)

(6) $\overline{BC} \cong \overline{CB}$
Reflexive Property of Congruence

(7) $\triangle BDC \cong \triangle CEB$
AAS
from (5), (2), (6)

(8) $\overline{BD} \cong \overline{CE}$
CPCTC
from (7)

∎

15. See the model proof.

PARAGRAPH PROOF

Given: $\overline{AB} \perp \overline{BD}$; $\overline{ED} \perp \overline{BD}$; C is the midpoint of $\overline{BD}$; $\overline{AC} \cong \overline{EC}$

Prove: $\angle A \cong \angle E$

Because $\overline{AB} \perp \overline{BD}$ and $\overline{ED} \perp \overline{BD}$, the angles $\angle ABC$ and $\angle EDC$ are right angles by the definition of perpendicular lines, so $\triangle ABC$ and $\triangle EDC$ are right triangles. Their hypotenuses $\overline{AC}$ and $\overline{EC}$ are congruent (given).

Since C is the midpoint of $\overline{BD}$, $\overline{BC} \cong \overline{DC}$ by the definition of midpoint; these are legs of the two right triangles. So $\triangle ABC \cong \triangle EDC$ by HL, and $\angle A \cong \angle E$ by CPCTC.

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UNIT 5

Relationships Within Triangles

In Unit 4 you learned to prove two triangles congruent. In this unit you look inside a single triangle and discover how its parts are related. Every triangle has four special centers: the circumcenter, where the perpendicular bisectors of the sides meet; the incenter, where the angle bisectors meet; the centroid, where the medians meet; and the orthocenter, where the altitudes meet. You will prove why the first three always exist, construct the circles about and in a triangle as Euclid does in Book IV, and use these points to solve practical problems, from placing a well equidistant from three farmhouses to balancing a cardboard triangle on a pencil.

The second half of the unit is about inequalities. The midsegment of a triangle is parallel to the third side and half as long. You will learn a new kind of proof, the indirect proof (proof by contradiction), and use it with Euclid I.16 to I.25 to prove that the longer side is opposite the larger angle, that any two sides of a triangle are together longer than the third, and that opening the angle between two fixed sides lengthens the third side (the Hinge Theorem).

For parents: this unit is proof-heavy and follows Euclid's Book I closely. Expect your student to write two-column, paragraph, flow and indirect proofs, and to use compass and straightedge. Asking your student to explain aloud why a statement is true, not just what the answer is, is the best help you can give.

PREREQUISITES

- Prove triangles congruent with SSS, SAS, ASA, AAS and HL, and use CPCTC (Lessons 4.3–4.7).
- Use the Isosceles Triangle Theorem and its converse (Lesson 4.6).
- Use the Triangle Angle Sum Theorem, the Exterior Angle Theorem and the Exterior Angle Inequality (Lesson 4.1).
- Use the angle pairs formed by parallel lines and a transversal (Unit 3).
- Use the slope, midpoint and distance formulas, and write equations of perpendicular lines (Lessons 2.2 and 3.5).
- Bisect a segment and an angle and drop a perpendicular with compass and straightedge (Lesson 2.5).

MASTERY CHECKLIST

- ☐ I can prove and use the Perpendicular Bisector Theorem and the Angle Bisector Theorem and their converses.
- ☐ I can find the distance from a point to a line by using the perpendicular segment.
- ☐ I can prove that the perpendicular bisectors and the angle bisectors of a triangle are concurrent, and find the circumcenter and incenter.
- ☐ I can construct the circumscribed and inscribed circles of a triangle with compass and straightedge.
- ☐ I can use the Centroid Theorem and find the centroid and orthocenter of a triangle in the coordinate plane.
- ☐ I can construct a median and an altitude of a triangle.
- ☐ I can prove the Triangle Midsegment Theorem with coordinates and use it to find lengths and angles.
- ☐ I can write an indirect proof by assuming the opposite of the conclusion and reaching a contradiction.
- ☐ I can order the sides and angles of a triangle using Euclid I.18 and I.19.
- ☐ I can decide whether three lengths form a triangle and find the possible lengths of a third side.

- ☐ I can use the Hinge Theorem and its converse to compare sides and angles of two triangles.

LESSONS IN THIS UNIT

5.1	**Perpendicular Bisectors and Angle Bisectors**	2 sessions of 45 minutes
5.2	**Circumcenter and Incenter**	2 sessions of 45 minutes
5.3	**Medians, Altitudes and the Centroid**	2 sessions of 45 minutes
5.4	**The Triangle Midsegment Theorem**	45 minutes
5.5	**Indirect Proof and Inequalities in One Triangle**	2 sessions of 45 minutes
5.6	**The Triangle Inequality and the Hinge Theorem**	2 sessions of 45 minutes

Unit 5 Test

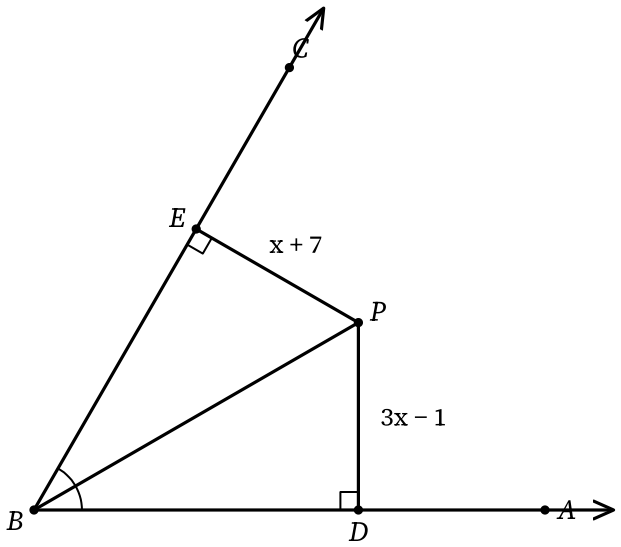
You have 50 minutes. No calculator is needed or allowed; give exact answers (simplest radical form where needed) unless a problem says otherwise. You will need a compass and a straightedge for Problem 14. Show your work for every computation. In every proof, give a reason for every statement, and do not assume anything from the look of a figure. The test is worth 100 points.

1. **5 pts**

   Point P lies on the perpendicular bisector of $\overline{AB}$. If $PA = 2y + 5$ and $PB = 4y - 3$, find PA.

2. **6 pts**

In the figure, P lies on the bisector of $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$ and $\overline{PE} \perp \overrightarrow{BC}$. If $PD = 3x - 1$ and $PE = x + 7$, find PD.

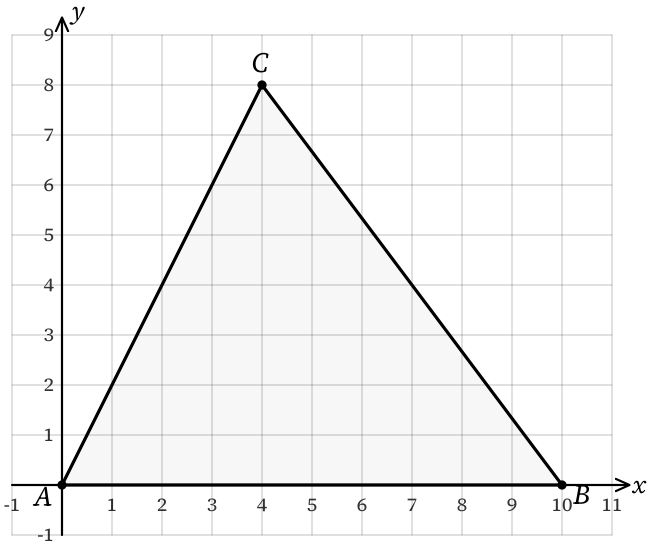


3. **4 pts**

Which point of concurrency of a triangle is equidistant from the three sides of the triangle?

4. **7 pts**

On a map with a grid in miles, three towns are at $A(0, 0)$, $B(10, 0)$ and $C(4, 8)$. A radio tower is to be built the same distance from all three towns. Find the coordinates of the tower.



5. **6 pts**

A triangular park has sides of 9 m, 12 m and 15 m, and it is a right triangle. The town wants to plant the largest possible circular flower bed inside it. Find the radius of the flower bed.

6. **5 pts**

G is the centroid of $\triangle ABC$, and the median $\overline{BM}$ has length 27. Find BG.

7. **7 pts**

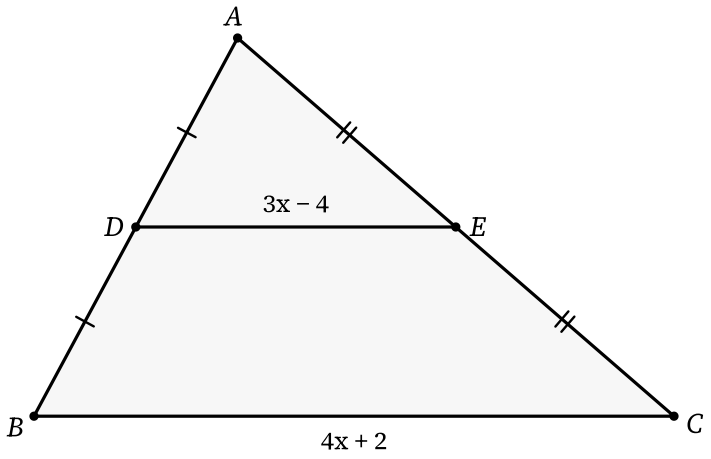
Find the orthocenter of the triangle with vertices $A(0, 0)$, $B(12, 0)$ and $C(3, 6)$.

8. **5 pts**

The legs of an A-frame swing set are 9 feet apart at the ground. A horizontal brace is bolted between the midpoints of the two legs. How long is the brace?

9. **6 pts**

In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, $DE = 3x - 4$ and $BC = 4x + 2$. Find BC.



10. **5 pts**

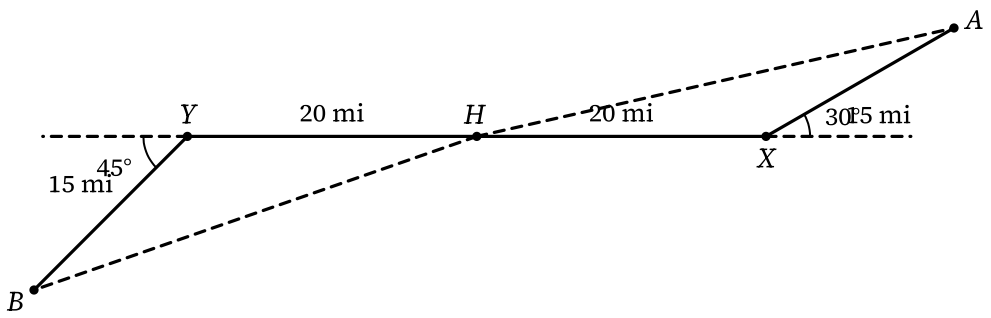
In $\triangle ABC$, $AB = 7$, $BC = 11$ and $AC = 9$. List the angles from smallest to largest.

11. **6 pts**

A triangular flower bed has two sides of 12 ft and 20 ft. The third side is a whole number of feet. What are the smallest and the largest possible lengths of the third side?

12. **6 pts**

Two ships leave harbor H. Ship A sails 20 miles due east, then turns 30° toward the north and sails 15 miles. Ship B sails 20 miles due west, then turns 45° toward the south and sails 15 miles, as in the figure. Which ship ends farther from the harbor? Explain with a theorem.



13. **8 pts**

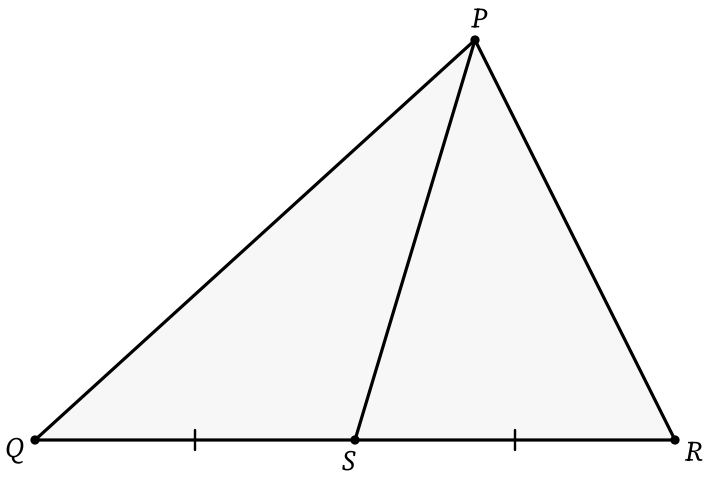
Explain why the three perpendicular bisectors of the sides of any triangle must meet at a single point. Your explanation should use the Perpendicular Bisector Theorem and its converse.

14. **6 pts**

Construction: draw a large scalene triangle ABC. With compass and straightedge, construct its centroid. List your steps and explain why the point you found is the centroid.

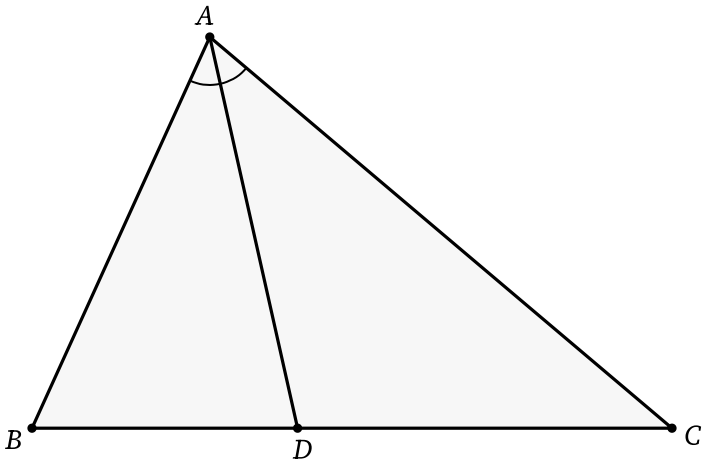
15. **8 pts**

In the figure, $\overline{PQ} \ncong \overline{PR}$ in $\triangle PQR$, and S is the midpoint of $\overline{QR}$. Write an indirect proof that $\overline{PS}$ is not perpendicular to $\overline{QR}$.



16. **10 pts**

In the figure, $\overrightarrow{AD}$ bisects $\angle BAC$ of $\triangle ABC$, with D on $\overline{BC}$. Prove that $AB > BD$. Write a two-column proof.



ANSWER KEY

1. $2y + 5 = 4y - 3$, so $y = 4$ and $PA = 13$.
2. By the Angle Bisector Theorem, $3x - 1 = x + 7$, so $x = 4$ and $PD = 11$.
3. The incenter.
4. The tower is at the circumcenter. The perpendicular bisector of $\overline{AB}$ is $x = 5$. Setting $PA^2 = PC^2$ for $P(5, y)$: $25 + y^2 = 1 + (8 - y)^2 = 65 - 16y + y^2$, so $16y = 40$ and $y = 2.5$. The tower is at $(5, 2.5)$.
5. The largest circle is the inscribed circle. Area $= \frac{1}{2}(9)(12) = 54$ m^2 and $s = \frac{9+12+15}{2} = 18$ m, so $r = \dfrac{54}{18} = 3$ m.
6. $BG = \frac{2}{3}(27) = 18$
7. The altitude from C is $x = 3$. The slope of $\overline{BC}$ is $\dfrac{6}{3 - 12} = -\dfrac{2}{3}$, so the altitude from A is $y = \frac{3}{2}x$. At $x = 3$, $y = 4.5$. The orthocenter is $(3, 4.5)$. (Check: the altitude from B, $y = -\frac{1}{2}(x - 12)$, also passes through $(3, 4.5)$.)
8. The brace is a midsegment of the triangle formed by the legs and the ground, so it is $\frac{1}{2}(9) = 4.5$ feet long.
9. $4x + 2 = 2(3x - 4) = 6x - 8$, so $x = 5$ and $BC = 22$ (and $DE = 11$).
10. $\angle C$ (opposite 7), $\angle B$ (opposite 9), $\angle A$ (opposite 11).
11. $20 - 12 < x < 20 + 12$, so $8 < x < 32$. The smallest whole-number length is 9 ft and the largest is 31 ft.
12. Ship A. Each ship's path makes a triangle with sides 20 and 15 miles. The included angle at ship A's turning point is $180^\circ - 30^\circ = 150^\circ$, and at ship B's it is $180^\circ - 45^\circ = 135^\circ$. By the Hinge Theorem, the side opposite the larger included angle is longer, so ship A is farther from the harbor.
13. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ meet at P (they are not parallel, because $\overline{AB}$ and $\overline{BC}$ are not on one line). Since P is on the perpendicular bisector of $\overline{AB}$, $PA = PB$; since it is on the perpendicular bisector of $\overline{BC}$, $PB = PC$ (Perpendicular Bisector Theorem). So $PA = PC$, and by the converse P is on the perpendicular bisector of $\overline{AC}$. All three lines pass through P.
14. Construct the perpendicular bisector of $\overline{BC}$ (equal arcs from B and C crossing at two points, joined by a line) to find the midpoint L of $\overline{BC}$; do the same for $\overline{AC}$ to find its midpoint M. Draw the medians $\overline{AL}$ and $\overline{BM}$; they meet at G. Since L and M are midpoints (Euclid I.10), $\overline{AL}$ and $\overline{BM}$ are medians, and by the

Centroid Theorem the medians meet at the centroid. Check: G should be two-thirds of the way from A to L, and the third median should pass through G.

15. See the model proof.

    INDIRECT PROOF

    Given: $\overline{PQ} \ncong \overline{PR}$; S is the midpoint of $\overline{QR}$

    Prove: $\overline{PS}$ is not perpendicular to $\overline{QR}$

    Assume temporarily that $\overline{PS} \perp \overline{QR}$.

    Then $\angle PSQ$ and $\angle PSR$ are right angles (definition of perpendicular lines), so $\angle PSQ \cong \angle PSR$ (all right angles are congruent). Since S is the midpoint of $\overline{QR}$, $\overline{QS} \cong \overline{RS}$ (definition of midpoint), and $\overline{PS} \cong \overline{PS}$ (Reflexive Property). So $\triangle PSQ \cong \triangle PSR$ by SAS, and $\overline{PQ} \cong \overline{PR}$ by CPCTC.

    This contradicts the given fact that $\overline{PQ} \ncong \overline{PR}$. Therefore the assumption is false, and $\overline{PS}$ is not perpendicular to $\overline{QR}$.

    ∎

16. See the model proof.

    TWO-COLUMN PROOF

    Given: $\overrightarrow{AD}$ bisects $\angle BAC$, D on $\overline{BC}$

    Prove: $AB > BD$

	STATEMENTS	REASONS
1	$\overrightarrow{AD}$ bisects $\angle BAC$	Given
2	$m\angle BAD = m\angle DAC$	Definition of angle bisector
3	$m\angle ADB > m\angle DAC$	Exterior Angle Inequality ($\angle ADB$ is an exterior angle of $\triangle ADC$)
4	$m\angle ADB > m\angle BAD$	Substitution (steps 2 and 3)
5	$AB > BD$	Triangle Larger Angle Theorem (in $\triangle ABD$, $\overline{AB}$ is opposite $\angle ADB$ and $\overline{BD}$ is opposite $\angle BAD$)

    ∎

5.1 Perpendicular Bisectors and Angle Bisectors

OBJECTIVES

- Prove and use the Perpendicular Bisector Theorem and its converse.
- Prove and use the Angle Bisector Theorem and its converse.
- Find the distance from a point to a line as the length of the perpendicular segment.
- Describe a set of points with locus language and locate points that are equidistant from two points or from two sides of an angle.

WATCH

Introduction to Locus (2 of 3: Perpendicular Bisectors)

Eddie Woo · 6:23

https://www.youtube.com/watch?v=oBdQ9WVktHw

> **WHEN TO WATCH**
> Before the Perpendicular Bisector Theorem: the set of points equidistant from two points.

Note: Eddie Woo teaches in Australia: he says 'locus' for the set of points. The backup uses a perpendicular-distance formula the course does not teach.

BACKUP VIDEO

Applications of Perpendicular Distance (3 of 4: Points Equidistant from 2 Lines)

Eddie Woo · 7:17

https://www.youtube.com/watch?v=eAXJq7GNkD0

After the Angle Bisector Theorem, as a stretch: points equidistant from two lines, done with coordinates.

KEY TERMS

equidistant

A point is equidistant from two figures when its distance to each of them is the same.

perpendicular bisector

The line (or ray or segment) that is perpendicular to a segment at its midpoint.

distance from a point to a line

The length of the perpendicular segment from the point to the line. If the point is on the line, the distance is 0.

angle bisector

The ray that divides an angle into two congruent angles.

locus

The set of all points that satisfy a given condition (plural: loci). For example, the locus of points 3 cm from a point O is the circle with center O and radius 3 cm.

CORE IDEAS

In Unit 2 you constructed the perpendicular bisector of a segment and the bisector of an angle, and in Lesson 4.5 you proved those constructions work. This lesson asks a deeper question: what is special about *every* point on those lines? The answer is a single word: **equidistant**.

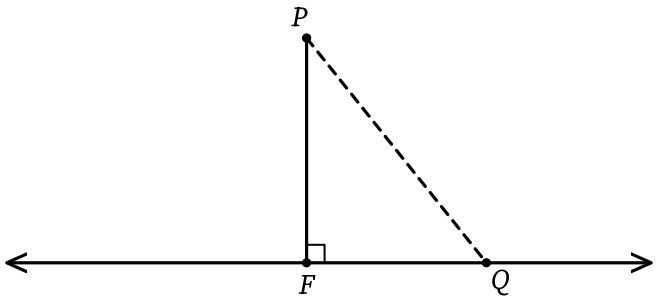
Take a segment $\overline{AB}$ with midpoint M and a point P on its perpendicular bisector. Draw $\overline{PA}$ and $\overline{PB}$. The two triangles $\triangle PMA$ and $\triangle PMB$ share the side $\overline{PM}$, have right angles at M, and have $AM = BM$. So they are congruent by SAS, and $PA = PB$. That is the **Perpendicular Bisector Theorem**. The converse is true too: if $PA = PB$, then P is on the perpendicular bisector. Together they say that the perpendicular bisector of $\overline{AB}$ is *exactly* the set of points equidistant from A and B, no more and no less.

Why do we need both directions? The theorem tells you that points on the line have the property. The converse tells you that no point *off* the line has it. When you are asked to find **all** points with a property, you need both, and locus language ("the set of all points such that ...") reminds you of this.

To talk about the distance from a point to a *side* of an angle, we need the distance from a point to a line. There are many segments from a point P to a line, but in Lesson 3.4 you learned that the perpendicular segment is the shortest one. So we define the distance from P to the line as the length of the perpendicular segment. A tape measure held slanted always reads too much.

Now take a point P on the bisector of $\angle ABC$ and drop perpendiculars $\overline{PD}$ and $\overline{PE}$ to the two sides. The right triangles $\triangle PDB$ and $\triangle PEB$ have congruent angles at B, right angles at D and E, and the common side $\overline{BP}$. By AAS they are congruent, so $PD = PE$. This is the **Angle Bisector Theorem**: a point on the bisector is equidistant from the sides. Its converse, proved with HL, says that a point in the interior of the angle that is equidistant from the sides lies on the bisector.

Notice the pattern: each theorem and its converse together make a biconditional. A point is on the perpendicular bisector of $\overline{AB}$ **if and only if** it is equidistant from A and B. A point in the interior of an angle is on its bisector **if and only if** it is equidistant from the sides. In the next lesson these two facts will let you prove that three special lines in every triangle meet at one point.

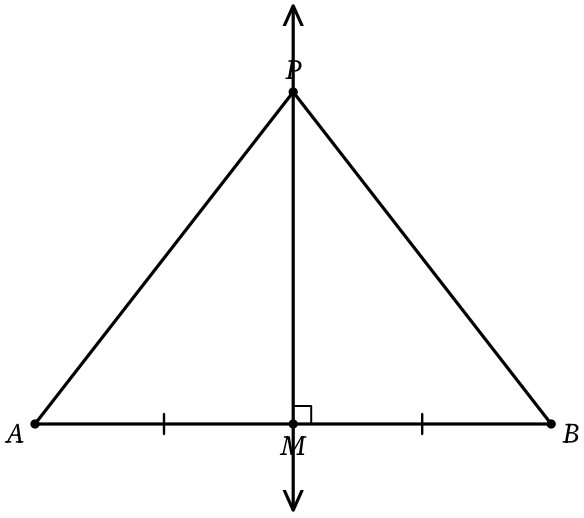


PF is the shortest segment from P to the line, so PF is the distance from P to the line.

POSTULATES AND THEOREMS

THEOREM **Perpendicular Bisector Theorem**

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.



TWO-COLUMN PROOF

Given: $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$ at M

Prove: $PA = PB$

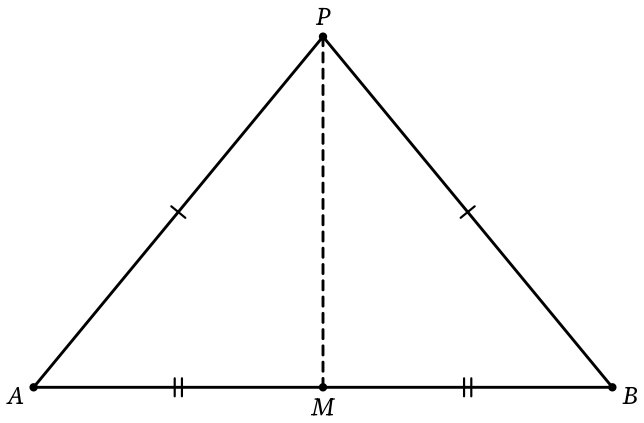
	STATEMENTS	REASONS
1	$\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$ at M	Given
2	M is the midpoint of $\overline{AB}$ and $\overleftrightarrow{PM} \perp \overline{AB}$	Definition of perpendicular bisector
3	$\overline{AM} \cong \overline{BM}$	Definition of midpoint
4	$\angle PMA$ and $\angle PMB$ are right angles	Definition of perpendicular lines
5	$\angle PMA \cong \angle PMB$	All right angles are congruent
6	$\overline{PM} \cong \overline{PM}$	Reflexive Property of Congruence
7	$\triangle PMA \cong \triangle PMB$	SAS
8	$\overline{PA} \cong \overline{PB}$	CPCTC
9	$PA = PB$	Definition of congruent segments

∎

If P is the midpoint M itself, then $PA = MA = MB = PB$ directly by the definition of midpoint, so the theorem holds for every point of the perpendicular bisector.

THEOREM **Converse of the Perpendicular Bisector Theorem**

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.



PARAGRAPH PROOF

Given: $PA = PB$

Prove: P is on the perpendicular bisector of $\overline{AB}$

If P lies on line AB, it cannot lie outside $\overline{AB}$: if, say, B were between A and P, then $PA = PB + AB > PB$ by the Segment Addition Postulate. So P is on $\overline{AB}$, and $PA = PB$ makes P the midpoint of $\overline{AB}$, and the midpoint is on the perpendicular bisector. So suppose P is not on line AB.

Let M be the midpoint of $\overline{AB}$ (every segment has exactly one midpoint) and draw $\overline{PM}$. Then $\overline{PA} \cong \overline{PB}$ (given), $\overline{AM} \cong \overline{BM}$ (definition of midpoint), and $\overline{PM} \cong \overline{PM}$ (Reflexive Property). So $\triangle PMA \cong \triangle PMB$ by SSS, and $\angle PMA \cong \angle PMB$ by CPCTC.

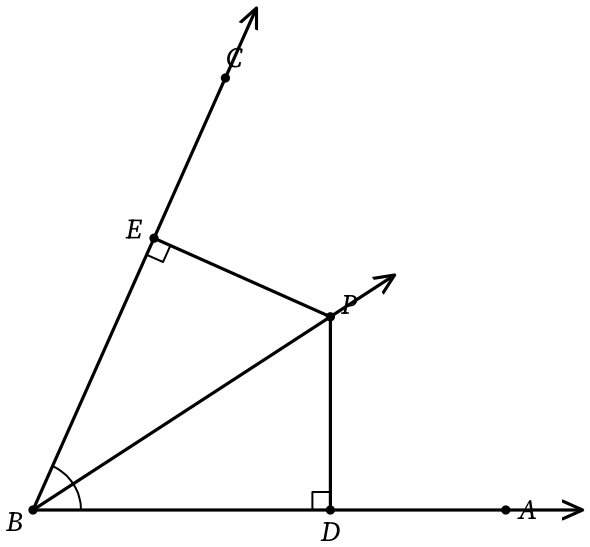
But $\angle PMA$ and $\angle PMB$ form a linear pair, and two congruent angles that form a linear pair are right angles (each is half of $180°$). So $\overleftrightarrow{PM} \perp \overline{AB}$ at its midpoint M, which means $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$ (there is only one line perpendicular to $\overline{AB}$ through M). Therefore P is on the perpendicular bisector of $\overline{AB}$.

∎

Together with the Perpendicular Bisector Theorem: the perpendicular bisector of a segment is the locus of all points equidistant from its endpoints.

THEOREM **Angle Bisector Theorem**

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.



FLOW PROOF

Given: $\overrightarrow{BP}$ bisects $\angle ABC$; $\overline{PD} \perp \overrightarrow{BA}$ at D and $\overline{PE} \perp \overrightarrow{BC}$ at E

Prove: $PD = PE$

(1) $\overrightarrow{BP}$ bisects $\angle ABC$
Given

(2) $\overline{PD} \perp \overrightarrow{BA}$ and $\overline{PE} \perp \overrightarrow{BC}$
Given

(3) $\angle PBD \cong \angle PBE$
Definition of angle bisector
from (1)

(4) $\angle PDB$ and $\angle PEB$ are right angles
Definition of perpendicular lines
from (2)

(5) $\angle PDB \cong \angle PEB$
All right angles are congruent
from (4)

(6) $\overline{BP} \cong \overline{BP}$
Reflexive Property of Congruence

(7) $\triangle PDB \cong \triangle PEB$
AAS
from (3), (5), (6)

(8) $\overline{PD} \cong \overline{PE}$
CPCTC
from (7)

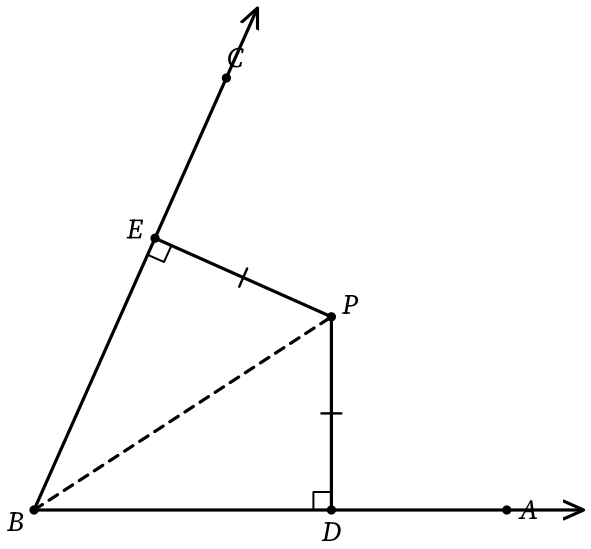
(9) $PD = PE$
Definition of congruent segments
from (8)

∎

PD and PE are the distances from P to the sides because the segments are perpendicular to the sides.

THEOREM **Converse of the Angle Bisector Theorem**

If a point in the interior of an angle is equidistant from the two sides of the angle, then it is on the bisector of the angle.



TWO-COLUMN PROOF

Given: P is in the interior of $\angle ABC$; $\overline{PD} \perp \overrightarrow{BA}$ at D and $\overline{PE} \perp \overrightarrow{BC}$ at E; $PD = PE$

Prove: $\overrightarrow{BP}$ bisects $\angle ABC$

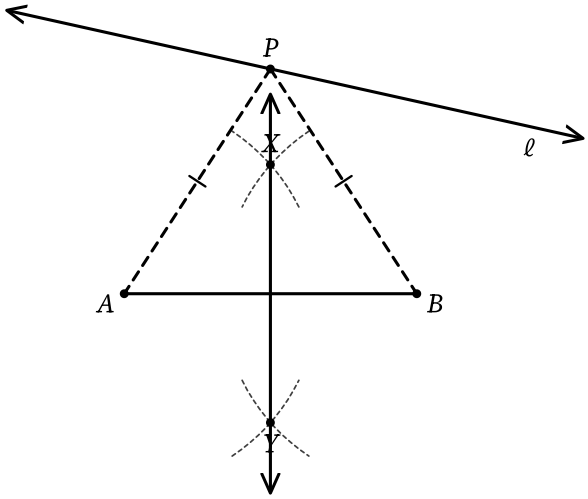
	STATEMENTS	REASONS
1	$\overline{PD} \perp \overrightarrow{BA}$ and $\overline{PE} \perp \overrightarrow{BC}$	Given
2	$\angle PDB$ and $\angle PEB$ are right angles	Definition of perpendicular lines
3	$\triangle PDB$ and $\triangle PEB$ are right triangles	Definition of right triangle
4	$PD = PE$	Given
5	$\overline{PD} \cong \overline{PE}$	Definition of congruent segments
6	$\overline{BP} \cong \overline{BP}$	Reflexive Property of Congruence
7	$\triangle PDB \cong \triangle PEB$	HL
8	$\angle PBD \cong \angle PBE$	CPCTC
9	P is in the interior of $\angle ABC$	Given
10	$\overrightarrow{BP}$ bisects $\angle ABC$	Definition of angle bisector

∎

The word interior matters: without it, the point could be on the bisector of the neighboring (supplementary) angle instead. Together with the Angle Bisector Theorem: the bisector of an angle is the locus of points in its interior that are equidistant from its sides.

CONSTRUCTIONS

CONSTRUCTION **Find the point on a line that is equidistant from two given points** Euclid I.10



1. You are given points A and B and a line ℓ. Open the compass to more than half of AB.
2. With the compass point on A, draw an arc above $\overline{AB}$ and an arc below it.
3. Without changing the compass width, put the compass point on B and draw two arcs that cross the first two. Label the crossings X and Y.
4. With the straightedge draw line XY. This is the perpendicular bisector of $\overline{AB}$.
5. Mark the point P where line XY meets line ℓ. Then $PA = PB$.

Why it works. Because the compass width was not changed, $XA = XB$ and $YA = YB$. By the Converse of the Perpendicular Bisector Theorem, X and Y both lie on the perpendicular bisector of $\overline{AB}$, and two points determine

a line, so line XY is that perpendicular bisector (this is Euclid's I.10). Since P lies on it, $PA = PB$ by the Perpendicular Bisector Theorem. If ℓ happens to be parallel to line XY, there is no such point; if ℓ is line XY itself, every point of ℓ works.

KEY FORMULAS

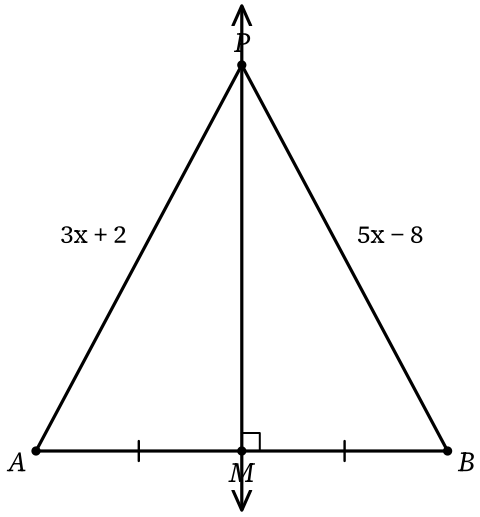
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

From history. The Greek geometers called a set of points with a common property a *topos*, a "place"; Latin writers translated the word as *locus*. Pappus of Alexandria reports that both Euclid and Apollonius wrote books on loci, though those books are now lost. Surveyors still think in loci: a boundary that must stay the same distance from two landmarks runs along a perpendicular bisector.

Euclid connection. Euclid bisects an angle in I.9 and a segment in I.10, and drops a perpendicular from a point to a line in I.12; his proofs of those constructions rest on the congruence propositions I.4 and I.8, just as ours do. The theorems of this lesson go one step further and describe every point on the bisectors as equidistant, an idea Euclid uses in Book IV to find the centers of circles about and in a triangle.

Practice habit. Recite the two theorem pairs as biconditionals until they are automatic: "on the perpendicular bisector if and only if equidistant from the endpoints" and "on the angle bisector if and only if equidistant from the sides." Before using either converse, mark the right angles on your diagram, because distance to a line always means the perpendicular distance.

Example 1. In the figure, line PM is the perpendicular bisector of $\overline{AB}$. If $PA = 3x + 2$ and $PB = 5x - 8$, find PA.

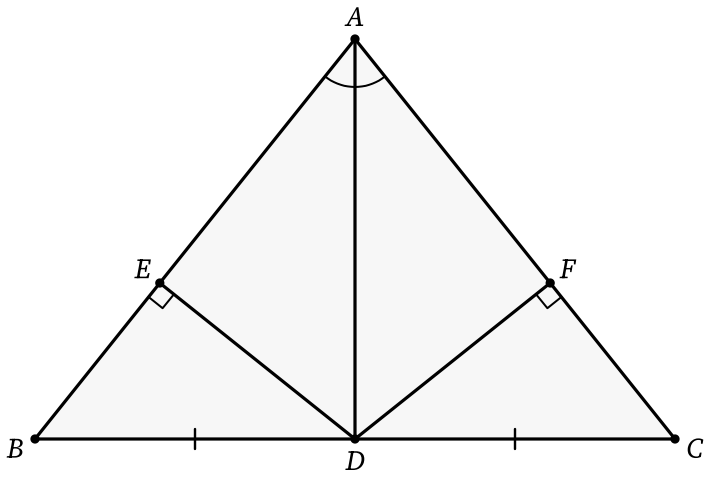


Show solution

1. P is on the perpendicular bisector of $\overline{AB}$, so by the Perpendicular Bisector Theorem $PA = PB$.
2. Set the expressions equal: $3x + 2 = 5x - 8$.
3. Subtract $3x$ and add 8 to both sides: $10 = 2x$, so $x = 5$.
4. Then $PA = 3(5) + 2 = 17$. Check: $PB = 5(5) - 8 = 17$.

Answer: $PA = 17$

Example 2. In $\triangle ABC$, $\overrightarrow{AD}$ bisects $\angle BAC$ with D on $\overline{BC}$, $\overline{DE} \perp \overline{AB}$ at E, $\overline{DF} \perp \overline{AC}$ at F, and $\overline{DB} \cong \overline{DC}$. Prove that $\overline{BE} \cong \overline{CF}$.



Show solution

1. Work backward: $\overline{BE}$ and $\overline{CF}$ are parts of the right triangles $\triangle DEB$ and $\triangle DFC$, so try to prove those triangles congruent.
2. Their hypotenuses $\overline{DB}$ and $\overline{DC}$ are given congruent.
3. D is on the bisector of $\angle BAC$, so the Angle Bisector Theorem gives the legs $DE = DF$.
4. Use HL, then CPCTC.

TWO-COLUMN PROOF

Given: $\overrightarrow{AD}$ bisects $\angle BAC$; $\overline{DE} \perp \overline{AB}$ at E and $\overline{DF} \perp \overline{AC}$ at F; $\overline{DB} \cong \overline{DC}$
Prove: $\overline{BE} \cong \overline{CF}$

	STATEMENTS	REASONS
1	$\overrightarrow{AD}$ bisects $\angle BAC$	Given
2	$\overline{DE} \perp \overline{AB}$ and $\overline{DF} \perp \overline{AC}$	Given
3	$DE = DF$	Angle Bisector Theorem
4	$\overline{DE} \cong \overline{DF}$	Definition of congruent segments
5	$\angle DEB$ and $\angle DFC$ are right angles	Definition of perpendicular lines

	STATEMENTS	REASONS
6	$\triangle DEB$ and $\triangle DFC$ are right triangles	Definition of right triangle
7	$\overline{DB} \cong \overline{DC}$	Given
8	$\triangle DEB \cong \triangle DFC$	HL
9	$\overline{BE} \cong \overline{CF}$	CPCTC

∎

Answer: See the model proof.

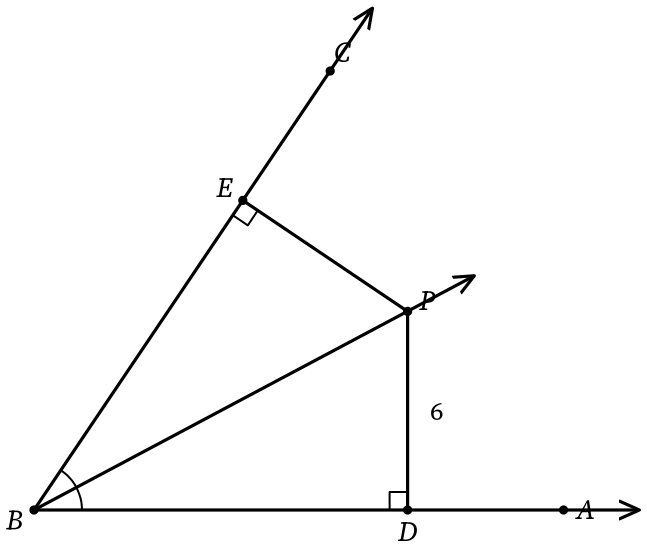
PRACTICE

1. **Easy** Point P lies on the perpendicular bisector of $\overline{AB}$, and $PA = 9.4$ cm. Find PB.

   Show answer

   $PB = 9.4$ cm (Perpendicular Bisector Theorem)

2. **Easy** In the figure, $\overrightarrow{BP}$ bisects $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$, $\overline{PE} \perp \overrightarrow{BC}$, and $PD = 6$. Find PE.



   Show answer

   $PE = 6$ (Angle Bisector Theorem)

3. **Easy** Find the distance from the point $P(3, 7)$ to the horizontal line $y = 2$.

Show answer

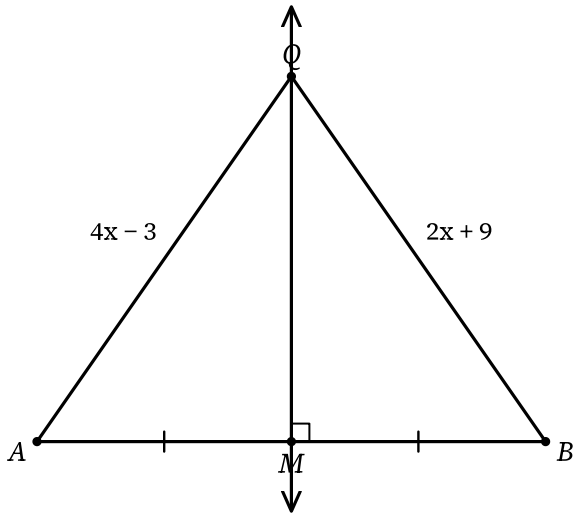
The perpendicular from P to the line is vertical and meets it at $(3, 2)$, so the distance is $7 - 2 = 5$ units.

4. **Easy** Complete the sentence with the name of a figure: the locus of all points in a plane that are equidistant from two points A and B is the ____ of $\overline{AB}$.

Show answer

the perpendicular bisector of $\overline{AB}$

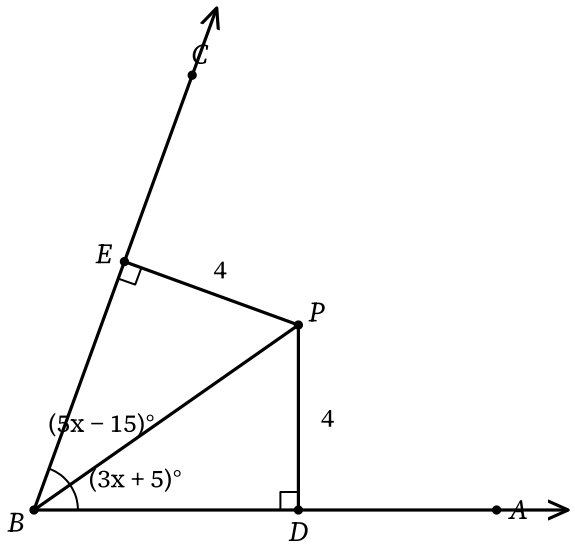
5. **Medium** In the figure, line QM is the perpendicular bisector of $\overline{AB}$, $QA = 4x - 3$ and $QB = 2x + 9$. Find QA.



Show answer

$4x - 3 = 2x + 9$ gives $x = 6$, so $QA = 21$.

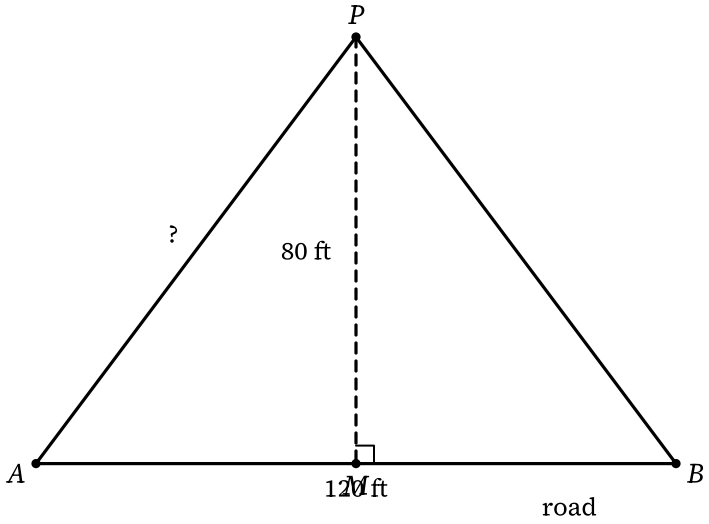
6. **Medium** In the figure, P is in the interior of $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$, $\overline{PE} \perp \overrightarrow{BC}$, and $PD = PE = 4$. Also $m\angle ABP = (3x + 5)^\circ$ and $m\angle PBC = (5x - 15)^\circ$. Find $m\angle ABC$.



Show answer

By the Converse of the Angle Bisector Theorem, $\overrightarrow{BP}$ bisects $\angle ABC$, so $3x + 5 = 5x - 15$, $x = 10$, and each half is 35°. So $m\angle ABC = 70^\circ$.

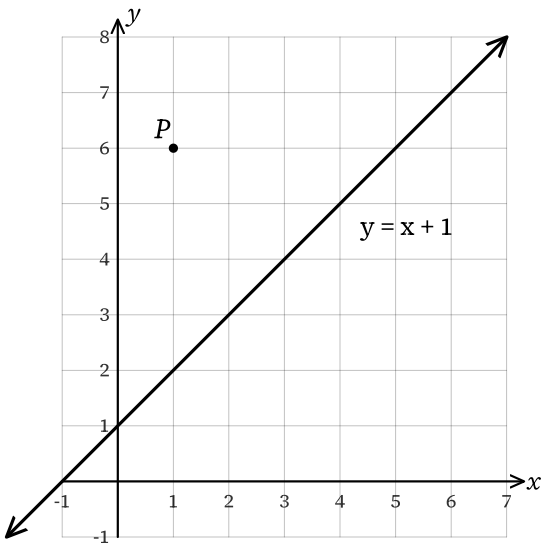
7. **Medium** The Miller house A and the Carter house B stand 120 feet apart on the edge of a straight road, so that $\overline{AB}$ runs along the road. The two families want to dig a shared well at a point P that is the same distance from both houses and 80 feet from the road, as in the figure. How far will the well be from each house?



Show answer

P is equidistant from A and B, so it is on the perpendicular bisector of $\overline{AB}$, which meets the road at a right angle at the midpoint M. So $PM = 80$ ft, $AM = 60$ ft, and $PA = \sqrt{60^2 + 80^2} = 100$ ft. The well is 100 feet from each house.

8. **Medium** Find the distance from $P(1, 6)$ to the line $y = x + 1$. Give an exact answer in simplest radical form.



Show answer

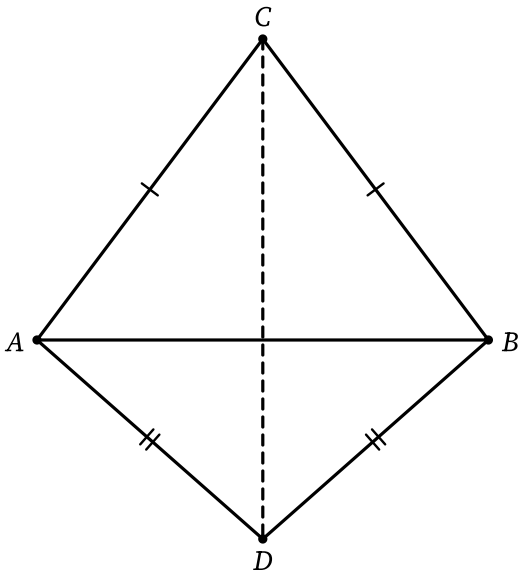
The perpendicular through P has slope -1: $y = -x + 7$. It meets $y = x + 1$ where $x + 1 = -x + 7$, so at $(3, 4)$. The distance is $\sqrt{(3-1)^2 + (4-6)^2} = \sqrt{8} = 2\sqrt{2}$ units.

9. **Medium** Grace measures from a point Q to the two sides of $\angle ABC$ and finds $QD = 5$ cm and $QE = 5$ cm, where D is on $\overrightarrow{BA}$ and E is on $\overrightarrow{BC}$. She concludes that Q is on the bisector of $\angle ABC$. Explain what else must be true before her conclusion follows, and why.

Show answer

The segments $\overline{QD}$ and $\overline{QE}$ must be perpendicular to the sides, because the distance from a point to a line is measured along the perpendicular; slanted segments of equal length prove nothing. Also Q must lie in the interior of $\angle ABC$. Then the Converse of the Angle Bisector Theorem applies.

10. Hard In the figure, $CA = CB$ and $DA = DB$, where C and D are two different points. Prove that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $CA = CB$; $DA = DB$; $C \neq D$

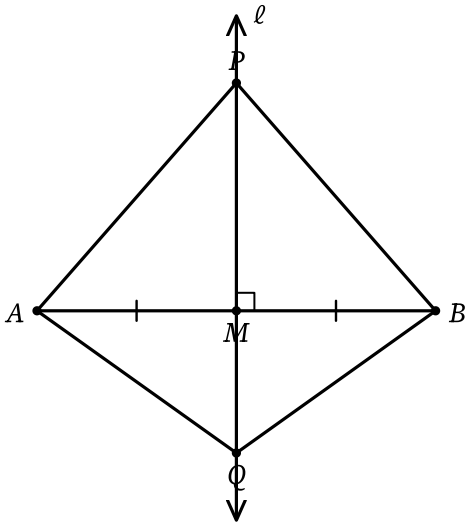
Prove: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$

Since $CA = CB$, point C is equidistant from A and B, so by the Converse of the Perpendicular Bisector Theorem C lies on the perpendicular bisector of $\overline{AB}$. For the same reason, since $DA = DB$, point D lies on the perpendicular bisector of $\overline{AB}$.

The perpendicular bisector of $\overline{AB}$ is a line, and C and D are two different points on it. Through two points there is exactly one line, so the perpendicular bisector is $\overleftrightarrow{CD}$. Therefore $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.

∎

11. Hard In the figure, M is the midpoint of $\overline{AB}$, line ℓ is perpendicular to $\overline{AB}$ at M, and P and Q lie on ℓ. Prove that $\triangle PAQ \cong \triangle PBQ$.



Show answer

See the model proof.

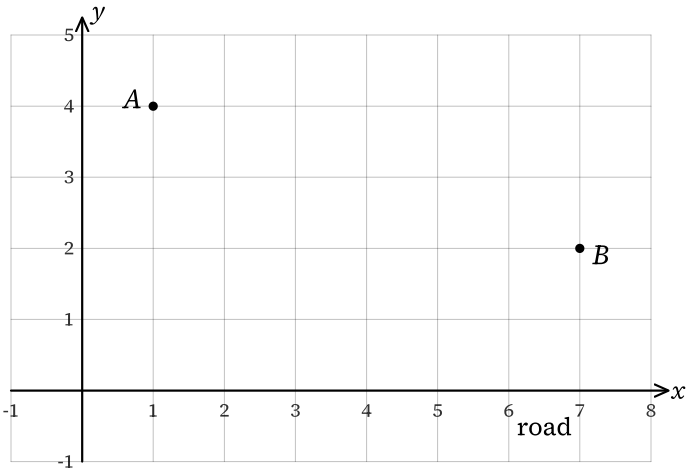
TWO-COLUMN PROOF

Given: M is the midpoint of $\overline{AB}$; $\ell \perp \overline{AB}$ at M; P and Q lie on ℓ

Prove: $\triangle PAQ \cong \triangle PBQ$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$ and $\ell \perp \overline{AB}$ at M	Given
2	ℓ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector
3	P and Q lie on ℓ	Given
4	$PA = PB$ and $QA = QB$	Perpendicular Bisector Theorem
5	$\overline{PA} \cong \overline{PB}$ and $\overline{QA} \cong \overline{QB}$	Definition of congruent segments
6	$\overline{PQ} \cong \overline{PQ}$	Reflexive Property of Congruence
7	$\triangle PAQ \cong \triangle PBQ$	SSS

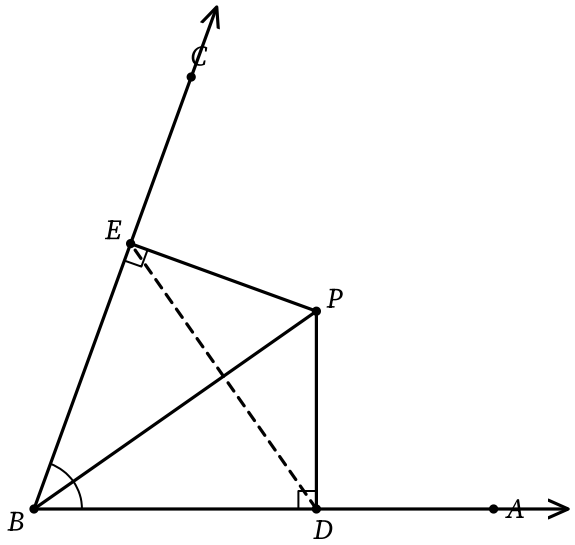
12. Hard On a map with a grid in miles, two farmhouses are at $A(1, 4)$ and $B(7, 2)$, and a straight county road runs along the x-axis. The farmers want a shared mailbox on the road that is the same distance from both farmhouses. Find its coordinates.



Show answer

Let the mailbox be $(x, 0)$. Setting the squared distances equal: $(x-1)^2 + 16 = (x-7)^2 + 4$, so $-2x + 17 = -14x + 53$, $12x = 36$, $x = 3$. The mailbox is at $(3, 0)$, $\sqrt{20} \approx 4.5$ miles from each farmhouse.

13. **Challenge** In the figure, $\overrightarrow{BP}$ bisects $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$ at D, and $\overline{PE} \perp \overrightarrow{BC}$ at E. Prove that $\overleftrightarrow{BP}$ is the perpendicular bisector of $\overline{DE}$. Write a flow proof.



Show answer

See the model proof: both P and B are equidistant from D and E.

FLOW PROOF

Given: $\overrightarrow{BP}$ bisects $\angle ABC$; $\overline{PD} \perp \overrightarrow{BA}$ at D and $\overline{PE} \perp \overrightarrow{BC}$ at E

Prove: $\overleftrightarrow{BP}$ is the perpendicular bisector of $\overline{DE}$

(1) $\overrightarrow{BP}$ bisects $\angle ABC$
Given

(2) $\overline{PD} \perp \overrightarrow{BA}$ and $\overline{PE} \perp \overrightarrow{BC}$
Given

(3) $PD = PE$
Angle Bisector Theorem
from (1), (2)

(4) P is on the perpendicular bisector of $\overline{DE}$
Converse of the Perpendicular Bisector Theorem
from (3)

(5) $\angle PDB$ and $\angle PEB$ are right angles, so $\triangle PDB$ and $\triangle PEB$ are right triangles
Definition of perpendicular lines; definition of right triangle
from (2)

(6) $\overline{PD} \cong \overline{PE}$
Definition of congruent segments
from (3)

(7) $\overline{BP} \cong \overline{BP}$
Reflexive Property of Congruence

(8) $\triangle PDB \cong \triangle PEB$
HL
from (5), (6), (7)

(9) $\overline{BD} \cong \overline{BE}$, so $BD = BE$
CPCTC; definition of congruent segments
from (8)

(10) B is on the perpendicular bisector of $\overline{DE}$
Converse of the Perpendicular Bisector Theorem
from (9)

(11) $\overleftrightarrow{BP}$ is the perpendicular bisector of $\overline{DE}$
Through two points there is exactly one line
from (4), (10)

∎

COMMON MISTAKES

- Measuring the distance from a point to a line along a slanted segment. The distance is always the length of the perpendicular segment; mark the right angle before you use it.
- Using the Converse of the Angle Bisector Theorem when the equal segments are not shown perpendicular to the sides, or when the point is not in the interior of the angle. Check both conditions first.
- Calling a line the perpendicular bisector because it passes through the midpoint. It must also be perpendicular; a line through the midpoint at a slant is only a bisector.
- Thinking one point is enough to locate a perpendicular bisector. One point equidistant from A and B only tells you one point on the line; you need two such points to draw it.

Make it a game · Walk the Locus. Place two cups on the floor 4 meters apart. Walk a path where you stay the same distance from both cups, checking with a string. You are tracing the perpendicular bisector. Next, stand in a room corner and walk a path equally distant from both walls. Say each theorem you used and why it works.

5.2 Circumcenter and Incenter

OBJECTIVES

- Prove that the perpendicular bisectors of the sides of a triangle meet at one point, the circumcenter, and that the angle bisectors meet at one point, the incenter.
- Construct the circumscribed and inscribed circles of a triangle with compass and straightedge.
- Predict where the circumcenter lies in acute, right and obtuse triangles.
- Use the circumcenter, the incenter and the inradius to solve coordinate and practical problems.

WATCH

Triangle Centres (Orthocentre, Centroid & Circumcentre)

Eddie Woo · 10:12

https://www.youtube.com/watch?v=RmECJRogR-4

> **WHEN TO WATCH**
> Before the circumcenter: the centres of a triangle and where each one lies.

Note: Eddie Woo teaches in Australia: he spells it 'circumcentre'. The primary also previews the centroid and orthocenter (Lesson 5.3).

BACKUP VIDEO

Incenter of a Triangle - Find Using Compass (Geometry)

Mario's Math Tutoring · 4:14

https://www.youtube.com/watch?v=JXwNu5J_-TU

With the incenter: bisecting all three angles with a compass, then inscribing the circle.

KEY TERMS

concurrent

Three or more lines are concurrent when they all pass through one point, called the point of concurrency.

circumcenter

The point where the three perpendicular bisectors of the sides of a triangle meet. It is equidistant from the three vertices.

circumscribed circle

The circle that passes through all three vertices of a triangle (its circumcircle). Its center is the circumcenter and its radius is the circumradius.

incenter

The point where the three angle bisectors of a triangle meet. It is equidistant from the three sides.

inscribed circle

The circle inside a triangle that touches each side at exactly one point (its incircle). Its center is the incenter and its radius, the inradius, is the distance from the incenter to each side.

semiperimeter

Half the perimeter of a triangle, usually written $s = \frac{a+b+c}{2}$.

CORE IDEAS

Two lines in a plane usually meet in one point, but three lines usually do not; they make a little triangle. So it is a real surprise when three lines always pass through one point. In every triangle this happens for the three perpendicular bisectors of the sides, and again for the three angle bisectors. Lines that pass through one point are called **concurrent**.

Why are the perpendicular bisectors concurrent? Let the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ meet at P. Because P is on the first, $PA = PB$; because it is on the second, $PB = PC$. So $PA = PC$, and by the converse from Lesson 5.1 P is also on the perpendicular bisector of $\overline{AC}$. The third line has no choice: it must pass through P. This point, the **circumcenter**, is the same distance from all three vertices, so a circle centered there passes through A, B and C.

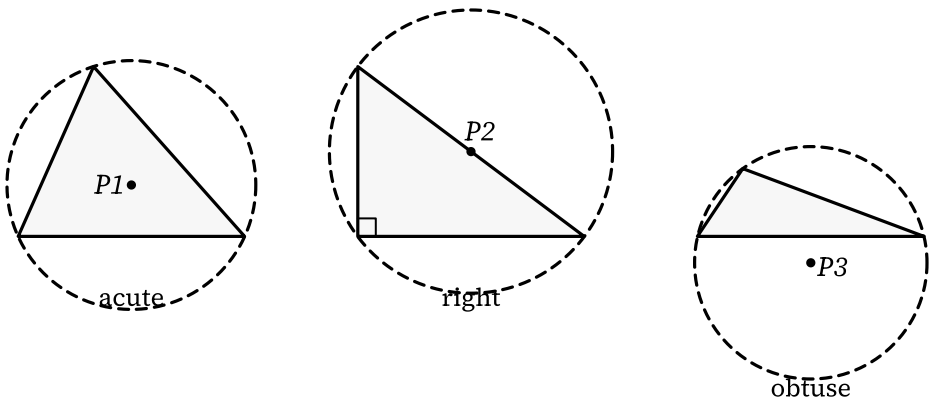
The angle bisectors work the same way with sides instead of vertices. Let the bisectors of $\angle A$ and $\angle B$ meet at I. Being on the bisector of $\angle A$, I is equidistant from sides $\overline{AB}$ and $\overline{AC}$; being on the bisector of $\angle B$, it is equidistant from $\overline{AB}$ and $\overline{BC}$. So it is equidistant from $\overline{AC}$ and $\overline{BC}$, and the converse of the Angle Bisector Theorem puts it on the bisector of $\angle C$. This point, the **incenter**, is the center of the **inscribed circle**.

Where is the circumcenter? Draw several triangles and you will see a pattern: it lies inside an acute triangle, exactly at the midpoint of the hypotenuse of a right triangle,

and outside an obtuse triangle (beyond the longest side). The incenter, by contrast, is always inside, because each angle bisector runs through the interior of its angle.

The inradius has a neat connection with area. Draw segments from the incenter I to the three vertices. They cut the triangle into three smaller triangles whose bases are the sides a, b, c and whose heights are all the inradius r (the perpendicular distances from I to the sides). Adding their areas gives $A = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = rs$, where s is the semiperimeter. So $r = \dfrac{A}{s}$.

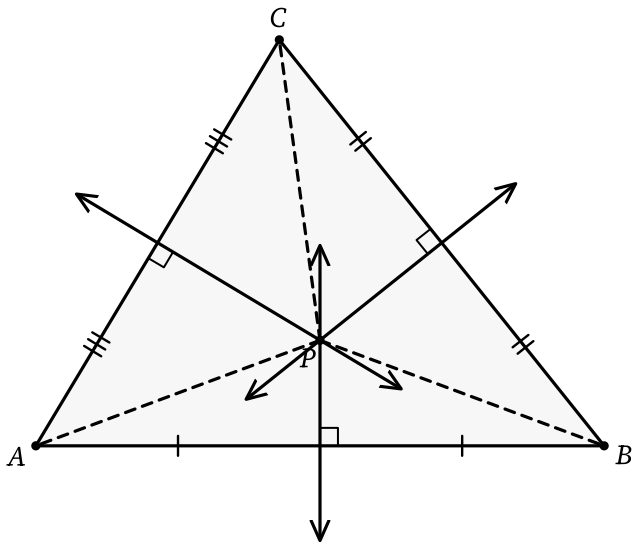
These points answer practical questions. A well that is the same distance from three farmhouses belongs at the circumcenter of the triangle they form. The largest round fountain that fits in a triangular courtyard is the inscribed circle, centered at the incenter. Keep straight which is which: **circumcenter, vertices, perpendicular bisectors**; **incenter, sides, angle bisectors**.



The circumcenter is inside an acute triangle, at the midpoint of the hypotenuse of a right triangle, and outside an obtuse triangle.

THEOREM **Concurrency of Perpendicular Bisectors (Circumcenter Theorem)** Euclid IV.5

The perpendicular bisectors of the sides of a triangle are concurrent at a point that is equidistant from the three vertices.



PARAGRAPH PROOF

Given: $\triangle ABC$

Prove: The perpendicular bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ meet at a point P with $PA = PB = PC$.

First, the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ meet. If they were parallel, then line AB, which is perpendicular to the first, would also be perpendicular to the second (a line perpendicular to one of two parallel lines is perpendicular to the other). Line BC is perpendicular to the second as well. Then lines AB and BC would be two different perpendiculars to the same line through the same point B, which is impossible; they would have to be one line, and A, B, C would be collinear, which is not true of a triangle. So the two perpendicular bisectors meet at a point; call it P.

Since P lies on the perpendicular bisector of $\overline{AB}$, $PA = PB$ by the Perpendicular Bisector Theorem. Since P lies on the perpendicular bisector of $\overline{BC}$, $PB = PC$ by the same theorem. By the Transitive Property of Equality, $PA = PC$.

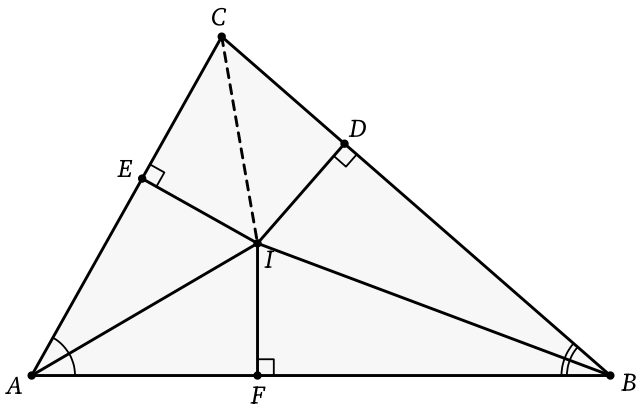
So P is equidistant from A and C, and by the Converse of the Perpendicular Bisector Theorem P lies on the perpendicular bisector of $\overline{AC}$. Therefore all three perpendicular bisectors pass through P, and $PA = PB = PC$.

∎

Because $PA = PB = PC$, the circle with center P through A also passes through B and C: the circumscribed circle.

THEOREM **Concurrency of Angle Bisectors (Incenter Theorem)** Euclid IV.4

The bisectors of the angles of a triangle are concurrent at a point that is equidistant from the three sides.



TWO-COLUMN PROOF

Given: In $\triangle ABC$, the bisectors of $\angle BAC$ and $\angle ABC$ meet at I inside the triangle; $\overline{ID} \perp \overline{BC}$ at D, $\overline{IE} \perp \overline{AC}$ at E, $\overline{IF} \perp \overline{AB}$ at F

Prove: $\overrightarrow{CI}$ bisects $\angle ACB$, and $ID = IE = IF$

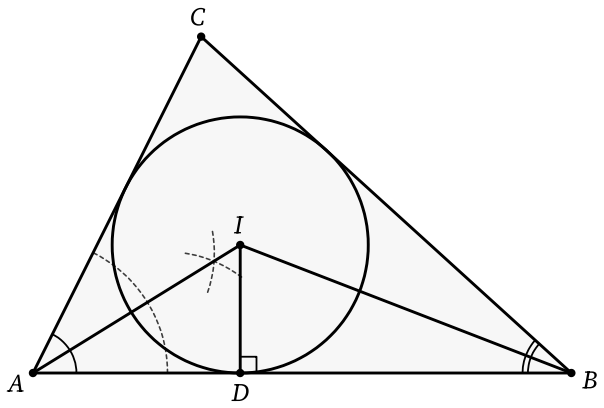
	STATEMENTS	REASONS
1	$\overrightarrow{AI}$ bisects $\angle BAC$ and $\overrightarrow{BI}$ bisects $\angle ABC$	Given
2	$\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AC}$, $\overline{IF} \perp \overline{AB}$	Given (each can be drawn by Euclid I.12)
3	$IF = IE$	Angle Bisector Theorem (I is on the bisector of $\angle BAC$)
4	$IF = ID$	Angle Bisector Theorem (I is on the bisector of $\angle ABC$)
5	$IE = ID$	Transitive Property of Equality
6	I is in the interior of $\angle ACB$	I is inside $\triangle ABC$
7	$\overrightarrow{CI}$ bisects $\angle ACB$, and $ID = IE = IF$	Converse of the Angle Bisector Theorem; steps 3–5

∎

The bisectors of two angles of a triangle always meet inside it, since each bisector runs through the interior of its angle and crosses the opposite side.

COROLLARY **The Inscribed Circle** Euclid IV.4

The circle with center at the incenter I and radius equal to the distance from I to a side meets each side of the triangle at exactly one point, the foot of the perpendicular from I.



PARAGRAPH PROOF

Given: I is the incenter of $\triangle ABC$; D, E, F are the feet of the perpendiculars from I to $\overline{AB}$, $\overline{BC}$, $\overline{CA}$; $r = ID$

Prove: The circle with center I and radius r meets each side at exactly one point.

By the Incenter Theorem, $ID = IE = IF = r$, so D, E and F all lie on the circle. Each foot lies on its side. Suppose D were not between A and B; say $D = A$ or A is between D and B (the case at B is the same). If $D = A$, then $\angle IAB$ is a right angle. Otherwise $\angle IAB$ is an exterior angle of right triangle $\triangle IDA$, so $m\angle IAB > m\angle IDA = 90°$ (Exterior Angle Inequality). Either way $m\angle IAB \geq 90°$. But $\overrightarrow{AI}$ bisects $\angle CAB$, so $m\angle IAB = \frac{1}{2}m\angle CAB < 90°$, a contradiction. The same argument works for E and F.

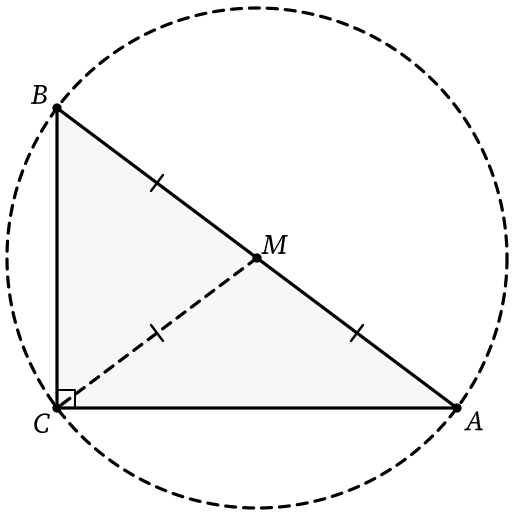
Let X be any other point of line AB. The perpendicular segment $\overline{ID}$ is the shortest segment from I to line AB (Lesson 3.4), so $IX > ID = r$, and X lies outside the circle. So the circle meets line AB only at D. The same argument with E and F works for the other two sides.

∎

In Unit 10 a line that meets a circle at exactly one point will be called a tangent.

THEOREM **Location of the Circumcenter** Euclid IV.5

The circumcenter of an acute triangle lies inside the triangle; the circumcenter of a right triangle is the midpoint of the hypotenuse; the circumcenter of an obtuse triangle lies outside the triangle.

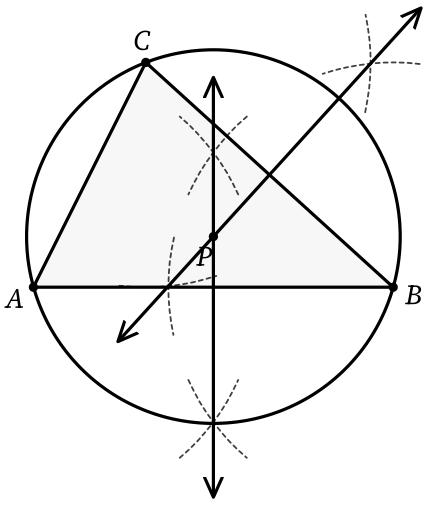


The right-triangle case (the midpoint of the hypotenuse is equidistant from all three vertices) is proved in Lesson 13.3; you can check it in any example with the distance formula. The acute and obtuse cases are accepted here from careful drawings; Euclid records all three cases in a note after IV.5.

CONSTRUCTIONS

CONSTRUCTION **Circumscribe a circle about a triangle** Euclid IV.5

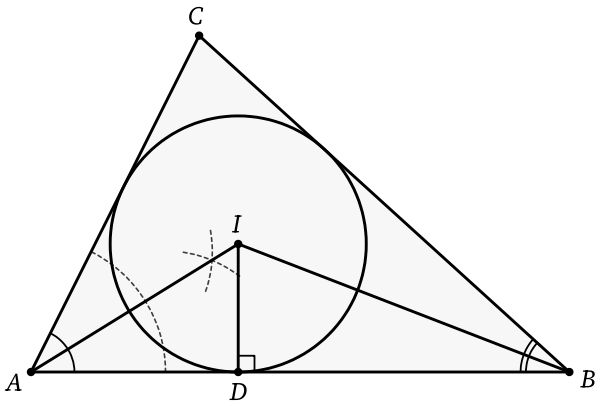
1. Given $\triangle ABC$. Construct the perpendicular bisector of $\overline{AB}$: with the compass wider than half of AB, draw arcs from A and from B above and below the segment, and draw the line through the two crossing points.
2. In the same way construct the perpendicular bisector of $\overline{BC}$.
3. Label the point where the two perpendicular bisectors meet P.
4. Put the compass point on P and open it to A. Draw the circle. It



passes through B and C as well.

Why it works. Each perpendicular bisector is constructed as in Euclid I.10 and Lesson 5.1. By the Circumcenter Theorem, P is equidistant from A, B and C, so the circle of radius PA passes through all three. A third perpendicular bisector is not needed, but drawing it is a good check: it should pass through P.

CONSTRUCTION **Inscribe a circle in a triangle** Euclid IV.4



1. Given $\triangle ABC$. Bisect $\angle A$: with the compass point on A, draw an arc crossing both sides; from the two crossing points draw arcs of equal radius that meet inside the angle; draw the ray from A through that meeting point.
2. Bisect $\angle B$ in the same way. Label the point where the two bisectors meet I.
3. Construct the perpendicular from I to side $\overline{AB}$ (Euclid I.12): an arc centered at I cuts line AB twice; bisect the segment between those two points. Label the foot of the perpendicular D.
4. Put the compass point on I, open it to D, and draw the

circle. It touches all three sides.

Why it works. Each angle bisector is constructed as in Euclid I.9. By the Incenter Theorem, I is equidistant from the three sides, and ID is that distance because $\overline{ID}$ is perpendicular to $\overline{AB}$. By the corollary above, the circle of radius ID touches each side at exactly one point.

KEY FORMULAS

$$PA = PB = PC \quad (\text{circumcenter } P)$$

$$ID = IE = IF = r \quad (\text{incenter } I)$$

$$A = rs, \qquad r = \frac{A}{s}, \qquad s = \frac{a+b+c}{2}$$

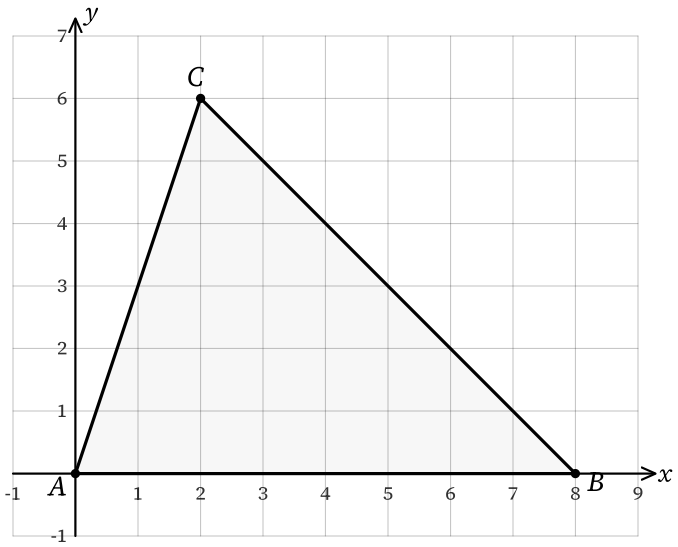
From history. Book IV of Euclid's Elements is about figures inscribed in and circumscribed about circles. Its fourth and fifth propositions inscribe a circle in a given triangle and circumscribe a circle about one, exactly the constructions of this lesson. Later in Book IV Euclid inscribes a regular pentagon, hexagon and fifteen-sided figure in a circle, the kind of figures the master masons of the cathedrals laid out with compass and cord when they designed rose windows.

Euclid connection. In IV.5 Euclid bisects two sides of the triangle at right angles and shows that the point where the bisectors meet is equally distant from all three vertices, the same argument as our proof; in a note after it he observes that the center falls inside, on a side, or outside according as the triangle's angle is acute, right or obtuse. In IV.4 he bisects two angles and drops perpendiculars to the sides, proving them equal by I.26 (our AAS), as in Lesson 5.1.

Practice habit. Say the pairs aloud until they are automatic: "circumcenter: perpendicular bisectors, equidistant from the vertices" and "incenter: angle bisectors, equidistant from the sides." When you construct, keep the compass width fixed for each pair of arcs, and check your work by drawing the third bisector through the point you found.

WORKED EXAMPLES

Example 1. On a map whose grid unit is 100 feet, three farmhouses stand at $A(0,0)$, $B(8,0)$ and $C(2,6)$. The families want to dig one well that is the same distance from all three houses. Find the coordinates of the well.

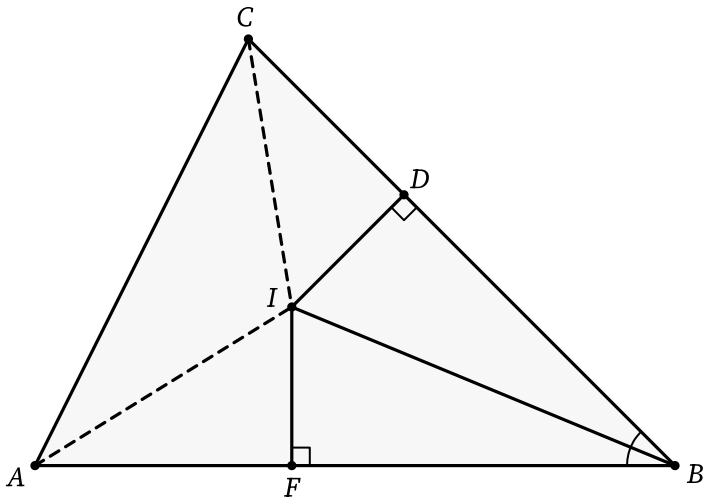


Show solution

1. A point equidistant from the three vertices is the circumcenter, where the perpendicular bisectors meet.
2. $\overline{AB}$ is horizontal with midpoint $(4,0)$, so its perpendicular bisector is the vertical line $x = 4$. The well is at $P(4, y)$ for some y.
3. Set $PA = PC$ using squared distances: $4^2 + y^2 = (4-2)^2 + (y-6)^2$, so $16 + y^2 = 4 + y^2 - 12y + 36$. Then $16 = 40 - 12y$, so $y = 2$.
4. Check: $PA = \sqrt{16+4} = \sqrt{20}$, $PB = \sqrt{16+4} = \sqrt{20}$, $PC = \sqrt{4+16} = \sqrt{20}$. The well is $\sqrt{20} = 2\sqrt{5} \approx 4.47$ units, about 447 feet, from each house.

Answer: The well belongs at $(4, 2)$.

Example 2. In $\triangle ABC$, I is the incenter, $\overline{ID} \perp \overline{BC}$ at D and $\overline{IF} \perp \overline{AB}$ at F. Prove that $BD = BF$.



Show solution

1. $\overline{BD}$ and $\overline{BF}$ are sides of the right triangles $\triangle IDB$ and $\triangle IFB$; prove those congruent.
2. Since I is the incenter, $\overrightarrow{BI}$ bisects $\angle ABC$, giving a pair of congruent angles at B.
3. The right angles and the shared side $\overline{BI}$ finish AAS.

TWO-COLUMN PROOF

Given: I is the incenter of $\triangle ABC$; $\overline{ID} \perp \overline{BC}$ at D and $\overline{IF} \perp \overline{AB}$ at F

Prove: $BD = BF$

	STATEMENTS	REASONS
1	I is the incenter of $\triangle ABC$	Given
2	$\overrightarrow{BI}$ bisects $\angle ABC$	Definition of incenter
3	$\angle IBF \cong \angle IBD$	Definition of angle bisector
4	$\overline{ID} \perp \overline{BC}$ and $\overline{IF} \perp \overline{AB}$	Given
5	$\angle IFB$ and $\angle IDB$ are right angles	Definition of perpendicular lines
6	$\angle IFB \cong \angle IDB$	All right angles are congruent

	STATEMENTS	REASONS
7	$\overline{BI} \cong \overline{BI}$	Reflexive Property of Congruence
8	$\triangle IFB \cong \triangle IDB$	AAS
9	$\overline{BF} \cong \overline{BD}$	CPCTC
10	$BD = BF$	Definition of congruent segments

∎

Answer: See the model proof.

PRACTICE

1. Easy P is the circumcenter of $\triangle ABC$ and $PA = 7$ cm. Find PC.

   Show answer

   $PC = 7$ cm, since the circumcenter is equidistant from the vertices.

2. Easy I is the incenter of $\triangle ABC$, and the distance from I to side $\overline{AB}$ is 3 cm. What is the distance from I to side $\overline{BC}$?

   Show answer

   3 cm, since the incenter is equidistant from the sides.

3. Easy A triangle has angles of $30°$, $50°$ and $100°$. Is its circumcenter inside, on, or outside the triangle?

   Show answer

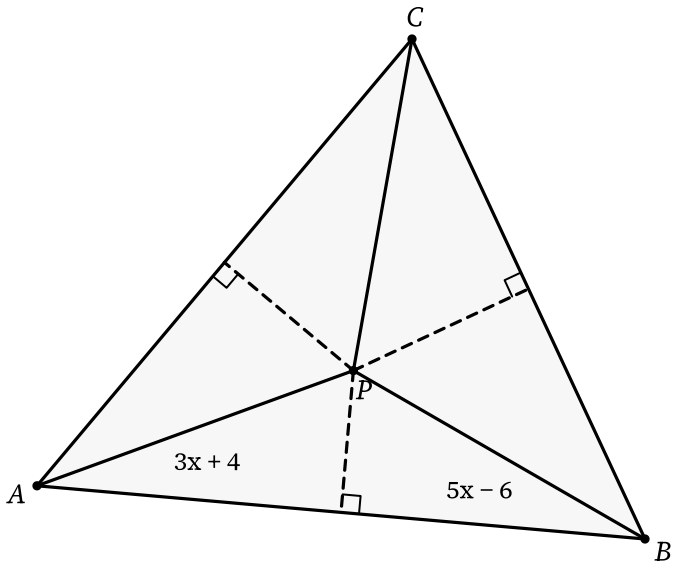
   Outside, because the triangle is obtuse.

4. Easy A right triangle has a hypotenuse of 26 cm. Find the radius of its circumscribed circle.

   Show answer

   The circumcenter is the midpoint of the hypotenuse, so the circumradius is 13 cm.

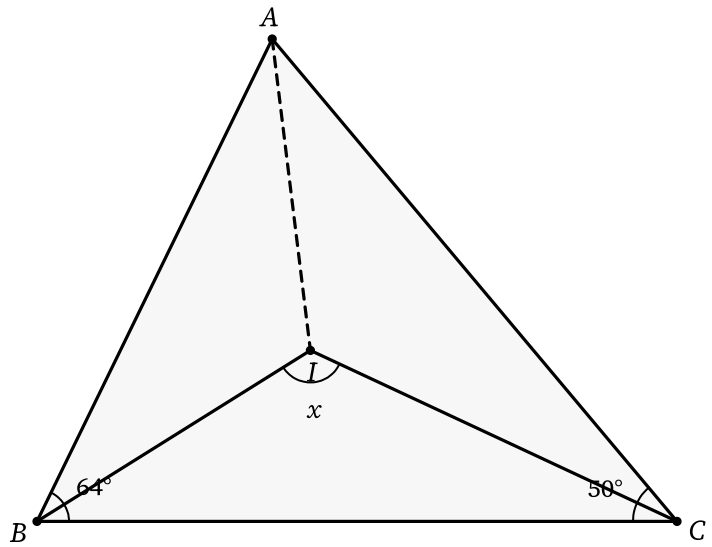
5. **Medium** In the figure, the perpendicular bisectors of the sides of $\triangle ABC$ meet at P. If $PA = 3x + 4$ and $PB = 5x - 6$, find PC.



Show answer

$PA = PB$, so $3x + 4 = 5x - 6$ and $x = 5$. Then $PA = 19$, and $PC = PA = 19$.

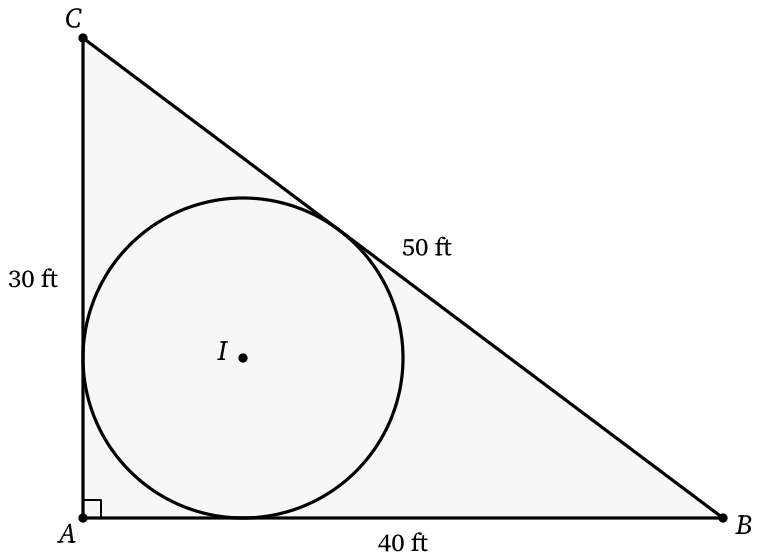
6. **Medium** In the figure, I is the incenter of $\triangle ABC$, $m\angle ABC = 64°$ and $m\angle ACB = 50°$. Find $m\angle BIC$.



Show answer

$\overrightarrow{BI}$ and $\overrightarrow{CI}$ are angle bisectors, so $m\angle IBC = 32°$ and $m\angle ICB = 25°$. In $\triangle BIC$, $m\angle BIC = 180° - 32° - 25° = 123°$.

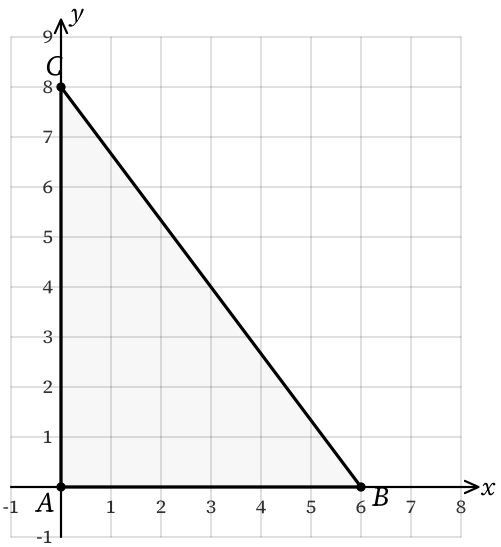
7. **Medium** A courtyard behind a church is a right triangle with sides 30 ft, 40 ft and 50 ft, as in the figure. The builders want the largest round fountain that fits inside it, which is the inscribed circle. Find the radius of the fountain.



Show answer

Area $A = \frac{1}{2}(30)(40) = 600$ sq ft and $s = \frac{30+40+50}{2} = 60$ ft, so $r = \dfrac{A}{s} = \dfrac{600}{60} = 10$ ft.

8. **Medium** Find the coordinates of the circumcenter of the triangle with vertices $A(0,0)$, $B(6,0)$ and $C(0,8)$, and check your answer with the distance formula.



Show answer

$\angle A$ is a right angle, so the circumcenter is the midpoint of the hypotenuse $\overline{BC}$: $(3,4)$. Check: its distance to A, B and C is $\sqrt{9+16} = 5$ each time.

9. **Medium** Construction: draw a large obtuse triangle. With compass and straightedge construct its circumscribed circle. Where does the center fall? Then construct the third perpendicular bisector as a check.

Show answer

Construct the perpendicular bisectors of two sides, mark their intersection P, and draw the circle with center P through a vertex. The center falls outside the triangle, beyond the longest side (opposite the obtuse angle), and the third perpendicular bisector passes through P.

10. Medium Explain why the incenter of a triangle is always inside the triangle, while the circumcenter can lie outside it.

Show answer

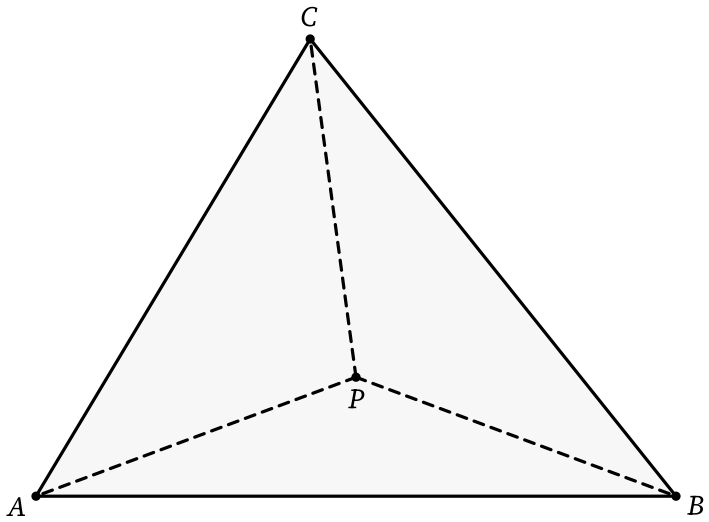
The incenter lies on each angle bisector, and an angle bisector runs through the interior of its angle; a point in the interior of all three angles of a triangle is inside the triangle. A perpendicular bisector is tied to the midpoint of a side, not to the angles, and nothing keeps it inside the triangle; in an obtuse triangle two of them cross beyond the longest side.

11. Hard A triangle has sides 13, 14 and 15 inches and an area of 84 square inches. Find the radius of its inscribed circle.

Show answer

$s = \dfrac{13 + 14 + 15}{2} = 21$, so $r = \dfrac{A}{s} = \dfrac{84}{21} = 4$ inches.

12. **Hard** In the figure, P is the circumcenter of $\triangle ABC$ and is not on $\overline{AB}$. Prove that $\angle PAB \cong \angle PBA$.



P is the circumcenter of triangle ABC.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: P is the circumcenter of $\triangle ABC$; P is not on $\overline{AB}$

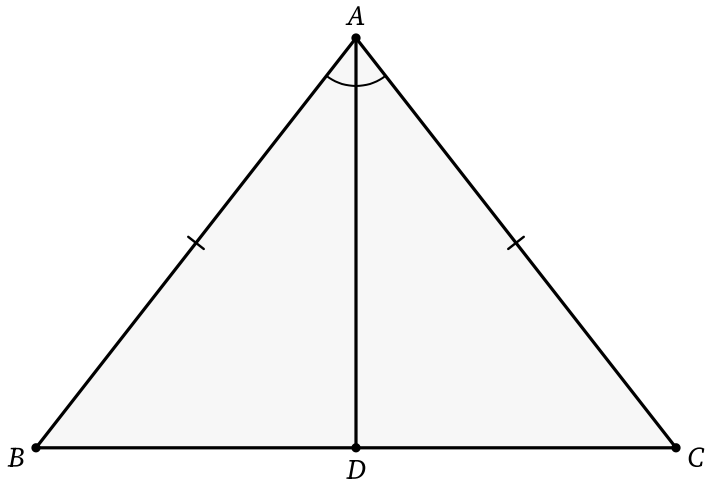
Prove: $\angle PAB \cong \angle PBA$

By the definition of circumcenter, P lies on the perpendicular bisector of $\overline{AB}$. By the Perpendicular Bisector Theorem, $PA = PB$, so $\overline{PA} \cong \overline{PB}$.

Since P is not on $\overline{AB}$, the points P, A, B form a triangle, and $\triangle PAB$ is isosceles with legs $\overline{PA}$ and $\overline{PB}$. By the Isosceles Triangle Theorem, the angles opposite the congruent sides are congruent: $\angle PBA \cong \angle PAB$. Therefore $\angle PAB \cong \angle PBA$.

∎

13. Hard In the figure, $\overline{AB} \cong \overline{AC}$ in $\triangle ABC$, and $\overrightarrow{AD}$ bisects $\angle BAC$ with D on $\overline{BC}$. Prove that the circumcenter of $\triangle ABC$ lies on $\overleftrightarrow{AD}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$; $\overrightarrow{AD}$ bisects $\angle BAC$, D on $\overline{BC}$

Prove: The circumcenter of $\triangle ABC$ lies on $\overleftrightarrow{AD}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$\overrightarrow{AD}$ bisects $\angle BAC$	Given
3	$\angle BAD \cong \angle CAD$	Definition of angle bisector
4	$\overline{AD} \cong \overline{AD}$	Reflexive Property of Congruence
5	$\triangle BAD \cong \triangle CAD$	SAS
6	$\overline{DB} \cong \overline{DC}$	CPCTC
7	$AB = AC$ and $DB = DC$	Definition of congruent segments
8	A and D lie on the perpendicular bisector of $\overline{BC}$	Converse of the Perpendicular Bisector Theorem
9	$\overleftrightarrow{AD}$ is the perpendicular bisector of $\overline{BC}$	Through two points there is exactly one line

	STATEMENTS	REASONS
10	The circumcenter of $\triangle ABC$ lies on the perpendicular bisector of $\overline{BC}$	Definition of circumcenter
11	The circumcenter of $\triangle ABC$ lies on $\overleftrightarrow{AD}$	Substitution (steps 9 and 10)

∎

14. **Challenge** An equilateral triangle has sides of 12 cm. Find the radius of its circumscribed circle in simplest radical form. (Hint: by the reasoning of Problem 13, each angle bisector is also a perpendicular bisector, so the incenter and circumcenter are the same point O. Find the altitude, then the inradius from $A = rs$.)

Show answer

The altitude from A meets $\overline{BC}$ at its midpoint D, so $AD = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$. The area is $\frac{1}{2}(12)(6\sqrt{3}) = 36\sqrt{3}$ and $s = 18$, so $r = OD = 2\sqrt{3}$. Since O lies on $\overline{AD}$, the circumradius is $OA = 6\sqrt{3} - 2\sqrt{3} = 4\sqrt{3}$ cm ≈ 6.9 cm.

COMMON MISTAKES

- Mixing up the two centers. The circumcenter comes from perpendicular bisectors and is equidistant from the vertices; the incenter comes from angle bisectors and is equidistant from the sides.
- Measuring the inradius from the incenter to a vertex. The inradius is the perpendicular distance from the incenter to a side; the distance to a vertex is longer.
- Assuming a perpendicular bisector of a side passes through the opposite vertex. That happens only when the two sides at that vertex are congruent; in general it does not.
- Expecting the circumcenter to be inside every triangle. Classify the triangle first: inside for acute, at the midpoint of the hypotenuse for right, outside for obtuse.

> **Make it a game · Predict the Center.** Cut three paper triangles: acute, right and obtuse. Before folding, predict where each circumcenter will land: inside, on a side or outside. Then fold each vertex onto another to find the perpendicular bisectors and check. Next, fold the angles in half for the incenter. One point per correct prediction.

5.3 Medians, Altitudes and the Centroid

OBJECTIVES

- Identify and draw the medians and altitudes of a triangle, including altitudes that fall outside an obtuse triangle.
- Use the Centroid Theorem to find lengths along a median.
- Find the centroid and the orthocenter of a triangle in the coordinate plane.
- Construct a median and an altitude with compass and straightedge.

WATCH

Concurrence of Medians (1 of 2: Proof via similar triangles)

Eddie Woo · 8:09

https://www.youtube.com/watch?v=L7mfmg0VAU0

> **WHEN TO WATCH**
> After the Centroid Theorem: a proof that the medians meet at one point.

Note: The primary proof uses similar triangles (Unit 8). The backup also reviews the incenter and circumcenter from Lesson 5.2.

BACKUP VIDEO

Incenter, Circumcenter, Centroid, Orthocenter (Properties & Diagrams)

Mario's Math Tutoring · 8:02

https://www.youtube.com/watch?v=p52H7YcoDO8

Before the altitudes and orthocenter: all four centers, and where each one lies in acute, right and obtuse triangles.

KEY TERMS

median of a triangle

A segment from a vertex to the midpoint of the opposite side. Every triangle has three medians.

centroid

The point where the three medians of a triangle meet. It is the triangle's balance point (center of gravity).

altitude of a triangle

The perpendicular segment from a vertex to the line containing the opposite side. Its length is the height of the triangle for that side.

center of gravity

The point at which a flat object of even thickness balances exactly; for a triangle it is the centroid.

orthocenter

The point where the three lines containing the altitudes of a triangle meet.

CORE IDEAS

A **median** joins a vertex to the midpoint of the opposite side. Why should anyone care about the midpoint? Because a median cuts a triangle into two triangles with equal area: they have equal bases (the two halves of the side) and the same height (the distance from the vertex to that side). Equal areas alone do not make a shape balance on a line, but a median does more. Cut the triangle into thin strips parallel to that side: the median passes through the middle of every strip (each strip is a small copy of the side, and in Unit 8 you will see why its midpoint lies on the median). Each strip balances on the median, so a cardboard triangle laid along a median on a ruler's edge balances. This strip argument is Archimedes' own.

If the triangle balances along each of its three medians, it must balance at a point on all three. This is the reasoning of Archimedes, and it suggests the **Centroid Theorem**: the three medians meet at one point, the **centroid** G, and G is two-thirds of the way from each vertex to the midpoint of the opposite side. So on median $\overline{AL}$, $AG = \frac{2}{3}AL$ and $GL = \frac{1}{3}AL$; the longer piece is always twice the shorter piece.

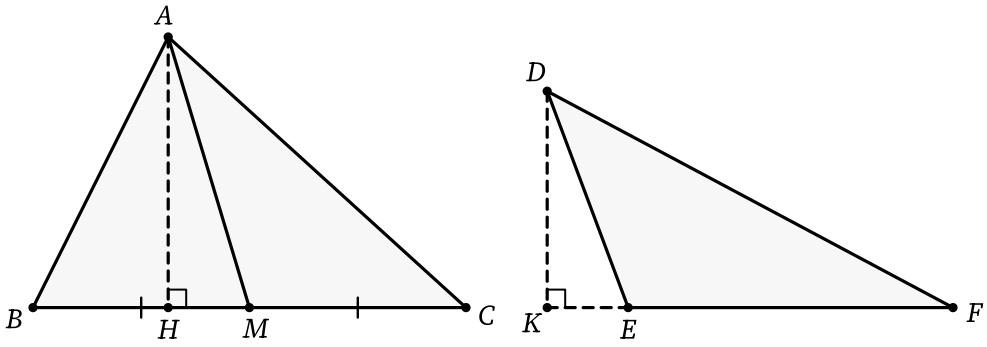
A balancing argument is persuasive, but it is not a proof. We prove the theorem with coordinates. Place A at the origin and $\overline{AB}$ on the x-axis, and write the coordinates with a factor of 6 so the midpoints and thirds come out without fractions. Then we check that one point is two-thirds of the way along each median. A useful result falls out: the centroid's coordinates are the averages of the vertices' coordinates.

An **altitude** is the perpendicular segment from a vertex to the line containing the opposite side. In an acute triangle all three altitudes are inside. In a right triangle two of the altitudes are the legs themselves. In an obtuse triangle, the altitudes from

the two acute vertices fall outside, so you must extend the opposite side with a dashed line to reach the foot of the perpendicular.

The lines containing the three altitudes also meet at one point, the **orthocenter** H. It is inside an acute triangle, at the right-angle vertex of a right triangle, and outside an obtuse triangle. To find it with coordinates, write the equations of two altitude lines (each is perpendicular to a side, so its slope is the negative reciprocal of that side's slope) and solve them together.

You now know four triangle centers: circumcenter, incenter, centroid and orthocenter. Keep them sorted by the lines that make them: perpendicular bisectors, angle bisectors, medians and altitudes. The centroid and incenter are always inside; the circumcenter and orthocenter can be outside.

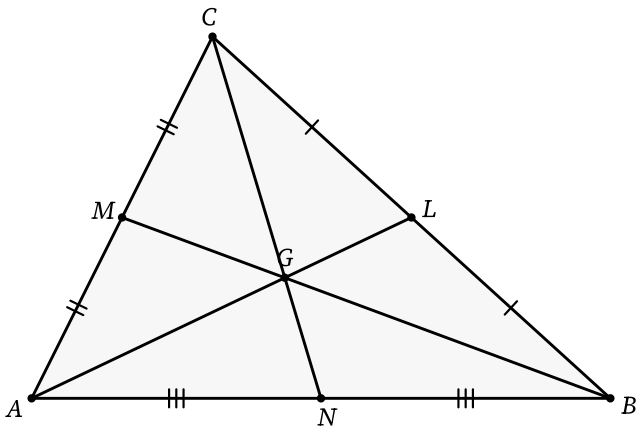


Left: median AM and altitude AH. Right: in an obtuse triangle the altitude from D meets the extension of side EF.

POSTULATES AND THEOREMS

THEOREM **Centroid Theorem (Concurrency of Medians)**

The medians of a triangle meet at a point, the centroid, that is two-thirds of the way from each vertex to the midpoint of the opposite side.



PARAGRAPH PROOF

Given: $\triangle ABC$ with medians $\overline{AL}$, $\overline{BM}$ and $\overline{CN}$ (L, M, N the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$)

Prove: The point G two-thirds of the way from A to L is also two-thirds of the way from B to M and from C to N.

Place the triangle on a coordinate plane with $A = (0, 0)$, $B = (6b, 0)$ and $C = (6c, 6d)$. Any triangle can be placed this way by putting one vertex at the origin and one side along the x-axis. By the midpoint formula, $L = (3b + 3c, 3d)$, $M = (3c, 3d)$ and $N = (3b, 0)$.

To go two-thirds of the way from one point to another, add two-thirds of the change in x and two-thirds of the change in y; the point reached lies on the segment, and by the distance formula its distance from the start is two-thirds of the whole length. From A to L the changes are $3b + 3c$ and $3d$, so the point two-thirds of the way is $G = (2b + 2c, 2d)$.

From $B = (6b, 0)$ to $M = (3c, 3d)$ the changes are $3c - 6b$ and $3d$; two-thirds of the way gives $(6b + 2c - 4b, 0 + 2d) = (2b + 2c, 2d) = G$. From $C = (6c, 6d)$ to $N = (3b, 0)$ the changes are $3b - 6c$ and $-6d$; two-thirds of the way gives $(6c + 2b - 4c, 6d - 4d) = (2b + 2c, 2d) = G$.

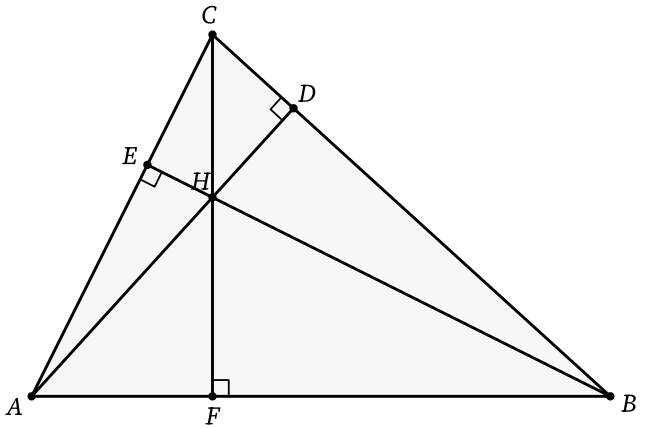
So the single point G lies on all three medians, two-thirds of the way from each vertex to the opposite midpoint. Notice that $G = \left(\dfrac{0 + 6b + 6c}{3}, \dfrac{0 + 0 + 6d}{3}\right)$: the centroid's coordinates are the averages of the vertices' coordinates.

∎

Archimedes reached the same point by a balancing argument in On the Equilibrium of Planes.

THEOREM **Concurrency of Altitudes**

The lines containing the three altitudes of a triangle meet at one point, the orthocenter.

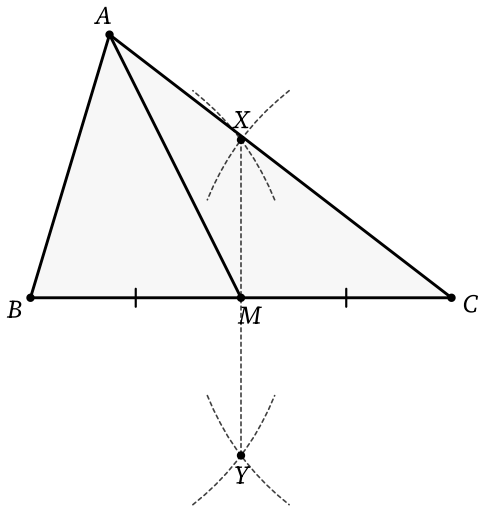


We accept this for now and check it in coordinate examples. After Unit 6 it has a short proof: through each vertex draw the line parallel to the opposite side; these lines form a larger triangle whose perpendicular bisectors are exactly the altitudes of the original triangle, and perpendicular bisectors are concurrent by the Circumcenter Theorem.

CONSTRUCTIONS

CONSTRUCTION **Construct a median of a triangle** Euclid I.10

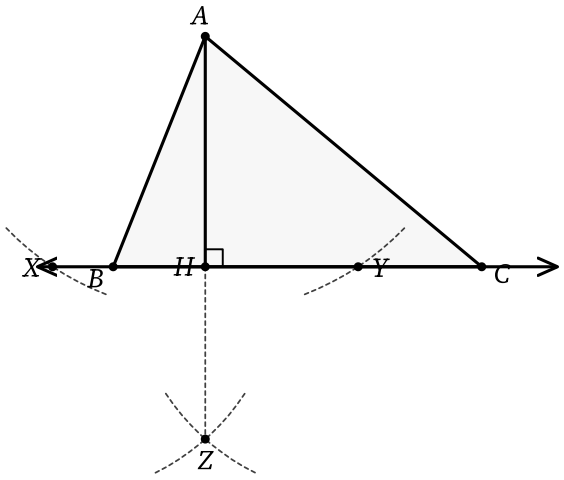
1. Given $\triangle ABC$, to construct the median from A. Open the compass to more than half of BC.
2. With the compass point on B, draw arcs above and below $\overline{BC}$. With the same width and the point on C, draw arcs crossing them at X and Y.
3. Draw $\overline{XY}$. It crosses $\overline{BC}$ at its midpoint M.
4. Draw $\overline{AM}$. It is the median from A.



Why it works. Since $XB = XC$ and $YB = YC$, line XY is the perpendicular bisector of $\overline{BC}$ (Converse of the Perpendicular Bisector Theorem, as in Euclid I.10), so M is the midpoint of $\overline{BC}$, and a segment from a vertex to the midpoint of the opposite side is a median.

CONSTRUCTION **Construct an altitude of a triangle** Euclid I.12

1. Given $\triangle ABC$, to construct the altitude from A. Draw line BC long enough (extend it if needed).
2. With the compass point on A, draw an arc that crosses line BC at two points X and Y.
3. With the compass point on X and then on Y, using one width greater than half of XY, draw two arcs that cross at Z on the side of line BC opposite A.
4. Draw line AZ. It meets line BC at H, and $\overline{AH}$ is the altitude from A.



Why it works. $AX = AY$ (same compass width) and $ZX = ZY$ (same compass width), so A and Z both lie on the perpendicular bisector of $\overline{XY}$ by the Converse of the Perpendicular Bisector Theorem. So line AZ is perpendicular to line BC, and $\overline{AH}$ is the perpendicular segment from A: the altitude. This is Euclid's I.12.

KEY FORMULAS

$$AG = \frac{2}{3}AL, \qquad GL = \frac{1}{3}AL, \qquad AG = 2 \cdot GL$$

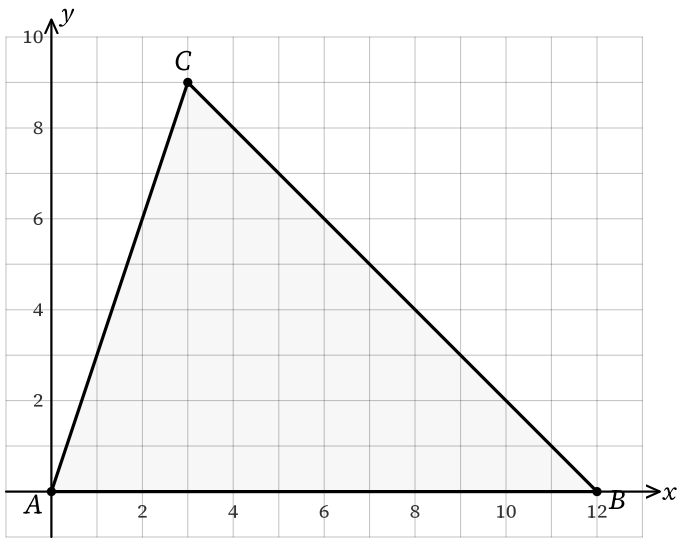
$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

From history. Archimedes of Syracuse (about 250 BC), in his work On the Equilibrium of Planes, proved that the center of gravity of a triangle lies on the line joining a vertex to the midpoint of the opposite side, and therefore at the point where these lines meet. He was one of the first to bring geometry and mechanics together. You can repeat his discovery at the kitchen table: cut a triangle from stiff cardboard, draw two medians, and balance it on a pencil point at their crossing.

Euclid connection. Euclid does not name medians, altitudes, the centroid or the orthocenter, but his constructions produce them: I.10 bisects a side, giving the midpoint a median needs, and I.12 draws the perpendicular from a point to a line, which from a vertex is an altitude. The balancing property of the centroid belongs to Archimedes, who built his mechanics on Euclid's geometry.

Practice habit. Draw a large figure and put tick marks on both halves of each side a median bisects; in an obtuse triangle, extend the side with a dashed line before you draw an altitude. Estimate as a check: the centroid always sits inside the triangle, about one-third of the way from each side toward the opposite vertex.

Example 1. Henry cuts a triangle of stiff cardboard and lays it on a grid whose unit is 1 inch, with corners at $A(0,0)$, $B(12,0)$ and $C(3,9)$. Where should he put the pencil point so the triangle balances?

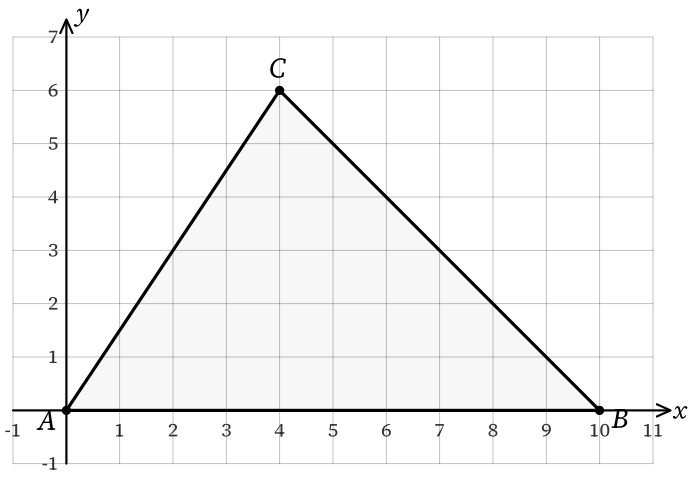


Show solution

1. The triangle balances at its centroid, where the medians meet.
2. The midpoint of $\overline{BC}$ is $L = \left(\dfrac{12+3}{2}, \dfrac{0+9}{2}\right) = (7.5, 4.5)$.
3. The centroid is two-thirds of the way from A to L: $\left(\frac{2}{3}(7.5), \frac{2}{3}(4.5)\right) = (5, 3)$.
4. Check with the averaging formula: $\left(\dfrac{0+12+3}{3}, \dfrac{0+0+9}{3}\right) = (5, 3)$.

Answer: At the centroid, $(5, 3)$.

Example 2. Find the orthocenter of the triangle with vertices $A(0, 0)$, $B(10, 0)$ and $C(4, 6)$.



Show solution

1. Side $\overline{AB}$ is horizontal, so the altitude from C is the vertical line $x = 4$.
2. The slope of $\overline{BC}$ is $\dfrac{6 - 0}{4 - 10} = -1$, so the altitude from A has slope 1 and passes through $(0, 0)$: $y = x$.
3. These meet where $x = 4$ and $y = 4$: $H = (4, 4)$.
4. Check with the third altitude: the slope of $\overline{AC}$ is $\dfrac{6}{4} = \dfrac{3}{2}$, so the altitude from B has slope $-\dfrac{2}{3}$: $y = -\dfrac{2}{3}(x - 10)$. At $x = 4$, $y = -\dfrac{2}{3}(-6) = 4$. It passes through $(4, 4)$ too.

Answer: $H = (4, 4)$

PRACTICE

1. Easy $\overline{AD}$ is a median of $\triangle ABC$ and $BD = 7$ cm. Find BC.

   Show answer

   D is the midpoint of $\overline{BC}$, so $BC = 2(7) = 14$ cm.

2. Easy G is the centroid of $\triangle ABC$, and the median $\overline{AL}$ has length 21 cm. Find AG.

   Show answer

   $AG = \frac{2}{3}(21) = 14$ cm.

3. **Easy** G is the centroid of $\triangle ABC$, and $\overline{CM}$ is a median with $GM = 4$. Find CM.

Show answer

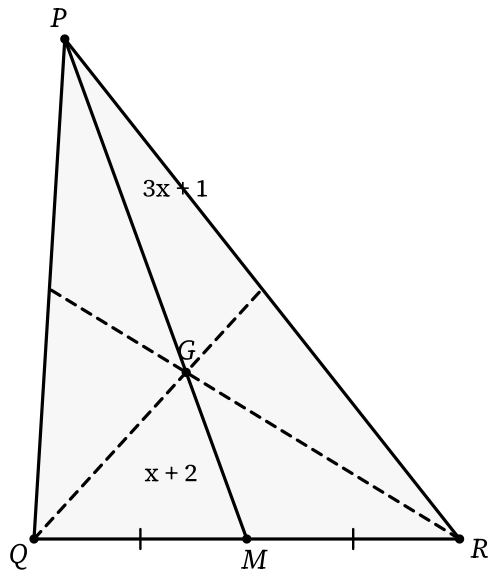
$CG = 2 \cdot GM = 8$, so $CM = 8 + 4 = 12$.

4. **Easy** In what kind of triangle (acute, right or obtuse) is the orthocenter at one of the vertices?

Show answer

A right triangle: the two legs are altitudes, and they meet at the right-angle vertex.

5. **Medium** In the figure, G is the centroid of $\triangle PQR$ and $\overline{PM}$ is a median. If $PG = 3x + 1$ and $GM = x + 2$, find PM.



Show answer

$PG = 2 \cdot GM$, so $3x + 1 = 2x + 4$ and $x = 3$. Then $PG = 10$, $GM = 5$ and $PM = 15$.

6. **Medium** Find the centroid of the triangle with vertices $A(-2, 1)$, $B(4, 7)$ and $C(7, -2)$.

Show answer

$$\left(\frac{-2 + 4 + 7}{3}, \frac{1 + 7 - 2}{3}\right) = (3, 2)$$

7. **Medium** Explain why a median of a triangle divides it into two triangles with equal areas. Then explain why the triangle balances along a median, and why that makes the centroid the balance point.

Show answer

If $\overline{AM}$ is a median of $\triangle ABC$, the triangles $\triangle ABM$ and $\triangle ACM$ have equal bases $BM = MC$ and the same height, the distance from A to line BC. Since a triangle's area is $\frac{1}{2}bh$, their areas are equal. Equal areas alone do not guarantee balance; the triangle balances along the median because the median passes through the midpoint of every thin strip of the triangle parallel to $\overline{BC}$, so each strip balances on it. The triangle therefore balances along each of its three medians, and the only point on all three, the centroid, is where it balances on a pencil point.

8. **Medium** A triangular garden has corners at $(0, 0)$, $(30, 0)$ and $(12, 24)$, measured in feet. Ruth wants to put a birdbath at the garden's centroid. Find its coordinates.

Show answer

$$\left(\frac{0 + 30 + 12}{3}, \frac{0 + 0 + 24}{3}\right) = (14, 8)$$

9. **Medium** For the triangle with vertices $A(1, 1)$, $B(9, 1)$ and $C(4, 7)$, find (a) the length of the altitude from C to $\overline{AB}$ and (b) the area of the triangle.

Show answer

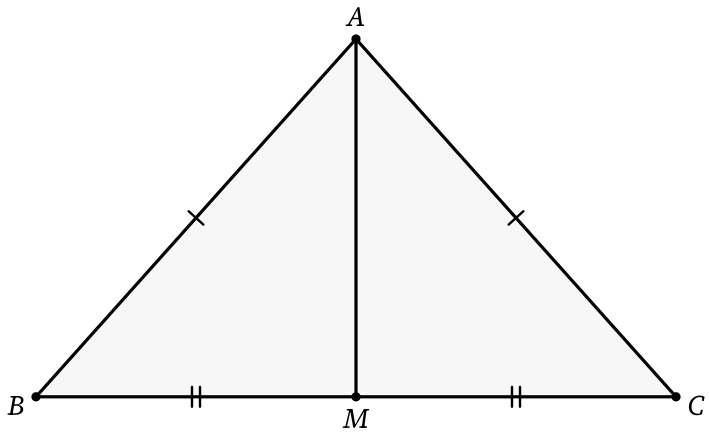
(a) $\overline{AB}$ lies on the line $y = 1$, so the altitude from C is vertical and has length $7 - 1 = 6$. (b) $AB = 8$, so the area is $\frac{1}{2}(8)(6) = 24$ square units.

10. **Hard** Find the orthocenter of the triangle with vertices $A(0, 0)$, $B(6, 0)$ and $C(4, 4)$.

Show answer

The altitude from C is $x = 4$. The slope of $\overline{BC}$ is -2, so the altitude from A is $y = \frac{1}{2}x$. They meet at $(4, 2)$. Check: the altitude from B is $y = -x + 6$, which also passes through $(4, 2)$.

11. **Hard** In the figure, $\overline{AB} \cong \overline{AC}$ and $\overline{AM}$ is a median of $\triangle ABC$. Prove that $\overline{AM}$ is also an altitude, that is, $\overline{AM} \perp \overline{BC}$.



Show answer

See the model proof.

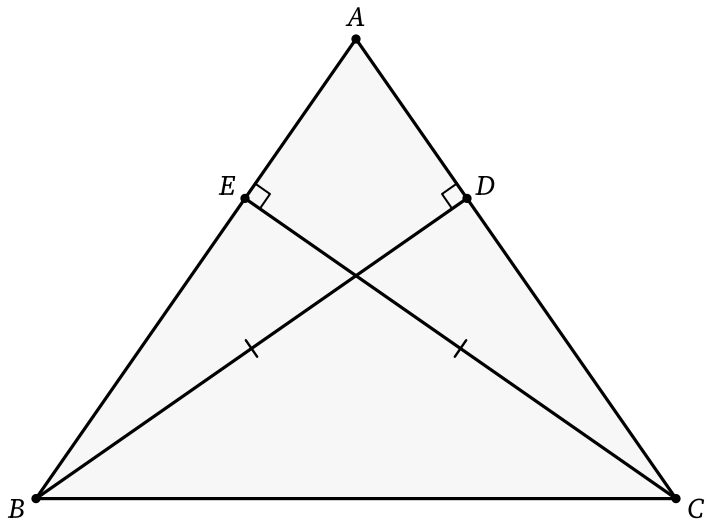
TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$; $\overline{AM}$ is a median of $\triangle ABC$

Prove: $\overline{AM} \perp \overline{BC}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$\overline{AM}$ is a median of $\triangle ABC$	Given
3	M is the midpoint of $\overline{BC}$	Definition of median
4	$\overline{BM} \cong \overline{CM}$	Definition of midpoint
5	$\overline{AM} \cong \overline{AM}$	Reflexive Property of Congruence
6	$\triangle ABM \cong \triangle ACM$	SSS
7	$\angle AMB \cong \angle AMC$	CPCTC
8	$\angle AMB$ and $\angle AMC$ form a linear pair	Definition of linear pair (B, M, C are collinear)
9	$\overline{AM} \perp \overline{BC}$	If two lines form a linear pair of congruent angles, they are perpendicular

12. **Hard** In the figure, $\overline{BD}$ and $\overline{CE}$ are altitudes of $\triangle ABC$, with D on $\overline{AC}$ and E on $\overline{AB}$, and $\overline{BD} \cong \overline{CE}$. Prove that $\overline{AB} \cong \overline{AC}$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{BD}$ and $\overline{CE}$ are altitudes of $\triangle ABC$, D on $\overline{AC}$, E on $\overline{AB}$; $\overline{BD} \cong \overline{CE}$

Prove: $\overline{AB} \cong \overline{AC}$

(1) $\overline{BD}$ and $\overline{CE}$ are altitudes
Given

(2) $\angle BDC$ and $\angle CEB$ are right angles
Definition of altitude and of perpendicular lines
from (1)

(3) $\triangle BDC$ and $\triangle CEB$ are right triangles
Definition of right triangle
from (2)

(4) $\overline{BD} \cong \overline{CE}$
Given

(5) $\overline{BC} \cong \overline{CB}$
Reflexive Property of Congruence

(6) $\triangle BDC \cong \triangle CEB$
HL

from (3), (4), (5)

(7) $\angle DCB \cong \angle EBC$

CPCTC

from (6)

(8) $\angle ACB \cong \angle ABC$

Same angles: D is on $\overline{AC}$ and E is on $\overline{AB}$

from (7)

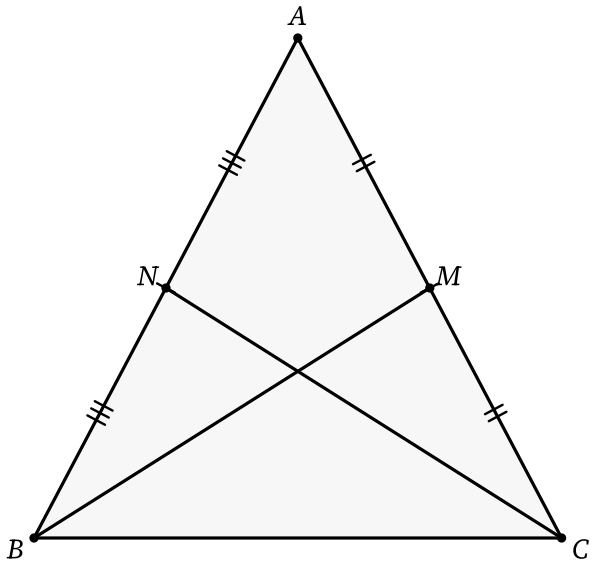
(9) $\overline{AB} \cong \overline{AC}$

Converse of the Isosceles Triangle Theorem

from (8)

∎

13. **Hard** In the figure, $\overline{AB} \cong \overline{AC}$, M is the midpoint of $\overline{AC}$ and N is the midpoint of $\overline{AB}$. Prove that the medians $\overline{BM}$ and $\overline{CN}$ are congruent. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{AC}$; M is the midpoint of $\overline{AC}$; N is the midpoint of $\overline{AB}$

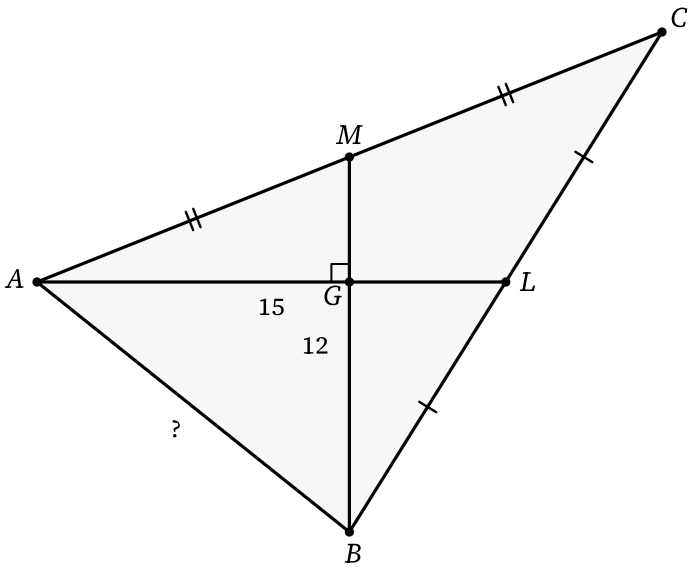
Prove: $\overline{BM} \cong \overline{CN}$

Since M is the midpoint of $\overline{AC}$, $AM = \frac{1}{2}AC$, and since N is the midpoint of $\overline{AB}$, $AN = \frac{1}{2}AB$ (definition of midpoint). We are given $AB = AC$, so $\frac{1}{2}AB = \frac{1}{2}AC$ by the Multiplication Property of Equality, and therefore $AN = AM$ by substitution.

Now compare $\triangle ABM$ and $\triangle ACN$. We have $\overline{AB} \cong \overline{AC}$ (given), $\angle BAM \cong \angle CAN$ (both are $\angle BAC$; Reflexive Property), and $\overline{AM} \cong \overline{AN}$ (just shown). So $\triangle ABM \cong \triangle ACN$ by SAS, and $\overline{BM} \cong \overline{CN}$ by CPCTC.

∎

14. **Challenge** In the figure, the medians $\overline{AL}$ and $\overline{BM}$ of $\triangle ABC$ are perpendicular and meet at the centroid G. If $AL = 15$ and $BM = 12$, find AB in simplest radical form.



Show answer

$AG = \frac{2}{3}(15) = 10$ and $BG = \frac{2}{3}(12) = 8$. $\triangle AGB$ has a right angle at G, so $AB = \sqrt{10^2 + 8^2} = \sqrt{164} = 2\sqrt{41} \approx 12.8$.

COMMON MISTAKES

- Confusing a median with an altitude. A median goes to the midpoint of the opposite side; an altitude goes to the opposite side at a right angle. They are the same segment only in special cases, such as from the vertex of an isosceles triangle.
- Using the two-thirds rule backward. The centroid is two-thirds of the way from the vertex, so the piece next to the vertex is the longer one: $AG = 2 \cdot GL$, not $GL = 2 \cdot AG$.
- Drawing an altitude of an obtuse triangle inside the triangle. From an acute vertex of an obtuse triangle, extend the opposite side and drop the perpendicular to the extension.
- Using the slope of a side instead of its negative reciprocal when writing the equation of an altitude. An altitude is perpendicular to the side, so the slopes multiply to -1.

> **Make it a game · Balance Point Duel.** Cut a triangle from a cereal box and mark where you think it balances. Draw its three medians on the back and balance it on a pencil tip. Closest guess wins a point. Then parent gives a median length, 15 centimeters, and you name the pieces the centroid makes (10 and 5). Explain why the medians meet.

5.4 The Triangle Midsegment Theorem

OBJECTIVES

- Identify the midsegments of a triangle and its midsegment triangle.
- Prove the Triangle Midsegment Theorem with coordinates.
- Use the theorem to find lengths, angle measures and perimeters, and to measure across an obstacle.

WATCH

Midpoints & Parallel Lines in Polygons (1 of 3: Any triangle)

Eddie Woo · 10:56

https://www.youtube.com/watch?v=eGY2rDFcAMY

> **WHEN TO WATCH**
> Before Example 1: the segment joining two midpoints of a triangle.

Note: Eddie Woo teaches in Australia: he says 'interval' for segment.

BACKUP VIDEO

Midpoints & Parallel Lines in Polygons (2 of 3: Any quadrilateral)

Eddie Woo · 7:56

https://www.youtube.com/watch?v=Ej_0PQ7wCzo

After the lesson, as enrichment: the midpoints of any quadrilateral (proved again in Lesson 13.3).

KEY TERMS

midsegment of a triangle
A segment that joins the midpoints of two sides of a triangle. Every triangle has three midsegments.

midsegment triangle
The triangle whose sides are the three midsegments of a triangle.

CORE IDEAS

Join the midpoints D and E of two sides $\overline{AB}$ and $\overline{AC}$ of a triangle, and you get a **midsegment** $\overline{DE}$. Draw a few triangles, measure, and you will notice two things every time: $\overline{DE}$ runs exactly parallel to the third side $\overline{BC}$, and it is exactly half as long. That is the **Triangle Midsegment Theorem**.

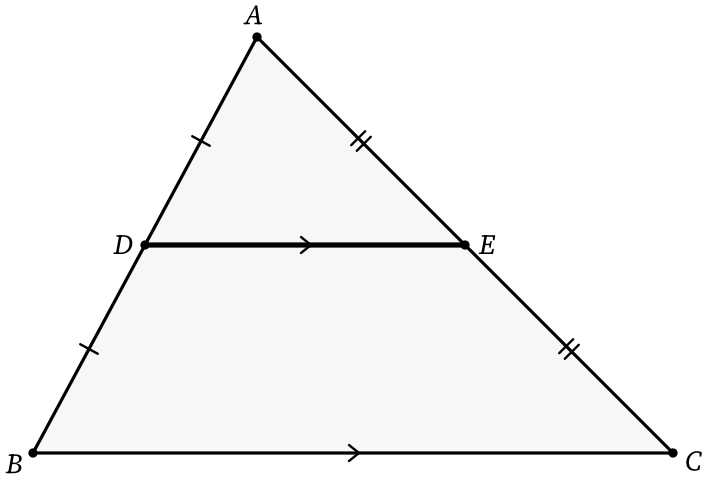
Measuring a few drawings suggests the theorem; it does not prove it. We prove it with coordinates. The trick is to choose coordinates that make the algebra easy: put B at the origin and C on the x-axis, and write every coordinate with a factor of 2, as $(2a, 2b)$ and $(2c, 0)$. Then the midpoints come out as whole expressions like (a, b), with no fractions. Any triangle can be placed this way, so the proof covers every triangle.

Why is the midsegment half as long and parallel? Picture the small triangle $\triangle ADE$ at the top. Every side of it is half the matching side of $\triangle ABC$: $AD = \frac{1}{2}AB$, $AE = \frac{1}{2}AC$, and, by the theorem, $DE = \frac{1}{2}BC$. So $\triangle ADE$ is a half-size copy of $\triangle ABC$ with the same angles. In Unit 8 you will call such triangles **similar**, and Euclid's VI.2 (for the parallel) together with VI.4 (for the length) will give a second proof.

Because $\overline{DE} \parallel \overline{BC}$, every angle fact about parallel lines from Unit 3 applies. For example, with transversal $\overleftrightarrow{AB}$, $\angle ADE$ and $\angle ABC$ are corresponding angles, so they are congruent.

The three midsegments together form the **midsegment triangle**. They cut the original triangle into four smaller triangles, and each small triangle has sides equal to half of the three original sides, so all four are congruent by SSS. The midsegment triangle has half the perimeter of the original triangle.

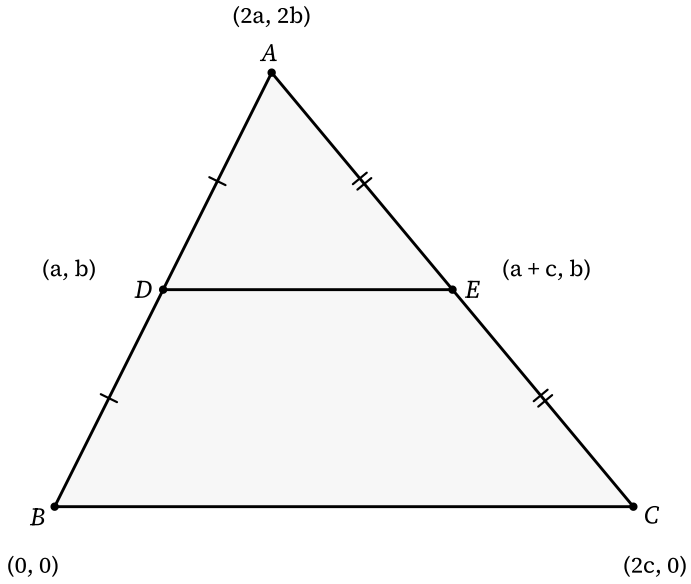
The theorem gives surveyors a way to measure across a pond, a marsh or a building they cannot walk through. To find AB, pick a point C from which you can walk straight to both A and B. Mark the midpoint D of $\overline{CA}$ and the midpoint E of $\overline{CB}$, measure DE on dry ground, and double it.



POSTULATES AND THEOREMS

THEOREM **Triangle Midsegment Theorem**

The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.



TWO-COLUMN PROOF

Given: D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{AC}$ in $\triangle ABC$

Prove: $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$

	STATEMENTS	REASONS
1	Place $B = (0, 0)$, $C = (2c, 0)$ and $A = (2a, 2b)$ with $c > 0$ and $b \neq 0$	Any triangle can be placed with one vertex at the origin and one side on the positive x-axis
2	$D = (a, b)$ and $E = (a + c, b)$	Given midpoints; Midpoint Formula

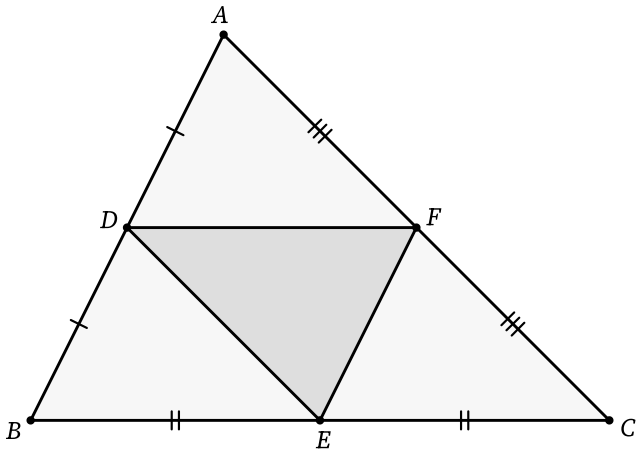
	STATEMENTS	REASONS
3	Slope of $\overline{DE} = \dfrac{b-b}{(a+c)-a} = 0$ and slope of $\overline{BC} = \dfrac{0-0}{2c-0} = 0$	Slope Formula
4	$\overline{DE} \parallel \overline{BC}$	Lines with equal slopes are parallel ($\overline{DE}$ lies on $y = b$, not on $y = 0$, since $b \neq 0$)
5	$DE = (a+c) - a = c$ and $BC = 2c - 0 = 2c$	Distance Formula (horizontal segments)
6	$DE = \frac{1}{2}BC$	Substitution Property of Equality
7	$\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$	Steps 4 and 6

∎

Proof idea with similar triangles (Unit 8): $\triangle ADE$ and $\triangle ABC$ share $\angle A$ and have sides in the ratio $1:2$, so they are similar, and the midsegment is half the third side.

COROLLARY **Midsegment Triangle Corollary**

The three midsegments of a triangle divide it into four congruent triangles, and the perimeter of the midsegment triangle is half the perimeter of the original triangle.



PARAGRAPH PROOF

Given: D, E, F are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of $\triangle ABC$

Prove: $\triangle ADF \cong \triangle DBE \cong \triangle FEC \cong \triangle EFD$, and $DE + EF + FD = \frac{1}{2}(AB + BC + CA)$

By the definition of midpoint, $AD = DB = \frac{1}{2}AB$, $BE = EC = \frac{1}{2}BC$ and $CF = FA = \frac{1}{2}CA$. By the Triangle Midsegment Theorem, $EF = \frac{1}{2}AB$, $FD = \frac{1}{2}BC$ and $DE = \frac{1}{2}CA$.

So each of the four triangles has one side equal to $\frac{1}{2}AB$, one equal to $\frac{1}{2}BC$ and one equal to $\frac{1}{2}CA$: in $\triangle ADF$ they are AD, DF, FA; in $\triangle DBE$, DB, BE, ED; in $\triangle FEC$, FE, EC, CF; in $\triangle EFD$, EF, FD, DE. By SSS, $\triangle ADF \cong \triangle DBE \cong \triangle FEC \cong \triangle EFD$.

Adding the three midsegment equations, $DE + EF + FD = \frac{1}{2}CA + \frac{1}{2}AB + \frac{1}{2}BC = \frac{1}{2}(AB + BC + CA)$.

∎

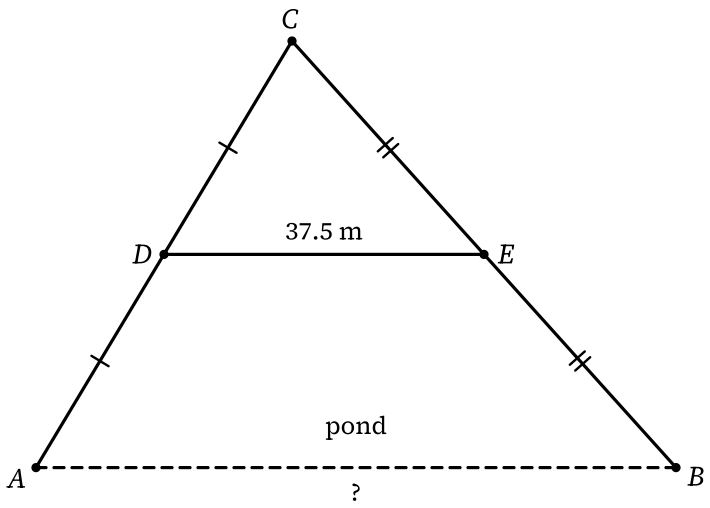
KEY FORMULAS

$$\overline{DE} \parallel \overline{BC}, \qquad DE = \frac{1}{2}BC, \qquad BC = 2 \cdot DE$$

Euclid connection. The parallel half of the Midsegment Theorem is a special case of Euclid's VI.2: a line parallel to one side of a triangle cuts the other two sides proportionally, and a line that cuts them proportionally is parallel to the third side. The 'half as long' half then follows from VI.4, since $\triangle ADE$ and $\triangle ABC$ are equiangular. Euclid proves VI.2 with areas of triangles on the same base and between the same parallels (I.37–I.39) rather than with coordinates; you will meet it in Lesson 8.5.

Practice habit. Recite the theorem with both conclusions every time: "parallel to the third side AND half as long." Before using it, mark both midpoints with tick marks and check that the midsegment is paired with the third side, the one it does not touch.

Example 1. To find the distance across a pond from stake A to stake B, Samuel sets a third stake C from which he can walk straight to both. He marks the midpoint D of $\overline{CA}$ and the midpoint E of $\overline{CB}$ and measures $DE = 37.5$ meters. How far is it across the pond from A to B?

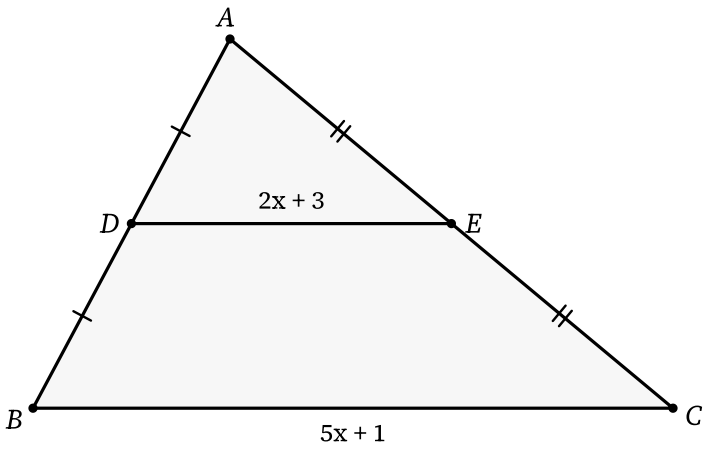


Show solution

1. $\overline{DE}$ joins the midpoints of two sides of $\triangle CAB$, so it is a midsegment, and the third side is $\overline{AB}$.
2. By the Triangle Midsegment Theorem, $DE = \frac{1}{2}AB$, so $AB = 2 \cdot DE$.
3. $AB = 2(37.5) = 75$ meters.

Answer: $AB = 75$ meters

Example 2. In $\triangle ABC$, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$. If $DE = 2x + 3$ and $BC = 5x + 1$, find DE and BC.



Show solution

1. $\overline{DE}$ is a midsegment, so $BC = 2 \cdot DE$.
2. Write the equation: $5x + 1 = 2(2x + 3) = 4x + 6$.
3. Subtract $4x$ and 1 from both sides: $x = 5$.
4. $DE = 2(5) + 3 = 13$ and $BC = 5(5) + 1 = 26$. Check: $26 = 2 \times 13$.

Answer: $DE = 13$ and $BC = 26$

PRACTICE

1. Easy $\overline{DE}$ is a midsegment of $\triangle ABC$ joining the midpoints of $\overline{AB}$ and $\overline{AC}$, and $BC = 18$ cm. Find DE.

   Show answer

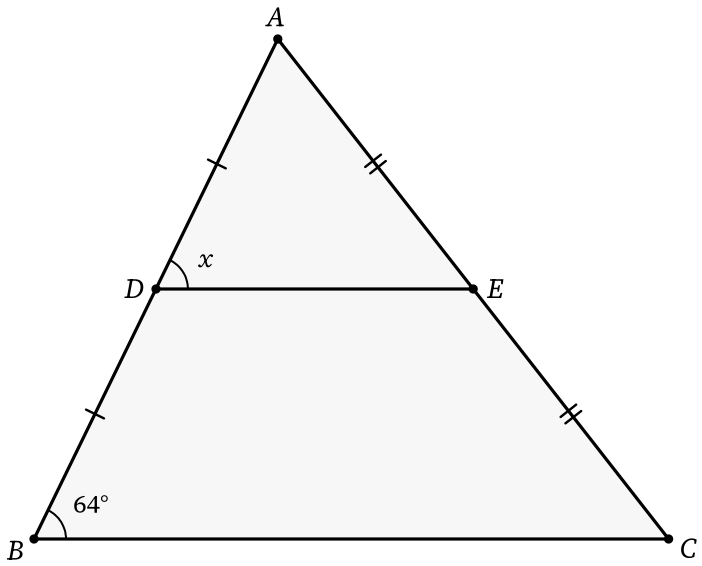
   $DE = \frac{1}{2}(18) = 9$ cm

2. Easy $\overline{DE}$ is a midsegment of $\triangle ABC$ joining the midpoints of $\overline{AB}$ and $\overline{AC}$, and $DE = 7.5$ in. Find BC.

   Show answer

   $BC = 2(7.5) = 15$ in.

3. **Easy** In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, and $m\angle ABC = 64°$. Find $m\angle ADE$.



Show answer

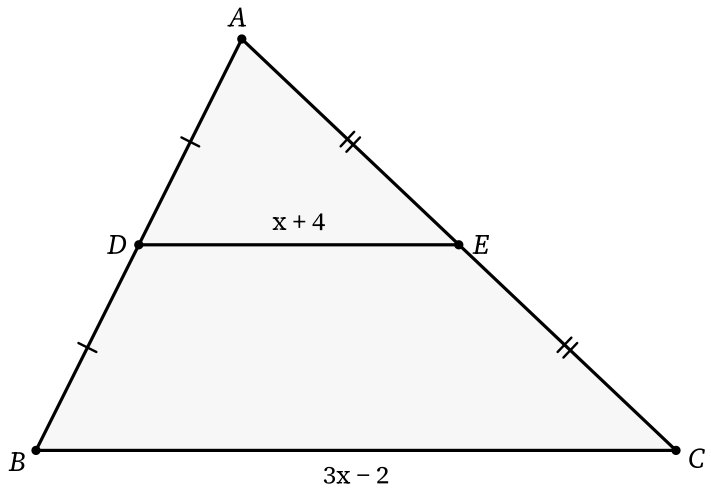
$\overline{DE} \parallel \overline{BC}$ by the Midsegment Theorem, so the corresponding angles $\angle ADE$ and $\angle ABC$ are congruent: $m\angle ADE = 64°$.

4. **Easy** A triangular stained-glass pane has sides of 10 in., 14 in. and 16 in. The artist sets strips of lead along its three midsegments. What total length of lead does this take?

Show answer

The midsegments are 5, 7 and 8 in., a total of 20 in. (half the perimeter of 40 in.).

5. **Medium** In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, $DE = x + 4$ and $BC = 3x - 2$. Find BC.



Show answer

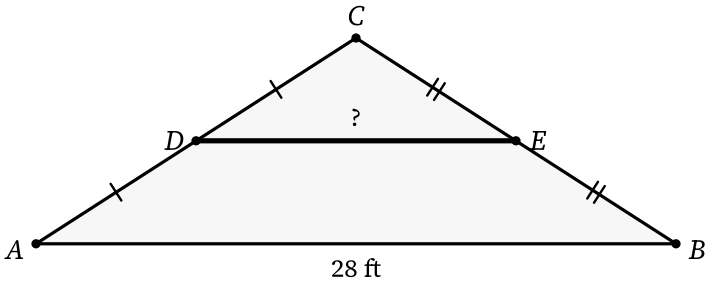
$3x - 2 = 2(x + 4) = 2x + 8$, so $x = 10$, $DE = 14$ and $BC = 28$.

6. **Medium** The vertices of $\triangle ABC$ are $A(2, 8)$, $B(0, 0)$ and $C(10, 2)$. D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$. Find DE in simplest radical form, and check that $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$.

Show answer

$D = (1, 4)$ and $E = (6, 5)$, so $DE = \sqrt{25 + 1} = \sqrt{26}$. Check: both slopes are $\frac{1}{5}$, and $BC = \sqrt{100 + 4} = \sqrt{104} = 2\sqrt{26}$.

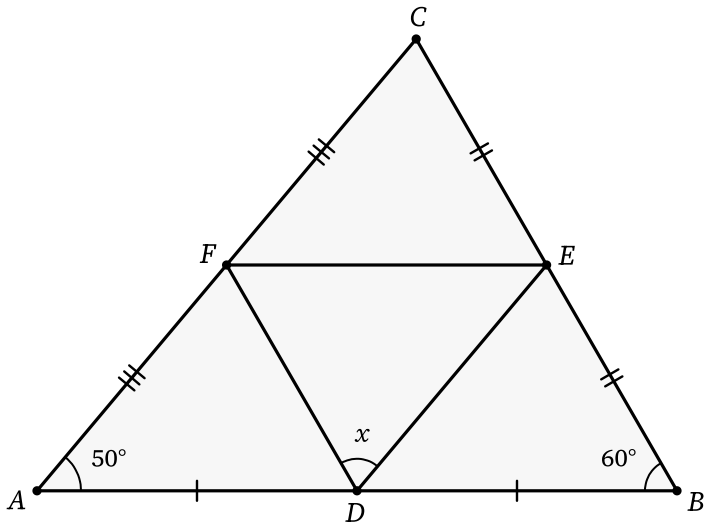
7. **Medium** A barn's roof truss is a triangle with a bottom chord 28 feet long. A horizontal collar tie is nailed between the midpoints of the two rafters, as in the figure. How long is the collar tie?



Show answer

The collar tie is a midsegment, so it is $\frac{1}{2}(28) = 14$ feet long.

8. **Medium** In the figure, D, E and F are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, $m\angle A = 50^\circ$ and $m\angle B = 60^\circ$. Find $m\angle EDF$.



Show answer

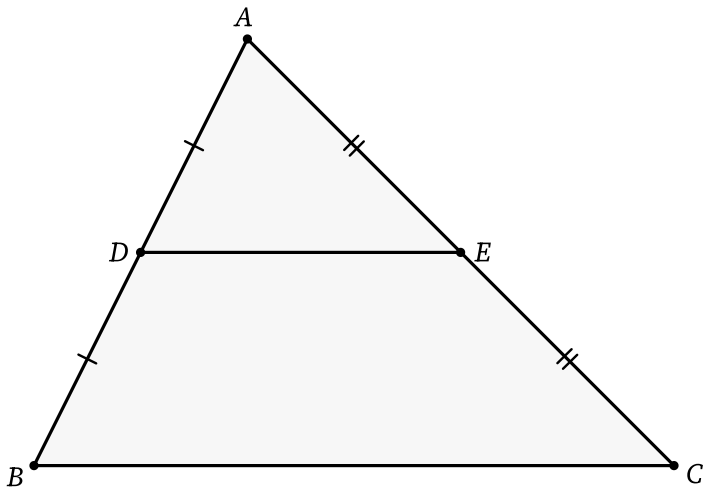
$\overline{DF} \parallel \overline{BC}$, so $m\angle ADF = m\angle B = 60^\circ$ (corresponding angles). $\overline{DE} \parallel \overline{AC}$, so $m\angle BDE = m\angle A = 50^\circ$. The three angles at D along line AB add to 180°, so $m\angle EDF = 180^\circ - 60^\circ - 50^\circ = 70^\circ$.

9. **Medium** In the coordinate proof of the Midsegment Theorem we placed the vertices at $(0, 0)$, $(2c, 0)$ and $(2a, 2b)$. Explain why this choice is allowed and why it is wise.

Show answer

It is allowed because any triangle can be slid and turned so that one vertex is at the origin and one side lies along the positive x-axis; the letters a, b, c can stand for any numbers, so no particular triangle is assumed. It is wise because the midpoint formula divides by 2, and the factors of 2 make the midpoints (a, b) and $(a + c, b)$ come out without fractions.

10. **Hard** In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$. Prove that $\angle ADE \cong \angle ABC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

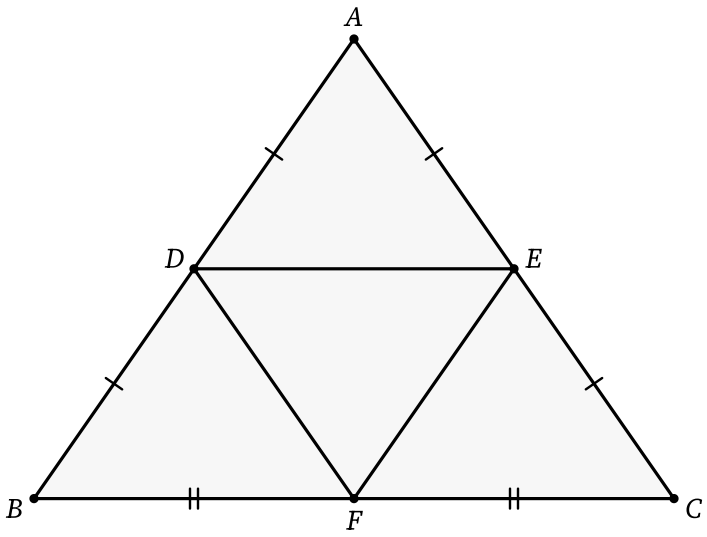
Given: D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{AC}$

Prove: $\angle ADE \cong \angle ABC$

	STATEMENTS	REASONS
1	D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{AC}$	Given
2	$\overline{DE}$ is a midsegment of $\triangle ABC$	Definition of midsegment
3	$\overline{DE} \parallel \overline{BC}$	Triangle Midsegment Theorem
4	$\angle ADE \cong \angle ABC$	Corresponding Angles Postulate (transversal $\overleftrightarrow{AB}$)

∎

11. Hard In the figure, $\overline{AB} \cong \overline{AC}$ in $\triangle ABC$, and D, E, F are the midpoints of $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. Prove that $\triangle DEF$ is isosceles. Write a paragraph proof.



Show answer

See the model proof: $FD = \frac{1}{2}AC$ and $FE = \frac{1}{2}AB$.

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{AC}$; D, E, F are the midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{BC}$

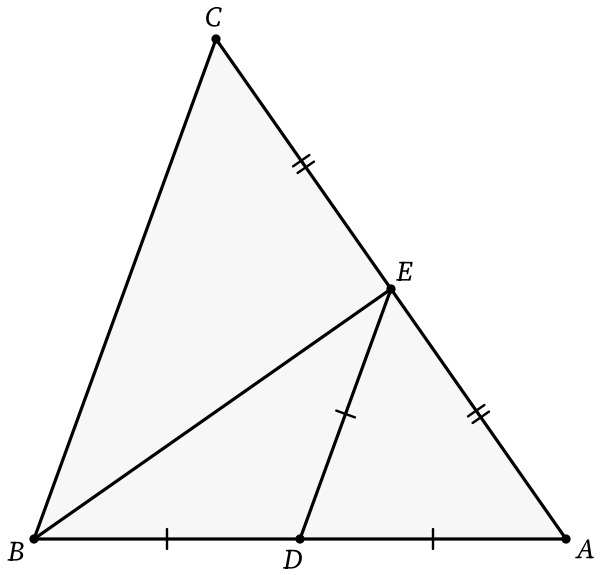
Prove: $\triangle DEF$ is isosceles

$\overline{FD}$ joins the midpoints of $\overline{BC}$ and $\overline{BA}$, so it is a midsegment of $\triangle ABC$, and by the Triangle Midsegment Theorem $FD = \frac{1}{2}AC$. Likewise $\overline{FE}$ joins the midpoints of $\overline{CB}$ and $\overline{CA}$, so $FE = \frac{1}{2}AB$.

We are given $AB = AC$, so $\frac{1}{2}AB = \frac{1}{2}AC$ by the Multiplication Property of Equality, and by substitution $FE = FD$. A triangle with two congruent sides is isosceles, so $\triangle DEF$ is isosceles.

∎

12. **Hard** In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$ in $\triangle ABC$, and $\overline{DE} \cong \overline{DB}$. Prove that $\overrightarrow{BE}$ bisects $\angle ABC$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$; $\overline{DE} \cong \overline{DB}$

Prove: $\overrightarrow{BE}$ bisects $\angle ABC$

(1) D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$
Given

(2) $\overline{DE} \parallel \overline{BC}$
Triangle Midsegment Theorem
from (1)

(3) $\angle DEB \cong \angle EBC$
Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BE}$)
from (2)

(4) $\overline{DE} \cong \overline{DB}$
Given

(5) $\angle DBE \cong \angle DEB$
Isosceles Triangle Theorem (in $\triangle DBE$)
from (4)

(6) $\angle DBE \cong \angle EBC$
Transitive Property of Congruence
from (3), (5)

(7) $\overrightarrow{BE}$ bisects $\angle ABC$
Definition of angle bisector ($\angle DBE$ is $\angle ABE$, and E is in the interior of $\angle ABC$)
from (6)

∎

13. **Challenge** The midpoints of the sides of $\triangle ABC$ are $(2, 3)$ on $\overline{AB}$, $(5, 7)$ on $\overline{BC}$ and $(8, 1)$ on $\overline{CA}$. Find the coordinates of A, B and C.

Show answer

Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$. The midpoint conditions give $x_1 + x_2 = 4$, $x_2 + x_3 = 10$, $x_3 + x_1 = 16$, so $x_1 + x_2 + x_3 = 15$ and $x_1 = 5$, $x_2 = -1$, $x_3 = 11$. Likewise $y_1 + y_2 = 6$, $y_2 + y_3 = 14$, $y_3 + y_1 = 2$, so the sum is 11 and $y_1 = -3$, $y_2 = 9$, $y_3 = 5$. So $A(5, -3)$, $B(-1, 9)$, $C(11, 5)$. (Check: the midpoint of $\overline{AB}$ is $(2, 3)$.)

COMMON MISTAKES

- Pairing the midsegment with the wrong side. A midsegment is parallel to and half of the one side it does not touch; it has nothing simple to say about the two sides whose midpoints it joins.
- Doubling when you should halve, or halving when you should double. The midsegment is the short one: $DE = \frac{1}{2}BC$ and $BC = 2 \cdot DE$. Check that your midsegment comes out shorter.
- Calling any segment across a triangle a midsegment because it looks parallel. Both endpoints must be shown to be midpoints before the theorem applies.

> **Make it a game · Midsegment Prediction.** Draw a triangle and mark two midpoints. Your parent measures the longest side secretly and tells you its length. Predict the midsegment's length and direction before measuring, then check with a ruler. Next, parent gives vertices like (0,0), (8,0), (2,6), and you prove the midsegment is half the base with coordinates.

5.5 Indirect Proof and Inequalities in One Triangle

OBJECTIVES

- Write indirect proofs by assuming the opposite of the conclusion and reaching a contradiction.
- Prove and use the theorems that the longer side is opposite the larger angle (Euclid I.18) and the larger angle is opposite the longer side (I.19).
- Order the sides and the angles of a triangle from smallest to largest.
- Use the Exterior Angle Inequality (I.16) in proofs.

WATCH

Proof by Contradiction (1 of 2: How does it work?)

Eddie Woo · 8:34

https://www.youtube.com/watch?v=rV9esU9gHO8

> **WHEN TO WATCH**
> Before the first indirect proof: how proof by contradiction works.

Note: Neither video proves the side-angle inequalities (Euclid I.18 and I.19); teach those from the Core ideas.

BACKUP VIDEO

Tangent & Radius are Perpendicular (Proof by Contradiction)

Eddie Woo · 12:43

https://www.youtube.com/watch?v=a7nfrqyhU6w

As enrichment after the practice set: an indirect proof in geometry (it uses tangents from Lesson 10.3).

KEY TERMS

indirect proof

A proof that assumes temporarily that the conclusion is false and shows that this assumption leads to a contradiction; also called proof by contradiction (Latin *reductio ad absurdum*).

contradiction

A pair of statements that cannot both be true, such as $AB = AC$ and $AB \neq AC$, or a statement that contradicts a given, a definition, a postulate or a proved theorem.

Trichotomy Property

For any two numbers a and b, exactly one of these is true: $a < b$, $a = b$, or $a > b$.

Comparison Property of Inequality

If $a = b + c$ and $c > 0$, then $a > b$. This is Euclid's Common Notion 5: the whole is greater than the part.

Transitive Property of Inequality

If $a > b$ and $b > c$, then $a > c$.

CORE IDEAS

Sometimes the clearest way to show that something must be true is to show that it cannot be false. In an **indirect proof** you assume, just for the sake of argument, that the conclusion is false. Then you reason carefully, step by valid step, until you reach something impossible: a statement that contradicts a given, a definition, a postulate, a theorem, or itself. Since valid reasoning from a true starting point can never reach a falsehood, the only thing that can be wrong is your temporary assumption. So the conclusion must be true.

An indirect proof has three parts. (1) Assume temporarily that the conclusion is false, and say exactly what that means; if the conclusion is $AC > AB$, the opposite is "$AC = AB$ or $AC < AB$," and you must rule out both cases (the Trichotomy Property). (2) Reason to a contradiction. (3) Conclude that the assumption was false, so the conclusion is true. Euclid uses this method often, first in I.6.

Proofs about inequalities use a few new tools. The **Comparison Property** says the whole is greater than any of its parts: if $m\angle ABC = m\angle ABD + m\angle DBC$, then $m\angle ABC > m\angle ABD$. The **Transitive Property of Inequality** chains comparisons: if $a > b$ and $b > c$, then $a > c$. And from Lesson 4.1 you have the **Exterior Angle**

Inequality (Euclid I.16): an exterior angle of a triangle is greater than either remote interior angle.

Now the main theorem. In $\triangle ABC$, suppose $AC > AB$. Why should the angle opposite $\overline{AC}$, which is $\angle B$, be larger than the angle opposite $\overline{AB}$, which is $\angle C$? Euclid's idea: on the longer side $\overline{AC}$, cut off $\overline{AD}$ equal to the shorter side $\overline{AB}$. Triangle ABD is isosceles, so its base angles are equal. One of them is part of $\angle B$, so $\angle B$ is bigger than it; the other is an exterior angle of $\triangle BDC$, so it is bigger than $\angle C$. Chain the two comparisons and $\angle B > \angle C$.

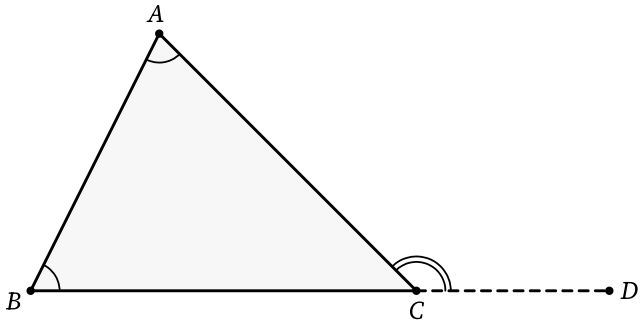
The converse, that the larger angle is opposite the longer side, is proved indirectly. If $\angle B$ is larger than $\angle C$ but AC were not longer than AB, then either $AC = AB$, which would make the angles equal (Isosceles Triangle Theorem), or $AC < AB$, which would make $\angle B$ smaller (the theorem just proved). Both contradict what we know.

Together these theorems say the order of the sides matches the order of the opposite angles. The longest side is opposite the largest angle, and the shortest side is opposite the smallest angle. In particular, the hypotenuse is the longest side of a right triangle. Keep the word **opposite** in mind: an angle is matched with the side it does not touch.

POSTULATES AND THEOREMS

THEOREM **Exterior Angle Inequality** Euclid I.16

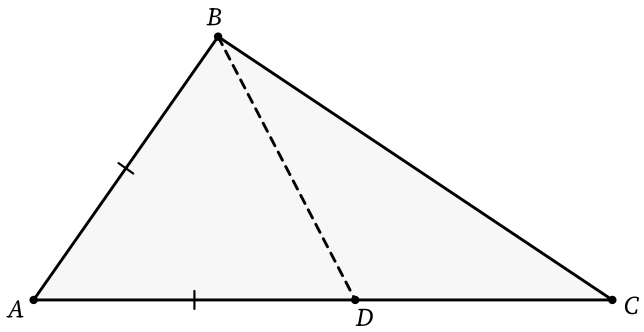
The measure of an exterior angle of a triangle is greater than the measure of either of its remote interior angles.



Proved in Lesson 4.1. Euclid proves it without the parallel postulate; it also follows at once from the Exterior Angle Theorem, since the exterior angle equals the sum of two remote interior angles, each with positive measure.

THEOREM **Triangle Longer Side Theorem** Euclid I.18

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.



TWO-COLUMN PROOF

Given: $\triangle ABC$ with $AC > AB$

Prove: $m\angle ABC > m\angle C$

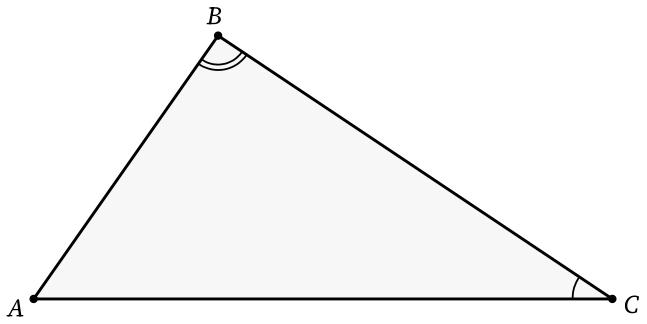
	STATEMENTS	REASONS
1	$AC > AB$	Given
2	There is a point D on $\overline{AC}$ with $AD = AB$, and D is between A and C	Ruler Postulate (since $AB < AC$); Euclid I.3
3	Draw $\overline{BD}$	Through two points there is exactly one line
4	$\overline{AD} \cong \overline{AB}$	Definition of congruent segments
5	$m\angle ABD = m\angle ADB$	Isosceles Triangle Theorem (in $\triangle ABD$)
6	$m\angle ADB > m\angle C$	Exterior Angle Inequality ($\angle ADB$ is an exterior angle of $\triangle BDC$)
7	$m\angle ABC = m\angle ABD + m\angle DBC$	Angle Addition Postulate (D is in the interior of $\angle ABC$)
8	$m\angle ABC > m\angle ABD$	Comparison Property of Inequality
9	$m\angle ABC > m\angle ADB$	Substitution (steps 5 and 8)
10	$m\angle ABC > m\angle C$	Transitive Property of Inequality (steps 9 and 6)

∎

Euclid I.18: "In any triangle the greater side subtends the greater angle."

THEOREM **Triangle Larger Angle Theorem** Euclid I.19

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.



INDIRECT PROOF

Given: $\triangle ABC$ with $m\angle B > m\angle C$

Prove: $AC > AB$

Assume temporarily that AC is not greater than AB. By the Trichotomy Property, then either $AC = AB$ or $AC < AB$.

Case 1: $AC = AB$. Then $\triangle ABC$ is isosceles, and by the Isosceles Triangle Theorem the angles opposite these sides are congruent: $m\angle B = m\angle C$. This contradicts the given fact that $m\angle B > m\angle C$.

Case 2: $AC < AB$. Then $\overline{AB}$ is the longer side, and by the Triangle Longer Side Theorem the angle opposite it is larger: $m\angle C > m\angle B$. This also contradicts $m\angle B > m\angle C$.

Both cases lead to contradictions, so the assumption is false. Therefore $AC > AB$.

∎

Euclid I.19: "In any triangle the greater angle is subtended by the greater side." Euclid proves it indirectly with the same two cases, using I.5 and I.18.

COROLLARY **Longest Side of a Right or Obtuse Triangle**

In a right triangle the hypotenuse is the longest side. In an obtuse triangle the side opposite the obtuse angle is the longest side.

PARAGRAPH PROOF

Given: $\triangle ABC$ with $m\angle C \geq 90°$

Prove: $AB > AC$ and $AB > BC$

By the Triangle Angle Sum Theorem, $m\angle A + m\angle B = 180^\circ - m\angle C \leq 90^\circ$. Each of $m\angle A$ and $m\angle B$ is positive, so each is less than 90°, and therefore less than $m\angle C$.

Since $m\angle C > m\angle B$, the Triangle Larger Angle Theorem gives $AB > AC$; since $m\angle C > m\angle A$, it gives $AB > BC$. So $\overline{AB}$, the side opposite the right or obtuse angle, is the longest side.

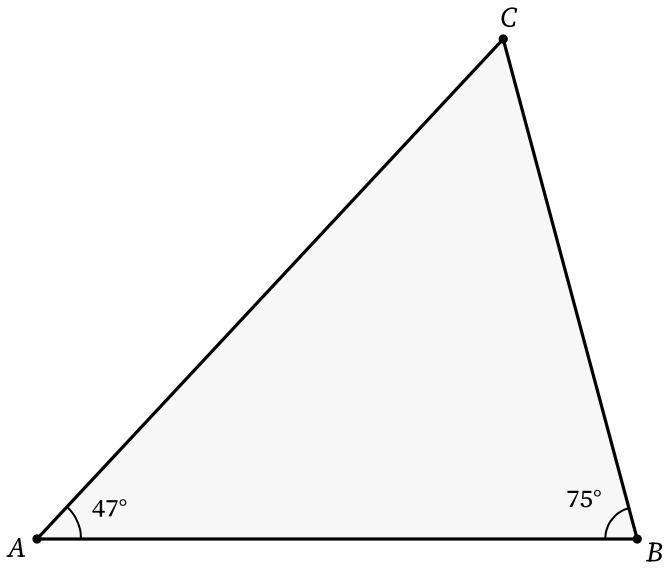
∎

From history. Mathematicians call proof by contradiction *reductio ad absurdum*, "reduction to the absurd." Euclid uses it many times in the Elements, beginning with I.6, the converse of the bridge of asses. In 1733 the Italian priest and mathematician Girolamo Saccheri tried to prove Euclid's parallel postulate this way, by assuming it false and hunting for a contradiction; he never found a true one, and a century later mathematicians understood why.

Euclid connection. This lesson follows Euclid closely. I.16 gives the Exterior Angle Inequality; I.18 cuts off from the greater side a part equal to the less (using I.3) and compares angles with I.5 and I.16, exactly as in our proof; and I.19 proves the converse indirectly, ruling out "equal" by I.5 and "less" by I.18.

Practice habit. Recite I.18 and I.19 together with the key word: "the longer side is opposite the larger angle, and the larger angle is opposite the longer side." When you begin an indirect proof, write "Assume temporarily that ..." and list every case of the opposite, including the equal case.

Example 1. The Miller family's triangular pasture has corners A, B and C. The fence lines meet at A at an angle of $47°$ and at B at an angle of $75°$. List the three sides from shortest to longest. Which side needs the most fencing?

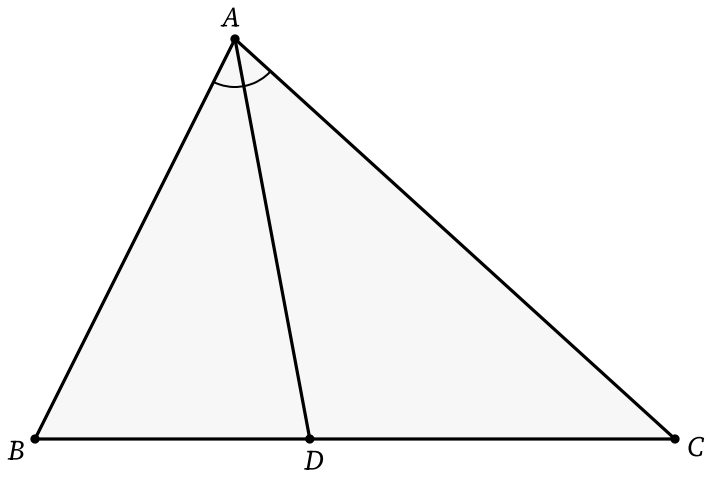


Show solution

1. Find the third angle: $m\angle C = 180° - 47° - 75° = 58°$.
2. Order the angles: $m\angle A = 47° < m\angle C = 58° < m\angle B = 75°$.
3. Match each angle with its opposite side: $\angle A$ is opposite $\overline{BC}$, $\angle C$ is opposite $\overline{AB}$, $\angle B$ is opposite $\overline{AC}$.
4. By the Triangle Larger Angle Theorem, the sides are in the same order: $BC < AB < AC$.

Answer: $BC < AB < AC$; side $\overline{AC}$ needs the most fencing.

Example 2. $\triangle ABC$ is scalene (no two sides are congruent), and $\overrightarrow{AD}$ bisects $\angle BAC$ with D on $\overline{BC}$. Prove that $\overline{AD}$ is not perpendicular to $\overline{BC}$.



Show solution

1. The conclusion is a "not" statement, which is a signal to try an indirect proof: assume $\overline{AD} \perp \overline{BC}$.
2. With the right angles, the bisected angle and the shared side, the two small triangles would be congruent by ASA.
3. Then $\overline{AB} \cong \overline{AC}$, which contradicts "scalene."

INDIRECT PROOF

Given: $\triangle ABC$ is scalene; $\overrightarrow{AD}$ bisects $\angle BAC$, D on $\overline{BC}$
Prove: $\overline{AD}$ is not perpendicular to $\overline{BC}$
Assume temporarily that $\overline{AD} \perp \overline{BC}$.

Then $\angle ADB$ and $\angle ADC$ are right angles (definition of perpendicular lines), so $\angle ADB \cong \angle ADC$ (all right angles are congruent). Since $\overrightarrow{AD}$ bisects $\angle BAC$, $\angle BAD \cong \angle CAD$ (definition of angle bisector). Also $\overline{AD} \cong \overline{AD}$ (Reflexive Property). So $\triangle ABD \cong \triangle ACD$ by ASA, and $\overline{AB} \cong \overline{AC}$ by CPCTC.

But $\triangle ABC$ is scalene, so no two of its sides are congruent. This is a contradiction.

Therefore the assumption is false, and $\overline{AD}$ is not perpendicular to $\overline{BC}$.

∎

Answer: See the model proof.

PRACTICE

1. **Easy** In $\triangle ABC$, $AB = 9$, $BC = 12$ and $AC = 7$. Name the largest angle of the triangle.

   Show answer

   $\angle A$, the angle opposite the longest side $\overline{BC}$.

2. **Easy** In $\triangle PQR$, $m\angle P = 40°$, $m\angle Q = 65°$ and $m\angle R = 75°$. Name the longest side.

   Show answer

   $\overline{PQ}$, the side opposite the largest angle $\angle R$.

3. **Easy** Write the first sentence of an indirect proof of the statement "$m\angle A < 90°$."

   Show answer

   Assume temporarily that $m\angle A \geq 90°$ (that is, $m\angle A = 90°$ or $m\angle A > 90°$).

4. **Easy** An exterior angle of a triangle measures $110°$. Can one of its two remote interior angles measure $115°$?

   Show answer

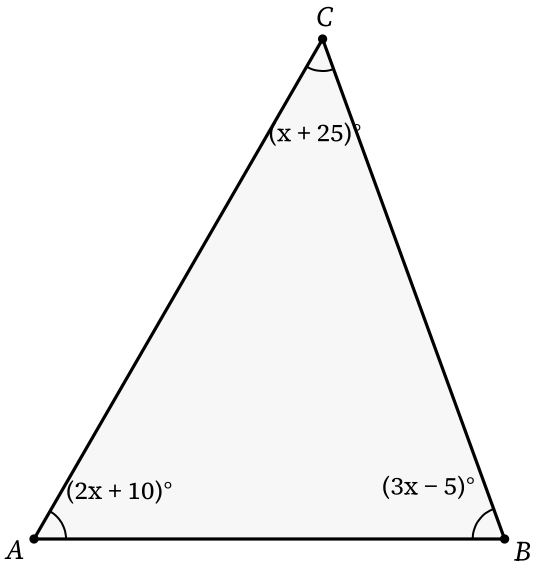
   No. By the Exterior Angle Inequality, each remote interior angle is less than $110°$.

5. **Medium** In $\triangle DEF$, $DE = 15$, $EF = 8$ and $DF = 13$. List the angles from smallest to largest.

   Show answer

   $\angle D$ (opposite 8), $\angle E$ (opposite 13), $\angle F$ (opposite 15).

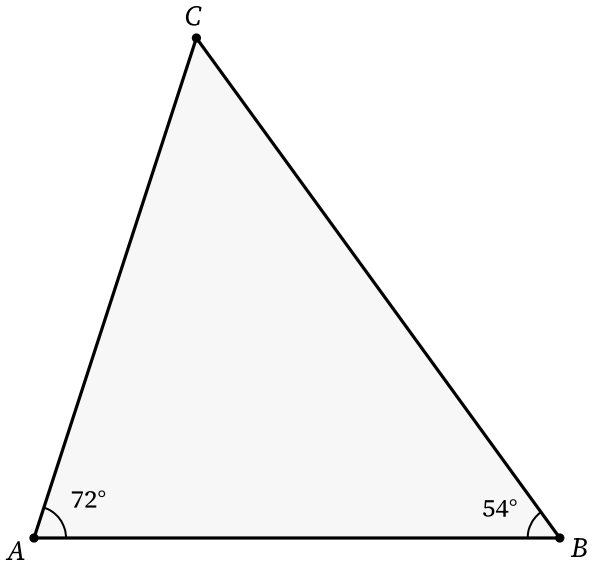
6. **Medium** In the figure, $m\angle A = (2x + 10)^\circ$, $m\angle B = (3x - 5)^\circ$ and $m\angle C = (x + 25)^\circ$. Find x, then name the longest side of $\triangle ABC$.



Show answer

$6x + 30 = 180$, so $x = 25$, and the angles are 60°, 70° and 50°. The largest angle is $\angle B$, so the longest side is $\overline{AC}$.

7. **Medium** Three villages A, B and C are joined by straight roads, as in the figure. The roads meet at A at an angle of $72°$ and at B at an angle of $54°$. Which road is the longest?



Show answer

$m\angle C = 180° - 72° - 54° = 54°$. The largest angle is $\angle A$, so the longest road is $\overline{BC}$. (Roads $\overline{AC}$ and $\overline{AB}$ are equal, since $\angle B \cong \angle C$.)

8. **Medium** In $\triangle ABC$, $AB = 10$ and $BC = 5$. Grace concludes that $m\angle C = 2 \cdot m\angle A$. Explain what you can and cannot conclude about $\angle C$ and $\angle A$.

Show answer

You can conclude only that $m\angle C > m\angle A$, because $\angle C$ is opposite the longer side $\overline{AB}$ (Triangle Longer Side Theorem). The theorem gives the order of the angles, not their ratio; doubling a side does not double the opposite angle (the angles of a triangle must still total $180°$).

9. **Medium** Write an indirect proof that a triangle cannot have two right angles.

Show answer

See the model proof.

INDIRECT PROOF

Given: $\triangle ABC$

Prove: $\triangle ABC$ does not have two right angles

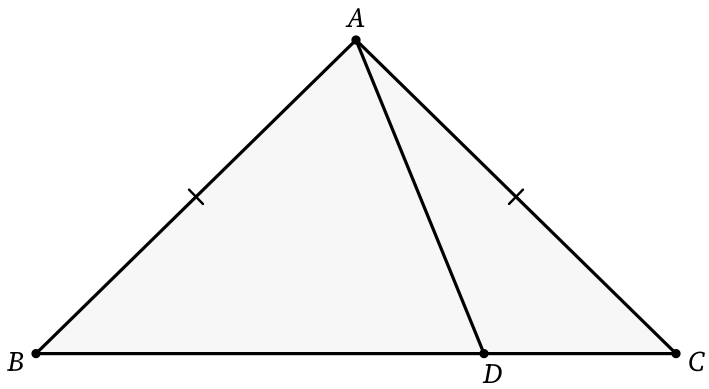
Assume temporarily that $\triangle ABC$ has two right angles, say $m\angle A = 90^\circ$ and $m\angle B = 90^\circ$.

By the Triangle Angle Sum Theorem, $m\angle A + m\angle B + m\angle C = 180^\circ$, so $90^\circ + 90^\circ + m\angle C = 180^\circ$ and $m\angle C = 0^\circ$. But every angle of a triangle has a positive measure. This is a contradiction.

Therefore the assumption is false, and a triangle cannot have two right angles. (Euclid I.17 says more: any two angles of a triangle together are less than two right angles.)

∎

10. **Hard** In the figure, $\overline{AB} \cong \overline{AC}$ and D is between B and C. Prove that $AD < AB$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$; D is between B and C

Prove: $AD < AB$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$m\angle B = m\angle C$	Isosceles Triangle Theorem
3	D is between B and C	Given
4	$m\angle ADB > m\angle C$	Exterior Angle Inequality ($\angle ADB$ is an exterior angle of $\triangle ADC$)
5	$m\angle ADB > m\angle B$	Substitution (steps 2 and 4)
6	$AD < AB$	Triangle Larger Angle Theorem (in $\triangle ABD$, $\overline{AB}$ is opposite $\angle ADB$ and $\overline{AD}$ is opposite $\angle B$)

∎

11. Hard Write an indirect proof: if two angles of a triangle are not congruent, then the sides opposite them are not congruent. Use $\triangle ABC$ with $\angle B \ncong \angle C$.

Show answer

See the model proof.

INDIRECT PROOF

Given: $\triangle ABC$ with $\angle B \ncong \angle C$

Prove: $\overline{AC} \ncong \overline{AB}$

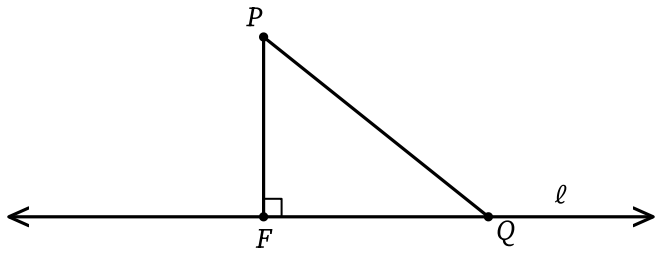
Assume temporarily that $\overline{AC} \cong \overline{AB}$.

Then $\triangle ABC$ is isosceles, and by the Isosceles Triangle Theorem the angles opposite these sides are congruent. The angle opposite $\overline{AC}$ is $\angle B$ and the angle opposite $\overline{AB}$ is $\angle C$, so $\angle B \cong \angle C$.

This contradicts the given fact that $\angle B \ncong \angle C$. Therefore the assumption is false, and $\overline{AC} \ncong \overline{AB}$.

∎

12. **Hard** In the figure, $\overline{PF} \perp \ell$ at F, and Q is any other point on ℓ. Prove that $PQ > PF$, so the perpendicular segment is the shortest segment from P to ℓ.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{PF} \perp \ell$ at F; Q is on ℓ, $Q \neq F$

Prove: $PQ > PF$

	STATEMENTS	REASONS
1	$\overline{PF} \perp \ell$ at F	Given
2	$m\angle PFQ = 90°$	Definition of perpendicular lines
3	$m\angle PFQ + m\angle FPQ + m\angle PQF = 180°$	Triangle Angle Sum Theorem
4	$m\angle FPQ + m\angle PQF = 90°$	Substitution and Subtraction Property of Equality
5	$m\angle PQF < 90°$	Comparison Property of Inequality ($m\angle FPQ > 0$)
6	$m\angle PFQ > m\angle PQF$	Substitution (steps 2 and 5)
7	$PQ > PF$	Triangle Larger Angle Theorem ($\overline{PQ}$ is opposite $\angle PFQ$ and $\overline{PF}$ is opposite $\angle PQF$)

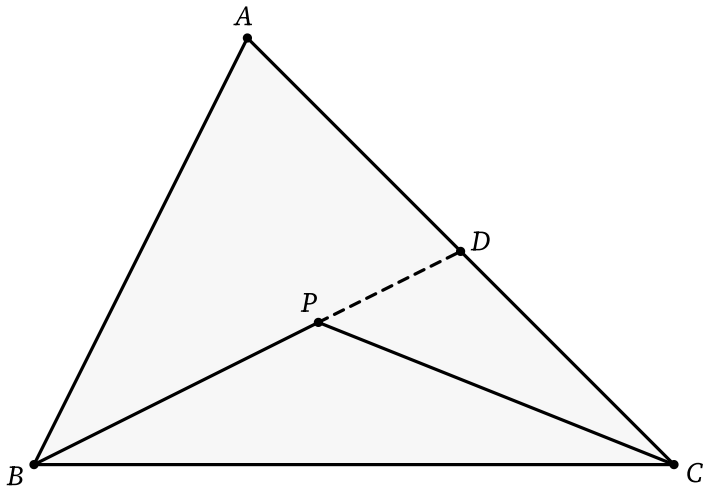
∎

13. Hard In $\triangle ABC$, $AB = 2x + 3$ and $BC = 5x - 9$, and $m\angle C > m\angle A$. Find all possible values of x.

Show answer

$\angle C$ is opposite $\overline{AB}$ and $\angle A$ is opposite $\overline{BC}$, so $AB > BC$: $2x + 3 > 5x - 9$, giving $x < 4$. Also $BC > 0$: $5x - 9 > 0$, giving $x > 1.8$. So $1.8 < x < 4$.

14. **Challenge** In the figure, P is a point inside $\triangle ABC$. Prove that $m\angle BPC > m\angle BAC$. (Hint: extend $\overline{BP}$ to meet $\overline{AC}$ at D, and use the Exterior Angle Inequality twice. This is part of Euclid I.21.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: P is in the interior of $\triangle ABC$

Prove: $m\angle BPC > m\angle BAC$

	STATEMENTS	REASONS
1	P is in the interior of $\triangle ABC$	Given
2	Extend $\overline{BP}$ to meet $\overline{AC}$ at D, with P between B and D	A ray from a vertex through an interior point crosses the opposite side
3	$m\angle BPC > m\angle PDC$	Exterior Angle Inequality ($\angle BPC$ is an exterior angle of $\triangle PDC$)
4	$m\angle BDC > m\angle BAD$	Exterior Angle Inequality ($\angle BDC$ is an exterior angle of $\triangle ABD$)
5	$\angle PDC$ is $\angle BDC$, and $\angle BAD$ is $\angle BAC$	Same angles (P is on $\overrightarrow{DB}$ and D is on $\overrightarrow{AC}$)

	STATEMENTS	REASONS
6	$m\angle BPC > m\angle BAC$	Substitution and Transitive Property of Inequality (steps 3, 4 and 5)

∎

COMMON MISTAKES

- Negating a statement incorrectly at the start of an indirect proof. The opposite of $AC > AB$ is $AC \leq AB$ (equal or less), not just $AC < AB$; forgetting the equal case leaves the proof incomplete.
- Matching an angle with a side it touches. Each angle is paired with the side opposite it, the one side not forming the angle; say "opposite" out loud as you match.
- Reading the order of sides from the look of the figure. Order sides only from given measures or proved facts; a figure can be drawn with misleading proportions.
- Thinking the theorems give ratios. They give only order: if one side is twice another, the opposite angle is larger, but not necessarily twice as large.

> **Make it a game · Assume the Opposite.** Prove each claim by contradiction, 60 seconds apiece: a triangle cannot have two right angles; a triangle cannot have two obtuse angles. Begin each with 'Assume the opposite' and reach a clash. Then parent draws a triangle with angles 40, 60 and 80 degrees, and you order the sides from shortest to longest without measuring.

5.6 The Triangle Inequality and the Hinge Theorem

OBJECTIVES

- Prove the Triangle Inequality Theorem and use it to decide whether three lengths can form a triangle.
- Find the range of possible lengths for the third side of a triangle.
- Prove and apply the Hinge Theorem (SAS Inequality) and its converse (SSS Inequality).

WATCH

Hinge Theorem (Geometry)

Mario's Math Tutoring · 3:42

https://www.youtube.com/watch?v=aEecHBieC1E

> **WHEN TO WATCH**
> With the Hinge Theorem: the theorem and its converse, with worked examples comparing sides and angles.

Note: The proof of the Hinge Theorem is taught from the Core ideas.

BACKUP VIDEO

Triangle Inequality Theorem Possible Values of x

Mario's Math Tutoring · 4:28

https://www.youtube.com/watch?v=RZkOM7jWf7o

With the Triangle Inequality Theorem: why any two sides must add to more than the third, then an algebra example.

KEY TERMS

Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

included angle

The angle formed by two sides of a triangle, at the vertex they share.

Hinge Theorem (SAS Inequality)

If two sides of one triangle are congruent to two sides of another, and the included angle of the first is larger, then the third side of the first is longer.

Converse of the Hinge Theorem (SSS Inequality)

If two sides of one triangle are congruent to two sides of another, and the third side of the first is longer, then the included angle of the first is larger.

CORE IDEAS

Can any three sticks form a triangle? Try 3 in., 4 in. and 9 in.: lay the 9-inch stick down and swing the other two from its ends. Even pointing straight at each other, they reach only $3 + 4 = 7$ inches and cannot meet. The **Triangle Inequality Theorem** says that in every triangle, any two sides together are longer than the third. A straight path is shorter than a path with a bend in it.

Euclid proves this as I.20, and his proof uses only the last lesson's theorems. In $\triangle ABC$, extend $\overline{BA}$ past A to a point D with $AD = AC$. Then $BD = BA + AC$. Triangle ACD is isosceles, so its base angles at C and D are equal, and the angle at C in the large triangle BCD is even bigger. In $\triangle BCD$ the larger angle is opposite the longer side (I.19), so $BD > BC$, that is, $BA + AC > BC$.

To test three lengths, you do not need all three inequalities: if the two shorter lengths add up to more than the longest, the other two inequalities are automatic. And if two sides are a and b, the third side c must satisfy $|a - b| < c < a + b$. For sides 6 and 10, the third side is between 4 and 16. In Lesson 4.3 you saw the other half of the story (Euclid I.22): when the lengths do pass this test, a triangle with those sides can always be constructed.

Now picture a hinge, or a drafting compass, or a folding ladder: two sides of fixed length joined at a vertex. As you open the angle between them, the gap between their free ends grows. That is the **Hinge Theorem** (Euclid I.24): if two triangles have two pairs of congruent sides, the one with the larger included angle has the longer third side. Its converse (I.25) runs the other way: the longer third side goes with the larger included angle.

The proof of the Hinge Theorem lays a copy of the smaller triangle inside the larger angle, and then uses an angle bisector, SAS and the Triangle Inequality. The converse is proved indirectly: if the included angle were not larger, it would be equal (SAS

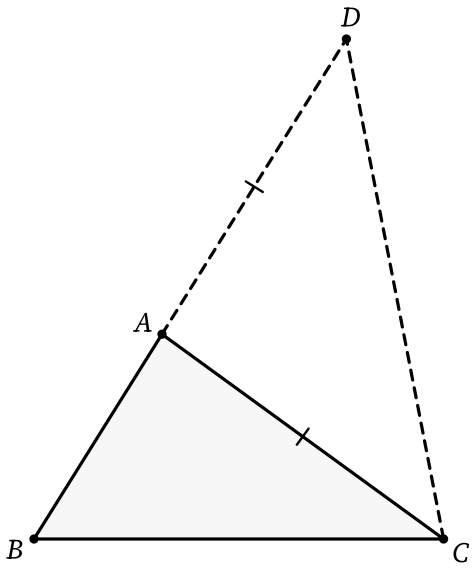
makes the third sides equal) or smaller (the Hinge Theorem makes the third side shorter), and both contradict what we know.

When you use either hinge theorem, check three things: two pairs of sides are congruent, you are comparing the **included** angles (the angles between those congruent sides), and the third sides are the ones opposite those angles.

POSTULATES AND THEOREMS

THEOREM **Triangle Inequality Theorem** Euclid I.20

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



TWO-COLUMN PROOF

Given: $\triangle ABC$

Prove: $AB + AC > BC$

	STATEMENTS	REASONS
1	$\triangle ABC$	Given
2	Extend $\overline{BA}$ past A to a point D with $AD = AC$	Ruler Postulate (Euclid I.3)
3	Draw $\overline{CD}$	Through two points there is exactly one line
4	$m\angle ACD = m\angle ADC$	Isosceles Triangle Theorem (in $\triangle ACD$, $\overline{AD} \cong \overline{AC}$)
5	$m\angle BCD = m\angle BCA + m\angle ACD$	Angle Addition Postulate (A is in the interior of $\angle BCD$, since A is between B and D)

	STATEMENTS	REASONS
6	$m\angle BCD > m\angle ACD$	Comparison Property of Inequality
7	$m\angle BCD > m\angle BDC$	Substitution (step 4; $\angle ADC$ is $\angle BDC$)
8	$BD > BC$	Triangle Larger Angle Theorem (in $\triangle BCD$)
9	$BD = BA + AD$	Segment Addition Postulate
10	$BD = BA + AC$	Substitution ($AD = AC$)
11	$AB + AC > BC$	Substitution (steps 8 and 10)

∎

The same argument, started from a different vertex, gives $AB + BC > AC$ and $AC + BC > AB$.

COROLLARY **Possible Lengths of the Third Side**

If two sides of a triangle have lengths a and b, then the third side c satisfies $|a - b| < c < a + b$.

PARAGRAPH PROOF

Given: A triangle with sides a, b, c

Prove: $|a - b| < c < a + b$

By the Triangle Inequality Theorem, $c < a + b$. Also $a < b + c$, so $c > a - b$ (Subtraction Property of Inequality), and $b < a + c$, so $c > b - a$.

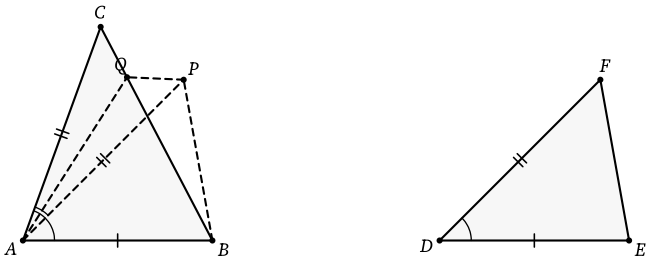
Since c is greater than both $a - b$ and $b - a$, and one of these numbers is $|a - b|$, we have $c > |a - b|$. Together, $|a - b| < c < a + b$.

∎

Conversely, any length strictly between $|a - b|$ and $a + b$ works: a triangle with those three sides can be constructed (Euclid I.22, Lesson 4.3).

THEOREM **Hinge Theorem (SAS Inequality)** Euclid I.24

If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.



PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$; $m\angle BAC > m\angle EDF$

Prove: $BC > EF$

Since $m\angle BAC > m\angle EDF$, we can copy $\angle EDF$ inside $\angle BAC$ (Protractor Postulate; Euclid I.23): draw a ray from A in the interior of $\angle BAC$ making $\angle BAP \cong \angle EDF$, and choose P on it with $AP = DF$. Then $\triangle ABP \cong \triangle DEF$ by SAS ($\overline{AB} \cong \overline{DE}$, $\angle BAP \cong \angle EDF$, $\overline{AP} \cong \overline{DF}$), so $BP = EF$ by CPCTC. Note also that $AP = DF = AC$.

If P happens to lie on $\overline{BC}$, then $BC = BP + PC > BP = EF$ by the Segment Addition Postulate and the Comparison Property, and we are done.

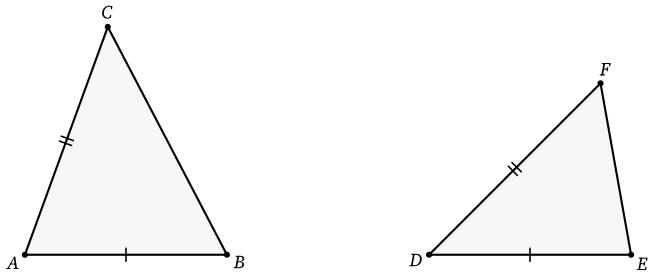
Otherwise, draw the bisector of $\angle PAC$. It lies inside $\angle BAC$, so it meets $\overline{BC}$ at a point Q. In $\triangle APQ$ and $\triangle ACQ$, $\overline{AP} \cong \overline{AC}$, $\angle PAQ \cong \angle CAQ$ (definition of angle bisector) and $\overline{AQ} \cong \overline{AQ}$, so $\triangle APQ \cong \triangle ACQ$ by SAS, and $PQ = CQ$ by CPCTC.

Now in $\triangle BPQ$, the Triangle Inequality gives $BQ + QP > BP$. Substituting $QP = QC$: $BQ + QC > BP$. Since Q is between B and C, $BQ + QC = BC$ (Segment Addition Postulate). So $BC > BP = EF$, that is, $BC > EF$.

∎

Euclid I.24: "If two triangles have two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base."

THEOREM **Converse of the Hinge Theorem (SSS Inequality)** Euclid I.25



If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

INDIRECT PROOF

Given: $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$; $BC > EF$

Prove: $m\angle BAC > m\angle EDF$

Assume temporarily that $m\angle BAC$ is not greater than $m\angle EDF$. Then either $m\angle BAC = m\angle EDF$ or $m\angle BAC < m\angle EDF$ (Trichotomy Property).

Case 1: $m\angle BAC = m\angle EDF$. Then $\triangle ABC \cong \triangle DEF$ by SAS, so $BC = EF$ by CPCTC. This contradicts $BC > EF$.

Case 2: $m\angle BAC < m\angle EDF$. Then by the Hinge Theorem, applied with the triangles' roles exchanged, $EF > BC$. This also contradicts $BC > EF$.

Both cases are impossible, so the assumption is false. Therefore $m\angle BAC > m\angle EDF$.

∎

Euclid proves I.25 indirectly in exactly this way, using I.4 for the equal case and I.24 for the smaller case.

KEY FORMULAS

$$a + b > c, \qquad a + c > b, \qquad b + c > a$$

$$|a - b| < c < a + b$$

From history. According to Proclus, the Epicurean philosophers mocked Euclid's I.20 as a theorem obvious even to a donkey: put hay at one corner of a triangle and the donkey at another, and he walks straight to the hay rather than going around by two sides. Proclus answered that seeing a thing is true is not the same as knowing why it is true, and the geometer's work is to give the reason. That is a good motto for this whole course.

Euclid connection. Euclid proves the Triangle Inequality in I.20 by extending one side, exactly as in our proof, using I.5 and I.19. I.22 then constructs a triangle from three lengths that pass the test, and I.24 and I.25 are the Hinge Theorem and its converse, with I.25 proved indirectly from I.4 and I.24.

Practice habit. Recite the Triangle Inequality as "any two sides together are longer than the third," and test three lengths by adding the two shortest and comparing with the longest. Before using a hinge theorem, mark the two pairs of congruent sides on the diagram and put an arc on the included angle in each triangle, so you are sure you are comparing the right angles.

WORKED EXAMPLES

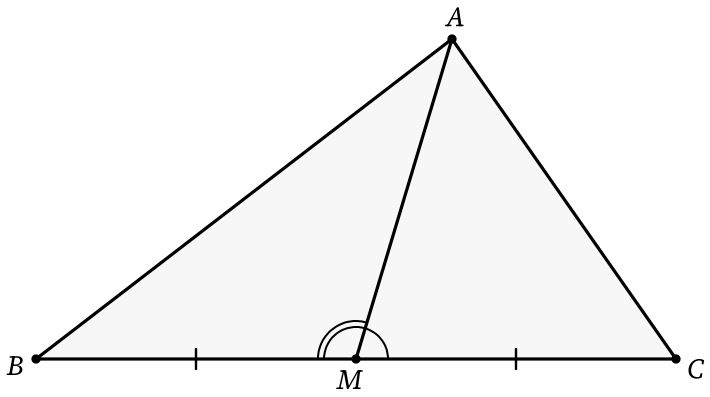
Example 1. Samuel is building a triangular frame for a raised garden bed. He has two boards, 6 feet and 10 feet long, for two of the sides. What lengths could the third board be?

Show solution

1. Let the third side be x feet. By the Triangle Inequality, $6 + 10 > x$, so $x < 16$.
2. Also $x + 6 > 10$, so $x > 4$. (The third inequality, $x + 10 > 6$, is true for every positive x.)
3. So $4 < x < 16$. A board of exactly 4 ft or 16 ft would make the frame collapse flat.

Answer: The third board must be longer than 4 ft and shorter than 16 ft: $4 < x < 16$.

Example 2. In $\triangle ABC$, $\overline{AM}$ is a median and $m\angle AMB > m\angle AMC$. Prove that $AB > AC$.



Show solution

1. The two triangles $\triangle AMB$ and $\triangle AMC$ share $\overline{AM}$, and the median gives $\overline{MB} \cong \overline{MC}$: two pairs of congruent sides.
2. The angles at M are the included angles of those sides, and $\angle AMB$ is the larger.
3. The Hinge Theorem says the side opposite the larger included angle is longer.

TWO-COLUMN PROOF

Given: $\overline{AM}$ is a median of $\triangle ABC$; $m\angle AMB > m\angle AMC$

Prove: $AB > AC$

	STATEMENTS	REASONS
1	$\overline{AM}$ is a median of $\triangle ABC$	Given
2	M is the midpoint of $\overline{BC}$	Definition of median
3	$\overline{MB} \cong \overline{MC}$	Definition of midpoint
4	$\overline{MA} \cong \overline{MA}$	Reflexive Property of Congruence
5	$m\angle AMB > m\angle AMC$	Given
6	$AB > AC$	Hinge Theorem (in $\triangle AMB$ and $\triangle AMC$)

∎

Answer: See the model proof.

PRACTICE

1. Easy Can segments of lengths 4 cm, 7 cm and 12 cm form a triangle?

   Show answer

   No: $4 + 7 = 11$, which is not greater than 12.

2. Easy Can segments of lengths 5 in., 8 in. and 12 in. form a triangle?

   Show answer

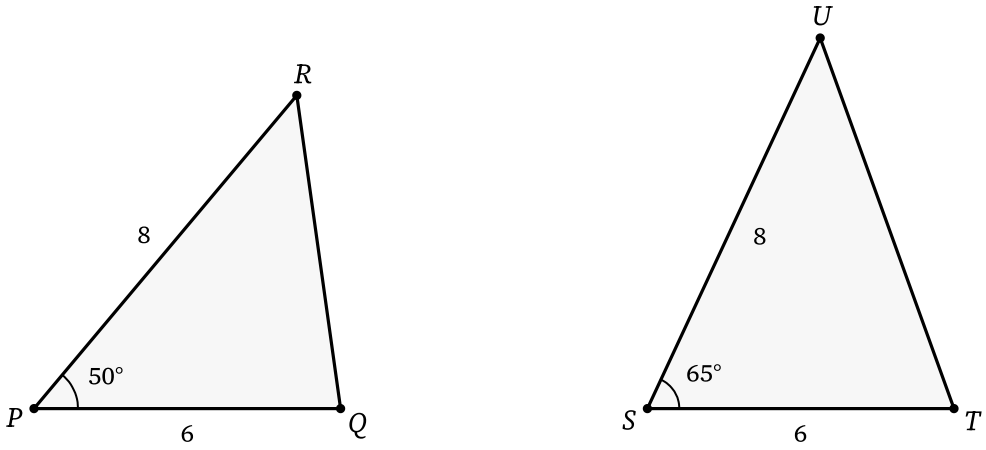
   Yes: the two shorter sides give $5 + 8 = 13 > 12$.

3. Easy Two sides of a triangle measure 9 m and 14 m. Write an inequality for the possible lengths x of the third side.

   Show answer

   $14 - 9 < x < 14 + 9$, so $5 < x < 23$ (meters).

4. Easy In the figure, $PQ = ST = 6$, $PR = SU = 8$, $m\angle P = 50°$ and $m\angle S = 65°$. Which is longer, QR or TU?



   Show answer

   TU, by the Hinge Theorem, since the included angle at S is larger.

5. **Medium** Explain why sticks of lengths 3 ft, 3 ft and 6 ft cannot form a triangle. What happens if you try?

Show answer

The two shorter sticks add to exactly 6 ft, not more than 6 ft, so the Triangle Inequality fails. If you lay the 6-ft stick down and swing the two 3-ft sticks from its ends, they can meet only by lying flat along it, so the "triangle" collapses into a segment with no inside.

6. **Medium** Ruth's house is 3 miles from the church, and the church is 5 miles from the market. The three places are not in a straight line. What are the possible distances d from Ruth's house to the market?

Show answer

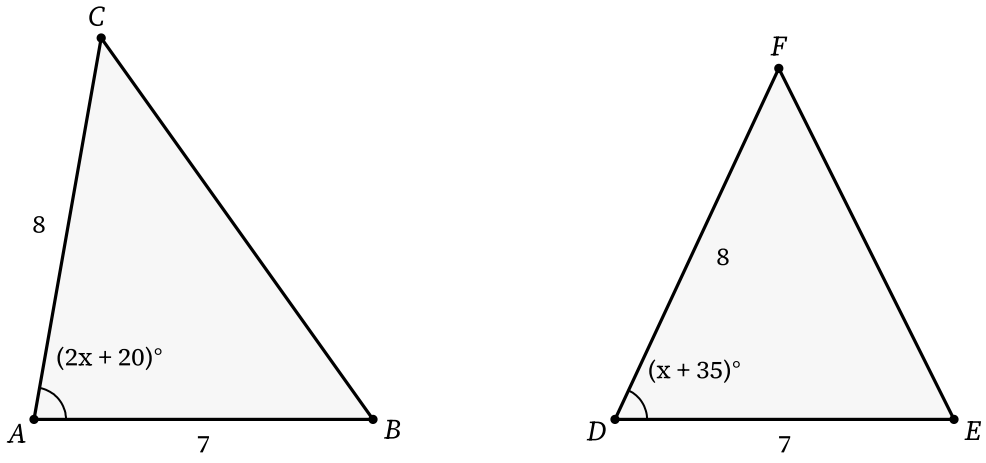
The three places form a triangle, so $5 - 3 < d < 5 + 3$: $2 < d < 8$ miles.

7. **Medium** In $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $BC = 10$ and $EF = 8$. Which angle is larger, $\angle A$ or $\angle D$?

Show answer

$\angle A$, by the Converse of the Hinge Theorem, since $BC > EF$.

8. **Medium** In the figure, $AB = DE = 7$, $AC = DF = 8$, and $BC > EF$. If $m\angle BAC = (2x + 20)^\circ$ and $m\angle EDF = (x + 35)^\circ$, find all possible values of x.



Show answer

By the Converse of the Hinge Theorem, $2x + 20 > x + 35$, so $x > 15$. An angle of a triangle is less than 180°, so $2x + 20 < 180$ and $x < 80$. Therefore $15 < x < 80$.

9. Medium Henry and Grace each set up a folding stepladder. Both ladders have legs 8 feet long. Henry opens his so the legs make a 30° angle at the top; Grace opens hers to 25°. Whose ladder has the wider spread between its feet? Name the theorem that tells you.

Show answer

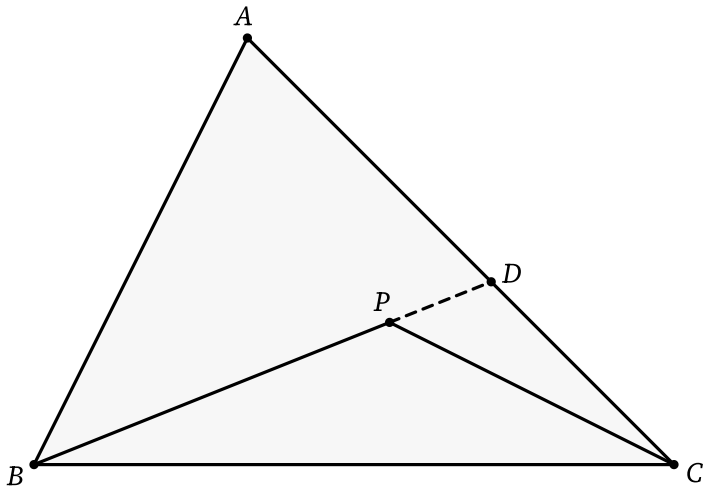
Henry's. The legs are congruent pairs, and his included angle is larger, so by the Hinge Theorem the distance between the feet is longer.

10. Hard The sides of a triangle have lengths $x + 2$, $x + 5$ and $2x + 1$. Find all possible values of x.

Show answer

$(x + 2) + (x + 5) > 2x + 1$ is always true. $(x + 2) + (2x + 1) > x + 5$ gives $x > 1$. $(x + 5) + (2x + 1) > x + 2$ gives $x > -2$. All sides are positive when $x > 1$. So $x > 1$.

11. **Hard** In the figure, P is a point inside $\triangle ABC$. Prove that $PB + PC < AB + AC$. (Hint: extend $\overline{BP}$ to meet $\overline{AC}$ at D. This is the first part of Euclid I.21.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: P is in the interior of $\triangle ABC$

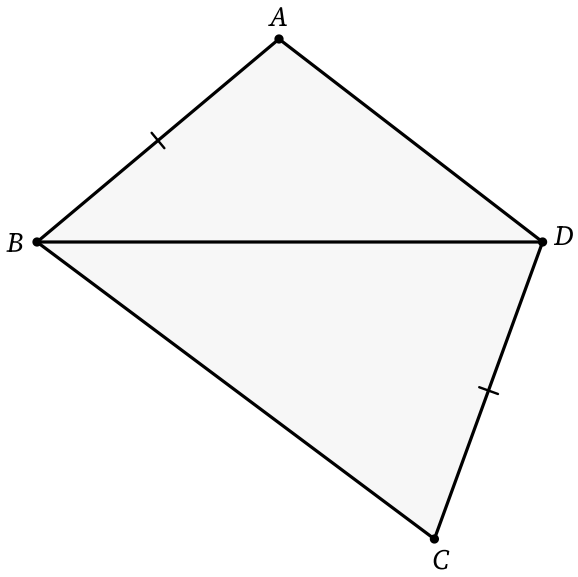
Prove: $PB + PC < AB + AC$

	STATEMENTS	REASONS
1	P is in the interior of $\triangle ABC$	Given
2	Extend $\overline{BP}$ to meet $\overline{AC}$ at D, with P between B and D	A ray from a vertex through an interior point crosses the opposite side
3	$AB + AD > BD$	Triangle Inequality Theorem (in $\triangle ABD$)
4	$BD = BP + PD$	Segment Addition Postulate
5	$AB + AD > BP + PD$	Substitution
6	$PD + DC > PC$	Triangle Inequality Theorem (in $\triangle PDC$)
7	$AB + AD + PD + DC > BP + PD + PC$	Addition Property of Inequality (steps 5 and 6)

	STATEMENTS	REASONS
8	$AB + AD + DC > BP + PC$	Subtraction Property of Inequality
9	$AD + DC = AC$	Segment Addition Postulate
10	$PB + PC < AB + AC$	Substitution (steps 8 and 9)

∎

12. **Hard** In the figure, quadrilateral $ABCD$ has diagonal $\overline{BD}$, $\overline{AB} \cong \overline{CD}$, and $BC > AD$. Prove that $m\angle CDB > m\angle ABD$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{AB} \cong \overline{CD}$; $BC > AD$

Prove: $m\angle CDB > m\angle ABD$

(1) $\overline{CD} \cong \overline{AB}$
Given

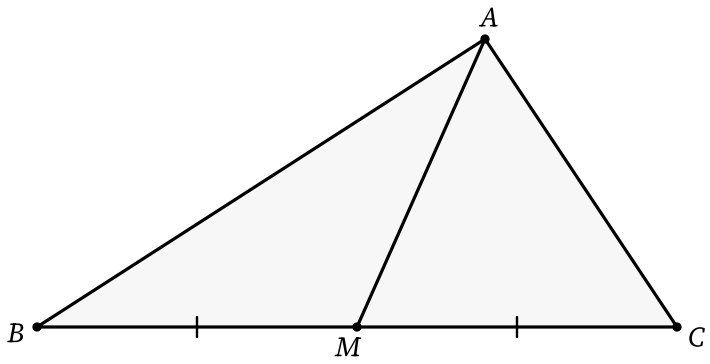
(2) $\overline{DB} \cong \overline{BD}$
Reflexive Property of Congruence

(3) $BC > AD$
Given

(4) $m\angle CDB > m\angle ABD$
Converse of the Hinge Theorem (in $\triangle CDB$ and $\triangle ABD$: $\angle CDB$ is included between $\overline{CD}$ and $\overline{DB}$ and is opposite $\overline{BC}$; $\angle ABD$ is included between $\overline{AB}$ and $\overline{BD}$ and is opposite $\overline{AD}$)
from (1), (2), (3)

∎

13. **Hard** In the figure, M is the midpoint of $\overline{BC}$ in $\triangle ABC$, and $AB > AC$. Prove that $\angle AMB$ is obtuse.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: M is the midpoint of $\overline{BC}$; $AB > AC$

Prove: $\angle AMB$ is obtuse

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{BC}$	Given
2	$\overline{MB} \cong \overline{MC}$	Definition of midpoint
3	$\overline{MA} \cong \overline{MA}$	Reflexive Property of Congruence
4	$AB > AC$	Given
5	$m\angle AMB > m\angle AMC$	Converse of the Hinge Theorem (in $\triangle AMB$ and $\triangle AMC$)
6	$m\angle AMB + m\angle AMC = 180°$	Linear Pair Theorem (linear pairs are supplementary)
7	$m\angle AMB + m\angle AMB > m\angle AMC + m\angle AMB$	Addition Property of Inequality (add $m\angle AMB$ to both sides of step 5)
8	$2 \cdot m\angle AMB > 180°$	Substitution (step 6)
9	$m\angle AMB > 90°$	Division Property of Inequality

	STATEMENTS	REASONS
10	$\angle AMB$ is obtuse	Definition of obtuse angle ($m\angle AMB < 180^\circ$ since it is one angle of a linear pair)

∎

14. **Challenge** Prove that in any quadrilateral $ABCD$, the sum of the lengths of the diagonals is less than the perimeter: $AC + BD < AB + BC + CD + DA$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Quadrilateral $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$

Prove: $AC + BD < AB + BC + CD + DA$

No three vertices of a quadrilateral are collinear, so each diagonal is a side of two triangles. By the Triangle Inequality Theorem: in $\triangle ABC$, $AC < AB + BC$; in $\triangle CDA$, $AC < CD + DA$; in $\triangle BCD$, $BD < BC + CD$; and in $\triangle DAB$, $BD < DA + AB$.

Adding the four inequalities (Addition Property of Inequality): $2AC + 2BD < 2AB + 2BC + 2CD + 2DA$. Dividing both sides by 2 (Division Property of Inequality): $AC + BD < AB + BC + CD + DA$.

∎

COMMON MISTAKES

- Checking only one of the three inequalities, or the wrong one. Add the two shortest lengths and compare with the longest; if that works, the others do too.
- Allowing the third side to equal $a + b$ or $|a - b|$. Those lengths make the figure collapse into a segment; the inequalities are strict.
- Using the Hinge Theorem with an angle that is not included between the congruent sides. Mark the two pairs of congruent sides first, then compare the angles at the vertices where they meet.
- Trying the Hinge Theorem when only one pair of sides is congruent. Both pairs must be congruent; otherwise no conclusion follows.

Make it a game · Can It Close?. Parent calls three lengths, such as 4, 7 and 12. Say yes or no within 5 seconds, and justify using the Triangle Inequality. Then parent gives two sides, and you state the range of the third. Finally, hinge two paper strips at different angles and predict which opening makes the longer third side, citing the Hinge Theorem.

UNIT 5

Relationships Within Triangles

In Unit 4 you learned to prove two triangles congruent. In this unit you look inside a single triangle and discover how its parts are related. Every triangle has four special centers: the circumcenter, where the perpendicular bisectors of the sides meet; the incenter, where the angle bisectors meet; the centroid, where the medians meet; and the orthocenter, where the altitudes meet. You will prove why the first three always exist, construct the circles about and in a triangle as Euclid does in Book IV, and use these points to solve practical problems, from placing a well equidistant from three farmhouses to balancing a cardboard triangle on a pencil.

The second half of the unit is about inequalities. The midsegment of a triangle is parallel to the third side and half as long. You will learn a new kind of proof, the indirect proof (proof by contradiction), and use it with Euclid I.16 to I.25 to prove that the longer side is opposite the larger angle, that any two sides of a triangle are together longer than the third, and that opening the angle between two fixed sides lengthens the third side (the Hinge Theorem).

For parents: this unit is proof-heavy and follows Euclid's Book I closely. Expect your student to write two-column, paragraph, flow and indirect proofs, and to use compass and straightedge. Asking your student to explain aloud why a statement is true, not just what the answer is, is the best help you can give.

PREREQUISITES

- Prove triangles congruent with SSS, SAS, ASA, AAS and HL, and use CPCTC (Lessons 4.3–4.7).
- Use the Isosceles Triangle Theorem and its converse (Lesson 4.6).
- Use the Triangle Angle Sum Theorem, the Exterior Angle Theorem and the Exterior Angle Inequality (Lesson 4.1).
- Use the angle pairs formed by parallel lines and a transversal (Unit 3).
- Use the slope, midpoint and distance formulas, and write equations of perpendicular lines (Lessons 2.2 and 3.5).
- Bisect a segment and an angle and drop a perpendicular with compass and straightedge (Lesson 2.5).

MASTERY CHECKLIST

- ☐ I can prove and use the Perpendicular Bisector Theorem and the Angle Bisector Theorem and their converses.
- ☐ I can find the distance from a point to a line by using the perpendicular segment.
- ☐ I can prove that the perpendicular bisectors and the angle bisectors of a triangle are concurrent, and find the circumcenter and incenter.
- ☐ I can construct the circumscribed and inscribed circles of a triangle with compass and straightedge.
- ☐ I can use the Centroid Theorem and find the centroid and orthocenter of a triangle in the coordinate plane.
- ☐ I can construct a median and an altitude of a triangle.
- ☐ I can prove the Triangle Midsegment Theorem with coordinates and use it to find lengths and angles.
- ☐ I can write an indirect proof by assuming the opposite of the conclusion and reaching a contradiction.
- ☐ I can order the sides and angles of a triangle using Euclid I.18 and I.19.
- ☐ I can decide whether three lengths form a triangle and find the possible lengths of a third side.

- ☐ I can use the Hinge Theorem and its converse to compare sides and angles of two triangles.

LESSONS IN THIS UNIT

5.1	**Perpendicular Bisectors and Angle Bisectors**	2 sessions of 45 minutes
5.2	**Circumcenter and Incenter**	2 sessions of 45 minutes
5.3	**Medians, Altitudes and the Centroid**	2 sessions of 45 minutes
5.4	**The Triangle Midsegment Theorem**	45 minutes
5.5	**Indirect Proof and Inequalities in One Triangle**	2 sessions of 45 minutes
5.6	**The Triangle Inequality and the Hinge Theorem**	2 sessions of 45 minutes

Unit 5 Test

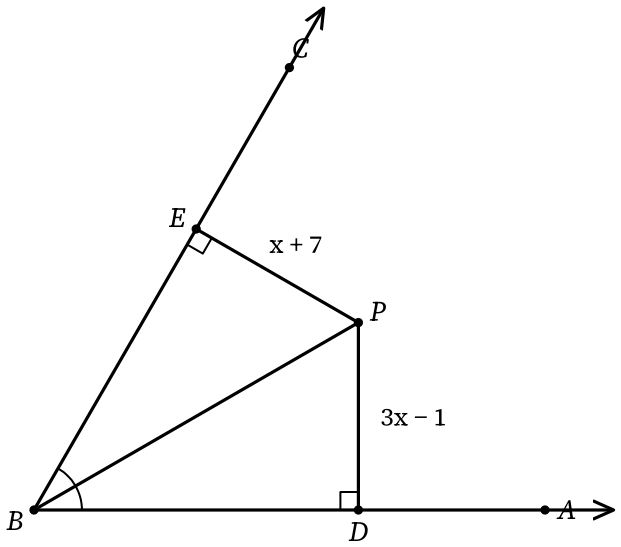
You have 50 minutes. No calculator is needed or allowed; give exact answers (simplest radical form where needed) unless a problem says otherwise. You will need a compass and a straightedge for Problem 14. Show your work for every computation. In every proof, give a reason for every statement, and do not assume anything from the look of a figure. The test is worth 100 points.

1. **5 pts**

   Point P lies on the perpendicular bisector of $\overline{AB}$. If $PA = 2y + 5$ and $PB = 4y - 3$, find PA.

2. **6 pts**

In the figure, P lies on the bisector of $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$ and $\overline{PE} \perp \overrightarrow{BC}$. If $PD = 3x - 1$ and $PE = x + 7$, find PD.

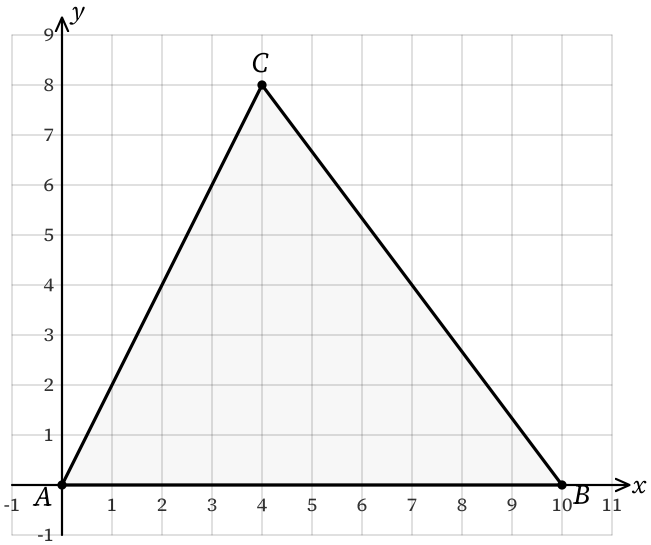


3. **4 pts**

Which point of concurrency of a triangle is equidistant from the three sides of the triangle?

4. **7 pts**

On a map with a grid in miles, three towns are at $A(0, 0)$, $B(10, 0)$ and $C(4, 8)$. A radio tower is to be built the same distance from all three towns. Find the coordinates of the tower.



5. **6 pts**

A triangular park has sides of 9 m, 12 m and 15 m, and it is a right triangle. The town wants to plant the largest possible circular flower bed inside it. Find the radius of the flower bed.

6. **5 pts**

G is the centroid of $\triangle ABC$, and the median $\overline{BM}$ has length 27. Find BG.

7. **7 pts**

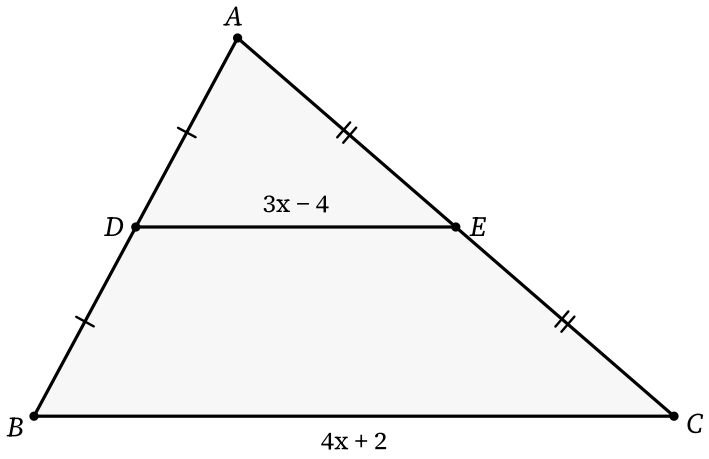
Find the orthocenter of the triangle with vertices $A(0, 0)$, $B(12, 0)$ and $C(3, 6)$.

8. **5 pts**

The legs of an A-frame swing set are 9 feet apart at the ground. A horizontal brace is bolted between the midpoints of the two legs. How long is the brace?

9. **6 pts**

In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, $DE = 3x - 4$ and $BC = 4x + 2$. Find BC.



10. **5 pts**

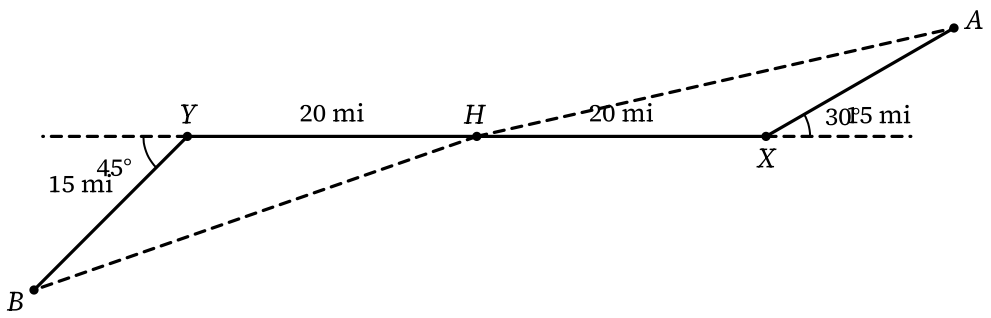
In $\triangle ABC$, $AB = 7$, $BC = 11$ and $AC = 9$. List the angles from smallest to largest.

11. **6 pts**

A triangular flower bed has two sides of 12 ft and 20 ft. The third side is a whole number of feet. What are the smallest and the largest possible lengths of the third side?

12. **6 pts**

Two ships leave harbor H. Ship A sails 20 miles due east, then turns 30° toward the north and sails 15 miles. Ship B sails 20 miles due west, then turns 45° toward the south and sails 15 miles, as in the figure. Which ship ends farther from the harbor? Explain with a theorem.



13. **8 pts**

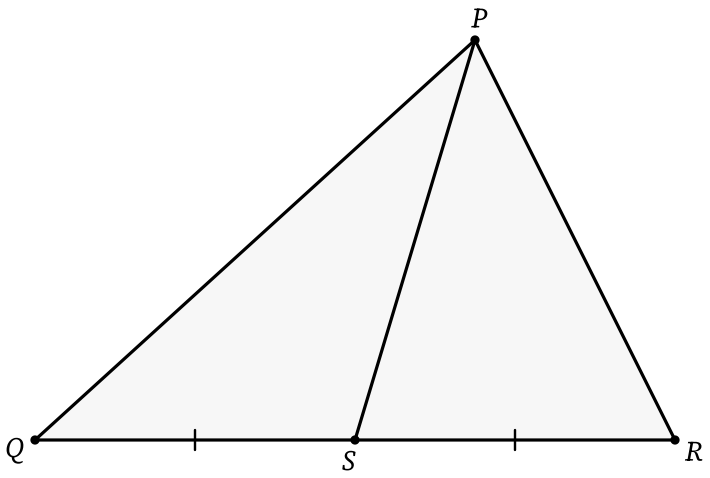
Explain why the three perpendicular bisectors of the sides of any triangle must meet at a single point. Your explanation should use the Perpendicular Bisector Theorem and its converse.

14. **6 pts**

Construction: draw a large scalene triangle ABC. With compass and straightedge, construct its centroid. List your steps and explain why the point you found is the centroid.

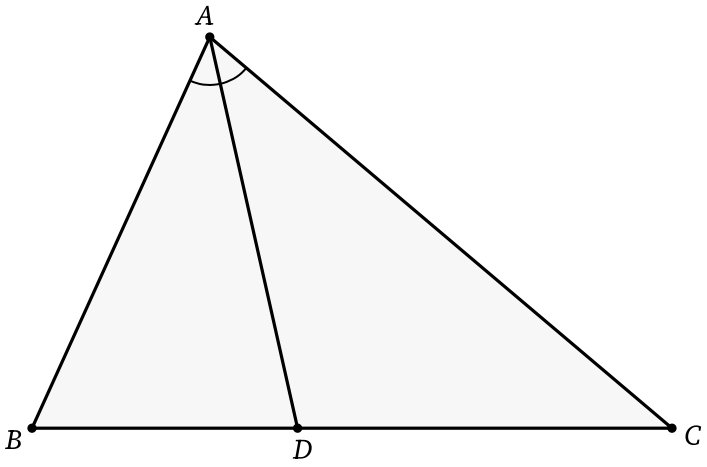
15. **8 pts**

In the figure, $\overline{PQ} \ncong \overline{PR}$ in $\triangle PQR$, and S is the midpoint of $\overline{QR}$. Write an indirect proof that $\overline{PS}$ is not perpendicular to $\overline{QR}$.



16. **10 pts**

In the figure, $\overrightarrow{AD}$ bisects $\angle BAC$ of $\triangle ABC$, with D on $\overline{BC}$. Prove that $AB > BD$. Write a two-column proof.



ANSWER KEY

1. $2y + 5 = 4y - 3$, so $y = 4$ and $PA = 13$.
2. By the Angle Bisector Theorem, $3x - 1 = x + 7$, so $x = 4$ and $PD = 11$.
3. The incenter.
4. The tower is at the circumcenter. The perpendicular bisector of $\overline{AB}$ is $x = 5$. Setting $PA^2 = PC^2$ for $P(5, y)$: $25 + y^2 = 1 + (8 - y)^2 = 65 - 16y + y^2$, so $16y = 40$ and $y = 2.5$. The tower is at $(5, 2.5)$.
5. The largest circle is the inscribed circle. Area $= \frac{1}{2}(9)(12) = 54$ m^2 and $s = \frac{9+12+15}{2} = 18$ m, so $r = \dfrac{54}{18} = 3$ m.
6. $BG = \frac{2}{3}(27) = 18$
7. The altitude from C is $x = 3$. The slope of $\overline{BC}$ is $\dfrac{6}{3 - 12} = -\dfrac{2}{3}$, so the altitude from A is $y = \frac{3}{2}x$. At $x = 3$, $y = 4.5$. The orthocenter is $(3, 4.5)$. (Check: the altitude from B, $y = -\frac{1}{2}(x - 12)$, also passes through $(3, 4.5)$.)
8. The brace is a midsegment of the triangle formed by the legs and the ground, so it is $\frac{1}{2}(9) = 4.5$ feet long.
9. $4x + 2 = 2(3x - 4) = 6x - 8$, so $x = 5$ and $BC = 22$ (and $DE = 11$).
10. $\angle C$ (opposite 7), $\angle B$ (opposite 9), $\angle A$ (opposite 11).
11. $20 - 12 < x < 20 + 12$, so $8 < x < 32$. The smallest whole-number length is 9 ft and the largest is 31 ft.
12. Ship A. Each ship's path makes a triangle with sides 20 and 15 miles. The included angle at ship A's turning point is $180^\circ - 30^\circ = 150^\circ$, and at ship B's it is $180^\circ - 45^\circ = 135^\circ$. By the Hinge Theorem, the side opposite the larger included angle is longer, so ship A is farther from the harbor.
13. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ meet at P (they are not parallel, because $\overline{AB}$ and $\overline{BC}$ are not on one line). Since P is on the perpendicular bisector of $\overline{AB}$, $PA = PB$; since it is on the perpendicular bisector of $\overline{BC}$, $PB = PC$ (Perpendicular Bisector Theorem). So $PA = PC$, and by the converse P is on the perpendicular bisector of $\overline{AC}$. All three lines pass through P.
14. Construct the perpendicular bisector of $\overline{BC}$ (equal arcs from B and C crossing at two points, joined by a line) to find the midpoint L of $\overline{BC}$; do the same for $\overline{AC}$ to find its midpoint M. Draw the medians $\overline{AL}$ and $\overline{BM}$; they meet at G. Since L and M are midpoints (Euclid I.10), $\overline{AL}$ and $\overline{BM}$ are medians, and by the

Centroid Theorem the medians meet at the centroid. Check: G should be two-thirds of the way from A to L, and the third median should pass through G.

15. See the model proof.

    INDIRECT PROOF

    Given: $\overline{PQ} \ncong \overline{PR}$; S is the midpoint of $\overline{QR}$

    Prove: $\overline{PS}$ is not perpendicular to $\overline{QR}$

    Assume temporarily that $\overline{PS} \perp \overline{QR}$.

    Then $\angle PSQ$ and $\angle PSR$ are right angles (definition of perpendicular lines), so $\angle PSQ \cong \angle PSR$ (all right angles are congruent). Since S is the midpoint of $\overline{QR}$, $\overline{QS} \cong \overline{RS}$ (definition of midpoint), and $\overline{PS} \cong \overline{PS}$ (Reflexive Property). So $\triangle PSQ \cong \triangle PSR$ by SAS, and $\overline{PQ} \cong \overline{PR}$ by CPCTC.

    This contradicts the given fact that $\overline{PQ} \ncong \overline{PR}$. Therefore the assumption is false, and $\overline{PS}$ is not perpendicular to $\overline{QR}$.

    ∎

16. See the model proof.

    TWO-COLUMN PROOF

    Given: $\overrightarrow{AD}$ bisects $\angle BAC$, D on $\overline{BC}$

    Prove: $AB > BD$

	STATEMENTS	REASONS
1	$\overrightarrow{AD}$ bisects $\angle BAC$	Given
2	$m\angle BAD = m\angle DAC$	Definition of angle bisector
3	$m\angle ADB > m\angle DAC$	Exterior Angle Inequality ($\angle ADB$ is an exterior angle of $\triangle ADC$)
4	$m\angle ADB > m\angle BAD$	Substitution (steps 2 and 3)
5	$AB > BD$	Triangle Larger Angle Theorem (in $\triangle ABD$, $\overline{AB}$ is opposite $\angle ADB$ and $\overline{BD}$ is opposite $\angle BAD$)

    ∎

UNIT 6

Quadrilaterals and Polygons

In this unit you take the triangle facts of Units 4 and 5 and use them to understand figures with more sides. Cutting a polygon into triangles gives the angle sum of any polygon, and drawing a diagonal in a quadrilateral turns almost every question about it into a question about congruent triangles. You will prove the properties of parallelograms just as Euclid did in I.34, then prove the converses that let you recognize a parallelogram, including Euclid's I.33.

Next you study the special members of the quadrilateral family: rectangles, rhombuses and squares (with Euclid's construction of a square, I.46), and then trapezoids and kites. Along the way you will see why builders measure the diagonals of a foundation, why a navigator's parallel rule and a scissor lift stay parallel, and which regular polygons can tile a floor or frame a rose window.

Parents: the unit is proof-heavy. Expect two-column, paragraph and flow proofs in every lesson after the first, and indirect proofs in several of them. A good check of your student's work is to ask, for any step, 'Which theorem says so?' Every step should have a named reason.

PREREQUISITES

- Use the Triangle Angle Sum Theorem and the Linear Pair Theorem (Lessons 2.4 and 4.1).
- Use the angle theorems for parallel lines and their converses, including Euclid I.27, I.28 and I.30 (Lessons 3.2 and 3.3).
- Prove triangles congruent with SSS, SAS, ASA, AAS and HL and use CPCTC (Lessons 4.3–4.7).
- Use the Isosceles Triangle Theorem and its converse (Lesson 4.6).
- Use the Perpendicular Bisector Theorem and its converse and the Triangle Midsegment Theorem (Lessons 5.1 and 5.4).
- Find slope, distance and midpoint on the coordinate plane and use the Pythagorean theorem numerically (Algebra I).

MASTERY CHECKLIST

- ☐ I can classify polygons as convex or concave and count their diagonals.
- ☐ I can prove and use the Polygon Interior Angle Sum Theorem and the Exterior Angle Sum Theorem.
- ☐ I can find each angle of a regular polygon and the number of sides from an angle, and decide which regular polygons tile a floor.
- ☐ I can construct a regular hexagon in a circle (Euclid IV.15) and a square on a segment (Euclid I.46) and explain why each construction works.
- ☐ I can prove that the opposite sides and angles of a parallelogram are congruent (Euclid I.34), that consecutive angles are supplementary and that the diagonals bisect each other.
- ☐ I can prove that a quadrilateral is a parallelogram with any of the four tests, including one pair of sides both congruent and parallel (Euclid I.33).
- ☐ I can show on the coordinate plane that a quadrilateral is a parallelogram, rectangle, rhombus, square or trapezoid using slope, distance and midpoints.
- ☐ I can prove and use the diagonal properties of rectangles and rhombuses and explain why

builders compare the diagonals of a foundation.

- ☐ I can prove and use the theorems on isosceles trapezoids and the Trapezoid Midsegment Theorem.
- ☐ I can prove that the diagonals of a kite are perpendicular and that exactly one pair of its opposite angles is congruent.
- ☐ I can write two-column, paragraph, flow and indirect proofs about quadrilaterals with a correct reason for every step.

LESSONS IN THIS UNIT

6.1	**Angles of Polygons**	45 minutes
6.2	**Properties of Parallelograms**	2 sessions of 45 minutes
6.3	**Proving That a Quadrilateral Is a Parallelogram**	2 sessions of 45 minutes
6.4	**Rectangles, Rhombuses and Squares**	2 sessions of 45 minutes
6.5	**Trapezoids and Kites**	2 sessions of 45 minutes

Unit 6 test

You have 55 minutes. No calculator is allowed; work every problem by hand and give exact answers (for example $8\sqrt{5}$) unless a problem asks you to round. Show all of your work. In every proof, give a reason for every step: a definition, postulate or theorem, not 'from the figure'. Bring a compass and straightedge for Problem 13. Figures are drawn to scale, but use only the given information and the marks on them, not measurements.

1. **5 pts**

   Find the sum of the measures of the interior angles of a convex 9-sided polygon (a nonagon).

2. **5 pts**

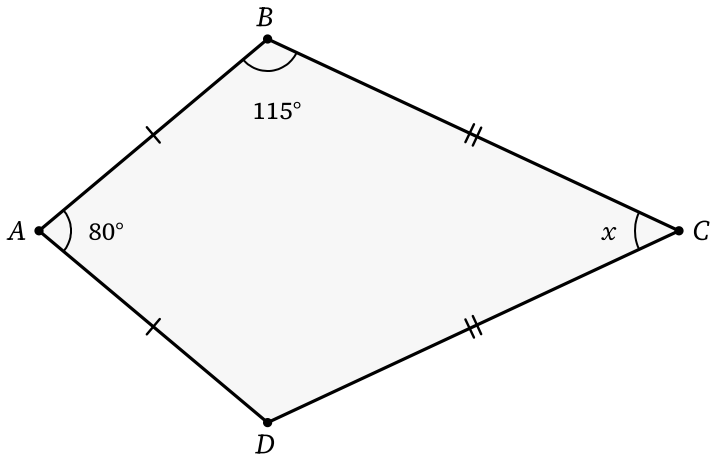
   Each exterior angle of a regular polygon measures $20°$. How many sides does the polygon have?

3. **5 pts**

   The diagonals of parallelogram $PQRS$ meet at T. If $PT = 3y - 4$ and $TR = y + 10$, find PR.

4. **5 pts**

   $ABCD$ in the figure is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$; $m\angle A = 80°$ and $m\angle B = 115°$. Find $m\angle C$.

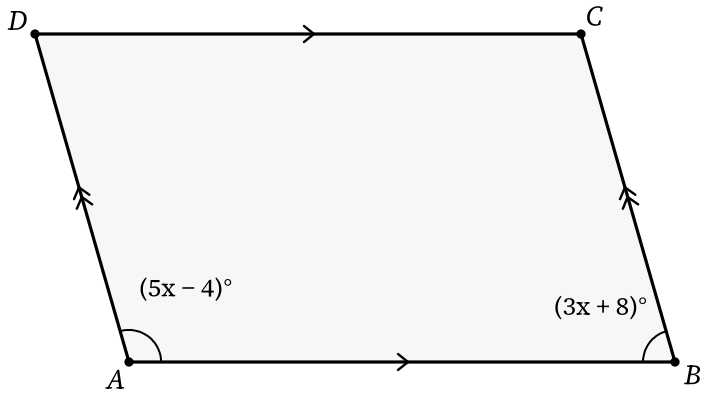


5. **5 pts**

   $ABCD$ is an isosceles trapezoid with bases $\overline{AB}$ and $\overline{DC}$. If $m\angle A = (3x + 5)°$ and $m\angle D = (5x + 15)°$, find x.

6. **6 pts**

In parallelogram $ABCD$ in the figure, $m\angle A = (5x - 4)^\circ$ and $m\angle B = (3x + 8)^\circ$. Find the measure of the larger of these two angles.



7. **6 pts**

The interior angles of a convex hexagon measure x°, $(x + 10)^\circ$, $(x + 20)^\circ$, $(x + 30)^\circ$, $(x + 40)^\circ$ and $(x + 50)^\circ$. Find the measure of the largest angle.

8. **6 pts**

A carpenter is making the wooden frame for a rose window in the shape of a regular 16-sided polygon. At what angle do two neighboring pieces of the frame meet (the interior angle)?

9. **5 pts**

One side of a barn roof is a trapezoid. Its bottom edge (the eave) is 32 ft long and its top edge (the ridge) is parallel to it. A row of shingles runs straight across the roof from the midpoint of one slanted edge to the midpoint of the other, and the row is 26 ft long. How long is the ridge?

10. **6 pts**

A floor tile is a rhombus whose diagonals measure 16 cm and 30 cm. Find the perimeter of the tile.

11. **6 pts**

Henry is building a rectangular deck 15 ft by 20 ft. He has set the frame so that the opposite sides measure 15 ft and 20 ft. What should each diagonal measure, and why does finding that both diagonals have this length prove that the corners are square?

12. **7 pts**

Quadrilateral $ABCD$ has vertices $A(-1, -2)$, $B(4, -1)$, $C(6, 3)$ and $D(1, 2)$. Classify it as precisely as possible: parallelogram, rectangle, rhombus or square. Justify your answer with slopes or distances.

13. **8 pts**

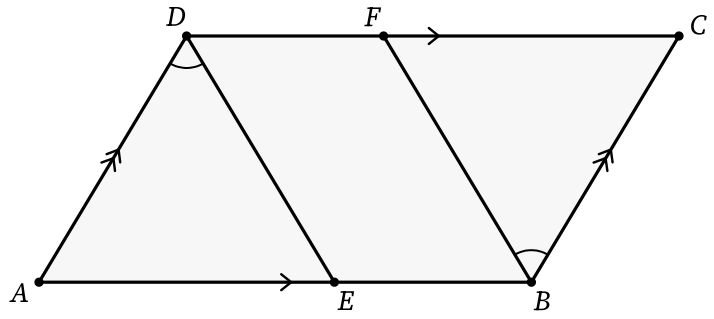
Draw a segment $\overline{AB}$ about 5 cm long. With compass and straightedge, construct a square with side $\overline{AB}$. List your steps and explain why the figure you constructed is a square.

14. **5 pts**

Prove indirectly: a convex quadrilateral cannot have four acute angles.

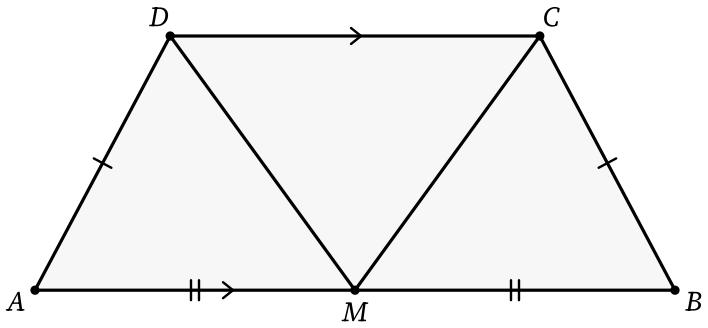
15. **10 pts**

Given: $ABCD$ is a parallelogram; E lies on $\overline{AB}$ and F lies on $\overline{CD}$, with $\angle ADE \cong \angle CBF$, as in the figure. Prove: $\overline{DE} \cong \overline{BF}$. Write a two-column proof.



16. **10 pts**

Given: $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$, and M is the midpoint of $\overline{AB}$, as in the figure. Prove: $\overline{MD} \cong \overline{MC}$. Write a flow proof.



ANSWER KEY

1. $(9-2)\cdot 180^\circ = 1260^\circ$
2. $360 \div 20 = 18$ sides
3. The diagonals bisect each other: $3y - 4 = y + 10$, so $y = 7$ and $PT = TR = 17$. Then $PR = 34$.
4. $\angle D \cong \angle B$, so $m\angle D = 115^\circ$. Then $m\angle C = 360^\circ - 80^\circ - 115^\circ - 115^\circ = 50^\circ$.
5. $\angle A$ and $\angle D$ lie along leg $\overline{AD}$, so they are supplementary: $8x + 20 = 180$, $x = 20$. (The angles are 65° and 115°.)
6. Consecutive angles are supplementary: $8x + 4 = 180$, so $x = 22$. Then $m\angle A = 106^\circ$ and $m\angle B = 74^\circ$; the larger is 106°.
7. The sum is $(6-2)\cdot 180^\circ = 720^\circ$: $6x + 150 = 720$, so $x = 95$. The largest angle is $95^\circ + 50^\circ = 145^\circ$.
8. $\dfrac{(16-2)\cdot 180^\circ}{16} = \dfrac{2520^\circ}{16} = 157.5^\circ$
9. The row is the midsegment: $\dfrac{32 + r}{2} = 26$, so $32 + r = 52$ and the ridge is $r = 20$ ft.
10. The diagonals of a rhombus bisect each other at right angles, making right triangles with legs 8 cm and 15 cm. Each side is $\sqrt{64 + 225} = \sqrt{289} = 17$ cm, so the perimeter is $4 \times 17 = 68$ cm.
11. Each diagonal should be $\sqrt{15^2 + 20^2} = \sqrt{625} = 25$ ft. Both pairs of opposite sides are congruent, so the frame is a parallelogram (Opposite Sides Converse); a parallelogram with congruent diagonals is a rectangle (Rectangle Diagonals Theorem), so all four corners are right angles.
12. Parallelogram. Slopes: $\overline{AB}$ and $\overline{DC}$ are both $\frac{1}{5}$; $\overline{BC}$ and $\overline{AD}$ are both 2; so both pairs of opposite sides are parallel. It is not a rhombus, because $AB = \sqrt{26}$ but $BC = \sqrt{20}$. It is not a rectangle, because the slopes $\frac{1}{5}$ and 2 do not multiply to -1, so $\angle B$ is not a right angle.
13. Steps (Euclid I.46): (1) Extend line AB past A. (2) Construct the perpendicular to line AB at A: arcs centered at A cut the line at P and Q; equal arcs from P and Q cross at R; draw $\overleftrightarrow{AR}$. (3) With compass set to AB, mark D on $\overleftrightarrow{AR}$ with $AD = AB$. (4) With the same width, swing arcs from B and from D that cross at C. (5) Draw $\overline{BC}$ and $\overline{DC}$. Why: all four sides equal AB, so $ABCD$ is a rhombus and therefore a parallelogram; $\angle A$ is a right angle by construction; consecutive angles of a parallelogram are supplementary and opposite angles congruent, so

all four angles are right angles. Four congruent sides and four right angles make a square.

14. See the model proof.

    INDIRECT PROOF

    Given: $ABCD$ is a convex quadrilateral

    Prove: $ABCD$ does not have four acute angles

    Assume, to the contrary, that all four angles of $ABCD$ are acute.
    Then each angle measures less than $90°$, so the sum of the four angles is less than $4 \cdot 90° = 360°$. But by the Polygon Interior Angle Sum Theorem the angles of a quadrilateral add to exactly $(4-2) \cdot 180° = 360°$. This is a contradiction.
    So the assumption is false: a convex quadrilateral cannot have four acute angles.

    ∎

15. See the model proof.

    TWO-COLUMN PROOF

    Given: $ABCD$ is a parallelogram; E on $\overline{AB}$, F on $\overline{CD}$; $\angle ADE \cong \angle CBF$

    Prove: $\overline{DE} \cong \overline{BF}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; E on $\overline{AB}$, F on $\overline{CD}$; $\angle ADE \cong \angle CBF$	Given
2	$\angle DAE \cong \angle BCF$	Parallelogram Opposite Angles Theorem ($\angle DAE$ is $\angle DAB$ and $\angle BCF$ is $\angle BCD$)
3	$\overline{AD} \cong \overline{CB}$	Parallelogram Opposite Sides Theorem
4	$\triangle ADE \cong \triangle CBF$	ASA
5	$\overline{DE} \cong \overline{BF}$	CPCTC

    ∎

16. See the model proof.

    FLOW PROOF

    Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$; M is the midpoint of $\overline{AB}$

    Prove: $\overline{MD} \cong \overline{MC}$

    (1) $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

Given

(2) M is the midpoint of $\overline{AB}$
Given

(3) $\overline{AM} \cong \overline{BM}$
Definition of midpoint
from (2)

(4) $\angle DAM \cong \angle CBM$
Isosceles Trapezoid Base Angles Theorem
from (1)

(5) $\triangle DAM \cong \triangle CBM$
SAS
from (1), (3), (4)

(6) $\overline{MD} \cong \overline{MC}$
CPCTC
from (5)

∎

6.1 Angles of Polygons

OBJECTIVES

- Classify polygons as convex or concave and count their diagonals.
- Prove the Polygon Interior Angle Sum Theorem by dividing a polygon into triangles, and use it to find unknown angles.
- Find each interior and exterior angle of a regular polygon, and find the number of sides from an angle.
- Prove and use the Exterior Angle Sum Theorem, and decide which regular polygons can tile a floor.

WATCH

Angle Sums of a Polygon (1 of 2: Interior angles)

Eddie Woo · 9:21

https://www.youtube.com/watch?v=frjiwP32PC0

> **WHEN TO WATCH**
> Before the Polygon Interior Angle Sum Theorem: dividing a polygon into triangles.

BACKUP VIDEO

Angle Sums of a Polygon (2 of 2: Exterior angles)

Eddie Woo · 4:44

https://www.youtube.com/watch?v=mq_9drgfudo

Before the exterior-angle examples.

KEY TERMS

polygon

A closed plane figure made of three or more segments (its sides), each side meeting exactly two other sides, one at each endpoint (its vertices), so that sides meet only at their endpoints and no two sides with a common endpoint are collinear. A polygon with n sides is an n-gon; it is named by listing its vertices in order.

convex polygon

A polygon in which no line that contains a side passes through the interior; every interior angle is less than 180°.

concave polygon

A polygon that is not convex; at least one interior angle is greater than 180°, so the polygon 'caves in'.

diagonal

A segment that joins two nonconsecutive vertices of a polygon.

regular polygon

A polygon that is both equilateral (all sides congruent) and equiangular (all interior angles congruent).

exterior angle of a polygon

The angle formed at a vertex by one side and the extension of the adjacent side; it forms a linear pair with the interior angle at that vertex.

CORE IDEAS

You already know the angles of every triangle add up to 180° (Lesson 4.1, Euclid I.32). In this lesson we push that one fact much further: it tells us the angle sum of a quadrilateral, a pentagon, a 100-gon, any polygon at all. A **polygon** is a closed figure made of segments that meet only at their endpoints. We name it by going around its vertices in order, so pentagon $ABCDE$ has sides $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EA}$.

A polygon is **convex** if no line through a side cuts into the polygon; then every interior angle is less than 180° and every diagonal lies inside. Otherwise it is **concave**: it has a 'dent', an angle greater than 180°, and some diagonal lies outside. A **diagonal** joins two vertices that are not next to each other. Each vertex of an n-gon can be joined by a diagonal to $n - 3$ others (not to itself and not to its two neighbors).

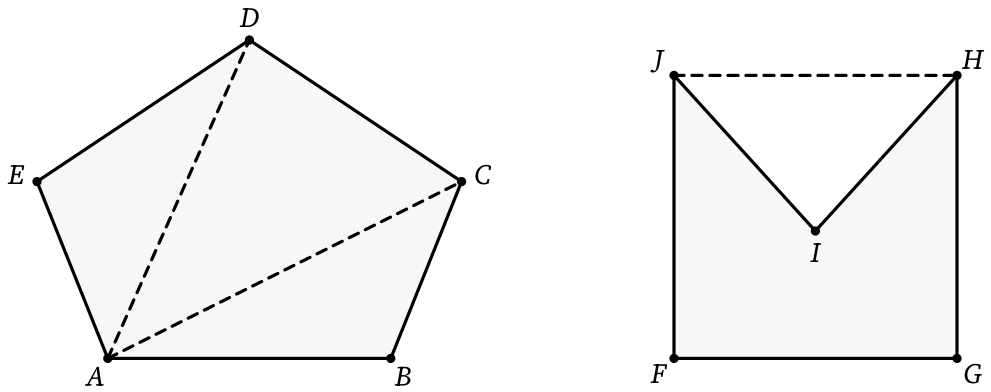
That gives $n(n-3)$ ends of diagonals, but each diagonal has two ends and was counted twice, so an n-gon has $\dfrac{n(n-3)}{2}$ diagonals.

Why is the angle sum $(n-2)\cdot 180^\circ$? Stand at one vertex of a convex n-gon and draw every diagonal from it. You draw $n-3$ diagonals, and they cut the polygon into $n-2$ triangles (one for each side that does not touch your vertex). The angles of those triangles fit together exactly to make up the angles of the polygon, so the polygon's angle sum is $n-2$ triangles' worth: $(n-2)\cdot 180^\circ$. A quadrilateral makes 2 triangles (360°), a pentagon 3 (540°), a hexagon 4 (720°).

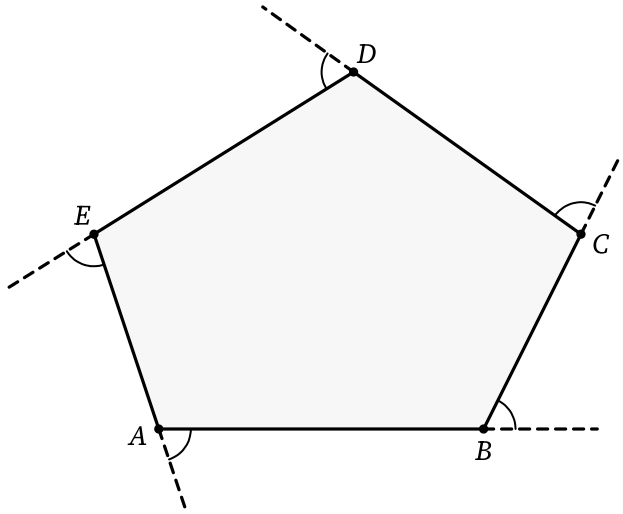
A **regular polygon** has all sides congruent *and* all angles congruent. Both conditions matter: a rhombus has equal sides but not always equal angles, and a rectangle has equal angles but not always equal sides. Because the n angles of a regular polygon are equal, each one is the sum divided by n: $\dfrac{(n-2)\cdot 180^\circ}{n}$. For a regular hexagon that is $\dfrac{720^\circ}{6} = 120^\circ$.

At each vertex, extend one side to form an **exterior angle**. It makes a linear pair with the interior angle, so the two add to 180°. Adding over all n vertices gives $n\cdot 180^\circ$ for interior plus exterior angles together; take away the interior sum $(n-2)\cdot 180^\circ$ and exactly 360° is left, no matter how many sides there are. You can feel why: walk all the way around a convex field, turning at each corner by the exterior angle. When you return to your start you are facing the same way again, so you have turned exactly once around: 360°.

Floor tiles and windows use these facts. Copies of one regular polygon can fill the space around a point with no gap or overlap only when its interior angle divides 360° evenly. Equilateral triangles ($6\times 60^\circ$), squares ($4\times 90^\circ$) and regular hexagons ($3\times 120^\circ$) work; regular pentagons (108°) do not, because $360 \div 108$ is not a whole number. Gothic rose windows are built the other way around, on a circle divided into equal parts, so their stone frames are regular polygons.



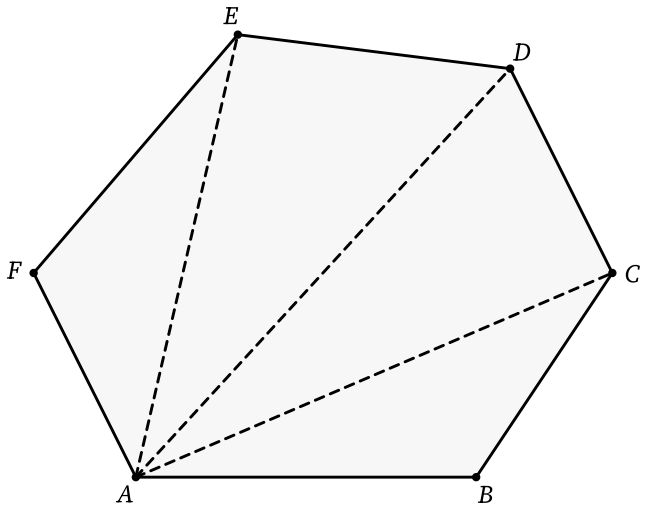
Left: a convex pentagon with two diagonals from A. Right: a concave pentagon; its diagonal HJ lies outside it.



One exterior angle at each vertex: each side is extended as you walk around.

THEOREM Polygon Interior Angle Sum Theorem

The sum of the measures of the interior angles of a convex polygon with n sides is $(n-2)\cdot 180^\circ$.



PARAGRAPH PROOF

Given: Convex polygon $V_1V_2V_3\ldots V_n$ with n sides

Prove: The sum of the measures of the interior angles is $(n-2)\cdot 180^\circ$.

From vertex V_1 draw the diagonals $\overline{V_1V_3}, \overline{V_1V_4}, \ldots, \overline{V_1V_{n-1}}$ (two points determine a line). V_1 cannot be joined by a diagonal to itself or to its neighbors V_2 and V_n, so there are $n-3$ diagonals. Because the polygon is convex, each of them lies inside the polygon, and together they cut it into the $n-2$ triangles $\triangle V_1V_2V_3, \triangle V_1V_3V_4, \ldots, \triangle V_1V_{n-1}V_n$, one for each of the $n-2$ sides that do not touch V_1. (In the figure, hexagon $ABCDEF$ is cut by three diagonals from A into four triangles.)

Every angle of these triangles is part of an interior angle of the polygon, and together they fill each interior angle exactly once. At V_2 and at V_n the polygon's angle is an angle of just one triangle. At V_1 and at each of $V_3, \ldots, V_{n-1}$ the diagonals split the polygon's angle into angles of neighboring triangles, and by the Angle Addition Postulate the polygon's angle equals the sum of those parts.

Therefore the sum of the interior angles of the polygon equals the sum of all the angles of the $n-2$ triangles. By the Triangle Angle Sum Theorem each triangle contributes 180°, so the sum is $(n-2)\cdot 180^\circ$.

∎

The theorem is also true for concave polygons (see Practice 14 for a quadrilateral), but this course proves it for convex polygons.

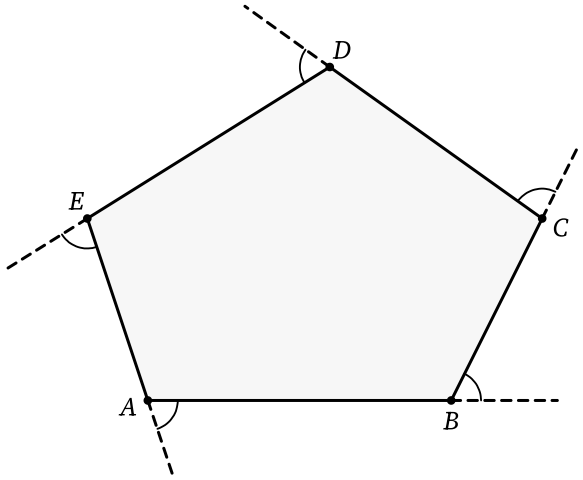
COROLLARY **Interior Angle of a Regular Polygon**

Each interior angle of a regular n-gon measures $\dfrac{(n-2)\cdot 180^\circ}{n}$.

The n angles are congruent and their sum is $(n-2)\cdot 180^\circ$, so each is the sum divided by n.

THEOREM **Polygon Exterior Angle Sum Theorem**

The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360°.



One exterior angle at each vertex: each side is extended as you walk around.

TWO-COLUMN PROOF

Given: A convex polygon with n sides; One exterior angle is drawn at each vertex

Prove: The sum of the exterior angles is 360°

	STATEMENTS	REASONS
1	A convex n-gon with one exterior angle at each vertex	Given
2	At each vertex the interior angle and the exterior angle form a linear pair	Definition of exterior angle of a polygon
3	At each vertex, (interior angle) + (exterior angle) $= 180^\circ$	Linear Pair Theorem (Euclid I.13)
4	(sum of interior angles) + (sum of exterior angles) $= n \cdot 180^\circ$	Addition Property of Equality (add the n equations)
5	Sum of interior angles $= (n-2)\cdot 180^\circ$	Polygon Interior Angle Sum Theorem

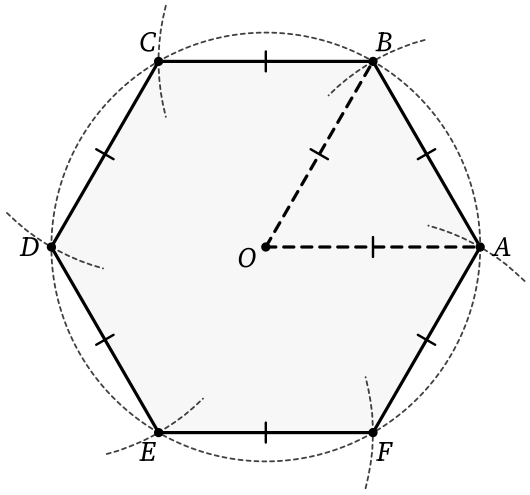
	STATEMENTS	REASONS
6	$(n-2)\cdot 180^\circ$ + (sum of exterior angles) $= n\cdot 180^\circ$	Substitution Property of Equality
7	Sum of exterior angles $= n\cdot 180^\circ - (n-2)\cdot 180^\circ$	Subtraction Property of Equality
8	The sum of the exterior angles is 360°	Distributive Property: $180n - 180n + 360 = 360$

∎

Corollary: each exterior angle of a regular n-gon measures $\dfrac{360^\circ}{n}$.

CONSTRUCTIONS

CONSTRUCTION **Inscribe a regular hexagon in a circle** Euclid IV.15



1. With center O, draw a circle of any radius. Mark a point A on the circle.
2. Keep the compass set to the radius. Put the point at A and swing an arc that crosses the circle at B.
3. Without changing the compass, put the point at B and mark C on the circle; then from C mark D, from D mark E, and from E mark F.
4. Check: an arc from F with the same width lands exactly on A.
5. With the straightedge draw $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}$ and $\overline{FA}$. Hexagon $ABCDEF$ is regular.

Why it works. $OA = OB = AB$ (all equal the radius), so $\triangle OAB$ is

equilateral, hence equiangular, and $m\angle AOB = 60^\circ$. The same is true of each of the six triangles around O, and $6 \times 60^\circ = 360^\circ$, so the sixth step closes exactly at A. Every side of the hexagon equals the radius, so it is equilateral; each angle of the hexagon, such as $\angle ABC$, is made of two 60° angles of neighboring triangles, so every angle is 120° and it is equiangular. Euclid adds that the side of the hexagon is equal to the radius of the circle.

KEY FORMULAS

$$\text{Sum of interior angles of an } n\text{-gon} = (n-2)\cdot 180^\circ$$

$$\text{Each interior angle of a regular } n\text{-gon} = \frac{(n-2)\cdot 180^\circ}{n}$$

$$\text{Sum of exterior angles (one at each vertex)} = 360^\circ$$

$$\text{Each exterior angle of a regular } n\text{-gon} = \frac{360^\circ}{n}$$

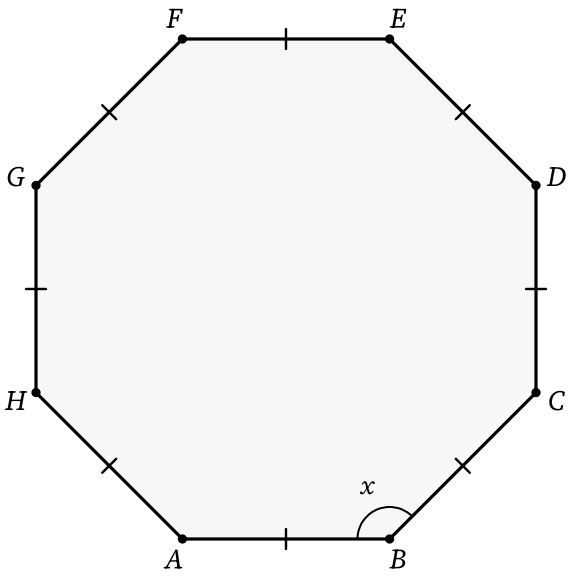
$$\text{Number of diagonals of an } n\text{-gon} = \frac{n(n-3)}{2}$$

From history. The great Gothic cathedrals built in France in the 1100s and 1200s, such as Chartres and Notre-Dame in Paris, are famous for their round rose windows. The stone tracery of a rose window is laid out on a circle divided into equal parts, so its framework is a regular polygon with spokes running to the center. Master masons drew such patterns full size with compass and cord, stepping equal arcs around a circle much as in Euclid's construction of the regular hexagon in this lesson.

Euclid connection. Euclid proves the triangle angle sum in I.32, and commentators on that proposition, beginning with Proclus, draw out its consequence for any rectilinear figure: the interior angles together equal twice as many right angles as the figure has sides, less four right angles, and the exterior angles together equal four right angles. In degrees, $(2n-4)\cdot 90^\circ = (n-2)\cdot 180^\circ$ and $4\cdot 90^\circ = 360^\circ$, which are this lesson's two theorems. Book IV of the Elements constructs regular polygons in circles, including the hexagon (IV.15).

Practice habit. Recite the two angle-sum theorems word for word until they are automatic: 'The interior angles of a convex n-gon add to $(n-2)\cdot 180^\circ$' and 'The exterior angles, one at each vertex, add to 360°.' Before you calculate, estimate: an interior angle of a regular polygon is always at least 60°, less than 180°, and grows as n grows.

Example 1. The Miller family is building a garden gazebo whose floor is a regular octagon. What is the measure of each interior angle of the floor, where two sides meet?



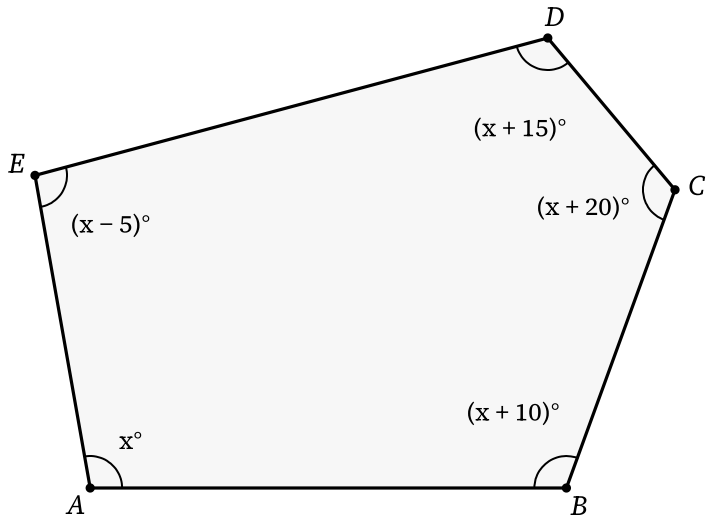
The gazebo floor, a regular octagon

Show solution

1. An octagon has $n = 8$ sides.
2. Sum of the interior angles: $(8 - 2) \cdot 180^\circ = 6 \cdot 180^\circ = 1080^\circ$.
3. The octagon is regular, so its 8 angles are congruent: $1080^\circ \div 8 = 135^\circ$.
4. Check with exterior angles: each exterior angle is $360^\circ \div 8 = 45^\circ$, and $180^\circ - 45^\circ = 135^\circ$. The answers agree.

Answer: 135°

Example 2. In pentagon $ABCDE$ in the figure, $m\angle A = x°$, $m\angle B = (x + 10)°$, $m\angle C = (x + 20)°$, $m\angle D = (x + 15)°$ and $m\angle E = (x - 5)°$. Find the measure of the largest angle of the pentagon.



Show solution

1. A pentagon has $n = 5$, so its angles add to $(5 - 2) \cdot 180° = 540°$.
2. Write the equation: $x + (x + 10) + (x + 20) + (x + 15) + (x - 5) = 540$.
3. Combine like terms: $5x + 40 = 540$, so $5x = 500$ and $x = 100$.
4. The angles are $100°, 110°, 120°, 115°, 95°$. Check: $100 + 110 + 120 + 115 + 95 = 540$. The largest is $m\angle C = 120°$.

Answer: $x = 100$, so the largest angle is $m\angle C = 120°$

PRACTICE

1. Easy Find the sum of the measures of the interior angles of a convex hexagon.

   Show answer

   $$(6 - 2) \cdot 180° = 720°$$

2. Easy Find the measure of each interior angle of a regular decagon (10 sides).

   Show answer

   $$\frac{(10 - 2) \cdot 180°}{10} = \frac{1440°}{10} = 144°$$

3. **Easy** The stone frame of a church's rose window is a regular 12-sided polygon. Find the measure of each interior angle of the frame.

Show answer

$$\frac{(12-2)\cdot 180^\circ}{12} = \frac{1800^\circ}{12} = 150^\circ$$

4. **Easy** How many diagonals does an octagon have?

Show answer

$$\frac{8(8-3)}{2} = \frac{40}{2} = 20 \text{ diagonals}$$

5. **Medium** Each exterior angle of a regular polygon measures 24°. How many sides does the polygon have?

Show answer

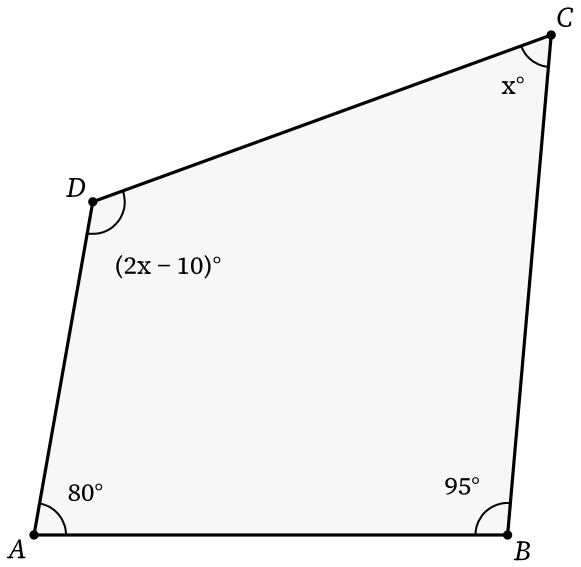
The exterior angles add to 360°, so $n = 360 \div 24 = 15$ sides.

6. **Medium** Each interior angle of a regular polygon measures 140°. How many sides does the polygon have?

Show answer

Each exterior angle is $180^\circ - 140^\circ = 40^\circ$, so $n = 360 \div 40 = 9$ sides.

7. **Medium** In quadrilateral $ABCD$ in the figure, $m\angle A = 80°$, $m\angle B = 95°$, $m\angle C = x°$ and $m\angle D = (2x - 10)°$. Find x.



Show answer

$80 + 95 + x + (2x - 10) = 360$, so $3x + 165 = 360$, $3x = 195$, $x = 65$. (Then $m\angle D = 120°$.)

8. **Medium** The exterior angles of a convex pentagon, one at each vertex, measure $x°$, $2x°$, $3x°$, $3x°$ and $3x°$. Find the measure of the largest interior angle of the pentagon.

Show answer

$12x = 360$, so $x = 30$. The smallest exterior angle, $30°$, sits at the largest interior angle: $180° - 30° = 150°$.

9. **Medium** Samuel wants to tile a floor using only regular pentagons, fitted edge to edge with corners meeting at points. Can copies of a regular pentagon fill the space around a point with no gap and no overlap? Explain why or why not.

Show answer

No. Each angle of a regular pentagon is $\frac{3 \cdot 180°}{5} = 108°$. Three pentagons fill $324°$, leaving a $36°$ gap, and four would need $432°$, more than the $360°$ around a point. Since $360 \div 108$ is not a whole number, pentagons cannot fit exactly.

10. Hard Each interior angle of a regular polygon is 5 times as large as each of its exterior angles. How many sides does the polygon have?

Show answer

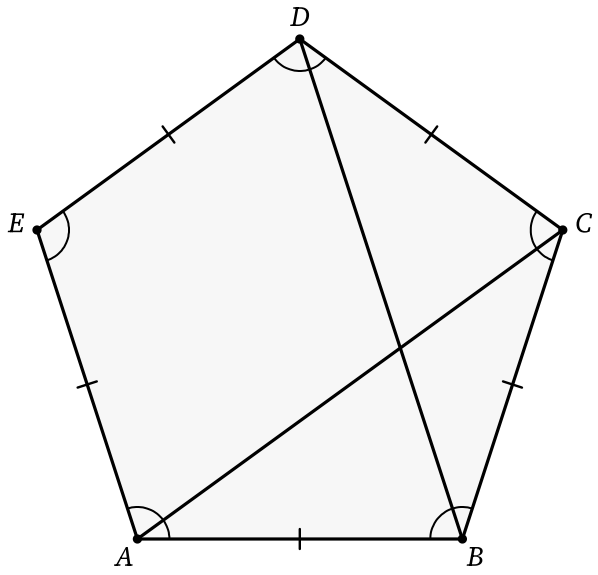
At each vertex, interior + exterior $= 180^\circ$. With exterior e and interior $5e$: $6e = 180$, $e = 30^\circ$, so $n = 360 \div 30 = 12$ sides.

11. Hard Grace is designing a kitchen floor of regular octagons and squares that meet edge to edge, with two octagons and one square around every corner point. Find the measure of each interior angle of a regular octagon, and show that the three tiles fit around a point with no gap or overlap.

Show answer

Each octagon angle is $\dfrac{6 \cdot 180^\circ}{8} = 135^\circ$. Around a point: $135^\circ + 135^\circ + 90^\circ = 360^\circ$, exactly a full turn, so the tiles fit.

12. Hard Given: $ABCDE$ is a regular pentagon, as in the figure. Prove: the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.



Show answer

See the model proof.

TWO-COLUMN PROOF

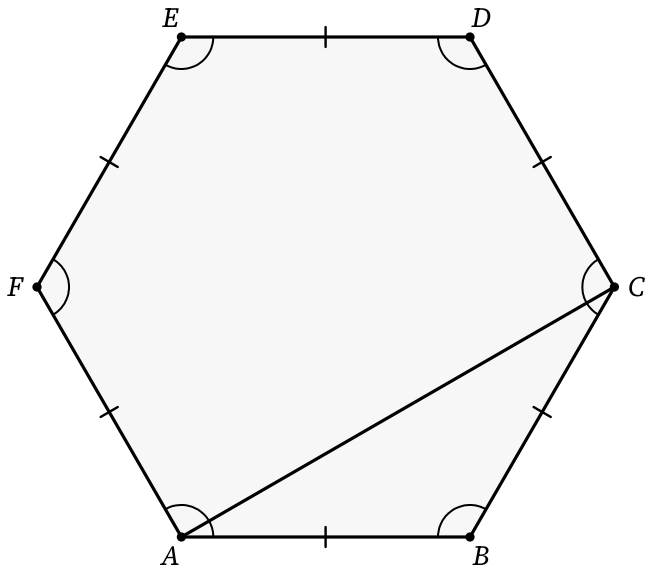
Given: $ABCDE$ is a regular pentagon

Prove: $\overline{AC} \cong \overline{BD}$

	STATEMENTS	REASONS
1	$ABCDE$ is a regular pentagon	Given
2	$\overline{AB} \cong \overline{DC}$	Definition of regular polygon (all sides congruent)
3	$\angle ABC \cong \angle DCB$	Definition of regular polygon (all angles congruent)
4	$\overline{BC} \cong \overline{CB}$	Reflexive Property of Congruence
5	$\triangle ABC \cong \triangle DCB$	SAS
6	$\overline{AC} \cong \overline{BD}$	CPCTC

∎

13. Hard Given: $ABCDEF$ is a regular hexagon, and diagonal $\overline{AC}$ is drawn, as in the figure. Prove: $m\angle BAC = 30°$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCDEF$ is a regular hexagon; $\overline{AC}$ is a diagonal

Prove: $m\angle BAC = 30°$

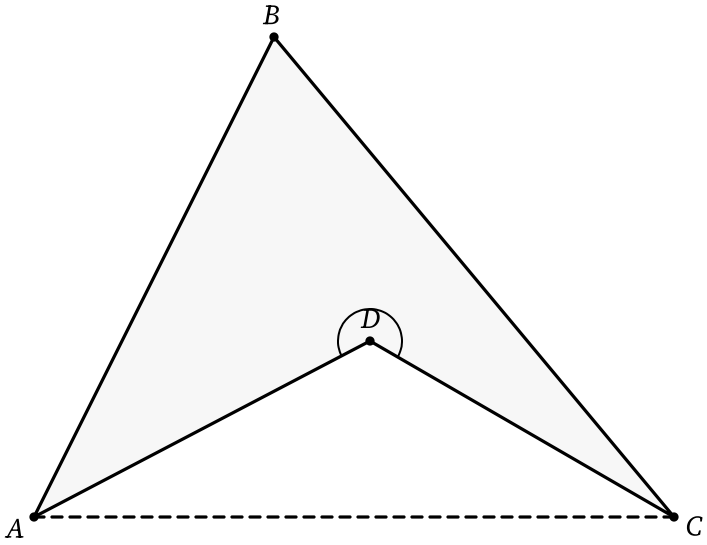
Each interior angle of a regular hexagon measures $\frac{(6-2)\cdot 180°}{6} = 120°$ (corollary of the Polygon Interior Angle Sum Theorem), so $m\angle ABC = 120°$.

All sides of a regular polygon are congruent, so $\overline{BA} \cong \overline{BC}$ and $\triangle ABC$ is isosceles with base $\overline{AC}$. By the Isosceles Triangle Theorem, its base angles are congruent: $m\angle BAC = m\angle BCA$.

By the Triangle Angle Sum Theorem, $m\angle BAC + m\angle BCA + 120° = 180°$, so $2\,m\angle BAC = 60°$ by substitution, and $m\angle BAC = 30°$ by the Division Property of Equality.

∎

14. **Challenge** Concave quadrilateral $ABCD$ in the figure has vertex D inside $\triangle ABC$, so its interior angle at D is the reflex angle, which measures $360^\circ - m\angle ADC$. Prove that the four interior angles of $ABCD$ still add to 360°: $m\angle DAB + m\angle ABC + m\angle BCD + (360^\circ - m\angle ADC) = 360^\circ$. (Hint: draw $\overline{AC}$.)



Show answer

See the model proof.

PARAGRAPH PROOF

Given: Quadrilateral $ABCD$ with D inside $\triangle ABC$; The interior angle at D measures $360^\circ - m\angle ADC$

Prove: $m\angle DAB + m\angle ABC + m\angle BCD + (360^\circ - m\angle ADC) = 360^\circ$

Draw $\overline{AC}$ (two points determine a line). Because D lies inside $\triangle ABC$, it lies in the interior of $\angle BAC$ and of $\angle BCA$. By the Angle Addition Postulate, $m\angle BAC = m\angle DAB + m\angle DAC$ and $m\angle BCA = m\angle BCD + m\angle DCA$, so $m\angle DAB = m\angle BAC - m\angle DAC$ and $m\angle BCD = m\angle BCA - m\angle DCA$.

Substitute these into the sum $S = m\angle DAB + m\angle ABC + m\angle BCD + (360^\circ - m\angle ADC)$ and regroup: $S = (m\angle BAC + m\angle ABC + m\angle BCA) - (m\angle DAC + m\angle DCA + m\angle ADC) + 360^\circ$.

The first group is the angle sum of $\triangle ABC$ and the second is the angle sum of $\triangle ADC$; by the Triangle Angle Sum Theorem each is 180°. So $S = 180^\circ - 180^\circ + 360^\circ = 360^\circ$.

∎

COMMON MISTAKES

- Using $n \cdot 180^\circ$ instead of $(n-2) \cdot 180^\circ$. Test your formula on a triangle: $n = 3$ must give 180°.
- Dividing the angle sum by n for a polygon that is not regular. Only a regular polygon has all angles equal; for any other polygon, write an equation with the given angles.
- Thinking the exterior angle sum grows with the number of sides. It is always 360° (one exterior angle at each vertex), which is why $\frac{360^\circ}{n}$ is the quickest way to an angle of a regular polygon.
- Counting diagonals as $n(n-3)$ and forgetting to divide by 2; every diagonal has two ends and was counted twice.

> **Make it a game · Turtle Walk.** Walk the outline of a regular polygon in the living room, turning at each corner. Parent calls the number of sides; you state the turn at each corner (360 divided by n), the interior angle, and the angle sum. Then sketch a polygon with seven sides, predict its diagonals, draw them to check, and name a concave shape.

6.2 Properties of Parallelograms

OBJECTIVES

- Prove that the opposite sides and opposite angles of a parallelogram are congruent (Euclid I.34).
- Prove that consecutive angles of a parallelogram are supplementary and that its diagonals bisect each other.
- Use the properties of parallelograms to find lengths, angle measures and missing vertices with algebra and coordinates.
- Write two-column, paragraph and flow proofs that use the properties of parallelograms.

WATCH

Using Quadrilateral Properties (2 of 2: Example Proof)

Eddie Woo · 6:57

https://www.youtube.com/watch?v=B9OshgQMRU0

> **WHEN TO WATCH**
> After the parallelogram theorems: using them as reasons to find angles.

Note: Eddie Woo teaches in Australia: he says 'co-interior angles' for same-side interior angles and 'pronumeral' for variable.

BACKUP VIDEO

What is a Parallelogram? | Quadrilaterals | Math with Mr. J

Math with Mr. J · 8:09

https://www.youtube.com/watch?v=NJUeESYpovI

Before the Core ideas, for review: what a parallelogram is.

KEY TERMS

parallelogram

A quadrilateral with both pairs of opposite sides parallel. Parallelogram $ABCD$ has $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$.

opposite sides and opposite angles

In quadrilateral $ABCD$, sides that do not share a vertex ($\overline{AB}$ and $\overline{CD}$; $\overline{BC}$ and $\overline{DA}$) and angles at vertices that are not next to each other ($\angle A$ and $\angle C$; $\angle B$ and $\angle D$).

consecutive angles

Two angles of a polygon whose vertices are the endpoints of the same side, such as $\angle A$ and $\angle B$ in quadrilateral $ABCD$.

CORE IDEAS

A **parallelogram** is a quadrilateral whose opposite sides are parallel. The letters go around the figure in order, so in parallelogram $ABCD$ side $\overline{AB}$ is opposite $\overline{DC}$ and $\overline{AD}$ is opposite $\overline{BC}$. When it is clear which polygon we mean, $\angle A$ stands for the polygon's angle at A, here $\angle DAB$.

The definition says only 'parallel'. Everything else we learn about a parallelogram must be proved, and Euclid shows the key move in I.34: draw a diagonal. Diagonal $\overline{AC}$ is a transversal of both pairs of parallel sides, so it makes two pairs of congruent alternate interior angles, and it is a side of both triangles it creates. By ASA the two triangles are congruent. From those congruent triangles come the facts that opposite sides are congruent and opposite angles are congruent.

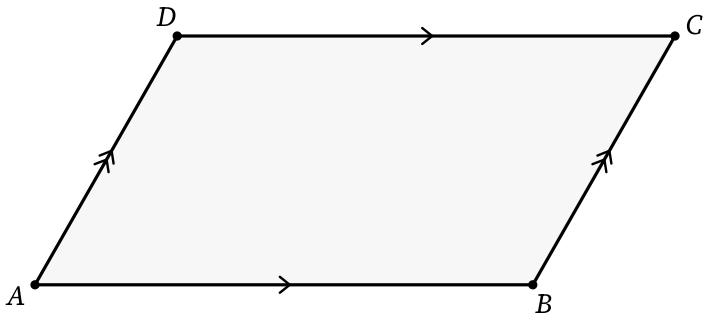
Consecutive angles, such as $\angle A$ and $\angle B$, are same-side interior angles for the parallel sides $\overline{AD}$ and $\overline{BC}$ cut by the transversal $\overline{AB}$, so they are supplementary. This gives a second, quick reason that opposite angles are congruent: $\angle A$ and $\angle C$ are both supplements of $\angle B$.

Draw both diagonals and they cross at a point E. Triangles ABE and CDE are congruent by ASA (alternate interior angles and the congruent opposite sides $\overline{AB}$ and $\overline{CD}$), so E is the midpoint of both diagonals: the diagonals **bisect each other**. Be careful: in general the diagonals are *not* congruent and *not* perpendicular. Those are

special properties of rectangles and rhombuses (Lesson 6.4), and you may not read them from a figure.

These four properties turn geometry into algebra. If two opposite sides are labeled $3x + 4$ and $5x - 8$, set them equal. If two consecutive angles are labeled, their sum is $180°$. On the coordinate plane, the two diagonals must share a midpoint, which lets you find a missing vertex.

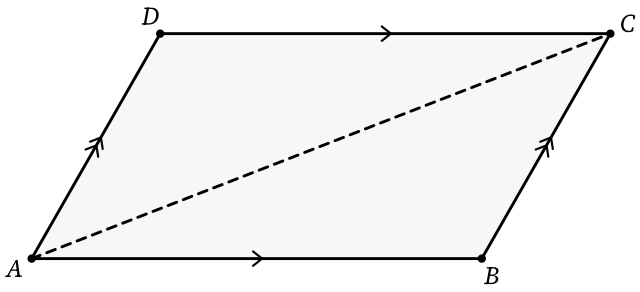
Notice the direction of every theorem in this lesson: *if* a quadrilateral is a parallelogram, *then* it has these properties. The converses (if a quadrilateral has these properties, then it is a parallelogram) are different statements that need their own proofs. That is the work of Lesson 6.3.



POSTULATES AND THEOREMS

THEOREM **Parallelogram Opposite Sides Theorem** Euclid I.34

If a quadrilateral is a parallelogram, then its opposite sides are congruent.



TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram
Prove: $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram	Given
2	$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	Definition of parallelogram

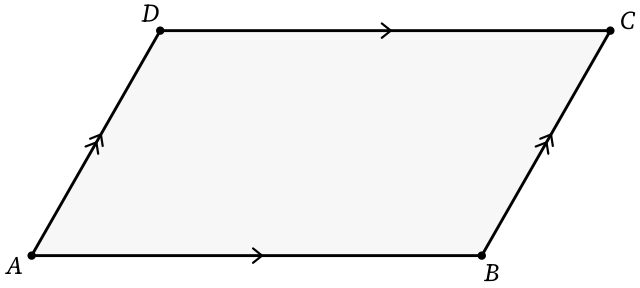
	STATEMENTS	REASONS
3	Draw diagonal $\overline{AC}$	Two points determine a line
4	$\angle BAC \cong \angle DCA$	Alternate Interior Angles Theorem ($\overline{AB} \parallel \overline{DC}$)
5	$\angle BCA \cong \angle DAC$	Alternate Interior Angles Theorem ($\overline{AD} \parallel \overline{BC}$)
6	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
7	$\triangle ABC \cong \triangle CDA$	ASA
8	$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$	CPCTC

∎

This is Euclid's own proof of I.34, which uses I.29 (alternate angles) and I.26 (ASA). The same pair of congruent triangles also gives $\angle B \cong \angle D$.

THEOREM **Parallelogram Consecutive Angles Theorem** Euclid I.29

If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.



PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram

Prove: Each pair of consecutive angles of $ABCD$ is supplementary

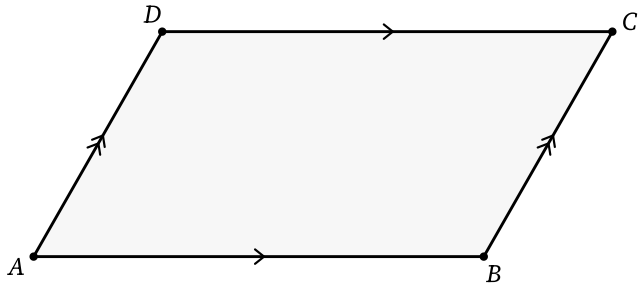
By the definition of a parallelogram, $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$. Angles $\angle A$ and $\angle B$ are same-side interior angles formed when the parallel lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are cut by the transversal $\overleftrightarrow{AB}$, so they are supplementary by the Same-Side Interior Angles Theorem.

In the same way, $\angle B$ and $\angle C$ are same-side interior angles for $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ with transversal $\overleftrightarrow{BC}$; $\angle C$ and $\angle D$ for $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$ with transversal $\overleftrightarrow{CD}$; and $\angle D$ and $\angle A$ for $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$ with transversal $\overleftrightarrow{DA}$. So each pair of consecutive angles is supplementary.

∎

THEOREM **Parallelogram Opposite Angles Theorem** Euclid I.34

If a quadrilateral is a parallelogram, then its opposite angles are congruent.



FLOW PROOF

Given: $ABCD$ is a parallelogram

Prove: $\angle A \cong \angle C$ and $\angle B \cong \angle D$

(1) $ABCD$ is a parallelogram
Given

(2) $\angle A$ and $\angle B$ are supplementary
Parallelogram Consecutive Angles Theorem
from (1)

(3) $\angle B$ and $\angle C$ are supplementary
Parallelogram Consecutive Angles Theorem
from (1)

(4) $\angle C$ and $\angle D$ are supplementary
Parallelogram Consecutive Angles Theorem
from (1)

(5) $\angle A \cong \angle C$
Congruent Supplements Theorem (both are supplements of $\angle B$)
from (2), (3)

(6) $\angle B \cong \angle D$
Congruent Supplements Theorem (both are supplements of $\angle C$)
from (3), (4)

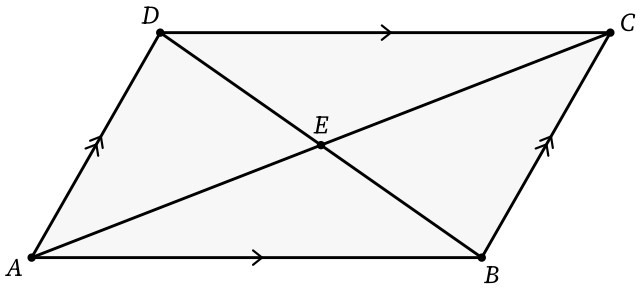
(7) $\angle A \cong \angle C$ and $\angle B \cong \angle D$
Steps 5 and 6 together
from (5), (6)

∎

Euclid proves this half of I.34 differently, from the congruent triangles of the first theorem and angle addition at A and C.

THEOREM **Parallelogram Diagonals Theorem**

If a quadrilateral is a parallelogram, then its diagonals bisect each other.



TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; Diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; $\overline{AC}$ and $\overline{BD}$ meet at E	Given
2	$\overline{AB} \parallel \overline{DC}$	Definition of parallelogram
3	$\overline{AB} \cong \overline{CD}$	Parallelogram Opposite Sides Theorem
4	$\angle BAE \cong \angle DCE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{AC}$)
5	$\angle ABE \cong \angle CDE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BD}$)
6	$\triangle ABE \cong \triangle CDE$	ASA
7	$\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$	CPCTC

∎

So E is the midpoint of each diagonal.

KEY FORMULAS

$$AB = CD, \quad BC = DA$$

$$m\angle A = m\angle C, \quad m\angle B = m\angle D, \quad m\angle A + m\angle B = 180^\circ$$

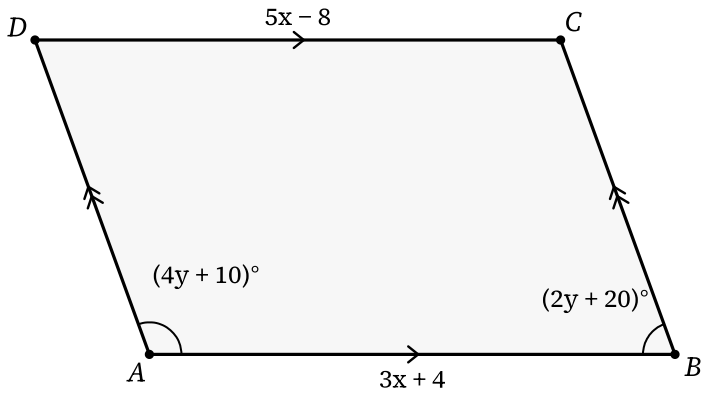
$$AE = EC, \quad BE = ED \quad (E \text{ the intersection of the diagonals})$$

From history. According to Proclus, the word 'parallelogram' (in Greek, 'bounded by parallel lines') was coined by Euclid himself, and it appears for the first time in I.34. Up to that point Euclid had proved facts about parallel lines one pair at a time; in I.34 he puts two pairs together and studies the figure they enclose, which he then uses again and again in the propositions on area that follow.

Euclid connection. Euclid I.34 reads: 'In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.' Euclid calls the diagonal the 'diameter'. His proof is exactly our first theorem: draw the diagonal, use alternate angles (I.29) and ASA (I.26). The last clause, that the diagonal cuts the area in half, waits for our area unit (Unit 11).

Practice habit. Mark every given on the diagram before you start: put parallel arrows on both pairs of opposite sides of a parallelogram. Then recite the four properties until they are automatic: opposite sides congruent, opposite angles congruent, consecutive angles supplementary, diagonals bisect each other.

Example 1. In parallelogram $ABCD$ in the figure, $AB = 3x + 4$, $CD = 5x - 8$, $m\angle A = (4y + 10)^\circ$ and $m\angle B = (2y + 20)^\circ$. Find x and y.

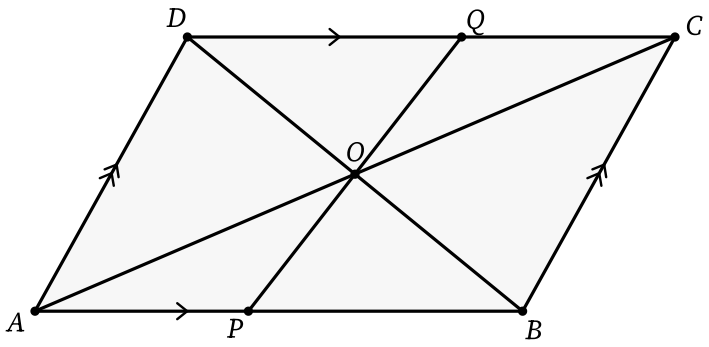


Show solution

1. Opposite sides of a parallelogram are congruent, so $3x + 4 = 5x - 8$. Then $12 = 2x$ and $x = 6$ (so $AB = CD = 22$).
2. Consecutive angles of a parallelogram are supplementary, so $(4y + 10) + (2y + 20) = 180$.
3. Then $6y + 30 = 180$, $6y = 150$, and $y = 25$.
4. Check: $m\angle A = 110^\circ$ and $m\angle B = 70^\circ$, which add to 180°.

Answer: $x = 6$ and $y = 25$

Example 2. Given: $ABCD$ is a parallelogram whose diagonals meet at O; a line through O meets $\overline{AB}$ at P and $\overline{CD}$ at Q, as in the figure. Prove: $\overline{OP} \cong \overline{OQ}$.



Show solution

1. Notice that O is the midpoint of diagonal $\overline{AC}$, because the diagonals of a parallelogram bisect each other.
2. Look for two triangles that contain $\overline{OP}$ and $\overline{OQ}$ and the halves of $\overline{AC}$: $\triangle AOP$ and $\triangle COQ$.
3. The parallel sides give alternate interior angles at A and C; the angles at O are vertical angles. Use ASA, then CPCTC.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; Diagonals $\overline{AC}$ and $\overline{BD}$ meet at O; P on $\overline{AB}$ and Q on $\overline{CD}$ with P, O, Q collinear

Prove: $\overline{OP} \cong \overline{OQ}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; its diagonals meet at O; P, O, Q are collinear with P on $\overline{AB}$ and Q on $\overline{CD}$	Given
2	$\overline{AO} \cong \overline{CO}$	Parallelogram Diagonals Theorem
3	$\overline{AB} \parallel \overline{DC}$	Definition of parallelogram
4	$\angle PAO \cong \angle QCO$	Alternate Interior Angles Theorem
5	$\angle AOP \cong \angle COQ$	Vertical Angles Theorem
6	$\triangle AOP \cong \triangle COQ$	ASA

	STATEMENTS	REASONS
7	$\overline{OP} \cong \overline{OQ}$	CPCTC

∎

Answer: See the proof: $\triangle AOP \cong \triangle COQ$ by ASA, so $\overline{OP} \cong \overline{OQ}$.

PRACTICE

1. Easy In parallelogram $PQRS$, $PQ = 12$ cm and $QR = 7$ cm. Find the perimeter of $PQRS$.

   Show answer

   Opposite sides are congruent, so $RS = 12$ cm and $SP = 7$ cm. Perimeter $= 12 + 7 + 12 + 7 = 38$ cm.

2. Easy In parallelogram $PQRS$, $m\angle P = 64°$. Find $m\angle Q$.

   Show answer

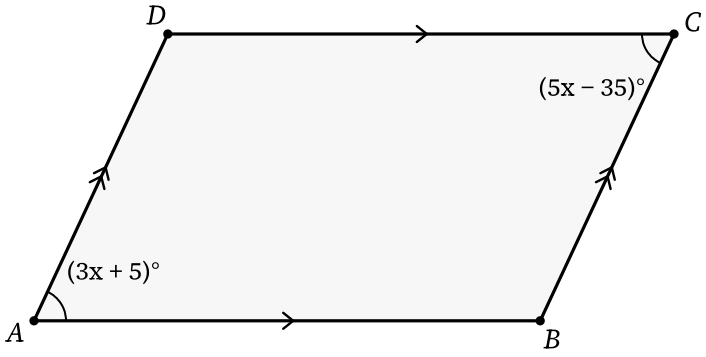
   Consecutive angles are supplementary: $m\angle Q = 180° - 64° = 116°$.

3. Easy The diagonals of parallelogram $PQRS$ meet at T, and $PT = 9$. Find PR.

   Show answer

   The diagonals bisect each other, so $TR = PT = 9$ and $PR = 18$.

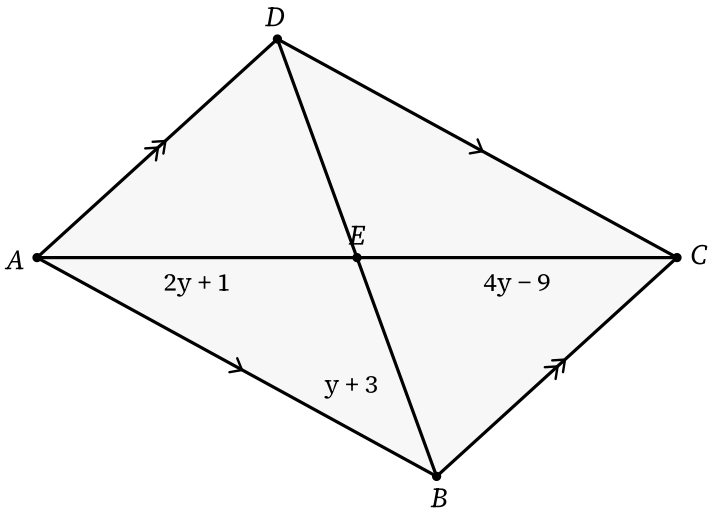
4. Easy In parallelogram $ABCD$ in the figure, $m\angle A = (3x + 5)°$ and $m\angle C = (5x - 35)°$. Find x.



   Show answer

   Opposite angles are congruent: $3x + 5 = 5x - 35$, so $40 = 2x$ and $x = 20$. (Each angle is $65°$.)

5. **Medium** The diagonals of parallelogram $ABCD$ meet at E, as in the figure. $AE = 2y + 1$, $EC = 4y - 9$ and $BE = y + 3$. Find y and BD.



Show answer

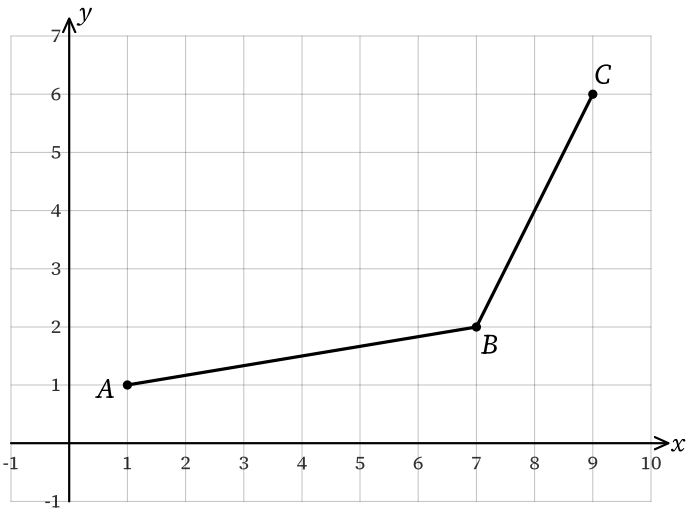
The diagonals bisect each other: $2y + 1 = 4y - 9$, so $y = 5$. Then $BE = 5 + 3 = 8$, so $BD = 2 \cdot 8 = 16$.

6. **Medium** The Miller family's garden plot is shaped like a parallelogram. One angle of the plot is 3 times as large as a consecutive angle. Find the measure of the larger angle.

Show answer

Consecutive angles are supplementary: $x + 3x = 180$, so $x = 45$. The larger angle is $3 \cdot 45^\circ = 135^\circ$.

7. **Medium** Three vertices of parallelogram $ABCD$ are $A(1, 1)$, $B(7, 2)$ and $C(9, 6)$, as in the figure. Find the coordinates of D. (Use the fact that the diagonals bisect each other.)



Show answer

The midpoint of $\overline{AC}$ is $\left(\frac{1+9}{2}, \frac{1+6}{2}\right) = (5, 3.5)$. It must also be the midpoint of $\overline{BD}$: $\frac{7+x}{2} = 5$ gives $x = 3$, and $\frac{2+y}{2} = 3.5$ gives $y = 5$. So $D(3, 5)$.

8. **Medium** Ruth is making a stained-glass pane shaped like a parallelogram with a lead border all around. The border is 46 inches long in all, and one side of the pane is 3 inches longer than the side next to it. How long is the longer side?

Show answer

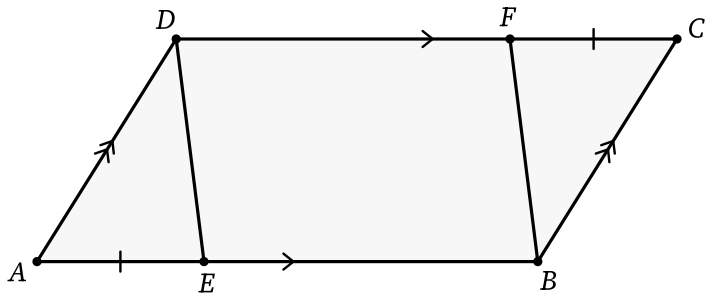
Opposite sides are congruent, so the perimeter is $2(x + (x + 3)) = 46$. Then $2x + 3 = 23$, $x = 10$, and the longer side is 13 inches.

9. **Medium** Grace says, 'The diagonals of a parallelogram bisect each other, so they must be congruent.' Is she right? Explain, using the parallelogram with vertices $(0, 0)$, $(6, 0)$, $(8, 3)$ and $(2, 3)$.

Show answer

No. Bisecting each other means each diagonal is cut in half; it says nothing about the two diagonals having equal length. In the given parallelogram one diagonal runs from $(0, 0)$ to $(8, 3)$ with length $\sqrt{64 + 9} = \sqrt{73} \approx 8.5$, and the other from $(6, 0)$ to $(2, 3)$ with length $\sqrt{16 + 9} = 5$.

10. **Hard** Given: $ABCD$ is a parallelogram; E lies on $\overline{AB}$ and F lies on $\overline{CD}$ with $\overline{AE} \cong \overline{CF}$, as in the figure. Prove: $\overline{DE} \cong \overline{BF}$.



Show answer

See the model proof.

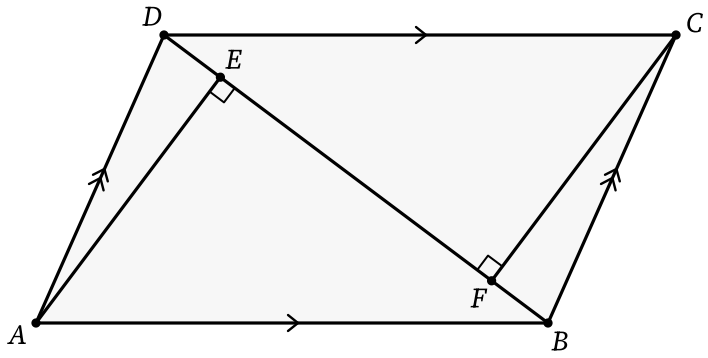
TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; E on $\overline{AB}$, F on $\overline{CD}$, $\overline{AE} \cong \overline{CF}$
Prove: $\overline{DE} \cong \overline{BF}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; $\overline{AE} \cong \overline{CF}$	Given
2	$\overline{DA} \cong \overline{BC}$	Parallelogram Opposite Sides Theorem
3	$\angle DAE \cong \angle BCF$	Parallelogram Opposite Angles Theorem ($\angle DAE$ is $\angle DAB$ and $\angle BCF$ is $\angle BCD$)
4	$\triangle DAE \cong \triangle BCF$	SAS
5	$\overline{DE} \cong \overline{BF}$	CPCTC

∎

11. **Hard** Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$; $\overline{AE} \perp \overline{BD}$ at E and $\overline{CF} \perp \overline{BD}$ at F, as in the figure. Prove: $\overline{AE} \cong \overline{CF}$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

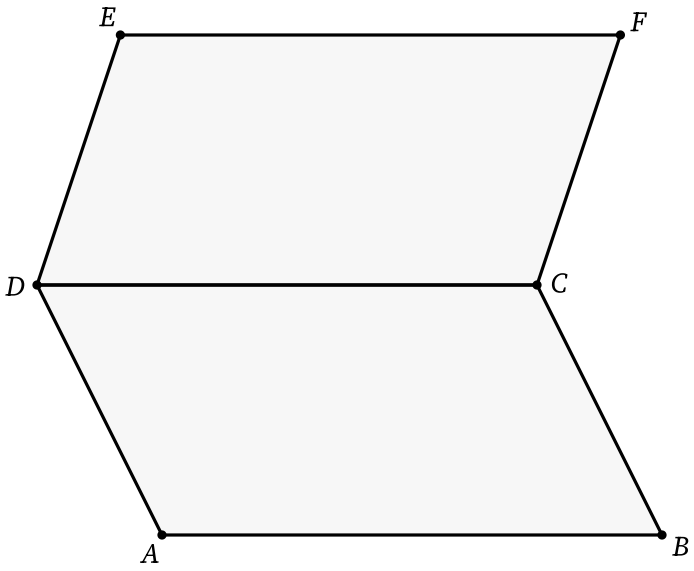
Given: $ABCD$ is a parallelogram; $\overline{AE} \perp \overline{BD}$ at E; $\overline{CF} \perp \overline{BD}$ at F
Prove: $\overline{AE} \cong \overline{CF}$
Because $\overline{AE} \perp \overline{BD}$ and $\overline{CF} \perp \overline{BD}$, the angles $\angle AEB$ and $\angle CFD$ are right angles (definition of perpendicular lines), so $\angle AEB \cong \angle CFD$ (all right angles are congruent).

Since $ABCD$ is a parallelogram, $\overline{AB} \parallel \overline{DC}$, and $\overleftrightarrow{BD}$ is a transversal; so the alternate interior angles $\angle ABE$ (which is $\angle ABD$) and $\angle CDF$ (which is $\angle CDB$) are congruent. Also $\overline{AB} \cong \overline{CD}$ because opposite sides of a parallelogram are congruent.

So $\triangle ABE \cong \triangle CDF$ by AAS (two angles and the non-included side $\overline{AB} \cong \overline{CD}$), and $\overline{AE} \cong \overline{CF}$ by CPCTC.

∎

12. **Hard** Given: $ABCD$ and $DCFE$ are parallelograms that share side $\overline{DC}$, as in the figure. Prove: $\overline{AB} \cong \overline{EF}$ and $\overline{AB} \parallel \overline{EF}$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $ABCD$ is a parallelogram; $DCFE$ is a parallelogram
Prove: $\overline{AB} \cong \overline{EF}$ and $\overline{AB} \parallel \overline{EF}$

(1) $ABCD$ is a parallelogram
Given

(2) $DCFE$ is a parallelogram
Given

(3) $\overline{AB} \cong \overline{DC}$
Parallelogram Opposite Sides Theorem
from (1)

(4) $\overline{DC} \cong \overline{EF}$
Parallelogram Opposite Sides Theorem
from (2)

(5) $\overline{AB} \cong \overline{EF}$
Transitive Property of Congruence
from (3), (4)

(6) $\overline{AB} \parallel \overline{DC}$
Definition of parallelogram
from (1)

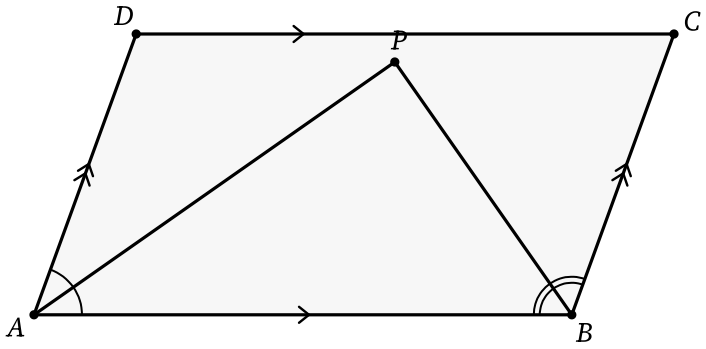
(7) $\overline{DC} \parallel \overline{EF}$
Definition of parallelogram
from (2)

(8) $\overline{AB} \parallel \overline{EF}$
Two lines parallel to the same line are parallel (Euclid I.30)
from (6), (7)

(9) $\overline{AB} \cong \overline{EF}$ and $\overline{AB} \parallel \overline{EF}$
Steps 5 and 8 together
from (5), (8)

∎

13. **Challenge** Given: $ABCD$ is a parallelogram; $\overrightarrow{AP}$ bisects $\angle DAB$ and $\overrightarrow{BP}$ bisects $\angle ABC$, and the bisectors meet at P, as in the figure. Prove: $\angle APB$ is a right angle.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram; $\overrightarrow{AP}$ bisects $\angle DAB$; $\overrightarrow{BP}$ bisects $\angle ABC$

Prove: $\angle APB$ is a right angle

Consecutive angles of a parallelogram are supplementary, so $m\angle DAB + m\angle ABC = 180°$.

Because $\overrightarrow{AP}$ bisects $\angle DAB$, $m\angle PAB = \frac{1}{2}m\angle DAB$; because $\overrightarrow{BP}$ bisects $\angle ABC$, $m\angle PBA = \frac{1}{2}m\angle ABC$ (definition of angle bisector). Adding and substituting, $m\angle PAB + m\angle PBA = \frac{1}{2}(m\angle DAB + m\angle ABC) = \frac{1}{2}(180°) = 90°$.

By the Triangle Angle Sum Theorem in $\triangle APB$, $m\angle APB = 180° - 90° = 90°$. So $\angle APB$ is a right angle.

∎

COMMON MISTAKES

- Assuming the diagonals of a parallelogram are congruent or perpendicular because they look that way. Only 'the diagonals bisect each other' is true of every parallelogram; check with numbers, as in Practice 9.
- Setting consecutive angles equal. Consecutive angles (sharing a side) are supplementary; only opposite angles are congruent. Ask yourself whether the two angles share a side before writing the equation.

- Reading the vertices out of order. In parallelogram $ABCD$, A is next to B and D and opposite C; sketch and label the figure in order before pairing sides or angles.
- Using these properties backward, to prove that a figure *is* a parallelogram. That needs the converse theorems of Lesson 6.3.

> **Make it a game · Fold and Test.** Cut a parallelogram from paper and fold it along both diagonals to see them bisect each other. Rotate it 180 degrees about their crossing and watch it land on itself. Then parent reads eight statements about quadrilaterals; you call each true or false for parallelograms and name the theorem behind every true answer.

6.3 Proving That a Quadrilateral Is a Parallelogram

OBJECTIVES

- Prove the converse theorems that show a quadrilateral is a parallelogram: both pairs of opposite sides congruent, both pairs of opposite angles congruent, diagonals bisecting each other, or one pair of sides both congruent and parallel (Euclid I.33).
- Decide whether given information is enough to guarantee a parallelogram, and give a counterexample when it is not.
- Show that a quadrilateral on the coordinate plane is a parallelogram using slope, distance or midpoints.
- Explain how parallelogram linkages such as the parallel rule and the scissor lift work.

WATCH

Quadrilaterals (Relationships, Properties & Sufficiency Conditions)

Eddie Woo · 12:53

https://www.youtube.com/watch?v=FshhyM06uLE

> **WHEN TO WATCH**
> After the converse theorems: the difference between a property and a condition that proves a shape.

Note: Eddie Woo teaches in Australia: he says 'sufficiency conditions' and uses the inclusive definitions: a trapezium has at least one pair of parallel sides and a rhombus is a kind of kite. This course uses exclusive definitions (Lesson 6.5).

BACKUP VIDEO

Quadrilateral Evolution (1 of 2: To the Parallelogram)

Eddie Woo · 8:43

https://www.youtube.com/watch?v=oQ3dYagTPsI

Before Example 1: how a quadrilateral becomes a parallelogram as conditions are added.

KEY TERMS

converse

The statement formed by exchanging the hypothesis and conclusion of a conditional. The converse of a true theorem may be true or false, so it needs its own proof.

parallel rule

A drafting and navigation tool made of two straightedges joined by two equal crossbars that pivot at their ends, so that the straightedges always stay parallel.

sufficient condition

A condition that, when it is true, guarantees the conclusion. Each theorem in this lesson names a sufficient condition for a quadrilateral to be a parallelogram.

CORE IDEAS

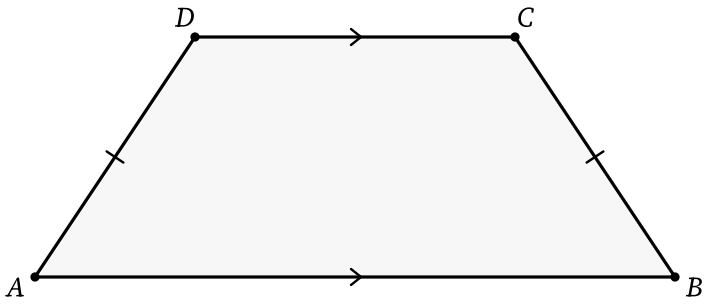
In Lesson 6.2 we went from 'parallelogram' to its properties. Now we go the other way: if a quadrilateral has a certain property, must it be a parallelogram? These are converses, and a converse is never automatically true; each one below gets its own proof. To show a quadrilateral is a parallelogram you may use the definition (both pairs of opposite sides parallel) or any one of these four tests: both pairs of opposite sides congruent; both pairs of opposite angles congruent; the diagonals bisect each other; one pair of opposite sides both congruent and parallel.

The proofs share one idea. Draw a diagonal (or use both) to make triangles, prove the triangles congruent with SSS or SAS, and use CPCTC to get a pair of congruent alternate interior angles. Then the Alternate Interior Angles Converse (Euclid I.27) says the sides are parallel, and the definition finishes the job. For the angle test, the polygon angle sum from Lesson 6.1 shows consecutive angles are supplementary, and the Same-Side Interior Angles Converse (I.28) gives the parallel sides.

Some tempting conditions are *not* enough. A quadrilateral with one pair of sides parallel and the *other* pair congruent can be an isosceles trapezoid, which is not a parallelogram (see the figure). The working test is 'the *same* pair both parallel and congruent'. Likewise one pair of congruent opposite sides alone, or one pair of congruent opposite angles alone, guarantees nothing. When you are tempted to use a test, say its exact words.

On the coordinate plane you can check a quadrilateral three ways: (1) find all four slopes and show both pairs of opposite sides are parallel; (2) show one pair of opposite sides has equal slopes *and* equal lengths (distance formula); or (3) show the two diagonals have the same midpoint, so they bisect each other. Method 3 is often the quickest.

These theorems explain useful machines. A navigator's parallel rule has two straightedges joined by two crossbars of equal length; since the straightedges between the pivots are also equal, both pairs of opposite sides stay congruent as it swings, so it is always a parallelogram and the edges stay parallel. A scissor lift is built from pairs of bars pinned at their midpoints; the bars are the diagonals of the quadrilateral formed by their ends, they bisect each other, so that quadrilateral is a parallelogram and the platform stays parallel to the floor.

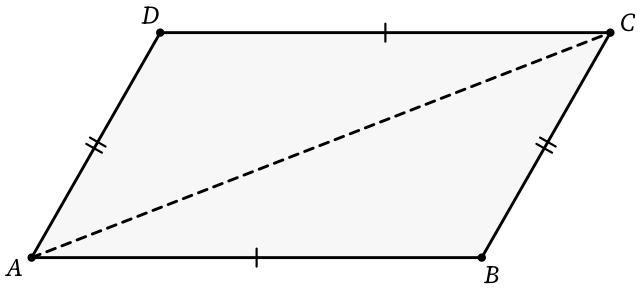


One pair of sides parallel and the other pair congruent, yet not a parallelogram.

POSTULATES AND THEOREMS

THEOREM **Opposite Sides Converse**

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{CD}$; $\overline{BC} \cong \overline{DA}$

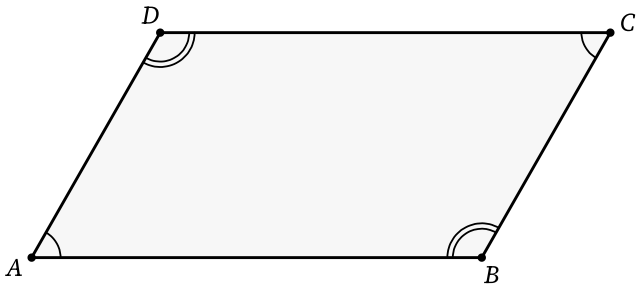
Prove: $ABCD$ is a parallelogram

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$	Given
2	Draw diagonal $\overline{AC}$	Two points determine a line
3	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
4	$\triangle ABC \cong \triangle CDA$	SSS
5	$\angle BAC \cong \angle DCA$	CPCTC
6	$\overline{AB} \parallel \overline{DC}$	Alternate Interior Angles Converse (Euclid I.27)
7	$\angle BCA \cong \angle DAC$	CPCTC
8	$\overline{AD} \parallel \overline{BC}$	Alternate Interior Angles Converse (Euclid I.27)
9	$ABCD$ is a parallelogram	Definition of parallelogram

∎

THEOREM **Opposite Angles Converse**

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



PARAGRAPH PROOF

Given: $\angle A \cong \angle C$; $\angle B \cong \angle D$

Prove: $ABCD$ is a parallelogram

Let $a = m\angle A = m\angle C$ and $b = m\angle B = m\angle D$ (definition of congruent angles). The angles of every quadrilateral add to $360°$ (Polygon Interior Angle Sum Theorem for a convex quadrilateral; Lesson 6.1, Practice 14 for a concave one), so $a + b + a + b = 360°$. Then $2a + 2b = 360°$ and, dividing by 2, $a + b = 180°$. So no angle of $ABCD$ is $180°$ or more (if a were, then $a + b > 180°$, and the same for b), and $ABCD$ is convex: each side's line has the rest of the quadrilateral on one side of it.

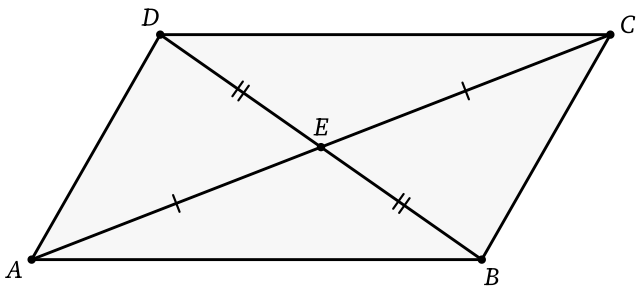
Angles $\angle A$ and $\angle B$ are same-side interior angles formed by $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ with transversal $\overleftrightarrow{AB}$. Since $m\angle A + m\angle B = a + b = 180°$, they are supplementary, so $\overline{AD} \parallel \overline{BC}$ by the Same-Side Interior Angles Converse (Euclid I.28).

Angles $\angle A$ and $\angle D$ are same-side interior angles formed by $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ with transversal $\overleftrightarrow{AD}$, and $m\angle A + m\angle D = a + b = 180°$. So $\overline{AB} \parallel \overline{DC}$ by the same converse. Both pairs of opposite sides are parallel, so $ABCD$ is a parallelogram by definition.

∎

THEOREM **Bisecting Diagonals Converse**

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



FLOW PROOF

Given: Diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other at E

Prove: $ABCD$ is a parallelogram

(1) $\overline{AC}$ and $\overline{BD}$ bisect each other at E
Given

(2) $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$
Definition of segment bisector (and of midpoint)
from (1)

(3) $\angle AEB \cong \angle CED$
Vertical Angles Theorem

(4) $\triangle AEB \cong \triangle CED$
SAS
from (2), (3)

(5) $\angle BAE \cong \angle DCE$
CPCTC
from (4)

(6) $\overline{AB} \parallel \overline{DC}$
Alternate Interior Angles Converse
from (5)

(7) $\angle AED \cong \angle CEB$
Vertical Angles Theorem

(8) $\triangle AED \cong \triangle CEB$
SAS
from (2), (7)

(9) $\angle DAE \cong \angle BCE$
CPCTC
from (8)

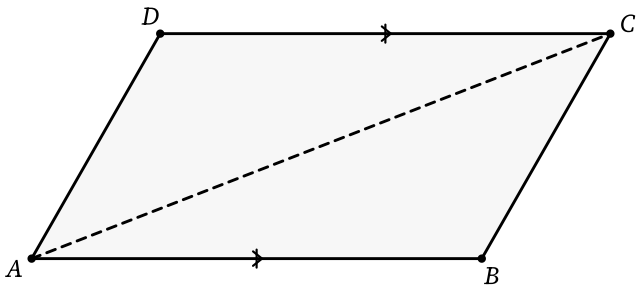
(10) $\overline{AD} \parallel \overline{BC}$
Alternate Interior Angles Converse
from (9)

(11) $ABCD$ is a parallelogram
Definition of parallelogram
from (6), (10)

∎

THEOREM **Congruent and Parallel Sides Theorem** Euclid I.33

If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram.



TWO-COLUMN PROOF

Given: $\overline{AB} \parallel \overline{DC}$; $\overline{AB} \cong \overline{DC}$

Prove: $ABCD$ is a parallelogram

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$	Given
2	Draw diagonal $\overline{AC}$	Two points determine a line

	STATEMENTS	REASONS
3	$\angle BAC \cong \angle DCA$	Alternate Interior Angles Theorem
4	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
5	$\triangle BAC \cong \triangle DCA$	SAS
6	$\angle BCA \cong \angle DAC$	CPCTC
7	$\overline{AD} \parallel \overline{BC}$	Alternate Interior Angles Converse (Euclid I.27)
8	$ABCD$ is a parallelogram	Definition of parallelogram

∎

This is Euclid's proof of I.33. CPCTC also gives $\overline{AD} \cong \overline{BC}$, which is Euclid's 'also equal'.

KEY FORMULAS

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

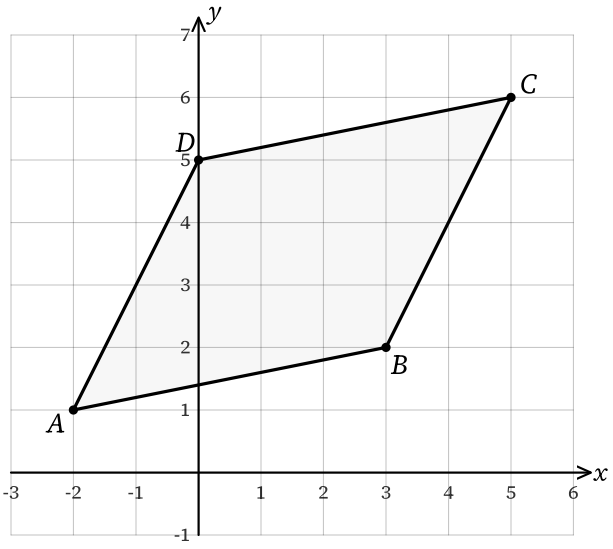
From history. Ship's navigators have long used a parallel rule to carry a compass direction from the compass rose printed on a chart over to their own position: they line one edge up with the direction, then 'walk' the rule across the chart, holding one straightedge still while swinging the other. The tool works only because its four pivots always form a parallelogram, which is exactly the Opposite Sides Converse.

Euclid connection. Euclid I.33 says: 'The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.' His proof is our fourth theorem: draw the diagonal, use alternate angles (I.29) and SAS (I.4), then I.27 for the parallel sides. Most of the proofs in this lesson end with Euclid's tests for parallel lines, I.27 and I.28.

Practice habit. Before you use a parallelogram test, say its exact words and check that it is one of the proved converses; 'one pair of sides parallel and the other pair congruent' sounds like a test but is not. In every proof, name the specific test as your reason, not just 'it is a parallelogram'.

WORKED EXAMPLES

Example 1. Quadrilateral $ABCD$ has vertices $A(-2, 1)$, $B(3, 2)$, $C(5, 6)$ and $D(0, 5)$, as in the figure. Is $ABCD$ a parallelogram? Justify your answer.

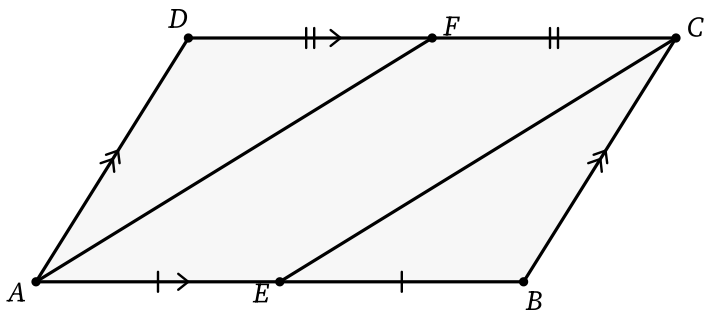


Show solution

1. Use one pair of opposite sides, $\overline{AB}$ and $\overline{DC}$, and show they are both parallel and congruent.
2. Slopes: $\overline{AB}$: $\dfrac{2-1}{3-(-2)} = \dfrac{1}{5}$; $\overline{DC}$: $\dfrac{6-5}{5-0} = \dfrac{1}{5}$. Equal slopes, so $\overline{AB} \parallel \overline{DC}$.
3. Lengths: $AB = \sqrt{5^2 + 1^2} = \sqrt{26}$ and $DC = \sqrt{5^2 + 1^2} = \sqrt{26}$, so $\overline{AB} \cong \overline{DC}$.
4. One pair of opposite sides is both congruent and parallel, so $ABCD$ is a parallelogram. Check another way: the midpoint of $\overline{AC}$ is $(1.5, 3.5)$ and the midpoint of $\overline{BD}$ is $(1.5, 3.5)$; the diagonals bisect each other.

Answer: Yes; $\overline{AB}$ and $\overline{DC}$ are both parallel (slope $\frac{1}{5}$) and congruent (length $\sqrt{26}$).

Example 2. Given: $ABCD$ is a parallelogram; E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{CD}$, as in the figure. Prove: quadrilateral $AECF$ is a parallelogram.



Show solution

1. Look at the sides of $AECF$ that lie along the parallelogram's sides: $\overline{AE}$ lies on $\overline{AB}$ and $\overline{FC}$ lies on $\overline{DC}$, so they are parallel.
2. Each is half of a pair of congruent opposite sides, so they are congruent too.
3. One pair of opposite sides both congruent and parallel: use the Congruent and Parallel Sides Theorem (Euclid I.33).

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; E is the midpoint of $\overline{AB}$; F is the midpoint of $\overline{CD}$

Prove: Quadrilateral $AECF$ is a parallelogram

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; E is the midpoint of $\overline{AB}$; F is the midpoint of $\overline{CD}$	Given
2	$\overline{AB} \parallel \overline{DC}$	Definition of parallelogram
3	$\overline{AE} \parallel \overline{FC}$	$\overline{AE}$ lies on $\overleftrightarrow{AB}$ and $\overline{FC}$ lies on $\overleftrightarrow{DC}$ (definition of parallel segments)
4	$AB = CD$	Parallelogram Opposite Sides Theorem
5	$AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$	Definition of midpoint and Segment Addition Postulate

	STATEMENTS	REASONS
6	$AE = CF$	Substitution Property of Equality
7	$\overline{AE} \cong \overline{CF}$	Definition of congruent segments
8	Quadrilateral $AECF$ is a parallelogram	Congruent and Parallel Sides Theorem (Euclid I.33)

∎

Answer: See the proof: $\overline{AE}$ and $\overline{CF}$ are both parallel and congruent.

PRACTICE

1. Easy In quadrilateral $WXYZ$, $WX = 9$, $XY = 6$, $YZ = 9$ and $ZW = 6$. Must $WXYZ$ be a parallelogram? Name the theorem that decides.

   Show answer

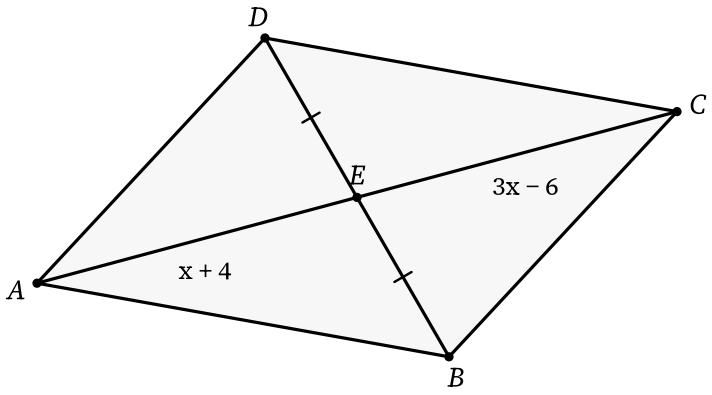
   Yes, by the Opposite Sides Converse: both pairs of opposite sides ($\overline{WX}, \overline{YZ}$ and $\overline{XY}, \overline{ZW}$) are congruent.

2. Easy Henry says, 'If one pair of opposite sides of a quadrilateral is parallel and the other pair is congruent, the quadrilateral must be a parallelogram.' Is he right? Explain.

   Show answer

   No. An isosceles trapezoid has one pair of parallel sides and congruent legs, yet its legs are not parallel, so it is not a parallelogram. The true test needs the *same* pair of sides to be both parallel and congruent.

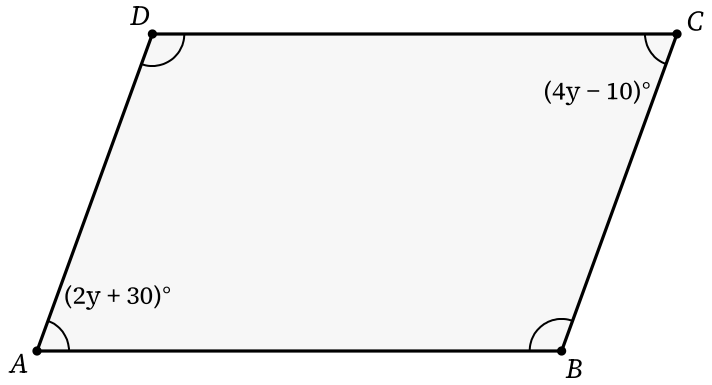
3. **Easy** In quadrilateral $ABCD$ in the figure, the diagonals meet at E, $\overline{BE} \cong \overline{DE}$, $AE = x + 4$ and $EC = 3x - 6$. For what value of x must $ABCD$ be a parallelogram?



Show answer

The diagonals must bisect each other: $x + 4 = 3x - 6$, so $10 = 2x$ and $x = 5$.

4. **Easy** In quadrilateral $ABCD$ in the figure, $\angle B \cong \angle D$, $m\angle A = (2y + 30)^\circ$ and $m\angle C = (4y - 10)^\circ$. For what value of y must $ABCD$ be a parallelogram?



Show answer

Both pairs of opposite angles must be congruent: $2y + 30 = 4y - 10$, so $40 = 2y$ and $y = 20$.

5. **Medium** Use midpoints to decide whether the quadrilateral with vertices $A(0,0)$, $B(5,1)$, $C(7,5)$ and $D(2,4)$ is a parallelogram.

Show answer

Yes. The midpoint of $\overline{AC}$ is $(3.5, 2.5)$ and the midpoint of $\overline{BD}$ is $\left(\frac{5+2}{2}, \frac{1+4}{2}\right) = (3.5, 2.5)$. The diagonals bisect each other.

6. **Medium** Use slopes to decide whether the quadrilateral with vertices $P(-3,-1)$, $Q(2,0)$, $R(4,4)$ and $S(-2,3)$ is a parallelogram.

Show answer

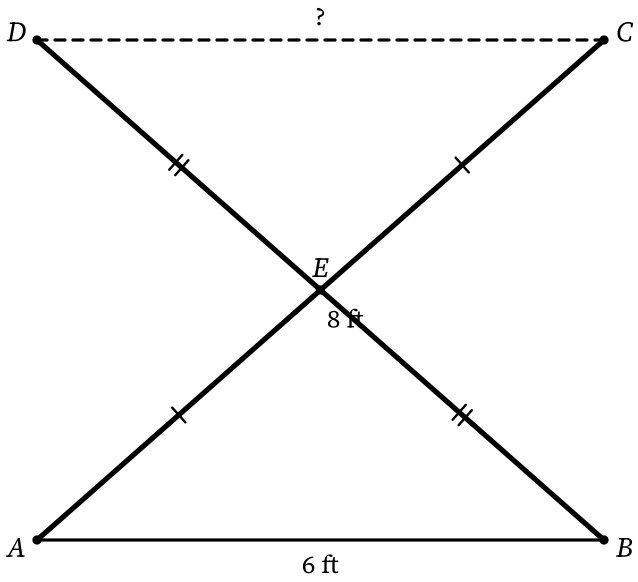
No. Slope of $\overline{PQ} = \frac{1}{5}$ but slope of $\overline{SR} = \frac{4-3}{4-(-2)} = \frac{1}{6}$; slope of $\overline{QR} = 2$ but slope of $\overline{PS} = 4$. Neither pair of opposite sides is parallel.

7. **Medium** A navigator's parallel rule has straightedges with pivots A, B on one and D, C on the other, where $AB = DC = 15$ cm, joined by crossbars $\overline{AD}$ and $\overline{BC}$, each 6 cm long. Explain why $\overline{AB}$ stays parallel to $\overline{DC}$ however the rule is opened.

Show answer

However the rule is opened, $AB = DC$ and $AD = BC$, so both pairs of opposite sides of $ABCD$ are congruent. By the Opposite Sides Converse, $ABCD$ is always a parallelogram, so $\overline{AB} \parallel \overline{DC}$ by the definition of a parallelogram.

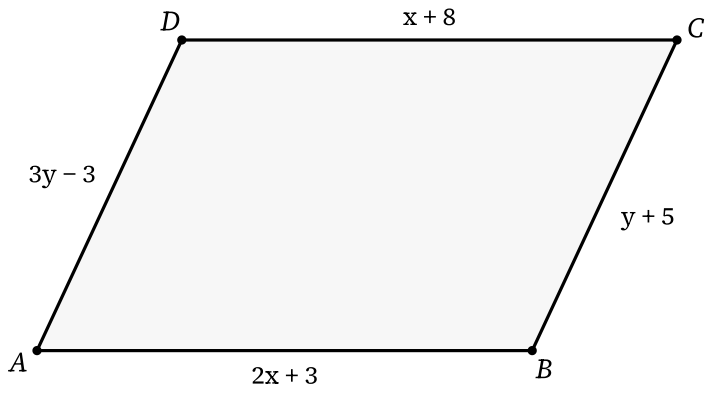
8. **Medium** A scissor lift is built from two steel bars $\overline{AC}$ and $\overline{BD}$, each 8 ft long, pinned together at their midpoints E, as in the figure. The bottom ends A and B are 6 ft apart on the floor, and the top ends D and C hold the platform. How far apart are the top ends D and C?



Show answer

The bars are the diagonals of $ABCD$ and they bisect each other at E, so $ABCD$ is a parallelogram (Bisecting Diagonals Converse). Opposite sides are congruent, so $DC = AB = 6$ ft (and the platform $\overline{DC}$ is parallel to the floor).

9. **Medium** In quadrilateral $ABCD$ in the figure, $AB = 2x + 3$, $DC = x + 8$, $AD = 3y - 3$ and $BC = y + 5$. Find the values of x and y that make $ABCD$ a parallelogram.



Show answer

Opposite sides congruent: $2x + 3 = x + 8$ gives $x = 5$; $3y - 3 = y + 5$ gives $2y = 8$, $y = 4$. (Then $AB = DC = 13$ and $AD = BC = 9$.)

10. **Hard** Given: quadrilateral $ABCD$ with $m\angle A = 70°$ and $m\angle B = 100°$. Prove indirectly that $ABCD$ is not a parallelogram.

Show answer

See the model proof.

INDIRECT PROOF

Given: Quadrilateral $ABCD$ with $m\angle A = 70°$ and $m\angle B = 100°$

Prove: $ABCD$ is not a parallelogram

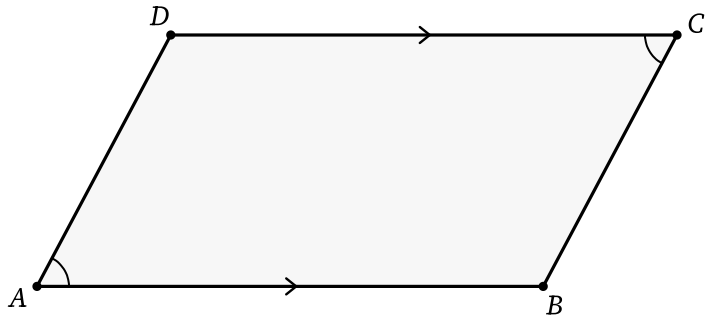
Assume, to the contrary, that $ABCD$ is a parallelogram.

Then its consecutive angles $\angle A$ and $\angle B$ are supplementary (Parallelogram Consecutive Angles Theorem), so $m\angle A + m\angle B = 180°$. But $m\angle A + m\angle B = 70° + 100° = 170° \neq 180°$. This contradicts the given angle measures.

So the assumption is false, and $ABCD$ is not a parallelogram.

∎

11. **Hard** Given: $\overline{AB} \parallel \overline{DC}$ and $\angle A \cong \angle C$ in quadrilateral $ABCD$, as in the figure. Prove: $ABCD$ is a parallelogram.



Show answer

See the model proof.

TWO-COLUMN PROOF

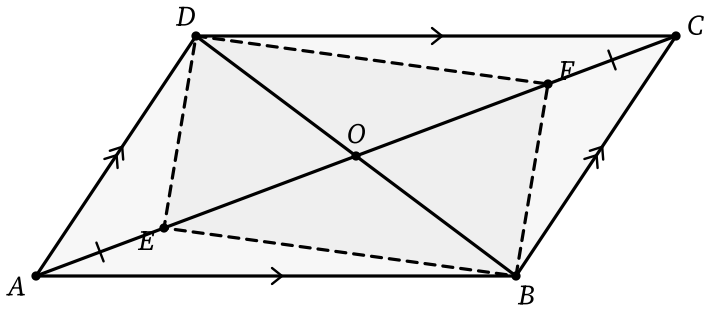
Given: $\overline{AB} \parallel \overline{DC}$; $\angle DAB \cong \angle BCD$

Prove: $ABCD$ is a parallelogram

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$; $\angle DAB \cong \angle BCD$	Given
2	$\angle DAB$ and $\angle ADC$ are supplementary	Same-Side Interior Angles Theorem (transversal $\overleftrightarrow{AD}$)
3	$m\angle DAB + m\angle ADC = 180^\circ$	Definition of supplementary angles
4	$m\angle DAB = m\angle BCD$	Definition of congruent angles
5	$m\angle BCD + m\angle ADC = 180^\circ$	Substitution Property of Equality
6	$\overline{AD} \parallel \overline{BC}$	Same-Side Interior Angles Converse (Euclid I.28; transversal $\overleftrightarrow{DC}$)
7	$ABCD$ is a parallelogram	Definition of parallelogram

∎

12. Hard Given: $ABCD$ is a parallelogram whose diagonals meet at O; E and F lie on $\overline{AC}$ with $\overline{AE} \cong \overline{CF}$, as in the figure. Prove: quadrilateral $BFDE$ is a parallelogram. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram with diagonals meeting at O; E and F on $\overline{AC}$ with $\overline{AE} \cong \overline{CF}$

Prove: Quadrilateral $BFDE$ is a parallelogram

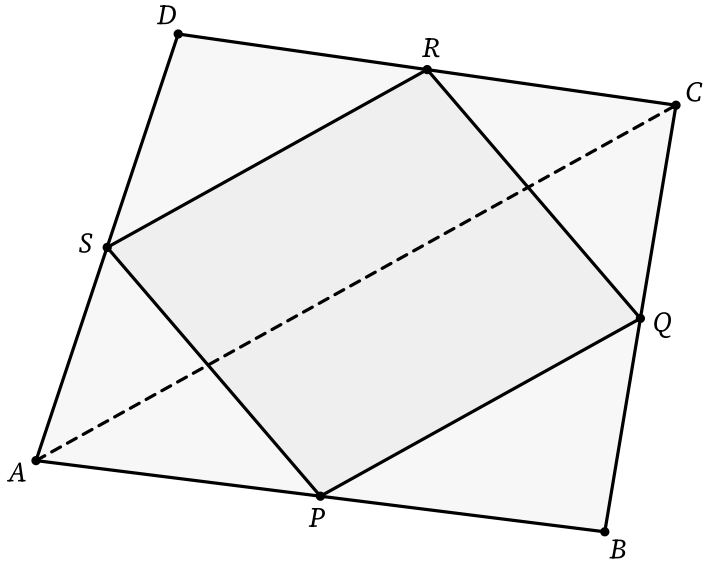
Because $ABCD$ is a parallelogram, its diagonals bisect each other, so $AO = CO$ and $BO = DO$.

As in the figure, E lies between A and O, and F lies between O and C. By the Segment Addition Postulate, $AO = AE + EO$ and $CO = CF + FO$, so $EO = AO - AE$ and $FO = CO - CF$. Since $AO = CO$ and $AE = CF$, substitution gives $EO = FO$.

Now look at quadrilateral $BFDE$. Its diagonals are $\overline{BD}$ and $\overline{EF}$, and they meet at O. Since $BO = DO$ and $EO = FO$, O is the midpoint of both diagonals: the diagonals of $BFDE$ bisect each other. By the Bisecting Diagonals Converse, $BFDE$ is a parallelogram.

∎

13. **Challenge** Given: $ABCD$ is any quadrilateral, and P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, as in the figure. Prove: $PQRS$ is a parallelogram. (Hint: draw diagonal $\overline{AC}$ and use the Triangle Midsegment Theorem.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a quadrilateral; P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$

Prove: $PQRS$ is a parallelogram

	STATEMENTS	REASONS
1	P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$	Given
2	Draw diagonal $\overline{AC}$	Two points determine a line
3	$\overline{PQ} \parallel \overline{AC}$ and $PQ = \frac{1}{2}AC$	Triangle Midsegment Theorem (in $\triangle ABC$)
4	$\overline{SR} \parallel \overline{AC}$ and $SR = \frac{1}{2}AC$	Triangle Midsegment Theorem (in $\triangle ADC$)
5	$\overline{PQ} \parallel \overline{SR}$	Two lines parallel to the same line are parallel (Euclid I.30)
6	$PQ = SR$	Substitution Property of Equality

	STATEMENTS	REASONS
7	$\overline{PQ} \cong \overline{SR}$	Definition of congruent segments
8	$PQRS$ is a parallelogram	Congruent and Parallel Sides Theorem (Euclid I.33)

∎

COMMON MISTAKES

- Using a property as a test when its converse has not been proved, or using a false 'test' such as one pair of sides parallel and the other pair congruent. Learn the four tests word for word.
- In coordinate problems, checking only one pair of slopes. Equal slopes for one pair show only that one pair is parallel (it might be a trapezoid); you also need the other pair parallel, or the same pair congruent, or a shared midpoint of the diagonals.
- Finding the midpoint of only one diagonal. The test needs both diagonals to have the *same* midpoint.
- Writing 'it is a parallelogram' as a reason. The reason must name which test was met, for example 'Opposite Sides Converse'.

> **Make it a game · Guaranteed or Not.** Parent reads conditions: 'One pair of sides is congruent and parallel.' 'Diagonals bisect each other.' 'Both pairs of opposite angles are congruent.' 'One pair of opposite sides is parallel and the other pair is congruent.' Say guaranteed parallelogram, or sketch a counterexample in 30 seconds. Then write two tricky conditions of your own.

6.4 Rectangles, Rhombuses and Squares

OBJECTIVES

- Define rectangles, rhombuses and squares and explain why each is a parallelogram and how they are related.
- Prove that a parallelogram is a rectangle exactly when its diagonals are congruent, and a rhombus exactly when its diagonals are perpendicular or each diagonal bisects a pair of opposite angles.
- Use these properties to find lengths and angles, classify figures on the coordinate plane, and square a building foundation.
- Construct a square on a given segment with compass and straightedge (Euclid I.46).

WATCH

Quadrilateral Evolution (2 of 2: To the Square)

Eddie Woo · 9:24

https://www.youtube.com/watch?v=sNMZNllkuiY

> **WHEN TO WATCH**
> Before the Core ideas: how the parallelogram becomes a rectangle, a rhombus and a square.

Note: The diagonal tests for rectangles and rhombuses are proved in the Core ideas.

BACKUP VIDEO

Is a Square a Rectangle? | Math with Mr. J

Math with Mr. J · 6:04

https://www.youtube.com/watch?v=mLF7_ZTeqYU

Before Example 1: why every square is a rectangle.

KEY TERMS

rectangle

A quadrilateral with four right angles.

rhombus

A quadrilateral with four congruent sides.

square

A quadrilateral with four congruent sides and four right angles.

CORE IDEAS

A **rectangle** has four right angles, a **rhombus** has four congruent sides, and a **square** has both. Each of them is a parallelogram, and we can prove it with Lesson 6.3. A rectangle's opposite angles are right angles, hence congruent, so it is a parallelogram by the Opposite Angles Converse. A rhombus's opposite sides are congruent, so it is a parallelogram by the Opposite Sides Converse. Therefore every property of parallelograms (opposite sides parallel and congruent, diagonals bisecting each other, and so on) belongs to rectangles, rhombuses and squares too.

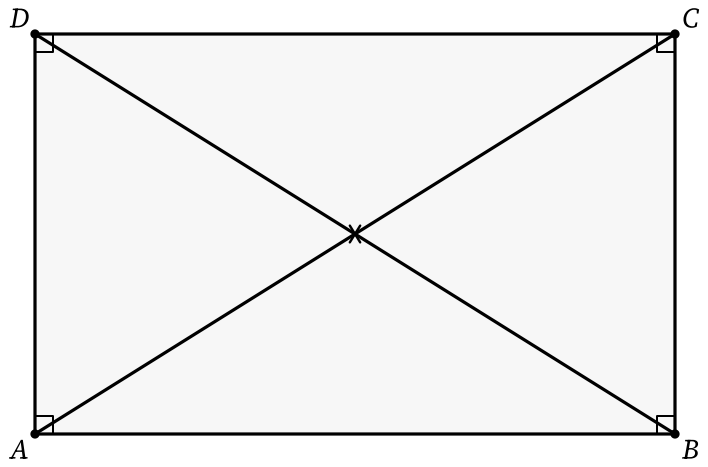
The definitions fit inside one another like a family tree. Every square is a rectangle (it has four right angles) and every square is a rhombus (it has four congruent sides); a square is exactly a figure that is both. But a rectangle need not be a rhombus, and a rhombus need not be a rectangle. When you classify a figure, give the most specific name that is proved.

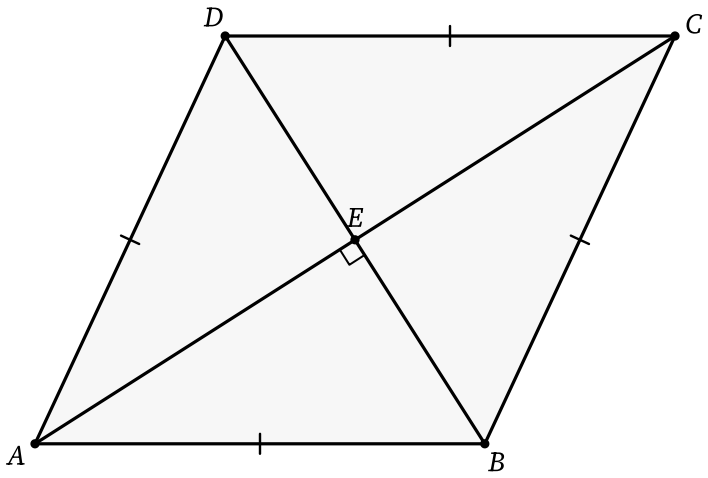
Rectangles have congruent diagonals. Why? In rectangle $ABCD$, triangles ABC and DCB share side $\overline{BC}$, have congruent opposite sides $\overline{AB}$ and $\overline{DC}$, and have right angles at B and C; by SAS they are congruent, so the diagonals $\overline{AC}$ and $\overline{DB}$ are congruent. The converse is also true for parallelograms, and builders use it every day: a frame with both pairs of opposite sides equal is a parallelogram, and if its two diagonals are also equal, its corners are square.

Rhombuses have perpendicular diagonals, and each diagonal bisects the angles at its ends. Why? Each vertex of a rhombus is the same distance from two other vertices, so SSS makes the triangles on either side of a diagonal congruent. Those congruent triangles give equal angles at the center (a linear pair of congruent angles, so right angles) and equal angles at the vertices (so the diagonal bisects them). Again the converses hold for parallelograms.

Read the diagonal tests carefully: each starts with 'a parallelogram'. A quadrilateral with congruent diagonals need not be a rectangle, and one with perpendicular diagonals need not be a rhombus (a kite has perpendicular diagonals). First prove the figure is a parallelogram, then apply the diagonal test.

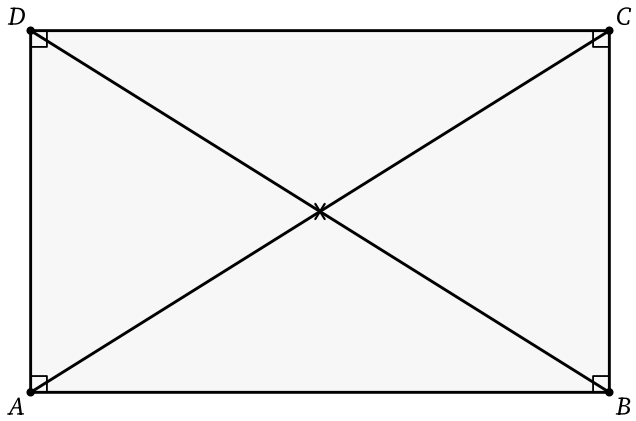
To construct a square on a segment $\overline{AB}$ (Euclid I.46), erect a perpendicular at A, mark $AD = AB$ on it, and swing equal arcs from B and D to find the fourth corner. The four sides are equal and one angle is right; a rhombus is a parallelogram, so its consecutive angles are supplementary and its opposite angles congruent, which makes every angle right.





THEOREM **Rectangle Diagonals Theorem**

A parallelogram is a rectangle if and only if its diagonals are congruent.



PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram

Prove: $ABCD$ is a rectangle if and only if $\overline{AC} \cong \overline{BD}$

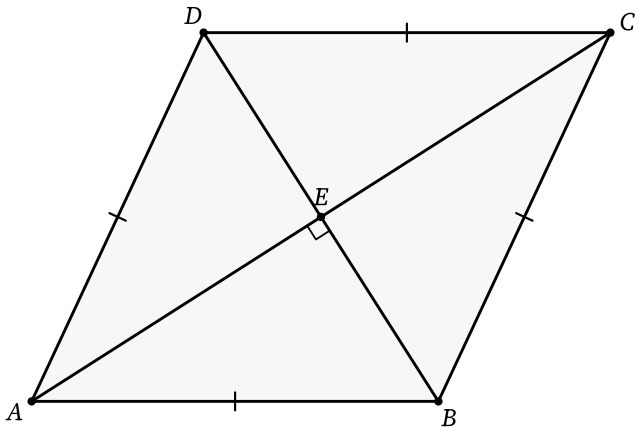
If $ABCD$ is a rectangle, then $\overline{AC} \cong \overline{BD}$. Angles $\angle ABC$ and $\angle DCB$ are right angles (definition of rectangle), so they are congruent (all right angles are congruent). Opposite sides of a parallelogram are congruent, so $\overline{AB} \cong \overline{DC}$, and $\overline{BC} \cong \overline{CB}$ by the Reflexive Property. So $\triangle ABC \cong \triangle DCB$ by SAS, and $\overline{AC} \cong \overline{DB}$ by CPCTC.

If $\overline{AC} \cong \overline{BD}$, then $ABCD$ is a rectangle. Since $ABCD$ is a parallelogram, $\overline{AB} \cong \overline{DC}$; also $\overline{BC} \cong \overline{CB}$ and $\overline{AC} \cong \overline{DB}$. So $\triangle ABC \cong \triangle DCB$ by SSS, and $\angle ABC \cong \angle DCB$ by CPCTC. But consecutive angles of a parallelogram are supplementary, so $m\angle ABC + m\angle DCB = 180°$. Two angles that are congruent and supplementary each measure $90°$, so $\angle ABC$ and $\angle DCB$ are right angles. Opposite angles of a parallelogram are congruent, so $\angle CDA$ (opposite $\angle ABC$) and $\angle DAB$ (opposite $\angle DCB$) are right angles too. All four angles are right angles, so $ABCD$ is a rectangle.

∎

THEOREM **Rhombus Diagonals Theorem**

If a parallelogram is a rhombus, then its diagonals are perpendicular.



TWO-COLUMN PROOF

Given: $ABCD$ is a rhombus; Diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

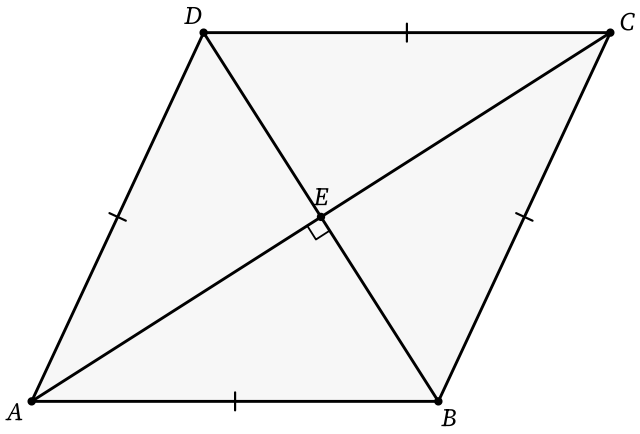
Prove: $\overline{AC} \perp \overline{BD}$

	STATEMENTS	REASONS
1	$ABCD$ is a rhombus; $\overline{AC}$ and $\overline{BD}$ meet at E	Given
2	$\overline{AB} \cong \overline{AD}$	Definition of rhombus
3	$ABCD$ is a parallelogram	Opposite Sides Converse (a rhombus has both pairs of opposite sides congruent)
4	$\overline{BE} \cong \overline{DE}$	Parallelogram Diagonals Theorem
5	$\overline{AE} \cong \overline{AE}$	Reflexive Property of Congruence
6	$\triangle ABE \cong \triangle ADE$	SSS
7	$\angle AEB \cong \angle AED$	CPCTC
8	$\angle AEB$ and $\angle AED$ form a linear pair	Definition of linear pair (B, E, D are collinear)
9	$\overline{AC} \perp \overline{BD}$	If two lines form a linear pair of congruent angles, the lines are perpendicular (Lesson 3.4)

∎

THEOREM **Perpendicular Diagonals Converse**

If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.



FLOW PROOF

Given: $ABCD$ is a parallelogram; $\overline{AC} \perp \overline{BD}$ at E

Prove: $ABCD$ is a rhombus

(1) $ABCD$ is a parallelogram; $\overline{AC} \perp \overline{BD}$ at E
Given

(2) $\overline{BE} \cong \overline{DE}$
Parallelogram Diagonals Theorem
from (1)

(3) $\angle AEB$ and $\angle AED$ are right angles
Definition of perpendicular lines
from (1)

(4) $\angle AEB \cong \angle AED$
All right angles are congruent
from (3)

(5) $\overline{AE} \cong \overline{AE}$
Reflexive Property of Congruence

(6) $\triangle AEB \cong \triangle AED$
SAS
from (2), (4), (5)

(7) $\overline{AB} \cong \overline{AD}$
CPCTC
from (6)

(8) $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$

Parallelogram Opposite Sides Theorem
from (1)

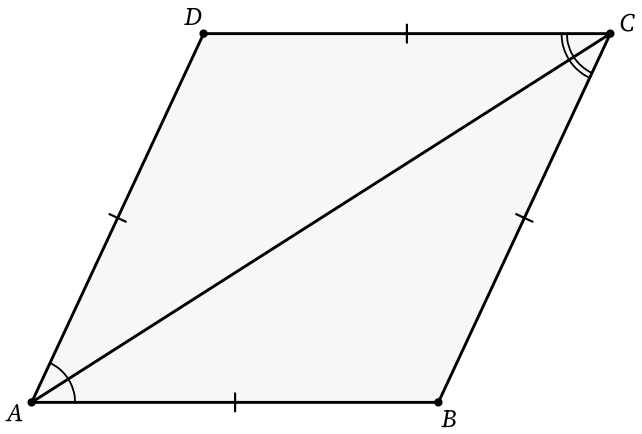
(9) $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$
Transitive Property of Congruence
from (7), (8)

(10) $ABCD$ is a rhombus
Definition of rhombus
from (9)

∎

THEOREM **Rhombus Angle Bisector Theorem**

A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.



PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram

Prove: $ABCD$ is a rhombus if and only if each diagonal bisects a pair of opposite angles

If $ABCD$ is a rhombus, its diagonals bisect its angles. All four sides are congruent, so in $\triangle ABC$ and $\triangle ADC$ we have $\overline{AB} \cong \overline{AD}$, $\overline{CB} \cong \overline{CD}$ and $\overline{AC} \cong \overline{AC}$; by SSS the triangles are congruent, and by CPCTC $\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$. So $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$. In the same way $\triangle BAD \cong \triangle BCD$ by SSS, so $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

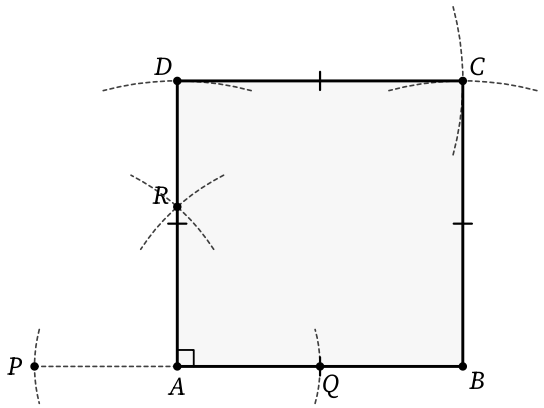
If a diagonal bisects an angle of parallelogram $ABCD$, it is a rhombus. (One diagonal bisecting one angle is already enough.) Suppose $\overline{AC}$ bisects $\angle BAD$, so $\angle BAC \cong \angle DAC$. Since $\overline{AB} \parallel \overline{DC}$, the alternate interior angles $\angle BAC$ and $\angle DCA$ are congruent. By the Transitive Property, $\angle DAC \cong \angle DCA$, so in $\triangle DAC$ the sides opposite these angles are congruent: $\overline{DC} \cong \overline{DA}$ (Converse of the Isosceles Triangle

Theorem, Euclid I.6). Opposite sides of a parallelogram are congruent, so $AB = DC$ and $BC = DA$; with $DC = DA$, all four sides are equal, and $ABCD$ is a rhombus.

∎

CONSTRUCTIONS

CONSTRUCTION **Construct a square on a given segment** Euclid I.46



1. Draw the given segment $\overline{AB}$, and extend line AB a little beyond A.
2. Construct the perpendicular to line AB at A: with center A, draw arcs crossing the line at P and Q on either side of A. With a wider compass setting, draw equal arcs from P and from Q that cross at R. Draw ray $\overrightarrow{AR}$.
3. Set the compass to length AB. With center A, mark point D on $\overrightarrow{AR}$, so $AD = AB$.
4. Keeping the compass at length AB, draw an arc centered at B and an arc centered at D, on the side away from A. They cross at C.
5. Draw $\overline{BC}$ and $\overline{DC}$. $ABCD$ is a square.

Why it works. All four sides were drawn with the same compass width, so $AB = BC = CD = DA$ and $ABCD$ is a rhombus, hence

a parallelogram. $\angle DAB$ is a right angle by the perpendicular construction (Euclid I.11). Consecutive angles of a parallelogram are supplementary, so $\angle B$ and $\angle D$ are right angles, and $\angle C$, opposite $\angle A$, is right too. Four congruent sides and four right angles make a square. Euclid finishes differently: after cutting off $AD = AB$ on the perpendicular, he draws parallels through D and B (I.31) and uses I.34 and I.29 to show the sides are equal and the angles right.

KEY FORMULAS

$$\text{Rectangle } ABCD: \ AC = BD$$

$$\text{Rhombus } ABCD: \ \overline{AC} \perp \overline{BD}$$

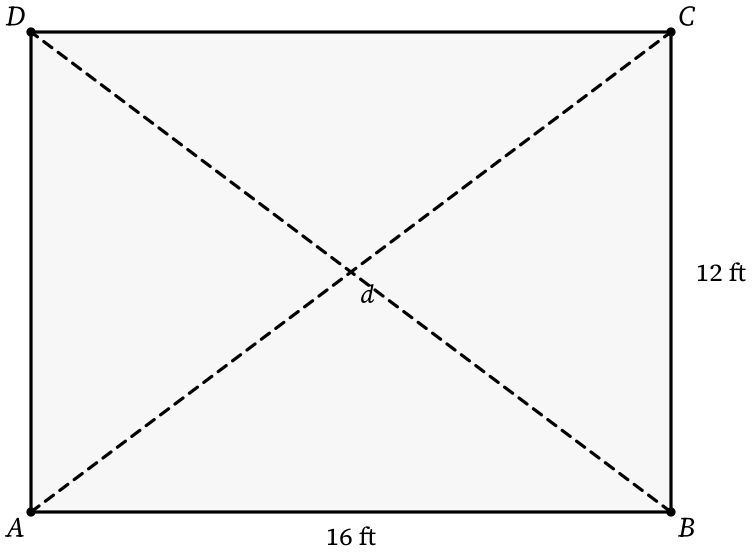
$$\text{Rectangle with sides } a, b: \ \text{diagonal} = \sqrt{a^2 + b^2}$$

From history. Carpenters and masons have long checked a rectangular layout by measuring its two diagonals. After the stakes or forms are set so that opposite sides have the right lengths, the builder stretches a tape from corner to corner both ways and shifts the frame until the two diagonals read the same. The first step makes a parallelogram; the second makes it a rectangle, by the Rectangle Diagonals Theorem.

Euclid connection. Euclid's Book I, Definition 22, reads: 'Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled.' Euclid's names exclude one another (for him a square is not a rhombus), while modern definitions include one another. Euclid needs his square at once: I.46 constructs it, and the very next proposition, I.47, is about the squares on the sides of a right triangle.

Practice habit. Recite the definitions of rectangle, rhombus and square and the three diagonal facts (rectangle: congruent; rhombus: perpendicular, and bisecting the angles) until automatic. Before using any diagonal test, write one line proving the figure is a parallelogram.

Example 1. The Miller family is laying out a shed foundation 16 ft long and 12 ft wide. Henry sets stakes so that the opposite sides measure 16 ft and 12 ft. How long must each diagonal be for the corners to be square, and why does checking that both diagonals have this length guarantee square corners?



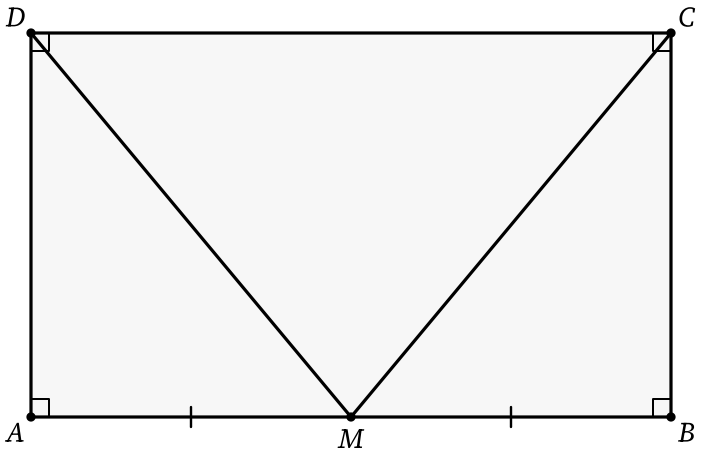
Shed foundation with diagonal strings

Show solution

1. Both pairs of opposite sides are congruent, so the layout $ABCD$ is a parallelogram (Opposite Sides Converse).
2. If the corners are square, $\triangle ABC$ has a right angle at B, so $AC^2 = 16^2 + 12^2 = 256 + 144 = 400$ and $AC = 20$ ft. The same is true of $\overline{BD}$.
3. Why it guarantees square corners: if both diagonals measure 20 ft, the parallelogram has congruent diagonals, so by the Rectangle Diagonals Theorem it is a rectangle, and all four corners are right angles.

Answer: Each diagonal must be 20 ft; a parallelogram with congruent diagonals is a rectangle.

Example 2. Given: $ABCD$ is a rectangle and M is the midpoint of $\overline{AB}$, as in the figure. Prove: $\overline{MD} \cong \overline{MC}$.



Show solution

1. The segments $\overline{MD}$ and $\overline{MC}$ are sides of $\triangle MAD$ and $\triangle MBC$.
2. The triangles have $\overline{AM} \cong \overline{BM}$ (midpoint), right angles at A and B (rectangle), and $\overline{AD} \cong \overline{BC}$ (a rectangle is a parallelogram).
3. Use SAS, then CPCTC.

TWO-COLUMN PROOF

Given: $ABCD$ is a rectangle; M is the midpoint of $\overline{AB}$
Prove: $\overline{MD} \cong \overline{MC}$

	STATEMENTS	REASONS
1	$ABCD$ is a rectangle; M is the midpoint of $\overline{AB}$	Given
2	$\overline{AM} \cong \overline{BM}$	Definition of midpoint
3	$\angle MAD$ and $\angle MBC$ are right angles	Definition of rectangle
4	$\angle MAD \cong \angle MBC$	All right angles are congruent
5	$ABCD$ is a parallelogram	Opposite Angles Converse (every rectangle is a parallelogram)
6	$\overline{AD} \cong \overline{BC}$	Parallelogram Opposite Sides Theorem
7	$\triangle MAD \cong \triangle MBC$	SAS

	STATEMENTS	REASONS
8	$\overline{MD} \cong \overline{MC}$	CPCTC

∎

Answer: See the proof: $\triangle MAD \cong \triangle MBC$ by SAS, so $\overline{MD} \cong \overline{MC}$.

PRACTICE

1. Easy True or false: every square is a rhombus.

Show answer

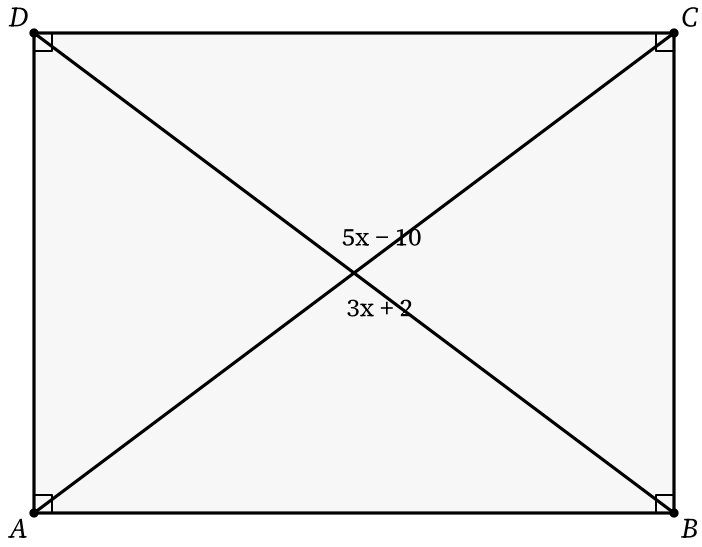
True. A square has four congruent sides, which is the definition of a rhombus.

2. Easy True or false: every rhombus is a square.

Show answer

False. A rhombus with angles of $60°$ and $120°$ has four congruent sides but no right angles.

3. Easy In rectangle $ABCD$ in the figure, $AC = 3x + 2$ and $BD = 5x - 10$. Find AC.



Show answer

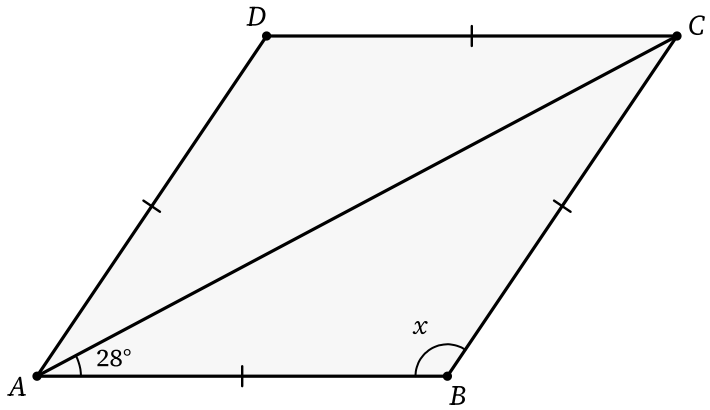
Diagonals of a rectangle are congruent: $3x + 2 = 5x - 10$, so $x = 6$ and $AC = 3(6) + 2 = 20$.

4. **Easy** Samuel builds a rectangular window frame 9 inches wide and 12 inches tall. How long is each diagonal brace?

Show answer

$\sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$ inches (both diagonals of a rectangle are equal).

5. **Medium** In rhombus $ABCD$ in the figure, $m\angle BAC = 28°$. Find $m\angle ABC$.



Show answer

Diagonal $\overline{AC}$ bisects $\angle BAD$, so $m\angle BAD = 56°$. Consecutive angles are supplementary: $m\angle ABC = 180° - 56° = 124°$.

6. **Medium** A stained-glass pane is a rhombus whose diagonals measure 10 inches and 24 inches. Find the length of each side of the pane.

Show answer

The diagonals bisect each other at right angles, making four right triangles with legs 5 in and 12 in. Each side is $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$ inches.

7. **Medium** Ruth's herb garden is a square 8 meters on a side. She lays a straight path along one diagonal. How long is the path? Give an exact answer in simplest radical form.

Show answer

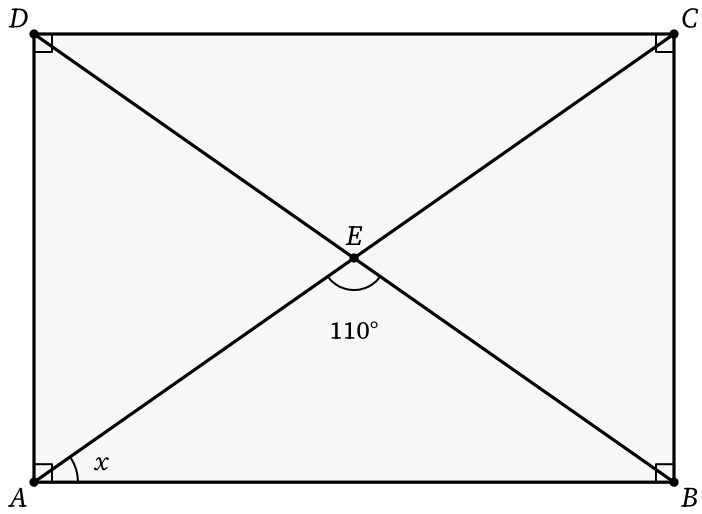
$\sqrt{8^2 + 8^2} = \sqrt{128} = 8\sqrt{2}$ meters (about 11.3 m).

8. **Medium** Henry measures a four-sided wooden frame and finds that its two diagonals are congruent. He concludes that the frame must be a rectangle. Is he right? Explain, using the quadrilateral with vertices $(0, 0)$, $(4, 0)$, $(5, 3)$ and $(-1, 3)$.

Show answer

No. The diagonal test applies only to parallelograms. In the given quadrilateral the diagonals from $(0, 0)$ to $(5, 3)$ and from $(4, 0)$ to $(-1, 3)$ both measure $\sqrt{34}$, yet the bottom side is 4 and the top side is 6, so it is not a parallelogram, much less a rectangle. Henry must first check that opposite sides are equal.

9. **Medium** The diagonals of rectangle $ABCD$ meet at E, and $m\angle AEB = 110^\circ$, as in the figure. Find $m\angle EAB$.



Show answer

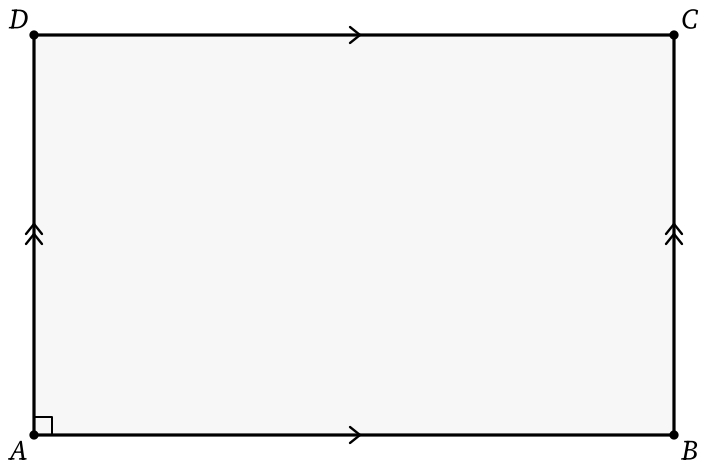
The diagonals of a rectangle are congruent and bisect each other, so $EA = EB$ and $\triangle AEB$ is isosceles. Its base angles are equal: $m\angle EAB = \dfrac{180^\circ - 110^\circ}{2} = 35^\circ$.

10. Hard Quadrilateral $ABCD$ has vertices $A(0,0)$, $B(5,0)$, $C(8,4)$ and $D(3,4)$. Classify it as precisely as possible: parallelogram, rectangle, rhombus or square. Justify your answer.

Show answer

Rhombus. $AB = 5$, $BC = \sqrt{9+16} = 5$, $CD = 5$, $DA = \sqrt{9+16} = 5$, so all four sides are congruent. The diagonals have lengths $AC = \sqrt{64+16} = \sqrt{80}$ and $BD = \sqrt{4+16} = \sqrt{20}$, which are not equal, so it is not a rectangle (and not a square). (Check: slopes of the diagonals are $\frac{1}{2}$ and -2, perpendicular as they must be.)

11. Hard Given: $ABCD$ is a parallelogram and $\angle A$ is a right angle, as in the figure. Prove: $ABCD$ is a rectangle. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram; $\angle A$ is a right angle

Prove: $ABCD$ is a rectangle

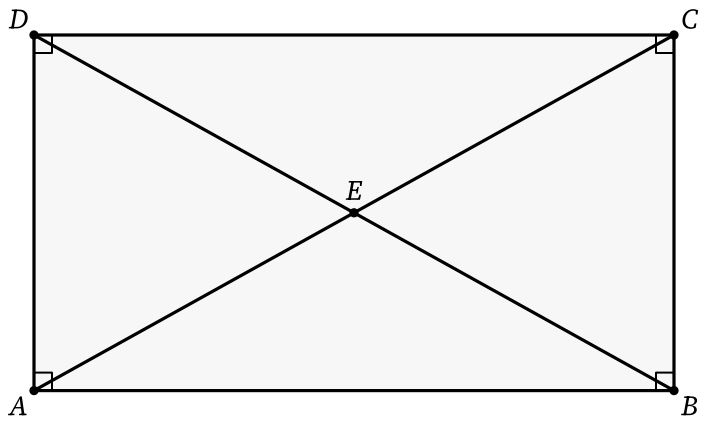
Opposite angles of a parallelogram are congruent, so $\angle C \cong \angle A$, and $\angle C$ is a right angle.

Consecutive angles of a parallelogram are supplementary, so $m\angle B = 180^\circ - m\angle A = 180^\circ - 90^\circ = 90^\circ$, and likewise $m\angle D = 180^\circ - 90^\circ = 90^\circ$.

All four angles are right angles, so $ABCD$ is a rectangle by definition.

∎

12. Hard Given: $ABCD$ is a rectangle whose diagonals meet at E, as in the figure. Prove: $\triangle AEB$ is isosceles.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a rectangle; Diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\triangle AEB$ is isosceles

	STATEMENTS	REASONS
1	$ABCD$ is a rectangle; $\overline{AC}$ and $\overline{BD}$ meet at E	Given
2	$AC = BD$	Rectangle Diagonals Theorem and definition of congruent segments
3	$ABCD$ is a parallelogram	Opposite Angles Converse (every rectangle is a parallelogram)
4	E is the midpoint of $\overline{AC}$ and of $\overline{BD}$	Parallelogram Diagonals Theorem
5	$AE = \frac{1}{2}AC$ and $BE = \frac{1}{2}BD$	Definition of midpoint
6	$\frac{1}{2}AC = \frac{1}{2}BD$	Multiplication Property of Equality
7	$AE = BE$	Substitution Property of Equality
8	$\triangle AEB$ is isosceles	Definition of isosceles triangle

∎

13. Hard Given: $ABCD$ is a rhombus that is not a square. Prove indirectly: the diagonals $\overline{AC}$ and $\overline{BD}$ are not congruent.

Show answer

See the model proof.

INDIRECT PROOF

Given: $ABCD$ is a rhombus; $ABCD$ is not a square

Prove: $\overline{AC} \ncong \overline{BD}$

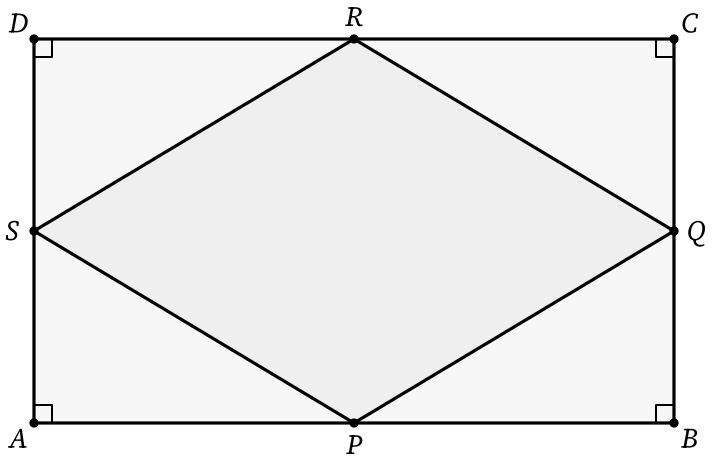
Assume, to the contrary, that $\overline{AC} \cong \overline{BD}$.

A rhombus is a parallelogram (Opposite Sides Converse). A parallelogram with congruent diagonals is a rectangle (Rectangle Diagonals Theorem), so $ABCD$ has four right angles. It also has four congruent sides, because it is a rhombus. Four congruent sides and four right angles make $ABCD$ a square by definition. This contradicts the given fact that $ABCD$ is not a square.

So the assumption is false, and $\overline{AC} \ncong \overline{BD}$.

∎

14. **Challenge** Given: $ABCD$ is a rectangle, and P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, as in the figure. Prove: $PQRS$ is a rhombus.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a rectangle; P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$
Prove: $PQRS$ is a rhombus
A rectangle is a parallelogram, so its opposite sides are congruent: $AB = DC$ and $BC = AD$. Its four angles are right angles, so they are all congruent.

Because P and R are midpoints of $\overline{AB}$ and $\overline{DC}$, $AP = PB = \frac{1}{2}AB$ and $CR = RD = \frac{1}{2}DC$; since $AB = DC$, these four lengths are equal. In the same way, Q and S give $BQ = QC = DS = SA$, all equal to $\frac{1}{2}BC = \frac{1}{2}AD$.

Now compare the four corner triangles $\triangle PAS$, $\triangle PBQ$, $\triangle RCQ$ and $\triangle RDS$. Each has a right angle included between a side of length $\frac{1}{2}AB$ and a side of length $\frac{1}{2}BC$. By SAS all four triangles are congruent, so by CPCTC their third sides are congruent: $\overline{PS} \cong \overline{PQ} \cong \overline{RQ} \cong \overline{RS}$.

A quadrilateral with four congruent sides is a rhombus, so $PQRS$ is a rhombus.

∎

COMMON MISTAKES

- Using a diagonal test on a quadrilateral that is not known to be a parallelogram. Congruent diagonals alone do not make a rectangle (Practice 8); first prove the figure is a parallelogram.

- Mixing up the diagonal properties: rectangles have congruent diagonals, rhombuses have perpendicular diagonals that bisect the angles. Only a square has both.
- Thinking a square is not a rectangle or not a rhombus. With our definitions a square is both; Euclid's older definitions were exclusive, but ours include one another.
- In a rhombus, assuming the diagonals are equal in length. They bisect each other and are perpendicular, but one is usually longer.

> **Make it a game · Mystery Quadrilateral.** Parent secretly picks parallelogram, rectangle, rhombus, square, trapezoid or kite. Ask yes-or-no questions about sides, angles and diagonals to identify it. Fewer questions wins; six or fewer is excellent. After each guess, explain how the shapes relate, such as why every square is also a rhombus and a rectangle. Swap roles.

6.5 Trapezoids and Kites

OBJECTIVES

- Identify the bases, legs, base angles and midsegment of a trapezoid, and the parts of a kite.
- Prove and use the theorems on isosceles trapezoids (congruent base angles, congruent diagonals) and the Trapezoid Midsegment Theorem.
- Prove and use the kite theorems: perpendicular diagonals and exactly one pair of congruent opposite angles.
- Classify quadrilaterals in the family tree, including on the coordinate plane with slope and distance.

WATCH

Math Antics - Quadrilaterals

mathantics · 9:50

https://www.youtube.com/watch?v=yiREqzDsMP8

> **WHEN TO WATCH**
> Before the Core ideas: the special quadrilaterals, including trapezoids and kites.

Note: The backup is by Eddie Woo, who teaches in Australia: he says 'trapezium' and uses inclusive definitions. In this course a trapezoid has exactly one pair of parallel sides and a kite has no pair of congruent opposite sides.

BACKUP VIDEO

Quadrilateral Family Tree

Eddie Woo · 11:31

https://www.youtube.com/watch?v=izyUfobZFGg

After the lesson: the quadrilateral family tree.

KEY TERMS

trapezoid

A quadrilateral with exactly one pair of parallel sides. The parallel sides are the **bases**; the other two sides are the **legs**.

base angles of a trapezoid

A pair of angles whose vertices are the endpoints of the same base. A trapezoid has two pairs of base angles.

isosceles trapezoid

A trapezoid whose legs are congruent.

midsegment of a trapezoid

The segment that joins the midpoints of the legs of a trapezoid.

kite

A quadrilateral with two pairs of consecutive congruent sides and no pair of congruent opposite sides. In this course kites are convex.

CORE IDEAS

A **trapezoid** has exactly one pair of parallel sides, its **bases**; the other two sides are its **legs**. (Some books allow 'at least one pair', which would make every parallelogram a trapezoid; in this course a trapezoid has exactly one pair.) Because the bases are parallel, each leg is a transversal, so the two angles along a leg are same-side interior angles and are supplementary. In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$: $m\angle A + m\angle D = 180^\circ$ and $m\angle B + m\angle C = 180^\circ$.

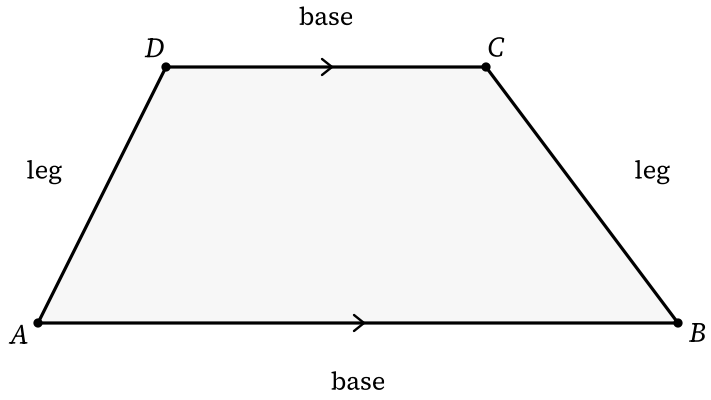
In an **isosceles trapezoid** the legs are congruent, and two more facts follow: each pair of base angles is congruent, and the diagonals are congruent. Why the base angles? Through one top corner draw a line parallel to the opposite leg. It cuts off a parallelogram and leaves an isosceles triangle, and the Isosceles Triangle Theorem (Euclid I.5) with corresponding angles does the rest. The converse is also true: a trapezoid with a pair of congruent base angles is isosceles (Practice 12).

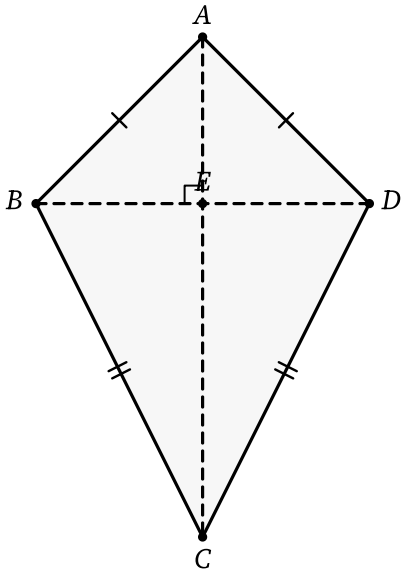
The **midsegment** of a trapezoid joins the midpoints of the legs. It is parallel to the bases and its length is the average of the bases: $\frac{1}{2}(b_1 + b_2)$. Why? Stretch the trapezoid into a triangle: draw a line from a top corner through the midpoint of the far leg until it meets the extended bottom base. A pair of congruent triangles shows the added piece equals the top base, and then the trapezoid's midsegment is a triangle midsegment (Lesson 5.4), half of the whole bottom: half of $b_1 + b_2$. The middle rung of a ladder whose sides lean in, or the middle shelf of a tapered bookcase, is a midsegment.

A **kite** has two pairs of consecutive congruent sides, like the outline of the toy. If $AB = AD$ and $CB = CD$, then A and C are each the same distance from B and D, so both lie on the perpendicular bisector of $\overline{BD}$ (Lesson 5.1). That makes diagonal

$\overline{AC}$ the perpendicular bisector of $\overline{BD}$: the diagonals are perpendicular, and $\overline{AC}$ cuts $\overline{BD}$ in half (but not the other way around). SSS also shows that the angles at B and D, between the unequal sides, are congruent, while the angles at A and C are not.

The quadrilateral family tree, in words: every quadrilateral with both pairs of opposite sides parallel is a parallelogram; a parallelogram with four right angles is a rectangle, one with four congruent sides is a rhombus, and one with both is a square. Separate branches hold the trapezoids (exactly one pair of parallel sides), with the isosceles trapezoids inside them, and the kites. On the coordinate plane, use slopes to decide which sides are parallel and the distance formula to decide which are congruent, then give the most specific name.

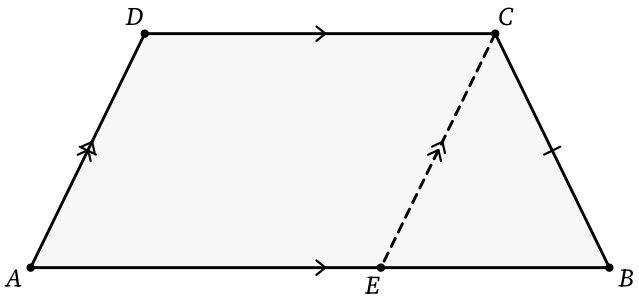




POSTULATES AND THEOREMS

THEOREM **Isosceles Trapezoid Base Angles Theorem**

If a trapezoid is isosceles, then each pair of base angles is congruent.



TWO-COLUMN PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, $\overline{AB}$ the longer base; $\overline{AD} \cong \overline{BC}$

Prove: $\angle A \cong \angle B$ and $\angle C \cong \angle D$

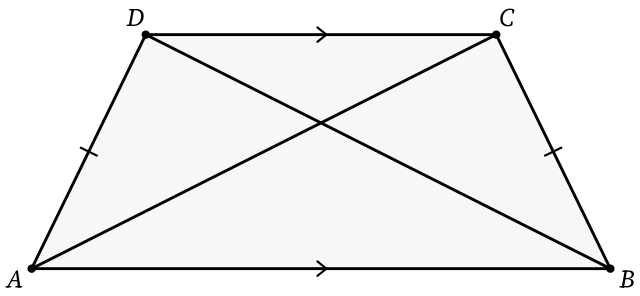
	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$; $\overline{AD} \cong \overline{BC}$	Given
2	Through C draw $\overleftrightarrow{CE} \parallel \overline{AD}$, meeting $\overline{AB}$ at E	Parallel Postulate (Playfair's axiom)
3	$AECD$ is a parallelogram	Definition of parallelogram ($\overline{AE} \parallel \overline{DC}$ and $\overline{EC} \parallel \overline{AD}$)
4	$\overline{AD} \cong \overline{EC}$	Parallelogram Opposite Sides Theorem
5	$\overline{EC} \cong \overline{BC}$	Transitive Property of Congruence
6	$\angle CEB \cong \angle B$	Isosceles Triangle Theorem (in $\triangle ECB$; Euclid I.5)
7	$\angle A \cong \angle CEB$	Corresponding Angles Postulate ($\overline{AD} \parallel \overline{EC}$, transversal $\overleftrightarrow{AB}$)
8	$\angle A \cong \angle B$	Transitive Property of Congruence
9	$\angle A$ and $\angle D$ are supplementary; $\angle B$ and $\angle C$ are supplementary	Same-Side Interior Angles Theorem ($\overline{AB} \parallel \overline{DC}$)
10	$\angle D \cong \angle C$	Congruent Supplements Theorem
11	$\angle A \cong \angle B$ and $\angle C \cong \angle D$	Steps 8 and 10 together

∎

The converse, that a trapezoid with a pair of congruent base angles is isosceles, is proved in Practice 12 by the same construction.

THEOREM **Isosceles Trapezoid Diagonals Theorem**

If a trapezoid is isosceles, then its diagonals are congruent.



FLOW PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \cong \overline{BC}$

Prove: $\overline{AC} \cong \overline{BD}$

(1) $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$
Given

(2) $\angle DAB \cong \angle CBA$
Isosceles Trapezoid Base Angles Theorem
from (1)

(3) $\overline{AB} \cong \overline{BA}$
Reflexive Property of Congruence

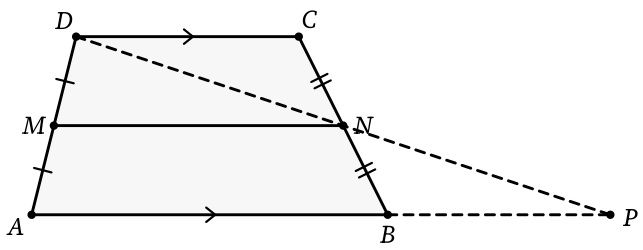
(4) $\triangle DAB \cong \triangle CBA$
SAS
from (1), (2), (3)

(5) $\overline{AC} \cong \overline{BD}$
CPCTC
from (4)

∎

THEOREM **Trapezoid Midsegment Theorem**

The midsegment of a trapezoid is parallel to the bases, and its length is half the sum of the lengths of the bases.



PARAGRAPH PROOF

Given: Trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{DC}$; M is the midpoint of $\overline{AD}$; N is the midpoint of $\overline{BC}$

Prove: $\overline{MN} \parallel \overline{AB}$, $\overline{MN} \parallel \overline{DC}$, and $MN = \frac{1}{2}(AB + DC)$

Draw $\overleftrightarrow{DN}$ (two points determine a line). It meets $\overleftrightarrow{AB}$ at some point P: by Playfair's axiom the only line through D parallel to $\overleftrightarrow{AB}$ is $\overleftrightarrow{DC}$, and $\overleftrightarrow{DN}$ is a different line because N is not on $\overleftrightarrow{DC}$. As in the figure, P lies on the extension of $\overline{AB}$ beyond B.

Compare $\triangle DCN$ and $\triangle PBN$. $\angle DCN \cong \angle PBN$ because they are alternate interior angles formed by the parallel lines $\overleftrightarrow{DC}$ and $\overleftrightarrow{AP}$ with transversal $\overleftrightarrow{BC}$. $\overline{CN} \cong \overline{BN}$ because N is the midpoint of $\overline{BC}$. $\angle DNC \cong \angle PNB$ because they are vertical angles. So $\triangle DCN \cong \triangle PBN$ by ASA, and by CPCTC $\overline{DN} \cong \overline{PN}$ and $\overline{DC} \cong \overline{PB}$.

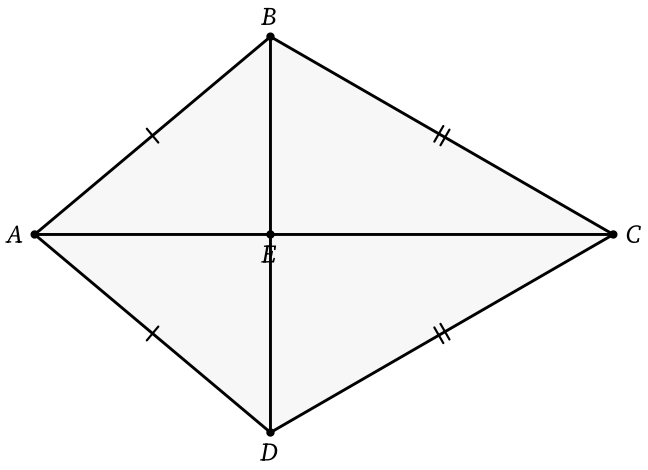
Now N is the midpoint of $\overline{DP}$, and M is the midpoint of $\overline{DA}$. So $\overline{MN}$ is a midsegment of $\triangle DAP$, and by the Triangle Midsegment Theorem $\overline{MN} \parallel \overline{AP}$ and $MN = \frac{1}{2}AP$. Line AP is line AB, so $\overline{MN} \parallel \overline{AB}$; and since $\overline{AB} \parallel \overline{DC}$, also $\overline{MN} \parallel \overline{DC}$ (two lines parallel to the same line are parallel).

Finally, B is between A and P, so $AP = AB + BP$ by the Segment Addition Postulate, and $BP = DC$. By substitution, $MN = \frac{1}{2}(AB + DC)$.

∎

THEOREM **Kite Theorem**

The diagonals of a kite are perpendicular, and exactly one pair of opposite angles of a kite is congruent (the pair between the noncongruent sides).



PARAGRAPH PROOF

Given: $ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$

Prove: $\overline{AC} \perp \overline{BD}$, $\angle ABC \cong \angle ADC$, and $\angle BAD \ncong \angle BCD$

Perpendicular diagonals. Since $AB = AD$, point A is equidistant from B and D, so it lies on the perpendicular bisector of $\overline{BD}$ (Converse of the Perpendicular Bisector

Theorem). Since $CB = CD$, point C lies on it too. Two points determine a line, so $\overleftrightarrow{AC}$ is the perpendicular bisector of $\overline{BD}$, and $\overline{AC} \perp \overline{BD}$.

One pair of congruent opposite angles. $\overline{AB} \cong \overline{AD}$, $\overline{CB} \cong \overline{CD}$ and $\overline{AC} \cong \overline{AC}$, so $\triangle ABC \cong \triangle ADC$ by SSS, and $\angle ABC \cong \angle ADC$ by CPCTC.

Only one pair (indirect). Suppose $\angle BAD \cong \angle BCD$ as well. Then both pairs of opposite angles of $ABCD$ would be congruent, so $ABCD$ would be a parallelogram (Opposite Angles Converse), and its opposite sides $\overline{AB}$ and $\overline{CD}$ would be congruent. But a kite has no pair of congruent opposite sides, by definition. This contradiction shows $\angle BAD \ncong \angle BCD$.

∎

KEY FORMULAS

$$\text{Trapezoid midsegment: } MN = \frac{b_1 + b_2}{2}$$

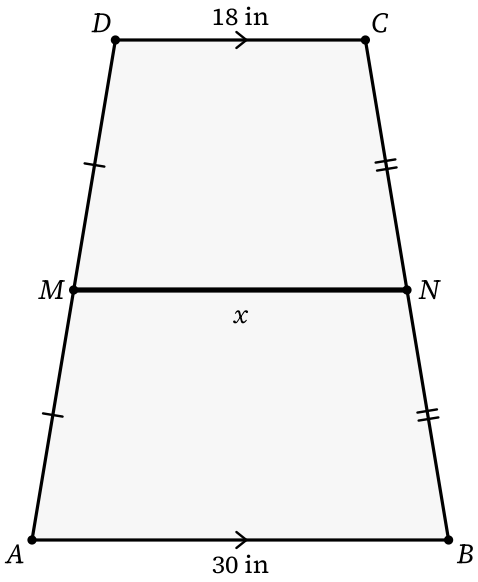
$$\text{Trapezoid } ABCD \text{ with } \overline{AB} \parallel \overline{DC}: \; m\angle A + m\angle D = 180^\circ, \; m\angle B + m\angle C = 180^\circ$$

From history. The word trapezoid comes from the Greek *trapezion*, 'little table'. Euclid used the name 'trapezia' for every quadrilateral that is not a square, oblong, rhombus or rhomboid; later writers narrowed it to quadrilaterals with one pair of parallel sides. The words still differ today: in Britain the shape of this lesson is called a trapezium.

Euclid connection. Euclid's Definition 22 of Book I ends: 'and let quadrilaterals other than these be called trapezia.' He proves no special theorems about trapezoids or kites, but every proof in this lesson is built from his: the Isosceles Triangle Theorem and its converse (I.5 and I.6), parallel lines (I.29 and I.30) and the parallelogram (I.34).

Practice habit. Before you start, mark the bases with parallel arrows and the congruent sides with ticks, and decide which pair of sides is parallel before calling any side a base. Recite until automatic: 'The midsegment of a trapezoid is parallel to the bases and equals half their sum' and 'The diagonals of a kite are perpendicular.'

Example 1. Samuel is building a bookshelf whose two sides lean inward, so its front is a trapezoid $ABCD$. The bottom shelf $\overline{AB}$ is 30 inches long and the top shelf $\overline{DC}$ is 18 inches long, and both are level (parallel). A middle shelf $\overline{MN}$ joins the midpoints of the two sides, as in the figure. How long must the middle shelf be?



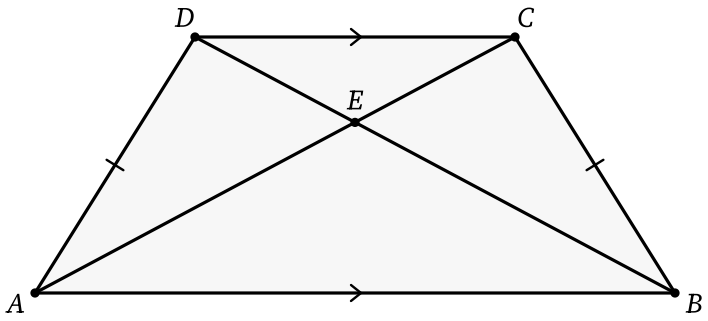
Front of the bookshelf

Show solution

1. $\overline{MN}$ joins the midpoints of the legs $\overline{AD}$ and $\overline{BC}$, so it is the midsegment of trapezoid $ABCD$.
2. By the Trapezoid Midsegment Theorem, $MN = \dfrac{30 + 18}{2} = \dfrac{48}{2} = 24$ inches.
3. Check: 24 inches lies between 18 and 30, as a middle shelf should, and the theorem also guarantees the shelf is level (parallel to the bases).

Answer: 24 inches

Example 2. Given: $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$; its diagonals meet at E, as in the figure. Prove: $\overline{AE} \cong \overline{BE}$.



Show solution

1. Work backward: $\overline{AE}$ and $\overline{BE}$ are sides of $\triangle ABE$, so it is enough to show $\angle EAB \cong \angle EBA$ (Converse of the Isosceles Triangle Theorem).
2. Those angles are angles of the overlapping triangles $\triangle DAB$ and $\triangle CBA$. Separate them in a sketch.
3. The triangles are congruent by SAS, using the congruent legs, the congruent base angles and the shared base.

TWO-COLUMN PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$; Diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\overline{AE} \cong \overline{BE}$

	STATEMENTS	REASONS
1	$ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$; the diagonals meet at E	Given
2	$\angle DAB \cong \angle CBA$	Isosceles Trapezoid Base Angles Theorem
3	$\overline{AB} \cong \overline{BA}$	Reflexive Property of Congruence
4	$\triangle DAB \cong \triangle CBA$	SAS
5	$\angle DBA \cong \angle CAB$	CPCTC
6	$\angle EBA \cong \angle EAB$	Same angles as in step 5 (E lies on $\overline{BD}$ and on $\overline{AC}$)

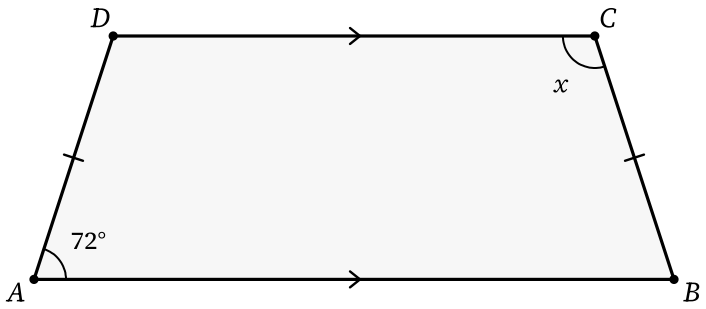
	STATEMENTS	REASONS
7	$\overline{AE} \cong \overline{BE}$	Converse of the Isosceles Triangle Theorem (Euclid I.6)

∎

Answer: See the proof: $\triangle DAB \cong \triangle CBA$ by SAS, so $\angle EAB \cong \angle EBA$ and $\overline{AE} \cong \overline{BE}$.

PRACTICE

1. **Easy** $ABCD$ in the figure is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $m\angle A = 72°$. Find $m\angle C$.



Show answer

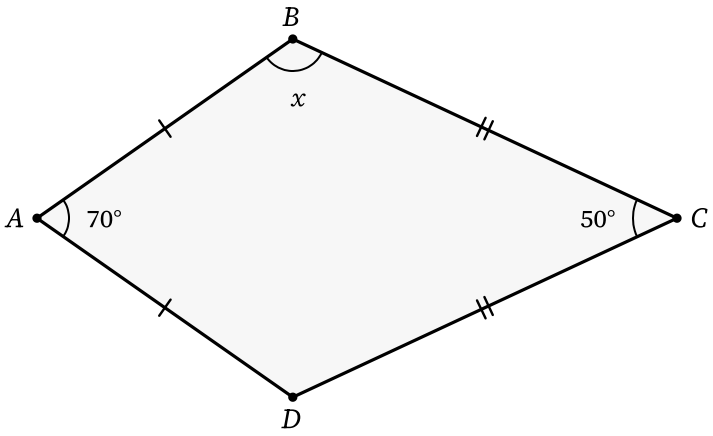
$\angle B \cong \angle A$, so $m\angle B = 72°$. Angles along leg $\overline{BC}$ are supplementary: $m\angle C = 180° - 72° = 108°$.

2. **Easy** The bases of a trapezoid measure 14 cm and 22 cm. Find the length of its midsegment.

Show answer

$$\frac{14 + 22}{2} = 18 \text{ cm}$$

3. **Easy** $ABCD$ in the figure is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$; $m\angle A = 70°$ and $m\angle C = 50°$. Find $m\angle B$.



Show answer

$\angle B \cong \angle D$ (Kite Theorem), and the angles add to $360°$: $2\,m\angle B = 360° - 70° - 50° = 240°$, so $m\angle B = 120°$.

4. **Easy** In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$, $m\angle A = 58°$ and $m\angle B = 74°$. Find $m\angle D$ and $m\angle C$.

Show answer

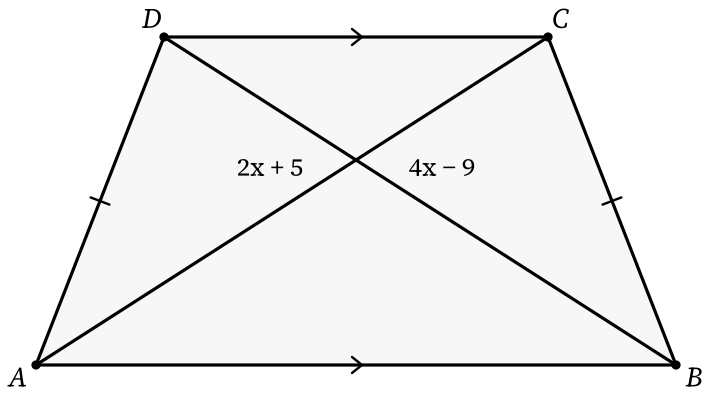
Angles along each leg are same-side interior angles, so they are supplementary: $m\angle D = 180° - 58° = 122°$ and $m\angle C = 180° - 74° = 106°$.

5. **Medium** Ruth's flower bed is a trapezoid. She lays a straight row of bricks from the midpoint of one leg to the midpoint of the other; the row is 12 ft long. The shorter of the two parallel sides is 9 ft. How long is the longer parallel side?

Show answer

$\dfrac{9 + b}{2} = 12$, so $9 + b = 24$ and $b = 15$ ft.

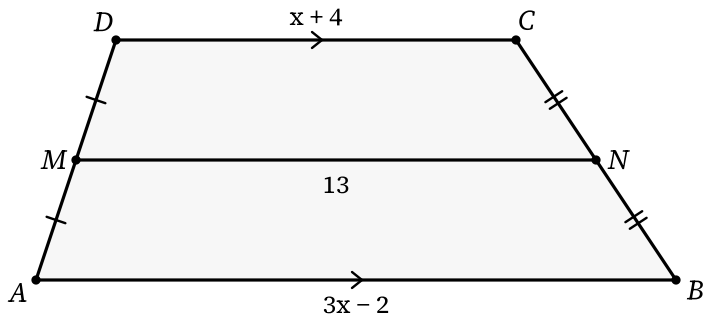
6. **Medium** $ABCD$ in the figure is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$. Its diagonals measure $AC = 2x + 5$ and $BD = 4x - 9$. Find x.



Show answer

The diagonals of an isosceles trapezoid are congruent: $2x + 5 = 4x - 9$, so $14 = 2x$ and $x = 7$. (Each diagonal is 19.)

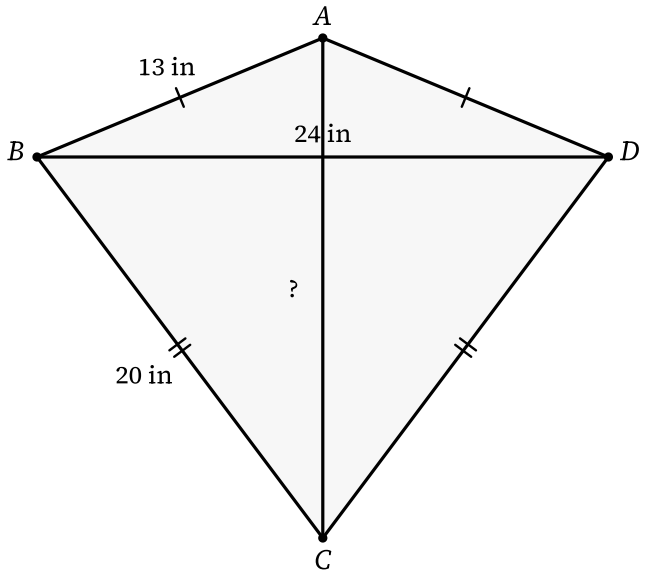
7. **Medium** In trapezoid $ABCD$ in the figure, $\overline{MN}$ is the midsegment, $AB = 3x - 2$, $DC = x + 4$ and $MN = 13$. Find x.



Show answer

$\frac{(3x - 2) + (x + 4)}{2} = 13$, so $4x + 2 = 26$, $4x = 24$ and $x = 6$. (The bases are 16 and 10.)

8. **Medium** Grace's kite $ABCD$ has sides $AB = AD = 13$ inches and $CB = CD = 20$ inches. The cross spar $\overline{BD}$ is 24 inches long, as in the figure. How long is the long spar $\overline{AC}$?



Show answer

By the Kite Theorem, $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$; let them meet at E, so $BE = 12$ and the angles at E are right. Then $AE = \sqrt{13^2 - 12^2} = \sqrt{25} = 5$ and $CE = \sqrt{20^2 - 12^2} = \sqrt{256} = 16$, so $AC = 5 + 16 = 21$ inches.

9. **Medium** Explain why a kite can never be a parallelogram.

Show answer

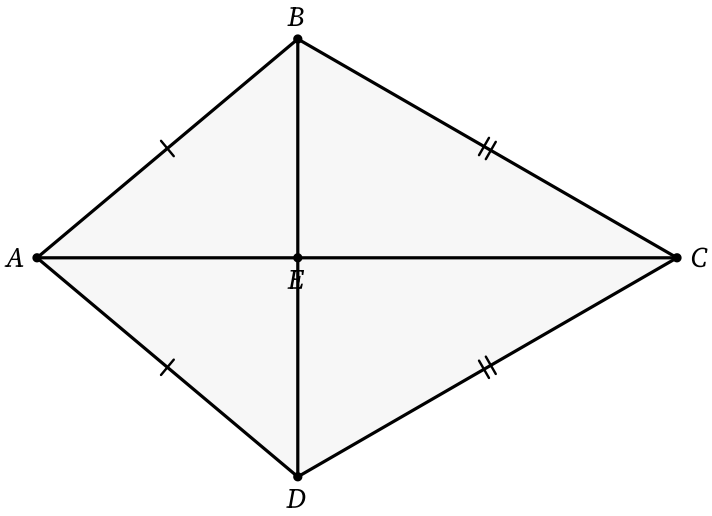
Opposite sides of every parallelogram are congruent (Parallelogram Opposite Sides Theorem). By definition, a kite has no pair of congruent opposite sides. So no quadrilateral can be both.

10. **Hard** Quadrilateral $ABCD$ has vertices $A(0, 0)$, $B(8, 0)$, $C(6, 4)$ and $D(2, 4)$. Classify it as precisely as possible and justify your answer.

Show answer

Isosceles trapezoid. Slopes: $\overline{AB}$ and $\overline{DC}$ both 0, so they are parallel; $\overline{AD}$ has slope 2 and $\overline{BC}$ has slope -2, so they are not parallel. Exactly one pair of parallel sides makes a trapezoid. The legs are $AD = \sqrt{4 + 16} = \sqrt{20}$ and $BC = \sqrt{4 + 16} = \sqrt{20}$, so it is isosceles.

11. Hard Given: $ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$, as in the figure. Prove: $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$.



Show answer

See the model proof.

TWO-COLUMN PROOF

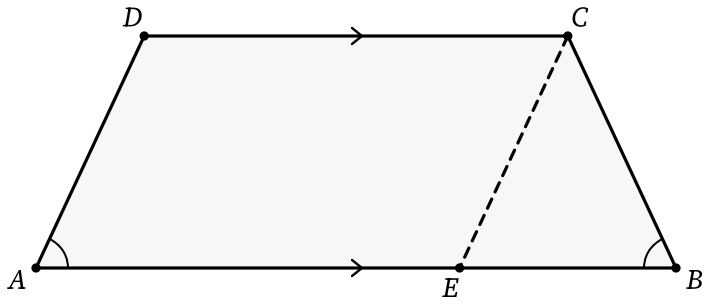
Given: $\overline{AB} \cong \overline{AD}$; $\overline{CB} \cong \overline{CD}$

Prove: $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$	Given
2	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
3	$\triangle ABC \cong \triangle ADC$	SSS
4	$\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$	CPCTC
5	$\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$	Definition of angle bisector

∎

12. Hard Given: trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$ and $\angle A \cong \angle B$. In the figure, $\overline{CE}$ has been drawn parallel to $\overline{AD}$ with E on $\overline{AB}$. Prove: $\overline{AD} \cong \overline{BC}$, so the trapezoid is isosceles. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$ and $\angle A \cong \angle B$; $\overline{CE} \parallel \overline{AD}$ with E on $\overline{AB}$
Prove: $\overline{AD} \cong \overline{BC}$

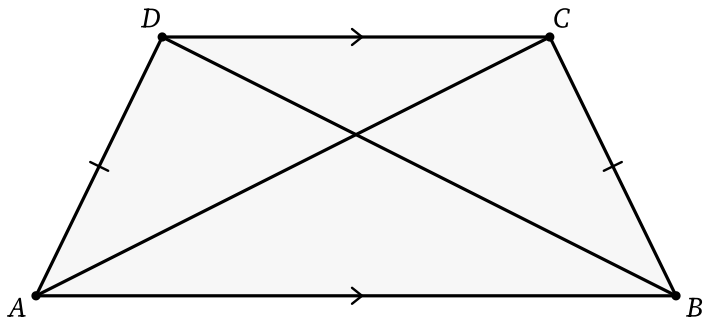
Quadrilateral $AECD$ has $\overline{AE} \parallel \overline{DC}$ (they lie on the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$) and $\overline{EC} \parallel \overline{AD}$ (given), so it is a parallelogram by definition, and its opposite sides are congruent: $\overline{AD} \cong \overline{EC}$.

Because $\overline{AD} \parallel \overline{EC}$ with transversal $\overleftrightarrow{AB}$, the corresponding angles $\angle A$ and $\angle CEB$ are congruent (Corresponding Angles Postulate). Since $\angle A \cong \angle B$, the Transitive Property gives $\angle CEB \cong \angle B$.

So in $\triangle ECB$ two angles are congruent, and by the Converse of the Isosceles Triangle Theorem (Euclid I.6) the sides opposite them are congruent: $\overline{EC} \cong \overline{BC}$. Combining with $\overline{AD} \cong \overline{EC}$ by the Transitive Property, $\overline{AD} \cong \overline{BC}$.

∎

13. Hard Given: $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$, as in the figure. Prove: $\angle ADB \cong \angle BCA$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \cong \overline{BC}$

Prove: $\angle ADB \cong \angle BCA$

(1) $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$
Given

(2) $\overline{DB} \cong \overline{CA}$
Isosceles Trapezoid Diagonals Theorem
from (1)

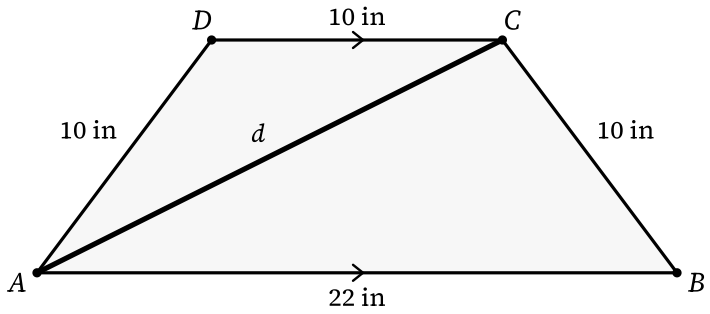
(3) $\overline{AB} \cong \overline{BA}$
Reflexive Property of Congruence

(4) $\triangle ADB \cong \triangle BCA$
SSS
from (1), (2), (3)

(5) $\angle ADB \cong \angle BCA$
CPCTC
from (4)

∎

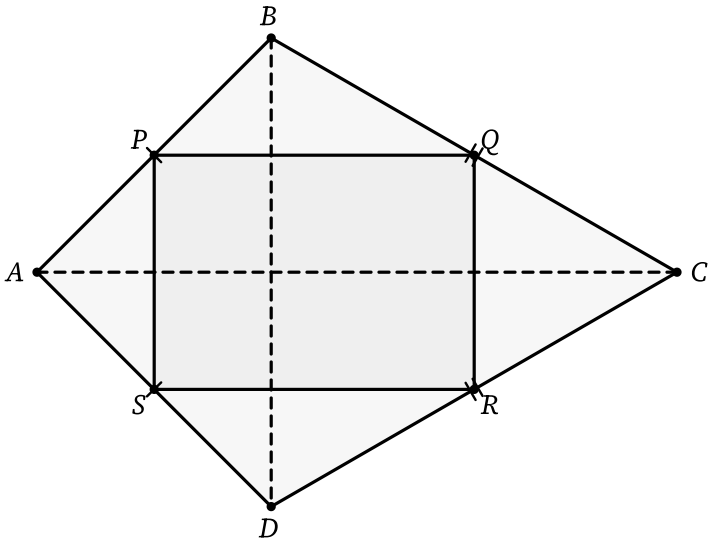
14. **Challenge** The frame of a stained-glass window is an isosceles trapezoid $ABCD$ with bases $AB = 22$ inches and $DC = 10$ inches and legs of 10 inches, as in the figure. A brace runs along diagonal $\overline{AC}$. Find the length of the brace, in simplest radical form. (Hint: drop perpendiculars from D and C to $\overline{AB}$.)



Show answer

Let the perpendiculars from D and C meet $\overline{AB}$ at F and G. $DCGF$ is a rectangle, so $FG = 10$, and the right triangles $\triangle AFD$ and $\triangle BGC$ are congruent by HL, so $AF = GB = \dfrac{22 - 10}{2} = 6$. The height is $CG = \sqrt{10^2 - 6^2} = 8$. Then $AG = 6 + 10 = 16$ and $AC = \sqrt{16^2 + 8^2} = \sqrt{320} = 8\sqrt{5}$ inches (about 17.9 in).

15. **Challenge** Given: $ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$, and P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, as in the figure. Prove: $PQRS$ is a rectangle.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$; P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$

Prove: $PQRS$ is a rectangle

Imagine the diagonals $\overline{AC}$ and $\overline{BD}$ drawn. In $\triangle ABC$, P and Q are midpoints of two sides, so $\overline{PQ} \parallel \overline{AC}$ by the Triangle Midsegment Theorem. In $\triangle ADC$, likewise $\overline{SR} \parallel \overline{AC}$. Two lines parallel to the same line are parallel, so $\overline{PQ} \parallel \overline{SR}$. In the same way, using $\triangle ABD$ and $\triangle CBD$, $\overline{PS} \parallel \overline{BD}$ and $\overline{QR} \parallel \overline{BD}$, so $\overline{PS} \parallel \overline{QR}$. Both pairs of opposite sides are parallel, so $PQRS$ is a parallelogram.

By the Kite Theorem, $\overline{AC} \perp \overline{BD}$. In a plane, a line perpendicular to one of two parallel lines is perpendicular to the other (Perpendicular Transversal Theorem). Since $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$ and $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AC}$, we get $\overleftrightarrow{BD} \perp \overleftrightarrow{PQ}$. Since $\overleftrightarrow{PQ} \perp \overleftrightarrow{BD}$ and $\overleftrightarrow{PS} \parallel \overleftrightarrow{BD}$, we get $\overleftrightarrow{PQ} \perp \overleftrightarrow{PS}$, so $\angle SPQ$ is a right angle.

A parallelogram with one right angle is a rectangle: the opposite angle is congruent to it, and each consecutive angle is its supplement, $180° - 90° = 90°$. So $PQRS$ is a rectangle.

COMMON MISTAKES

- Taking the midsegment of a trapezoid to be half of one base, or the sum of the bases. It is the *average* of the two bases, so it always lies between them in length.
- Using isosceles-trapezoid facts (congruent base angles, congruent diagonals) for a trapezoid whose legs are not known to be congruent. Check for leg ticks first.
- In a kite, marking the wrong pair of angles congruent. The congruent angles are the ones *between* a short side and a long side ($\angle B$ and $\angle D$ when $AB = AD$ and $CB = CD$), not the angles where the equal sides meet.
- Assuming both diagonals of a kite are bisected. Only the diagonal joining the vertices where unequal sides meet ($\overline{BD}$) is bisected; the other one, $\overline{AC}$, usually is not.

> **Make it a game · Sketch to Spec.** Parent reads descriptions and you sketch and label each in 30 seconds: an isosceles trapezoid with bases 14 and 8; a kite with perpendicular diagonals; a trapezoid whose midsegment is the average of its bases. Parent checks every label. Then answer: what is the midsegment's length, and which angles are congruent?

UNIT 6

Quadrilaterals and Polygons

In this unit you take the triangle facts of Units 4 and 5 and use them to understand figures with more sides. Cutting a polygon into triangles gives the angle sum of any polygon, and drawing a diagonal in a quadrilateral turns almost every question about it into a question about congruent triangles. You will prove the properties of parallelograms just as Euclid did in I.34, then prove the converses that let you recognize a parallelogram, including Euclid's I.33.

Next you study the special members of the quadrilateral family: rectangles, rhombuses and squares (with Euclid's construction of a square, I.46), and then trapezoids and kites. Along the way you will see why builders measure the diagonals of a foundation, why a navigator's parallel rule and a scissor lift stay parallel, and which regular polygons can tile a floor or frame a rose window.

Parents: the unit is proof-heavy. Expect two-column, paragraph and flow proofs in every lesson after the first, and indirect proofs in several of them. A good check of your student's work is to ask, for any step, 'Which theorem says so?' Every step should have a named reason.

PREREQUISITES

- Use the Triangle Angle Sum Theorem and the Linear Pair Theorem (Lessons 2.4 and 4.1).
- Use the angle theorems for parallel lines and their converses, including Euclid I.27, I.28 and I.30 (Lessons 3.2 and 3.3).
- Prove triangles congruent with SSS, SAS, ASA, AAS and HL and use CPCTC (Lessons 4.3–4.7).
- Use the Isosceles Triangle Theorem and its converse (Lesson 4.6).
- Use the Perpendicular Bisector Theorem and its converse and the Triangle Midsegment Theorem (Lessons 5.1 and 5.4).
- Find slope, distance and midpoint on the coordinate plane and use the Pythagorean theorem numerically (Algebra I).

MASTERY CHECKLIST

- ☐ I can classify polygons as convex or concave and count their diagonals.
- ☐ I can prove and use the Polygon Interior Angle Sum Theorem and the Exterior Angle Sum Theorem.
- ☐ I can find each angle of a regular polygon and the number of sides from an angle, and decide which regular polygons tile a floor.
- ☐ I can construct a regular hexagon in a circle (Euclid IV.15) and a square on a segment (Euclid I.46) and explain why each construction works.
- ☐ I can prove that the opposite sides and angles of a parallelogram are congruent (Euclid I.34), that consecutive angles are supplementary and that the diagonals bisect each other.
- ☐ I can prove that a quadrilateral is a parallelogram with any of the four tests, including one pair of sides both congruent and parallel (Euclid I.33).
- ☐ I can show on the coordinate plane that a quadrilateral is a parallelogram, rectangle, rhombus, square or trapezoid using slope, distance and midpoints.
- ☐ I can prove and use the diagonal properties of rectangles and rhombuses and explain why

builders compare the diagonals of a foundation.

- ☐ I can prove and use the theorems on isosceles trapezoids and the Trapezoid Midsegment Theorem.
- ☐ I can prove that the diagonals of a kite are perpendicular and that exactly one pair of its opposite angles is congruent.
- ☐ I can write two-column, paragraph, flow and indirect proofs about quadrilaterals with a correct reason for every step.

LESSONS IN THIS UNIT

6.1	**Angles of Polygons**	45 minutes
6.2	**Properties of Parallelograms**	2 sessions of 45 minutes
6.3	**Proving That a Quadrilateral Is a Parallelogram**	2 sessions of 45 minutes
6.4	**Rectangles, Rhombuses and Squares**	2 sessions of 45 minutes
6.5	**Trapezoids and Kites**	2 sessions of 45 minutes

Unit 6 test

You have 55 minutes. No calculator is allowed; work every problem by hand and give exact answers (for example $8\sqrt{5}$) unless a problem asks you to round. Show all of your work. In every proof, give a reason for every step: a definition, postulate or theorem, not 'from the figure'. Bring a compass and straightedge for Problem 13. Figures are drawn to scale, but use only the given information and the marks on them, not measurements.

1. **5 pts**

   Find the sum of the measures of the interior angles of a convex 9-sided polygon (a nonagon).

2. **5 pts**

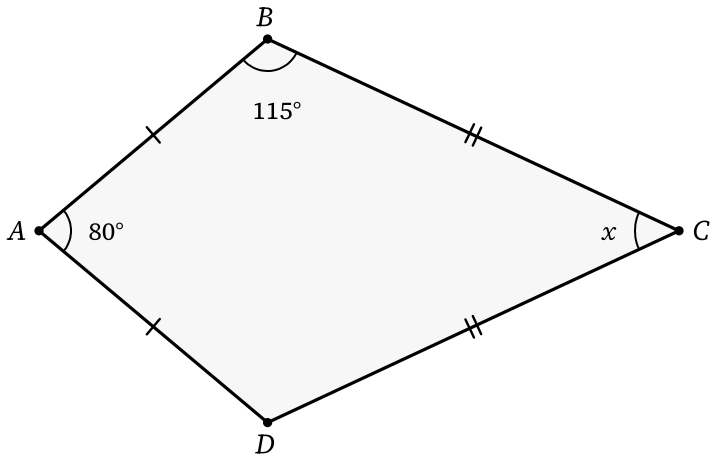
   Each exterior angle of a regular polygon measures $20°$. How many sides does the polygon have?

3. **5 pts**

   The diagonals of parallelogram $PQRS$ meet at T. If $PT = 3y - 4$ and $TR = y + 10$, find PR.

4. **5 pts**

   $ABCD$ in the figure is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$; $m\angle A = 80°$ and $m\angle B = 115°$. Find $m\angle C$.

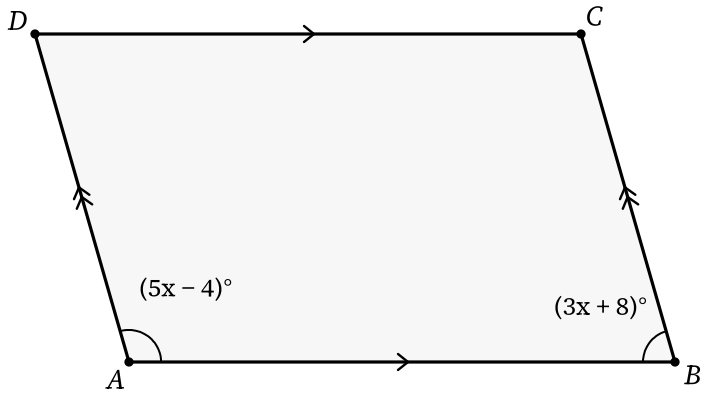


5. **5 pts**

   $ABCD$ is an isosceles trapezoid with bases $\overline{AB}$ and $\overline{DC}$. If $m\angle A = (3x + 5)°$ and $m\angle D = (5x + 15)°$, find x.

6. **6 pts**

In parallelogram $ABCD$ in the figure, $m\angle A = (5x - 4)^\circ$ and $m\angle B = (3x + 8)^\circ$. Find the measure of the larger of these two angles.



7. **6 pts**

The interior angles of a convex hexagon measure x°, $(x + 10)^\circ$, $(x + 20)^\circ$, $(x + 30)^\circ$, $(x + 40)^\circ$ and $(x + 50)^\circ$. Find the measure of the largest angle.

8. **6 pts**

A carpenter is making the wooden frame for a rose window in the shape of a regular 16-sided polygon. At what angle do two neighboring pieces of the frame meet (the interior angle)?

9. **5 pts**

One side of a barn roof is a trapezoid. Its bottom edge (the eave) is 32 ft long and its top edge (the ridge) is parallel to it. A row of shingles runs straight across the roof from the midpoint of one slanted edge to the midpoint of the other, and the row is 26 ft long. How long is the ridge?

10. **6 pts**

A floor tile is a rhombus whose diagonals measure 16 cm and 30 cm. Find the perimeter of the tile.

11. **6 pts**

Henry is building a rectangular deck 15 ft by 20 ft. He has set the frame so that the opposite sides measure 15 ft and 20 ft. What should each diagonal measure, and why does finding that both diagonals have this length prove that the corners are square?

12. **7 pts**

Quadrilateral $ABCD$ has vertices $A(-1, -2)$, $B(4, -1)$, $C(6, 3)$ and $D(1, 2)$. Classify it as precisely as possible: parallelogram, rectangle, rhombus or square. Justify your answer with slopes or distances.

13. **8 pts**

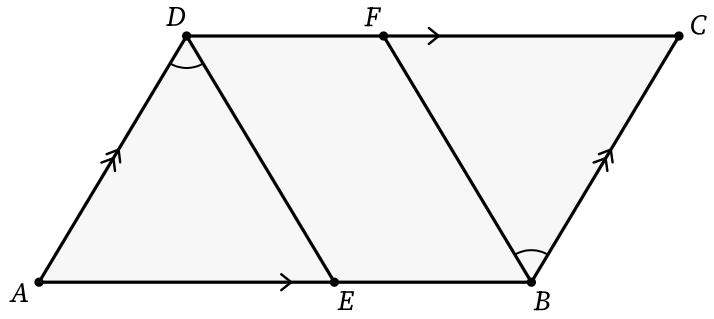
Draw a segment $\overline{AB}$ about 5 cm long. With compass and straightedge, construct a square with side $\overline{AB}$. List your steps and explain why the figure you constructed is a square.

14. **5 pts**

Prove indirectly: a convex quadrilateral cannot have four acute angles.

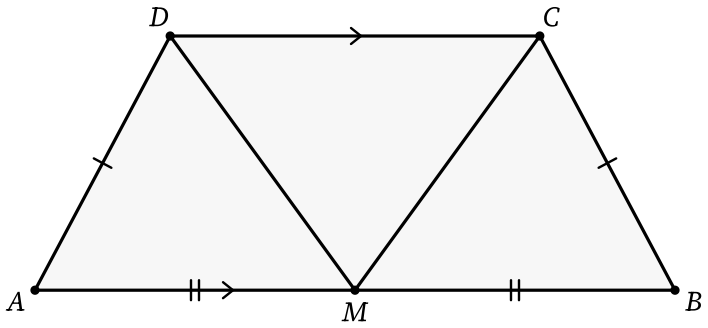
15. **10 pts**

Given: $ABCD$ is a parallelogram; E lies on $\overline{AB}$ and F lies on $\overline{CD}$, with $\angle ADE \cong \angle CBF$, as in the figure. Prove: $\overline{DE} \cong \overline{BF}$. Write a two-column proof.



16. **10 pts**

Given: $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$, and M is the midpoint of $\overline{AB}$, as in the figure. Prove: $\overline{MD} \cong \overline{MC}$. Write a flow proof.



ANSWER KEY

1. $(9-2)\cdot 180^\circ = 1260^\circ$
2. $360 \div 20 = 18$ sides
3. The diagonals bisect each other: $3y - 4 = y + 10$, so $y = 7$ and $PT = TR = 17$. Then $PR = 34$.
4. $\angle D \cong \angle B$, so $m\angle D = 115^\circ$. Then $m\angle C = 360^\circ - 80^\circ - 115^\circ - 115^\circ = 50^\circ$.
5. $\angle A$ and $\angle D$ lie along leg $\overline{AD}$, so they are supplementary: $8x + 20 = 180$, $x = 20$. (The angles are 65° and 115°.)
6. Consecutive angles are supplementary: $8x + 4 = 180$, so $x = 22$. Then $m\angle A = 106^\circ$ and $m\angle B = 74^\circ$; the larger is 106°.
7. The sum is $(6-2)\cdot 180^\circ = 720^\circ$: $6x + 150 = 720$, so $x = 95$. The largest angle is $95^\circ + 50^\circ = 145^\circ$.
8. $\dfrac{(16-2)\cdot 180^\circ}{16} = \dfrac{2520^\circ}{16} = 157.5^\circ$
9. The row is the midsegment: $\dfrac{32 + r}{2} = 26$, so $32 + r = 52$ and the ridge is $r = 20$ ft.
10. The diagonals of a rhombus bisect each other at right angles, making right triangles with legs 8 cm and 15 cm. Each side is $\sqrt{64 + 225} = \sqrt{289} = 17$ cm, so the perimeter is $4 \times 17 = 68$ cm.
11. Each diagonal should be $\sqrt{15^2 + 20^2} = \sqrt{625} = 25$ ft. Both pairs of opposite sides are congruent, so the frame is a parallelogram (Opposite Sides Converse); a parallelogram with congruent diagonals is a rectangle (Rectangle Diagonals Theorem), so all four corners are right angles.
12. Parallelogram. Slopes: $\overline{AB}$ and $\overline{DC}$ are both $\frac{1}{5}$; $\overline{BC}$ and $\overline{AD}$ are both 2; so both pairs of opposite sides are parallel. It is not a rhombus, because $AB = \sqrt{26}$ but $BC = \sqrt{20}$. It is not a rectangle, because the slopes $\frac{1}{5}$ and 2 do not multiply to -1, so $\angle B$ is not a right angle.
13. Steps (Euclid I.46): (1) Extend line AB past A. (2) Construct the perpendicular to line AB at A: arcs centered at A cut the line at P and Q; equal arcs from P and Q cross at R; draw $\overleftrightarrow{AR}$. (3) With compass set to AB, mark D on $\overleftrightarrow{AR}$ with $AD = AB$. (4) With the same width, swing arcs from B and from D that cross at C. (5) Draw $\overline{BC}$ and $\overline{DC}$. Why: all four sides equal AB, so $ABCD$ is a rhombus and therefore a parallelogram; $\angle A$ is a right angle by construction; consecutive angles of a parallelogram are supplementary and opposite angles congruent, so

all four angles are right angles. Four congruent sides and four right angles make a square.

14. See the model proof.

    INDIRECT PROOF

    Given: $ABCD$ is a convex quadrilateral

    Prove: $ABCD$ does not have four acute angles

    Assume, to the contrary, that all four angles of $ABCD$ are acute.
    Then each angle measures less than $90°$, so the sum of the four angles is less than $4 \cdot 90° = 360°$. But by the Polygon Interior Angle Sum Theorem the angles of a quadrilateral add to exactly $(4-2) \cdot 180° = 360°$. This is a contradiction.
    So the assumption is false: a convex quadrilateral cannot have four acute angles.

    ∎

15. See the model proof.

    TWO-COLUMN PROOF

    Given: $ABCD$ is a parallelogram; E on $\overline{AB}$, F on $\overline{CD}$; $\angle ADE \cong \angle CBF$

    Prove: $\overline{DE} \cong \overline{BF}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; E on $\overline{AB}$, F on $\overline{CD}$; $\angle ADE \cong \angle CBF$	Given
2	$\angle DAE \cong \angle BCF$	Parallelogram Opposite Angles Theorem ($\angle DAE$ is $\angle DAB$ and $\angle BCF$ is $\angle BCD$)
3	$\overline{AD} \cong \overline{CB}$	Parallelogram Opposite Sides Theorem
4	$\triangle ADE \cong \triangle CBF$	ASA
5	$\overline{DE} \cong \overline{BF}$	CPCTC

    ∎

16. See the model proof.

    FLOW PROOF

    Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$; M is the midpoint of $\overline{AB}$

    Prove: $\overline{MD} \cong \overline{MC}$

    (1) $ABCD$ is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

Given

(2) M is the midpoint of $\overline{AB}$
Given

(3) $\overline{AM} \cong \overline{BM}$
Definition of midpoint
from (2)

(4) $\angle DAM \cong \angle CBM$
Isosceles Trapezoid Base Angles Theorem
from (1)

(5) $\triangle DAM \cong \triangle CBM$
SAS
from (1), (3), (4)

(6) $\overline{MD} \cong \overline{MC}$
CPCTC
from (5)

∎

UNIT 7

Transformations

In this unit geometry starts to move. A transformation is a rule that sends every point of the plane to an image point. You will study the rigid motions, which are translations (slides), reflections (flips) and rotations (turns), and prove that each one preserves distance and angle measure. Along the way you will meet symmetry in letters, regular polygons, Gothic church windows and rose windows, and Heron of Alexandria's reflection trick for finding the shortest path.

Next you will combine motions. Two reflections make a translation or a rotation, and footprints in the snow make a glide reflection. This leads to a precise meaning for congruence: two figures are congruent when rigid motions carry one onto the other. That puts into modern language the way Euclid "applied" one triangle to another in Book I. The unit ends with dilations, which keep shape but change size. They are the mathematics of maps, blueprints and scale models, and they prepare the way for similarity in Unit 8.

Parents: most of the work is on the coordinate plane with simple rules such as $(x, y) \to (-y, x)$, together with short proofs that use the congruence tests of Unit 4 and the parallelogram facts of Unit 6. Graph paper, a compass and a straightedge are all the tools needed.

PREREQUISITES

- Plot points and use the distance, midpoint and slope formulas (Algebra I; reviewed in Lessons 2.2 and 3.5).
- Recognize parallel and perpendicular lines from their slopes (Lesson 3.5).
- Prove triangles congruent with SSS, SAS, ASA and AAS, and use CPCTC (Lessons 4.3–4.5).
- Use the Perpendicular Bisector Theorem and its converse (Lesson 5.1).
- Use the properties of parallelograms and the tests for a parallelogram (Lessons 6.2–6.3).
- Copy a segment, copy an angle and drop a perpendicular with compass and straightedge (Lessons 2.2 and 2.5).

MASTERY CHECKLIST

- ☐ I can translate a figure by a vector $\langle a, b \rangle$ and find the vector from a point and its image.
- ☐ I can explain why a transformation that preserves distance also preserves angle measure.
- ☐ I can reflect points in the x-axis, the y-axis, the line $y = x$ and other lines, and construct a reflection with compass and straightedge.
- ☐ I can find lines of symmetry and use the reflection trick to find a shortest path that touches a line.
- ☐ I can rotate points through 90°, 180° and 270° about the origin or another center

er, and prove that rotations preserve distance.

- ☐ I can find the order and angle of rotational symmetry of a regular polygon, a wheel or a rose window.
- ☐ I can find the image under a composition of rigid motions, and I know that two reflections make a translation or a rotation through twice the angle between the mirrors.
- ☐ I can describe a sequence of rigid motions that maps one congruent figure onto another, and explain how this connects to Euclid's proofs of I.4 and I.8.
- ☐ I can dilate a figure with a given center and scale factor, and find

the center and scale factor from a figure and its image.

- ☐ I can explain why a dilation multiplies lengths by $|k|$, keeps angle measures and maps lines to parallel lines, and I can use scale factors with maps and blueprints.

LESSONS IN THIS UNIT

7.1	**Translations and Vectors**	45 minutes
7.2	**Reflections and Symmetry**	45 minutes
7.3	**Rotations and Rotational Symmetry**	2 sessions of 45 minutes
7.4	**Compositions and Congruence**	2 sessions of 45 minutes
7.5	**Dilations**	2 sessions of 45 minutes

Unit 7 Test

Allow 45–60 minutes. No calculator is needed; give exact answers, in simplest radical form where needed, unless a problem says otherwise. Show your work for every computation. In each proof, give a reason for every statement. Problem 15 needs a compass and a straightedge. Label units in word problems.

1. **5 pts**
   Find the image of the point $(5, -1)$ under the translation by the vector $\langle -3, 8 \rangle$.

2. **5 pts**
   A translation maps $A(-4, 2)$ to $A'(3, -1)$. (a) Write the vector of the translation. (b) Find the image of $B(1, 1)$.

3. **6 pts**
   Reflect the point $(-2, 7)$ in (a) the x-axis, (b) the y-axis, (c) the line $y = x$.

4. **6 pts**

Rotate the point $(-3, 4)$ about the origin through (a) 90°, (b) 180°, (c) 270°.

5. **6 pts**

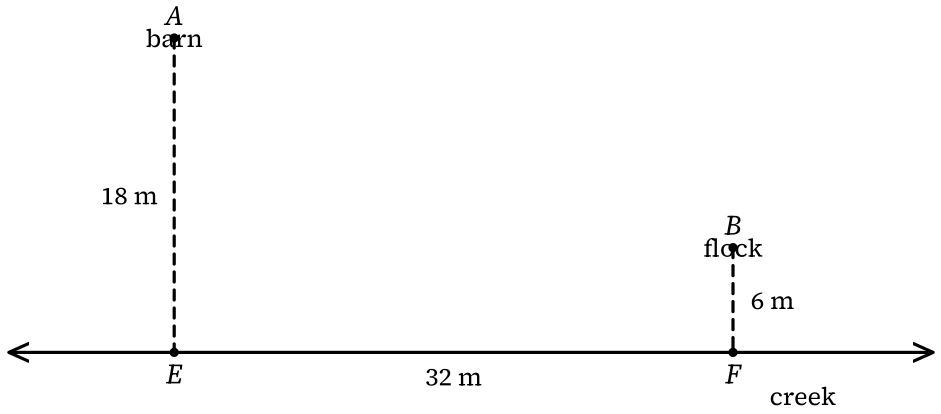
The stone tracery of a church's rose window is laid out on a regular 10-sided polygon. (a) How many lines of symmetry does a regular 10-sided polygon have? (b) What is the smallest angle of rotation that maps it onto itself?

6. **7 pts**

Segment $\overline{AB}$ has endpoints $A(-2, 4)$ and $B(6, 0)$. Dilate it with center at the origin and scale factor $\dfrac{3}{2}$. Find A' and B', and find $A'B'$ in simplest radical form.

7. **8 pts**

A sheepdog runs from the barn to a straight creek for a drink and then on to the flock. On a plan in meters, the creek is the x-axis, the barn is at $A(0, 18)$ and the flock is at $B(32, 6)$, as in the figure. (a) At what point on the creek should the dog drink to make its run as short as possible? (b) How long is that run?



8. **7 pts**

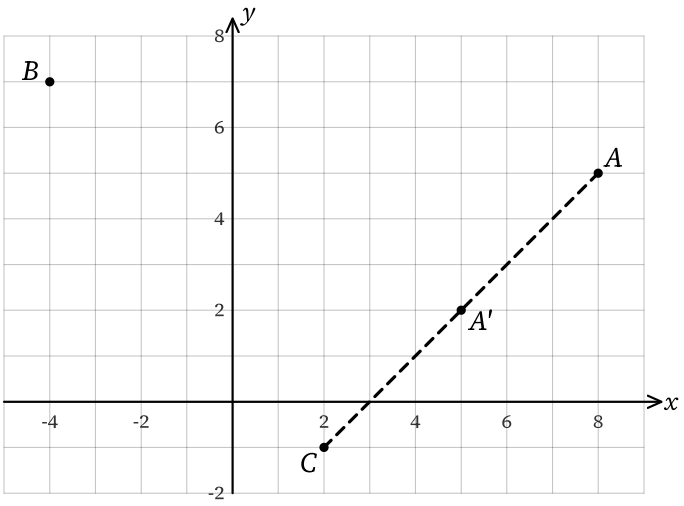
Reflect $P(2, 5)$ in the line $x = -1$, and then reflect the image in the line $x = 3$. (a) Give the final image. (b) Write the vector of the single translation that does the same job.

9. **6 pts**

Lines ℓ and m meet at point O. The reflection in ℓ followed by the reflection in m is a rotation about O. Through what angle does it rotate if the angle from ℓ to m is (a) 40°, (b) 90°?

10. **7 pts**

In the figure, a dilation with center $C(2, -1)$ maps $A(8, 5)$ to $A'(5, 2)$. (a) Find the scale factor. (b) Find the image of $B(-4, 7)$.



11. **6 pts**

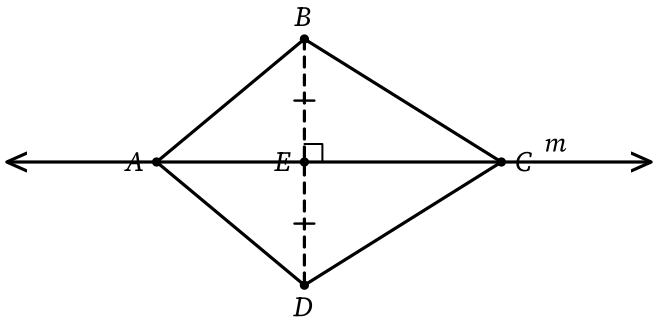
Henry builds a model of the family's barn at a scale factor of $\frac{1}{24}$. The real barn's roof ridge is 30 feet high and 48 feet long. How high and how long is the ridge on the model, in inches?

12. **6 pts**

Ruth says a dilation with scale factor 2 is a rigid motion because the image has the same shape as the original. Is she right? Explain why or why not.

13. **10 pts**

Given: line m is the perpendicular bisector of $\overline{BD}$, and A and C are different points on m, as in the figure. Prove: $\triangle ABC \cong \triangle ADC$.

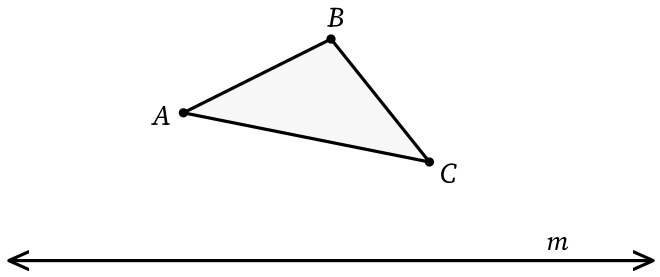


14. **7 pts**

Given: a dilation with center O and scale factor $k \neq 0$ maps A to A' and B to B', where $A \neq B$. Prove indirectly: $A' \neq B'$.

15. **8 pts**

Trace the figure, which shows $\triangle ABC$ and line m. With compass and straightedge, construct the image of $\triangle ABC$ under the reflection in m. Write the steps you used for vertex A, and explain why your construction works.



ANSWER KEY

1. $(2, 7)$
2. (a) $\langle 7, -3 \rangle$ (b) $B'(8, -2)$
3. (a) $(-2, -7)$ (b) $(2, 7)$ (c) $(7, -2)$
4. (a) $(-4, -3)$ (b) $(3, -4)$ (c) $(4, 3)$
5. (a) 10 (b) $\dfrac{360^\circ}{10} = 36^\circ$
6. $A'(-3, 6)$, $B'(9, 0)$; $A'B' = \sqrt{12^2 + 6^2} = \sqrt{180} = 6\sqrt{5}$ (which is $\dfrac{3}{2}$ of $AB = 4\sqrt{5}$)
7. (a) Reflect the barn to $A'(0, -18)$. Line $A'B$ has slope $\dfrac{24}{32} = \dfrac{3}{4}$: $y = \dfrac{3}{4}x - 18$, which meets the creek at $(24, 0)$. (b) $A'B = \sqrt{32^2 + 24^2} = 40$ m (check: $30 + 10 = 40$)
8. (a) $(2, 5) \to (-4, 5) \to (10, 5)$ (b) $\langle 8, 0 \rangle$, twice the distance 4 between the lines
9. (a) 80° (b) 180°
10. (a) Relative to C, A is $\langle 6, 6 \rangle$ and A' is $\langle 3, 3 \rangle$, so $k = \dfrac{1}{2}$. (b) B is $\langle -6, 8 \rangle$ from C; half of that is $\langle -3, 4 \rangle$, so $B'(-1, 3)$.
11. 30 ft $= 360$ in, and $360 \div 24 = 15$ in high; 48 ft $= 576$ in, and $576 \div 24 = 24$ in long
12. No. A rigid motion must preserve distance, but this dilation doubles every length; a segment of length 3 becomes length 6. The dilation does keep the shape, because it preserves angle measure, but the image is larger, so it is not congruent to the original. Only dilations with $|k| = 1$ are rigid motions.
13. See the model proof.

    TWO-COLUMN PROOF

    Given: Line m is the perpendicular bisector of $\overline{BD}$; A and C are different points on m

    Prove: $\triangle ABC \cong \triangle ADC$

	STATEMENTS	REASONS
1	m is the perpendicular bisector of $\overline{BD}$; A and C are different points on m	Given
2	The reflection in m maps B to D	Definition of reflection
3	The reflection in m maps A to A and C to C	Definition of reflection (points on the mirror line are fixed)

	STATEMENTS	REASONS
4	$AB = AD$ and $CB = CD$	Reflections preserve distance
5	$\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$	Definition of congruent segments
6	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
7	$\triangle ABC \cong \triangle ADC$	SSS

∎

14. See the model proof.

    INDIRECT PROOF

    Given: A dilation with center O and scale factor $k \neq 0$ maps A to A' and B to B'; $A \neq B$

    Prove: $A' \neq B'$

    Assume the opposite: $A' = B'$. Then $A'B' = 0$.
    By the Dilation Theorem, $A'B' = |k| \cdot AB$. Since $A \neq B$, $AB > 0$, and since $k \neq 0$, $|k| > 0$, so $|k| \cdot AB > 0$. Then $0 = A'B' > 0$, which is impossible.
    The assumption leads to a contradiction, so it is false. Therefore $A' \neq B'$.

    ∎

15. Steps for A: (1) With the compass point on A, draw an arc crossing m at two points X and Y. (2) With the same width, draw arcs centered at X and at Y on the other side of m; label their crossing A'. Repeat for B and C and join A', B', C'. Why: $XA = XA'$ and $YA = YA'$, so X and Y both lie on the perpendicular bisector of $\overline{AA'}$ (Converse of the Perpendicular Bisector Theorem). Two points determine a line, so m is the perpendicular bisector of $\overline{AA'}$, which is the definition of the reflection of A in m.

7.1 Translations and Vectors

OBJECTIVES

- Describe a transformation as a function of the plane, using the words preimage and image and prime notation.
- Translate points and figures by a vector $\langle a, b \rangle$ with the rule $(x, y) \to (x + a,\ y + b)$.
- Find the vector of a translation from a point and its image.
- Explain why translations and other rigid motions preserve distance and angle measure.

WATCH

Congruence Transformations (1 of 3: Translation)

Eddie Woo · 5:47

https://www.youtube.com/watch?v=LWc8b_A6BPQ

> **WHEN TO WATCH**
> Before the Core ideas: what a translation is.

Note: Eddie Woo teaches in Australia: he says 'number plane' for coordinate plane. Vectors ⟨a, b⟩ are taught from the Core ideas.

BACKUP VIDEO

Transformations on the Number Plane (1 of 2: Translation)

Eddie Woo · 4:48

https://www.youtube.com/watch?v=yoliJSZVDnA

Before Example 1: translating points on the coordinate plane.

KEY TERMS

transformation

A rule that matches every point of the plane with exactly one point, called its image. It is a function whose inputs and outputs are points.

preimage and image

The preimage is the original point or figure; the image is the point or figure it is sent to. The image of A is written A' (read "A prime").

translation

A transformation that moves every point the same distance in the same direction.

vector

A quantity with a length and a direction, written $\langle a, b\rangle$: a move of a units horizontally and b units vertically. It is drawn as an arrow.

rigid motion (isometry)

A transformation that preserves distance: for every two points A and B, $A'B' = AB$.

CORE IDEAS

A **transformation** is a rule that matches every point of the plane with exactly one point, its **image**. The original point is the **preimage**. A transformation is a function, just like $f(x) = 2x + 1$ in algebra, except that the inputs and outputs are points. We name images with a prime: the image of A is A', and the image of $\triangle ABC$ is $\triangle A'B'C'$.

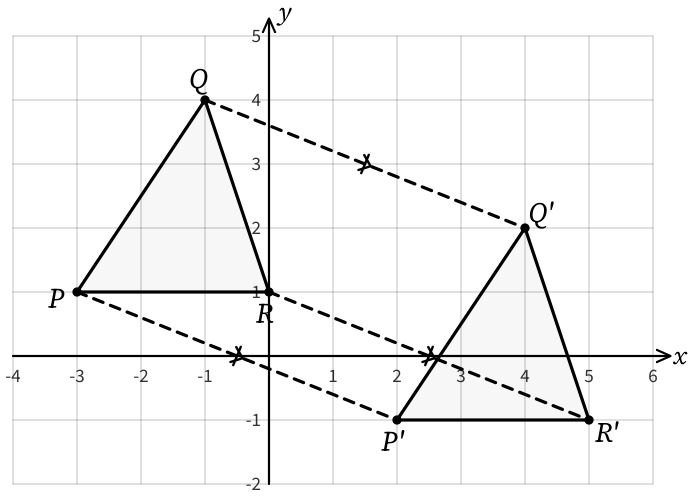
A **translation** slides every point the same distance in the same direction, the way you slide a book across a table without turning it. The distance and direction are recorded by a **vector** $\langle a, b\rangle$: move a units horizontally (right if $a > 0$, left if $a < 0$) and b units vertically (up if $b > 0$, down if $b < 0$). The arrow from $(0, 0)$ to (a, b) is the hypotenuse of a right triangle with legs $|a|$ and $|b|$, so by the Pythagorean theorem every point moves the distance $\sqrt{a^2 + b^2}$.

On a coordinate plane, moving a units right adds a to the x-coordinate and moving b units up adds b to the y-coordinate. That is WHY the translation by $\langle a, b\rangle$ has the rule $(x, y) \to (x + a,\ y + b)$. To find the vector from a point and its image, subtract: image minus preimage. If $A(2, 5)$ goes to $A'(-1, 9)$, the vector is $\langle -1 - 2,\ 9 - 5\rangle = \langle -3, 4\rangle$, and every other point moves by that same vector.

A **rigid motion** (or **isometry**, from Greek words meaning "equal measure") is a transformation that preserves distance: $A'B' = AB$ for every two points. A translation is a rigid motion. Why? The segments $\overline{AA'}$ and $\overline{BB'}$ are parallel and equally long, so $ABB'A'$ is a parallelogram, and opposite sides of a parallelogram are congruent. That is Euclid's I.33 and I.34, and the full proof is below.

A transformation that preserves distance automatically preserves angle measure. The reason is SSS: if the three sides of a triangle keep their lengths, the image triangle is congruent to the original, so its angles are congruent too. So a rigid motion moves a figure without stretching, shrinking or bending it, and the image is congruent to the preimage. This makes precise the phrase "same size and shape" from Lesson 4.2.

A translation also keeps every segment parallel to its image (in the parallelogram $ABB'A'$, the opposite sides $\overline{AB}$ and $\overline{A'B'}$ are parallel), and it never flips a figure over: if $A \to B \to C$ runs counterclockwise, so does $A' \to B' \to C'$. In the next lessons you will meet rigid motions that flip figures (reflections) and turn them (rotations).

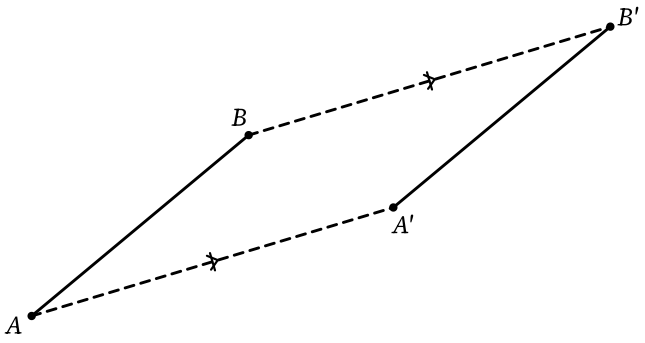


Every point moves by the same vector $\langle 5, -2 \rangle$.

POSTULATES AND THEOREMS

THEOREM **Translations Preserve Distance** Euclid I.33

If a translation maps A to A' and B to B', then $A'B' = AB$. So every translation is a rigid motion.



PARAGRAPH PROOF

Given: A translation by the vector $\langle a, b\rangle$ maps A to A' and B to B'; A, B and A' are not collinear

Prove: $A'B' = AB$

By the definition of a translation, every point moves by the same vector $\langle a, b\rangle$. So $AA' = BB' = \sqrt{a^2 + b^2}$, and the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ both have slope $\dfrac{b}{a}$ (or are both vertical when $a = 0$). They are different lines, because B is not on line AA'. Different lines with equal slopes, or two vertical lines, are parallel (Lesson 3.5), so $\overline{AA'} \parallel \overline{BB'}$.

Both moves point the same way, so $ABB'A'$ is a quadrilateral in which one pair of opposite sides, $\overline{AA'}$ and $\overline{BB'}$, is both congruent and parallel. A quadrilateral with one pair of opposite sides both congruent and parallel is a parallelogram (Lesson 6.3; Euclid I.33).

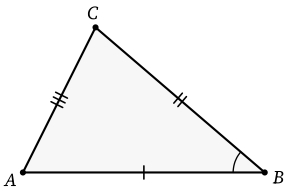
Opposite sides of a parallelogram are congruent (Lesson 6.2; Euclid I.34). Therefore $\overline{A'B'} \cong \overline{AB}$, that is, $A'B' = AB$.

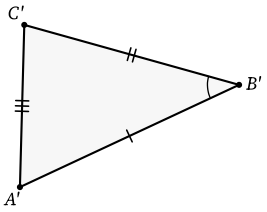
∎

If A, B and A' are collinear, the whole line slides along itself: A and B each move the same distance in the same direction along it, so the distance between them does not change. You can also check the theorem with the distance formula: adding a to both x-coordinates and b to both y-coordinates does not change the differences $x_2 - x_1$ and $y_2 - y_1$.

THEOREM **Rigid Motions Preserve Angle Measure** Euclid I.8

If a transformation preserves distance, then it maps every angle to a congruent angle.





TWO-COLUMN PROOF

Given: A transformation preserves distance and maps A, B, C to A', B', C'; A, B, C are not collinear

Prove: $\angle A'B'C' \cong \angle ABC$

	STATEMENTS	REASONS
1	The transformation preserves distance and maps A, B, C to A', B', C'; A, B, C are not	Given

	STATEMENTS	REASONS
	collinear	
2	$A'B' = AB$, $B'C' = BC$, $A'C' = AC$	Definition of preserving distance
3	A', B', C' are not collinear	Triangle Inequality Theorem: each side of $\triangle ABC$ is less than the sum of the other two, so by substitution no one of $A'B'$, $B'C'$, $A'C'$ equals the sum of the other two, as it would for collinear points (Segment Addition Postulate)
4	$\overline{A'B'} \cong \overline{AB}$, $\overline{B'C'} \cong \overline{BC}$, $\overline{A'C'} \cong \overline{AC}$	Definition of congruent segments
5	$\triangle A'B'C' \cong \triangle ABC$	SSS
6	$\angle A'B'C' \cong \angle ABC$	CPCTC

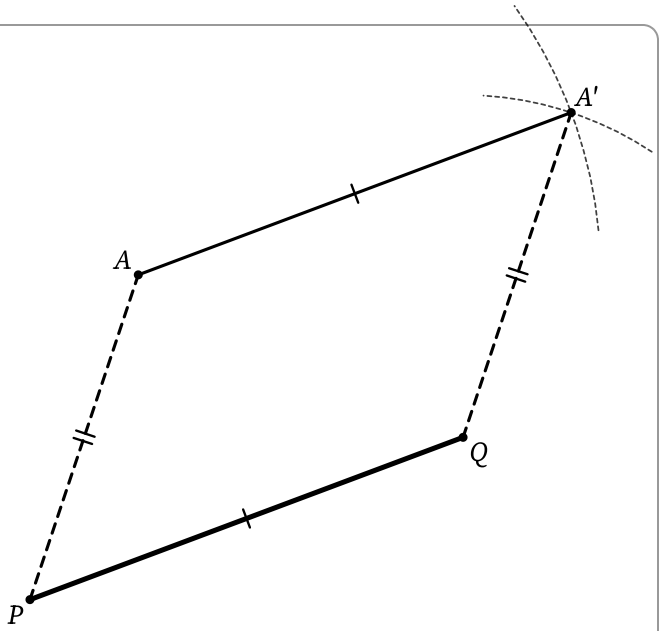
∎

The same congruence also gives $\angle A' \cong \angle A$ and $\angle C' \cong \angle C$. This is Euclid's I.8 at work: triangles whose three sides are equal have equal angles.

CONSTRUCTIONS

CONSTRUCTION **Translate a point by a given vector**

1. You are given the vector as an arrow from P to Q, and a point A to translate.
2. Set the compass to the length PQ. With the point on A, draw an arc on the side toward which the arrow points.
3. Set the compass to the length PA. With the point on Q, draw an arc that crosses the first arc



on the opposite side of line AQ from P. Label the crossing A'.

4. Draw $\overline{AA'}$. The point A' is the image of A under the translation.

Why it works. By construction $AA' = PQ$ and $QA' = PA$, so quadrilateral $PQA'A$ has both pairs of opposite sides congruent and is a parallelogram (Lesson 6.3). (Because P and A' are on opposite sides of $\overleftrightarrow{AQ}$, the diagonals $\overline{PA'}$ and $\overline{AQ}$ cross, so $PQA'A$ is a true quadrilateral and not a crossed figure.) Therefore $\overline{AA'}$ is parallel to $\overline{PQ}$ and just as long, pointing the same way: A has moved by exactly the vector from P to Q.

KEY FORMULAS

$$(x, y) \to (x + a,\ y + b)$$

$$\text{vector from } A(x, y) \text{ to } A'(x', y') = \langle x' - x,\ y' - y \rangle$$

$$\text{length of } \langle a, b \rangle = \sqrt{a^2 + b^2}$$

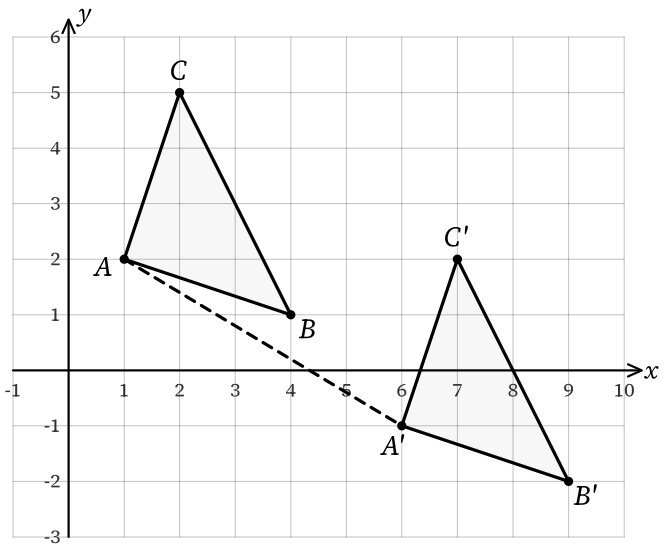
From history. Euclid moved figures in only a few proofs and did not study motions for their own sake. In 1872 the German mathematician Felix Klein proposed describing each kind of geometry by its transformations and by the properties those transformations leave unchanged. For the geometry of this course, the transformations are the rigid motions, and the unchanged properties are lengths and angle measures.

Euclid connection. Euclid I.33 states that the straight lines joining the ends of equal and parallel straight lines in the same directions are themselves equal and parallel; in modern words, a translation keeps every segment the same length and parallel to its image. Euclid proves I.4 by "applying" one triangle to another, moving it without changing it, which is the idea behind rigid motions (Lesson 7.4).

Practice habit. Find a vector as image minus preimage, then check it by moving one point on the grid: count across first, then up or down. Recite until automatic: "A rigid motion preserves distance and angle measure."

WORKED EXAMPLES

Example 1. Translate $\triangle ABC$ with $A(1,2)$, $B(4,1)$ and $C(2,5)$ by the vector $\langle 5,-3\rangle$. Give the coordinates of the image and check that $A'B' = AB$.

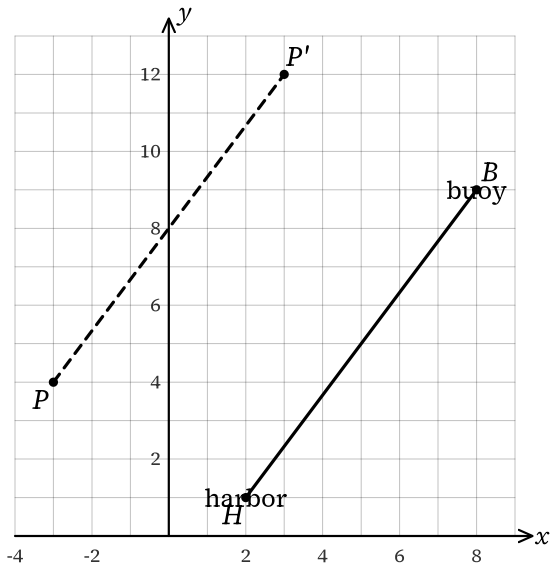


Show solution

1. The rule is $(x, y) \to (x+5,\ y-3)$: add 5 to each x-coordinate and subtract 3 from each y-coordinate.
2. $A(1,2) \to A'(6,-1)$, $B(4,1) \to B'(9,-2)$, $C(2,5) \to C'(7,2)$.
3. Check with the distance formula: $AB = \sqrt{(4-1)^2 + (1-2)^2} = \sqrt{10}$ and $A'B' = \sqrt{(9-6)^2 + (-2+1)^2} = \sqrt{10}$. The lengths agree, as the theorem promises.

Answer: $A'(6,-1)$, $B'(9,-2)$, $C'(7,2)$; $A'B' = AB = \sqrt{10}$

Example 2. On a harbor map whose grid units are miles, a fishing boat sails straight from the harbor $H(2, 1)$ to a buoy $B(8, 9)$. (a) Write the vector of the trip. (b) How far does the boat sail? (c) A second boat starts at $P(-3, 4)$ and makes exactly the same move. Where does it end?



Show solution

1. The vector is image minus preimage: $\langle 8 - 2,\ 9 - 1\rangle = \langle 6, 8\rangle$.
2. The length of the vector is $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$, so the boat sails 10 miles.
3. "The same move" is the same translation, so add the vector to P: $P'(-3 + 6,\ 4 + 8) = P'(3, 12)$. The second boat also sails 10 miles, because a translation moves every point the same distance.

Answer: (a) $\langle 6, 8\rangle$ (b) 10 miles (c) $P'(3, 12)$

PRACTICE

1. Easy Find the image of $P(3, -2)$ under the translation by the vector $\langle -4, 6\rangle$.

   Show answer

   $P'(-1, 4)$

2. Easy Write the vector and the coordinate rule for the translation that moves every point 5 units left and 2 units up.

   Show answer

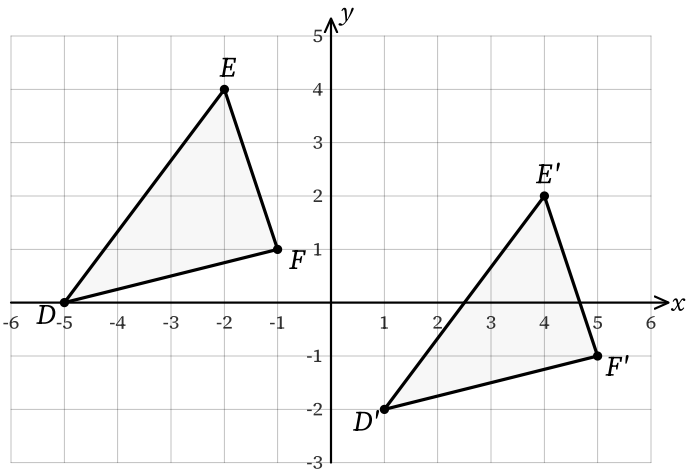
   The vector is $\langle -5, 2\rangle$, so the rule is $(x, y) \to (x - 5,\ y + 2)$.

3. **Easy** A translation maps $A(2, 7)$ to $A'(-1, 3)$. (a) Write the vector of the translation. (b) Find the image of the origin $O(0, 0)$.

Show answer

(a) $\langle -3, -4 \rangle$ (b) $O'(-3, -4)$

4. **Easy** In the figure, $\triangle D'E'F'$ is the image of $\triangle DEF$ under a translation. (a) Write the vector of the translation. (b) Find $D'E'$.



Show answer

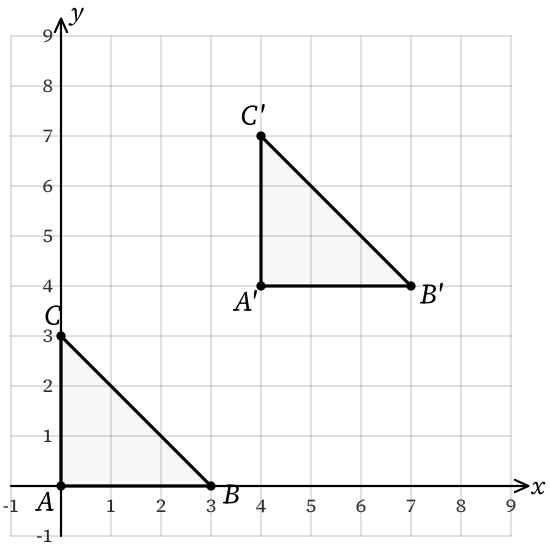
(a) $D(-5, 0) \to D'(1, -2)$, so the vector is $\langle 6, -2 \rangle$. (b) $D'E' = DE = \sqrt{3^2 + 4^2} = 5$

5. **Medium** Segment $\overline{AB}$ has endpoints $A(-2, 1)$ and $B(4, 9)$. Translate it by $\langle 3, -5 \rangle$. Give A' and B', and find the length $A'B'$ with the distance formula.

Show answer

$A'(1, -4)$, $B'(7, 4)$; $A'B' = \sqrt{6^2 + 8^2} = 10$, the same as $AB = \sqrt{6^2 + 8^2} = 10$

6. **Medium** Ruth designs a quilt on grid paper where each unit is 1 inch. One triangular patch is $\triangle ABC$ with $A(0,0)$, $B(3,0)$ and $C(0,3)$. In the figure, she has translated it by $\langle 4,4\rangle$ to make the copy $\triangle A'B'C'$. She will translate that copy by $\langle 4,4\rangle$ again to make a second copy. (a) Where is the image of B in the second copy? (b) What single translation takes the original patch straight to the second copy?



Show answer

(a) $(3,0) \to (7,4) \to (11,8)$ (b) $\langle 8,8\rangle$

7. **Medium** A figure is translated by $\langle 2,-7\rangle$ and then the image is translated by $\langle -5,3\rangle$. (a) What single vector does the same job? (b) How far does each point move in all?

Show answer

(a) $\langle 2+(-5),\ -7+3\rangle = \langle -3,-4\rangle$ (b) $\sqrt{9+16} = 5$ units

8. **Medium** A sailboat with its sails lowered drifts with a steady current that carries it 3 km east and 4 km north every hour. On a map with axes in kilometers it starts at $(-2,5)$. (a) Where is it after 2 hours? (b) How far has it drifted?

Show answer

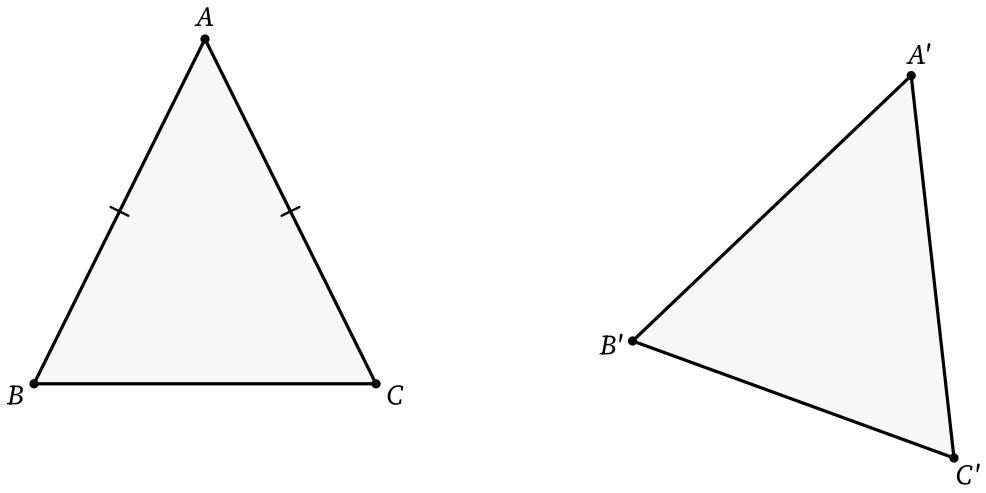
(a) In 2 hours the vector is $\langle 6,8\rangle$, so it is at $(4,13)$. (b) $\sqrt{36+64} = 10$ km

9. **Medium** Grace says the transformation $(x, y) \to (x + 3,\ 2y)$ is a translation because it adds 3 to x. Is it a translation? Is it even a rigid motion? Explain why, using the points $O(0, 0)$ and $P(0, 1)$.

Show answer

No and no. $O \to O'(3, 0)$ and $P \to P'(3, 2)$, so $OP = 1$ but $O'P' = 2$. The rule does not preserve distance, so it is not a rigid motion; every translation is a rigid motion, so it cannot be a translation. (It stretches figures vertically.)

10. **Hard** Given: $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$, and a rigid motion maps $\triangle ABC$ to $\triangle A'B'C'$, as in the figure. Prove: $\triangle A'B'C'$ is isosceles.



Show answer

See the model proof.

TWO-COLUMN PROOF

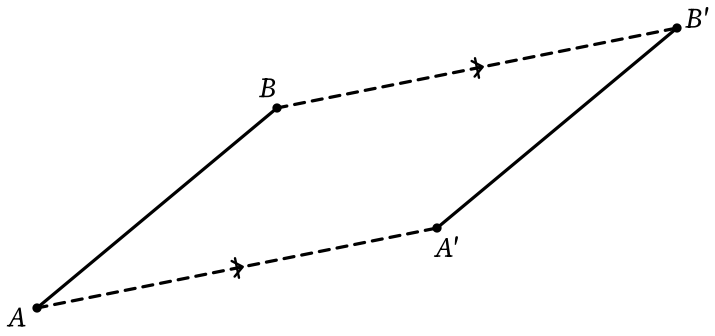
Given: $\overline{AB} \cong \overline{AC}$; A rigid motion maps $\triangle ABC$ to $\triangle A'B'C'$

Prove: $\triangle A'B'C'$ is isosceles

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	A rigid motion maps $\triangle ABC$ to $\triangle A'B'C'$	Given
3	$A'B' = AB$ and $A'C' = AC$	Definition of rigid motion (preserves distance)
4	$AB = AC$	Definition of congruent segments
5	$A'B' = A'C'$	Substitution Property of Equality
6	$\overline{A'B'} \cong \overline{A'C'}$	Definition of congruent segments
7	$\triangle A'B'C'$ is isosceles	Definition of isosceles triangle

∎

11. **Hard** Given: a translation maps A to A' and B to B', and A, B, A' are not collinear, as in the figure. Prove with a flow proof: $\overline{A'B'} \parallel \overline{AB}$.



Show answer

See the model proof.

FLOW PROOF

Given: A translation maps A to A' and B to B'; A, B, A' are not collinear

Prove: $\overline{A'B'} \parallel \overline{AB}$

(1) A translation maps A to A' and B to B'; A, B, A' are not collinear
Given

(2) $AA' = BB'$
Definition of translation (every point moves the same distance)
from (1)

(3) $\overline{AA'} \parallel \overline{BB'}$
Definition of translation (every point moves in the same direction, so the lines have equal slopes); they are different lines because B is not on line AA'
from (1)

(4) $ABB'A'$ is a parallelogram
One pair of opposite sides both congruent and parallel (Lesson 6.3; Euclid I.33)
from (2), (3)

(5) $\overline{A'B'} \parallel \overline{AB}$
Definition of parallelogram
from (4)

∎

12. Hard Henry is redrawing the plan of a garden fence on a grid whose units are meters. The whole fence is moved by one translation. The corner stake at $P(2, -1)$ moves to $P'(7, 11)$. (a) Write the vector. (b) How far does every stake move? (c) Another stake ends up at $(0, 0)$. Where did it start?

Show answer

(a) $\langle 5, 12 \rangle$ (b) $\sqrt{25 + 144} = 13$ m (c) Undo the move: $(0 - 5,\ 0 - 12) = (-5, -12)$

13. **Challenge** Given: the translation $(x, y) \to (x + a,\ y + b)$ maps $A(x_1, y_1)$ to A', $B(x_2, y_2)$ to B', and M to M', where M is the midpoint of $\overline{AB}$. Prove: M' is the midpoint of $\overline{A'B'}$.

Show answer

See the model proof.

TWO-COLUMN PROOF

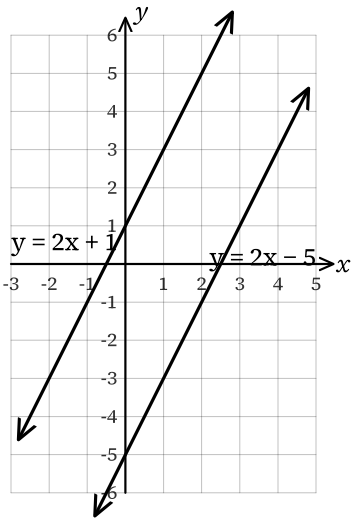
Given: The translation $(x, y) \to (x + a,\ y + b)$ maps $A(x_1, y_1)$ to A', $B(x_2, y_2)$ to B', and M to M'; M is the midpoint of $\overline{AB}$

Prove: M' is the midpoint of $\overline{A'B'}$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$; the translation maps A, B, M to A', B', M'	Given
2	$M = \left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right)$	Midpoint Formula
3	$M' = \left(\frac{x_1 + x_2}{2} + a,\ \frac{y_1 + y_2}{2} + b\right)$	Rule of the translation
4	$A' = (x_1 + a,\ y_1 + b)$ and $B' = (x_2 + a,\ y_2 + b)$	Rule of the translation
5	The midpoint of $\overline{A'B'}$ is $\left(\frac{(x_1 + a) + (x_2 + a)}{2},\ \frac{(y_1 + b) + (y_2 + b)}{2}\right)$	Midpoint Formula
6	The midpoint of $\overline{A'B'}$ is $\left(\frac{x_1 + x_2}{2} + a,\ \frac{y_1 + y_2}{2} + b\right)$	Combine like terms; $\frac{2a}{2} = a$ and $\frac{2b}{2} = b$ (Distributive Property)
7	M' is the midpoint of $\overline{A'B'}$	Substitution Property of Equality (rows 3 and 6 give the same point)

∎

14. **Challenge** The figure shows the parallel lines $y = 2x + 1$ and $y = 2x - 5$. (a) Does the translation by $\langle 3, 0 \rangle$ map the first line onto the second? (b) Find the vector $\langle a, b \rangle$, perpendicular to the lines, of a translation that maps the first line onto the second, and give its length in simplest radical form.



Show answer

(a) Yes: a point $(t,\ 2t + 1)$ goes to $(t + 3,\ 2t + 1)$, and $2(t + 3) - 5 = 2t + 1$, so the image is on the second line. (b) A perpendicular has slope $-\dfrac{1}{2}$, direction $\langle 2, -1 \rangle$. From $(0, 1)$ go to $(2s,\ 1 - s)$ and require $1 - s = 2(2s) - 5$, so $s = \dfrac{6}{5}$. The vector is $\left\langle \dfrac{12}{5}, -\dfrac{6}{5} \right\rangle$, of length $\dfrac{6}{5}\sqrt{5} = \dfrac{6\sqrt{5}}{5}$ (about 2.68 units). A translation keeps slope, so the whole first line lands on the second.

COMMON MISTAKES

- Subtracting in the wrong order. The vector from A to A' is image minus preimage, $\langle x' - x,\ y' - y \rangle$; check by adding the vector back to A and seeing that you land on A'.
- Adding the first number of $\langle a, b \rangle$ to the y-coordinate. The first number is always the horizontal move and goes with x.
- Confusing a vector with a point. The vector $\langle 3, 4 \rangle$ is a move (3 right, 4 up) that can start anywhere; the point $(3, 4)$ is one location.
- Assuming any rule that adds numbers is a translation. A rule such as $(x, y) \to (x + 3,\ 2y)$ stretches figures; test a rule by checking whether distances stay the

same.

> **Make it a game · Tile Translations.** Use floor tiles with one marked as the origin. Parent gives a point and a vector, such as (2, -3) translated by <4, 1>. Compute the image coordinates first, then step onto the tile to check. Use prime notation each time. Translate a triangle's three corners, and explain why a translation preserves distance.

7.2 Reflections and Symmetry

OBJECTIVES

- Reflect points and figures in a line, including the axes and the line $y = x$, and explain why the mirror line is the perpendicular bisector of each segment joining a point to its image.
- Prove that reflections preserve distance, and construct a reflection with compass and straightedge.
- Identify lines of symmetry in letters, polygons and church windows.
- Use the reflection trick to find the shortest path from one point to a line and on to another point.

WATCH

Congruence Transformations (3 of 3: Reflection)

Eddie Woo · 7:01

https://www.youtube.com/watch?v=1u302YEJbhk

> **WHEN TO WATCH**
> Before Example 1: reflection in a line.

Note: Heron's shortest-path trick is taught from the lesson.

BACKUP VIDEO

Reflectional Symmetry

Eddie Woo · 11:00

https://www.youtube.com/watch?v=LzCl9bDQWCc

Before the symmetry examples: lines of symmetry.

KEY TERMS

reflection

The reflection in line m maps each point P on m to itself and each point P not on m to the point P' for which m is the perpendicular bisector of $\overline{PP'}$.

line of reflection (mirror line)

The line m in which a figure is reflected.

line symmetry

A figure has line symmetry if a reflection in some line maps the figure onto itself; that line is a line of symmetry.

orientation

The direction, clockwise or counterclockwise, in which a figure's vertices are listed in order. A reflection reverses orientation.

CORE IDEAS

Stand in front of a flat mirror and your image appears straight across from you, exactly as far behind the glass as you are in front of it. That is the definition of a **reflection** in a line m: for each point P not on m, the image P' is the point for which m is the perpendicular bisector of $\overline{PP'}$. So $\overline{PP'} \perp m$, and m cuts $\overline{PP'}$ into two equal halves. Points on m do not move at all.

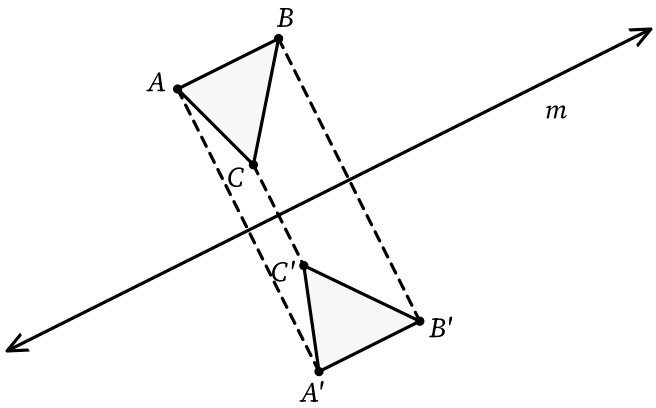
To reflect a point on a grid, measure its distance to the mirror along a perpendicular and go the same distance on the other side. In the x-axis, (x, y) stays at the same x and its height y becomes $-y$, so the rule is $(x, y) \to (x, -y)$. In the y-axis the rule is $(x, y) \to (-x, y)$. In the line $y = x$ the rule is $(x, y) \to (y, x)$. WHY: the midpoint of the segment from (x, y) to (y, x) is $\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$, which is on $y = x$, and the segment has slope $\frac{x-y}{y-x} = -1$, which is perpendicular to the slope 1 of the mirror. So $y = x$ is the perpendicular bisector, exactly as the definition requires.

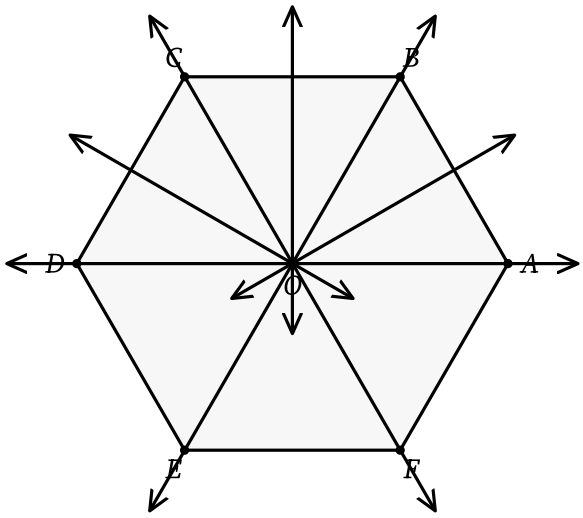
Every point on the mirror line is equidistant from P and P' (the Mirror Line Theorem below, which is the Perpendicular Bisector Theorem of Lesson 5.1 seen through a mirror). From it we prove that reflections preserve distance, so a reflection is a rigid motion: the image of a figure is congruent to it.

A reflection does change one thing: **orientation.** If you read the vertices $A \to B \to C$ counterclockwise, the image vertices $A' \to B' \to C'$ run clockwise, just as your right hand looks like a left hand in a mirror. Translations keep orientation; reflections reverse it.

A figure has **line symmetry** if a reflection maps it onto itself. A regular polygon with n sides has n lines of symmetry, all through the center. If n is odd, each one passes through a vertex and the midpoint of the opposite side. If n is even, half of them pass through two opposite vertices and half through the midpoints of two opposite sides. Builders of Gothic churches drew their windows symmetric about a vertical center line, and a rose window often has many lines of symmetry.

The **reflection trick** solves shortest-path puzzles. To go from A to a line and then to B (both on the same side), reflect A to A'. Every path $A \to X \to B$ with X on the line has the same length as $A' \to X \to B$, and that is shortest when it is straight. So aim straight from A' to B, and the point where you cross the line is the best one. Theorem 3 proves it.

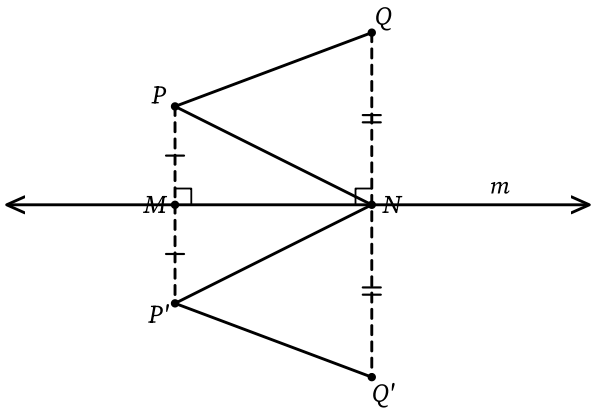




A regular hexagon has 6 lines of symmetry.

THEOREM **Mirror Line Theorem**

If P' is the reflection of P in line m, $\overline{PP'}$ meets m at M, and N is any other point of m, then $NP = NP'$ and $\angle PNM \cong \angle P'NM$.



TWO-COLUMN PROOF

Given: P' is the reflection of P in line m; $\overline{PP'}$ meets m at M; N is on m, $N \neq M$

Prove: $NP = NP'$ and $\angle PNM \cong \angle P'NM$

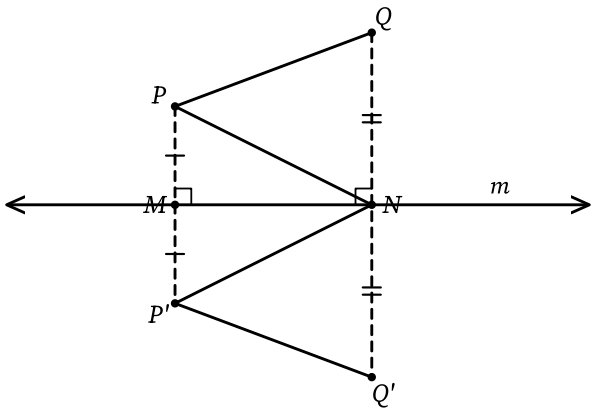
	STATEMENTS	REASONS
1	P' is the reflection of P in m; $\overline{PP'}$ meets m at M; N is on m, $N \neq M$	Given
2	m is the perpendicular bisector of $\overline{PP'}$	Definition of reflection
3	$\overline{PM} \cong \overline{P'M}$	Definition of bisector (M is the midpoint of $\overline{PP'}$)
4	$\angle PMN$ and $\angle P'MN$ are right angles	Definition of perpendicular lines
5	$\angle PMN \cong \angle P'MN$	All right angles are congruent
6	$\overline{NM} \cong \overline{NM}$	Reflexive Property of Congruence
7	$\triangle PMN \cong \triangle P'MN$	SAS
8	$\overline{NP} \cong \overline{NP'}$ and $\angle PNM \cong \angle P'NM$	CPCTC
9	$NP = NP'$ and $\angle PNM \cong \angle P'NM$	Definition of congruent segments

∎

The first half is the Perpendicular Bisector Theorem of Lesson 5.1. The second half says the mirror line makes equal angles with $\overline{NP}$ and $\overline{NP'}$.

THEOREM **Reflections Preserve Distance**

If the reflection in line m maps P to P' and Q to Q', then $P'Q' = PQ$. So every reflection is a rigid motion.



TWO-COLUMN PROOF

Given: The reflection in m maps P to P' and Q to Q'; P and Q are on the same side of m; $\overline{PP'}$ meets m at M, $\overline{QQ'}$ meets m at N, and $M \neq N$

Prove: $P'Q' = PQ$

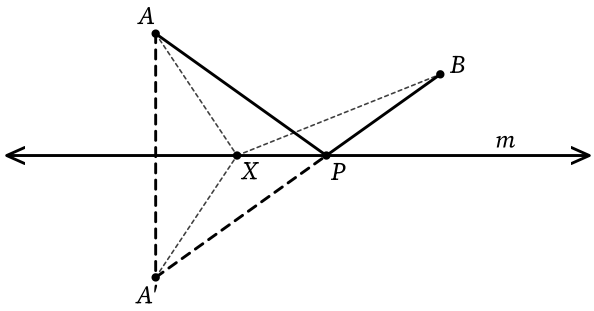
	STATEMENTS	REASONS
1	The reflection in m maps P to P' and Q to Q'; $\overline{PP'}$ meets m at M and $\overline{QQ'}$ meets m at N, $M \neq N$	Given
2	$\overline{PN} \cong \overline{P'N}$ and $\angle PNM \cong \angle P'NM$	Mirror Line Theorem (N is on m)
3	m is the perpendicular bisector of $\overline{QQ'}$, so $\overline{QN} \cong \overline{Q'N}$	Definition of reflection; definition of bisector
4	$\angle MNQ$ and $\angle MNQ'$ are right angles	Definition of perpendicular lines
5	$m\angle PNM + m\angle PNQ = 90°$ and $m\angle P'NM + m\angle P'NQ' = 90°$	Angle Addition Postulate (P is in the interior of $\angle MNQ$ and P' in the interior of $\angle MNQ'$)
6	$\angle PNQ \cong \angle P'NQ'$	Congruent Complements Theorem (rows 2 and 5)
7	$\triangle PNQ \cong \triangle P'NQ'$	SAS (rows 2, 6, 3)
8	$P'Q' = PQ$	CPCTC and definition of congruent segments

∎

This proof covers two points on the same side of m that are not on one perpendicular to m. If one point is on m, the Mirror Line Theorem gives the result at once (that point is its own image). The remaining cases, points on opposite sides or on one perpendicular, are proved the same way with small changes.

THEOREM **Heron's Shortest Path Theorem**

Let A and B be on the same side of line m, let A' be the reflection of A in m, and let $\overline{A'B}$ meet m at P. Then for every other point X of m, $AP + PB < AX + XB$.



PARAGRAPH PROOF

Given: A and B are on the same side of line m; A' is the reflection of A in m; $\overline{A'B}$ meets m at P; X is on m, $X \neq P$

Prove: $AP + PB < AX + XB$

The midpoint of $\overline{AA'}$ lies on m, so A' is on the opposite side of m from A, and therefore from B. So $\overline{A'B}$ does cross m, at the point P, and P is between A' and B.

Both P and X lie on the mirror line, so $AP = A'P$ and $AX = A'X$: by the Mirror Line Theorem for a point other than the midpoint of $\overline{AA'}$, and by the definition of reflection if the point is that midpoint. Then $AP + PB = A'P + PB = A'B$ by the Segment Addition Postulate.

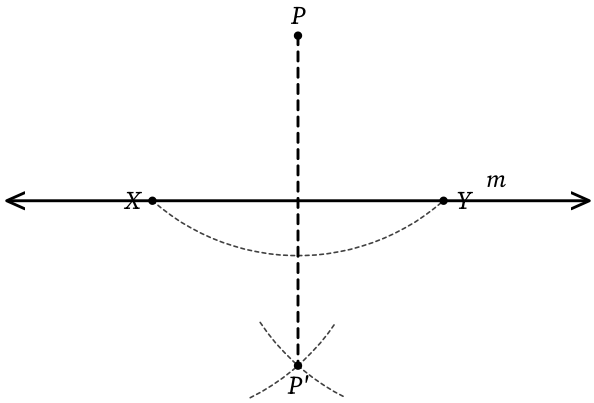
Line $A'B$ meets m only at P, because two different lines meet in at most one point. Since $X \neq P$, X is not on line $A'B$, so A', X and B form a triangle. By the Triangle Inequality Theorem (Euclid I.20), $A'X + XB > A'B$. Substituting, $AX + XB = A'X + XB > A'B = AP + PB$. So the path through P is shorter than the path through any other point of m.

∎

Practice 13 shows that the shortest path makes equal angles with the mirror line, the law of reflection for light.

CONSTRUCTION **Reflect a point in a line** Euclid I.12

1. Place the compass point on P and draw an arc that crosses line m at two points. Label them X and Y.
2. Keep the same compass width. Place the point on X and draw an arc on the other side of m.
3. Keep the same width. Place the point on Y and draw an arc that crosses the arc from step 2. Label the crossing P'.
4. P' is the reflection of P. If you draw $\overline{PP'}$, it is perpendicular to m.



Why it works. All four arcs have the same radius, so $XP = XP'$ and $YP = YP'$. By the Converse of the Perpendicular Bisector Theorem (Lesson 5.1), X and Y both lie on the perpendicular bisector of $\overline{PP'}$. Two points determine a line, so line $m = \overleftrightarrow{XY}$ is that perpendicular bisector, which is exactly the definition of the reflection of P in m. Euclid's I.12 starts the same way: a circle about the point cuts the line at two points.

KEY FORMULAS

$$\text{reflection in the } x\text{-axis: } (x, y) \to (x, -y)$$

$$\text{reflection in the } y\text{-axis: } (x, y) \to (-x, y)$$

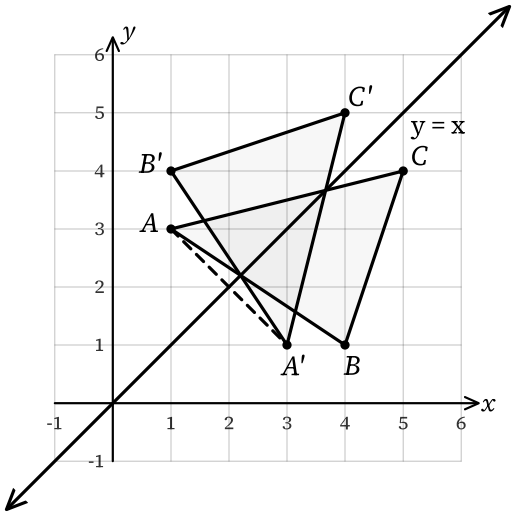
$$\text{reflection in the line } y = x : (x, y) \to (y, x)$$

From history. Heron of Alexandria, who lived in the first century AD, wrote a book on mirrors. In it he explained why light striking a flat mirror leaves at the same angle at which it arrives, by showing that this path is the shortest one from the object to the mirror and on to the eye. His argument is the reflection trick of this lesson.

Euclid connection. Euclid's I.12 (drawing a perpendicular to a line from a point not on it) and I.10 (bisecting a segment) are exactly the tools needed to locate a mirror image. Heron's shortest-path argument rests on Euclid's I.20: in any triangle two sides taken together are greater than the third.

Practice habit. Before reflecting, draw the mirror line clearly and measure each point's distance to it along a perpendicular; the image is the same distance on the other side. Recite the definition until automatic: "The mirror line is the perpendicular bisector of every segment joining a point to its image."

Example 1. Reflect $\triangle ABC$ with $A(1,3)$, $B(4,1)$ and $C(5,4)$ in the line $y = x$. Then check, for A and A', that the line $y = x$ is the perpendicular bisector of $\overline{AA'}$.

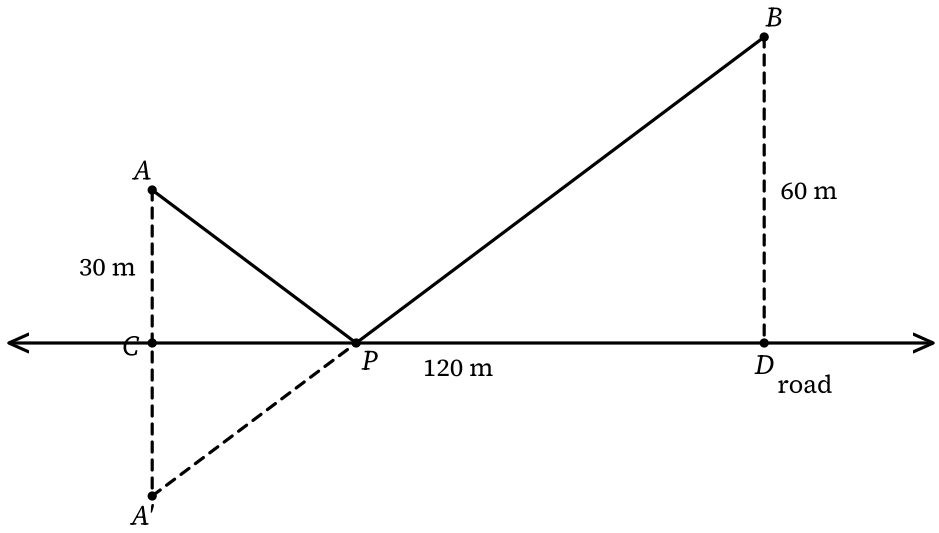


Show solution

1. The rule for reflection in $y = x$ is $(x, y) \to (y, x)$: swap the coordinates.
2. $A(1,3) \to A'(3,1)$, $B(4,1) \to B'(1,4)$, $C(5,4) \to C'(4,5)$.
3. Check: the midpoint of $\overline{AA'}$ is $\left(\dfrac{1+3}{2}, \dfrac{3+1}{2}\right) = (2,2)$, which is on $y = x$. The slope of $\overline{AA'}$ is $\dfrac{1-3}{3-1} = -1$, and $(-1)(1) = -1$, so $\overline{AA'}$ is perpendicular to the mirror.

Answer: $A'(3,1)$, $B'(1,4)$, $C'(4,5)$

Example 2. Two farmhouses stand on the same side of a straight road. On a plan in meters, the road is the x-axis and the houses are at $A(0, 30)$ and $B(120, 60)$. The two families will share one water pump P beside the road and lay pipe from A to P and from P to B. Where should the pump go to use the least pipe, and how much pipe is needed?



Show solution

1. Reflect A in the road: $A'(0, -30)$. By Heron's Shortest Path Theorem the best point is where $\overline{A'B}$ crosses the road.
2. The slope of $\overline{A'B}$ is $\dfrac{60 - (-30)}{120 - 0} = \dfrac{90}{120} = \dfrac{3}{4}$, so its equation is $y = \dfrac{3}{4}x - 30$. Setting $y = 0$ gives $x = 40$, so $P(40, 0)$.
3. The total pipe is $A'B = \sqrt{120^2 + 90^2} = \sqrt{22{,}500} = 150$ m.
4. Check: $AP = \sqrt{40^2 + 30^2} = 50$ and $PB = \sqrt{80^2 + 60^2} = 100$, and $50 + 100 = 150$.

Answer: At $P(40, 0)$, 40 m along the road from the point nearest house A; 150 m of pipe

PRACTICE

1. Easy Reflect the point $(4, -3)$ in (a) the x-axis, (b) the y-axis, (c) the line $y = x$.

   Show answer

   (a) $(4, 3)$ (b) $(-4, -3)$ (c) $(-3, 4)$

2. **Easy** How many lines of symmetry does each figure have? (a) a square (b) a rectangle that is not a square (c) an equilateral triangle (d) a regular hexagon

Show answer

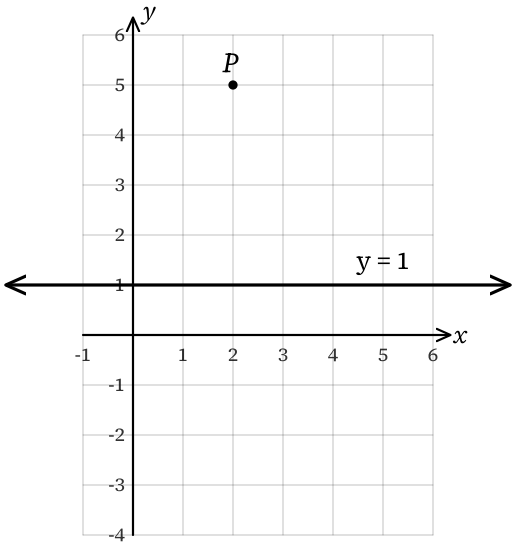
(a) 4 (b) 2 (c) 3 (d) 6

3. **Easy** Written as plain block capitals, which of the letters A, B, H, N, S have at least one line of symmetry?

Show answer

A (a vertical line), B (a horizontal line) and H (both). N and S have none.

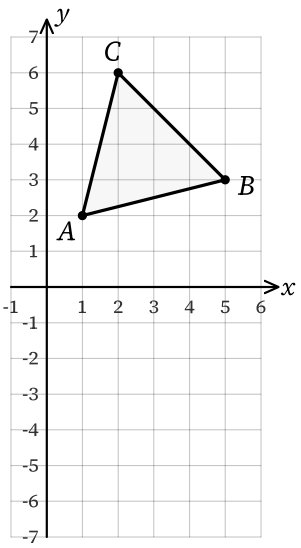
4. **Easy** In the figure, reflect $P(2, 5)$ in the line $y = 1$. Give the coordinates of P'.



Show answer

P is 4 units above the line, so P' is 4 units below it: $P'(2, -3)$

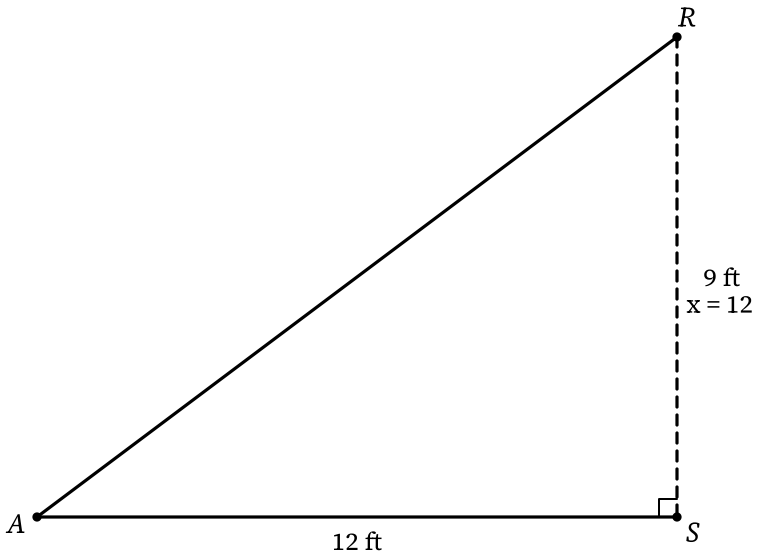
5. **Medium** In the figure, reflect $\triangle ABC$ with $A(1, 2)$, $B(5, 3)$ and $C(2, 6)$ in the x-axis. Give the image vertices.



Show answer

$A'(1, -2)$, $B'(5, -3)$, $C'(2, -6)$

6. **Medium** Henry is framing a barn roof. On his plan, in feet, the left rafter runs from $A(0, 0)$ to the ridge $R(12, 9)$, and the roof is symmetric about the vertical line $x = 12$ through the ridge, as in the figure. (a) Where does the right rafter end (the reflection of A)? (b) How long is each rafter?



Show answer

(a) $(24, 0)$ (b) $\sqrt{12^2 + 9^2} = 15$ ft

7. **Medium** A reflection maps $A(2, 1)$ to $A'(6, 5)$. Write an equation of the line of reflection.

Show answer

The mirror is the perpendicular bisector of $\overline{AA'}$. Midpoint $(4, 3)$; slope of $\overline{AA'}$ is 1, so the mirror has slope -1: $y = -x + 7$

8. **Medium** The stained-glass window of a small chapel is drawn on a grid in feet and is symmetric about its center line $x = 3$. A lead joint is at $(1.2, 7.5)$. Where is the matching joint on the other side?

Show answer

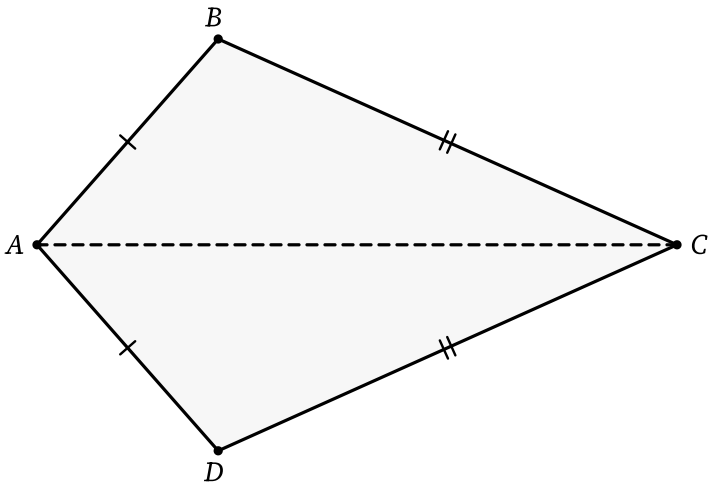
$(1.2, 7.5)$ is 1.8 ft left of the line, so its match is 1.8 ft right: $(4.8, 7.5)$

9. Medium Samuel reflects a point P (not on line m) in m and gets P'. Then he reflects P' in m. Explain why he gets P back.

Show answer

By the definition of reflection, m is the perpendicular bisector of $\overline{PP'}$. That is the same segment as $\overline{P'P}$, so the point whose segment with P' has m as its perpendicular bisector is P. Reflecting twice in the same line returns every point to where it started.

10. Hard Given: quadrilateral $ABCD$ (a kite) with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$, as in the figure. Prove with a paragraph proof: the reflection in line AC maps B to D.



Show answer

See the model proof. (So a kite has line symmetry about the diagonal $\overline{AC}$.)

PARAGRAPH PROOF

Given: $\overline{AB} \cong \overline{AD}$; $\overline{CB} \cong \overline{CD}$

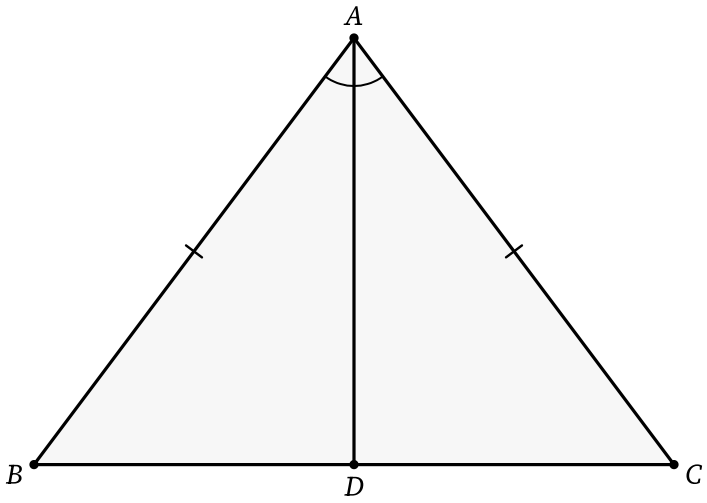
Prove: The reflection in $\overleftrightarrow{AC}$ maps B to D

Since $AB = AD$, point A is equidistant from B and D, so A lies on the perpendicular bisector of $\overline{BD}$ by the Converse of the Perpendicular Bisector Theorem (Lesson 5.1). In the same way, $CB = CD$, so C also lies on the perpendicular bisector of $\overline{BD}$.

A and C are different points, and through two points there is exactly one line, so $\overleftrightarrow{AC}$ is the perpendicular bisector of $\overline{BD}$. By the definition of reflection, the reflection in $\overleftrightarrow{AC}$ maps B to D.

∎

11. Hard Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$; $\overrightarrow{AD}$ bisects $\angle BAC$ and meets $\overline{BC}$ at D, as in the figure. Prove: the reflection in $\overleftrightarrow{AD}$ maps B to C.



Show answer

See the model proof. (So an isosceles triangle has line symmetry about the bisector of its vertex angle.)

TWO-COLUMN PROOF

Given: $\overline{AB} \cong \overline{AC}$; $\overrightarrow{AD}$ bisects $\angle BAC$; D is on $\overline{BC}$

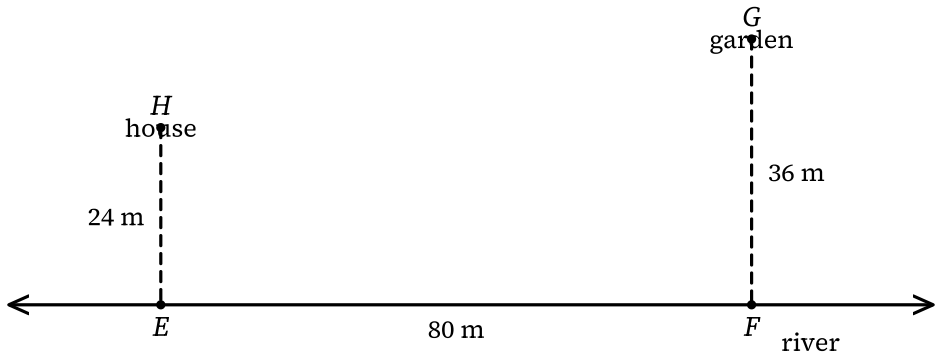
Prove: The reflection in $\overleftrightarrow{AD}$ maps B to C

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$; $\overrightarrow{AD}$ bisects $\angle BAC$; D is on $\overline{BC}$	Given
2	$\angle BAD \cong \angle CAD$	Definition of angle bisector
3	$\overline{AD} \cong \overline{AD}$	Reflexive Property of Congruence
4	$\triangle BAD \cong \triangle CAD$	SAS
5	$\overline{BD} \cong \overline{CD}$ and $\angle ADB \cong \angle ADC$	CPCTC
6	$\angle ADB$ and $\angle ADC$ form a linear pair	Definition of linear pair (D is between B and C)
7	$\overleftrightarrow{AD} \perp \overline{BC}$	If the angles of a linear pair are congruent, the lines are perpendicular (Lesson 3.4)

	STATEMENTS	REASONS
8	$\overleftrightarrow{AD}$ is the perpendicular bisector of $\overline{BC}$	Definition of perpendicular bisector (rows 5 and 7)
9	The reflection in $\overleftrightarrow{AD}$ maps B to C	Definition of reflection

∎

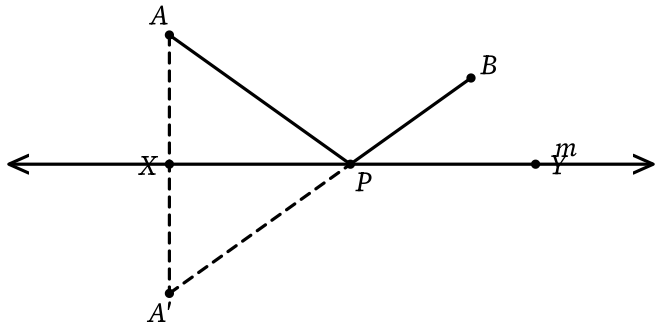
12. **Hard** Samuel walks from his house to a straight river to fill a bucket and then carries it to the garden. On a plan in meters, the river is the x-axis, the house is at $H(0, 24)$ and the garden is at $G(80, 36)$, as in the figure. (a) At what point on the river should he fill the bucket to walk the shortest total distance? (b) What is that distance?



Show answer

(a) Reflect H to $H'(0, -24)$. Line $H'G$ has slope $\dfrac{60}{80} = \dfrac{3}{4}$: $y = \dfrac{3}{4}x - 24$, which crosses the river at $(32, 0)$. (b) $H'G = \sqrt{80^2 + 60^2} = 100$ m (check: $40 + 60 = 100$)

13. **Hard** Given: A and B are on the same side of line m; A' is the reflection of A in m, and $\overline{AA'}$ meets m at X; $\overline{A'B}$ meets m at P, with $P \neq X$; Y is on m with P between X and Y, as in the figure. Prove with a paragraph proof: $\angle APX \cong \angle BPY$ (the angle of incidence equals the angle of reflection).



Show answer

See the model proof.

PARAGRAPH PROOF

Given: A and B are on the same side of line m; A' is the reflection of A in m; $\overline{AA'}$ meets m at X; $\overline{A'B}$ meets m at P, $P \neq X$; Y is on m with P between X and Y

Prove: $\angle APX \cong \angle BPY$

P is a point of the mirror line m other than X, where $\overline{AA'}$ meets m. By the Mirror Line Theorem, $\angle APX \cong \angle A'PX$.

P is between A' and B, so $\overrightarrow{PA'}$ and $\overrightarrow{PB}$ are opposite rays; P is between X and Y, so $\overrightarrow{PX}$ and $\overrightarrow{PY}$ are opposite rays. Therefore $\angle A'PX$ and $\angle BPY$ are vertical angles, and $\angle A'PX \cong \angle BPY$ by the Vertical Angles Theorem.

By the Transitive Property of Congruence, $\angle APX \cong \angle BPY$.

∎

14. **Challenge** Find the reflection of $P(6, 1)$ in the line $y = 2x - 1$. (Hint: find where the perpendicular from P meets the line.)

Show answer

The perpendicular through P has slope $-\frac{1}{2}$: $y = -\frac{1}{2}x + 4$. It meets $y = 2x - 1$ where $2x - 1 = -\frac{1}{2}x + 4$, so $x = 2$, $y = 3$: $M(2, 3)$. M is the midpoint of $\overline{PP'}$, so $P' = (2 \cdot 2 - 6,\ 2 \cdot 3 - 1) = (-2, 5)$.

COMMON MISTAKES

- Mixing up the axis rules. Reflecting in the x-axis changes the sign of y (the point flips up or down); reflecting in the y-axis changes the sign of x. Picture the flip before writing the rule.
- Changing signs when reflecting in $y = x$. That rule only swaps the coordinates: $(4, -3) \to (-3, 4)$.
- Measuring from the wrong line. To reflect in $y = 1$, count the distance to the line $y = 1$, not to the x-axis.
- Assuming every line through the center of a figure is a line of symmetry. A diagonal of a rectangle that is not a square is not a line of symmetry; test by asking whether the reflection lands every vertex on a vertex.

> **Make it a game · Mirror Letters.** In 60 seconds, write as many capital letters as you can that have line symmetry, and state how many lines each has. Then parent names a point and a mirror line, such as (3, 5) reflected in y = x, and you give the image. Explain why the mirror line is the perpendicular bisector of the segment joining the two.

7.3 Rotations and Rotational Symmetry

OBJECTIVES

- Rotate points and figures about a point through a given angle, with counterclockwise as the positive direction.
- Use and justify the rules for rotations of $90°$, $180°$ and $270°$ about the origin, and rotate about other centers.
- Prove that rotations preserve distance, and construct a rotation image with compass and straightedge.
- Find the order and angle of rotational symmetry of regular polygons, rose windows and wheels.

WATCH

Congruence Transformations (2 of 3: Rotation)

Eddie Woo · 5:44

https://www.youtube.com/watch?v=Gl-aQ-TBZZs

> **WHEN TO WATCH**
> Before Example 1: rotation about a point.

Note: Eddie Woo teaches in Australia: he says 'number plane' for coordinate plane.

BACKUP VIDEO

Transformations on the Number Plane (2 of 2: Rotation & Reflection)

Eddie Woo · 10:49

https://www.youtube.com/watch?v=rxMIedM844Y

Before the coordinate rules: rotations on the coordinate plane.

KEY TERMS

rotation

The rotation about point O through θ maps O to itself and each other point P to the point P' with $OP' = OP$ reached by turning $\overrightarrow{OP}$ through θ, counterclockwise when $\theta > 0$ and clockwise when $\theta < 0$. For $0^\circ < |\theta| \leq 180^\circ$ this means $m\angle POP' = |\theta|$; a turn through more than 180° is the same as the turn the other way through $360^\circ - |\theta|$ (so a 270° rotation is the same as a -90° rotation).

center of rotation

The fixed point O about which a rotation turns the plane.

angle of rotation

The angle θ through which every ray from the center is turned.

rotational symmetry

A figure has rotational symmetry if a rotation through an angle between 0° and 360° maps the figure onto itself.

order of rotational symmetry

The number of different positions in one full turn in which the figure fits onto itself. If the order is n, the smallest angle of rotational symmetry is $\frac{360^\circ}{n}$.

point symmetry

Rotational symmetry through 180°.

CORE IDEAS

Think of a wheel turning on its axle. The axle stays put; every other point stays the same distance from the axle and turns through the same angle. That is a **rotation**: about a center O through an angle θ, each point P goes to the point P' with $OP' = OP$ reached by turning through θ; for turns up to 180°, $m\angle POP' = \theta$. Mathematicians measure turning counterclockwise as positive, the same way directions are measured from east toward north, so a rotation of -90° is a quarter turn clockwise.

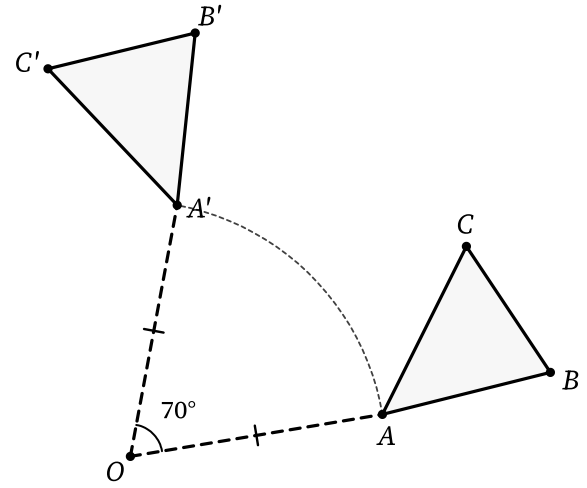
On a grid, a 90° rotation about the origin follows the rule $(x, y) \to (-y, x)$. WHY: the point $P(a, b)$ sits at the corner of a right triangle with legs $|a|$ and $|b|$. A quarter turn stands that triangle up, so the horizontal leg becomes vertical and the vertical leg becomes horizontal, pointing left. The image $(-b, a)$ is the same distance $\sqrt{a^2 + b^2}$ from O, and the slopes $\dfrac{b}{a}$ and $\dfrac{a}{-b}$ multiply to -1, so $\overline{OP} \perp \overline{OP'}$. Theorem 2 gives the full proof.

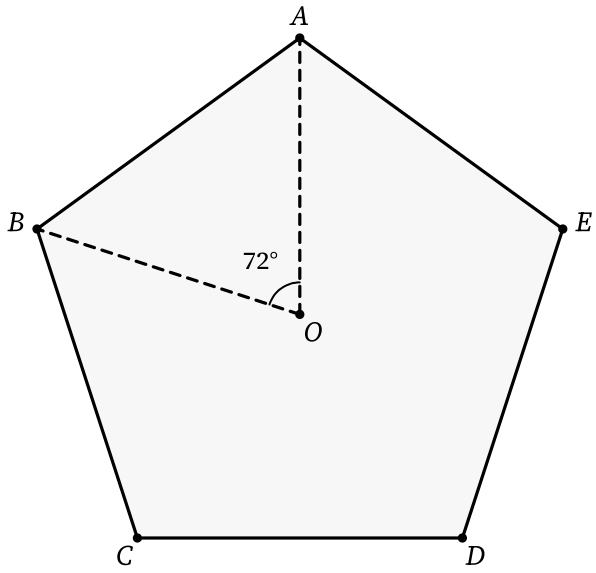
Turning 90° twice is turning 180°, and three times is 270°. Applying the rule again gives $(x, y) \to (-y, x) \to (-x, -y) \to (y, -x)$. So 180°: $(x, y) \to (-x, -y)$, and 270° (the same as -90°): $(x, y) \to (y, -x)$. Test any rule on $(1, 0)$: it should land on $(0, 1)$, $(-1, 0)$ or $(0, -1)$.

To rotate about a center C other than the origin, work relative to C: subtract C's coordinates from the point, apply the rule, then add C's coordinates back. This works because subtracting moves C to the origin by a translation, and adding moves it back.

Rotations preserve distance (Theorem 1, proved with SAS), so a rotation is a rigid motion and the image of a figure is congruent to it. Unlike a reflection, a rotation keeps orientation: counterclockwise vertices stay counterclockwise.

A figure has **rotational symmetry** if some rotation through an angle between 0° and 360° maps it onto itself. A regular polygon with n sides fits onto itself in n positions during one full turn, so its **order** is n and its smallest angle of symmetry is $\dfrac{360^\circ}{n}$; every multiple of that angle also works. A wagon wheel with 12 spokes has order 12 and angle 30°. Rotational symmetry through 180° is called point symmetry; the letters N, S and Z have it.

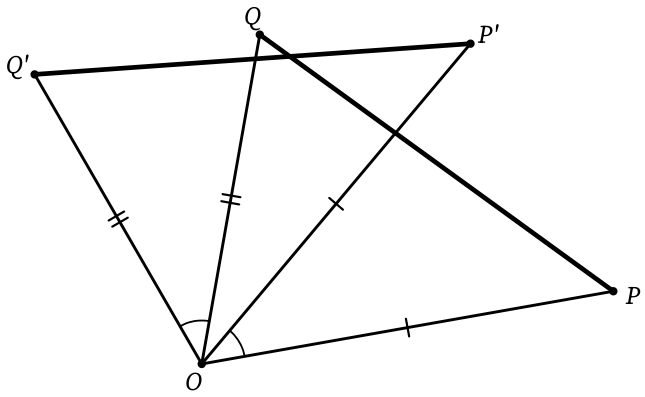




A regular pentagon fits onto itself after every turn of 72° about its center.

THEOREM **Rotations Preserve Distance**

If a rotation about O maps P to P' and Q to Q', then $P'Q' = PQ$. So every rotation is a rigid motion.



TWO-COLUMN PROOF

Given: The rotation about O through θ maps P to P' and Q to Q'; O, P, Q are not collinear, and $\overrightarrow{OP'}$ lies in the interior of $\angle POQ$ (as in the figure)

Prove: $P'Q' = PQ$

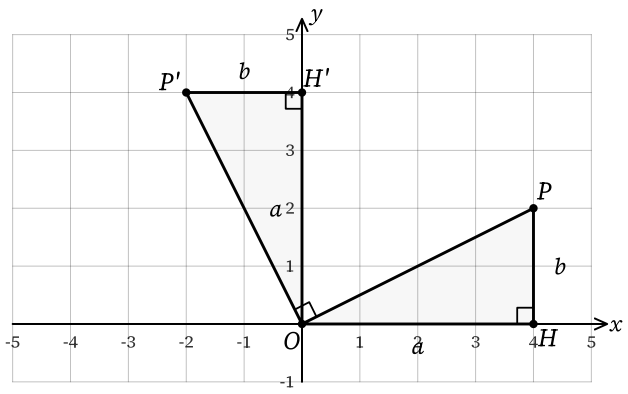
	STATEMENTS	REASONS
1	The rotation about O through θ maps P to P' and Q to Q'	Given
2	$OP' = OP$ and $OQ' = OQ$	Definition of rotation
3	$m\angle POP' = \theta$ and $m\angle QOQ' = \theta$	Definition of rotation
4	$m\angle POQ = m\angle POP' + m\angle P'OQ$	Angle Addition Postulate
5	$m\angle P'OQ' = m\angle P'OQ + m\angle QOQ'$	Angle Addition Postulate
6	$m\angle POQ = \theta + m\angle P'OQ$ and $m\angle P'OQ' = m\angle P'OQ + \theta$	Substitution Property of Equality (row 3)
7	$m\angle POQ = m\angle P'OQ'$	Transitive Property of Equality
8	$\triangle POQ \cong \triangle P'OQ'$	SAS (rows 2 and 7)
9	$P'Q' = PQ$	CPCTC and definition of congruent segments

∎

If the rotation angle is larger than $\angle POQ$, the same proof works with subtraction in place of addition. If P, Q and O are collinear, the images lie on one line through O and the result follows from the Segment Addition Postulate.

THEOREM **Rule for a 90° Rotation About the Origin**

The rotation of $90°$ *about the origin maps every point* (x, y) *to* $(-y, x)$.



PARAGRAPH PROOF

Given: $P(a, b)$ is a point other than the origin O; Q is the point $(-b, a)$

Prove: Q is the image of P under the rotation of $90°$ about O

Distance: by the distance formula, $OQ = \sqrt{(-b)^2 + a^2} = \sqrt{a^2 + b^2} = OP$.

Right angle: if $a \neq 0$ and $b \neq 0$, the slope of $\overline{OP}$ is $\frac{b}{a}$ and the slope of $\overline{OQ}$ is $\frac{a}{-b}$. Their product is $\frac{b}{a} \cdot \frac{a}{-b} = -1$, so $\overline{OP} \perp \overline{OQ}$ (Lesson 3.5). If $a = 0$ or $b = 0$, one of P, Q lies on one axis and the other on the other axis, so again $\overline{OP} \perp \overline{OQ}$. Either way $m\angle POQ = 90°$.

Direction: exactly two points are at distance OP from O on the line through O perpendicular to $\overline{OP}$, namely $(-b, a)$ and $(b, -a)$. The point $(-b, a)$ is the one on the left as you walk from O toward P (for $P(1, 0)$ it is $(0, 1)$, a quarter turn counterclockwise from the positive x-axis), so it is reached by turning counterclockwise. By the definition of rotation, $Q(-b, a)$ is the image of P.

∎

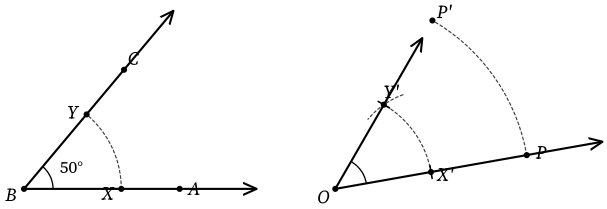
The other point, $(b, -a)$*, is the image under a* $90°$ *clockwise turn, the rotation of* $270°$.

COROLLARY **Rules for 180° and 270° Rotations About the Origin**

The rotation of $180°$ *about the origin maps* (x, y) *to* $(-x, -y)$*, and the rotation of* $270°$ *maps* (x, y) *to* $(y, -x)$.

A $180°$ *turn is two* $90°$ *turns:* $(x, y) \to (-y, x) \to (-x, -y)$*. A* $270°$ *turn is three:* $(-x, -y) \to (y, -x)$*. Each point keeps its distance from the center and the angles add.*

CONSTRUCTION **Rotate a point about a center through a given angle** Euclid I.23



1. You are given the center O, the point P, and the angle $\angle ABC$ to rotate through. Draw $\overrightarrow{OP}$.
2. With the compass point on B, draw an arc crossing $\overrightarrow{BA}$ at X and $\overrightarrow{BC}$ at Y. With the same width and the point on O, draw an arc crossing $\overrightarrow{OP}$ at X' and continuing counterclockwise.
3. Set the compass to XY. With the point on X', mark Y' on the arc from step 2, on the counterclockwise side. Draw $\overrightarrow{OY'}$. You have copied $\angle ABC$ to $\angle X'OY'$ (Euclid I.23).
4. Set the compass to OP. With the point on O, mark the point P' where this circle crosses $\overrightarrow{OY'}$ (Euclid I.3). P' is the image of P.

Why it works. The copied angle is congruent to $\angle ABC$ because $\triangle X'OY' \cong \triangle XBY$ by SSS (Lesson 4.3; this is how the angle-copying construction was justified in Lesson 4.5), so $m\angle POP' = m\angle ABC$, measured counterclockwise. The last arc

makes $OP' = OP$. These are exactly the two conditions in the definition of rotation.

KEY FORMULAS

$$90^\circ : (x, y) \to (-y,\ x)$$

$$180^\circ : (x, y) \to (-x,\ -y)$$

$$270^\circ : (x, y) \to (y,\ -x)$$

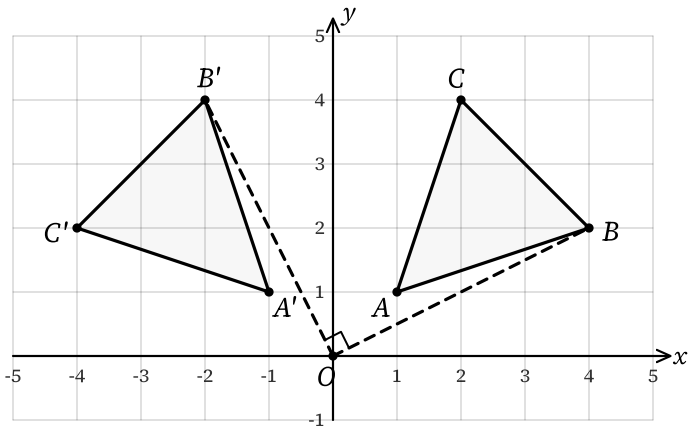
$$\text{smallest angle of rotational symmetry of order } n = \frac{360^\circ}{n}$$

From history. The great rose windows of the Gothic cathedrals of France, built in the 1200s, are wheels of stone and stained glass. Master masons laid them out with compass and cord by dividing a circle into equal parts, often 12 or 16, so each window has rotational symmetry: turn it through one of those parts and the pattern fits onto itself.

Euclid connection. Euclid never rotates a figure, but Book IV inscribes regular polygons in a circle, among them the square (IV.6), the pentagon (IV.11), the hexagon (IV.15) and the fifteen-sided polygon (IV.16), and each has rotational symmetry about the circle's center. Constructing a rotation uses Euclid's I.23 (copying an angle) and I.3 (cutting off a segment equal to a given one).

Practice habit. Mark the center of rotation first and draw a light segment from it to each point; check that each image is the same distance from the center. Recite the three rules until automatic, and test each one on $(1, 0)$: 90° sends it to $(0, 1)$, 180° to $(-1, 0)$, 270° to $(0, -1)$.

Example 1. Rotate $\triangle ABC$ with $A(1,1)$, $B(4,2)$ and $C(2,4)$ through 90° about the origin. Then check that $OB' = OB$ and $\overline{OB} \perp \overline{OB'}$.

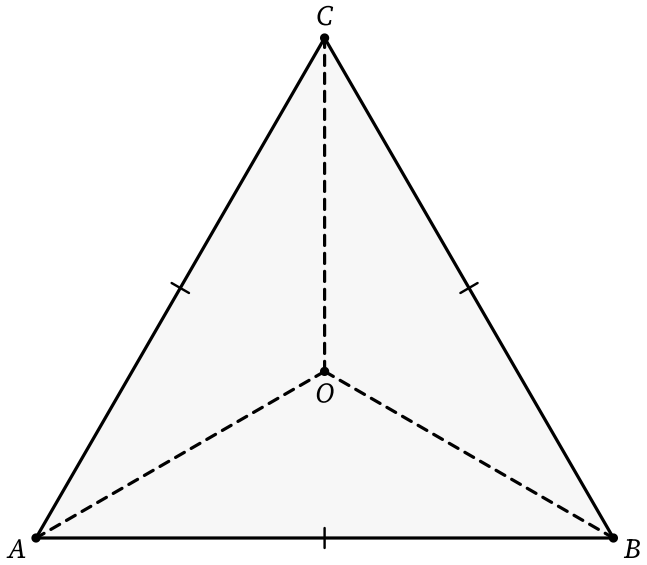


Show solution

1. Use the rule $(x, y) \to (-y, x)$: $A(1,1) \to A'(-1,1)$, $B(4,2) \to B'(-2,4)$, $C(2,4) \to C'(-4,2)$.
2. Distance: $OB = \sqrt{16+4} = \sqrt{20}$ and $OB' = \sqrt{4+16} = \sqrt{20}$.
3. Angle: the slope of $\overline{OB}$ is $\dfrac{2}{4} = \dfrac{1}{2}$ and the slope of $\overline{OB'}$ is $\dfrac{4}{-2} = -2$. The product is -1, so the segments are perpendicular.

Answer: $A'(-1,1)$, $B'(-2,4)$, $C'(-4,2)$

Example 2. Given: equilateral $\triangle ABC$, with vertices labeled counterclockwise, and its circumcenter O, as in the figure. Prove: the rotation of $120°$ about O maps A to B, B to C, and C to A. (So the triangle has rotational symmetry of order 3.)



Show solution

1. A rotation about O needs two things: equal distances from O, and the right angle of turn.
2. The circumcenter gives equal distances: $OA = OB = OC$.
3. With $AB = BC = CA$, the three triangles at O are congruent by SSS, so the three angles at O are congruent; together they make a full turn of $360°$, so each is $120°$.

TWO-COLUMN PROOF

Given: $\triangle ABC$ is equilateral, labeled counterclockwise; O is the circumcenter of $\triangle ABC$

Prove: The rotation of $120°$ about O maps A to B, B to C, and C to A

	STATEMENTS	REASONS
1	$\triangle ABC$ is equilateral, labeled counterclockwise; O is its circumcenter	Given
2	$OA = OB = OC$	The circumcenter is equidistant from the vertices (Lesson 5.2)

	STATEMENTS	REASONS
3	$AB = BC = CA$	Definition of equilateral triangle
4	$\triangle AOB \cong \triangle BOC \cong \triangle COA$	SSS
5	$\angle AOB \cong \angle BOC \cong \angle COA$	CPCTC
6	$m\angle AOB + m\angle BOC + m\angle COA = 360°$	Angle Addition Postulate: the three angles fill the full turn around O, which lies inside the triangle because an equilateral triangle is acute (Lesson 5.2)
7	$3 \cdot m\angle AOB = 360°$, so $m\angle AOB = 120°$	Substitution and Division Properties of Equality
8	$m\angle AOB = m\angle BOC = m\angle COA = 120°$	Definition of congruent angles (rows 5 and 7)
9	The rotation of $120°$ about O maps A to B, B to C, and C to A	Definition of rotation (rows 2 and 8; counterclockwise because the vertices are labeled counterclockwise)

∎

Answer: See the proof.

PRACTICE

1. Easy Rotate the point $(5, 2)$ about the origin through (a) $90°$, (b) $180°$, (c) $270°$.

   Show answer

   (a) $(-2, 5)$ (b) $(-5, -2)$ (c) $(2, -5)$

2. Easy Give the order of rotational symmetry and the smallest angle of rotational symmetry of (a) a regular pentagon, (b) a regular octagon.

   Show answer

   (a) order 5, $72°$ (b) order 8, $45°$

3. **Easy** Written as plain block capitals, which of the letters N, A, S, H have 180° rotational symmetry (point symmetry)?

Show answer

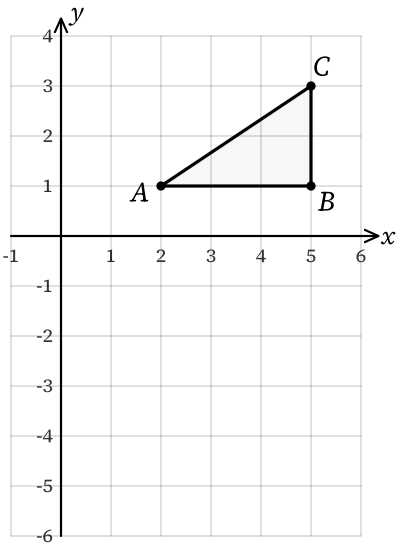
N, S and H. (Turned upside down, A does not match itself.)

4. **Easy** The minute hand of the kitchen clock moves from 12:00 to 12:20. Through how many degrees does it turn, and with our sign convention, what is the angle of this rotation?

Show answer

20 minutes is $\dfrac{20}{60} = \dfrac{1}{3}$ of a turn, 120°. Clocks turn clockwise, so the rotation is -120°.

5. **Medium** In the figure, rotate $\triangle ABC$ with $A(2,1)$, $B(5,1)$ and $C(5,3)$ through 270° about the origin. Give the image vertices.



Show answer

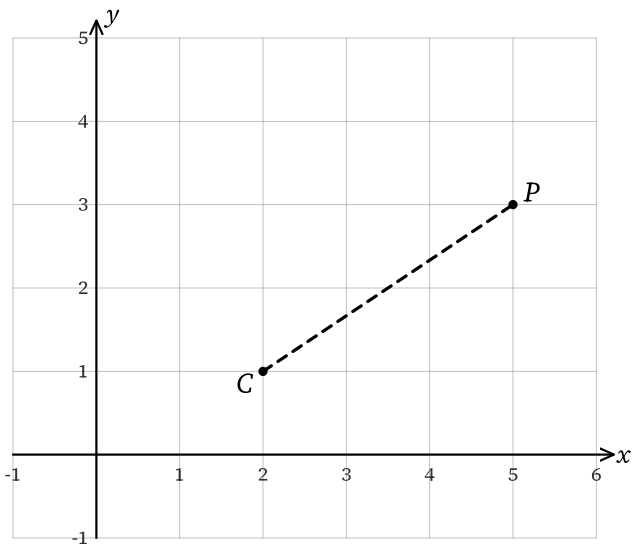
Rule $(x, y) \to (y, -x)$: $A'(1,-2)$, $B'(1,-5)$, $C'(3,-5)$

6. **Medium** The rotation of 180° about the origin maps $P(-3,4)$ to Q. Find Q and the length PQ.

Show answer

$Q(3,-4)$; $PQ = \sqrt{6^2 + 8^2} = 10$ (the origin is the midpoint of $\overline{PQ}$)

7. **Medium** In the figure, rotate $P(5, 3)$ through $90°$ about the center $C(2, 1)$.



Show answer

Relative to C, P is $(3, 2)$. Rotating gives $(-2, 3)$. Adding C back: $P'(0, 4)$. (Check: $CP = CP' = \sqrt{13}$.)

8. **Medium** The water wheel of a grain mill has 16 paddles equally spaced around it. (a) What is the smallest angle of rotation that maps the wheel onto itself? (b) If the wheel turns $202.5°$, how many paddle positions has it moved?

Show answer

(a) $\dfrac{360°}{16} = 22.5°$ (b) $\dfrac{202.5}{22.5} = 9$ positions

9. **Medium** A church's rose window has 12 identical petals evenly spaced around its center. (a) What is its order of rotational symmetry? (b) What is its smallest angle of rotational symmetry? (c) Does a rotation of $135°$ map the window onto itself?

Show answer

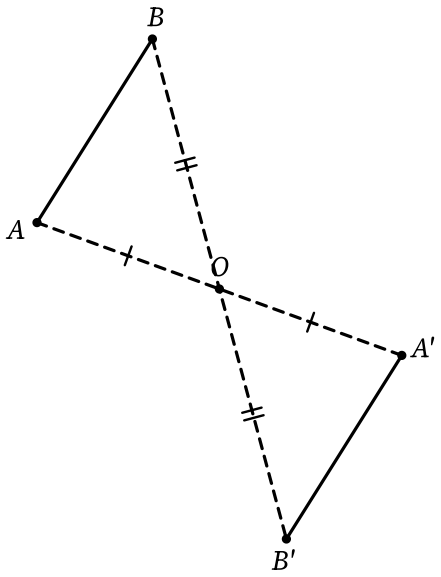
(a) 12 (b) $30°$ (c) No: $135 \div 30 = 4.5$ is not a whole number of steps.

10. **Medium** Explain why every regular polygon with an even number of sides has point symmetry, but a regular polygon with an odd number of sides does not.

Show answer

A regular n-gon maps onto itself exactly under rotations through whole-number multiples of $\dfrac{360^\circ}{n}$. If n is even, $180^\circ = \dfrac{n}{2} \cdot \dfrac{360^\circ}{n}$ is such a multiple, since $\dfrac{n}{2}$ is a whole number. If n is odd, $\dfrac{180^\circ}{360^\circ/n} = \dfrac{n}{2}$ is not a whole number, so 180° is not a multiple. (For example, a half turn of an equilateral triangle about its center sends each vertex to a point beyond the opposite side, where there is no vertex.)

11. Hard Given: the rotation of $180°$ about O maps A to A' and B to B', and O is not on $\overleftrightarrow{AB}$, as in the figure. Prove: $\overline{AB} \parallel \overline{A'B'}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: The rotation of $180°$ about O maps A to A' and B to B'; O is not on $\overleftrightarrow{AB}$

Prove: $\overline{AB} \parallel \overline{A'B'}$

	STATEMENTS	REASONS
1	The rotation of $180°$ about O maps A to A' and B to B'; O is not on $\overleftrightarrow{AB}$	Given
2	$OA = OA'$ and $OB = OB'$	Definition of rotation
3	$m\angle AOA' = 180°$ and $m\angle BOB' = 180°$	Definition of rotation
4	O is between A and A', and between B and B'	Definition of straight angle
5	O is the midpoint of $\overline{AA'}$ and of $\overline{BB'}$	Definition of midpoint (rows 2 and 4)
6	The diagonals $\overline{AA'}$ and $\overline{BB'}$ of quadrilateral $ABA'B'$ bisect each other	Definition of segment bisector
7	$ABA'B'$ is a parallelogram	If the diagonals of a quadrilateral bisect each other,

	STATEMENTS	REASONS
		it is a parallelogram (Lesson 6.3)
8	$\overline{AB} \parallel \overline{A'B'}$	Definition of parallelogram

∎

12. Hard Given: a rotation about O maps A to A' and B to B', and $A \neq B$. Prove indirectly: $A' \neq B'$.

Show answer

See the model proof. (A rotation never sends two different points to the same place.)

INDIRECT PROOF

Given: A rotation about O maps A to A' and B to B'; $A \neq B$

Prove: $A' \neq B'$

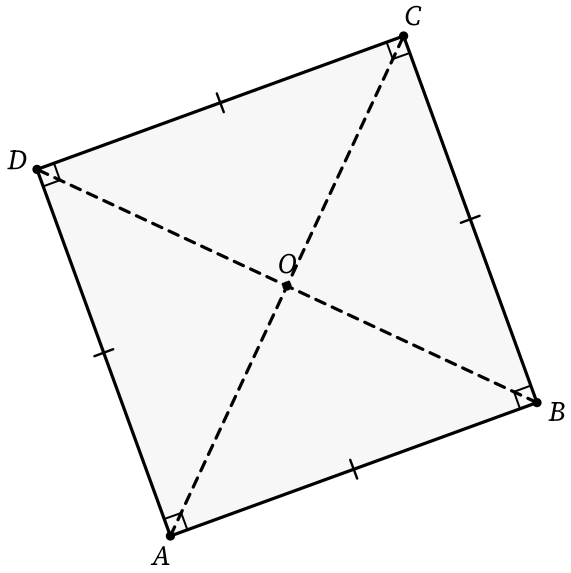
Assume the opposite: $A' = B'$. Then $A'B' = 0$.

Rotations preserve distance (Theorem 1), so $A'B' = AB$. Since A and B are different points, $AB > 0$ (Ruler Postulate). So $0 = A'B' = AB > 0$, which is impossible.

The assumption led to a contradiction, so it is false. Therefore $A' \neq B'$.

∎

13. Hard Given: square $ABCD$, labeled counterclockwise, whose diagonals meet at O, as in the figure. Prove with a paragraph proof: the rotation of 90° about O maps A to B.



Show answer

See the model proof. (The same argument maps B to C, C to D and D to A, so a square has rotational symmetry of order 4.)

PARAGRAPH PROOF

Given: $ABCD$ is a square, labeled counterclockwise; $\overline{AC}$ and $\overline{BD}$ meet at O

Prove: The rotation of 90° about O maps A to B

A square is a parallelogram, so its diagonals bisect each other (Lesson 6.2): $OA = \frac{1}{2}AC$ and $OB = \frac{1}{2}BD$. A square is a rectangle, so its diagonals are congruent (Lesson 6.4): $AC = BD$. Therefore $OA = \frac{1}{2}AC = \frac{1}{2}BD = OB$.

A square is also a rhombus, so its diagonals are perpendicular (Lesson 6.4), and $m\angle AOB = 90^\circ$.

Because the vertices are labeled counterclockwise, B is reached from A by turning counterclockwise about O. So $OB = OA$ and $m\angle AOB = 90^\circ$, measured counterclockwise, and by the definition of rotation the 90° rotation about O maps A to B.

∎

14. **Challenge** A rotation of 90° (counterclockwise) about an unknown center $C(h, k)$ maps $P(1, 2)$ to $P'(5, 4)$. Find C.

Show answer

Relative to C, P is $(1 - h,\ 2 - k)$, and the 90° rule turns this into $(k - 2,\ 1 - h)$. Adding C back must give P': $h + k - 2 = 5$ and $k + 1 - h = 4$. So $h + k = 7$ and $k - h = 3$, giving $k = 5$, $h = 2$: $C(2, 5)$. Check: $\overrightarrow{CP} = \langle -1, -3 \rangle$ and $\overrightarrow{CP'} = \langle 3, -1 \rangle$ have equal length $\sqrt{10}$, and turning $\langle -1, -3 \rangle$ by 90° gives $\langle 3, -1 \rangle$.

COMMON MISTAKES

- Turning the wrong way. A positive angle means counterclockwise; a clock's hands turn through negative angles.
- Mixing up the 90° and 270° rules. Test the rule on $(1, 0)$: a 90° turn must land on $(0, 1)$, so the rule is $(x, y) \to (-y, x)$.
- Using the origin rules when the center is somewhere else. Subtract the center first, apply the rule, then add the center back.
- Miscounting the order of symmetry. The order counts the positions in one full turn in which the figure fits, including the starting one; a square fits at 90°, 180°, 270° and 360°, so its order is 4.

> **Make it a game · Spin the Shape.** Cut paper shapes: square, equilateral triangle, rectangle, regular pentagon, a letter S. Pin each at its center with a pencil. Predict the order of rotational symmetry before you spin it, and score a point per correct prediction. Then parent calls a point and a turn: rotate (3, 2) by 90, 180 or 270 degrees about the origin.

7.4 Compositions and Congruence

OBJECTIVES

- Find the image of a figure under a composition of rigid motions, and show that the order of the motions can matter.
- Prove that two reflections in parallel lines make a translation and two reflections in intersecting lines make a rotation through twice the angle between them.
- Describe and apply glide reflections.
- Define congruence by rigid motions, find a sequence of rigid motions that maps one congruent figure onto another, and explain how SSS, SAS and ASA follow from rigid motions.

WATCH

Transformation & Rotation by Repeated Reflection

Eddie Woo · 7:46

https://www.youtube.com/watch?v=XekUKlGeA_g

> **WHEN TO WATCH**
> After the theorems on two reflections: a rotation made from repeated reflections.

Note: Glide reflections and congruence by rigid motions are taught from the Core ideas.

BACKUP VIDEO

Composite Transformations

Eddie Woo · 11:21

https://www.youtube.com/watch?v=49Z124F74Pc

Before the practice set: carrying out one transformation after another.

KEY TERMS

composition

Doing one transformation and then another to the image. The composition of T followed by S sends P to $S(T(P))$.

glide reflection

A translation along a line ℓ (by a vector parallel to ℓ) followed by the reflection in ℓ.

congruent figures (rigid-motion definition)

Two figures are congruent if some sequence of rigid motions maps one onto the other.

identity transformation

The transformation that leaves every point where it is, such as a rotation of $360°$ or a translation by $\langle 0, 0 \rangle$.

CORE IDEAS

A **composition** applies one transformation and then another to the result. Write each stage: $A \to A_1 \to A_2$. Since each rigid motion preserves distance, $A_2B_2 = A_1B_1 = AB$, so a composition of rigid motions is again a rigid motion. But order can matter: reflecting $(1, 4)$ in $y = x$ and then translating by $\langle 2, 0 \rangle$ gives $(6, 1)$, while translating first and then reflecting gives $(4, 3)$.

Reflect a point in a line ℓ, and then reflect the image in a parallel line m. The two flips undo each other's reversal of orientation, and every point ends up moved straight across, perpendicular to the lines, through twice the distance between them. So **two reflections in parallel lines make a translation** by a vector perpendicular to the lines, of length $2d$, pointing from ℓ toward m (Theorem 1).

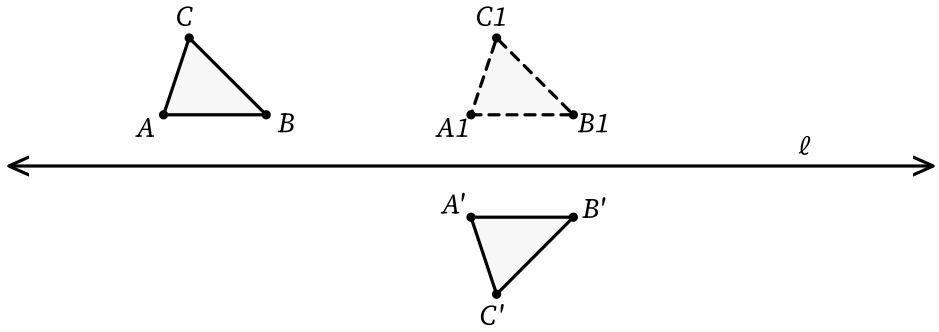
If instead ℓ and m meet at O at an angle θ, each reflection keeps the point's distance from O, and the angles to the mirrors double up. So **two reflections in intersecting lines make a rotation** about O through 2θ, turning in the direction from ℓ toward m (Theorem 2). This is how a kaleidoscope works: two mirrors at $30°$ turn the picture through $60°$, giving six turned copies around the center, with a mirror-image copy between each pair (twelve wedges in all).

A **glide reflection** is a translation along a line followed by the reflection in that line. Footprints in the snow form a glide reflection pattern: each left print, slid forward and flipped across the path, becomes the next right print. Like a reflection, a glide reflection reverses orientation; unlike a reflection, it leaves no point fixed.

Now we can say exactly what congruence means: two figures are **congruent** when a sequence of rigid motions maps one onto the other. Theorem 3 shows that for triangles this agrees with Unit 4: if the three pairs of sides are equal, a translation, a rotation, and if needed a reflection, carry one triangle onto the other. That argument is SSS in the language of rigid motions. SAS works the same way: after translating A to D and turning $\overrightarrow{AB}$ onto $\overrightarrow{DE}$ (B lands on E because $AB = DE$), reflect in $\overleftrightarrow{DE}$ if

needed so C is on F's side; then the equal angles at D put C's image on $\overrightarrow{DF}$, and $AC = DF$ puts it exactly on F. For ASA, the two equal angles put C's image on both rays from D and E, which meet in only one point, F.

A word of care about the logic. In this course we took SAS as our starting point (Euclid argued for it in I.4 by \"applying\" one triangle to the other, and Hilbert later made a form of it an axiom) and proved the other tests from it, then used congruent triangles to prove that reflections and rotations preserve distance. Many modern books start the other way: they postulate that rigid motions preserve distance and angle measure and then prove SAS, ASA and SSS as above. Either starting point builds the same geometry; what you must not do is use each to prove the other in one chain.

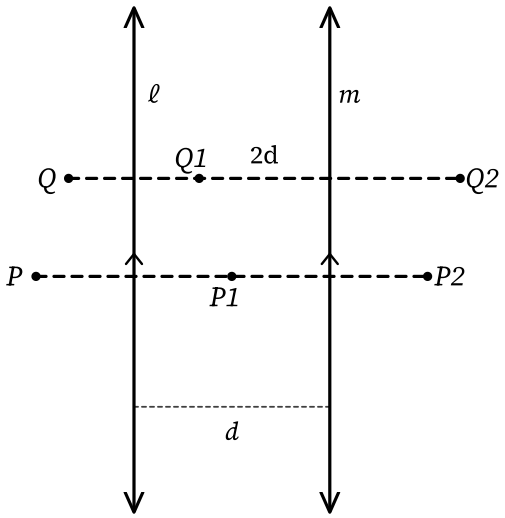


A glide reflection: slide along ℓ, then reflect in ℓ.

POSTULATES AND THEOREMS

THEOREM **Reflections in Parallel Lines**

If $\ell \parallel m$ and the distance between them is d, then the reflection in ℓ followed by the reflection in m is the translation by a vector perpendicular to the lines, of length $2d$, pointing from ℓ toward m.



PARAGRAPH PROOF

Given: $\ell \parallel m$, with distance d between them; The reflection in ℓ maps P to P_1, and the reflection in m maps P_1 to P_2

Prove: P_2 is the image of P under the translation of length $2d$, perpendicular to the lines, from ℓ toward m

Let t be the line through P perpendicular to ℓ. Because $\ell \parallel m$, t is also perpendicular to m (a line perpendicular to one of two parallel lines is perpendicular to the other, by the Corresponding Angles Postulate). By the definition of reflection, P_1 lies on the perpendicular to ℓ through P, which is t. Then P_2 lies on the perpendicular to m through P_1; there is only one such line, and t is one, so P_2 is on t too.

Place a number line on t (Ruler Postulate) with 0 where t crosses ℓ and d where it crosses m. Let P have coordinate p. The crossing point 0 is the midpoint of $\overline{PP_1}$, so P_1 has coordinate $-p$, since $\frac{p + (-p)}{2} = 0$. The crossing point d is the midpoint of $\overline{P_1P_2}$, so P_2 has coordinate $2d - (-p) = p + 2d$, since $\frac{-p + (p + 2d)}{2} = d$. (If P is on ℓ, then $p = 0$ and the same formulas hold.)

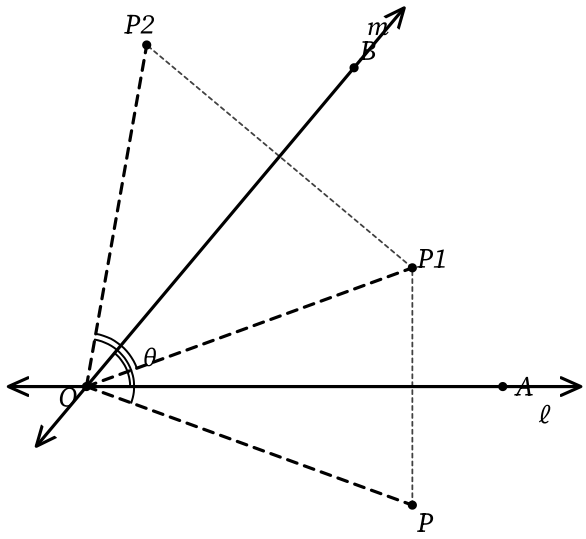
So every point P moves the distance $2d$ along a line perpendicular to ℓ, in the direction from ℓ toward m. Every point moves the same distance in the same direction, which is the definition of a translation.

∎

Reversing the order (first m, then ℓ) gives the translation of the same length in the opposite direction.

THEOREM **Reflections in Intersecting Lines**

If lines ℓ and m meet at O and the angle from ℓ to m is θ, then the reflection in ℓ followed by the reflection in m is the rotation about O through 2θ.



TWO-COLUMN PROOF

Given: Lines ℓ and m meet at O; A is on ℓ, B is on m, and $m\angle AOB = \theta$; The reflection in ℓ maps P to P_1, and the reflection in m maps P_1 to P_2; P and P_1 are positioned as in the figure (P_1 in the interior of $\angle AOB$), and $\theta \leq 90°$

Prove: $OP_2 = OP$ and $m\angle POP_2 = 2\theta$, so P_2 is the image of P under the rotation about O through 2θ

	STATEMENTS	REASONS
1	ℓ and m meet at O; the reflections map P to P_1 and P_1 to P_2; $m\angle AOB = \theta$	Given
2	ℓ is the perpendicular bisector of $\overline{PP_1}$; m is the perpendicular bisector of $\overline{P_1P_2}$	Definition of reflection
3	$OP = OP_1$ and $OP_1 = OP_2$	Perpendicular Bisector Theorem (O is on ℓ and on m)
4	$OP = OP_2$	Transitive Property of Equality
5	$m\angle POA = m\angle AOP_1$ and $m\angle P_1OB = m\angle BOP_2$	Mirror Line Theorem (Lesson 7.2)
6	$m\angle POP_2 = m\angle POA + m\angle AOP_1 + m\angle P_1OB + m\angle BOP_2$	Angle Addition Postulate
7	$m\angle POP_2 = 2\,m\angle AOP_1 + 2\,m\angle P_1OB$	Substitution Property of Equality (row 5)

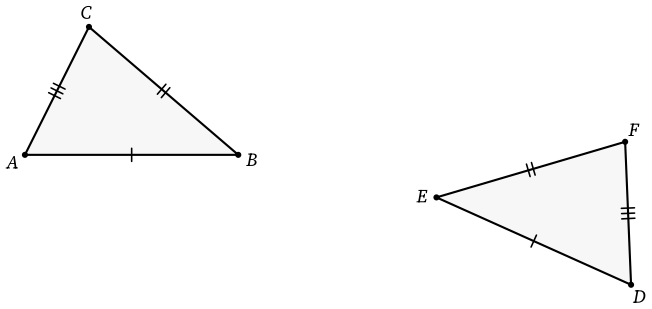
	STATEMENTS	REASONS
8	$m\angle AOP_1 + m\angle P_1OB = m\angle AOB = \theta$	Angle Addition Postulate
9	$m\angle POP_2 = 2(m\angle AOP_1 + m\angle P_1OB) = 2\theta$	Distributive Property and Substitution (row 8)
10	$OP_2 = OP$ and $m\angle POP_2 = 2\theta$, so P_2 is the image of P under the rotation about O through 2θ	Definition of rotation (rows 4 and 9; the turn runs from ℓ toward m)

∎

Other positions of P are handled the same way with subtraction in place of addition. Reversing the order of the reflections gives the rotation through -2θ. If $\theta > 90°$, then $2\theta > 180°$; the same argument shows that P_2 is reached by turning $\overrightarrow{OP}$ counterclockwise through 2θ, which is the rotation through 2θ (the same as the rotation through $2\theta - 360°$).

THEOREM **Congruence and Rigid Motions Theorem** Euclid I.8

Two triangles are congruent exactly when a sequence of rigid motions maps one onto the other. In fact, if $AB = DE$, $BC = EF$ and $CA = FD$, then a translation, a rotation and, if needed, a reflection map A to D, B to E and C to F.



PARAGRAPH PROOF

Given: $AB = DE, BC = EF, CA = FD$

Prove: A sequence of at most three rigid motions maps A to D, B to E, and C to F

Translate. The translation by the vector from A to D maps A to D; call the images of B and C B_1 and C_1. Translations preserve distance, so $DB_1 = AB$, $DC_1 = AC$ and $B_1C_1 = BC$.

Rotate. Rotate about D through the angle from $\overrightarrow{DB_1}$ to $\overrightarrow{DE}$ (skip this if the rays already agree). D stays fixed, and B_1 goes to a point B_2 on $\overrightarrow{DE}$ with $DB_2 = DB_1 = AB = DE$. On a ray there is only one point at a given distance from its endpoint (Ruler Postulate), so $B_2 = E$. Call the image of C_1 C_2. Rotations preserve distance, so $DC_2 = AC = DF$ and $EC_2 = BC = EF$.

Reflect if needed. If $C_2 = F$, we are done. Otherwise C_2 and F are different points with $DC_2 = DF$ and $EC_2 = EF$, so D and E are each equidistant from C_2 and F. By the Converse of the Perpendicular Bisector Theorem, both lie on the perpendicular bisector of $\overline{C_2F}$; two points determine a line, so $\overleftrightarrow{DE}$ is that bisector. The reflection in $\overleftrightarrow{DE}$ then maps C_2 to F and leaves D and E, which are on the mirror, where they are.

So the composition maps A, B, C to D, E, F. It also carries each side onto the matching side: a point X between A and B has $AX + XB = AB$, so its image X' has $DX' + X'E = DE$, and X' must lie on $\overline{DE}$, since otherwise the Triangle Inequality or the Segment Addition Postulate would make the sum larger. Conversely, if rigid motions map one triangle onto another, they keep every side length and angle measure (Lesson 7.1), so the triangles are congruent by the definition in Lesson 4.2.

∎

The proof uses only the three side lengths, so it is SSS told in the language of rigid motions. Euclid proved SAS (I.4) and SSS (I.8) by the same kind of moving, which he called "applying" one triangle to the other.

KEY FORMULAS

$$\text{reflect in } \ell,\ \text{then in } m \parallel \ell \ (\text{distance } d) = \text{translation of length } 2d$$

$$\text{reflect in } \ell,\ \text{then in } m \ (\text{meeting at angle } \theta) = \text{rotation through } 2\theta$$

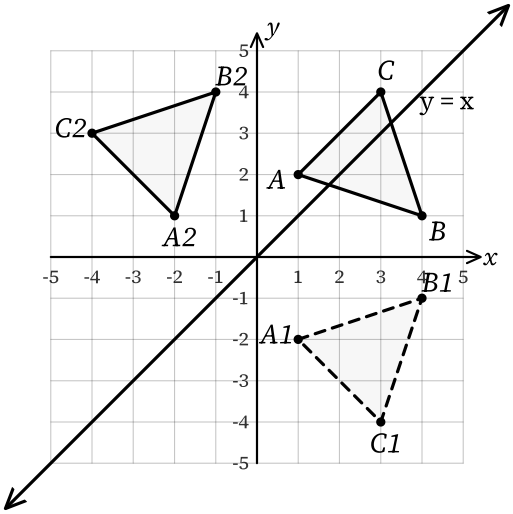
From history. Euclid proved I.4 by laying one triangle on the other, but he never stated an axiom that allows figures to be moved, and later readers noticed the gap. David Hilbert, in his Foundations of Geometry (1899), took a form of SAS as an axiom instead. Modern textbooks often take the other route and build congruence on rigid motions, an approach shaped by Felix Klein's 1872 idea of studying geometry through its transformations.

Euclid connection. In I.4 Euclid writes, in effect: if the triangle be applied to the other triangle, with the point A placed on D and the line AB on DE, then B will coincide with E, because AB is equal to DE. Theorem 3 does exactly this with a translation, a rotation and a reflection, turning Euclid's "application" into precise steps.

Practice habit. Write the motions of a composition in order and apply them one at a time, recording each stage ($A \to A_1 \to A_2$). Recite until automatic: "Two reflections in parallel lines make a translation of twice the distance; two reflections in intersecting lines make a rotation of twice the angle."

WORKED EXAMPLES

Example 1. Reflect $\triangle ABC$ with $A(1,2)$, $B(4,1)$ and $C(3,4)$ in the x-axis, and then reflect the image in the line $y = x$. Give the final image, and name the single rigid motion that does the same job.

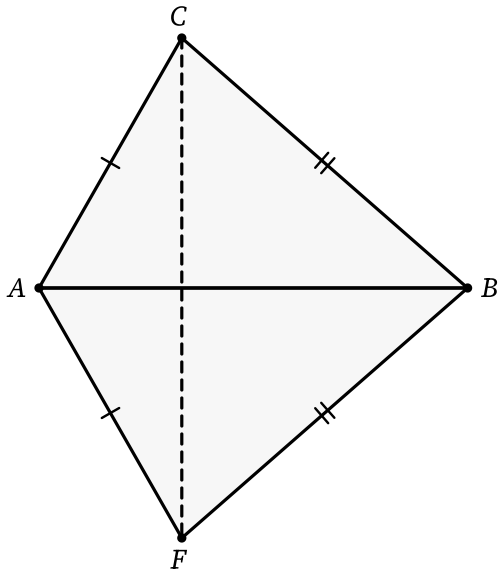


Show solution

1. Reflect in the x-axis, $(x, y) \to (x, -y)$: $A_1(1, -2)$, $B_1(4, -1)$, $C_1(3, -4)$.
2. Reflect in $y = x$, $(x, y) \to (y, x)$: $A_2(-2, 1)$, $B_2(-1, 4)$, $C_2(-4, 3)$.
3. Together, $(x, y) \to (x, -y) \to (-y, x)$, which is the rule for a 90° rotation about the origin.
4. This agrees with Theorem 2: the x-axis and $y = x$ meet at the origin, and the angle from the x-axis to $y = x$ is 45°, so the composition is a rotation through $2 \times 45^\circ = 90^\circ$.

Answer: $A_2(-2, 1)$, $B_2(-1, 4)$, $C_2(-4, 3)$; a rotation of 90° about the origin

Example 2. Given: $\triangle ABC$ and $\triangle ABF$ share side $\overline{AB}$; $AC = AF$ and $BC = BF$; C and F are on opposite sides of $\overleftrightarrow{AB}$, as in the figure. Prove: the reflection in $\overleftrightarrow{AB}$ maps C to F. (This is the final step in the proof of Theorem 3.)



Show solution

1. To show a reflection maps C to F, show that the mirror line is the perpendicular bisector of $\overline{CF}$.
2. A and B are each equidistant from C and F, so each lies on the perpendicular bisector.
3. Two points determine a line, so that bisector is $\overleftrightarrow{AB}$.

TWO-COLUMN PROOF

Given: $AC = AF$ and $BC = BF$; C and F are on opposite sides of $\overleftrightarrow{AB}$

Prove: The reflection in $\overleftrightarrow{AB}$ maps C to F

	STATEMENTS	REASONS
1	$AC = AF$ and $BC = BF$; C and F are on opposite sides of $\overleftrightarrow{AB}$	Given
2	$C \neq F$	Points on opposite sides of a line are different
3	A is on the perpendicular bisector of $\overline{CF}$	Converse of the Perpendicular Bisector Theorem

	STATEMENTS	REASONS
4	B is on the perpendicular bisector of $\overline{CF}$	Converse of the Perpendicular Bisector Theorem
5	$\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CF}$	Through two points there is exactly one line
6	The reflection in $\overleftrightarrow{AB}$ maps C to F	Definition of reflection

∎

Answer: See the proof.

PRACTICE

1. **Easy** A figure is translated by $\langle 3, -1\rangle$ and then by $\langle -7, 4\rangle$. (a) What single translation does the same job? (b) Where does the point $(2, 2)$ end up?

   Show answer

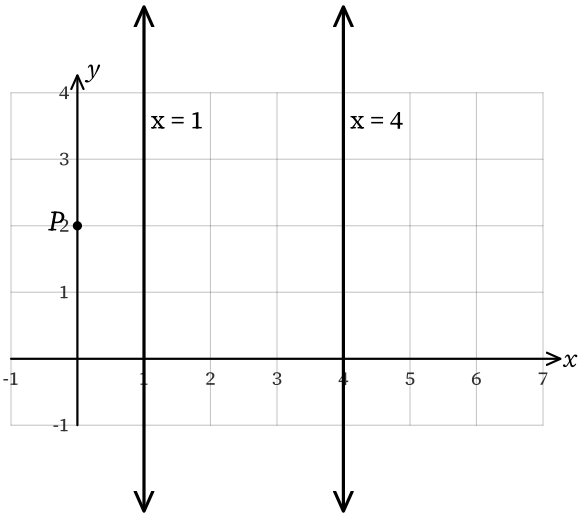
   (a) $\langle -4, 3\rangle$ (b) $(-2, 5)$

2. **Easy** Reflect $P(3, 5)$ in the x-axis and then reflect the image in the y-axis. Give the final image and name the single rigid motion that does the same job.

   Show answer

   $(3, 5) \to (3, -5) \to (-3, -5)$; a rotation of 180° about the origin (the axes meet at 90°, and $2 \times 90^\circ = 180^\circ$)

3. **Easy** In the figure, reflect $P(0, 2)$ in the line $x = 1$ and then reflect the image in the line $x = 4$. Give the final image and the vector of the single translation that does the same job.



Show answer

$(0, 2) \to (2, 2) \to (6, 2)$; the translation $\langle 6, 0 \rangle$, twice the distance 3 between the lines

4. **Easy** Lines ℓ and m meet at O, and the angle from ℓ to m is 35°. The reflection in ℓ followed by the reflection in m is a rotation about O. Through what angle?

Show answer

$2 \times 35^\circ = 70^\circ$

5. **Medium** Let r be the reflection in $y = x$ and T the translation by $\langle 2, 0 \rangle$. Find the image of $P(1, 4)$ under (a) r followed by T, (b) T followed by r. Does the order matter?

Show answer

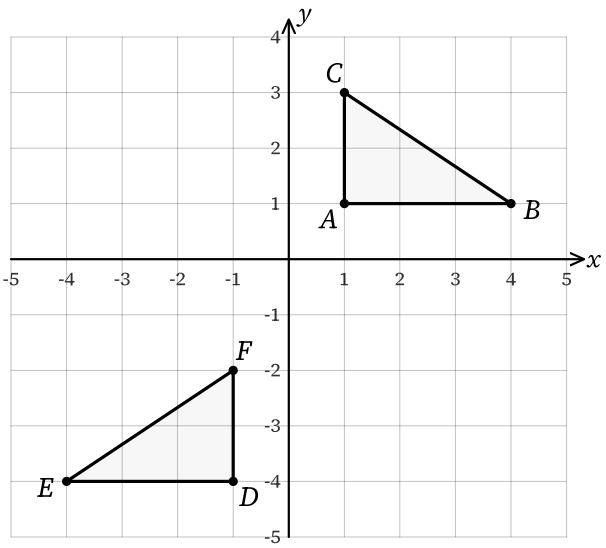
(a) $(1, 4) \to (4, 1) \to (6, 1)$ (b) $(1, 4) \to (3, 4) \to (4, 3)$. The results differ, so the order matters.

6. **Medium** Henry's footprints in the snow form a glide reflection pattern. On a grid in inches, with his path along the x-axis, each print is the image of the one before under the translation by $\langle 14, 0\rangle$ followed by the reflection in the x-axis. The heel of one print is at $(1, 3)$. (a) Where are the heels of the next two prints? (b) What single rigid motion maps a print to the print two steps later?

Show answer

(a) $(1, 3) \to (15, 3) \to (15, -3)$, then $(15, -3) \to (29, -3) \to (29, 3)$ (b) the two reflections cancel, leaving the translation $\langle 28, 0\rangle$

7. **Medium** In the figure, $\triangle ABC$ has vertices $A(1, 1)$, $B(4, 1)$, $C(1, 3)$ and $\triangle DEF$ has vertices $D(-1, -4)$, $E(-4, -4)$, $F(-1, -2)$. Describe a sequence of rigid motions that maps $\triangle ABC$ onto $\triangle DEF$ with $A \to D$, $B \to E$, $C \to F$. What does this show about the triangles?



Show answer

One answer: reflect in the y-axis, giving $(-1, 1)$, $(-4, 1)$, $(-1, 3)$, then translate by $\langle 0, -5\rangle$, giving D, E, F. Since rigid motions map one triangle onto the other, $\triangle ABC \cong \triangle DEF$. (Other correct sequences exist.)

8. **Medium** A kaleidoscope has two long mirrors that meet at an angle of 30°. (a) A bead's image formed by reflecting in one mirror and then the other is the bead turned through what angle about the line where the mirrors meet? (b) How many times must this rotation be repeated to bring the bead back to its starting place?

   Show answer

   (a) $2 \times 30^\circ = 60^\circ$ (b) $360 \div 60 = 6$ times

9. **Medium** (a) In a dance studio, large mirrors hang on two parallel opposite walls 15 feet apart. The image of a lamp formed by reflecting first in one mirror and then in the other is the lamp translated how far? (b) Two hinged mirrors meet at 65°. Reflecting in one and then the other turns a picture through what angle?

   Show answer

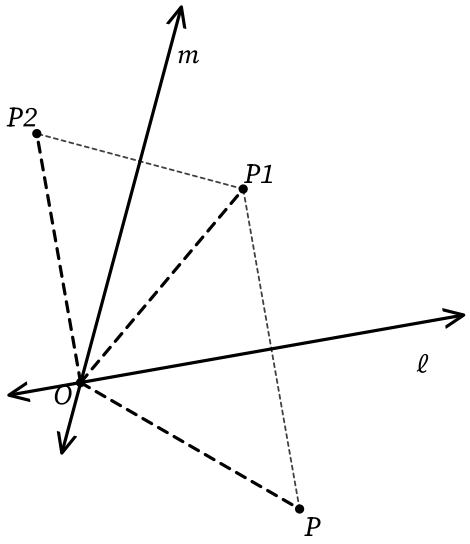
   (a) $2 \times 15 = 30$ ft (b) $2 \times 65^\circ = 130^\circ$

10. **Medium** Explain why the composition of two rigid motions is always a rigid motion.

    Show answer

    Let the first motion map A, B to A_1, B_1 and the second map those to A_2, B_2. Each preserves distance, so $A_1B_1 = AB$ and $A_2B_2 = A_1B_1$. By the Transitive Property, $A_2B_2 = AB$ for every pair of points, so the composition preserves distance, which is the definition of a rigid motion.

11. **Hard** Given: lines ℓ and m meet at O; the reflection in ℓ maps P to P_1, and the reflection in m maps P_1 to P_2, as in the figure. Prove with a flow proof: $OP = OP_2$.



Show answer

See the model proof.

FLOW PROOF

Given: Lines ℓ and m meet at O; The reflection in ℓ maps P to P_1; the reflection in m maps P_1 to P_2

Prove: $OP = OP_2$

(1) ℓ and m meet at O; the reflection in ℓ maps P to P_1 and the reflection in m maps P_1 to P_2
Given

(2) ℓ is the perpendicular bisector of $\overline{PP_1}$ (or $P_1 = P$ if P is on ℓ)
Definition of reflection
from (1)

(3) m is the perpendicular bisector of $\overline{P_1P_2}$ (or $P_2 = P_1$ if P_1 is on m)
Definition of reflection
from (1)

(4) $OP = OP_1$
Perpendicular Bisector Theorem (O is on ℓ)
from (2)

(5) $OP_1 = OP_2$
Perpendicular Bisector Theorem (O is on m)
from (3)

(6) $OP = OP_2$
Transitive Property of Equality
from (4), (5)

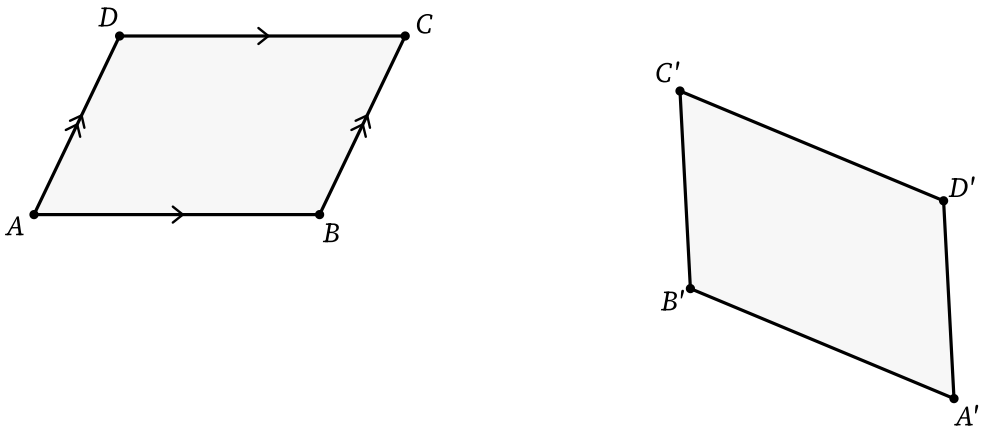
∎

12. **Hard** Rotate $P(2, 5)$ through 90° about the origin, and then rotate the image through 180° about the origin. Give the final image and name the single rotation that does the same job.

Show answer

$(2, 5) \to (-5, 2) \to (5, -2)$; a rotation of 270° about the origin (check: the 270° rule $(x, y) \to (y, -x)$ gives $(5, -2)$)

13. Hard Given: $ABCD$ is a parallelogram, and a rigid motion maps A, B, C, D to the vertices of quadrilateral $A'B'C'D'$, as in the figure. Prove: $A'B'C'D'$ is a parallelogram.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; A rigid motion maps A, B, C, D to the vertices A', B', C', D' of a quadrilateral

Prove: $A'B'C'D'$ is a parallelogram

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; a rigid motion maps A, B, C, D to A', B', C', D'	Given
2	$AB = CD$ and $BC = DA$	Opposite sides of a parallelogram are congruent (Lesson 6.2)
3	$A'B' = AB$, $B'C' = BC$, $C'D' = CD$, $D'A' = DA$	Definition of rigid motion
4	$A'B' = C'D'$ and $B'C' = D'A'$	Substitution Property of Equality
5	$A'B'C'D'$ is a parallelogram	If both pairs of opposite sides of a quadrilateral are congruent, it is a parallelogram (Lesson 6.3)

∎

14. Hard A glide reflection is the translation by a vector $\vec{v} \neq \langle 0, 0 \rangle$ parallel to line ℓ, followed by the reflection in ℓ. Prove indirectly: a glide reflection leaves no point fixed.

Show answer

See the model proof.

INDIRECT PROOF

Given: A glide reflection: the translation by $\vec{v} \neq \langle 0, 0 \rangle$, parallel to ℓ, followed by the reflection in ℓ

Prove: No point is mapped to itself

Assume the opposite: some point P is mapped to itself. Let P_1 be the image of P under the translation. Then the reflection in ℓ maps P_1 to P. Since $\vec{v}$ is not zero, $PP_1 > 0$, so $P_1 \neq P$.

Case 1: P_1 is on ℓ. Points of ℓ are their own reflections, so the reflection of P_1 is P_1, and $P = P_1$. This contradicts $P_1 \neq P$.

Case 2: P_1 is not on ℓ. Then ℓ is the perpendicular bisector of $\overline{PP_1}$, so $\overleftrightarrow{PP_1}$ is perpendicular to ℓ and meets it. But $\overrightarrow{PP_1}$ has the direction of $\vec{v}$, which is parallel to ℓ, and $\overleftrightarrow{PP_1}$ is not ℓ itself because P_1 is not on ℓ; so $\overleftrightarrow{PP_1} \parallel \ell$, and parallel lines never meet. This is a contradiction.

Every case leads to a contradiction, so the assumption is false: a glide reflection has no fixed point.

∎

15. Challenge A glide reflection maps $A(1, 3)$ to $A'(5, -3)$ and $B(2, 1)$ to $B'(6, -1)$. Find the line of the glide reflection and the translation vector. (Hint: where is the midpoint of a point and its image?)

Show answer

The translation slides a point parallel to ℓ, keeping its distance to ℓ; the reflection then puts it the same distance on the other side. So the midpoint of each point and its image lies on ℓ. The midpoints are $(3, 0)$ and $(4, 0)$, so ℓ is the x-axis, $y = 0$. Reflecting A' back in the x-axis gives $(5, 3)$, so the vector is $\langle 5 - 1,\ 3 - 3 \rangle = \langle 4, 0 \rangle$. Check with B: $(2, 1) \to (6, 1) \to (6, -1)$.

COMMON MISTAKES

- Applying the motions in the wrong order. A composition is done in the order stated; reversing it can give a different image (Practice 5).
- Forgetting to double. Mirrors 3 units apart give a translation of 6 units, and mirrors meeting at 35° give a rotation of 70°.
- Getting the direction backward. Reflecting in ℓ and then m moves or turns points from ℓ toward m; reflecting in the other order goes the opposite way.
- Deciding that two figures are not congruent because one is flipped. Reflections are rigid motions, so a mirror-image triangle is still congruent.

> **Make it a game · Does Order Matter?.** Cut a paper F and place it on graph paper. Parent names two motions, such as a slide then a flip, or a flip then a slide. Predict whether the final images will match, then try both orders to find out. Finally, fold two paper lines as mirrors and discover whether two reflections make a rotation or a translation.

7.5 Dilations

OBJECTIVES

- Dilate points and figures with a given center and scale factor, using the rule $(x, y) \to (kx, ky)$ when the center is the origin.
- Find the center and scale factor of a dilation and classify it as an enlargement or a reduction.
- Prove that a dilation multiplies lengths by $|k|$, maps each line to a parallel line or to itself, and preserves angle measure.
- Use dilations to solve problems about scale drawings, maps and blueprints.

WATCH

Transformations - Math Antics

mathantics · 12:25

https://www.youtube.com/watch?v=61OPqIp9xkA

> **WHEN TO WATCH**
> Before the Core ideas: the four transformations; the dilation part comes last.

Note: The backup is by Eddie Woo, who teaches in Australia: he says 'enlargement factor' for scale factor.

BACKUP VIDEO

Finding Sides of Similar Things (1 of 2: Similar Figures & finding corresponding lengths)

Eddie Woo · 10:14

https://www.youtube.com/watch?v=7kWFEG8UK-Y

Before Example 2: scale factors, including fractions.

KEY TERMS

dilation

The dilation with center O and scale factor $k \neq 0$ maps O to itself and each other point P to the point P' with $OP' = |k| \cdot OP$, on $\overrightarrow{OP}$ if $k > 0$ and on the ray opposite $\overrightarrow{OP}$ if $k < 0$.

center of dilation

The fixed point O from which all distances are stretched or shrunk.

scale factor

The number k by which a dilation multiplies distances from the center (in size, $|k|$).

enlargement and reduction

A dilation is an enlargement if $|k| > 1$ (the image is larger) and a reduction if $0 < |k| < 1$ (the image is smaller).

scale drawing

A drawing, map or blueprint whose lengths are all the same multiple of the real lengths; that multiple is the scale factor.

CORE IDEAS

Rigid motions move figures without changing their size. A **dilation** keeps the shape but changes the size, like a picture thrown on a wall by a projector. Choose a center O and a positive scale factor k. Each point P slides along the ray from O through P until it is k times as far from O: $OP' = k \cdot OP$. (Negative scale factors come below; in general $OP' = |k| \cdot OP$.) With $k = 2$ every point goes twice as far out; with $k = \frac{1}{2}$ every point comes halfway in. The center does not move.

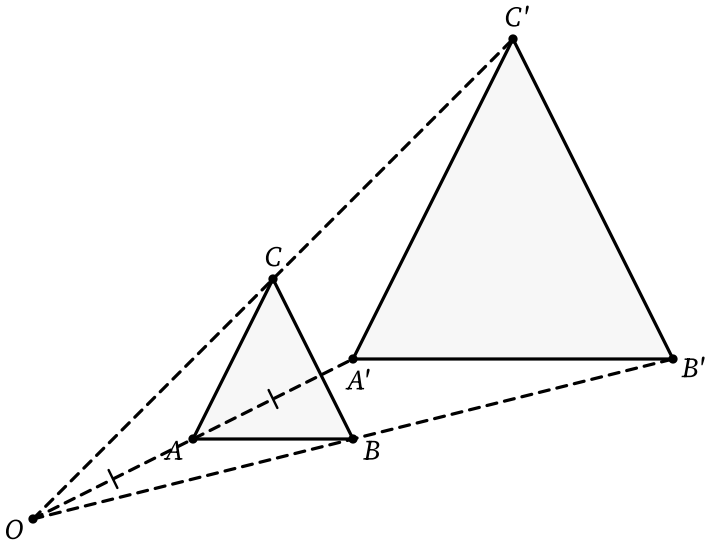
With the center at the origin the rule is $(x, y) \to (kx, ky)$. WHY: the point (ka, kb) has slope $\frac{kb}{ka} = \frac{b}{a}$ from the origin, so it lies on the same line through O as (a, b), and its distance from O is $\sqrt{k^2a^2 + k^2b^2} = |k|\sqrt{a^2 + b^2}$. When the center C is elsewhere, work relative to C: subtract C, multiply by k, then add C back.

A dilation multiplies every length by $|k|$, not just distances from the center. The reason is that the differences $x_2 - x_1$ and $y_2 - y_1$ in the distance formula are all multiplied by k, so $A'B' = |k| \cdot AB$. The same differences give the slope, and the factor k cancels, so each segment's image has the same slope: **a dilation maps every line to a parallel line**, or to the same line if the line passes through the center.

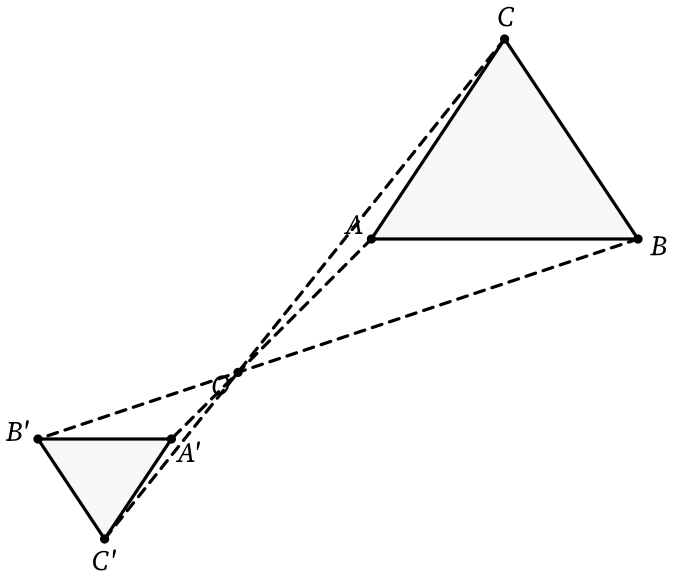
Because each side of an angle keeps its direction, a dilation preserves angle measure (Theorem 3). So the image is the same shape as the preimage, with every length multiplied by $|k|$. A dilation is not a rigid motion unless $|k| = 1$. In Unit 8 you will see that a dilation followed by rigid motions is exactly what it means for two figures to be similar.

A negative scale factor sends each point through the center to the other side: with $k = -2$, the point P goes to the point twice as far from O on the opposite ray. The rule $(x, y) \to (kx, ky)$ still works, and lengths are multiplied by $|k| = 2$. A dilation with $k = -1$ is the same as a rotation of 180° about the center.

Maps, blueprints and scale models are dilations. A blueprint drawn at "$\frac{1}{4}$ inch = 1 foot" has scale factor $\frac{1}{4} \div 12 = \frac{1}{48}$, since 1 foot is 12 inches. Every length on the drawing is $\frac{1}{48}$ of the real length, and every angle on the drawing equals the real angle, which is why a carpenter can measure angles straight from the plan.



Scale factor 2: each point goes twice as far from O along its ray.



Scale factor −1/2: each point goes through O to the other side, half as far.

POSTULATES AND THEOREMS

THEOREM **Dilation Rule for Center at the Origin**

The dilation with center at the origin and scale factor k maps every point (x, y) to (kx, ky).

PARAGRAPH PROOF

Given: $P(a, b)$ is a point other than the origin O; Q is the point (ka, kb), $k \neq 0$

Prove: Q is the image of P under the dilation with center O and scale factor k

Distance: by the distance formula, $OQ = \sqrt{(ka)^2 + (kb)^2} = \sqrt{k^2(a^2 + b^2)} = |k|\sqrt{a^2 + b^2} = |k| \cdot OP$.

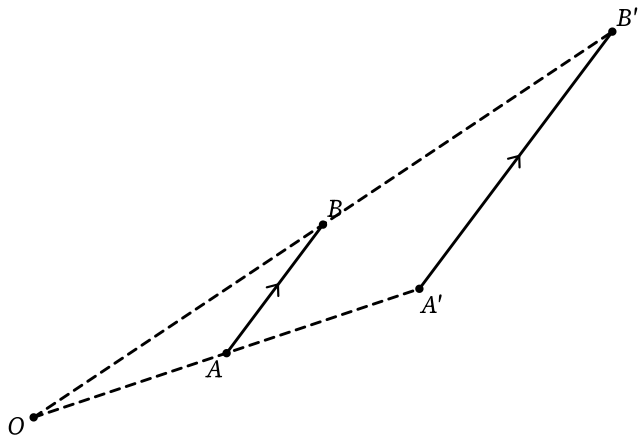
Line: if $a \neq 0$, the slope of $\overline{OQ}$ is $\dfrac{kb}{ka} = \dfrac{b}{a}$, the slope of $\overline{OP}$; two lines through O with the same slope are the same line. If $a = 0$, both P and Q are on the y-axis. Either way Q is on line OP.

Ray: if $k > 0$, then ka and kb have the same signs as a and b, so Q is on the same side of O as P, on $\overrightarrow{OP}$. If $k < 0$, the signs are reversed, so Q is on the opposite ray. These are exactly the conditions in the definition of the dilation, so Q is the image of P.

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THEOREM **Dilation Theorem**

A dilation with scale factor k maps every segment $\overline{AB}$ to a segment $\overline{A'B'}$ with $A'B' = |k| \cdot AB$, and maps line AB to a line parallel to it or to the same line.



TWO-COLUMN PROOF

Given: The dilation with center at the origin and scale factor k maps $A(x_1, y_1)$ to A' and $B(x_2, y_2)$ to B'

Prove: $A'B' = |k| \cdot AB$, and $\overleftrightarrow{A'B'}$ is parallel to $\overleftrightarrow{AB}$ or is the same line

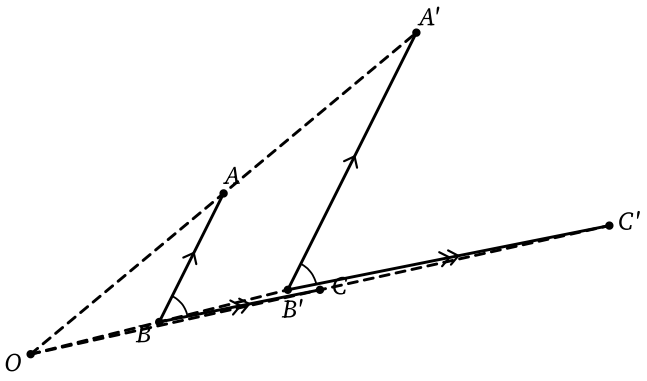
	STATEMENTS	REASONS
1	The dilation with center at the origin and scale factor k maps $A(x_1, y_1)$ to A' and $B(x_2, y_2)$ to B'	Given
2	$A' = (kx_1,\ ky_1)$ and $B' = (kx_2,\ ky_2)$	Dilation Rule for Center at the Origin
3	$A'B' = \sqrt{(kx_2 - kx_1)^2 + (ky_2 - ky_1)^2}$	Distance Formula
4	$A'B' = \sqrt{k^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]}$	Distributive Property; $(kd)^2 = k^2d^2$
5	$A'B' = \lvert k\rvert\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \lvert k\rvert \cdot AB$	$\sqrt{k^2} = \lvert k\rvert$; Distance Formula
6	If $x_1 \neq x_2$: slope of $\overline{A'B'} = \dfrac{k(y_2 - y_1)}{k(x_2 - x_1)} = \dfrac{y_2 - y_1}{x_2 - x_1}$ = slope of $\overline{AB}$; if $x_1 = x_2$, both segments are vertical	Slope Formula; divide out $k \neq 0$
7	$A'B' = \lvert k\rvert \cdot AB$, and $\overleftrightarrow{A'B'}$ is parallel to $\overleftrightarrow{AB}$ or is the same line	Row 5; lines with equal slopes, or two vertical lines, are parallel or the same line (Lesson 3.5)

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We put the center at the origin. Any dilation can be described this way by choosing axes with the origin at its center, and lengths and parallelism do not depend on where the axes are placed.

THEOREM **Dilations Preserve Angle Measure**

A dilation maps every angle to a congruent angle.



PARAGRAPH PROOF

Given: The dilation with center O and scale factor $k > 0$ maps A, B, C (not collinear) to A', B', C'

Prove: $\angle A'B'C' \cong \angle ABC$

Place the origin at O, so the dilation multiplies every coordinate by k. Then the move from B' to A' is k times the move from B to A: in vector form, $A' - B' = k(A - B)$. Likewise $C' - B' = k(C - B)$.

Now translate by the vector from B to B'. This translation maps B to B', A to $A_1 = B' + (A - B)$, and C to $C_1 = B' + (C - B)$, so it maps $\angle ABC$ to $\angle A_1B'C_1$. Since $A' = B' + k(A - B)$ with $k > 0$, the point A' lies on $\overrightarrow{B'A_1}$, so $\overrightarrow{B'A'}$ and $\overrightarrow{B'A_1}$ are the same ray. In the same way $\overrightarrow{B'C'}$ is $\overrightarrow{B'C_1}$. So $\angle A_1B'C_1$ is $\angle A'B'C'$.

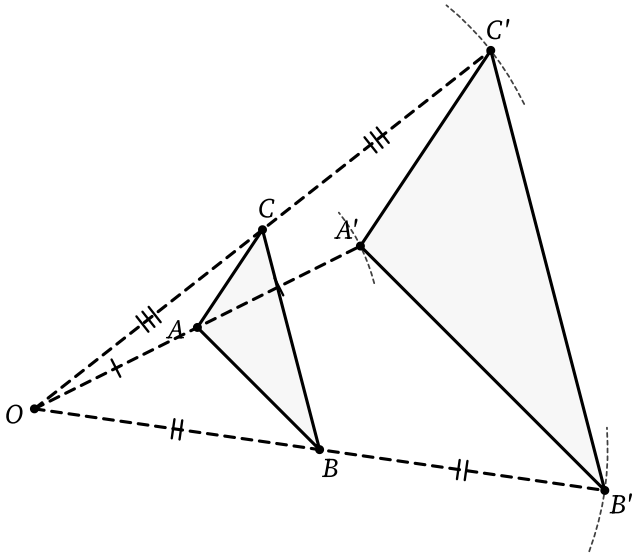
Translations preserve angle measure (Lesson 7.1), so $\angle A'B'C' \cong \angle ABC$.

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If $k < 0$, the dilation by k is the dilation by $|k|$ followed by the rotation of $180°$ about the center, because $(x, y) \to (|k|x, |k|y) \to (-|k|x, -|k|y) = (kx, ky)$. Both preserve angle measure, so the composition does too.

CONSTRUCTIONS

CONSTRUCTION **Dilate a triangle with scale factor 2 (or 1/2)** Euclid I.3



1. Given $\triangle ABC$ and a center O, draw rays from O through A, through B and through C.
2. Set the compass to the length OA. Put the point on A and mark A' on $\overrightarrow{OA}$ beyond A, so that $AA' = OA$ (Euclid I.3).
3. Do the same for B and C to mark B' and C'.
4. Draw $\triangle A'B'C'$. It is the image of $\triangle ABC$ under the dilation with center O and scale factor

2. For scale factor $\frac{1}{2}$, instead bisect $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ (Euclid I.10) and join the midpoints.

Why it works. A' is on $\overrightarrow{OA}$ with A between O and A', so by the Segment Addition Postulate $OA' = OA + AA' = OA + OA = 2 \cdot OA$. That is the definition of the dilation with center O and scale factor 2, and the same holds for B' and C'. (For $\frac{1}{2}$, the midpoint M of $\overline{OA}$ is on $\overrightarrow{OA}$ with $OM = \frac{1}{2}OA$.)

KEY FORMULAS

$$(x, y) \to (kx,\ ky) \quad \text{(center at the origin)}$$

$$A'B' = |k| \cdot AB$$

$$k = \frac{\text{image length}}{\text{preimage length}} \quad (k > 0)$$

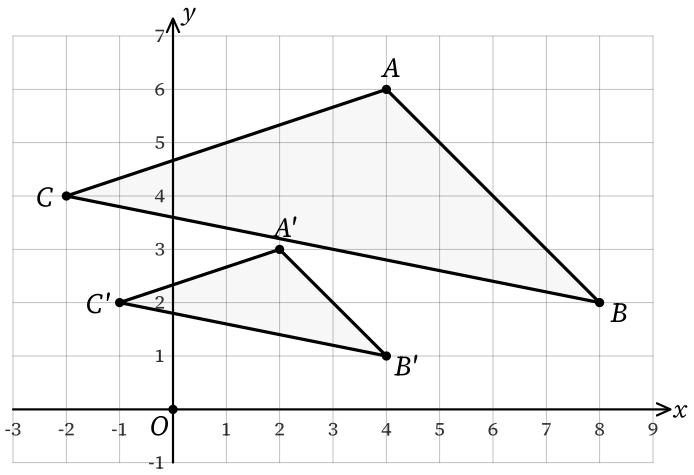
From history. The pantograph is a hinged parallelogram of rods used to copy a drawing at a larger or smaller size. It is credited to the astronomer Christoph Scheiner, who described it in a book published in 1631. One point of the instrument is pinned to the table as the center; as one point traces the original drawing, a pencil at another point draws its dilation.

Euclid connection. Euclid has no dilations, but Book VI on proportion treats the same idea; VI.2 states that a straight line drawn parallel to one side of a triangle cuts the other two sides proportionally, which is the heart of the similarity work in Unit 8. Constructing a dilation uses I.3 (cutting off a segment equal to a given one) and I.10 (bisecting a segment).

Practice habit. Always measure from the center, not from the figure: subtract the center, multiply by k, then add the center back. Before you finish, estimate: the image should be larger if $|k| > 1$ and smaller if $|k| < 1$, on the same side of the center if $k > 0$ and on the opposite side if $k < 0$.

WORKED EXAMPLES

Example 1. Dilate $\triangle ABC$ with $A(4, 6)$, $B(8, 2)$ and $C(-2, 4)$ using center at the origin and scale factor $k = \dfrac{1}{2}$. Then compare $A'B'$ with AB.

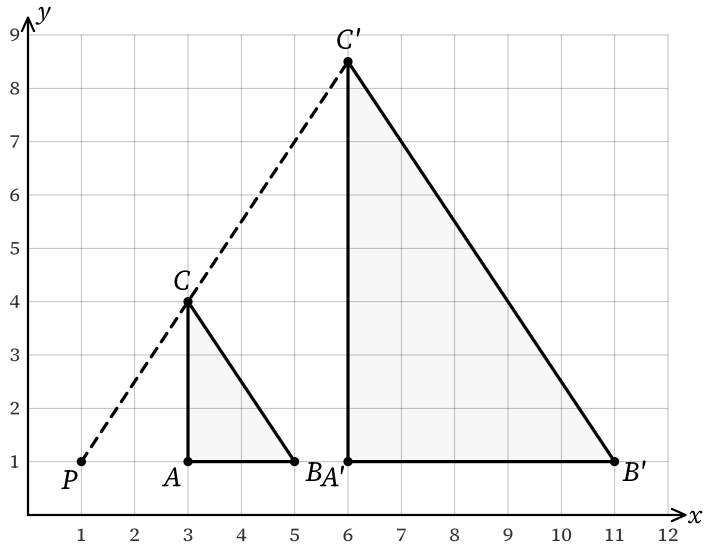


Show solution

1. Multiply each coordinate by $\dfrac{1}{2}$: $A'(2, 3)$, $B'(4, 1)$, $C'(-1, 2)$.
2. $AB = \sqrt{(8-4)^2 + (2-6)^2} = \sqrt{32} = 4\sqrt{2}$ and $A'B' = \sqrt{(4-2)^2 + (1-3)^2} = \sqrt{8} = 2\sqrt{2}$.
3. So $A'B' = \dfrac{1}{2} \cdot AB$, as the Dilation Theorem promises. Both segments have slope -1, so $\overline{A'B'} \parallel \overline{AB}$. This is a reduction.

Answer: $A'(2, 3)$, $B'(4, 1)$, $C'(-1, 2)$; $A'B' = 2\sqrt{2}$, half of $AB = 4\sqrt{2}$

Example 2. Grace is designing a stained-glass window. On her grid, in inches, a small triangular pane has vertices $A(3,1)$, $B(5,1)$ and $C(3,4)$. She wants a larger pane that is the dilation of this one with center $P(1,1)$ and scale factor 2.5. (a) Find the vertices of the larger pane. (b) Find the perimeter of each pane to the nearest tenth of an inch.



Show solution

1. Work relative to the center $P(1,1)$. A is $\langle 2,0\rangle$ from P; times 2.5 gives $\langle 5,0\rangle$, so $A'(6,1)$. B is $\langle 4,0\rangle$ from P, giving $\langle 10,0\rangle$ and $B'(11,1)$. C is $\langle 2,3\rangle$ from P, giving $\langle 5,7.5\rangle$ and $C'(6,8.5)$.
2. Small pane: $AB = 2$, $AC = 3$, $BC = \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.606$. Perimeter $= 5 + \sqrt{13} \approx 8.606$, about 8.6 in.
3. Large pane: every length is multiplied by 2.5, so the perimeter is $2.5(5 + \sqrt{13}) \approx 21.514$, about 21.5 in. Check directly: $A'B' = 5$, $A'C' = 7.5$, $B'C' = \sqrt{5^2 + 7.5^2} \approx 9.014$, and $5 + 7.5 + 9.014 = 21.514$.

Answer: (a) $A'(6,1)$, $B'(11,1)$, $C'(6,8.5)$ (b) about 8.6 in and 21.5 in

PRACTICE

1. **Easy** Dilate the point $(4, -6)$ with center at the origin and scale factor (a) $\frac{1}{2}$, (b) 3.

   Show answer

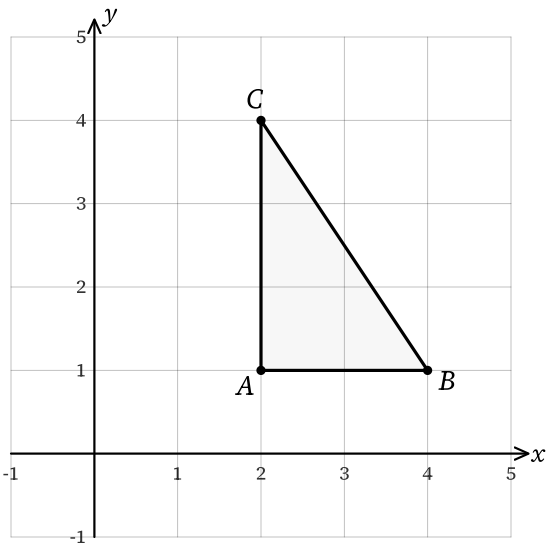
   (a) $(2, -3)$ (b) $(12, -18)$

2. **Easy** A dilation with a positive scale factor maps a segment 6 cm long to a segment 15 cm long. (a) What is the scale factor? (b) Is the dilation an enlargement or a reduction?

   Show answer

   (a) $k = \frac{15}{6} = 2.5$ (b) an enlargement

3. **Easy** In the figure, dilate $\triangle ABC$ with $A(2, 1)$, $B(4, 1)$ and $C(2, 4)$ using center at the origin and scale factor 3. Give the image vertices.



   Show answer

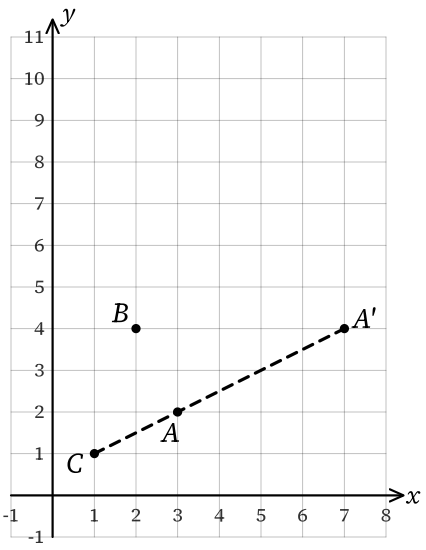
   $A'(6, 3)$, $B'(12, 3)$, $C'(6, 12)$

4. **Easy** A triangle has perimeter 18 cm and one angle of 50°. It is dilated with scale factor $\frac{2}{3}$. (a) What is the perimeter of the image? (b) What is the measure of the image of the 50° angle?

Show answer

(a) $\frac{2}{3} \times 18 = 12$ cm (b) 50°, because dilations preserve angle measure

5. **Medium** In the figure, a dilation with center $C(1, 1)$ maps $A(3, 2)$ to $A'(7, 4)$. (a) Find the scale factor. (b) Find the image of $B(2, 4)$.



Show answer

(a) Relative to C, A is $\langle 2, 1\rangle$ and A' is $\langle 6, 3\rangle$, so $k = 3$. (b) B is $\langle 1, 3\rangle$ from C; times 3 gives $\langle 3, 9\rangle$, so $B'(4, 10)$.

6. **Medium** The blueprint for a church fellowship hall uses the scale $\frac{1}{4}$ inch = 1 foot. The hall is a rectangle 60 feet long and 36 feet wide. (a) What is the scale factor of the blueprint, as a fraction? (b) What are the length and width of the hall on the blueprint, in inches?

Show answer

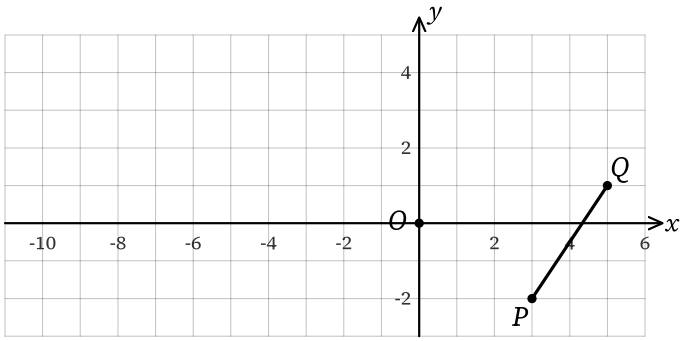
(a) $\frac{1}{4}$ in for every 12 in, so $k = \frac{1}{48}$ (b) $60 \times \frac{1}{4} = 15$ in long and $36 \times \frac{1}{4} = 9$ in wide

7. **Medium** Ruth enlarges a 4 in by 6 in family photograph with scale factor 2.5. What are the length and width of the enlargement? Are its corners still right angles?

Show answer

$2.5 \times 4 = 10$ in by $2.5 \times 6 = 15$ in; yes, the corners are still 90° because dilations preserve angle measure

8. **Medium** In the figure, segment $\overline{PQ}$ has endpoints $P(3, -2)$ and $Q(5, 1)$. Dilate it with center at the origin and scale factor -2. (a) Find P' and Q'. (b) Find $P'Q'$ in simplest radical form. (c) Is $\overline{P'Q'} \parallel \overline{PQ}$?



Show answer

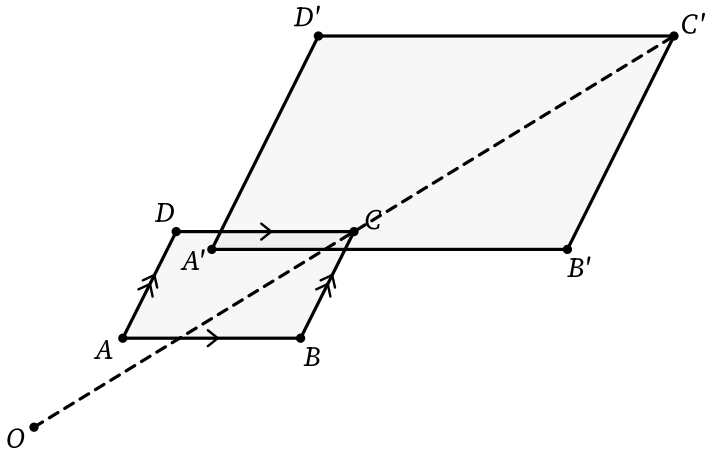
(a) $P'(-6, 4)$, $Q'(-10, -2)$ (b) $P'Q' = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$, twice $PQ = \sqrt{13}$ (c) Yes: both slopes are $\dfrac{3}{2}$.

9. **Medium** Explain why a dilation with scale factor 3 is not a rigid motion, even though it preserves angle measure. Which dilations are rigid motions?

Show answer

A rigid motion must preserve distance, but a dilation with scale factor 3 multiplies every distance by 3 (for example, a segment of length 2 becomes length 6). Only dilations with $|k| = 1$ preserve distance: $k = 1$ leaves every point where it is, and $k = -1$ is the rotation of 180° about the center.

10. Hard Given: $ABCD$ is a parallelogram, and a dilation with scale factor k maps A, B, C, D to the vertices of quadrilateral $A'B'C'D'$, as in the figure. Prove with a flow proof: $A'B'C'D'$ is a parallelogram.



Show answer

See the model proof.

FLOW PROOF

Given: $ABCD$ is a parallelogram; A dilation with scale factor k maps A, B, C, D to the vertices A', B', C', D' of a quadrilateral

Prove: $A'B'C'D'$ is a parallelogram

(1) $ABCD$ is a parallelogram; a dilation with scale factor k maps A, B, C, D to the vertices of quadrilateral $A'B'C'D'$
Given

(2) $AB = CD$ and $BC = DA$
Opposite sides of a parallelogram are congruent (Lesson 6.2)
from (1)

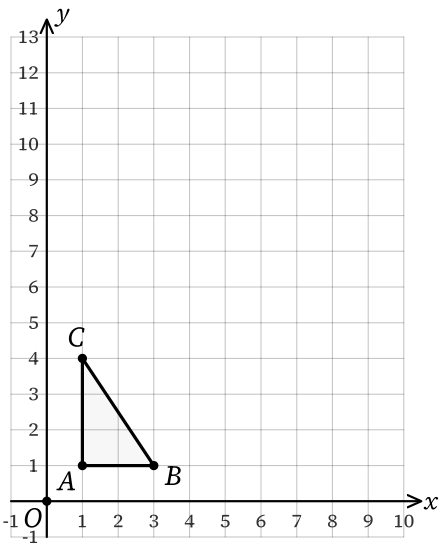
(3) $A'B' = |k| \cdot AB$, $B'C' = |k| \cdot BC$, $C'D' = |k| \cdot CD$, $D'A' = |k| \cdot DA$
Dilation Theorem
from (1)

(4) $A'B' = C'D'$ and $B'C' = D'A'$
Substitution Property of Equality
from (2), (3)

(5) $A'B'C'D'$ is a parallelogram
If both pairs of opposite sides of a quadrilateral are congruent, it is a parallelogram (Lesson 6.3)

from (4)

11. **Hard** Samuel uses a pantograph to enlarge a drawing with scale factor 3; the pivot is the center, placed at the origin of a grid in inches. He traces a right triangle with vertices $(1, 1)$, $(3, 1)$ and $(1, 4)$. (a) Give the vertices of the enlarged triangle. (b) Find the area of the original and of the enlargement. (c) How many times as large is the enlargement's area?



Show answer

(a) $(3, 3)$, $(9, 3)$, $(3, 12)$ (b) Original legs 2 and 3: area $\frac{1}{2}(2)(3) = 3$ sq in. Enlargement legs 6 and 9: area $\frac{1}{2}(6)(9) = 27$ sq in. (c) 9 times, which is 3^2: each of the two dimensions is tripled.

12. Hard Given: a dilation with center O and scale factor k, where $k \neq 1$. Prove indirectly: no point other than O is mapped to itself.

Show answer

See the model proof.

INDIRECT PROOF

Given: A dilation with center O and scale factor k, $k \neq 0$ and $k \neq 1$

Prove: No point $P \neq O$ is mapped to itself

Assume the opposite: some point $P \neq O$ has $P' = P$.

By the definition of dilation, $OP' = |k| \cdot OP$, so $OP = |k| \cdot OP$. Since $P \neq O$, $OP > 0$, and dividing by OP gives $|k| = 1$. So $k = 1$ or $k = -1$.

$k = 1$ is ruled out by the given. If $k = -1$, then P' lies on the ray opposite $\overrightarrow{OP}$; opposite rays share only the point O, and $P \neq O$, so $P' \neq P$. Either way we contradict $P' = P$.

So the assumption is false: the only point a dilation with $k \neq 1$ leaves fixed is its center.

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13. Hard Given: the dilation with center at the origin and scale factor k maps $A(x_1, y_1)$ to A', $B(x_2, y_2)$ to B', and M to M', where M is the midpoint of $\overline{AB}$. Prove: M' is the midpoint of $\overline{A'B'}$.

Show answer

See the model proof.

TWO-COLUMN PROOF

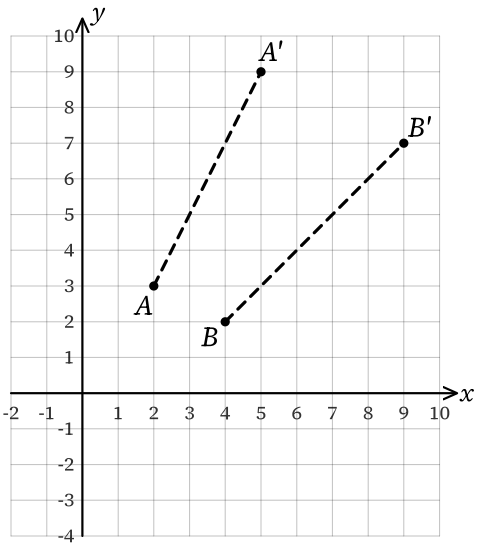
Given: The dilation with center at the origin and scale factor k maps $A(x_1, y_1)$ to A', $B(x_2, y_2)$ to B', and M to M'; M is the midpoint of $\overline{AB}$

Prove: M' is the midpoint of $\overline{A'B'}$

	STATEMENTS	REASONS
1	M is the midpoint of $\overline{AB}$; the dilation maps A, B, M to A', B', M'	Given
2	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	Midpoint Formula
3	$M' = \left(\frac{k(x_1 + x_2)}{2}, \frac{k(y_1 + y_2)}{2}\right)$	Dilation Rule for Center at the Origin
4	$A' = (kx_1,\ ky_1)$ and $B' = (kx_2,\ ky_2)$	Dilation Rule for Center at the Origin
5	The midpoint of $\overline{A'B'}$ is $\left(\frac{kx_1 + kx_2}{2}, \frac{ky_1 + ky_2}{2}\right) = \left(\frac{k(x_1 + x_2)}{2}, \frac{k(y_1 + y_2)}{2}\right)$	Midpoint Formula; Distributive Property
6	M' is the midpoint of $\overline{A'B'}$	Substitution Property of Equality (rows 3 and 5 give the same point)

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14. **Challenge** In the figure, a dilation maps $A(2, 3)$ to $A'(5, 9)$ and $B(4, 2)$ to $B'(9, 7)$. Find the center and the scale factor. (Hint: each point and its image lie on a line through the center.)



Show answer

Line AA' has slope 2: $y = 2x - 1$. Line BB' has slope 1: $y = x - 2$. They meet where $2x - 1 = x - 2$, at $C(-1, -3)$. From C, A is $\langle 3, 6\rangle$ and A' is $\langle 6, 12\rangle$, so $k = 2$. (Check: B is $\langle 5, 5\rangle$ from C and B' is $\langle 10, 10\rangle$.)

15. **Challenge** A dilation with center at the origin maps the line $y = 2x + 3$ onto the line $y = 2x + 12$. Find the scale factor.

Show answer

The point $(0, 3)$ goes to $(0, 3k)$, which must lie on $y = 2x + 12$: $3k = 12$, so $k = 4$. Check: $(x,\ 2x + 3) \to (4x,\ 8x + 12)$, and $2(4x) + 12 = 8x + 12$, so every image point is on the second line.

COMMON MISTAKES

- Multiplying the coordinates by k when the center is not the origin. Subtract the center first, multiply, then add the center back (Practice 5).
- Writing the scale factor upside down. k is image length divided by preimage length; an enlargement must have $|k| > 1$.
- Forgetting that a negative scale factor puts the image on the other side of the center. Lengths are multiplied by $|k|$, never by a negative number.

- Multiplying areas by k. Lengths are multiplied by $|k|$, but areas are multiplied by k^2, as Practice 11 shows.

> **Make it a game · Zoom In, Zoom Out.** Parent names a triangle on graph paper and a scale factor: 3, then 1/2. Write the image coordinates for a dilation centered at the origin within 20 seconds. Then parent draws a figure and its image, and you find the center by drawing lines through corresponding points, and state the scale factor and whether it is an enlargement.

UNIT 7

Transformations

In this unit geometry starts to move. A transformation is a rule that sends every point of the plane to an image point. You will study the rigid motions, which are translations (slides), reflections (flips) and rotations (turns), and prove that each one preserves distance and angle measure. Along the way you will meet symmetry in letters, regular polygons, Gothic church windows and rose windows, and Heron of Alexandria's reflection trick for finding the shortest path.

Next you will combine motions. Two reflections make a translation or a rotation, and footprints in the snow make a glide reflection. This leads to a precise meaning for congruence: two figures are congruent when rigid motions carry one onto the other. That puts into modern language the way Euclid "applied" one triangle to another in Book I. The unit ends with dilations, which keep shape but change size. They are the mathematics of maps, blueprints and scale models, and they prepare the way for similarity in Unit 8.

Parents: most of the work is on the coordinate plane with simple rules such as $(x, y) \to (-y, x)$, together with short proofs that use the congruence tests of Unit 4 and the parallelogram facts of Unit 6. Graph paper, a compass and a straightedge are all the tools needed.

PREREQUISITES

- Plot points and use the distance, midpoint and slope formulas (Algebra I; reviewed in Lessons 2.2 and 3.5).
- Recognize parallel and perpendicular lines from their slopes (Lesson 3.5).
- Prove triangles congruent with SSS, SAS, ASA and AAS, and use CPCTC (Lessons 4.3–4.5).
- Use the Perpendicular Bisector Theorem and its converse (Lesson 5.1).
- Use the properties of parallelograms and the tests for a parallelogram (Lessons 6.2–6.3).
- Copy a segment, copy an angle and drop a perpendicular with compass and straightedge (Lessons 2.2 and 2.5).

MASTERY CHECKLIST

- ☐ I can translate a figure by a vector $\langle a, b \rangle$ and find the vector from a point and its image.
- ☐ I can explain why a transformation that preserves distance also preserves angle measure.
- ☐ I can reflect points in the x-axis, the y-axis, the line $y = x$ and other lines, and construct a reflection with compass and straightedge.
- ☐ I can find lines of symmetry and use the reflection trick to find a shortest path that touches a line.
- ☐ I can rotate points through $90°$, $180°$ and $270°$ about the origin or another center

er, and prove that rotations preserve distance.

- ☐ I can find the order and angle of rotational symmetry of a regular polygon, a wheel or a rose window.
- ☐ I can find the image under a composition of rigid motions, and I know that two reflections make a translation or a rotation through twice the angle between the mirrors.
- ☐ I can describe a sequence of rigid motions that maps one congruent figure onto another, and explain how this connects to Euclid's proofs of I.4 and I.8.
- ☐ I can dilate a figure with a given center and scale factor, and find

the center and scale factor from a figure and its image.

- ☐ I can explain why a dilation multiplies lengths by $|k|$, keeps angle measures and maps lines to parallel lines, and I can use scale factors with maps and blueprints.

LESSONS IN THIS UNIT

7.1	**Translations and Vectors**	45 minutes
7.2	**Reflections and Symmetry**	45 minutes
7.3	**Rotations and Rotational Symmetry**	2 sessions of 45 minutes
7.4	**Compositions and Congruence**	2 sessions of 45 minutes
7.5	**Dilations**	2 sessions of 45 minutes

Unit 7 Test

Allow 45–60 minutes. No calculator is needed; give exact answers, in simplest radical form where needed, unless a problem says otherwise. Show your work for every computation. In each proof, give a reason for every statement. Problem 15 needs a compass and a straightedge. Label units in word problems.

1. **5 pts**
   Find the image of the point $(5, -1)$ under the translation by the vector $\langle -3, 8 \rangle$.

2. **5 pts**
   A translation maps $A(-4, 2)$ to $A'(3, -1)$. (a) Write the vector of the translation. (b) Find the image of $B(1, 1)$.

3. **6 pts**
   Reflect the point $(-2, 7)$ in (a) the x-axis, (b) the y-axis, (c) the line $y = x$.

4. **6 pts**

Rotate the point $(-3, 4)$ about the origin through (a) 90°, (b) 180°, (c) 270°.

5. **6 pts**

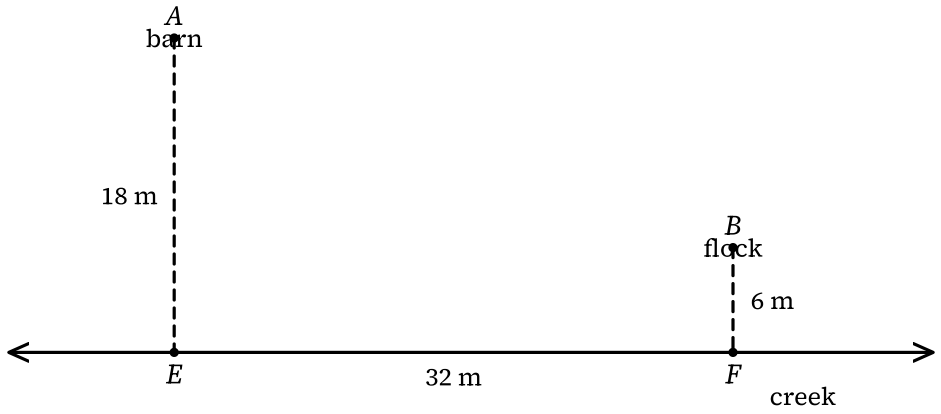
The stone tracery of a church's rose window is laid out on a regular 10-sided polygon. (a) How many lines of symmetry does a regular 10-sided polygon have? (b) What is the smallest angle of rotation that maps it onto itself?

6. **7 pts**

Segment $\overline{AB}$ has endpoints $A(-2, 4)$ and $B(6, 0)$. Dilate it with center at the origin and scale factor $\dfrac{3}{2}$. Find A' and B', and find $A'B'$ in simplest radical form.

7. **8 pts**

A sheepdog runs from the barn to a straight creek for a drink and then on to the flock. On a plan in meters, the creek is the x-axis, the barn is at $A(0, 18)$ and the flock is at $B(32, 6)$, as in the figure. (a) At what point on the creek should the dog drink to make its run as short as possible? (b) How long is that run?



8. **7 pts**

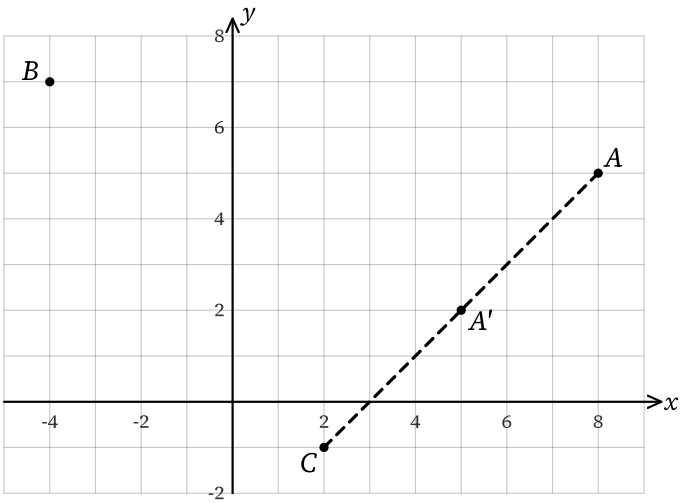
Reflect $P(2, 5)$ in the line $x = -1$, and then reflect the image in the line $x = 3$. (a) Give the final image. (b) Write the vector of the single translation that does the same job.

9. **6 pts**

Lines ℓ and m meet at point O. The reflection in ℓ followed by the reflection in m is a rotation about O. Through what angle does it rotate if the angle from ℓ to m is (a) 40°, (b) 90°?

10. **7 pts**

In the figure, a dilation with center $C(2, -1)$ maps $A(8, 5)$ to $A'(5, 2)$. (a) Find the scale factor. (b) Find the image of $B(-4, 7)$.



11. **6 pts**

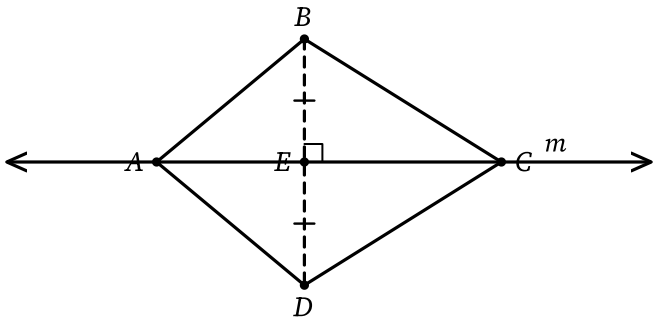
Henry builds a model of the family's barn at a scale factor of $\frac{1}{24}$. The real barn's roof ridge is 30 feet high and 48 feet long. How high and how long is the ridge on the model, in inches?

12. **6 pts**

Ruth says a dilation with scale factor 2 is a rigid motion because the image has the same shape as the original. Is she right? Explain why or why not.

13. **10 pts**

Given: line m is the perpendicular bisector of $\overline{BD}$, and A and C are different points on m, as in the figure. Prove: $\triangle ABC \cong \triangle ADC$.

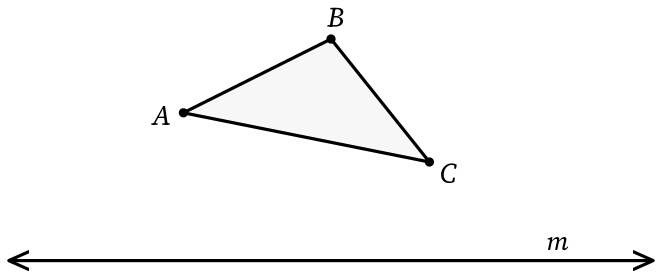


14. **7 pts**

Given: a dilation with center O and scale factor $k \neq 0$ maps A to A' and B to B', where $A \neq B$. Prove indirectly: $A' \neq B'$.

15. **8 pts**

Trace the figure, which shows $\triangle ABC$ and line m. With compass and straightedge, construct the image of $\triangle ABC$ under the reflection in m. Write the steps you used for vertex A, and explain why your construction works.



ANSWER KEY

1. $(2, 7)$
2. (a) $\langle 7, -3 \rangle$ (b) $B'(8, -2)$
3. (a) $(-2, -7)$ (b) $(2, 7)$ (c) $(7, -2)$
4. (a) $(-4, -3)$ (b) $(3, -4)$ (c) $(4, 3)$
5. (a) 10 (b) $\dfrac{360^\circ}{10} = 36^\circ$
6. $A'(-3, 6)$, $B'(9, 0)$; $A'B' = \sqrt{12^2 + 6^2} = \sqrt{180} = 6\sqrt{5}$ (which is $\dfrac{3}{2}$ of $AB = 4\sqrt{5}$)
7. (a) Reflect the barn to $A'(0, -18)$. Line $A'B$ has slope $\dfrac{24}{32} = \dfrac{3}{4}$: $y = \dfrac{3}{4}x - 18$, which meets the creek at $(24, 0)$. (b) $A'B = \sqrt{32^2 + 24^2} = 40$ m (check: $30 + 10 = 40$)
8. (a) $(2, 5) \to (-4, 5) \to (10, 5)$ (b) $\langle 8, 0 \rangle$, twice the distance 4 between the lines
9. (a) 80° (b) 180°
10. (a) Relative to C, A is $\langle 6, 6 \rangle$ and A' is $\langle 3, 3 \rangle$, so $k = \dfrac{1}{2}$. (b) B is $\langle -6, 8 \rangle$ from C; half of that is $\langle -3, 4 \rangle$, so $B'(-1, 3)$.
11. 30 ft $= 360$ in, and $360 \div 24 = 15$ in high; 48 ft $= 576$ in, and $576 \div 24 = 24$ in long
12. No. A rigid motion must preserve distance, but this dilation doubles every length; a segment of length 3 becomes length 6. The dilation does keep the shape, because it preserves angle measure, but the image is larger, so it is not congruent to the original. Only dilations with $|k| = 1$ are rigid motions.
13. See the model proof.

    TWO-COLUMN PROOF

    Given: Line m is the perpendicular bisector of $\overline{BD}$; A and C are different points on m

    Prove: $\triangle ABC \cong \triangle ADC$

	STATEMENTS	REASONS
1	m is the perpendicular bisector of $\overline{BD}$; A and C are different points on m	Given
2	The reflection in m maps B to D	Definition of reflection
3	The reflection in m maps A to A and C to C	Definition of reflection (points on the mirror line are fixed)

	STATEMENTS	REASONS
4	$AB = AD$ and $CB = CD$	Reflections preserve distance
5	$\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$	Definition of congruent segments
6	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
7	$\triangle ABC \cong \triangle ADC$	SSS

∎

14. See the model proof.

    INDIRECT PROOF

    Given: A dilation with center O and scale factor $k \neq 0$ maps A to A' and B to B'; $A \neq B$

    Prove: $A' \neq B'$

    Assume the opposite: $A' = B'$. Then $A'B' = 0$.
    By the Dilation Theorem, $A'B' = |k| \cdot AB$. Since $A \neq B$, $AB > 0$, and since $k \neq 0$, $|k| > 0$, so $|k| \cdot AB > 0$. Then $0 = A'B' > 0$, which is impossible.
    The assumption leads to a contradiction, so it is false. Therefore $A' \neq B'$.

    ∎

15. Steps for A: (1) With the compass point on A, draw an arc crossing m at two points X and Y. (2) With the same width, draw arcs centered at X and at Y on the other side of m; label their crossing A'. Repeat for B and C and join A', B', C'. Why: $XA = XA'$ and $YA = YA'$, so X and Y both lie on the perpendicular bisector of $\overline{AA'}$ (Converse of the Perpendicular Bisector Theorem). Two points determine a line, so m is the perpendicular bisector of $\overline{AA'}$, which is the definition of the reflection of A in m.

UNIT 8

Similarity

Congruent figures have the same size and shape; similar figures have the same shape but may differ in size. In this unit you will learn exactly what "same shape" means: corresponding angles congruent and corresponding sides proportional, or, in the language of Unit 7, a dilation followed by rigid motions. You will begin with the algebra of ratios and proportions, meet the geometric mean and the golden ratio, and then learn three short tests (AA, SSS and SAS) for proving triangles similar.

This unit follows Book VI of Euclid's Elements, where the theory of proportion from Book V is applied to figures: equiangular triangles have proportional sides (VI.4), a line parallel to a side of a triangle cuts the other sides proportionally (VI.2), an angle bisector divides the opposite side in the ratio of the other sides (VI.3), and the mean proportional is constructed in a semicircle (VI.13). You will write two-column, paragraph and flow proofs with these theorems and use compass and straightedge to construct a geometric mean and to divide a segment into equal parts.

Similarity is also one of the most practical ideas in geometry. With it you will find the height of a steeple from its shadow, the width of a river without crossing it, the real size of a room from a blueprint, and the frontage of lots between parallel streets. Parents: a student who masters this unit is ready for right-triangle trigonometry in Unit 9, which is built entirely on similar triangles.

PREREQUISITES

- Solve linear and quadratic equations, including equations with fractions (Algebra I).
- Use the Pythagorean Theorem numerically and simplify square roots (Algebra I).
- Use parallel-line angle relationships: corresponding, alternate interior and vertical angles (Lessons 2.4, 3.2, 3.3).
- Prove triangles congruent with SSS, SAS, ASA and AAS and use CPCTC (Lessons 4.3–4.5).
- Use the Isosceles Triangle Theorem and its converse and the Triangle Midsegment Theorem (Lessons 4.6, 5.4).
- Describe dilations and rigid motions and their effect on lengths and angles (Lessons 7.1 and 7.5).

MASTERY CHECKLIST

- ☐ I can simplify ratios, use extended ratios, and solve proportions with the Cross Products Property.
- ☐ I can prove the basic properties of proportions with the properties of equality.
- ☐ I can find the geometric mean of two numbers and construct it in a semicircle (Euclid VI.13).
- ☐ I can explain the golden ratio as Euclid's extreme and mean ratio and find its exact value.
- ☐ I can define similar polygons, find a scale factor, and find missing sides, angles and perimeters of similar figures.
- ☐ I can solve problems with maps, scale models and blueprints.
- ☐ I can state the AA Similarity Postulate and the SSS and SAS Similarity Theorems, and explain why SSA is not a test.
- ☐ I can use similar triangles for indirect measurement with shadows, mirrors and surveying.
- ☐ I can write two-column, paragraph and flow proofs that triangles are similar, and prove that products of lengths are equal.
- ☐ I can use the Triangle Proportionality Theorem and its converse, the Three Parallel Lines Theorem and the Triangle Angle Bisector Theorem.

- ☐ I can divide a segment into equal parts or in a given ratio with compass and straightedge (Euclid VI.9 and VI.10).

LESSONS IN THIS UNIT

8.1	**Ratios, Proportions and the Geometric Mean**	45 minutes
8.2	**Similar Polygons**	45 minutes
8.3	**Proving Triangles Similar: AA**	2 sessions of 45 minutes
8.4	**Proving Triangles Similar: SSS and SAS**	2 sessions of 45 minutes
8.5	**Proportions in Triangles**	2 sessions of 45 minutes

Unit 8 test

You have 50 minutes. No calculator is needed or allowed. Show your work for every problem. Give exact answers unless a problem asks you to round, and include units in answers to word problems. In every proof, give a reason for every statement. You will need a compass and a straightedge for problem 14. Figures are drawn to scale, but use only the given information and the marks on the figure in your reasoning.

1. **5 pts**
   Solve the proportion $\frac{x-1}{5} = \frac{x+5}{15}$.

2. **5 pts**
   Find the geometric mean of 8 and 18.

3. **5 pts**
   A rectangular stained-glass window for a church has width and height in the ratio $2:5$. Its perimeter is 28 ft. Find its width and its height.

4. **6 pts**

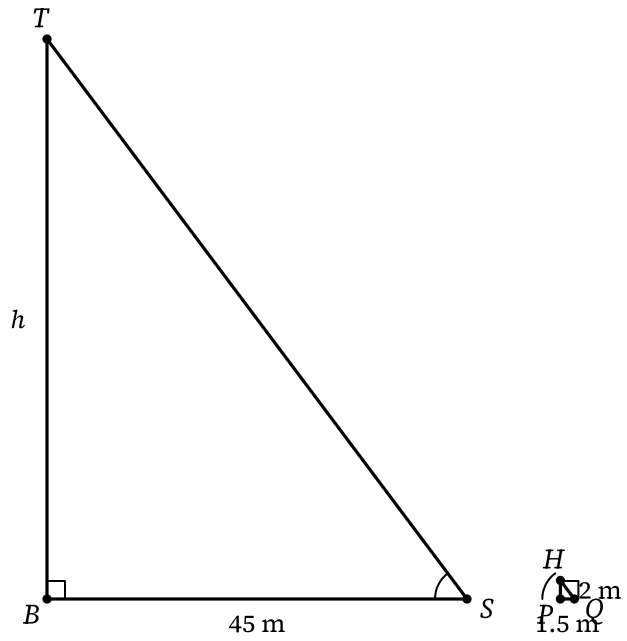
   Pentagon $ABCDE \sim$ pentagon $PQRST$. $AB = 12$, $PQ = 8$, $DE = 9$ and $m\angle C = 110°$. Find ST and $m\angle R$.

5. **5 pts**

   Henry builds a model of a covered bridge at a scale of $1 : 40$. The model is 2.5 ft long. (a) How long is the real bridge, in feet? (b) The real bridge is 16 ft tall. How tall is the model, in inches?

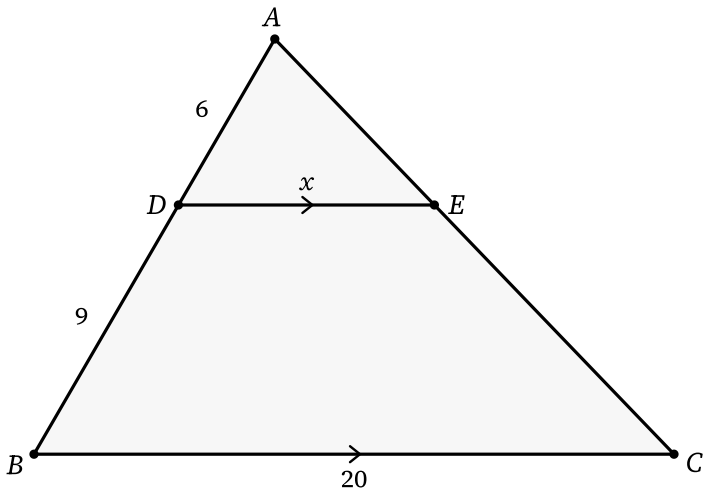
6. **7 pts**

   At the same moment, a lighthouse casts a shadow 45 m long and a 2-m post casts a shadow 1.5 m long, as in the figure. Both stand straight up on level ground. How tall is the lighthouse? Name the similarity test that justifies your proportion.



7. **5 pts**

In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = 6$, $DB = 9$ and $BC = 20$. Find DE.

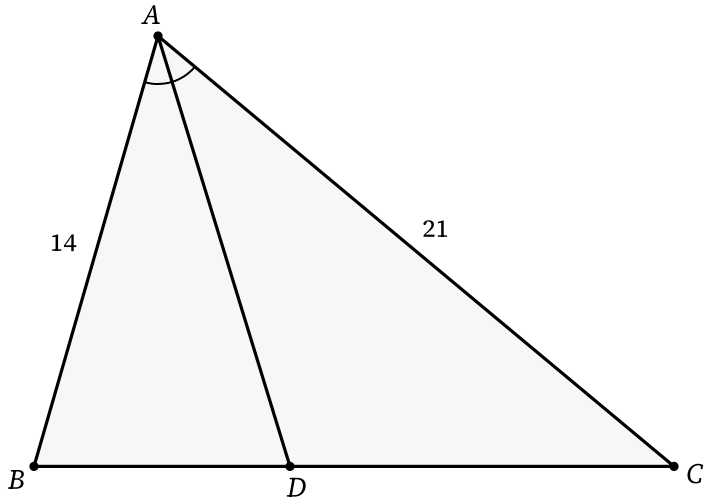


8. **7 pts**

One triangle has sides 6, 10 and 14; another has sides 9, 15 and 21. (a) Are the triangles similar? (b) If so, which test proves it? (c) What is the scale factor from the first triangle to the second?

9. **7 pts**

In the figure, $\overline{AD}$ bisects $\angle BAC$, $AB = 14$, $AC = 21$ and $BC = 20$. Find BD.



10. **7 pts**

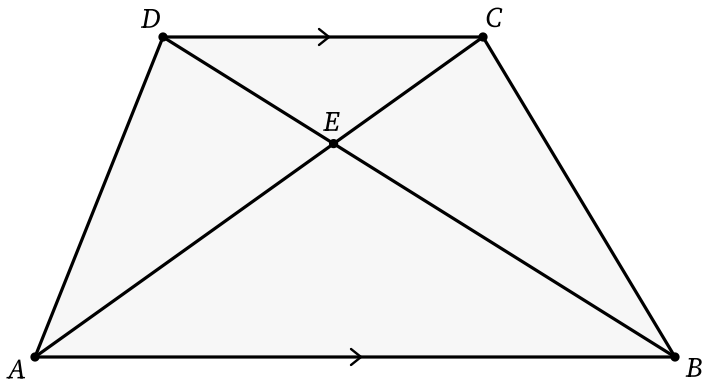
Three parallel streets cross First Avenue and Second Avenue. Along First Avenue, the two blocks between the streets measure 90 m and 135 m. Along Second Avenue, the two blocks between the same streets measure 300 m altogether. How long is each block along Second Avenue?

11. **6 pts**

Explain why any two isosceles right triangles are similar, but two isosceles triangles in general need not be similar. Give a specific example for the second part.

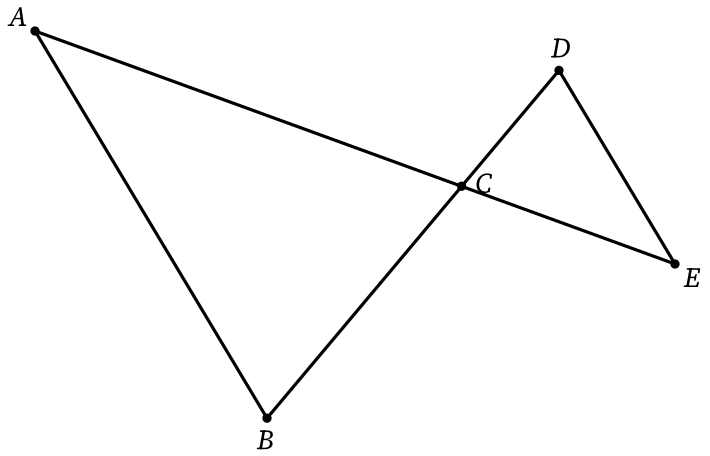
12. **10 pts**

In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and its diagonals $\overline{AC}$ and $\overline{BD}$ meet at E. Write a two-column proof that $AE \cdot DE = BE \cdot CE$.



13. **10 pts**

In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, $AC = 2 \cdot EC$ and $BC = 2 \cdot DC$. Write a paragraph proof that $AB = 2 \cdot DE$.

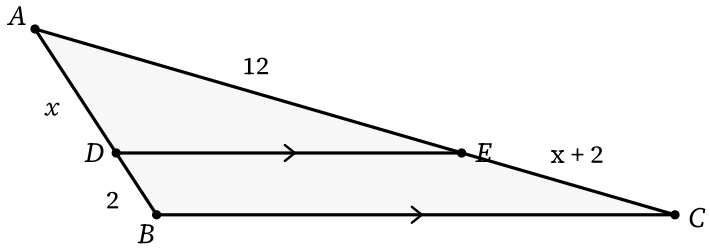


14. **8 pts**

Construction. Draw a segment 4 cm long and a segment 9 cm long. With compass and straightedge, construct a segment whose length is their geometric mean. List your steps, explain why the construction works, and state the length your segment should have.

15. **7 pts**

In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = x$, $DB = 2$, $AE = 12$ and $EC = x + 2$. Find x.



ANSWER KEY

1. $15(x-1) = 5(x+5)$, so $15x - 15 = 5x + 25$, $10x = 40$ and $x = 4$.
2. $\sqrt{8 \cdot 18} = \sqrt{144} = 12$
3. $2(2x + 5x) = 28$, so $14x = 28$ and $x = 2$: width 4 ft, height 10 ft.
4. The scale factor from $ABCDE$ to $PQRST$ is $\frac{8}{12} = \frac{2}{3}$, so $ST = \frac{2}{3} \cdot 9 = 6$; $\angle R$ corresponds to $\angle C$, so $m\angle R = 110°$.
5. (a) $2.5 \times 40 = 100$ ft. (b) $\frac{16}{40} = 0.4$ ft $= 4.8$ in.
6. The right angles are congruent and the sun's rays make congruent angles with the ground, so the triangles are similar by AA. $\frac{h}{2} = \frac{45}{1.5}$, so $h = 60$ m.
7. $\triangle ADE \sim \triangle ABC$ and $AB = 15$, so $\frac{DE}{20} = \frac{6}{15}$ and $DE = 8$.
8. (a) Yes. (b) SSS: $\frac{9}{6} = \frac{15}{10} = \frac{21}{14}$. (c) $\frac{3}{2}$.
9. $\frac{BD}{DC} = \frac{14}{21} = \frac{2}{3}$, so $BD = \frac{2}{5} \cdot 20 = 8$.
10. The parallel streets cut the avenues proportionally: $\frac{x}{300 - x} = \frac{90}{135} = \frac{2}{3}$, so $3x = 600 - 2x$ and $x = 120$. The blocks are 120 m and 180 m.
11. In an isosceles right triangle the base angles are congruent and complementary, so each is $45°$; any two such triangles have angles $90°$ and $45°$ in common, so they are similar by AA. A general isosceles triangle's base angles depend on its vertex angle: a triangle with angles $40°, 70°, 70°$ and one with angles $100°, 40°, 40°$ are both isosceles, but their angles are not congruent in pairs, so they are not similar.
12. See the model proof.

    TWO-COLUMN PROOF

    Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

    Prove: $AE \cdot DE = BE \cdot CE$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$	Given
2	$\angle BAE \cong \angle DCE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{AC}$)
3	$\angle ABE \cong \angle CDE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BD}$)

	STATEMENTS	REASONS
4	$\triangle ABE \sim \triangle CDE$	AA Similarity Postulate
5	$\dfrac{AE}{CE} = \dfrac{BE}{DE}$	Definition of similar polygons
6	$AE \cdot DE = BE \cdot CE$	Cross Products Property

∎

13. See the model proof.

PARAGRAPH PROOF

Given: $\overline{AE}$ and $\overline{BD}$ intersect at C; $AC = 2 \cdot EC$; $BC = 2 \cdot DC$

Prove: $AB = 2 \cdot DE$

Dividing the given equations (Division Property of Equality), $\dfrac{AC}{EC} = 2$ and $\dfrac{BC}{DC} = 2$, so $\dfrac{AC}{EC} = \dfrac{BC}{DC}$ by the Transitive Property of Equality. $\angle ACB \cong \angle ECD$ by the Vertical Angles Theorem, and these are the angles included between the proportional sides. So $\triangle ACB \sim \triangle ECD$ by the SAS Similarity Theorem.

Corresponding sides of similar triangles are proportional, so $\dfrac{AB}{ED} = \dfrac{AC}{EC} = 2$. By the Multiplication Property of Equality, $AB = 2 \cdot DE$.

∎

14. Lay the segments end to end on a line: $\overline{AD}$ = 4 cm and $\overline{DB}$ = 9 cm (copy segments, I.3). Bisect $\overline{AB}$ to find its midpoint M (I.10) and draw the semicircle with center M through A and B. Construct the perpendicular to $\overline{AB}$ at D (I.11), meeting the semicircle at C. Then $\overline{DC}$ is the geometric mean, because $\angle ACB$ is a right angle and $DC^2 = AD \cdot DB$ (the Mean Proportional in a Semicircle theorem, Euclid VI.13). Its length is $\sqrt{4 \cdot 9} = 6$ cm.
15. By the Triangle Proportionality Theorem, $\dfrac{x}{2} = \dfrac{12}{x+2}$, so $x^2 + 2x - 24 = 0$, $(x+6)(x-4) = 0$, and since $x > 0$, $x = 4$.

8.1 Ratios, Proportions and the Geometric Mean

OBJECTIVES

- Write and simplify ratios and extended ratios, and use them to solve practical problems.
- Solve proportions with the Cross Products Property and explain why the properties of proportions are true.
- Find the geometric mean of two numbers and construct it with compass and straightedge in a semicircle (Euclid VI.13).
- Describe the golden ratio as Euclid's extreme and mean ratio and compute its exact value.

WATCH

Math Antics - Proportions

mathantics · 10:30

https://www.youtube.com/watch?v=USmit5zUGas

> **WHEN TO WATCH**
> Before Example 1: solving proportions.

Note: Neither video constructs the geometric mean in a semicircle; teach that from the lesson.

BACKUP VIDEO

Calculating the Golden Ratio Geometrically

Eddie Woo · 12:29

https://www.youtube.com/watch?v=AXvmhCXT7-s

After the golden-ratio example: the golden ratio built with geometry.

KEY TERMS

ratio

A comparison of two quantities by division. The ratio of a to b ($b \neq 0$) is written $a : b$ or $\frac{a}{b}$. Both quantities must be measured in the same unit.

extended ratio

A comparison of three or more quantities, such as $a : b : c$, meaning the quantities are ax, bx and cx for some positive number x.

proportion

An equation stating that two ratios are equal, $\frac{a}{b} = \frac{c}{d}$. The numbers a and d are the extremes; b and c are the means.

geometric mean (mean proportional)

For positive numbers a and b, the positive number x with $\frac{a}{x} = \frac{x}{b}$. Then $x^2 = ab$, so $x = \sqrt{ab}$.

extreme and mean ratio (golden ratio)

A segment is cut in extreme and mean ratio when the whole is to the longer part as the longer part is to the shorter part. That common ratio is $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

CORE IDEAS

A **ratio** compares two quantities by dividing. If a barn door is 4 ft wide and 8 ft tall, the ratio of width to height is $4 : 8$, which simplifies to $1 : 2$. Simplify a ratio the way you simplify a fraction, but first make the units match: 2 ft to 8 in is not $2 : 8$; it is 24 in to 8 in, or $3 : 1$. Why? A ratio asks how many times one quantity fits into the other, and that only makes sense when both are counted in the same unit.

An **extended ratio** such as $5 : 5 : 8$ says the three quantities are $5x$, $5x$ and $8x$ for one common number x. Writing the parts this way turns a word problem into an ordinary equation: add the parts, set the sum equal to the total, and solve for x.

A **proportion** says two ratios are equal: $\frac{a}{b} = \frac{c}{d}$. The key tool is the **Cross Products Property**: the proportion is true exactly when $ad = bc$. Why does it work? Multiplying both sides of the proportion by bd clears both denominators, and dividing $ad = bc$ by bd brings the proportion back. Because each step can be undone, the two statements say the same thing. From it come the other properties you will use all unit: you may take reciprocals, swap the means, or add each denominator to its numerator, and the proportion stays true.

The **geometric mean** of two positive numbers a and b is the number x that sits "in proportion" between them: $\frac{a}{x} = \frac{x}{b}$. Cross products give $x^2 = ab$, so $x = \sqrt{ab}$ (we take the positive root because x will be a length). For 4 and 9 the geometric mean is $\sqrt{36} = 6$, and indeed $\frac{4}{6} = \frac{6}{9}$. Euclid calls this length the **mean proportional**.

Euclid shows how to *construct* the mean proportional (VI.13): lay the two lengths a and b end to end to make a segment, draw a semicircle on the whole segment, and raise a perpendicular at the point where the two lengths meet. The perpendicular, up to the semicircle, has length $\sqrt{ab}$. Below we prove this with tools you already have: the isosceles triangle theorem, the triangle angle sum and the Pythagorean Theorem.

One proportion is so famous it has its own name. Euclid (VI.30) asks how to cut a segment so that the whole is to the longer part as the longer part is to the shorter part; he calls this **extreme and mean ratio**. If the longer part is r times the shorter, then $\dfrac{r+1}{r} = \dfrac{r}{1}$, so $r^2 = r + 1$, and the quadratic formula gives $r = \dfrac{1+\sqrt{5}}{2} \approx 1.618$. Today it is called the golden ratio. It is a genuine and beautiful piece of geometry; Euclid uses it to build the regular pentagon. But be careful with popular claims that it is hidden everywhere: many of them do not hold up when the measurements are checked.

POSTULATES AND THEOREMS

THEOREM **Cross Products Property** Euclid VI.16

If $b \neq 0$ and $d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$.

TWO-COLUMN PROOF

Given: $\dfrac{a}{b} = \dfrac{c}{d}$; $b \neq 0$, $d \neq 0$

Prove: $ad = bc$

	STATEMENTS	REASONS
1	$\dfrac{a}{b} = \dfrac{c}{d}$	Given
2	$bd \cdot \dfrac{a}{b} = bd \cdot \dfrac{c}{d}$	Multiplication Property of Equality
3	$da = bc$	Simplify ($\dfrac{b}{b} = 1$ and $\dfrac{d}{d} = 1$)
4	$ad = bc$	Commutative Property of Multiplication

∎

For the converse, start from $ad = bc$ and divide both sides by bd (Division Property of Equality) to get $\dfrac{a}{b} = \dfrac{c}{d}$. Euclid's VI.16 is the same fact for lengths: if four straight lines are proportional,

the rectangle contained by the extremes equals the rectangle contained by the means, and conversely.

THEOREM **Properties of Proportions**

If $\frac{a}{b} = \frac{c}{d}$ *with* a, b, c, d *all nonzero, then (1)* $\frac{b}{a} = \frac{d}{c}$ *(reciprocals), (2)* $\frac{a}{c} = \frac{b}{d}$ *(interchange the means), and (3)* $\frac{a+b}{b} = \frac{c+d}{d}$ *(add each denominator to its numerator).*

PARAGRAPH PROOF

Given: $\frac{a}{b} = \frac{c}{d}$; $a, b, c, d \neq 0$

Prove: $\frac{b}{a} = \frac{d}{c}$, $\frac{a}{c} = \frac{b}{d}$ and $\frac{a+b}{b} = \frac{c+d}{d}$

By the Cross Products Property, $\frac{a}{b} = \frac{c}{d}$ gives $ad = bc$.

(1) Divide both sides of $ad = bc$ by ac (Division Property of Equality; $ac \neq 0$). This gives $\frac{d}{c} = \frac{b}{a}$, and by the Symmetric Property, $\frac{b}{a} = \frac{d}{c}$.

(2) Divide both sides of $ad = bc$ by cd instead. This gives $\frac{a}{c} = \frac{b}{d}$.

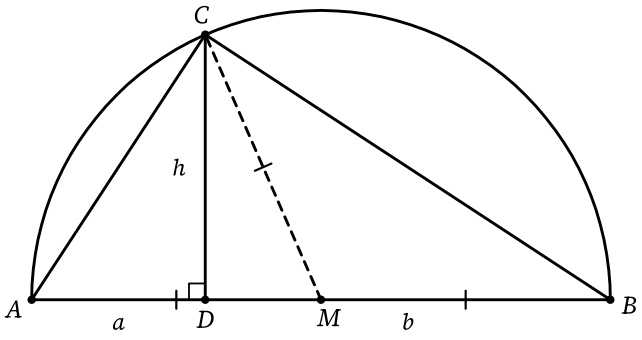
(3) Add 1 to both sides of the given proportion (Addition Property of Equality): $\frac{a}{b} + 1 = \frac{c}{d} + 1$. Since $1 = \frac{b}{b}$ and $1 = \frac{d}{d}$, adding fractions gives $\frac{a+b}{b} = \frac{c+d}{d}$.

∎

Euclid proves these facts for magnitudes in Book V: property (2) is alternation (V.16) and property (3) is composition (V.18).

THEOREM **Mean Proportional in a Semicircle** Euclid VI.13

Let $\overline{AB}$ *be a diameter of a circle with center* M*, let* D *be a point of* $\overline{AB}$*, and let* C *be a point of the circle with* $\overline{CD} \perp \overline{AB}$*. If* $AD = a$*,* $DB = b$ *and* $CD = h$*, then* $h^2 = ab$*; that is,* $CD = \sqrt{AD \cdot DB}$.



TWO-COLUMN PROOF

Given: $\overline{AB}$ is a diameter of a circle with center M; C is on the circle; $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$; $AD = a$, $DB = b$, $CD = h$

Prove: $h^2 = ab$

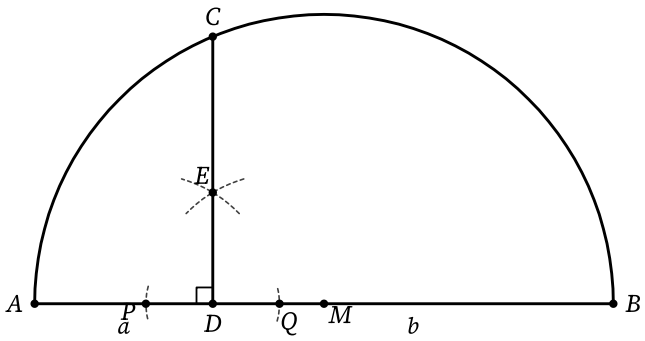
	STATEMENTS	REASONS
1	$\overline{AB}$ is a diameter with center M; C is on the circle; $\overline{CD} \perp \overline{AB}$	Given
2	$MA = MC$ and $MB = MC$	Definition of circle (all radii are equal)
3	$m\angle MAC = m\angle MCA$	Isosceles Triangle Theorem ($\triangle AMC$)
4	$m\angle MBC = m\angle MCB$	Isosceles Triangle Theorem ($\triangle BMC$)
5	$m\angle MAC + m\angle ACB + m\angle MBC = 180^\circ$	Triangle Angle Sum Theorem ($\triangle ABC$; M is on $\overline{AB}$)
6	$m\angle ACB = m\angle MCA + m\angle MCB$	Angle Addition Postulate (M is between A and B)
7	$m\angle MCA + m\angle ACB + m\angle MCB = 180^\circ$	Substitution (rows 3 and 4 into row 5)
8	$2 \cdot m\angle ACB = 180^\circ$	Substitution (row 6 into row 7)
9	$m\angle ACB = 90^\circ$	Division Property of Equality
10	$AC^2 + BC^2 = AB^2 = (a+b)^2$	Pythagorean Theorem (right $\triangle ACB$) and Segment Addition Postulate
11	$AC^2 = a^2 + h^2$ and $BC^2 = b^2 + h^2$	Pythagorean Theorem (right triangles ADC and BDC)
12	$a^2 + h^2 + b^2 + h^2 = a^2 + 2ab + b^2$	Substitution (row 11 into row 10) and squaring $a + b$
13	$2h^2 = 2ab$	Subtraction Property of Equality
14	$h^2 = ab$	Division Property of Equality

∎

Since h is a length, $h = \sqrt{ab}$. The proof uses the Pythagorean Theorem, which you know from Algebra I and which we prove in Lesson 9.1 (Euclid I.47); its proof does not use this theorem, so nothing is circular. Euclid's own proof of VI.13 uses similar triangles; you will see that route in Lesson 9.3.

CONSTRUCTIONS

CONSTRUCTION **Construct the mean proportional (geometric mean) of two segments** Euclid VI.13



1. On a straight line, copy the two given segments end to end (Euclid I.3): $\overline{AD}$ of length a and $\overline{DB}$ of length b, with D between A and B.
2. Bisect $\overline{AB}$ (Euclid I.10) to find its midpoint M.
3. Put the compass point on M, open it to MA, and draw a semicircle on $\overline{AB}$.
4. Construct the perpendicular to $\overline{AB}$ at D (Euclid I.11): with the compass point on D, mark P and Q on $\overline{AB}$ equally far from D; then, with a wider opening kept the same, swing arcs from P and from Q that cross at E, on the same side of $\overline{AB}$ as the semicircle. Draw ray $\overrightarrow{DE}$.
5. Label C where the ray meets the semicircle. Segment $\overline{DC}$ is the mean proportional: $DC = \sqrt{ab}$.

Why it works. By the Mean Proportional in a Semicircle theorem proved above, $DC^2 = AD \cdot DB = ab$, so $\dfrac{a}{DC} = \dfrac{DC}{b}$. The perpendicular construction (I.11) is justified by congruent triangles (Lesson 4.5).

KEY FORMULAS

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc \quad (b, d \neq 0)$$

$$\text{geometric mean of } a \text{ and } b: \quad \frac{a}{x} = \frac{x}{b}, \quad x = \sqrt{ab}$$

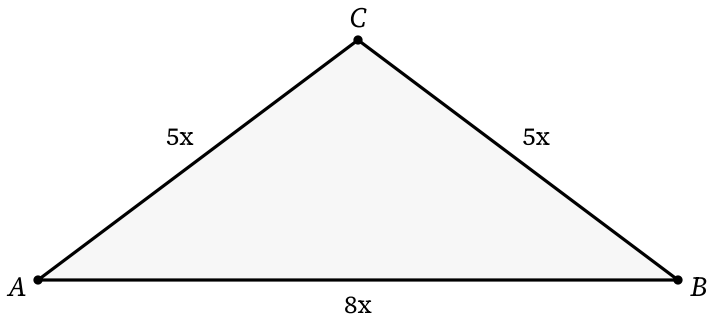
$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

From history. Book V of the Elements, the general theory of proportion, is usually credited to Eudoxus of Cnidus, an older contemporary of Aristotle. His theory works even for lengths whose ratio is not a ratio of whole numbers, such as the diagonal and side of a square, and Book VI applies it to figures. Euclid's name for the golden ratio is "extreme and mean ratio"; the word "golden" was attached to it many centuries later. Popular books sometimes claim it governs the Parthenon or the human body, but careful measurements usually do not support such claims.

Euclid connection. Euclid builds his whole theory of similar figures in Book VI on the proportions of Book V. Proposition VI.13, "to two given straight lines to find a mean proportional," is exactly the semicircle construction of this lesson, and VI.16 is the Cross Products Property stated with rectangles.

Practice habit. Recite the Cross Products Property and the definition of the geometric mean until they are automatic. Before writing any ratio, check that both quantities are in the same unit, and after solving a proportion, substitute your answer back to check it.

Example 1. The three sides of a triangular roof truss are in the extended ratio $5 : 5 : 8$, and the perimeter of the truss is 54 ft. Find the length of each side.

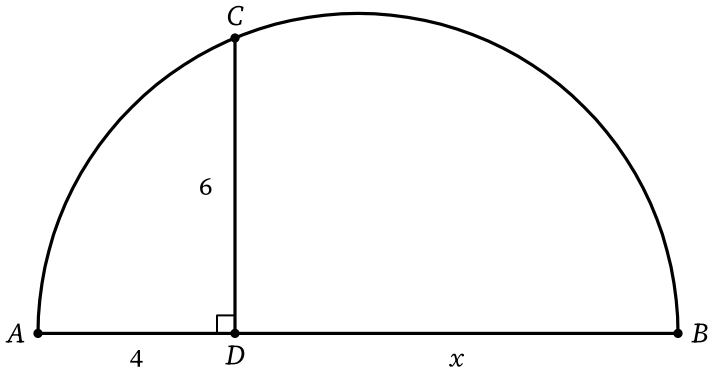


Show solution

1. Write the sides as $5x$, $5x$ and $8x$ feet, using one common number x.
2. The perimeter is the sum of the sides: $5x + 5x + 8x = 54$, so $18x = 54$ and $x = 3$.
3. The sides are $5(3) = 15$ ft, $5(3) = 15$ ft and $8(3) = 24$ ft.
4. Check: $15 + 15 + 24 = 54$, and $15 : 15 : 24$ simplifies to $5 : 5 : 8$.

Answer: 15 ft, 15 ft and 24 ft

Example 2. In the figure, $\overline{AB}$ is the diameter of a semicircle, C is on the semicircle, and $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$. If $AD = 4$ and $CD = 6$, find DB and the diameter AB.



Show solution

1. By the Mean Proportional in a Semicircle theorem, $CD^2 = AD \cdot DB$.
2. Substitute: $6^2 = 4 \cdot DB$, so $36 = 4 \cdot DB$ and $DB = 9$.
3. By the Segment Addition Postulate, $AB = AD + DB = 4 + 9 = 13$.
4. Check: $\frac{4}{6} = \frac{2}{3}$ and $\frac{6}{9} = \frac{2}{3}$, so 6 is indeed the geometric mean of 4 and 9.

Answer: $DB = 9$ and $AB = 13$

PRACTICE

1. Easy A rectangular garden is 18 ft wide and 24 ft long. Write the ratio of the width to the length as a fraction in simplest form.

   Show answer

   $\frac{3}{4}$

2. Easy Solve the proportion $\frac{x}{6} = \frac{10}{15}$.

   Show answer

   $x = 4$

3. Easy The angle measures of a triangle are in the extended ratio $2 : 3 : 4$. Find the three angle measures.

Show answer

40°, 60° and 80° (since $2x + 3x + 4x = 180$ gives $x = 20$)

4. Easy Find the geometric mean of 4 and 25.

Show answer

$\sqrt{4 \cdot 25} = \sqrt{100} = 10$

5. Medium Henry cuts a 72-inch board into three pieces whose lengths are in the ratio $2 : 3 : 7$. How long is each piece?

Show answer

12 in, 18 in and 42 in (since $12x = 72$ gives $x = 6$)

6. Medium Solve the proportion $\dfrac{x + 3}{4} = \dfrac{2x - 1}{6}$.

Show answer

$x = 11$ (cross products: $6x + 18 = 8x - 4$)

7. Medium Find the geometric mean of 6 and 15. Give the answer in simplest radical form.

Show answer

$\sqrt{90} = 3\sqrt{10}$

8. Medium Ruth is planning a quilt with 120 squares. The ratio of blue squares to white squares is $3 : 5$, and every square is blue or white. How many blue squares does she need?

Show answer

45 blue squares ($\dfrac{3}{8} \times 120 = 45$)

9. Medium If $\dfrac{x}{y} = \dfrac{5}{2}$, find the value of $\dfrac{x + y}{y}$. Name the property of proportions you used.

Show answer

$\dfrac{7}{2}$, by adding each denominator to its numerator: $\dfrac{x + y}{y} = \dfrac{5 + 2}{2}$

10. Hard Prove: if $\frac{a}{b} = \frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, then $\frac{a-b}{b} = \frac{c-d}{d}$. Write a two-column proof.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\frac{a}{b} = \frac{c}{d}; b \neq 0, d \neq 0$

Prove: $\frac{a-b}{b} = \frac{c-d}{d}$

	STATEMENTS	REASONS
1	$\frac{a}{b} = \frac{c}{d}$	Given
2	$\frac{a}{b} - 1 = \frac{c}{d} - 1$	Subtraction Property of Equality
3	$\frac{a}{b} - \frac{b}{b} = \frac{c}{d} - \frac{d}{d}$	Substitution ($1 = \frac{b}{b}$ and $1 = \frac{d}{d}$)
4	$\frac{a-b}{b} = \frac{c-d}{d}$	Subtracting fractions with a common denominator

∎

11. Hard Prove: if $\frac{a}{b} = \frac{c}{d}$ with $b \neq 0$, $d \neq 0$ and $b + d \neq 0$, then $\frac{a + c}{b + d} = \frac{a}{b}$. (Hint: call the common value of the two ratios k.)

Show answer

See the model proof.

TWO-COLUMN PROOF

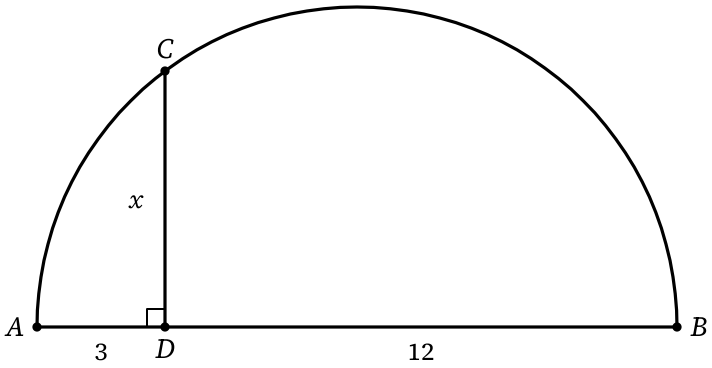
Given: $\frac{a}{b} = \frac{c}{d}$; $b \neq 0$, $d \neq 0$, $b + d \neq 0$

Prove: $\frac{a + c}{b + d} = \frac{a}{b}$

	STATEMENTS	REASONS
1	$\frac{a}{b} = \frac{c}{d}$	Given
2	Let $k = \frac{a}{b}$; then $\frac{c}{d} = k$	Naming the common value; Substitution
3	$a = kb$	Multiplication Property of Equality
4	$c = kd$	Multiplication Property of Equality
5	$a + c = kb + kd$	Addition Property of Equality
6	$a + c = k(b + d)$	Distributive Property
7	$\frac{a + c}{b + d} = k$	Division Property of Equality ($b + d \neq 0$)
8	$\frac{a + c}{b + d} = \frac{a}{b}$	Substitution

∎

12. Hard In the figure, $\overline{AB}$ is the diameter of a semicircle, C is on the semicircle, and $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$. If $AD = 3$ and $DB = 12$, find CD.



Show answer

$CD = \sqrt{3 \cdot 12} = \sqrt{36} = 6$

13. Hard Samuel is building a window frame shaped like a golden rectangle: the ratio of its length to its width is the golden ratio, about 1.618. The width is 40 inches. How long should the frame be? Round to the nearest tenth of an inch.

Show answer

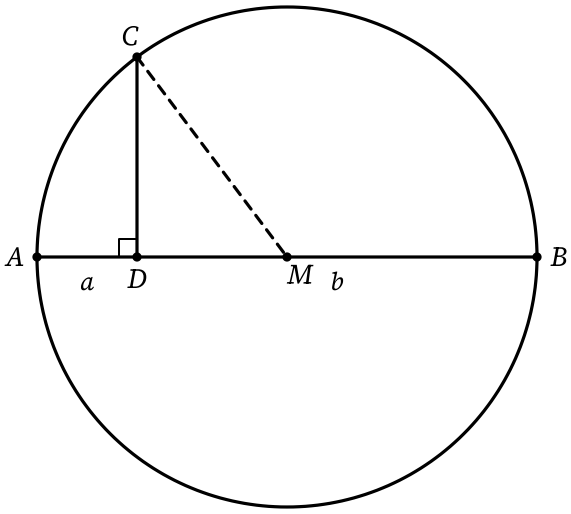
$40 \times 1.618 = 64.72$, about 64.7 in

14. Challenge Segment $\overline{AB}$ is 10 units long. Point P on $\overline{AB}$ cuts it in extreme and mean ratio, with AP the longer part, so that $\dfrac{AB}{AP} = \dfrac{AP}{PB}$. Find AP exactly, in simplest radical form.

Show answer

Let $AP = x$, so $PB = 10 - x$. Then $x^2 = 10(10 - x)$, so $x^2 + 10x - 100 = 0$ and $x = \dfrac{-10 + \sqrt{500}}{2} = 5\sqrt{5} - 5 \approx 6.18$.

15. **Challenge** Explain why the geometric mean of two positive numbers is never larger than their average. In the figure, $\overline{AB}$ is a diameter of the circle with center M, C is on the circle, $\overline{CD} \perp \overline{AB}$ at D, $AD = a$ and $DB = b$. Prove that $\sqrt{ab} \leq \frac{a+b}{2}$.



Show answer

See the model proof: $CD = \sqrt{ab}$ is a perpendicular segment and $CM = \frac{a+b}{2}$ is a radius, and the perpendicular is the shortest segment from C to line AB.

PARAGRAPH PROOF

Given: $\overline{AB}$ is a diameter of a circle with center M; C is on the circle; $\overline{CD} \perp \overline{AB}$ at D; $AD = a$, $DB = b$

Prove: $\sqrt{ab} \leq \frac{a+b}{2}$

By the Mean Proportional in a Semicircle theorem, $CD = \sqrt{ab}$. Since $AB = a + b$ (Segment Addition Postulate) and M is the center, the radius is $CM = \frac{a+b}{2}$.

If $D = M$, then $CD = CM$, so $\sqrt{ab} = \frac{a+b}{2}$. If $D \neq M$, then $\overline{CD}$ is the perpendicular segment from C to line AB and $\overline{CM}$ is another segment from C to that line. The perpendicular segment from a point to a line is the shortest segment from the point to the line (Lesson 3.4), so $CD < CM$, that is, $\sqrt{ab} < \frac{a+b}{2}$.

In either case $\sqrt{ab} \leq \frac{a+b}{2}$. (Equality happens exactly when $D = M$, that is, when $a = b$.)

COMMON MISTAKES

- Mixing units in a ratio, such as writing 2 ft to 8 in as $2:8$. Convert to one unit first: 24 in to 8 in is $3:1$.
- Confusing the geometric mean $\sqrt{ab}$ with the ordinary average $\dfrac{a+b}{2}$. For 4 and 9 the geometric mean is 6, not 6.5; check by testing $\dfrac{4}{6}=\dfrac{6}{9}$.
- Losing parentheses when cross-multiplying, as in turning $\dfrac{x+3}{4}=\dfrac{2x-1}{6}$ into $6x+3=8x-1$. Put each numerator in parentheses: $6(x+3)=4(2x-1)$.
- Giving a negative geometric mean. A length is positive, so from $x^2=ab$ take $x=\sqrt{ab}$ only.

> **Make it a game · Golden Rectangle Hunt.** Hunt through the house for five rectangles: an index card, a book, a window, a door, a picture frame. Measure the longer and shorter sides and divide to find each ratio. Which comes closest to the golden ratio, about 1.618? Then solve for x in 4 : x = x : 9, finding the geometric mean.

8.2 Similar Polygons

OBJECTIVES

- Define similar polygons by corresponding angles and proportional sides, and relate this to a dilation followed by rigid motions.
- Write similarity statements and find the scale factor, missing side lengths and missing angle measures of similar polygons.
- Prove that the perimeters of similar polygons are in the ratio of the scale factor.
- Solve problems with scale models, maps and blueprints.

WATCH

Similar Figures

Eddie Woo · 12:02

https://www.youtube.com/watch?v=ZtvxdMX0cbg

> **WHEN TO WATCH**
> Before the Core ideas: what makes figures similar.

Note: Eddie Woo teaches in Australia: he says 'enlargement' for dilation and 'pronumeral' for variable.

BACKUP VIDEO

Similar Objects (Using a ratio to find an unknown length of a similar object)

Eddie Woo · 6:07

https://www.youtube.com/watch?v=b19fxjuExt8

Before the practice set: finding an unknown length in similar objects.

KEY TERMS

similar polygons

Two polygons whose vertices can be matched so that every pair of corresponding angles is congruent and every pair of corresponding sides is in the same ratio. We write $ABCD \sim EFGH$.

scale factor

The common ratio of corresponding side lengths of similar polygons, image to original. If $ABCD \sim EFGH$ with $\frac{EF}{AB} = \frac{3}{2}$, the scale factor from $ABCD$ to $EFGH$ is $\frac{3}{2}$.

similarity transformation

A dilation followed by zero or more rigid motions (translations, reflections, rotations).

similarity statement

A statement such as $\triangle ABC \sim \triangle DEF$ that lists corresponding vertices in the same order, so $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$.

scale (of a drawing or model)

The ratio of a length on a drawing or model to the corresponding real length, such as 1 in : 8 ft or 1 : 48.

CORE IDEAS

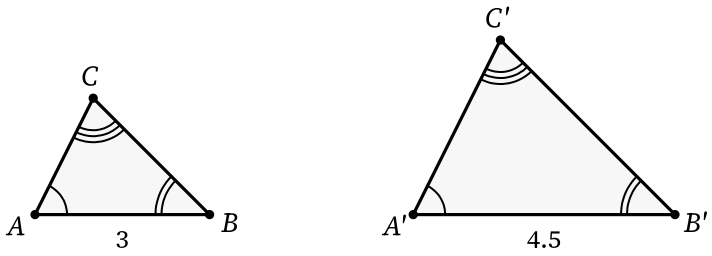
Two figures are **similar** when they have the same shape but not necessarily the same size: a photograph and its enlargement, a blueprint and the house, a scale model and the ship. To make "same shape" precise, Euclid's definition (Book VI, Def. 1) says similar figures have "their angles severally equal and the sides about the equal angles proportional." For polygons: match the vertices so that every pair of corresponding angles is congruent and every pair of corresponding sides has the same ratio k, the **scale factor**.

Both conditions are needed. A square and a long rectangle both have four right angles, but they do not have the same shape, because their sides are not proportional. A square and a tilted rhombus with the same side length have all sides in the ratio 1, but their angles differ. So check angles *and* sides, unless a theorem (like the ones in Lessons 8.3 and 8.4 for triangles) tells you that fewer checks are enough.

In Unit 7 you met dilations. A dilation with scale factor k multiplies every length by k and keeps every angle measure, and rigid motions keep both lengths and angles. So if you dilate a polygon and then slide, turn or flip it, the image is similar to the original with scale factor k. This is the modern way to say "same shape": two figures are similar when a **similarity transformation** (a dilation followed by rigid motions) maps one onto the other.

Order matters in a **similarity statement**. Writing $ABCD \sim EFGH$ promises that $\angle A \cong \angle E$, $\angle B \cong \angle F$, and so on, and that $\frac{EF}{AB} = \frac{FG}{BC} = \frac{GH}{CD} = \frac{HE}{DA}$. Read corresponding parts straight off the letters, and always say which direction your scale factor goes (from $ABCD$ to $EFGH$, or the reverse, which is the reciprocal).

Perimeter is a length, so it scales like a length: if every side is multiplied by k, the sum of the sides is multiplied by k too (the Distributive Property). That is why a model built at scale $1:15$ has a perimeter exactly $\frac{1}{15}$ of the real one. (Areas behave differently; you will see why in Lesson 11.6.)

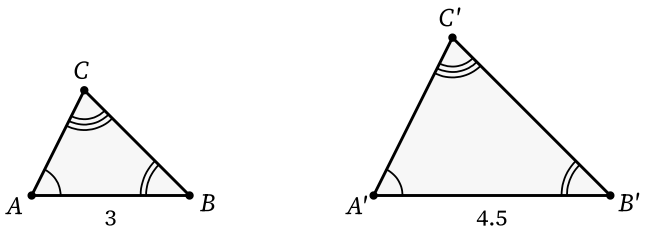


A dilation with scale factor 1.5 followed by a translation maps triangle ABC to triangle A'B'C'.

POSTULATES AND THEOREMS

THEOREM **Similarity Transformation Theorem**

If a polygon is mapped by a dilation with scale factor $k > 0$ followed by any rigid motions, then the image is similar to the original polygon with scale factor k.



A dilation with scale factor 1.5 followed by a translation maps triangle ABC to triangle A'B'C'.

PARAGRAPH PROOF

Given: Polygon $ABC\ldots$ is mapped by a dilation with scale factor k to $A_1B_1C_1\ldots$, which is then mapped by rigid motions to $A'B'C'\ldots$

Prove: $ABC\ldots \sim A'B'C'\ldots$ with scale factor k

A dilation with scale factor k multiplies every length by k and preserves angle measure (Lesson 7.5). So $A_1B_1 = k \cdot AB$, $B_1C_1 = k \cdot BC$, and so on for every side, and $m\angle A_1 = m\angle A$, $m\angle B_1 = m\angle B$, and so on for every angle.

A rigid motion preserves distance and angle measure (Lesson 7.1), and so does a composition of rigid motions. So $A'B' = A_1B_1 = k \cdot AB$, $B'C' = B_1C_1 = k \cdot BC$, and so on, and $m\angle A' = m\angle A_1 = m\angle A$, and so on.

Therefore every pair of corresponding angles is congruent, and dividing, $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \cdots = k$. By the definition of similar polygons, $ABC\ldots \sim A'B'C'\ldots$ with scale factor k.

∎

The reverse is also true: if two polygons are similar with scale factor k, a dilation by k produces a copy congruent to the second polygon, and a rigid motion then carries it onto the second. So "similar" and "related by a similarity transformation" mean the same thing.

THEOREM **Perimeters of Similar Polygons**

If two polygons are similar with scale factor k, then the ratio of their perimeters is k.

PARAGRAPH PROOF

Given: Polygon P has sides $a_1, a_2, \ldots, a_n$; Polygon $Q \sim P$ with scale factor k from P to Q

Prove: $\dfrac{\text{perimeter of } Q}{\text{perimeter of } P} = k$

By the definition of similar polygons, each side of Q is k times the corresponding side of P, so the sides of Q are $ka_1, ka_2, \ldots, ka_n$ (Multiplication Property of Equality).

The perimeter of Q is $ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n)$ by the Distributive Property, and $a_1 + a_2 + \cdots + a_n$ is the perimeter of P.

Dividing both sides by the perimeter of P (Division Property of Equality) gives $\dfrac{\text{perimeter of } Q}{\text{perimeter of } P} = k$.

∎

KEY FORMULAS

$$ABCD \sim EFGH \implies \frac{EF}{AB} = \frac{FG}{BC} = \frac{GH}{CD} = \frac{HE}{DA} = k$$

$$\frac{\text{perimeter of image}}{\text{perimeter of original}} = k$$

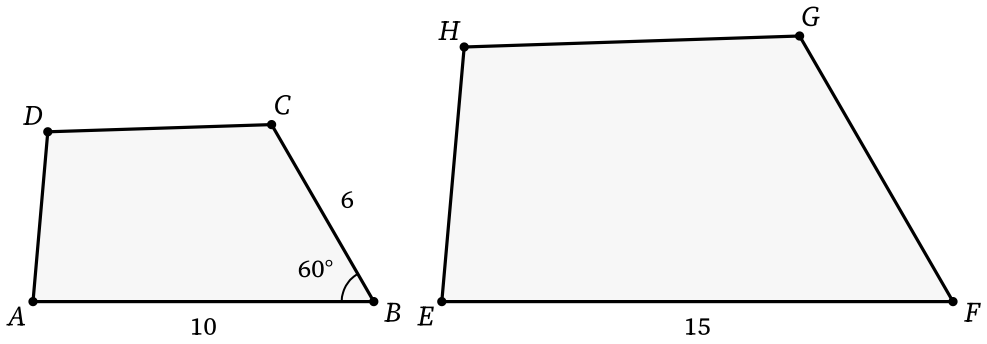
From history. The word "blueprint" comes from a copying process invented by the astronomer John Herschel in 1842, which printed white lines on a blue background. For about a century architects and shipbuilders copied their scale drawings this way. Every one of those drawings depends on the idea of this lesson: a plan is similar to the building it describes.

Euclid connection. Euclid's Book VI opens with the definition of similar rectilinear figures: those that "have their angles severally equal and the sides about the equal angles proportional." Our definition of similar polygons is exactly his; the language of dilations is a modern way of saying the same thing.

Practice habit. Recite the definition of similar polygons until it is automatic: corresponding angles congruent AND corresponding sides proportional. Before computing, write the similarity statement and line up corresponding sides in a ratio with the same polygon on top every time.

WORKED EXAMPLES

Example 1. In the figure, quadrilateral $ABCD \sim$ quadrilateral $EFGH$. $AB = 10$, $BC = 6$, $m\angle B = 60°$ and $EF = 15$. Find the scale factor from $ABCD$ to $EFGH$, the length FG, and $m\angle F$.

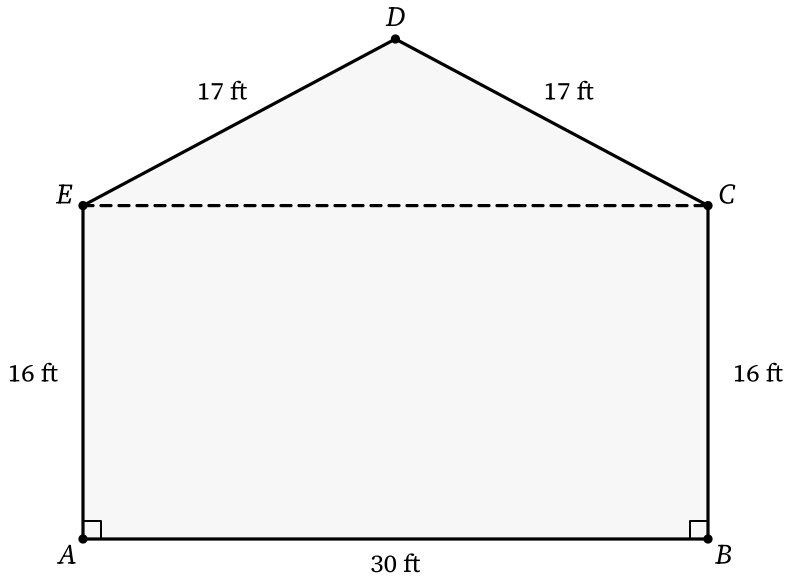


Show solution

1. From the similarity statement, $\overline{AB}$ corresponds to $\overline{EF}$, so the scale factor from $ABCD$ to $EFGH$ is $\dfrac{EF}{AB} = \dfrac{15}{10} = \dfrac{3}{2}$.
2. $\overline{FG}$ corresponds to $\overline{BC}$, so $FG = \dfrac{3}{2} \cdot 6 = 9$.
3. $\angle F$ corresponds to $\angle B$, and corresponding angles of similar polygons are congruent, so $m\angle F = 60°$. Notice that the angle does NOT get multiplied by the scale factor.

Answer: Scale factor $\dfrac{3}{2}$, $FG = 9$, $m\angle F = 60°$

Example 2. The front wall of the Millers' barn is the pentagon $ABCDE$ in the figure: a rectangle 30 ft wide and 16 ft tall topped by a gable whose two roof edges are each 17 ft long. Grace builds a similar model of the wall whose base is 24 inches long. How long is each roof edge of the model, and what is the model wall's perimeter, both in inches?



Show solution

1. Use one unit. The real base is 30 ft $= 360$ in, so the scale factor from the real wall to the model is $\dfrac{24}{360} = \dfrac{1}{15}$.
2. A real roof edge is 17 ft $= 204$ in, so a model roof edge is $\dfrac{204}{15} = 13.6$ in.
3. The real perimeter is $30 + 16 + 17 + 17 + 16 = 96$ ft $= 1152$ in. By the Perimeters of Similar Polygons theorem, the model's perimeter is $\dfrac{1152}{15} = 76.8$ in.
4. Check: the model's sides are 24, 12.8, 13.6, 13.6 and 12.8 inches, and their sum is 76.8 in.

Answer: Each roof edge is 13.6 in; the perimeter is 76.8 in.

PRACTICE

1. **Easy** $\triangle ABC \sim \triangle DEF$, $AB = 4$ and $DE = 10$. What is the scale factor from $\triangle ABC$ to $\triangle DEF$?

   Show answer

   $$\frac{10}{4} = \frac{5}{2}$$

2. **Easy** $\triangle PQR \sim \triangle XYZ$, $m\angle P = 48^\circ$ and $m\angle Q = 71^\circ$. Find $m\angle Z$.

   Show answer

   $\angle Z$ corresponds to $\angle R$, and $m\angle R = 180^\circ - 48^\circ - 71^\circ = 61^\circ$, so $m\angle Z = 61^\circ$.

3. **Easy** Two pentagons are similar, and the scale factor from the smaller to the larger is $\frac{5}{2}$. The perimeter of the smaller pentagon is 18 cm. Find the perimeter of the larger pentagon.

   Show answer

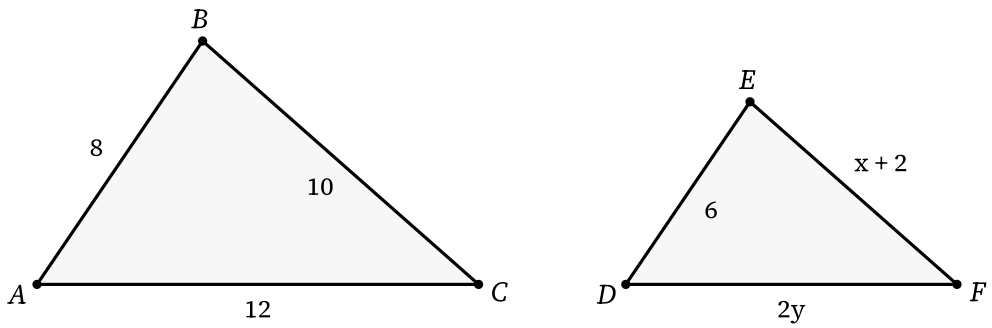
   $$18 \cdot \frac{5}{2} = 45 \text{ cm}$$

4. **Easy** Is a 4-inch by 6-inch photograph similar to a 6-inch by 9-inch enlargement of it? Answer yes or no and give the ratios you compared.

   Show answer

   Yes: all angles are right angles, and $\frac{6}{4} = \frac{9}{6} = \frac{3}{2}$.

5. **Medium** In the figure, $\triangle ABC \sim \triangle DEF$ with $AB = 8$, $BC = 10$, $AC = 12$, $DE = 6$, $EF = x + 2$ and $DF = 2y$. Find x and y.



   Show answer

   The scale factor from $\triangle ABC$ to $\triangle DEF$ is $\frac{6}{8} = \frac{3}{4}$. Then $x + 2 = \frac{3}{4}(10) = 7.5$, so $x = 5.5$, and $2y = \frac{3}{4}(12) = 9$, so $y = 4.5$.

6. **Medium** On a map with scale 1 inch : 25 miles, two towns are 3.4 inches apart. How far apart are the towns in miles?

   Show answer

   $3.4 \times 25 = 85$ miles

7. **Medium** A model sailing ship is built at a scale of $1 : 48$. The real ship's mainmast is 36 ft tall. How tall is the model's mainmast, in inches?

   Show answer

   $\frac{36}{48} = 0.75$ ft $= 9$ in

8. **Medium** A triangle has sides 4 cm, 6 cm and 7 cm. A similar triangle has perimeter 51 cm. Find the lengths of its sides, shortest first.

   Show answer

   The first perimeter is 17 cm, so the scale factor is $\frac{51}{17} = 3$; the sides are 12 cm, 18 cm and 21 cm.

9. **Medium** Are all rectangles similar to each other? Answer yes or no and explain, using a specific example.

   Show answer

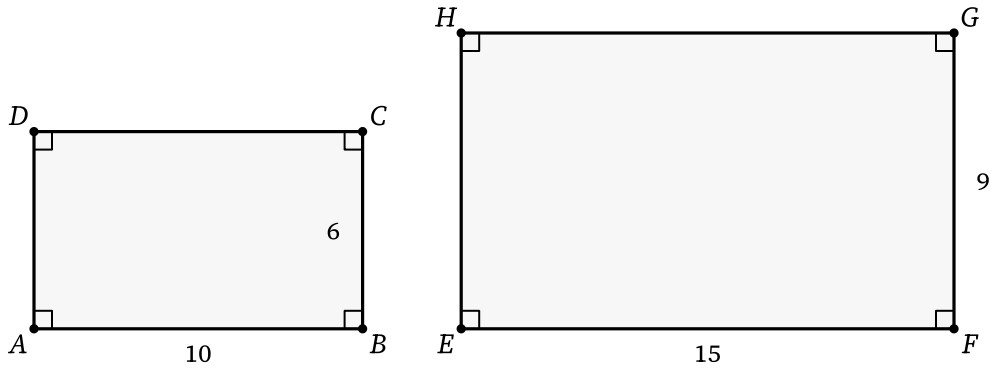
   No. All rectangles have four congruent right angles, but their sides need not be proportional. A 2-by-2 square and a 2-by-6 rectangle: $\frac{2}{2} = 1$ but $\frac{6}{2} = 3$, so the sides are not in one common ratio, and the definition of similar polygons fails.

10. **Medium** A house blueprint uses the scale $\frac{1}{4}$ inch $= 1$ foot. On the blueprint, the rectangular family room is 5.5 inches by 4 inches. What is the perimeter of the real room, in feet?

    Show answer

    Each inch on the plan is 4 ft, so the room is 22 ft by 16 ft and its perimeter is $2(22 + 16) = 76$ ft.

11. **Hard** In the figure, $ABCD$ and $EFGH$ are rectangles with $AB = 10$, $BC = 6$, $EF = 15$ and $FG = 9$. Write a two-column proof that $ABCD \sim EFGH$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ and $EFGH$ are rectangles; $AB = 10$, $BC = 6$, $EF = 15$, $FG = 9$

Prove: $ABCD \sim EFGH$

	STATEMENTS	REASONS
1	$ABCD$ and $EFGH$ are rectangles; $AB = 10$, $BC = 6$, $EF = 15$, $FG = 9$	Given
2	$\angle A, \angle B, \angle C, \angle D, \angle E, \angle F, \angle G, \angle H$ are right angles	Definition of rectangle
3	$\angle A \cong \angle E, \angle B \cong \angle F, \angle C \cong \angle G, \angle D \cong \angle H$	All right angles are congruent
4	$CD = 10, DA = 6, GH = 15, HE = 9$	Opposite sides of a parallelogram are congruent (a rectangle is a parallelogram)
5	$\frac{EF}{AB} = \frac{15}{10} = \frac{3}{2}, \frac{FG}{BC} = \frac{9}{6} = \frac{3}{2}, \frac{GH}{CD} = \frac{15}{10} = \frac{3}{2}$, $\frac{HE}{DA} = \frac{9}{6} = \frac{3}{2}$	Substitution and simplifying
6	$ABCD \sim EFGH$	Definition of similar polygons

∎

12. Hard Prove that any two equilateral triangles are similar. Let $\triangle ABC$ be equilateral with side s and $\triangle DEF$ be equilateral with side t. Write a paragraph proof.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\triangle ABC$ is equilateral with side s; $\triangle DEF$ is equilateral with side t

Prove: $\triangle ABC \sim \triangle DEF$

An equilateral triangle is equiangular (Lesson 4.6), and each angle of an equiangular triangle measures $60°$ (Lesson 4.1). So $m\angle A = m\angle B = m\angle C = 60°$ and $m\angle D = m\angle E = m\angle F = 60°$, and therefore $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.

Every side of $\triangle ABC$ has length s and every side of $\triangle DEF$ has length t, so $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA} = \frac{t}{s}$.

Corresponding angles are congruent and corresponding sides are proportional, so $\triangle ABC \sim \triangle DEF$ by the definition of similar polygons, with scale factor $\frac{t}{s}$.

∎

13. Hard Prove that similarity is transitive for triangles: if $\triangle ABC \sim \triangle DEF$ with scale factor k and $\triangle DEF \sim \triangle GHI$ with scale factor m, then $\triangle ABC \sim \triangle GHI$ with scale factor mk.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\triangle ABC \sim \triangle DEF$ with $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA} = k$; $\triangle DEF \sim \triangle GHI$ with $\frac{GH}{DE} = \frac{HI}{EF} = \frac{IG}{FD} = m$
Prove: $\triangle ABC \sim \triangle GHI$

	STATEMENTS	REASONS
1	$\triangle ABC \sim \triangle DEF$ with scale factor k; $\triangle DEF \sim \triangle GHI$ with scale factor m	Given
2	$\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$	Definition of similar polygons
3	$\angle D \cong \angle G, \angle E \cong \angle H, \angle F \cong \angle I$	Definition of similar polygons
4	$\angle A \cong \angle G, \angle B \cong \angle H, \angle C \cong \angle I$	Transitive Property of Congruence
5	$DE = k \cdot AB, EF = k \cdot BC, FD = k \cdot CA$	Multiplication Property of Equality
6	$GH = m \cdot DE, HI = m \cdot EF, IG = m \cdot FD$	Multiplication Property of Equality
7	$GH = mk \cdot AB, HI = mk \cdot BC, IG = mk \cdot CA$	Substitution
8	$\frac{GH}{AB} = \frac{HI}{BC} = \frac{IG}{CA} = mk$	Division Property of Equality
9	$\triangle ABC \sim \triangle GHI$	Definition of similar polygons

∎

14. **Challenge** A rectangle has length a and width b, with $b < a < 2b$. A b-by-b square is cut off one end, leaving a smaller rectangle with sides b and $a - b$, whose longer side is b. For which value of $\dfrac{a}{b}$ is the smaller rectangle similar to the original one? Give the exact value.

Show answer

Similarity requires $\dfrac{a}{b} = \dfrac{b}{a-b}$. With $r = \dfrac{a}{b}$ this is $r = \dfrac{1}{r-1}$, so $r^2 - r - 1 = 0$ and $r = \dfrac{1+\sqrt{5}}{2}$, the golden ratio.

COMMON MISTAKES

- Matching sides in the wrong order. Use the similarity statement: in $\triangle ABC \sim \triangle DEF$, $\overline{BC}$ goes with $\overline{EF}$ because B, C and E, F hold the same places.
- Multiplying angle measures by the scale factor. Corresponding angles of similar polygons are congruent; only lengths are multiplied.
- Using the scale factor in the wrong direction. Say "from the small figure to the large" or the reverse, and remember the two scale factors are reciprocals.
- Comparing measurements in different units, such as a 24-inch model and a 30-foot barn. Convert first: $30 \text{ ft} = 360 \text{ in}$.

> **Make it a game · Scale Model Duel.** Pick a toy car or small figure. Suppose the real object is 4 meters long. Find the scale factor using a ruler, then predict the width of the real object before measuring the toy's width. Write a similarity statement for the model and the real shape. Closest prediction scores. Repeat with another toy or object.

8.3 Proving Triangles Similar: AA

OBJECTIVES

- State the Angle-Angle (AA) Similarity Postulate and explain why two pairs of angles are enough for triangles.
- Prove triangles similar with AA in two-column, flow and paragraph proofs.
- Use similar triangles to find missing lengths, including heights found indirectly with shadows and mirrors.

WATCH

HSC Geometry Question (1 of 2: Proving & using triangle similarity)

Eddie Woo · 7:50

https://www.youtube.com/watch?v=nl7dciuUxb4

> **WHEN TO WATCH**
> After the AA postulate: a similarity proof that uses two pairs of angles, then a proportion.

Note: Eddie Woo teaches in Australia: he says 'equiangular' for AA similarity. Shadow and mirror problems are taught from the lesson.

BACKUP VIDEO

Tricky Similarity Question

Eddie Woo · 11:46

https://www.youtube.com/watch?v=uXsIWGgJL2M

As a challenge after the practice set: a golden triangle proved similar with AA.

KEY TERMS

Angle-Angle (AA) Similarity Postulate

If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

indirect measurement

Finding a length that is hard to measure directly (the height of a steeple, the width of a river) by measuring other lengths and using similar triangles.

equiangular triangles

Euclid's name for triangles whose angles are congruent in pairs; by the AA Postulate, such triangles are similar.

CORE IDEAS

For polygons in general, you must check every angle and every ratio of sides to show similarity (Lesson 8.2). Triangles are special: their shape is fixed by their angles. That is why a much shorter test works.

Why are two angles enough? First, if two angles of one triangle are congruent to two angles of another, the third angles are congruent too (Third Angles Theorem, Lesson 4.2), because each triangle's angles add to 180°. Second, think of a dilation: dilate $\triangle ABC$ by the scale factor $k = \dfrac{DE}{AB}$. The image has a side exactly as long as $\overline{DE}$ and the same angles as $\triangle ABC$, so by ASA it is congruent to $\triangle DEF$. A dilation followed by a rigid motion maps one triangle onto the other, so they are similar. In this course we take this as a postulate, the **AA Similarity Postulate**; Euclid proves it as VI.4 from his theory of proportion.

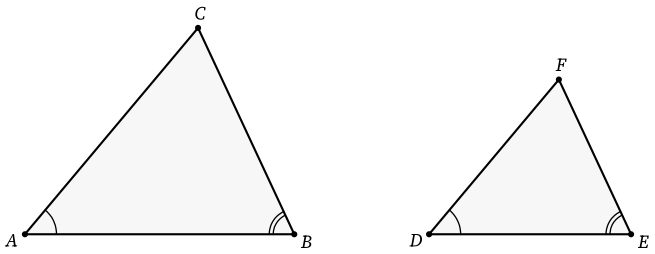
To use AA, look for angle pairs you can PROVE congruent: a shared angle (Reflexive Property), vertical angles, right angles, angles formed by parallel lines (corresponding or alternate interior angles), or angles given as congruent. Mark them on the figure, then write the similarity statement with corresponding vertices in the same order.

A very common case: a line parallel to one side of a triangle cuts off a smaller triangle. The parallel line creates congruent corresponding angles, and the two triangles share the vertex angle, so AA applies. Another: two right triangles with one pair of congruent acute angles are always similar.

Indirect measurement. At one moment the sun's rays are (for practical purposes) parallel, so the rays make congruent angles with level ground everywhere nearby. A post and its shadow and a steeple and its shadow form two right triangles with a congruent acute angle, so they are similar, and one proportion gives the steeple's height. A mirror works the same way: light leaves a mirror at the same angle it arrives, so the two right triangles on either side of the mirror are similar.

POSTULATE **Angle-Angle (AA) Similarity Postulate** Euclid VI.4

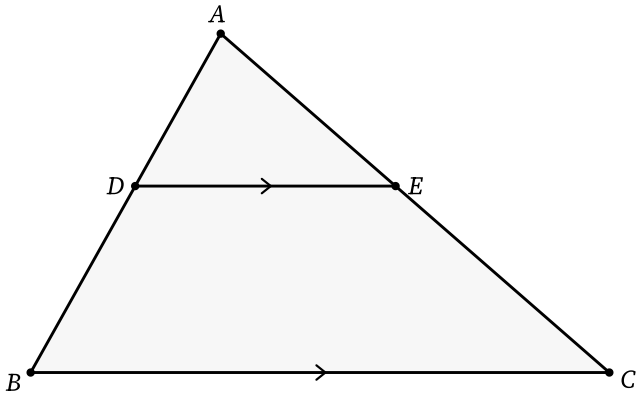
If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.



Euclid proves this as VI.4: "In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles." We accept it as a postulate; it agrees with dilations, since dilating one triangle to match a side of the other produces a congruent copy by ASA.

THEOREM **Parallel Line Similarity Theorem**

If a line parallel to one side of a triangle meets the other two sides at different points, it cuts off a triangle similar to the original triangle.



TWO-COLUMN PROOF

Given: $\triangle ABC$; D is on $\overline{AB}$ and E is on $\overline{AC}$; $\overline{DE} \parallel \overline{BC}$

Prove: $\triangle ADE \sim \triangle ABC$

	STATEMENTS	REASONS
1	$\overline{DE} \parallel \overline{BC}$	Given
2	$\angle ADE \cong \angle ABC$	Corresponding Angles Postulate (transversal $\overleftrightarrow{AB}$)
3	$\angle DAE \cong \angle BAC$	Reflexive Property of Congruence (same angle)
4	$\triangle ADE \sim \triangle ABC$	AA Similarity Postulate

∎

COROLLARY **Right Triangle Similarity Corollary**

If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then the triangles are similar.

PARAGRAPH PROOF

Given: $\triangle ABC$ and $\triangle DEF$ with right angles $\angle C$ and $\angle F$; $\angle A \cong \angle D$

Prove: $\triangle ABC \sim \triangle DEF$

Since $\angle C$ and $\angle F$ are right angles, $\angle C \cong \angle F$ because all right angles are congruent. We are given $\angle A \cong \angle D$. Two angles of $\triangle ABC$ are congruent to two angles of $\triangle DEF$, so $\triangle ABC \sim \triangle DEF$ by the AA Similarity Postulate.

∎

KEY FORMULAS

$$\angle A \cong \angle D,\ \angle B \cong \angle E \implies \triangle ABC \sim \triangle DEF \implies \frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

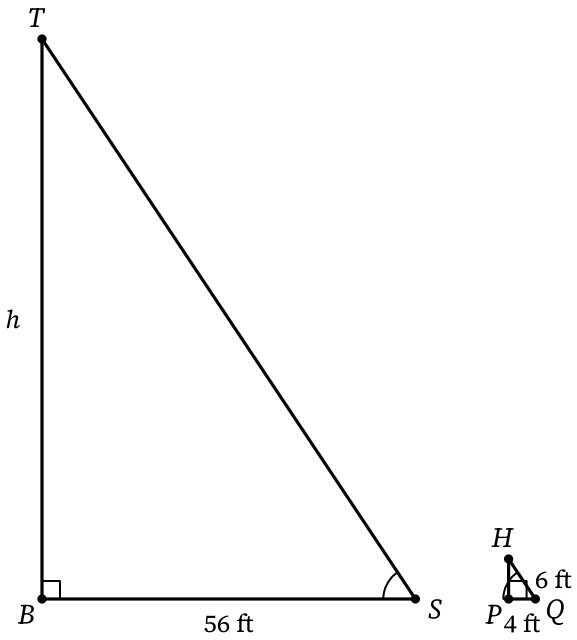
$$\frac{\text{height of object}}{\text{length of its shadow}} = \frac{\text{height of post}}{\text{length of its shadow}}$$

From history. Thales of Miletus, often called the first Greek mathematician, is said to have measured the height of the Great Pyramid of Egypt by its shadow. Later Greek writers report the method in two forms: wait until a person's shadow equals the person's height, then measure the pyramid's shadow; or compare the pyramid's shadow with the shadow of an upright stick at the same moment. Either way the idea is the AA similarity of this lesson.

Euclid connection. Euclid's VI.4 is the foundation of Book VI: equiangular triangles have proportional sides, and the corresponding sides are the ones opposite the equal angles. Our AA Postulate is the same statement, since two pairs of equal angles force the third pair to be equal (I.32).

Practice habit. Recite the AA Similarity Postulate until it is automatic, and mark every pair of congruent angles on the figure (with a reason: vertical, shared, right, parallel lines) before writing the similarity statement. Write the statement so that the congruent angles appear in the same positions.

Example 1. On a sunny afternoon, Samuel, who is 6 ft tall, casts a shadow 4 ft long. At the same moment the church steeple casts a shadow 56 ft long on level ground. Both Samuel and the steeple stand straight up. How tall is the steeple?

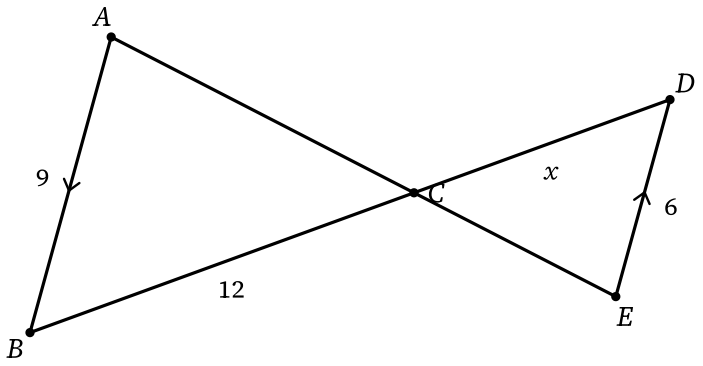


Show solution

1. Each height and its shadow form a right triangle with the right angle at the ground ($\angle TBS$ and $\angle HPQ$).
2. The sun's rays $\overline{TS}$ and $\overline{HQ}$ are parallel, so they make congruent angles with the level ground: $\angle TSB \cong \angle HQP$. With the right angles, $\triangle TBS \sim \triangle HPQ$ by AA.
3. Corresponding sides are proportional: $\dfrac{TB}{HP} = \dfrac{BS}{PQ}$, so $\dfrac{h}{6} = \dfrac{56}{4}$.
4. Then $h = 6 \cdot 14 = 84$. Check: $\dfrac{84}{56} = 1.5$ and $\dfrac{6}{4} = 1.5$.

Answer: The steeple is 84 ft tall.

Example 2. In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, and $\overline{AB} \parallel \overline{DE}$. Prove that $\triangle ABC \sim \triangle EDC$. Then, given $AB = 9$, $DE = 6$ and $BC = 12$, find CD.



Show solution

1. Plan: the parallel lines give alternate interior angles, and the crossing segments give vertical angles. Two angle pairs mean AA.
2. Line up the correspondence: $A \leftrightarrow E$, $B \leftrightarrow D$, $C \leftrightarrow C$.
3. After proving the similarity, use $\dfrac{DC}{BC} = \dfrac{ED}{AB}$: $\dfrac{x}{12} = \dfrac{6}{9}$, so $9x = 72$ and $x = 8$.

TWO-COLUMN PROOF

Given: $\overline{AE}$ and $\overline{BD}$ intersect at C; $\overline{AB} \parallel \overline{DE}$

Prove: $\triangle ABC \sim \triangle EDC$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DE}$	Given
2	$\angle BAC \cong \angle DEC$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{AE}$)
3	$\angle ACB \cong \angle ECD$	Vertical Angles Theorem
4	$\triangle ABC \sim \triangle EDC$	AA Similarity Postulate

∎

Answer: The triangles are similar by AA (see the proof), and $CD = 8$.

PRACTICE

1. **Easy** One triangle has angles of 40° and 75°. Another triangle has angles of 75° and 65°. Are the triangles similar? Answer yes or no and explain.

   Show answer

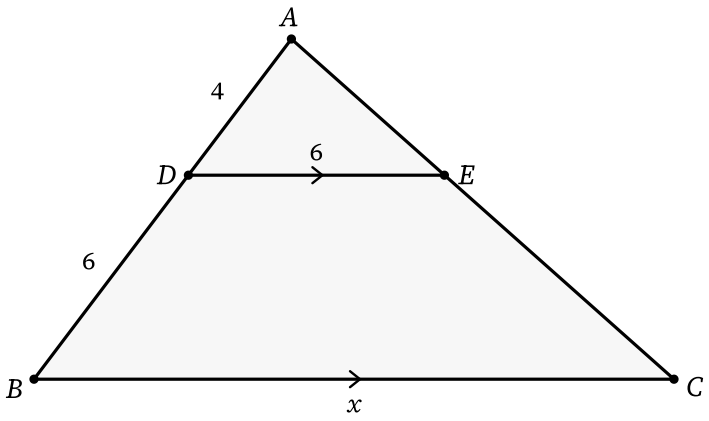
   Yes. The first triangle's third angle is $180^\circ - 40^\circ - 75^\circ = 65^\circ$, so both triangles have angles 75° and 65°; by AA they are similar.

2. **Easy** One triangle has angles of 50° and 60°. Another has angles of 60° and 80°. Are the triangles similar? Answer yes or no and explain.

   Show answer

   No. The angles are $50^\circ, 60^\circ, 70^\circ$ and $60^\circ, 80^\circ, 40^\circ$; only one pair of angles matches.

3. **Easy** In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = 4$, $DB = 6$ and $DE = 6$. Find BC.



   Show answer

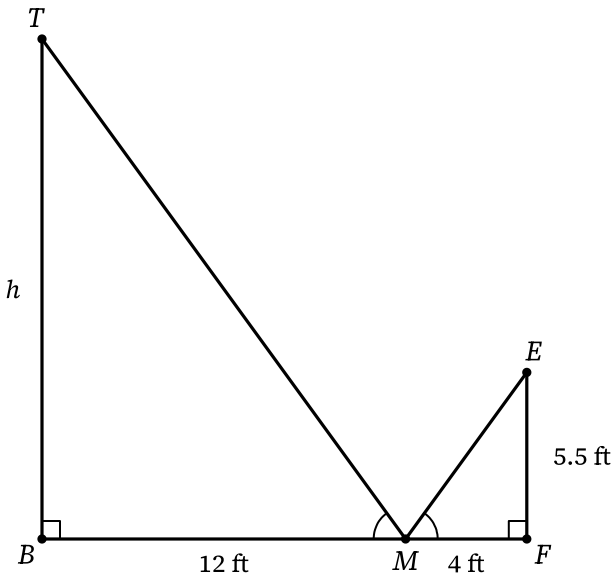
   $\triangle ADE \sim \triangle ABC$, and $AB = 10$, so $\dfrac{BC}{6} = \dfrac{10}{4}$ and $BC = 15$.

4. **Easy** A 5-ft fence post casts a 3-ft shadow at the same moment that a flagpole casts a 21-ft shadow. Both stand straight up on level ground. How tall is the flagpole?

   Show answer

   $\dfrac{h}{5} = \dfrac{21}{3}$, so $h = 35$ ft.

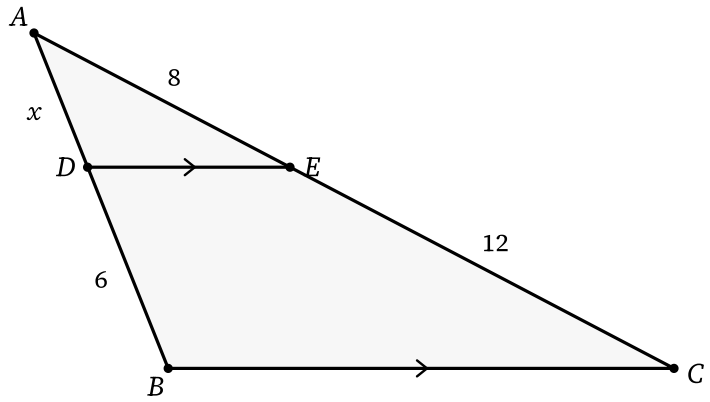
5. **Medium** Henry places a small mirror flat on level ground 12 ft from the base of a tree. He backs up until he sees the top of the tree in the mirror; then he is standing 4 ft from the mirror, and his eyes are 5.5 ft above the ground. Light leaves a mirror at the same angle it arrives, so $\angle TMB \cong \angle EMF$ in the figure. How tall is the tree?



Show answer

$\triangle TBM \sim \triangle EFM$ by AA (right angles and the mirror angles), so $\frac{h}{5.5} = \frac{12}{4}$ and $h = 16.5$ ft.

6. **Medium** In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = x$, $DB = 6$, $AE = 8$ and $EC = 12$. Use similar triangles to find x.



Show answer

$\triangle ADE \sim \triangle ABC$, so $\dfrac{AD}{AB} = \dfrac{AE}{AC}$: $\dfrac{x}{x+6} = \dfrac{8}{20}$. Then $20x = 8x + 48$ and $x = 4$.

7. **Medium** A straight roof rafter rises from the top of a wall. At a horizontal distance of 4 ft from the wall, the rafter is 1.5 ft above the top of the wall. How high above the top of the wall is it at a horizontal distance of 16 ft? Explain which triangles are similar and why.

Show answer

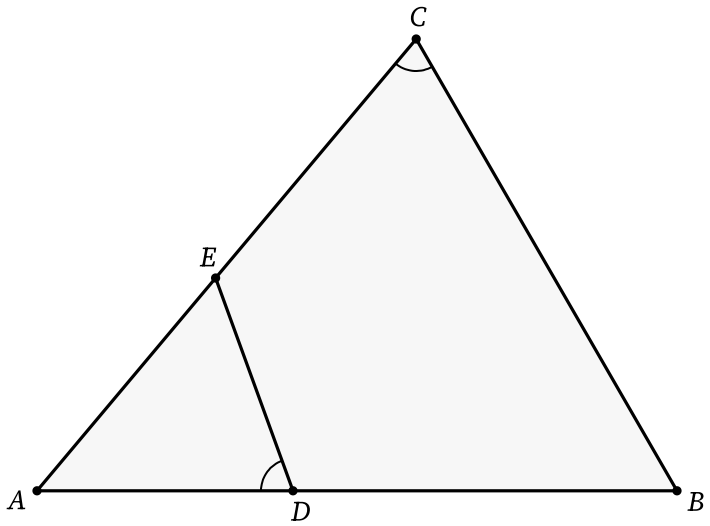
6 ft. Drop vertical segments from the rafter to the level line through the top of the wall; the two right triangles share the angle where the rafter meets the wall, so they are similar by AA, and $\dfrac{h}{1.5} = \dfrac{16}{4}$ gives $h = 6$.

8. **Medium** Explain why any two isosceles right triangles are similar.

Show answer

In an isosceles right triangle the base angles are congruent (Isosceles Triangle Theorem) and the two acute angles are complementary, so each measures 45°. Any two such triangles have two pairs of congruent angles (90° and 45°), so they are similar by AA.

9. **Medium** In the figure, D is on $\overline{AB}$, E is on $\overline{AC}$, and $\angle ADE \cong \angle ACB$. Write a two-column proof that $\triangle ADE \sim \triangle ACB$.



Show answer

See the model proof.

TWO-COLUMN PROOF

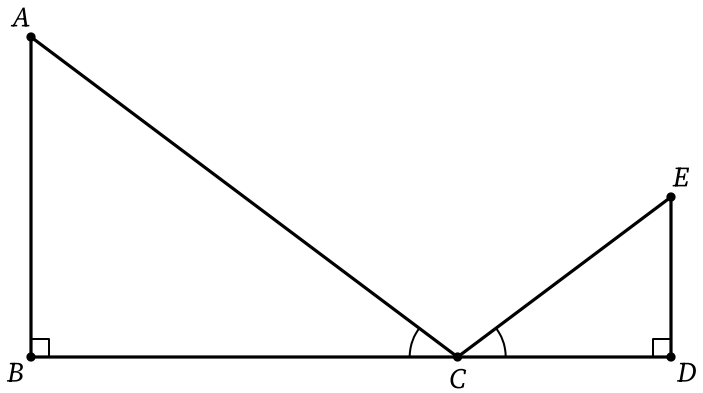
Given: D is on $\overline{AB}$ and E is on $\overline{AC}$; $\angle ADE \cong \angle ACB$

Prove: $\triangle ADE \sim \triangle ACB$

	STATEMENTS	REASONS
1	$\angle ADE \cong \angle ACB$	Given
2	$\angle DAE \cong \angle CAB$	Reflexive Property of Congruence (same angle)
3	$\triangle ADE \sim \triangle ACB$	AA Similarity Postulate

∎

10. **Hard** In the figure, $\overline{AB} \perp \overline{BD}$, $\overline{ED} \perp \overline{BD}$, C is on $\overline{BD}$, and $\angle ACB \cong \angle ECD$. Write a flow proof that $\triangle ABC \sim \triangle EDC$. (This is why the mirror method works.)



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{AB} \perp \overline{BD}$; $\overline{ED} \perp \overline{BD}$; $\angle ACB \cong \angle ECD$

Prove: $\triangle ABC \sim \triangle EDC$

(1) $\overline{AB} \perp \overline{BD}$
Given

(2) $\overline{ED} \perp \overline{BD}$
Given

(3) $\angle ACB \cong \angle ECD$
Given

(4) $\angle ABC$ is a right angle
Definition of perpendicular lines
from (1)

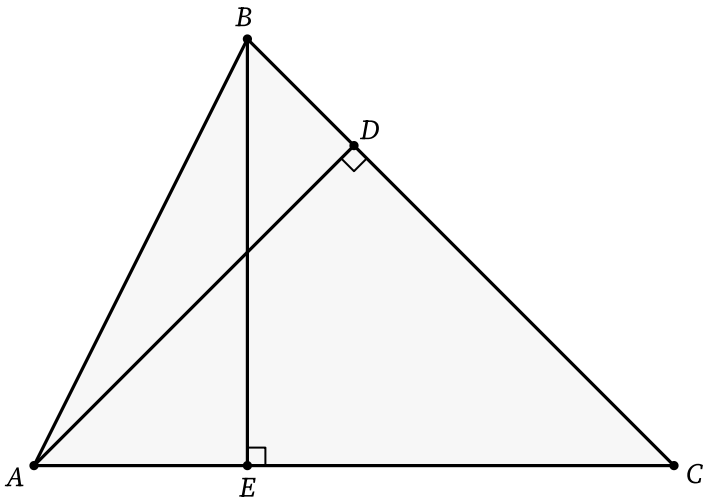
(5) $\angle EDC$ is a right angle
Definition of perpendicular lines
from (2)

(6) $\angle ABC \cong \angle EDC$
All right angles are congruent
from (4), (5)

(7) $\triangle ABC \sim \triangle EDC$
AA Similarity Postulate

from (3), (6)

11. **Hard** In acute $\triangle ABC$ in the figure, $\overline{AD}$ is the altitude to $\overline{BC}$ and $\overline{BE}$ is the altitude to $\overline{AC}$. Write a paragraph proof that $\triangle ADC \sim \triangle BEC$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\triangle ABC$ is acute; $\overline{AD}$ is the altitude to $\overline{BC}$; $\overline{BE}$ is the altitude to $\overline{AC}$

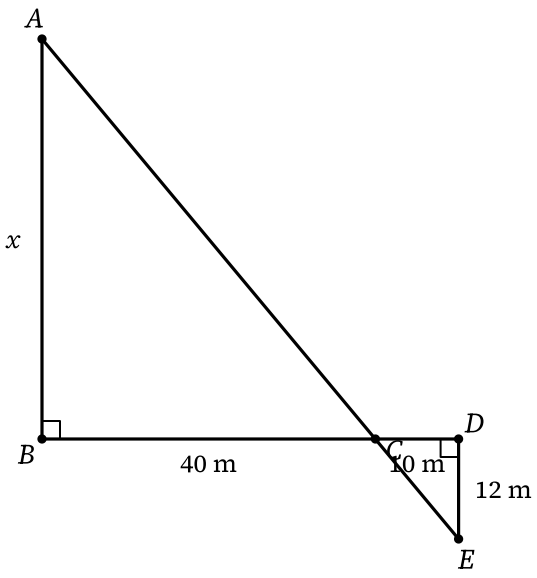
Prove: $\triangle ADC \sim \triangle BEC$

By the definition of altitude, $\overline{AD} \perp \overline{BC}$ and $\overline{BE} \perp \overline{AC}$, so $\angle ADC$ and $\angle BEC$ are right angles (definition of perpendicular lines). All right angles are congruent, so $\angle ADC \cong \angle BEC$.

Since D lies on $\overline{CB}$ and E lies on $\overline{CA}$, $\angle ACD$ and $\angle BCE$ are both the angle $\angle ACB$ of the triangle, so $\angle ACD \cong \angle BCE$ by the Reflexive Property of Congruence.

Two angles of $\triangle ADC$ are congruent to two angles of $\triangle BEC$, so $\triangle ADC \sim \triangle BEC$ by the AA Similarity Postulate.

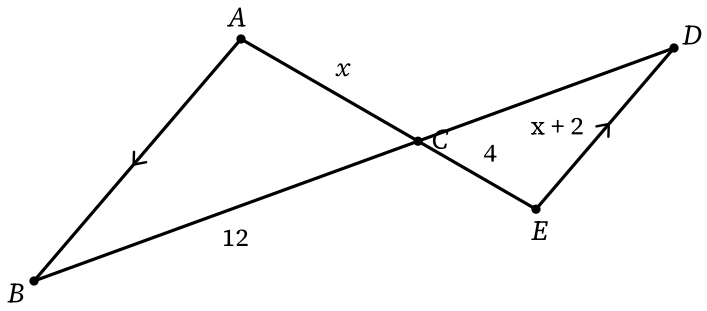
12. **Hard** To find the width of a river, a surveyor picks a tree at A on the far bank and a point B on the near bank directly across from it. She walks 40 m along the bank, perpendicular to $\overline{AB}$, to a stake at C, then 10 m farther to D. From D she walks at a right angle away from the river to the point E where E, C and A line up; $DE = 12$ m. How wide is the river (AB)?



Show answer

$\triangle ABC \sim \triangle EDC$ by AA (right angles at B and D, vertical angles at C), so $\dfrac{AB}{12} = \dfrac{40}{10}$ and $AB = 48$ m.

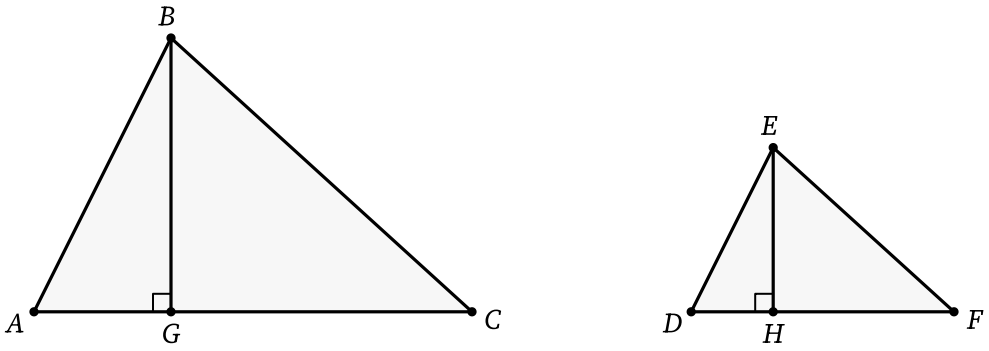
13. **Hard** In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, $\overline{AB} \parallel \overline{DE}$, $AC = x$, $CE = 4$, $BC = 12$ and $CD = x + 2$. Find x.



Show answer

$\triangle ABC \sim \triangle EDC$ by AA, so $\dfrac{AC}{EC} = \dfrac{BC}{DC}$: $\dfrac{x}{4} = \dfrac{12}{x+2}$. Then $x^2 + 2x - 48 = 0$, $(x+8)(x-6) = 0$, and since $x > 0$, $x = 6$.

14. **Challenge** Acute triangles ABC and DEF in the figure are similar, $\triangle ABC \sim \triangle DEF$. $\overline{BG}$ is the altitude from B to $\overline{AC}$ and $\overline{EH}$ is the altitude from E to $\overline{DF}$. Prove that $\frac{BG}{EH} = \frac{AB}{DE}$: corresponding altitudes are in the same ratio as corresponding sides.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\triangle ABC \sim \triangle DEF$, both acute; $\overline{BG}$ is the altitude to $\overline{AC}$; $\overline{EH}$ is the altitude to $\overline{DF}$

Prove: $\frac{BG}{EH} = \frac{AB}{DE}$

	STATEMENTS	REASONS
1	$\triangle ABC \sim \triangle DEF$	Given
2	$\angle BAC \cong \angle EDF$	Definition of similar polygons
3	$\angle BAG \cong \angle EDH$	Same angles as in row 2 (G is on $\overline{AC}$, H is on $\overline{DF}$)
4	$\overline{BG} \perp \overline{AC}$ and $\overline{EH} \perp \overline{DF}$	Definition of altitude
5	$\angle BGA$ and $\angle EHD$ are right angles	Definition of perpendicular lines
6	$\angle BGA \cong \angle EHD$	All right angles are congruent
7	$\triangle ABG \sim \triangle DEH$	AA Similarity Postulate
8	$\frac{BG}{EH} = \frac{AB}{DE}$	Definition of similar polygons

∎

COMMON MISTAKES

- Writing the similarity statement in the wrong order, such as $\triangle ABC \sim \triangle DEC$ when $\angle A$ matches $\angle E$. List the vertices so that congruent angles hold the same positions, then read the proportion from the statement.
- Assuming angles are congruent because they look alike. Each pair needs a reason: given, vertical angles, a shared angle, right angles, or angles formed by parallel lines.
- In nested triangles, using the short piece DB where the whole side AB belongs. The corresponding sides are AD and AB (whole sides of the two triangles), not AD and DB.
- In shadow problems, comparing an object's height with the other object's shadow. Set up the ratio as height : shadow for BOTH objects.

> **Make it a game · AA Match-Up.** Make eight triangle cards with two angles marked, in four similar pairs, such as 40 and 60 degrees against 60 and 80. Lay them face down and flip two at a time. Keep a pair only if the triangles are similar by AA, then state the third angle of each and the scale factor if lengths are given.

8.4 Proving Triangles Similar: SSS and SAS

OBJECTIVES

- State and prove the SSS and SAS Similarity Theorems (Euclid VI.5 and VI.6).
- Choose among AA, SSS and SAS to decide whether two triangles are similar, and explain why SSA is not a test.
- Write similarity proofs in two-column and paragraph form, including proofs that products of segment lengths are equal.

WATCH

Interesting HSC Geometry Proof (1 of 2: Triangle Similarity)

Eddie Woo · 7:17

https://www.youtube.com/watch?v=OYAI04H9oaE

> **WHEN TO WATCH**
> After the SSS and SAS Similarity Theorems: the four similarity tests side by side with the congruence tests.

Note: Eddie Woo teaches in Australia: he says 'equiangular' for AA and 'RHS' for a right-triangle test the course does not use.

BACKUP VIDEO

Triangle Similarity (quick review)

Eddie Woo · 11:17

https://www.youtube.com/watch?v=0-Uy1ehLkqQ

Before the practice set: a review of every similarity test.

KEY TERMS

Side-Side-Side (SSS) Similarity Theorem

If the three sides of one triangle are proportional to the three sides of another triangle, then the triangles are similar.

Side-Angle-Side (SAS) Similarity Theorem

If an angle of one triangle is congruent to an angle of another triangle, and the sides including those angles are proportional, then the triangles are similar.

included angle

The angle formed by two sides of a triangle; for sides $\overline{AB}$ and $\overline{AC}$ it is $\angle A$.

CORE IDEAS

AA uses only angles. Sometimes you know sides instead: a carpenter can measure the three edges of two triangular gables, but not easily their angles. Two more tests let you prove similarity from sides. **SSS Similarity:** if all three pairs of corresponding sides have the same ratio, the triangles are similar. **SAS Similarity:** if one pair of angles is congruent and the sides that form those angles are in the same ratio, the triangles are similar.

Why are they true? The idea of each proof is to build a copy. Inside the larger triangle, mark off a segment as long as the corresponding side of the smaller triangle, and draw a line parallel to the third side. That cuts off a small triangle that is similar to the large one (Lesson 8.3), and with a little algebra it turns out to be CONGRUENT to the other triangle (by SSS or SAS congruence). Similarity then passes from one triangle to the other. Euclid proves VI.5 and VI.6 in the same spirit: on a side of one triangle he builds a triangle equiangular with the other one, and then shows it is equal in every part to the first.

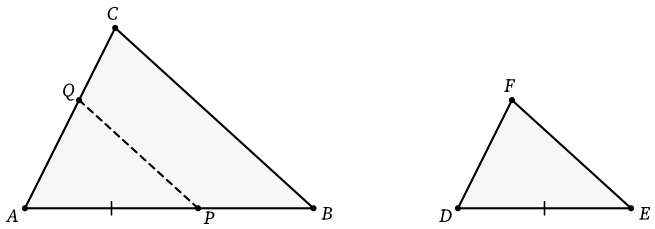
The angle in SAS must be the **included** angle, between the two proportional sides. Just as SSA is not a congruence test (Lesson 4.3), SSA is not a similarity test: a side of fixed length can swing to two different positions and close two triangles of different shapes.

To check SSS, pair the sides by size: shortest with shortest, middle with middle, longest with longest, and compute all three ratios. If even one ratio differs, the triangles are NOT similar. The order of vertices in your similarity statement must follow this pairing.

Similarity is a powerful way to prove that products of lengths are equal. If $\triangle ADE \sim \triangle ACB$, then $\frac{AD}{AC} = \frac{AE}{AB}$, and the Cross Products Property gives $AD \cdot AB = AE \cdot AC$. Working backward, to prove a product statement, rewrite it as a proportion and look for two similar triangles whose sides appear in it.

THEOREM **SSS Similarity Theorem** Euclid VI.5

If the corresponding side lengths of two triangles are proportional, then the triangles are similar.



PARAGRAPH PROOF

Given: $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA} = k$

Prove: $\triangle ABC \sim \triangle DEF$

If $k = 1$, the corresponding sides are congruent, so $\triangle ABC \cong \triangle DEF$ by SSS, and congruent triangles are similar with scale factor 1. If $k > 1$, exchange the names of the two triangles; so we may suppose $k < 1$, that is, $DE < AB$.

Choose P on $\overline{AB}$ with $AP = DE$ (Ruler Postulate), and through P draw the line parallel to $\overline{BC}$ (Parallel Postulate), meeting $\overline{AC}$ at Q. By the Parallel Line Similarity Theorem (Lesson 8.3), $\triangle APQ \sim \triangle ABC$, so $\frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC}$.

Since $AP = DE$, $\frac{AP}{AB} = \frac{DE}{AB} = k$. Therefore $\frac{PQ}{BC} = k$ and $\frac{AQ}{AC} = k$, so $PQ = k \cdot BC = EF$ and $AQ = k \cdot CA = FD$. Now $\triangle APQ$ and $\triangle DEF$ have $AP = DE$, $PQ = EF$ and $AQ = DF$, so $\triangle APQ \cong \triangle DEF$ by SSS.

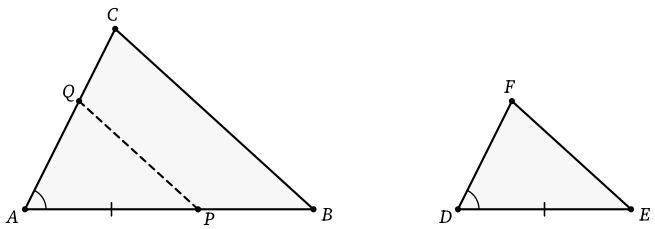
By CPCTC, $\angle PAQ \cong \angle D$ and $\angle APQ \cong \angle E$. But $\angle PAQ$ is $\angle A$, and $\angle APQ \cong \angle B$ by the Corresponding Angles Postulate ($\overline{PQ} \parallel \overline{BC}$). By the Transitive Property of Congruence, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. So $\triangle ABC \sim \triangle DEF$ by the AA Similarity Postulate.

∎

Euclid VI.5: "If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend."

THEOREM **SAS Similarity Theorem** Euclid VI.6

If an angle of one triangle is congruent to an angle of a second triangle, and the lengths of the



sides including these angles are proportional, then the triangles are similar.

TWO-COLUMN PROOF

Given: $\angle A \cong \angle D$; $\dfrac{DE}{AB} = \dfrac{DF}{AC}$, with $DE < AB$

Prove: $\triangle ABC \sim \triangle DEF$

	STATEMENTS	REASONS
1	$\angle A \cong \angle D$; $\dfrac{DE}{AB} = \dfrac{DF}{AC}$	Given
2	Choose P on $\overline{AB}$ with $AP = DE$	Ruler Postulate
3	Draw the line through P parallel to $\overline{BC}$, meeting $\overline{AC}$ at Q	Parallel Postulate (Playfair's Axiom)
4	$\triangle APQ \sim \triangle ABC$	Parallel Line Similarity Theorem
5	$\dfrac{AQ}{AC} = \dfrac{AP}{AB}$	Definition of similar polygons
6	$\dfrac{AQ}{AC} = \dfrac{DE}{AB}$	Substitution ($AP = DE$)
7	$\dfrac{AQ}{AC} = \dfrac{DF}{AC}$	Transitive Property of Equality (rows 1 and 6)
8	$AQ = DF$	Multiplication Property of Equality
9	$\triangle APQ \cong \triangle DEF$	SAS ($AP = DE$, $\angle PAQ \cong \angle D$, $AQ = DF$)
10	$\angle APQ \cong \angle E$	CPCTC
11	$\angle APQ \cong \angle B$	Corresponding Angles Postulate ($\overline{PQ} \parallel \overline{BC}$)
12	$\angle B \cong \angle E$	Transitive Property of Congruence
13	$\triangle ABC \sim \triangle DEF$	AA Similarity Postulate

∎

If $DE > AB$, the same proof works with the two triangles exchanged; if $DE = AB$, then $DF = AC$ and the triangles are congruent by SAS, hence similar. Euclid VI.6: "If two triangles

have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular."

KEY FORMULAS

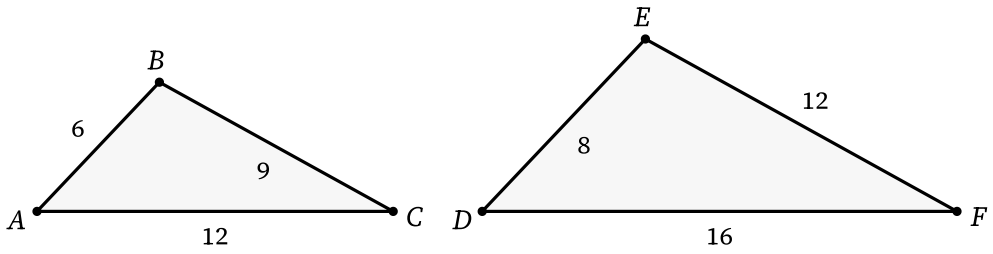
$$\text{SSS:}\quad \frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA} \implies \triangle ABC \sim \triangle DEF$$

$$\text{SAS:}\quad \angle A \cong \angle D \text{ and } \frac{DE}{AB} = \frac{DF}{AC} \implies \triangle ABC \sim \triangle DEF$$

Euclid connection. Euclid's VI.5 and VI.6 are our SSS and SAS Similarity Theorems. In both, Euclid constructs on a side of one triangle a triangle equiangular with the other one, then shows it is equal in every part to the triangle on that side, much as our proof builds $\triangle APQ$ and proves it congruent to $\triangle DEF$.

Practice habit. Recite the three similarity tests (AA, SSS, SAS) with their exact wording, stressing the word "included" in SAS. Plan each proof backward: write the ratio or product you must reach, then find the two triangles whose sides appear in it.

Example 1. In the figure, $\triangle ABC$ has $AB = 6$, $BC = 9$ and $AC = 12$, and $\triangle DEF$ has $DE = 8$, $EF = 12$ and $DF = 16$. Show that the triangles are similar and find the scale factor from $\triangle ABC$ to $\triangle DEF$.

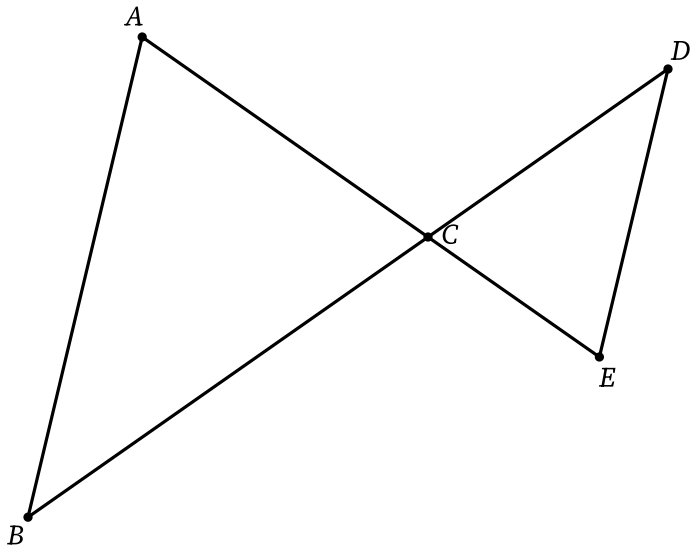


Show solution

1. Pair the sides by size: shortest $AB = 6$ with $DE = 8$, middle $BC = 9$ with $EF = 12$, longest $AC = 12$ with $DF = 16$.
2. Compute the ratios: $\frac{8}{6} = \frac{4}{3}$, $\frac{12}{9} = \frac{4}{3}$, $\frac{16}{12} = \frac{4}{3}$.
3. All three ratios are equal, so $\triangle ABC \sim \triangle DEF$ by the SSS Similarity Theorem, with scale factor $\frac{4}{3}$.

Answer: $\triangle ABC \sim \triangle DEF$ by SSS; the scale factor is $\frac{4}{3}$.

Example 2. In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, and $AC \cdot DC = BC \cdot EC$. Prove that $\overline{AB} \parallel \overline{DE}$.



Show solution

1. Work backward: to prove lines parallel, it is enough to find congruent alternate interior angles, such as $\angle CAB$ and $\angle CED$.
2. Those angles would be corresponding angles of $\triangle ACB$ and $\triangle ECD$, so try to prove these triangles similar.
3. Turn the product into a proportion, $\dfrac{AC}{EC} = \dfrac{BC}{DC}$; with the vertical angles at C this is SAS Similarity.

TWO-COLUMN PROOF

Given: $\overline{AE}$ and $\overline{BD}$ intersect at C; $AC \cdot DC = BC \cdot EC$

Prove: $\overline{AB} \parallel \overline{DE}$

	STATEMENTS	REASONS
1	$\overline{AE}$ and $\overline{BD}$ intersect at C; $AC \cdot DC = BC \cdot EC$	Given
2	$\dfrac{AC}{EC} = \dfrac{BC}{DC}$	Division Property of Equality (divide both sides by $EC \cdot DC$)
3	$\angle ACB \cong \angle ECD$	Vertical Angles Theorem
4	$\triangle ACB \sim \triangle ECD$	SAS Similarity Theorem

	STATEMENTS	REASONS
5	$\angle CAB \cong \angle CED$	Definition of similar polygons
6	$\overline{AB} \parallel \overline{DE}$	Converse of the Alternate Interior Angles Theorem

∎

Answer: See the model proof.

PRACTICE

1. Easy A triangle has sides 3, 4 and 6. Another triangle has sides 9, 12 and 18. Show that the triangles are similar and find the scale factor from the smaller to the larger.

   Show answer

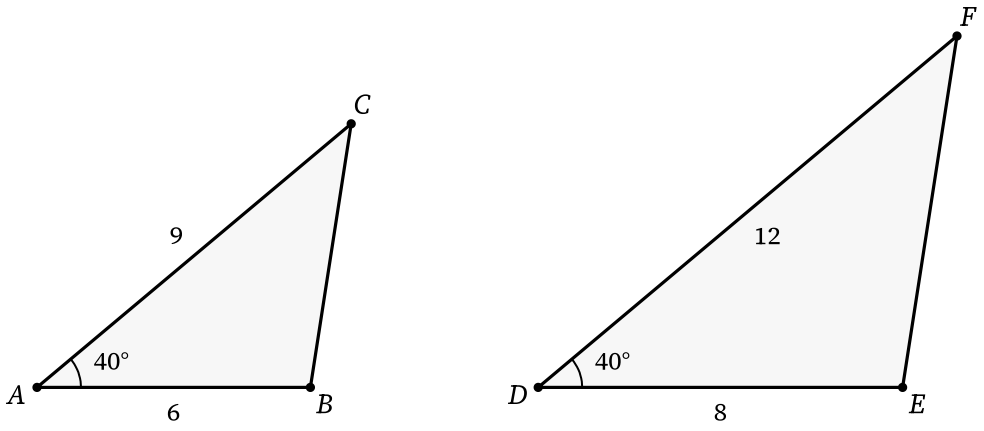
   $\frac{9}{3} = \frac{12}{4} = \frac{18}{6} = 3$, so they are similar by SSS with scale factor 3.

2. Easy A triangle has sides 5, 7 and 9. Another triangle has sides 10, 14 and 20. Are the triangles similar? Answer yes or no and explain.

   Show answer

   No: $\frac{10}{5} = 2$ and $\frac{14}{7} = 2$, but $\frac{20}{9} \neq 2$.

3. Easy In the figure, $m\angle A = m\angle D = 40°$, $AB = 6$, $AC = 9$, $DE = 8$ and $DF = 12$. Which similarity test proves $\triangle ABC \sim \triangle DEF$: AA, SSS or SAS?



   Show answer

   SAS: $\angle A \cong \angle D$ and $\frac{8}{6} = \frac{12}{9} = \frac{4}{3}$ for the sides including those angles.

4. **Easy** In $\triangle ABC$ and $\triangle DEF$, $m\angle B = m\angle E = 30°$, $AB = 6$, $AC = 4$, $DE = 12$ and $DF = 8$. Can you conclude that $\triangle ABC \sim \triangle DEF$? Answer yes or no and explain.

Show answer

No. The congruent angles $\angle B$ and $\angle E$ are not included between the proportional sides ($\overline{AB}$ and $\overline{AC}$ meet at A, not at B). This is SSA, which is not a similarity test: with $AB = 6$ and $m\angle B = 30°$, a side of length 4 from A can reach ray BC at two different points, so two triangles of different shapes fit these measurements.

5. **Medium** A triangle has sides 4, 6 and 8. A second triangle has sides 6, 9 and x, where x is its longest side. Find x so that the triangles are similar.

Show answer

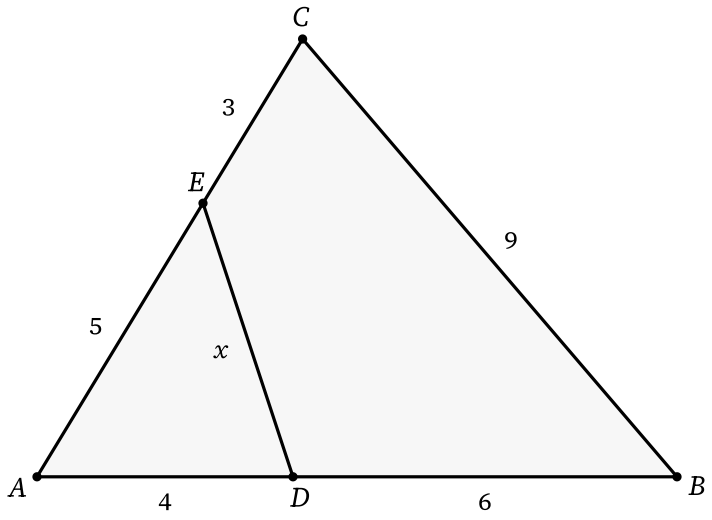
The scale factor is $\frac{6}{4} = \frac{9}{6} = \frac{3}{2}$, so $x = \frac{3}{2} \cdot 8 = 12$.

6. **Medium** The triangular gable of a house has sides 12 ft, 12 ft and 20 ft. The gable of a garden shed has sides 9 ft, 9 ft and 15 ft. Are the two gables similar? Answer yes or no and name the test.

Show answer

Yes, by SSS: $\frac{9}{12} = \frac{9}{12} = \frac{15}{20} = \frac{3}{4}$.

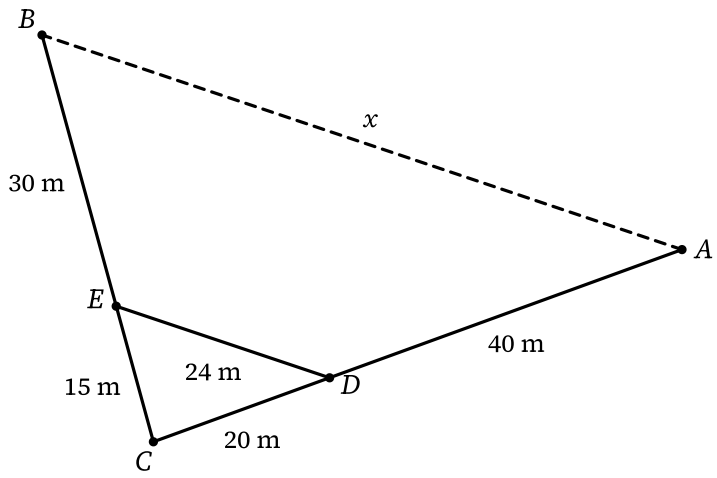
7. **Medium** In the figure, D is on $\overline{AB}$ and E is on $\overline{AC}$, with $AD = 4$, $DB = 6$, $AE = 5$, $EC = 3$ and $BC = 9$. Show that $\triangle ADE \sim \triangle ACB$ and find DE.



Show answer

$AB = 10$ and $AC = 8$, so $\dfrac{AD}{AC} = \dfrac{4}{8} = \dfrac{1}{2}$ and $\dfrac{AE}{AB} = \dfrac{5}{10} = \dfrac{1}{2}$; with the shared $\angle A$, $\triangle ADE \sim \triangle ACB$ by SAS. Then $DE = \dfrac{1}{2} \cdot BC = 4.5$.

8. **Medium** To find the distance AB across a pond, Grace stands at C and measures $CA = 60$ m and $CB = 45$ m. She marks D on $\overline{CA}$ with $CD = 20$ m and E on $\overline{CB}$ with $CE = 15$ m, and measures $DE = 24$ m. Find AB.



Show answer

$\dfrac{CD}{CA} = \dfrac{20}{60} = \dfrac{1}{3}$ and $\dfrac{CE}{CB} = \dfrac{15}{45} = \dfrac{1}{3}$, and $\angle C$ is shared, so $\triangle CDE \sim \triangle CAB$ by SAS. Then $AB = 3 \cdot 24 = 72$ m.

9. **Medium** $\triangle ABC \sim \triangle DEF$ with $AB = x + 1$, $BC = 8$, $CA = 10$, $DE = 9$, $EF = 12$ and $FD = y$. Find x and y.

Show answer

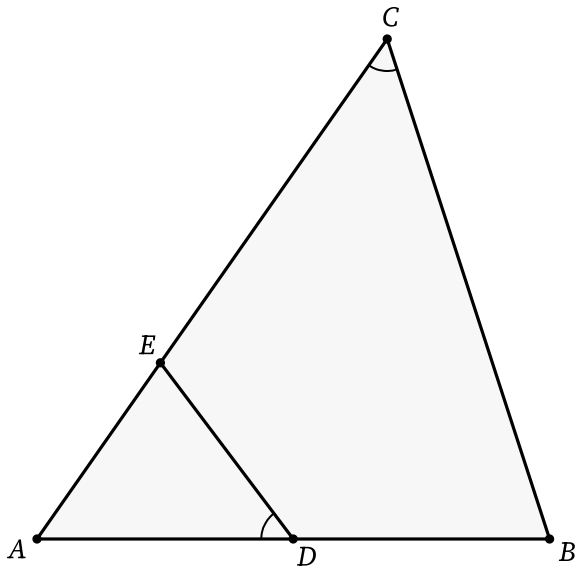
The scale factor from $\triangle ABC$ to $\triangle DEF$ is $\frac{12}{8} = \frac{3}{2}$. Then $x + 1 = \frac{2}{3} \cdot 9 = 6$, so $x = 5$, and $y = \frac{3}{2} \cdot 10 = 15$.

10. **Medium** Henry tests whether a triangle with sides 4, 6 and 8 is similar to one with sides 12, 9 and 6 by computing $\frac{12}{4}$, $\frac{9}{6}$ and $\frac{6}{8}$, and he concludes the triangles are not similar. Explain his error and give the correct conclusion.

Show answer

He paired the sides in the order they were listed instead of by size. Corresponding sides must be shortest with shortest, middle with middle, longest with longest: $\frac{6}{4} = \frac{9}{6} = \frac{12}{8} = \frac{3}{2}$, so the triangles ARE similar by SSS.

11. **Hard** In the figure, D is on $\overline{AB}$, E is on $\overline{AC}$, and $\angle ADE \cong \angle ACB$. Prove that $AD \cdot AB = AE \cdot AC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

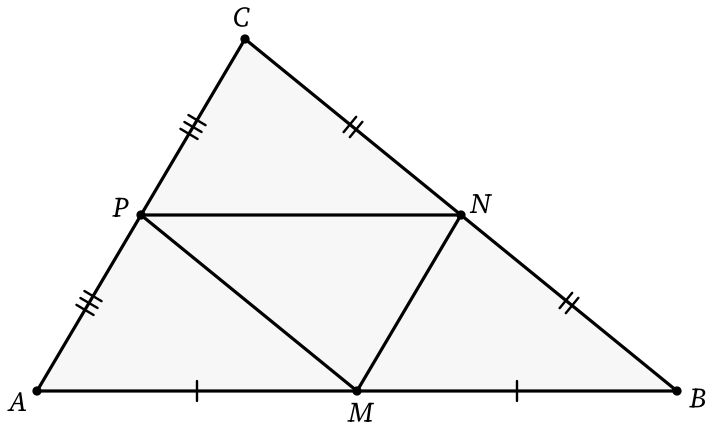
Given: D is on $\overline{AB}$ and E is on $\overline{AC}$; $\angle ADE \cong \angle ACB$

Prove: $AD \cdot AB = AE \cdot AC$

	STATEMENTS	REASONS
1	$\angle ADE \cong \angle ACB$	Given
2	$\angle DAE \cong \angle CAB$	Reflexive Property of Congruence (same angle)
3	$\triangle ADE \sim \triangle ACB$	AA Similarity Postulate
4	$\frac{AD}{AC} = \frac{AE}{AB}$	Definition of similar polygons
5	$AD \cdot AB = AE \cdot AC$	Cross Products Property

∎

12. Hard In the figure, M, N and P are the midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ of $\triangle ABC$. Write a paragraph proof that $\triangle NPM \sim \triangle ABC$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: M, N, P are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$

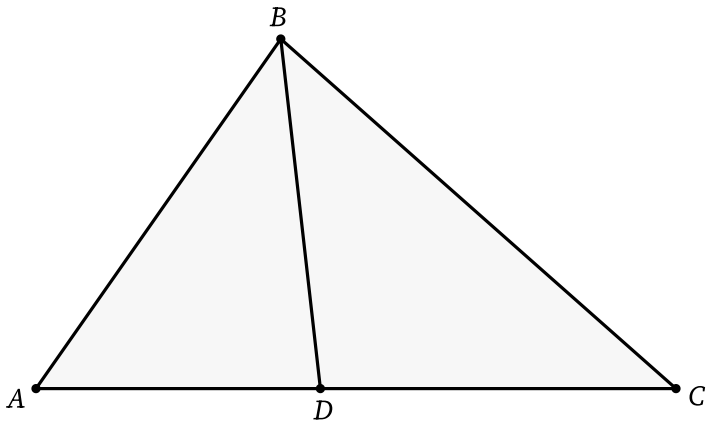
Prove: $\triangle NPM \sim \triangle ABC$

Segment $\overline{NP}$ joins the midpoints of $\overline{BC}$ and $\overline{CA}$, so by the Triangle Midsegment Theorem (Lesson 5.4), $NP = \frac{1}{2}AB$. In the same way, $\overline{PM}$ joins the midpoints of $\overline{CA}$ and $\overline{AB}$, so $PM = \frac{1}{2}BC$, and $\overline{MN}$ joins the midpoints of $\overline{AB}$ and $\overline{BC}$, so $MN = \frac{1}{2}CA$.

Dividing (Division Property of Equality), $\frac{NP}{AB} = \frac{PM}{BC} = \frac{MN}{CA} = \frac{1}{2}$. The three sides of $\triangle NPM$ are proportional to the corresponding sides of $\triangle ABC$, so $\triangle NPM \sim \triangle ABC$ by the SSS Similarity Theorem, with scale factor $\frac{1}{2}$.

∎

13. **Challenge** In the figure, D is on $\overline{AC}$ of $\triangle ABC$ and $AB^2 = AD \cdot AC$. Prove that $\angle ABD \cong \angle ACB$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: D is on $\overline{AC}$; $AB^2 = AD \cdot AC$

Prove: $\angle ABD \cong \angle ACB$

	STATEMENTS	REASONS
1	$AB^2 = AD \cdot AC$	Given
2	$\frac{AB}{AC} = \frac{AD}{AB}$	Division Property of Equality (divide both sides by $AB \cdot AC$)
3	$\angle BAD \cong \angle CAB$	Reflexive Property of Congruence (same angle, since D is on $\overline{AC}$)
4	$\triangle ABD \sim \triangle ACB$	SAS Similarity Theorem
5	$\angle ABD \cong \angle ACB$	Definition of similar polygons

∎

14. **Challenge** A triangle has sides 8, 12 and 18. A second triangle is similar to it, and two of its sides are 12 and 18. Find every possible length of the third side of the second triangle.

Show answer

If 12 and 18 match 8 and 12, the scale factor is $\frac{3}{2}$ and the third side is $\frac{3}{2} \cdot 18 = 27$. If 12 and 18 match 12 and 18, the scale factor is 1 and the third side is 8. (Matching them with 8 and 18 gives ratios $\frac{3}{2}$ and 1, which disagree.) So the third side is 8 or 27.

COMMON MISTAKES

- Using SAS Similarity with an angle that is not between the two proportional sides. Mark the two sides on the figure and check that the angle is where they meet; otherwise it is SSA, which proves nothing.
- Pairing sides in the order they are listed instead of by size. Sort each triangle's sides from shortest to longest before forming ratios.
- Checking only two of the three ratios for SSS. All three must be equal.
- In overlapping triangles such as $\triangle ADE$ and $\triangle ACB$, assuming $\overline{AD}$ matches $\overline{AB}$. Let the congruent or proportional parts decide the correspondence, and redraw the two triangles separately.

> **Make it a game · Similarity Verdict Sprint.** Parent reads two triangles' sides, such as 3, 4, 5 and 6, 8, 10, or 4, 6, 9 and 6, 9, 12. Compare ratios aloud and call the verdict: SSS, SAS, AA or not similar, in 10 seconds. Trap alert: sometimes the data is SSA. Try ten pairs and record your score, then beat it tomorrow.

8.5 Proportions in Triangles

OBJECTIVES

- Prove and use the Triangle Proportionality (side-splitter) Theorem and its converse (Euclid VI.2).
- Use the fact that three parallel lines cut any two transversals proportionally.
- Prove and use the Triangle Angle Bisector Theorem (Euclid VI.3).
- Divide a segment into equal parts or in a given ratio with compass and straightedge (Euclid VI.9 and VI.10).

WATCH

Ratios of Intercepts on Transversals (1 of 2: Setting up the problem)

Eddie Woo · 11:37

https://www.youtube.com/watch?v=tHqG0lwly1o

> **WHEN TO WATCH**
> Before the Triangle Proportionality Theorem: parallel lines cutting transversals in the same ratio.

Note: Eddie Woo teaches in Australia: he says 'intercepts on transversals'. The Triangle Angle Bisector Theorem and dividing a segment are taught from the lesson.

BACKUP VIDEO

Ratios of Intercepts on Transversals (2 of 2: Constructing the proof)

Eddie Woo · 8:34

https://www.youtube.com/watch?v=T6Pl9ggOKH8

After the theorem: building the proof.

KEY TERMS

Triangle Proportionality Theorem (side-splitter)

If a line parallel to one side of a triangle meets the other two sides at different points, it divides those sides proportionally.

Converse of the Triangle Proportionality Theorem

If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Triangle Angle Bisector Theorem

An angle bisector of a triangle divides the opposite side into two segments whose lengths are proportional to the lengths of the other two sides.

transversal

A line that crosses two or more other lines at different points.

CORE IDEAS

In Lesson 8.3 you proved that a line parallel to one side of a triangle cuts off a smaller similar triangle, so $\frac{AD}{AB} = \frac{AE}{AC}$, comparing a piece with a WHOLE side. Often it is more useful to compare the two pieces of each side: $\frac{AD}{DB} = \frac{AE}{EC}$. This is the **Triangle Proportionality Theorem**, sometimes called the side-splitter theorem. Its proof is the similarity from Lesson 8.3 plus the Segment Addition Postulate and a property of proportions.

The converse is also true: if a segment cuts two sides of a triangle in the same ratio, it must be parallel to the third side. This gives you a new way to prove lines parallel, using only lengths. Euclid states both directions in one proposition, VI.2.

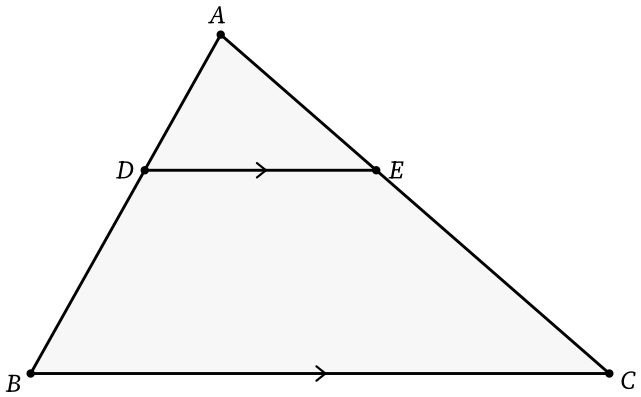
Extend the idea to several parallel lines. If three (or more) parallel lines cross two transversals, they cut the transversals proportionally. Why? Draw a diagonal from the first transversal to the second; it splits the picture into two triangles, and the side-splitter theorem applies to each. This is exactly the situation of town lots between parallel streets: the frontages along one road are proportional to the frontages along any other road that crosses the same streets.

An angle bisector also divides the opposite side in a predictable ratio: in $\triangle ABC$, if $\overline{AD}$ bisects $\angle A$, then $\frac{BD}{DC} = \frac{AB}{AC}$. The longer side gets the longer piece. Euclid's proof (VI.3) is clever: draw a line through C parallel to the bisector. It makes an isosceles triangle, and then the side-splitter theorem finishes the job.

These theorems give a construction you could not do before: dividing a segment into any number of equal parts. Mark equal steps along a second ray from one endpoint, join the last mark to the other endpoint, and draw parallels through the other marks. Because parallel lines cut transversals proportionally, the equal steps on the ray become equal steps on the segment.

THEOREM **Triangle Proportionality Theorem** Euclid VI.2

If a line parallel to one side of a triangle intersects the other two sides at different points, then it divides the two sides proportionally: in $\triangle ABC$ with D on $\overline{AB}$, E on $\overline{AC}$ and $\overline{DE} \parallel \overline{BC}$, $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.



TWO-COLUMN PROOF

Given: D is on $\overline{AB}$ and E is on $\overline{AC}$; $\overline{DE} \parallel \overline{BC}$

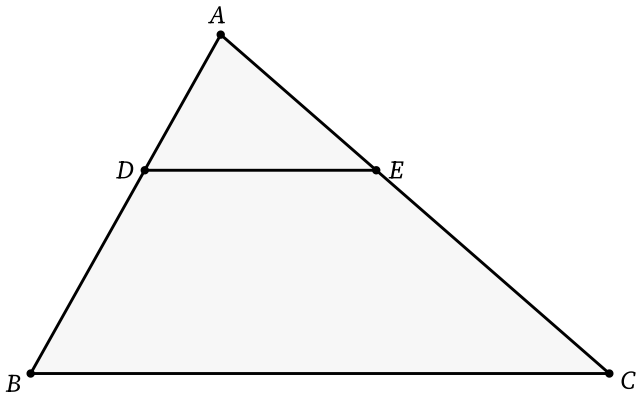
Prove: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

	STATEMENTS	REASONS
1	$\overline{DE} \parallel \overline{BC}$	Given
2	$\angle ADE \cong \angle ABC$	Corresponding Angles Postulate
3	$\angle DAE \cong \angle BAC$	Reflexive Property of Congruence
4	$\triangle ADE \sim \triangle ABC$	AA Similarity Postulate
5	$\dfrac{AB}{AD} = \dfrac{AC}{AE}$	Definition of similar polygons
6	$AB = AD + DB$ and $AC = AE + EC$	Segment Addition Postulate
7	$\dfrac{AD + DB}{AD} = \dfrac{AE + EC}{AE}$	Substitution
8	$1 + \dfrac{DB}{AD} = 1 + \dfrac{EC}{AE}$	Writing each fraction as a sum: $\dfrac{x+y}{x} = \dfrac{x}{x} + \dfrac{y}{x} = 1 + \dfrac{y}{x}$
9	$\dfrac{DB}{AD} = \dfrac{EC}{AE}$	Subtraction Property of Equality
10	$\dfrac{AD}{DB} = \dfrac{AE}{EC}$	Properties of Proportions (reciprocals)

∎

THEOREM **Converse of the Triangle Proportionality Theorem** Euclid VI.2

If a line divides two sides of a triangle proportionally, then it is parallel to the third side: if D is on $\overline{AB}$, E is on $\overline{AC}$ and $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, then $\overline{DE} \parallel \overline{BC}$.



PARAGRAPH PROOF

Given: D is on $\overline{AB}$ and E is on $\overline{AC}$; $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Prove: $\overline{DE} \parallel \overline{BC}$

Taking reciprocals (Properties of Proportions), $\dfrac{DB}{AD} = \dfrac{EC}{AE}$. Adding 1 to both sides (Addition Property of Equality) and writing $1 = \dfrac{AD}{AD} = \dfrac{AE}{AE}$ gives $\dfrac{AD + DB}{AD} = \dfrac{AE + EC}{AE}$. By the Segment Addition Postulate this says $\dfrac{AB}{AD} = \dfrac{AC}{AE}$, and taking reciprocals again, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.

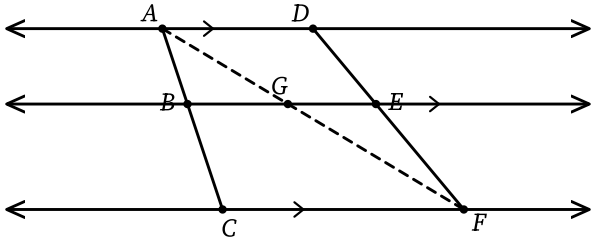
Also $\angle DAE \cong \angle BAC$ by the Reflexive Property of Congruence. So two sides of $\triangle ADE$ are proportional to two sides of $\triangle ABC$ and the included angles are congruent; by the SAS Similarity Theorem, $\triangle ADE \sim \triangle ABC$.

Corresponding angles of similar triangles are congruent, so $\angle ADE \cong \angle ABC$. These are corresponding angles formed by $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ with transversal $\overleftrightarrow{AB}$, so $\overline{DE} \parallel \overline{BC}$ by the Converse of the Corresponding Angles Postulate.

∎

COROLLARY **Three Parallel Lines Theorem**

If three parallel lines intersect two transversals, then they divide the transversals proportionally: if parallel lines ℓ_1, ℓ_2, ℓ_3 meet one



transversal at A, B, C and another at D, E, F, then $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

PARAGRAPH PROOF

Given: $\ell_1 \parallel \ell_2 \parallel \ell_3$; one transversal meets them at A, B, C and another at D, E, F

Prove: $\dfrac{AB}{BC} = \dfrac{DE}{EF}$

Draw the auxiliary segment $\overline{AF}$ (Postulate: two points determine a line), and let it meet ℓ_2 at G.

In $\triangle ACF$, $\overline{BG}$ lies on ℓ_2, which is parallel to ℓ_3 and so to $\overline{CF}$. By the Triangle Proportionality Theorem, $\dfrac{AB}{BC} = \dfrac{AG}{GF}$.

In $\triangle FAD$, $\overline{GE}$ lies on ℓ_2, which is parallel to ℓ_1 and so to $\overline{AD}$. By the Triangle Proportionality Theorem, $\dfrac{FG}{GA} = \dfrac{FE}{ED}$, and taking reciprocals, $\dfrac{AG}{GF} = \dfrac{DE}{EF}$.

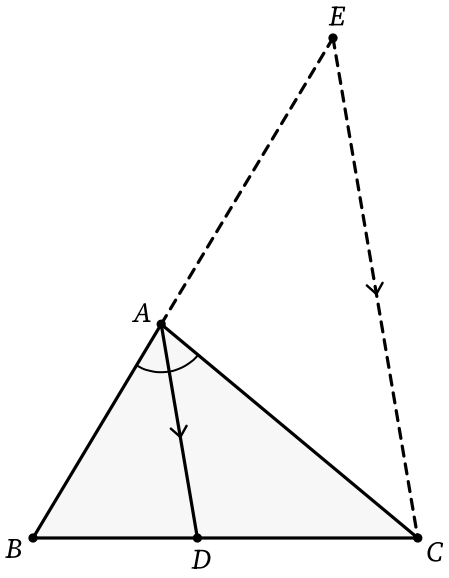
By the Transitive Property of Equality, $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

∎

The same reasoning works for any number of parallel lines, one pair of neighboring lines at a time.

THEOREM **Triangle Angle Bisector Theorem** Euclid VI.3

If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides: if $\overrightarrow{AD}$ bisects $\angle BAC$ with D on $\overline{BC}$, then $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.



TWO-COLUMN PROOF

Given: $\overrightarrow{AD}$ bisects $\angle BAC$, with D on $\overline{BC}$

Prove: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

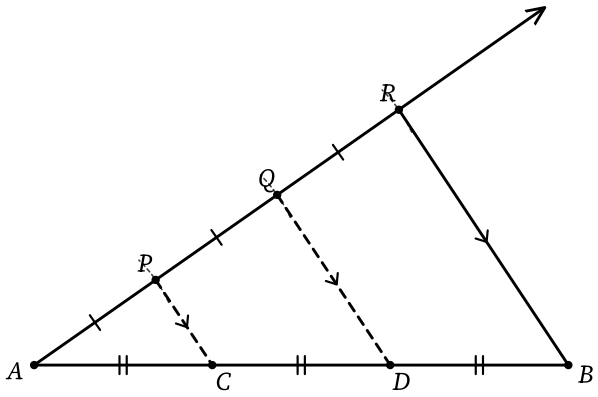
	STATEMENTS	REASONS
1	$\overrightarrow{AD}$ bisects $\angle BAC$	Given
2	Draw the line through C parallel to $\overleftrightarrow{AD}$, meeting $\overleftrightarrow{BA}$ at E (beyond A)	Parallel Postulate (Playfair's Axiom)
3	$\angle BAD \cong \angle DAC$	Definition of angle bisector
4	$\angle BAD \cong \angle AEC$	Corresponding Angles Postulate ($\overline{AD} \parallel \overline{EC}$, transversal $\overleftrightarrow{BE}$)
5	$\angle DAC \cong \angle ACE$	Alternate Interior Angles Theorem ($\overline{AD} \parallel \overline{EC}$, transversal $\overleftrightarrow{AC}$)
6	$\angle AEC \cong \angle ACE$	Transitive Property of Congruence (rows 3, 4, 5)
7	$AE = AC$	Converse of the Isosceles Triangle Theorem
8	$\dfrac{BD}{DC} = \dfrac{BA}{AE}$	Triangle Proportionality Theorem ($\triangle BEC$ with $\overline{AD} \parallel \overline{EC}$)
9	$\dfrac{BD}{DC} = \dfrac{AB}{AC}$	Substitution ($AE = AC$)

∎

This is Euclid's own proof of VI.3: "If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle."

CONSTRUCTION **Divide a segment into three equal parts** Euclid VI.9

1. Given $\overline{AB}$, draw a ray from A that is not on line AB.
2. Keep the compass at one fixed width. Starting at A, mark three equal steps along the ray: P, then Q, then R, so that $AP = PQ = QR$.
3. Draw $\overline{RB}$.
4. Through Q and through P, construct lines parallel to $\overline{RB}$ by copying $\angle ARB$ (Euclid I.31). Let them meet $\overline{AB}$ at D and C.
5. Then $AC = CD = DB$: the points C and D divide $\overline{AB}$ into three equal parts.



Why it works. The lines $\overleftrightarrow{PC}$, $\overleftrightarrow{QD}$ and $\overleftrightarrow{RB}$ are parallel, so by the Three Parallel Lines Theorem (applied to neighboring lines, with the Triangle Proportionality Theorem for the piece at A), the pieces of $\overline{AB}$ are in the same ratio as the equal pieces AP, PQ, QR of the ray. With five equal steps and the parallel through the second mark, the same method divides $\overline{AB}$ in the ratio $2 : 3$; Euclid gives the general method in VI.9 and VI.10.

KEY FORMULAS

$$\overline{DE} \parallel \overline{BC} \iff \frac{AD}{DB} = \frac{AE}{EC}$$

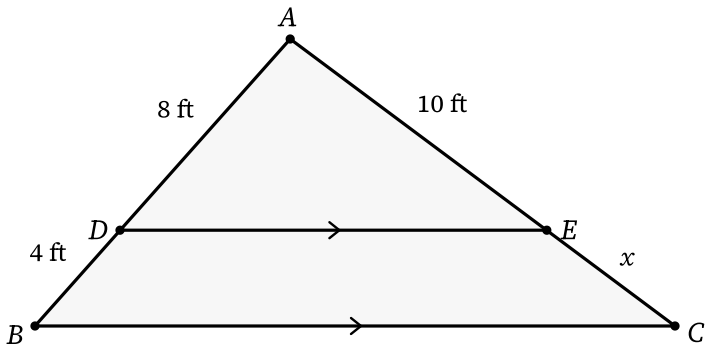
$$\overrightarrow{AD} \text{ bisects } \angle A \implies \frac{BD}{DC} = \frac{AB}{AC}$$

From history. In 1620 the English mathematician Edmund Gunter introduced a surveyor's chain 66 feet long, divided into 100 links. For more than two centuries surveyors in England and America measured farms, roads and town lots with it. Dividing the frontage of lots that run between parallel streets is exactly the kind of problem the theorems of this lesson solve.

Euclid connection. Euclid's VI.2 states both the Triangle Proportionality Theorem and its converse, and VI.3 is the Triangle Angle Bisector Theorem, proved, as here, with a parallel line that creates an isosceles triangle. VI.9 ("from a given straight line to cut off a prescribed part") and VI.10 ("to cut a given uncut straight line similarly to a given cut straight line") are the constructions for dividing a segment.

Practice habit. Recite the Triangle Proportionality Theorem, its converse and the Triangle Angle Bisector Theorem until they are automatic. When you set up a proportion, say which pieces you are comparing (part to part, or part to whole) and keep the same order on both sides.

Example 1. A roof truss is triangle ABC, with a straight brace $\overline{DE}$ parallel to the bottom chord $\overline{BC}$; D is on rafter $\overline{AB}$ and E is on rafter $\overline{AC}$. $AD = 8$ ft, $DB = 4$ ft and $AE = 10$ ft. How long is EC?

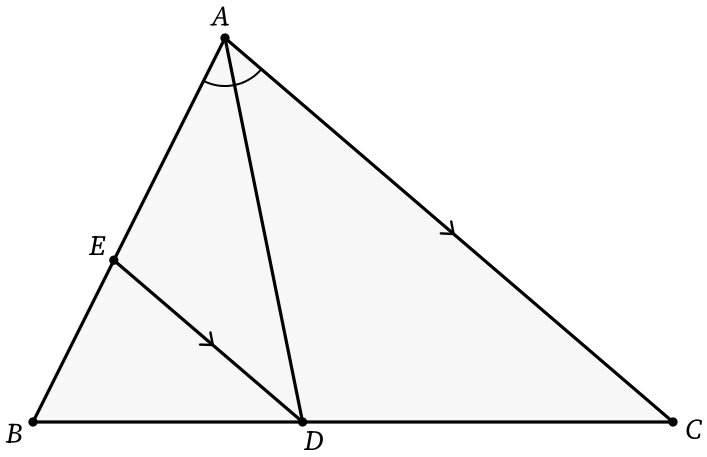


Show solution

1. Since $\overline{DE} \parallel \overline{BC}$, the Triangle Proportionality Theorem gives $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.
2. Substitute: $\dfrac{8}{4} = \dfrac{10}{x}$.
3. Cross products: $8x = 40$, so $x = 5$.
4. Check: $\dfrac{8}{4} = 2$ and $\dfrac{10}{5} = 2$.

Answer: $EC = 5$ ft

Example 2. In the figure, $\overline{AD}$ bisects $\angle BAC$ of $\triangle ABC$ with D on $\overline{BC}$, and $\overline{DE} \parallel \overline{AC}$ with E on $\overline{AB}$. Prove that $\dfrac{BE}{EA} = \dfrac{AB}{AC}$.



Show solution

1. Notice the two theorems at work: the angle bisector splits $\overline{BC}$ in the ratio $\dfrac{AB}{AC}$, and the parallel $\overline{DE}$ copies the split of $\overline{BC}$ onto $\overline{BA}$.
2. Use the Triangle Angle Bisector Theorem in $\triangle ABC$ and the Triangle Proportionality Theorem in $\triangle BAC$ with $\overline{ED} \parallel \overline{AC}$.
3. Join the two proportions with the Transitive Property.

TWO-COLUMN PROOF

Given: $\overline{AD}$ bisects $\angle BAC$, D on $\overline{BC}$; $\overline{DE} \parallel \overline{AC}$, E on $\overline{AB}$

Prove: $\dfrac{BE}{EA} = \dfrac{AB}{AC}$

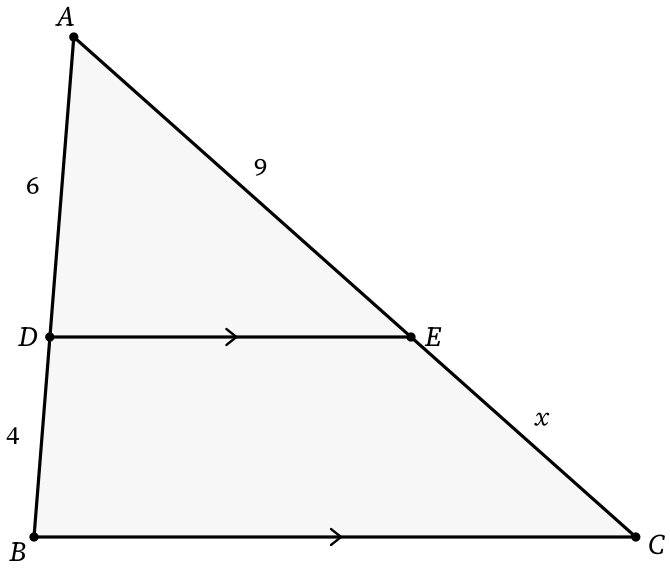
	STATEMENTS	REASONS
1	$\overline{AD}$ bisects $\angle BAC$	Given
2	$\dfrac{BD}{DC} = \dfrac{AB}{AC}$	Triangle Angle Bisector Theorem
3	$\overline{DE} \parallel \overline{AC}$	Given
4	$\dfrac{BE}{EA} = \dfrac{BD}{DC}$	Triangle Proportionality Theorem ($\triangle BAC$)
5	$\dfrac{BE}{EA} = \dfrac{AB}{AC}$	Transitive Property of Equality

∎

Answer: See the model proof.

PRACTICE

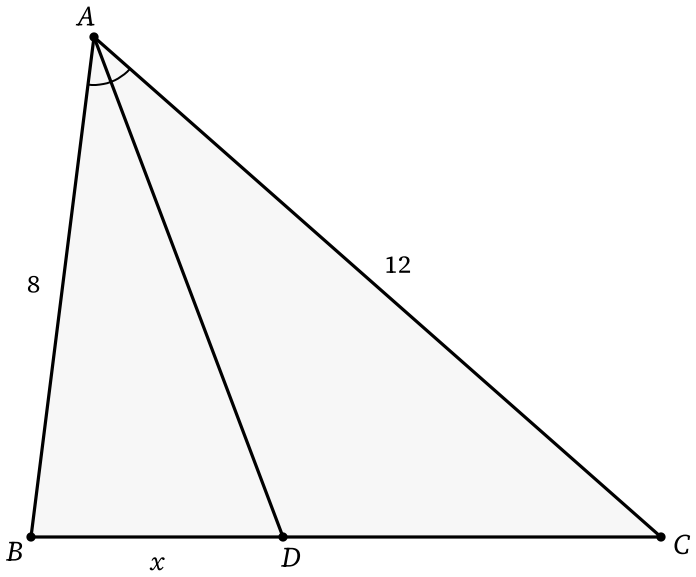
1. **Easy** In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = 6$, $DB = 4$ and $AE = 9$. Find EC.



Show answer

$\frac{6}{4} = \frac{9}{x}$, so $6x = 36$ and $EC = 6$.

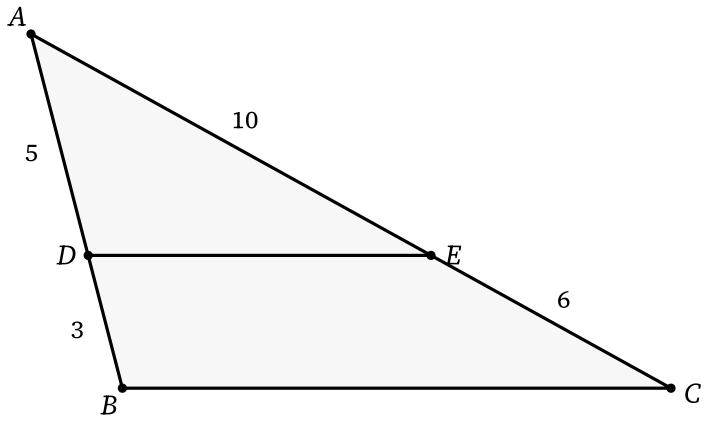
2. **Easy** In the figure, $\overline{AD}$ bisects $\angle BAC$, $AB = 8$, $AC = 12$ and $BC = 10$. Find BD.



Show answer

$\frac{BD}{DC} = \frac{8}{12} = \frac{2}{3}$, so $BD = \frac{2}{5} \cdot 10 = 4$.

3. **Easy** In the figure, $AD = 5$, $DB = 3$, $AE = 10$ and $EC = 6$. Is $\overline{DE} \parallel \overline{BC}$? Answer yes or no and name the theorem you used.



Show answer

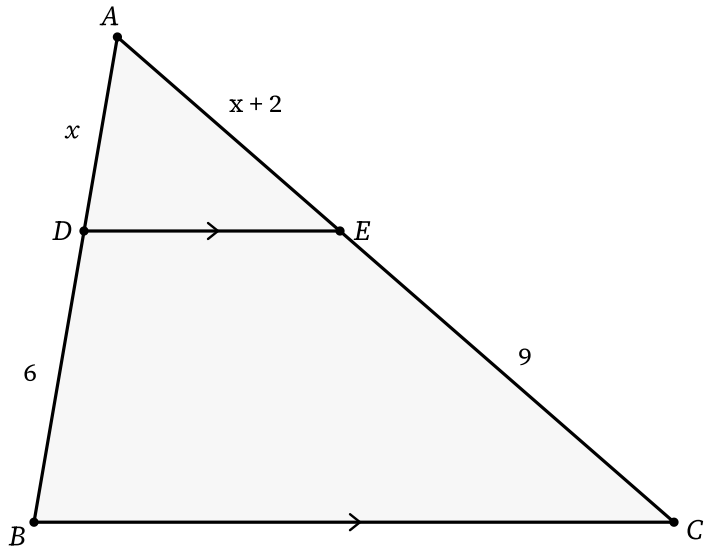
Yes. $\frac{5}{3} = \frac{10}{6}$, so by the Converse of the Triangle Proportionality Theorem, $\overline{DE} \parallel \overline{BC}$.

4. **Easy** Three parallel streets cross Elm Road and Maple Road. Along Elm Road, the blocks between the streets measure 120 ft and 180 ft. Along Maple Road, the first block (between the same first two streets) measures 100 ft. How long is the second block along Maple Road?

Show answer

$\frac{120}{180} = \frac{100}{x}$, so $120x = 18000$ and $x = 150$ ft.

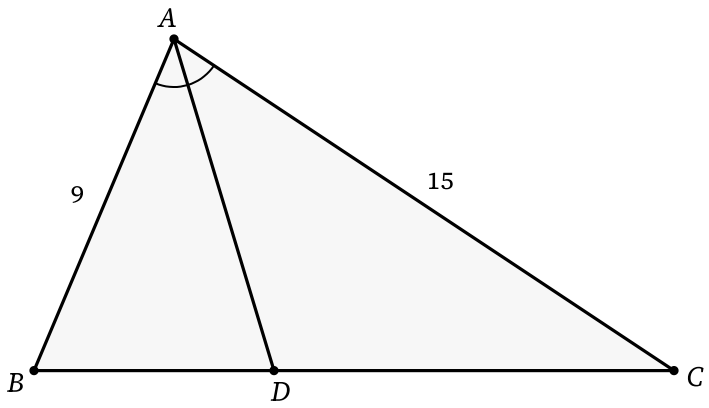
5. **Medium** In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = x$, $DB = 6$, $AE = x + 2$ and $EC = 9$. Find x.



Show answer

$\frac{x}{6} = \frac{x+2}{9}$, so $9x = 6x + 12$ and $x = 4$.

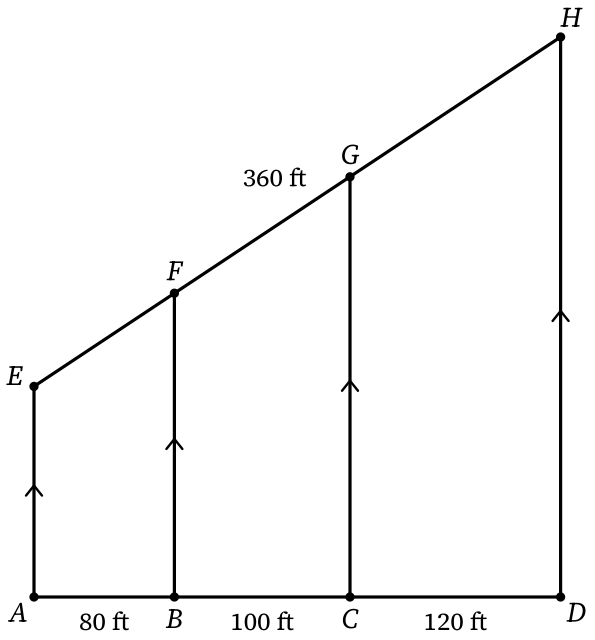
6. **Medium** In the figure, $\overline{AD}$ bisects $\angle BAC$, $AB = 9$, $AC = 15$ and $BC = 16$. Find BD and DC.



Show answer

$\frac{BD}{DC} = \frac{9}{15} = \frac{3}{5}$, so $BD = \frac{3}{8} \cdot 16 = 6$ and $DC = 10$.

7. **Medium** Three lots lie between Back Street and Front Street, separated by parallel side lines, as in the figure. Their frontages on Back Street are 80 ft, 100 ft and 120 ft. The total frontage on Front Street is 360 ft. Find each lot's frontage on Front Street, EF, FG and GH.



Back Street runs along ABCD; Front Street runs along EFGH.

Show answer

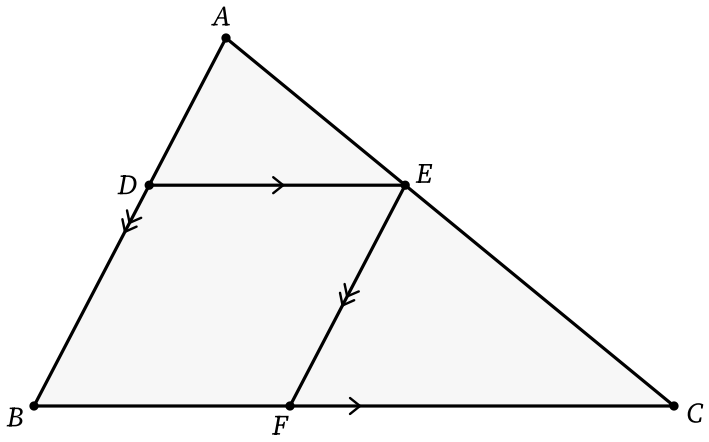
The parallel side lines cut the two streets proportionally. The Back Street total is 300 ft, so each Front Street frontage is $\frac{360}{300} = 1.2$ times the Back Street frontage: $EF = 96$ ft, $FG = 120$ ft, $GH = 144$ ft.

8. **Medium** In $\triangle ABC$, D is on $\overline{AB}$ and E is on $\overline{AC}$, with $AD = 4$, $DB = 6$, $AE = 5$ and $EC = 7$. Ruth thinks $\overline{DE}$ looks parallel to $\overline{BC}$. Is it? Answer yes or no and explain why.

Show answer

No. $\frac{AD}{DB} = \frac{4}{6} = \frac{2}{3}$ but $\frac{AE}{EC} = \frac{5}{7}$, and $\frac{2}{3} \neq \frac{5}{7}$. If $\overline{DE}$ were parallel to $\overline{BC}$, the Triangle Proportionality Theorem would force these ratios to be equal, so $\overline{DE}$ is not parallel to $\overline{BC}$, however it looks.

9. Hard In the figure, D is on $\overline{AB}$ and E is on $\overline{AC}$ of $\triangle ABC$ with $\overline{DE} \parallel \overline{BC}$, and F is on $\overline{BC}$ with $\overline{EF} \parallel \overline{AB}$. Prove that $\dfrac{AD}{DB} = \dfrac{BF}{FC}$.



Show answer

See the model proof.

TWO-COLUMN PROOF

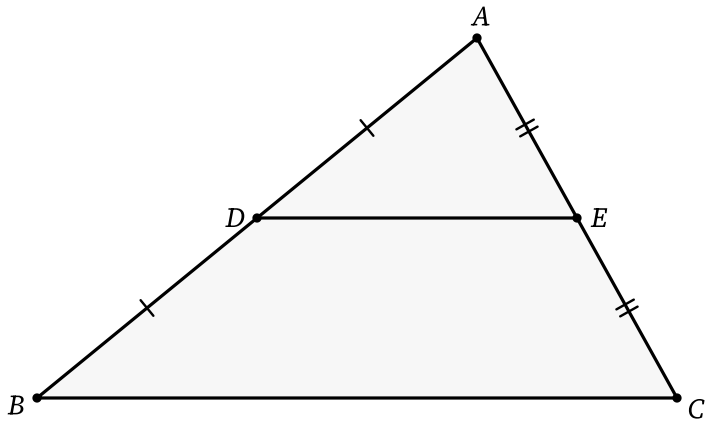
Given: $\overline{DE} \parallel \overline{BC}$, D on $\overline{AB}$, E on $\overline{AC}$; $\overline{EF} \parallel \overline{AB}$, F on $\overline{BC}$

Prove: $\dfrac{AD}{DB} = \dfrac{BF}{FC}$

	STATEMENTS	REASONS
1	$\overline{DE} \parallel \overline{BC}$	Given
2	$\dfrac{AD}{DB} = \dfrac{AE}{EC}$	Triangle Proportionality Theorem ($\triangle ABC$)
3	$\overline{EF} \parallel \overline{AB}$	Given
4	$\dfrac{CE}{EA} = \dfrac{CF}{FB}$	Triangle Proportionality Theorem ($\triangle CAB$)
5	$\dfrac{AE}{EC} = \dfrac{BF}{FC}$	Properties of Proportions (reciprocals)
6	$\dfrac{AD}{DB} = \dfrac{BF}{FC}$	Transitive Property of Equality

∎

10. Hard In the figure, D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$. Write a flow proof that $\overline{DE} \parallel \overline{BC}$, using the Converse of the Triangle Proportionality Theorem. (This is a new proof of part of the Midsegment Theorem.)



Show answer

See the model proof.

FLOW PROOF

Given: D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{AC}$

Prove: $\overline{DE} \parallel \overline{BC}$

(1) D is the midpoint of $\overline{AB}$
Given

(2) E is the midpoint of $\overline{AC}$
Given

(3) $AD = DB$
Definition of midpoint
from (1)

(4) $AE = EC$
Definition of midpoint
from (2)

(5) $\dfrac{AD}{DB} = 1$
Division Property of Equality
from (3)

(6) $\dfrac{AE}{EC} = 1$
Division Property of Equality

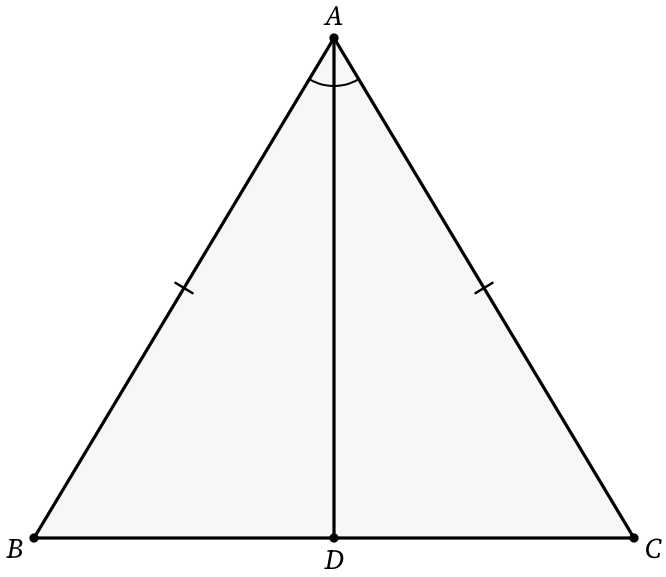
from (4)

(7) $\frac{AD}{DB} = \frac{AE}{EC}$
Transitive Property of Equality
from (5), (6)

(8) $\overline{DE} \parallel \overline{BC}$
Converse of the Triangle Proportionality Theorem
from (7)

∎

11. Hard In the figure, $\triangle ABC$ has $AB = AC$, and $\overline{AD}$ bisects $\angle BAC$ with D on $\overline{BC}$. Use the Triangle Angle Bisector Theorem to write a paragraph proof that D is the midpoint of $\overline{BC}$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $AB = AC$; $\overline{AD}$ bisects $\angle BAC$, D on $\overline{BC}$

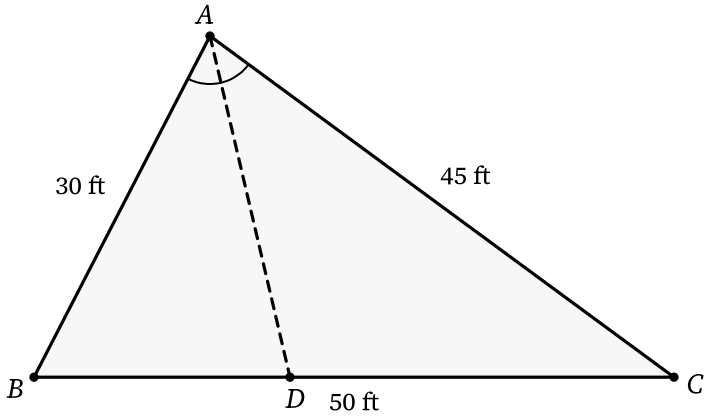
Prove: D is the midpoint of $\overline{BC}$

By the Triangle Angle Bisector Theorem, $\frac{BD}{DC} = \frac{AB}{AC}$. Since $AB = AC$, $\frac{AB}{AC} = 1$, so by substitution $\frac{BD}{DC} = 1$.

Multiplying both sides by DC (Multiplication Property of Equality) gives $BD = DC$. Since D is on $\overline{BC}$ and $BD = DC$, D is the midpoint of $\overline{BC}$ by the definition of midpoint.

∎

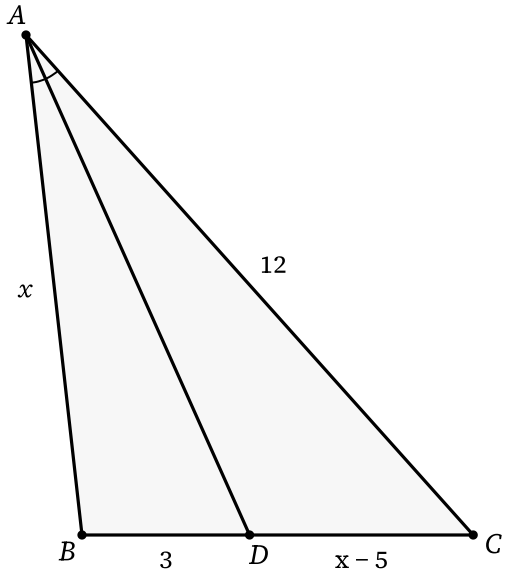
12. **Hard** The Miller family's triangular garden ABC has fences $AB = 30$ ft, $AC = 45$ ft and $BC = 50$ ft. A straight path starts at corner A, bisects the angle there, and ends at point D on fence $\overline{BC}$. How far is D from B, and how far from C?



Show answer

$\frac{BD}{DC} = \frac{30}{45} = \frac{2}{3}$, so $BD = \frac{2}{5} \cdot 50 = 20$ ft and $DC = 30$ ft.

13. **Challenge** In the figure, $\overline{AD}$ bisects $\angle BAC$, $AB = x$, $AC = 12$, $BD = 3$ and $DC = x - 5$. Find x.



Show answer

$\frac{3}{x-5} = \frac{x}{12}$, so $x^2 - 5x = 36$, $x^2 - 5x - 36 = 0$, $(x-9)(x+4) = 0$. Since $x > 5$, $x = 9$.

14. **Challenge** Describe how to divide a given segment $\overline{AB}$ with compass and straightedge at a point C so that $AC : CB = 2 : 3$, and explain why your construction works.

Show answer

Draw a ray from A not on line AB. With a fixed compass width, mark five equal steps P_1, P_2, P_3, P_4, P_5 along it from A. Draw $\overline{P_5B}$. Through P_2 construct the line parallel to $\overline{P_5B}$ (copy $\angle AP_5B$ at P_2, Euclid I.31); it meets $\overline{AB}$ at C. Why: in $\triangle AP_5B$, $\overline{P_2C} \parallel \overline{P_5B}$, so by the Triangle Proportionality Theorem $\dfrac{AC}{CB} = \dfrac{AP_2}{P_2P_5} = \dfrac{2}{3}$.

COMMON MISTAKES

- Mixing part-to-part and part-to-whole ratios, such as $\dfrac{AD}{DB} = \dfrac{AE}{AC}$. Compare piece to piece on both sides ($\dfrac{AD}{DB} = \dfrac{AE}{EC}$) or piece to whole on both sides ($\dfrac{AD}{AB} = \dfrac{AE}{AC}$).
- Using the side-splitter proportion to find the parallel segment itself. $\dfrac{DE}{BC}$ equals $\dfrac{AD}{AB}$ (whole sides of the similar triangles), not $\dfrac{AD}{DB}$.
- In the Angle Bisector Theorem, pairing each piece of the base with the wrong side. The piece next to B goes with the side next to B: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.
- Concluding that a segment is parallel because it looks parallel. Check the ratios and cite the Converse of the Triangle Proportionality Theorem.

> **Make it a game · Split Without a Ruler.** Draw a segment on plain paper. Divide it into five equal parts using only lined notebook paper: lay the segment's end on the first line, slant it to touch the sixth line, and mark where the ruled lines cross. Then solve: DE parallel to BC, AD = 4, DB = 6, AE = 6; find EC.

UNIT 8

Similarity

Congruent figures have the same size and shape; similar figures have the same shape but may differ in size. In this unit you will learn exactly what "same shape" means: corresponding angles congruent and corresponding sides proportional, or, in the language of Unit 7, a dilation followed by rigid motions. You will begin with the algebra of ratios and proportions, meet the geometric mean and the golden ratio, and then learn three short tests (AA, SSS and SAS) for proving triangles similar.

This unit follows Book VI of Euclid's Elements, where the theory of proportion from Book V is applied to figures: equiangular triangles have proportional sides (VI.4), a line parallel to a side of a triangle cuts the other sides proportionally (VI.2), an angle bisector divides the opposite side in the ratio of the other sides (VI.3), and the mean proportional is constructed in a semicircle (VI.13). You will write two-column, paragraph and flow proofs with these theorems and use compass and straightedge to construct a geometric mean and to divide a segment into equal parts.

Similarity is also one of the most practical ideas in geometry. With it you will find the height of a steeple from its shadow, the width of a river without crossing it, the real size of a room from a blueprint, and the frontage of lots between parallel streets. Parents: a student who masters this unit is ready for right-triangle trigonometry in Unit 9, which is built entirely on similar triangles.

PREREQUISITES

- Solve linear and quadratic equations, including equations with fractions (Algebra I).
- Use the Pythagorean Theorem numerically and simplify square roots (Algebra I).
- Use parallel-line angle relationships: corresponding, alternate interior and vertical angles (Lessons 2.4, 3.2, 3.3).
- Prove triangles congruent with SSS, SAS, ASA and AAS and use CPCTC (Lessons 4.3–4.5).
- Use the Isosceles Triangle Theorem and its converse and the Triangle Midsegment Theorem (Lessons 4.6, 5.4).
- Describe dilations and rigid motions and their effect on lengths and angles (Lessons 7.1 and 7.5).

MASTERY CHECKLIST

- ☐ I can simplify ratios, use extended ratios, and solve proportions with the Cross Products Property.
- ☐ I can prove the basic properties of proportions with the properties of equality.
- ☐ I can find the geometric mean of two numbers and construct it in a semicircle (Euclid VI.13).
- ☐ I can explain the golden ratio as Euclid's extreme and mean ratio and find its exact value.
- ☐ I can define similar polygons, find a scale factor, and find missing sides, angles and perimeters of similar figures.
- ☐ I can solve problems with maps, scale models and blueprints.
- ☐ I can state the AA Similarity Postulate and the SSS and SAS Similarity Theorems, and explain why SSA is not a test.
- ☐ I can use similar triangles for indirect measurement with shadows, mirrors and surveying.
- ☐ I can write two-column, paragraph and flow proofs that triangles are similar, and prove that products of lengths are equal.
- ☐ I can use the Triangle Proportionality Theorem and its converse, the Three Parallel Lines Theorem and the Triangle Angle Bisector Theorem.

- ☐ I can divide a segment into equal parts or in a given ratio with compass and straightedge (Euclid VI.9 and VI.10).

LESSONS IN THIS UNIT

8.1	**Ratios, Proportions and the Geometric Mean**	45 minutes
8.2	**Similar Polygons**	45 minutes
8.3	**Proving Triangles Similar: AA**	2 sessions of 45 minutes
8.4	**Proving Triangles Similar: SSS and SAS**	2 sessions of 45 minutes
8.5	**Proportions in Triangles**	2 sessions of 45 minutes

Unit 8 Test

You have 50 minutes. No calculator is needed or allowed. Show your work for every problem. Give exact answers unless a problem asks you to round, and include units in answers to word problems. In every proof, give a reason for every statement. You will need a compass and a straightedge for problem 14. Figures are drawn to scale, but use only the given information and the marks on the figure in your reasoning.

1. **5 pts**
   Solve the proportion $\frac{x-1}{5} = \frac{x+5}{15}$.

2. **5 pts**
   Find the geometric mean of 8 and 18.

3. **5 pts**
   A rectangular stained-glass window for a church has width and height in the ratio $2:5$. Its perimeter is 28 ft. Find its width and its height.

4. **6 pts**

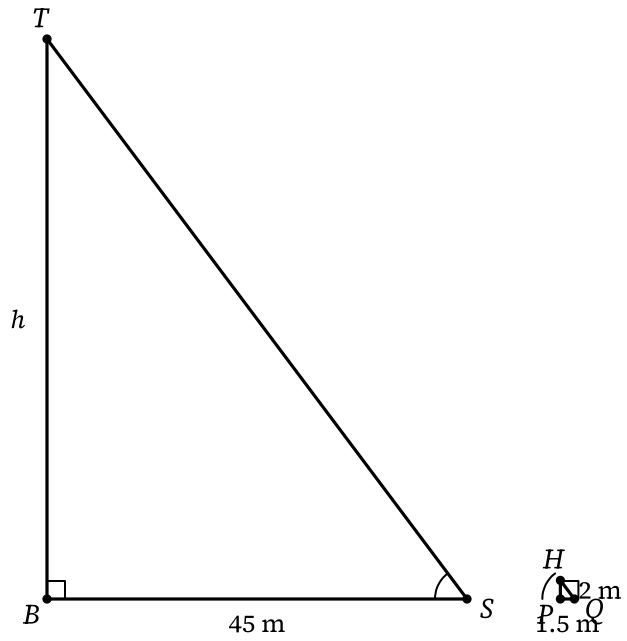
   Pentagon $ABCDE \sim$ pentagon $PQRST$. $AB = 12$, $PQ = 8$, $DE = 9$ and $m\angle C = 110°$. Find ST and $m\angle R$.

5. **5 pts**

   Henry builds a model of a covered bridge at a scale of $1 : 40$. The model is 2.5 ft long. (a) How long is the real bridge, in feet? (b) The real bridge is 16 ft tall. How tall is the model, in inches?

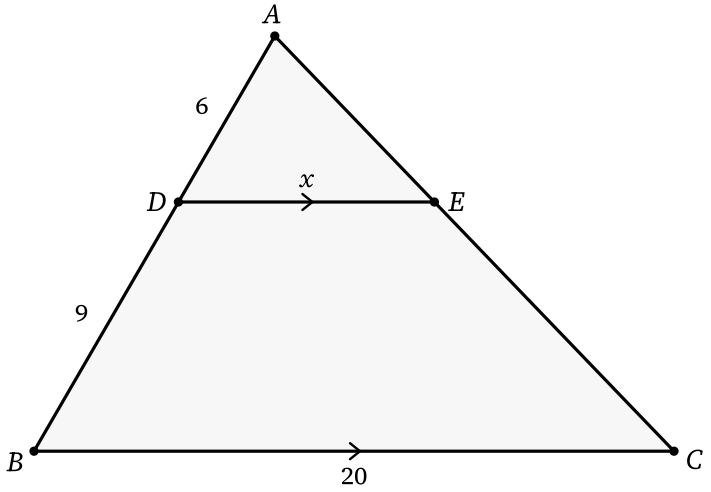
6. **7 pts**

   At the same moment, a lighthouse casts a shadow 45 m long and a 2-m post casts a shadow 1.5 m long, as in the figure. Both stand straight up on level ground. How tall is the lighthouse? Name the similarity test that justifies your proportion.



7. **5 pts**

In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = 6$, $DB = 9$ and $BC = 20$. Find DE.

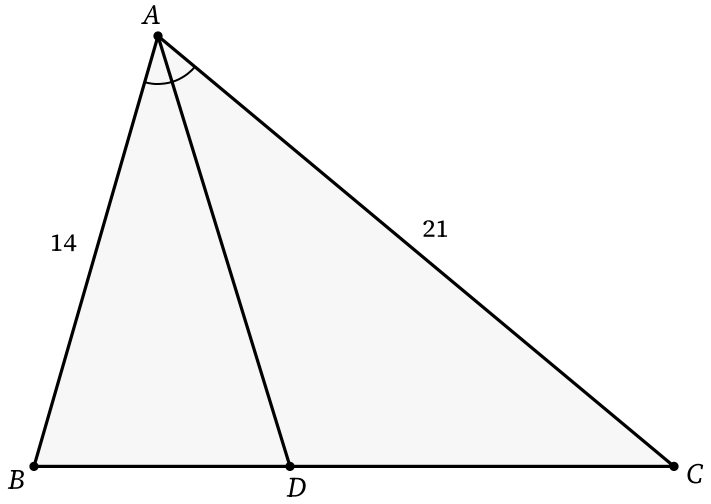


8. **7 pts**

One triangle has sides 6, 10 and 14; another has sides 9, 15 and 21. (a) Are the triangles similar? (b) If so, which test proves it? (c) What is the scale factor from the first triangle to the second?

9. **7 pts**

In the figure, $\overline{AD}$ bisects $\angle BAC$, $AB = 14$, $AC = 21$ and $BC = 20$. Find BD.



10. **7 pts**

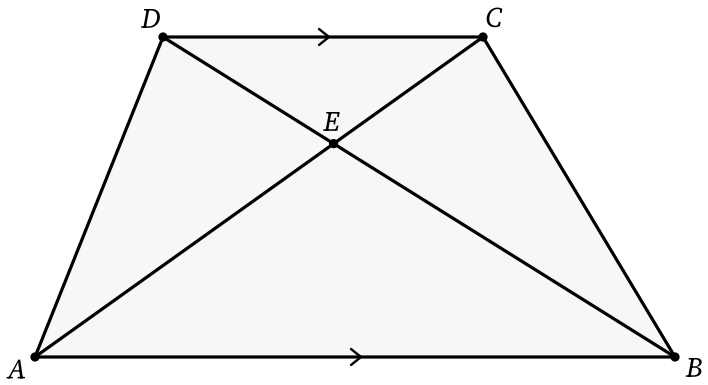
Three parallel streets cross First Avenue and Second Avenue. Along First Avenue, the two blocks between the streets measure 90 m and 135 m. Along Second Avenue, the two blocks between the same streets measure 300 m altogether. How long is each block along Second Avenue?

11. **6 pts**

Explain why any two isosceles right triangles are similar, but two isosceles triangles in general need not be similar. Give a specific example for the second part.

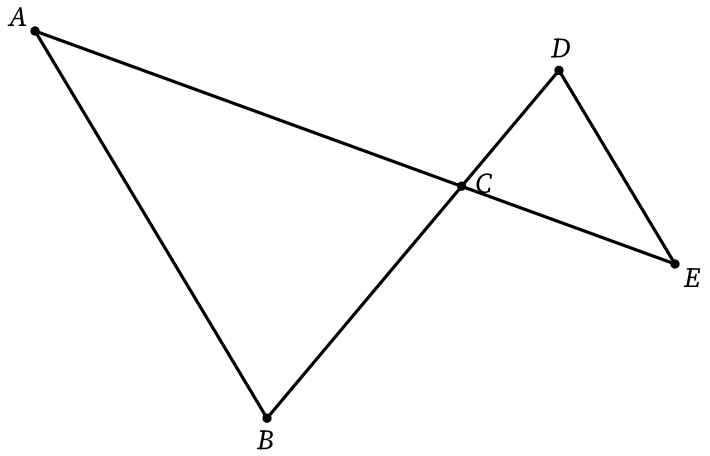
12. **10 pts**

In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and its diagonals $\overline{AC}$ and $\overline{BD}$ meet at E. Write a two-column proof that $AE \cdot DE = BE \cdot CE$.



13. **10 pts**

In the figure, $\overline{AE}$ and $\overline{BD}$ intersect at C, $AC = 2 \cdot EC$ and $BC = 2 \cdot DC$. Write a paragraph proof that $AB = 2 \cdot DE$.

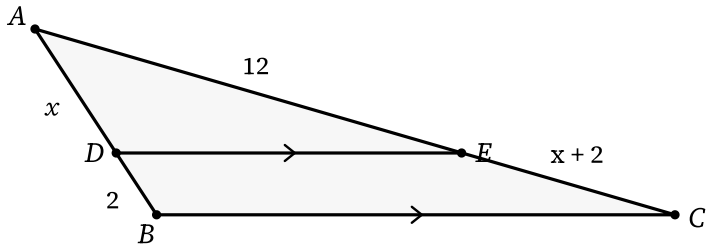


14. **8 pts**

Construction. Draw a segment 4 cm long and a segment 9 cm long. With compass and straightedge, construct a segment whose length is their geometric mean. List your steps, explain why the construction works, and state the length your segment should have.

15. **7 pts**

In the figure, $\overline{DE} \parallel \overline{BC}$, $AD = x$, $DB = 2$, $AE = 12$ and $EC = x + 2$. Find x.



ANSWER KEY

1. $15(x-1) = 5(x+5)$, so $15x - 15 = 5x + 25$, $10x = 40$ and $x = 4$.
2. $\sqrt{8 \cdot 18} = \sqrt{144} = 12$
3. $2(2x + 5x) = 28$, so $14x = 28$ and $x = 2$: width 4 ft, height 10 ft.
4. The scale factor from $ABCDE$ to $PQRST$ is $\dfrac{8}{12} = \dfrac{2}{3}$, so $ST = \dfrac{2}{3} \cdot 9 = 6$; $\angle R$ corresponds to $\angle C$, so $m\angle R = 110°$.
5. (a) $2.5 \times 40 = 100$ ft. (b) $\dfrac{16}{40} = 0.4$ ft $= 4.8$ in.
6. The right angles are congruent and the sun's rays make congruent angles with the ground, so the triangles are similar by AA. $\dfrac{h}{2} = \dfrac{45}{1.5}$, so $h = 60$ m.
7. $\triangle ADE \sim \triangle ABC$ and $AB = 15$, so $\dfrac{DE}{20} = \dfrac{6}{15}$ and $DE = 8$.
8. (a) Yes. (b) SSS: $\dfrac{9}{6} = \dfrac{15}{10} = \dfrac{21}{14}$. (c) $\dfrac{3}{2}$.
9. $\dfrac{BD}{DC} = \dfrac{14}{21} = \dfrac{2}{3}$, so $BD = \dfrac{2}{5} \cdot 20 = 8$.
10. The parallel streets cut the avenues proportionally: $\dfrac{x}{300 - x} = \dfrac{90}{135} = \dfrac{2}{3}$, so $3x = 600 - 2x$ and $x = 120$. The blocks are 120 m and 180 m.
11. In an isosceles right triangle the base angles are congruent and complementary, so each is $45°$; any two such triangles have angles $90°$ and $45°$ in common, so they are similar by AA. A general isosceles triangle's base angles depend on its vertex angle: a triangle with angles $40°, 70°, 70°$ and one with angles $100°, 40°, 40°$ are both isosceles, but their angles are not congruent in pairs, so they are not similar.
12. See the model proof.

    TWO-COLUMN PROOF

    Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

    Prove: $AE \cdot DE = BE \cdot CE$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$	Given
2	$\angle BAE \cong \angle DCE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{AC}$)
3	$\angle ABE \cong \angle CDE$	Alternate Interior Angles Theorem (transversal $\overleftrightarrow{BD}$)

	STATEMENTS	REASONS
4	$\triangle ABE \sim \triangle CDE$	AA Similarity Postulate
5	$\dfrac{AE}{CE} = \dfrac{BE}{DE}$	Definition of similar polygons
6	$AE \cdot DE = BE \cdot CE$	Cross Products Property

∎

13. See the model proof.

PARAGRAPH PROOF

Given: $\overline{AE}$ and $\overline{BD}$ intersect at C; $AC = 2 \cdot EC$; $BC = 2 \cdot DC$

Prove: $AB = 2 \cdot DE$

Dividing the given equations (Division Property of Equality), $\dfrac{AC}{EC} = 2$ and $\dfrac{BC}{DC} = 2$, so $\dfrac{AC}{EC} = \dfrac{BC}{DC}$ by the Transitive Property of Equality. $\angle ACB \cong \angle ECD$ by the Vertical Angles Theorem, and these are the angles included between the proportional sides. So $\triangle ACB \sim \triangle ECD$ by the SAS Similarity Theorem.

Corresponding sides of similar triangles are proportional, so $\dfrac{AB}{ED} = \dfrac{AC}{EC} = 2$. By the Multiplication Property of Equality, $AB = 2 \cdot DE$.

∎

14. Lay the segments end to end on a line: $\overline{AD}$ = 4 cm and $\overline{DB}$ = 9 cm (copy segments, I.3). Bisect $\overline{AB}$ to find its midpoint M (I.10) and draw the semicircle with center M through A and B. Construct the perpendicular to $\overline{AB}$ at D (I.11), meeting the semicircle at C. Then $\overline{DC}$ is the geometric mean, because $\angle ACB$ is a right angle and $DC^2 = AD \cdot DB$ (the Mean Proportional in a Semicircle theorem, Euclid VI.13). Its length is $\sqrt{4 \cdot 9} = 6$ cm.
15. By the Triangle Proportionality Theorem, $\dfrac{x}{2} = \dfrac{12}{x+2}$, so $x^2 + 2x - 24 = 0$, $(x+6)(x-4) = 0$, and since $x > 0$, $x = 4$.

UNIT 9

Right Triangles and Trigonometry

This unit is about the most useful triangle of all: the right triangle. It begins with the Pythagorean Theorem, the great proposition that closes Book I of Euclid's Elements (I.47), and its converse (I.48). Students study three different proofs: Euclid's own windmill proof by areas, a rearrangement proof and a proof with similar triangles. They then use the theorem and its converse to find lengths, test for right angles and classify triangles as acute, right or obtuse.

Next, similar triangles (Unit 8) are put to work: the altitude to the hypotenuse creates three similar right triangles (Euclid VI.8), which leads to the geometric-mean relationships. Then come the two special right triangles, the $45°$-$45°$-$90°$ triangle (half a square) and the $30°$-$60°$-$90°$ triangle (half an equilateral triangle).

The unit ends with right-triangle trigonometry. Because all right triangles with the same acute angle are similar, the ratios of their sides depend only on the angle; these ratios are the tangent, sine and cosine. Students learn to use a scientific calculator (allowed from this unit on) to find sides and angles, and they apply trigonometry the way surveyors, builders and navigators do: to find the heights of steeples and cliffs, the widths of rivers and the distances to ships.

PREREQUISITES

- Using the Pythagorean Theorem numerically, simplifying square roots and rationalizing a one-term denominator (Algebra I)
- Similar triangles: AA, SSS and SAS Similarity and proportions from similar triangles (Lessons 8.3–8.4)
- The Triangle Inequality and the Hinge Theorem and its converse (Lesson 5.6)
- The Isosceles Triangle Theorem and its converse; equilateral triangles are equiangular (Lesson 4.6)
- Alternate Interior Angles Theorem for parallel lines (Lesson 3.2)
- Writing two-column, paragraph, flow and indirect proofs (Units 2–8)

MASTERY CHECKLIST

- ☐ I can state the Pythagorean Theorem in symbols and in Euclid's words, and explain Euclid's proof, a rearrangement proof and a similar-triangles proof.
- ☐ I can find a missing side of a right triangle and give the answer in simplest radical form or rounded as asked.
- ☐ I can prove the Converse of the Pythagorean Theorem and use it to decide whether a triangle is right.
- ☐ I can classify a triangle as acute, right or obtuse by comparing c^2 with $a^2 + b^2$, after checking the Triangle Inequality.
- ☐ I can prove that the altitude to the hypotenuse forms three similar right triangles and use the geometric-mean relationships for the altitude and the legs.
- ☐ I can derive and use the side ratios of $45°$-$45°$-$90°$ and $30°$-$60°$-$90°$ triangles, and construct these triangles with compass and straightedge.

- ☐ I can explain why the trigonometric ratios depend only on the angle, and define $\sin A$, $\cos A$, and $\tan A$.
- ☐ I can use SOH-CAH-TOA and a calculator in degree mode to find missing sides, and inverse ratios to find missing angles.
- ☐ I can use the cofunction relationship $\sin A = \cos(90^\circ - A)$ and the exact values for 30°, 45°, and 60°.
- ☐ I can prove that angles of elevation and depression are congruent and solve height-and-distance problems, including two-observation problems.

LESSONS IN THIS UNIT

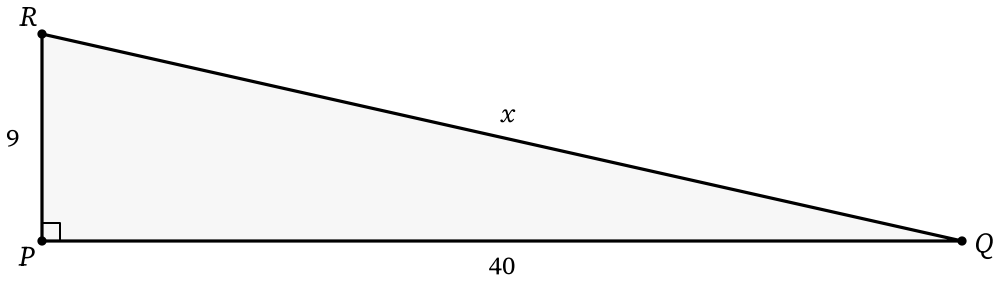
9.1	**The Pythagorean Theorem and Its Proofs**	2 sessions of 45 minutes
9.2	**The Converse of the Pythagorean Theorem**	45 minutes
9.3	**Similar Right Triangles and the Geometric Mean**	2 sessions of 45 minutes
9.4	**Special Right Triangles**	2 sessions of 45 minutes
9.5	**The Tangent Ratio**	2 sessions of 45 minutes
9.6	**The Sine and Cosine Ratios**	2 sessions of 45 minutes

Unit 9 test

You have 60 minutes. A scientific calculator is allowed; make sure it is in degree mode. You will need a compass and straightedge for problem 13. Show all your work. Give exact answers in simplest radical form when a problem asks for them; otherwise round as each problem directs, using full calculator values until the final step. In every proof, give a reason for every step. Figures are drawn to scale, but use only the information given in words and marked on the figure.

1. **5 pts**

   In the figure, right $\triangle PQR$ has legs $PQ = 40$ and $PR = 9$. Find the hypotenuse x.



2. **5 pts**

   A right triangle has a hypotenuse of 14 and one leg of 6. Find the other leg in simplest radical form.

3. **6 pts**

   Classify the triangle with sides 11, 13 and 17 as acute, right or obtuse. Show the comparison you use.

4. **6 pts**

   Samuel is building a deck. To check a corner he marks 12 ft along one side and 16 ft along the other, and the distance between the marks is 20 ft 3 in. Is the corner angle acute, right or obtuse?

5. **6 pts**

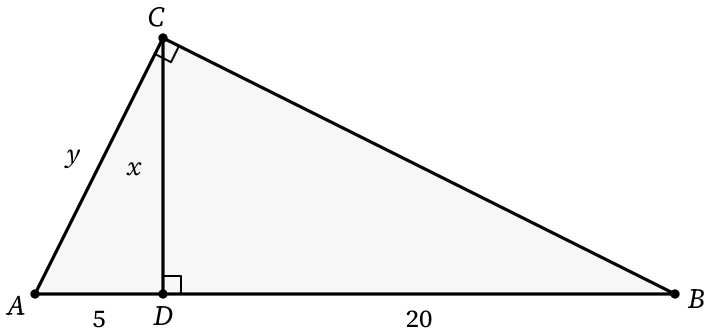
The diagonal of a square is 14 cm. Find the length of a side in simplest radical form.

6. **6 pts**

A rafter makes a 30° angle with the horizontal ceiling joist, and the peak of the roof is 6 ft above the joist. How long is the horizontal run from the end of the rafter to the point directly below the peak? Give the exact answer in simplest radical form.

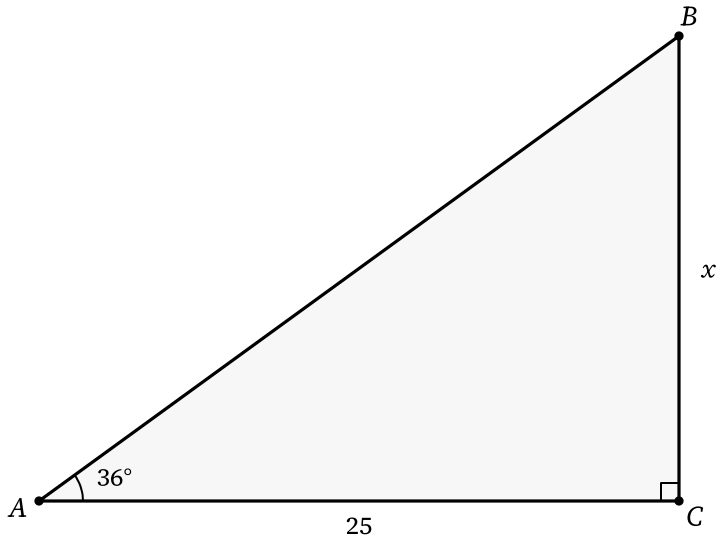
7. **7 pts**

In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $AD = 5$ and $DB = 20$. Find $x = CD$ and $y = AC$. Give y in simplest radical form.



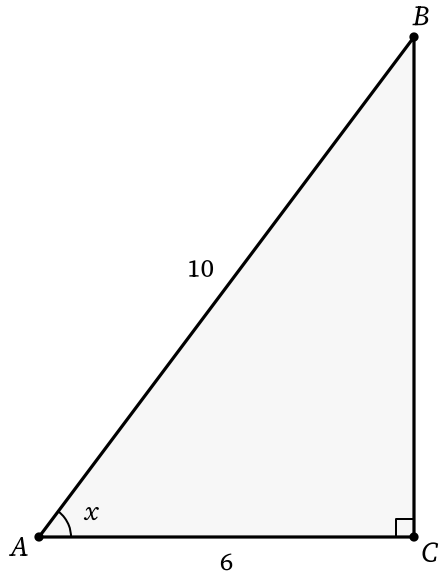
8. **6 pts**

In the figure, right $\triangle ABC$ has its right angle at C, $m\angle A = 36°$ and $AC = 25$. Find $x = BC$ to the nearest tenth.



9. **6 pts**

In the figure, right $\triangle ABC$ has its right angle at C, $AC = 6$ and hypotenuse $AB = 10$. Find $x = m\angle A$ to the nearest tenth of a degree.



10. **5 pts**

A 14-foot ladder leans against the wall of a barn and makes a 72° angle with the level ground. How high up the wall does it reach? Round to the nearest tenth of a foot.

11. **5 pts**

Explain why a carpenter who measures 6 ft and 8 ft along two walls from a corner and finds exactly 10 ft between the marks knows the corner is square. Then explain why, on a large foundation, a carpenter would rather use 6-8-10 (or 9-12-15) than 3-4-5.

12. **7 pts**

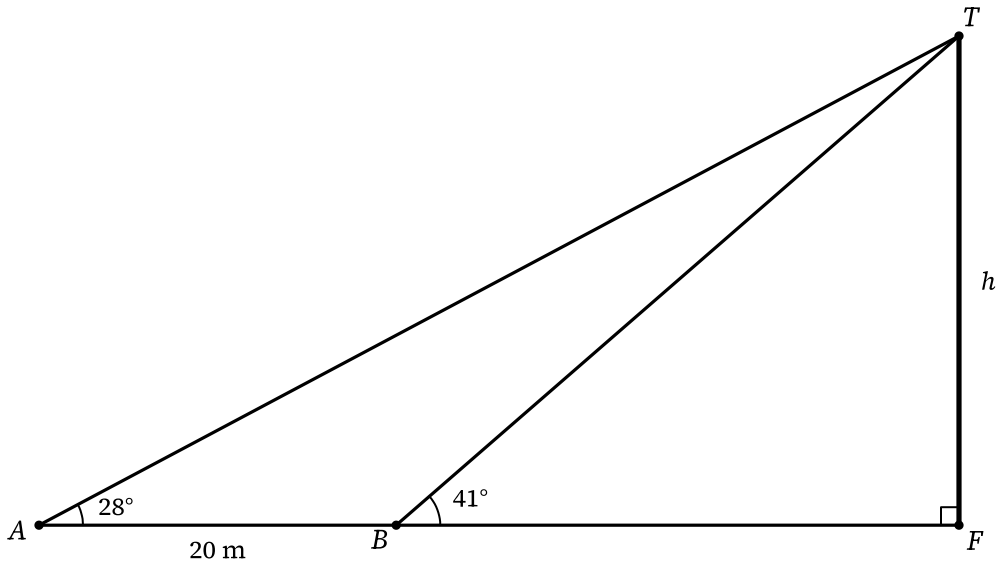
From a fishing boat, the angle of elevation to the top of a lighthouse that stands 35 m above sea level is 6°. How far is the boat from the base of the lighthouse? Round to the nearest meter.

13. **6 pts**

You are given a segment $\overline{AB}$. Using only a compass and straightedge, construct an isosceles right triangle (a 45°-45°-90° triangle) with $\overline{AB}$ as one leg. List your steps and explain why the triangle has angles of 45°, 45° and 90°.

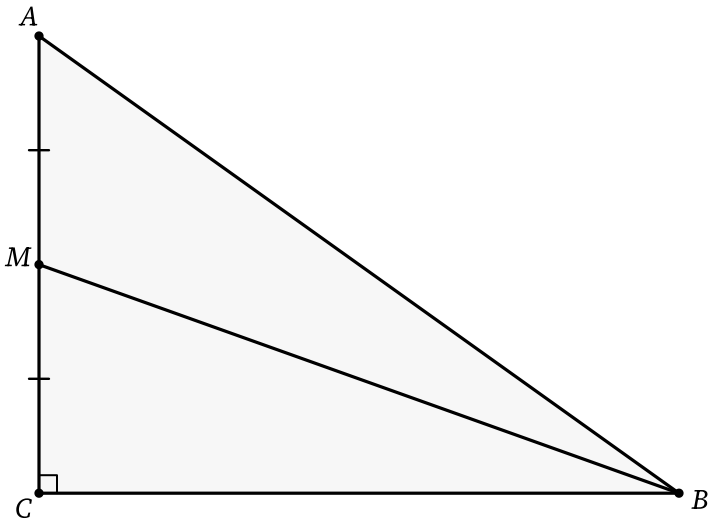
14. **8 pts**

From point A on level ground, the angle of elevation to the top T of a church tower is 28°. Grace walks 20 m straight toward the tower to point B, where the angle of elevation is 41°. Find the height h of the tower to the nearest tenth of a meter.



15. **8 pts**

In the figure, $\triangle ABC$ has a right angle at C, and M is the midpoint of $\overline{AC}$. Prove that $AB^2 - BM^2 = 3\,CM^2$.



16. **8 pts**

In right $\triangle ABC$ with the right angle at C, prove that $\sin A \cdot \tan B = \cos A$.

ANSWER KEY

1. $x = \sqrt{81 + 1600} = \sqrt{1681} = 41$
2. $\sqrt{196 - 36} = \sqrt{160} = 4\sqrt{10}$
3. Acute: $11 + 13 > 17$, so the triangle exists, and $17^2 = 289 < 11^2 + 13^2 = 290$.
4. Obtuse: 20 ft 3 in $= 20.25$ ft, and $20.25^2 = 410.0625 > 12^2 + 16^2 = 400$.
5. $\dfrac{14}{\sqrt{2}} = 7\sqrt{2}$ cm
6. The rise of 6 ft is the short leg (opposite 30°), so the run is the long leg: $6\sqrt{3}$ ft ≈ 10.4 ft.
7. $x = \sqrt{5 \cdot 20} = 10$; $y = \sqrt{AB \cdot AD} = \sqrt{25 \cdot 5} = 5\sqrt{5}$
8. $x = 25 \tan 36^\circ \approx 18.2$
9. $\cos A = \dfrac{6}{10}$, so $x = \cos^{-1}(0.6) \approx 53.1^\circ$
10. $14 \sin 72^\circ \approx 13.3$ ft
11. The three measurements form a triangle with $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, so by the Converse of the Pythagorean Theorem the angle opposite the 10-ft side, the corner, is a right angle. Longer measurements are better because a small error in reading the tape is a smaller fraction of a long length, and the marks lie farther along the walls, so an angle that is slightly off makes a larger, easier-to-see difference in the diagonal.
12. $d = \dfrac{35}{\tan 6^\circ} \approx 333$ m
13. Steps: (1) Extend $\overline{BA}$ beyond A. With center A, draw an arc crossing the line at two points X and Y on either side of A. (2) With centers X and Y and a larger equal radius, draw arcs that cross at P; draw $\overleftrightarrow{AP}$, which is perpendicular to $\overline{AB}$ at A (Euclid I.11). (3) With center A and radius AB, draw an arc crossing $\overleftrightarrow{AP}$ at C. (4) Draw $\overline{BC}$. Why: $\angle BAC$ is a right angle by construction, and $AC = AB$ because they are radii of the same circle, so $\triangle BAC$ is isosceles. Its base angles $\angle B$ and $\angle C$ are congruent (Isosceles Triangle Theorem, Euclid I.5) and complementary (acute angles of a right triangle), so each is 45°.
14. Let $x = BF$. Then $h = x \tan 41^\circ = (x + 20) \tan 28^\circ$, so $x = \dfrac{20 \tan 28^\circ}{\tan 41^\circ - \tan 28^\circ} \approx 31.50$ m and $h \approx 31.50 \tan 41^\circ \approx 27.4$ m.
15. See the model proof.

    TWO-COLUMN PROOF

    Given: $\angle ACB$ is a right angle; M is the midpoint of $\overline{AC}$

Prove: $AB^2 - BM^2 = 3\,CM^2$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; M is the midpoint of $\overline{AC}$	Given
2	$AB^2 = AC^2 + BC^2$	Pythagorean Theorem (right $\triangle ACB$)
3	$\angle MCB$ is a right angle	$\angle MCB$ and $\angle ACB$ are the same angle, since M lies on $\overrightarrow{CA}$
4	$BM^2 = CM^2 + BC^2$	Pythagorean Theorem (right $\triangle MCB$)
5	$AB^2 - BM^2 = AC^2 - CM^2$	Subtraction Property of Equality (steps 2 and 4)
6	$AM = MC$, so $AC = AM + MC = 2\,CM$	Definition of midpoint; Segment Addition Postulate; Substitution
7	$AC^2 = 4\,CM^2$	Multiplication Property of Equality (square both sides)
8	$AB^2 - BM^2 = 3\,CM^2$	Substitution ($4\,CM^2 - CM^2 = 3\,CM^2$)

∎

16. See the model proof.

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$

Prove: $\sin A \cdot \tan B = \cos A$

By the definition of sine, $\sin A = \dfrac{BC}{AB}$, because $\overline{BC}$ is the leg opposite $\angle A$ and $\overline{AB}$ is the hypotenuse. For $\angle B$, the opposite leg is $\overline{AC}$ and the adjacent leg is $\overline{BC}$, so by the definition of tangent $\tan B = \dfrac{AC}{BC}$.

Multiplying, $\sin A \cdot \tan B = \dfrac{BC}{AB} \cdot \dfrac{AC}{BC} = \dfrac{AC}{AB}$, since $BC \neq 0$. By the definition of cosine, $\cos A = \dfrac{AC}{AB}$, because $\overline{AC}$ is the leg adjacent to $\angle A$. Therefore, by substitution, $\sin A \cdot \tan B = \cos A$.

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9.1 The Pythagorean Theorem and Its Proofs

OBJECTIVES

- State the Pythagorean Theorem in symbols and as Euclid's fact about the areas of the squares on the sides of a right triangle.
- Explain three proofs of the theorem: Euclid's windmill proof (I.47), a rearrangement proof, and a proof with similar triangles.
- Use the theorem to find a missing side of a right triangle, giving exact answers in simplest radical form and rounded answers when asked.
- Write proofs that use the Pythagorean Theorem as a reason.

WATCH

Visual Proof of Pythagoras' Theorem

Eddie Woo · 11:14

https://www.youtube.com/watch?v=tTHhBE5lYTg

> **WHEN TO WATCH**
> With the rearrangement proof: four copies of a right triangle inside a square.

Note: Euclid's windmill proof and the similar-triangles proof are taught from the lesson.

BACKUP VIDEO

Pythagoras' Theorem (2 of 3: Dissection Proof)

Eddie Woo · 9:43

https://www.youtube.com/watch?v=EzCWuNZpBaQ

With the area proofs: a second, dissection proof.

KEY TERMS

Legs

The two sides of a right triangle that form the right angle.

Hypotenuse

The side of a right triangle opposite the right angle. It is always the longest side.

Pythagorean Theorem

In a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$.

Square on a side

Euclid's phrase for the square built with that side as one of its sides; its area is the square of the side's length.

Simplest radical form

A square root written with no perfect-square factor left under the radical sign and no radical in a denominator, such as $6\sqrt{2}$ instead of $\sqrt{72}$.

CORE IDEAS

You met the Pythagorean Theorem in Algebra I as a rule about numbers: in a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$. The **hypotenuse** is the side opposite the right angle, and the **legs** are the two sides that form it. Euclid states the same fact as a fact about areas. In his words (I.47): "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle." Build a square outward on each side of a right triangle: the big square has exactly as much area as the two smaller squares put together.

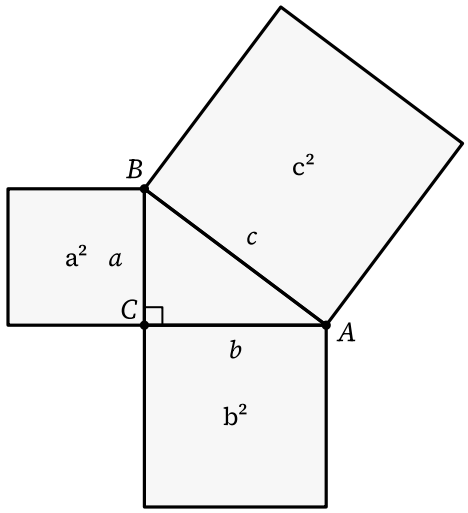
Why prove something you already use? You could draw a hundred right triangles, measure them, and find $a^2 + b^2 = c^2$ every time. That is inductive reasoning (Lesson 1.2): it suggests the rule but cannot guarantee it for the next triangle. A proof shows that the rule must hold for every right triangle, because it follows from postulates and theorems we already accept. People have found hundreds of proofs of this theorem; you will study three.

The rearrangement proof. Place four copies of a right triangle with legs a and b inside a square of side $a + b$, one in each corner. Their hypotenuses outline a tilted quadrilateral with four sides of length c. Each of its corners is a right angle, because at that corner two acute angles of the triangle (which add to 90°) and the corner angle together make a straight angle of 180°. So the big square is made of a square of area c^2 and four triangles of area $\frac{1}{2}ab$ each. Counting the area two ways gives $(a + b)^2 = c^2 + 2ab$; since $(a + b)^2 = a^2 + 2ab + b^2$, subtracting $2ab$ leaves $a^2 + b^2 = c^2$.

The similar-triangles proof. Let $\triangle ABC$ have its right angle at C, as in the figure, and draw the altitude $\overline{CD}$ from C to the hypotenuse $\overline{AB}$. It cuts the triangle into two smaller right triangles, and each shares an acute angle with the original, so each is similar to the original by AA (Lesson 8.3). The proportions give $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$. Adding, $AC^2 + BC^2 = AB(AD + BD) = AB \cdot AB = AB^2$. Lesson 9.3 studies these three similar triangles in detail.

Euclid's proof (the "windmill"). Euclid builds the three squares and, from the right-angle vertex (C in our figure; Euclid calls it A), draws a line parallel to the two sides of the big square that meet the hypotenuse. That line splits the big square into two rectangles, and he shows that each rectangle has the same area as one of the small squares. The key is a pair of triangles that are congruent by SAS (his I.4). One of them is half of a rectangle and the other is half of a small square, because a triangle has half the area of a rectangle with the same base and height (Euclid's I.41). Equal halves mean equal wholes. Euclid's proof needs no algebra and no similarity, which is why it can stand at the end of Book I.

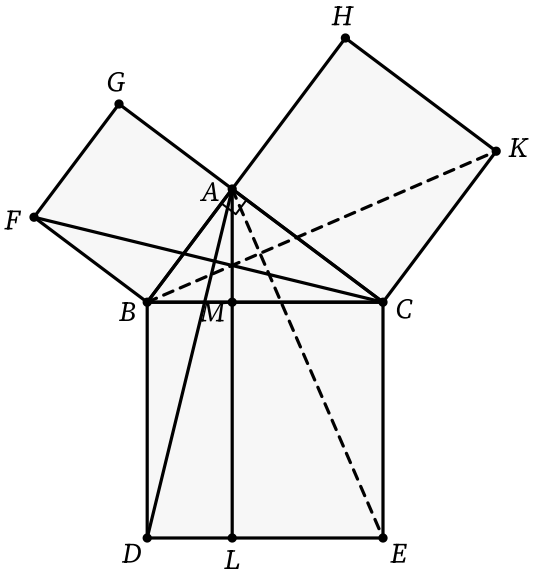
Using the theorem. First find the hypotenuse: it is opposite the right angle. To find the hypotenuse, add the squares of the legs: $c = \sqrt{a^2 + b^2}$. To find a leg, subtract: $a = \sqrt{c^2 - b^2}$. Give exact answers in **simplest radical form** by taking out the largest perfect-square factor, for example $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$. From this unit on you may use a scientific calculator for decimal approximations when a problem asks you to round.



The square on the hypotenuse equals the squares on the legs: $a^2 + b^2 = c^2$.

THEOREM **Pythagorean Theorem (Euclid's proof)** Euclid I.47

In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs: $a^2 + b^2 = c^2$. Equivalently, the square built on the hypotenuse has the same area as the squares built on the two legs together.



PARAGRAPH PROOF

Given: $\triangle ABC$ with right angle $\angle BAC$; Squares $BCED$, $ABFG$ and $ACKH$ are built outward on $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$; $\overline{AL} \parallel \overline{BD}$, meeting $\overline{BC}$ at M and $\overline{DE}$ at L

Prove: Area of square $BCED$ = area of square $ABFG$ + area of square $ACKH$, that is, $BC^2 = AB^2 + AC^2$

First, G, A and C lie on one line. The angle $\angle BAG$ is an angle of square $ABFG$ and $\angle BAC$ is given as a right angle, so these adjacent angles, on opposite sides of $\overline{AB}$, add to $180°$. Therefore their outer sides $\overrightarrow{AG}$ and $\overrightarrow{AC}$ are opposite rays (Euclid I.14). In the same way B, A and H lie on one line.

Next, compare $\triangle ABD$ and $\triangle FBC$. $BD = BC$ and $BA = BF$ because the sides of a square are congruent. $\angle CBD$ and $\angle FBA$ are right angles of the squares, so by the Angle Addition Postulate $m\angle ABD = m\angle ABC + 90° = m\angle FBC$. So $\triangle ABD \cong \triangle FBC$ by SAS (Euclid I.4), and congruent triangles have equal areas (equal bases and equal heights).

$BMLD$ is a rectangle: $\overline{ML} \parallel \overline{BD}$, and $\overline{BC}$ and $\overline{DE}$ are both perpendicular to $\overline{BD}$, so all four of its angles are right angles. Since A lies on line ML, which is parallel to $\overline{BD}$, the height of $\triangle ABD$ from A to base $\overline{BD}$ equals the distance between these parallel lines, which is BM. So area $\triangle ABD = \frac{1}{2} \cdot BD \cdot BM$, half the area of rectangle $BMLD$.

Likewise, C lies on line GA (first step), and $\overleftrightarrow{GA} \parallel \overline{BF}$ because they contain opposite sides of square $ABFG$. So the height of $\triangle FBC$ from C to base $\overline{BF}$ equals the distance between these parallel lines, which is AB. So area $\triangle FBC = \frac{1}{2} \cdot BF \cdot AB$, half the area of square $ABFG$.

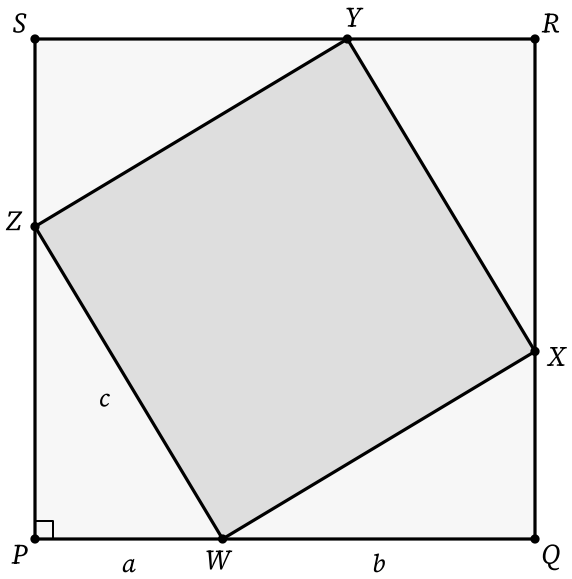
The rectangle is double $\triangle ABD$ and the square is double $\triangle FBC$, and doubles of equal areas are equal, so area of rectangle $BMLD$ = area of square $ABFG$. Joining $\overline{AE}$ and $\overline{BK}$ and repeating the argument with $\triangle ACE \cong \triangle KCB$ shows that area of rectangle $CMLE$ = area of square $ACKH$. The two rectangles together make up square $BCED$ (Area Addition), so area $BCED$ = area $ABFG$ + area $ACKH$, that is, $BC^2 = AB^2 + AC^2$.

∎

This is the proof found in the Elements. The figure is often called the windmill or the bride's chair.

THEOREM **Pythagorean Theorem (rearrangement proof)**

If a right triangle has legs a and b and hypotenuse c, then $a^2 + b^2 = c^2$.



TWO-COLUMN PROOF

Given: A right triangle has legs a and b and hypotenuse c; Square $PQRS$ has side $a + b$; W, X, Y, Z lie on $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{SP}$ with $PW = QX = RY = SZ = a$
Prove: $a^2 + b^2 = c^2$

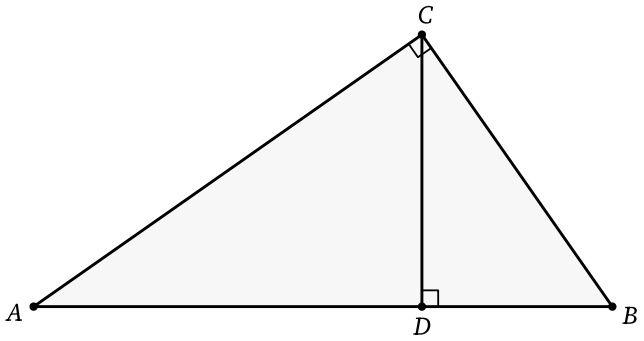
	STATEMENTS	REASONS
1	$PQRS$ is a square with side $a + b$; $PW = QX = RY = SZ = a$	Given

	STATEMENTS	REASONS
2	$WQ = XR = YS = ZP = b$	Segment Addition Postulate and Subtraction Property of Equality
3	$\triangle ZPW$, $\triangle WQX$, $\triangle XRY$ and $\triangle YSZ$ are each congruent to the given right triangle	SAS (legs a and b with the included right angle of the square)
4	$ZW = WX = XY = YZ = c$	CPCTC
5	$m\angle PWZ + m\angle PZW = 90^\circ$	Acute angles of a right triangle are complementary
6	$m\angle QWX = m\angle PZW$	CPCTC (each is the angle opposite the leg of length a)
7	$m\angle PWZ + m\angle ZWX + m\angle QWX = 180^\circ$	Angle Addition Postulate ($\angle PWQ$ is a straight angle)
8	$m\angle PWZ + m\angle ZWX + m\angle PZW = 180^\circ$	Substitution Property of Equality
9	$m\angle ZWX = 90^\circ$	Substitution (step 5) and Subtraction Property of Equality
10	$WXYZ$ is a square with side c	Steps 4–9 repeated at X, Y, Z; definition of a square
11	Area of $PQRS$ = area of $WXYZ$ + 4· (area of one triangle)	Area Addition (the five pieces fill the square without overlapping)
12	$(a+b)^2 = c^2 + 4 \cdot \frac{1}{2}ab$	Area of a square and of a triangle; Substitution
13	$a^2 + 2ab + b^2 = c^2 + 2ab$	Distributive Property (expand and simplify)
14	$a^2 + b^2 = c^2$	Subtraction Property of Equality

∎

THEOREM **Pythagorean Theorem (proof by similar triangles)**

In right triangle ABC with right angle C, $AC^2 + BC^2 = AB^2$.



FLOW PROOF

Given: $\triangle ABC$ with right angle $\angle ACB$; $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$

Prove: $AC^2 + BC^2 = AB^2$

(1) $\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$
Given

(2) $\angle ADC$ and $\angle BDC$ are right angles
Definition of perpendicular lines
from (1)

(3) $\angle ADC \cong \angle ACB$ and $\angle BDC \cong \angle ACB$
All right angles are congruent
from (1), (2)

(4) $\angle A \cong \angle A$ and $\angle B \cong \angle B$
Reflexive Property of Congruence

(5) $\triangle ACD \sim \triangle ABC$ and $\triangle CBD \sim \triangle ABC$
AA Similarity
from (3), (4)

(6) $\dfrac{AD}{AC} = \dfrac{AC}{AB}$ and $\dfrac{BD}{BC} = \dfrac{BC}{AB}$
Corresponding sides of similar triangles are proportional
from (5)

(7) $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$
Cross Products Property
from (6)

(8) $AC^2 + BC^2 = AB \cdot AD + AB \cdot BD = AB(AD + BD)$
Addition Property of Equality; Distributive Property
from (7)

(9) $AD + BD = AB$
Segment Addition Postulate (D is between A and B)

(10) $AC^2 + BC^2 = AB^2$
Substitution Property of Equality
from (8), (9)

∎

Lesson 9.3 proves in detail that the altitude to the hypotenuse makes triangles similar to the original.

KEY FORMULAS

$$a^2 + b^2 = c^2$$

$$c = \sqrt{a^2 + b^2}$$

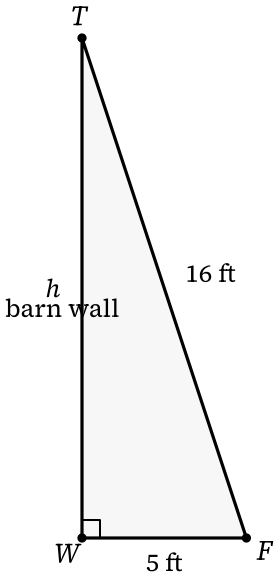
$$a = \sqrt{c^2 - b^2}$$

From history. The theorem carries the name of Pythagoras of Samos (sixth century BC), but no writing of Pythagoras survives, and the stories of his discovering it come from much later writers. The relationship was known long before him: the Babylonian clay tablet called Plimpton 322, from about 1800 BC, lists pairs of numbers that are generally understood to come from Pythagorean triples. In 1876 James A. Garfield, then a member of Congress and later President of the United States, published a short proof using a trapezoid built from two copies of a right triangle and an isosceles right triangle; his figure is exactly half of the rearrangement figure in this lesson.

Euclid connection. Proposition I.47 is the high point of Book I of the Elements: SAS (I.4), the theory of parallels (I.27–I.31) and the fact that a parallelogram is double a triangle on the same base and between the same parallels (I.41) are all used to reach it. In Book VI Euclid proves a more general version with similar figures (VI.31): any similar figures built on the three sides of a right triangle behave like the squares.

Practice habit. Recite the Pythagorean Theorem both in Euclid's words and in symbols until it is automatic. Before using it, mark the right angle and name the hypotenuse out loud; when you finish, check that the hypotenuse came out as the longest side.

Example 1. Grace leans a 16-foot ladder against the wall of the family barn. The foot of the ladder is 5 feet from the wall on level ground. How high up the wall does the ladder reach? Give the exact answer in simplest radical form and the answer to the nearest tenth of a foot.

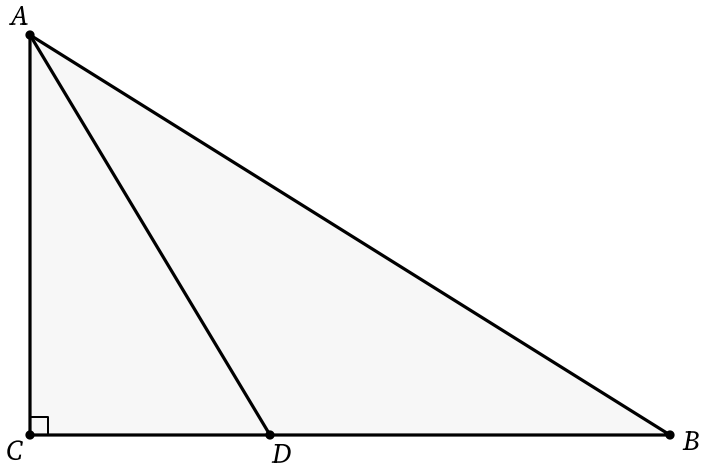


Show solution

1. The wall meets level ground at a right angle, so the ladder is the hypotenuse ($c = 16$) and the distance along the ground is a leg ($a = 5$). Let h be the height on the wall.
2. By the Pythagorean Theorem, $5^2 + h^2 = 16^2$, so $h^2 = 256 - 25 = 231$.
3. Since $231 = 3 \times 7 \times 11$ has no perfect-square factor, $h = \sqrt{231}$ is already in simplest radical form.
4. With a calculator, $\sqrt{231} \approx 15.199$, so the ladder reaches about 15.2 feet. Check: $15.2 < 16$, as it must be, since the hypotenuse is the longest side.

Answer: $\sqrt{231}$ ft ≈ 15.2 ft

Example 2. In the figure, $\triangle ABC$ has a right angle at C, and D is a point on $\overline{CB}$ other than C. Prove that $AB^2 - AD^2 = CB^2 - CD^2$.



Show solution

1. Notice two right triangles that share the leg $\overline{AC}$: $\triangle ACB$ and $\triangle ACD$. Both have the right angle at C.
2. Write the Pythagorean Theorem for each triangle.
3. Subtract the two equations; the shared AC^2 cancels.

TWO-COLUMN PROOF

Given: $\angle ACB$ is a right angle; D is on $\overline{CB}$, $D \neq C$
Prove: $AB^2 - AD^2 = CB^2 - CD^2$

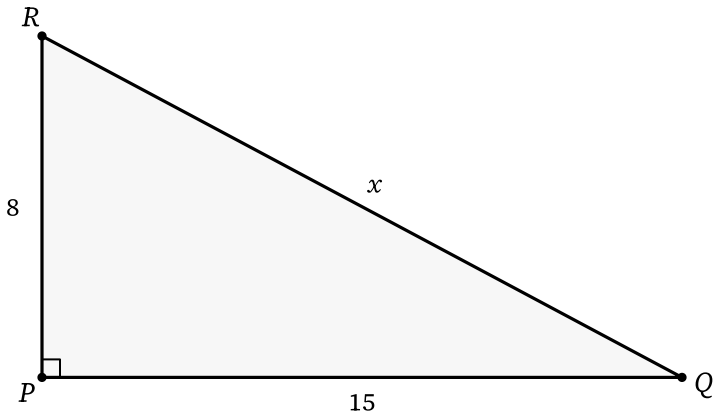
	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; D is on $\overline{CB}$	Given
2	$AB^2 = AC^2 + CB^2$	Pythagorean Theorem (right $\triangle ACB$)
3	$\angle ACD$ is a right angle	$\angle ACD$ and $\angle ACB$ are the same angle, since D lies on $\overrightarrow{CB}$
4	$AD^2 = AC^2 + CD^2$	Pythagorean Theorem (right $\triangle ACD$)
5	$AB^2 - AD^2 = (AC^2 + CB^2) - (AC^2 + CD^2)$	Subtraction Property of Equality
6	$AB^2 - AD^2 = CB^2 - CD^2$	Simplify (combine like terms)

∎

Answer: See the model proof.

PRACTICE

1. **Easy** In the figure, right triangle PQR has legs $PQ = 15$ and $PR = 8$. Find the length x of the hypotenuse.



Show answer

$x = 17$ (since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$)

2. **Easy** A right triangle has a hypotenuse of 25 cm and one leg of 7 cm. Find the length of the other leg.

Show answer

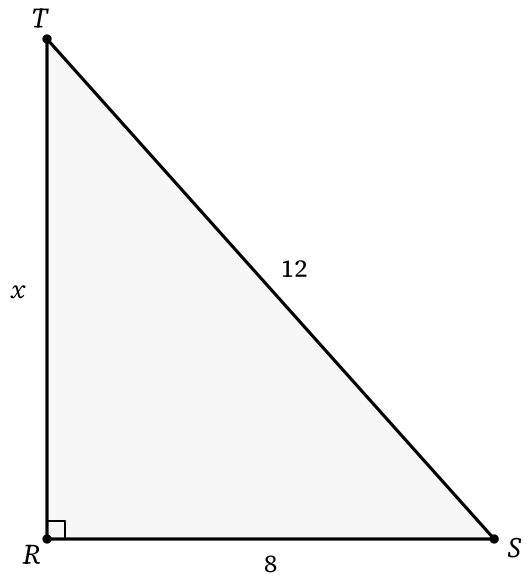
24 cm (since $25^2 - 7^2 = 625 - 49 = 576 = 24^2$)

3. **Easy** The legs of a right triangle are 4 and 6. Find the hypotenuse in simplest radical form.

Show answer

$\sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$

4. **Easy** In the figure, right triangle RST has hypotenuse $ST = 12$ and leg $RS = 8$. Find the other leg x in simplest radical form.



Show answer

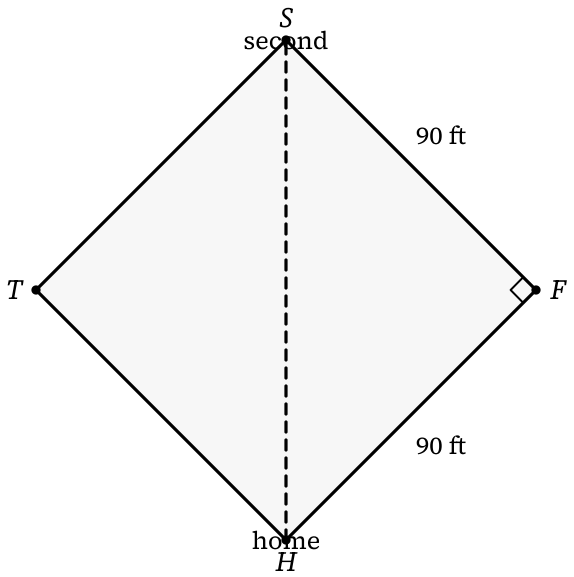
$$x = \sqrt{144 - 64} = \sqrt{80} = 4\sqrt{5}$$

5. **Medium** Samuel's vegetable garden is a rectangle 24 feet long and 10 feet wide. He wants a straight path from one corner to the opposite corner. How long is the path?

Show answer

26 ft (since $24^2 + 10^2 = 576 + 100 = 676 = 26^2$)

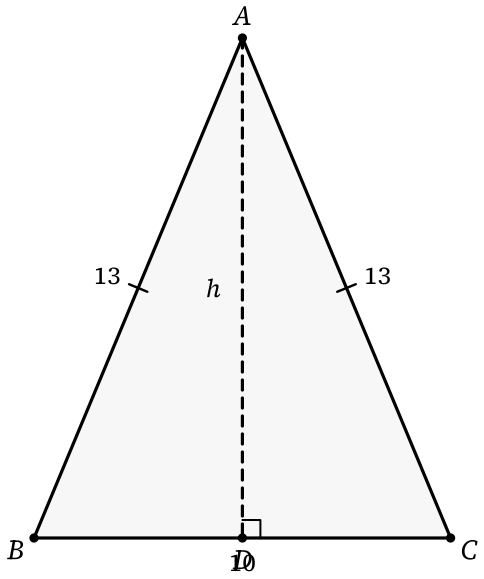
6. **Medium** A baseball diamond is a square 90 feet on a side. How far does the catcher throw from home plate to second base (the opposite corner)? Round to the nearest tenth of a foot.



Show answer

$90\sqrt{2} \approx 127.3$ ft

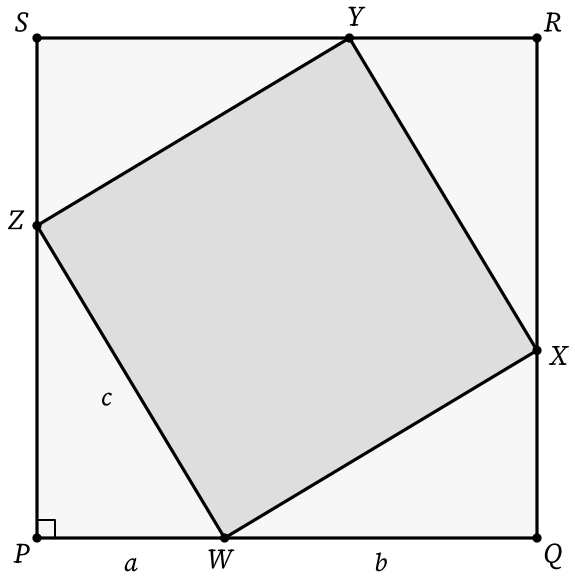
7. **Medium** In the figure, isosceles triangle ABC has legs $AB = AC = 13$ and base $BC = 10$. The altitude $\overline{AD}$ is drawn to the base. Find its length h. (Recall from Unit 4 that the altitude to the base of an isosceles triangle bisects the base.)



Show answer

$BD = 5$, so $h = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$

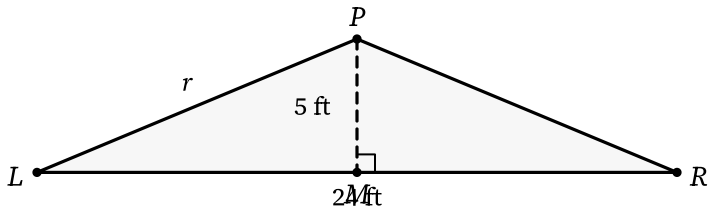
8. **Medium** Explain why, in the rearrangement proof, the inner quadrilateral $WXYZ$ is a square and not just a rhombus. Which fact about right triangles is the key?



Show answer

Its four sides are all hypotenuses of congruent triangles, so they are equal (that alone makes a rhombus). Each of its angles is a right angle because, at a corner such as W, the angle of $WXYZ$ together with two acute angles of the right triangle forms a straight angle of $180°$, and the two acute angles of a right triangle add to $90°$ (they are complementary). So each corner angle is $180° - 90° = 90°$, and a rhombus with right angles is a square.

9. **Medium** The roof of the Miller family's shed spans 24 feet from wall to wall, and the ridge is directly above the middle of the span, 5 feet higher than the tops of the walls. How long is each rafter from the top of a wall to the ridge (with no overhang)?



Show answer

Half the span is 12 ft, so the rafter is $\sqrt{12^2 + 5^2} = \sqrt{169} = 13$ ft.

10. Hard Prove that in a right triangle the hypotenuse is longer than each leg. Use the Pythagorean Theorem. (Given: $\triangle ABC$ with right angle $\angle C$. Prove: $AB > BC$ and $AB > AC$.)

Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\triangle ABC$ with right angle $\angle C$

Prove: $AB > BC$ and $AB > AC$

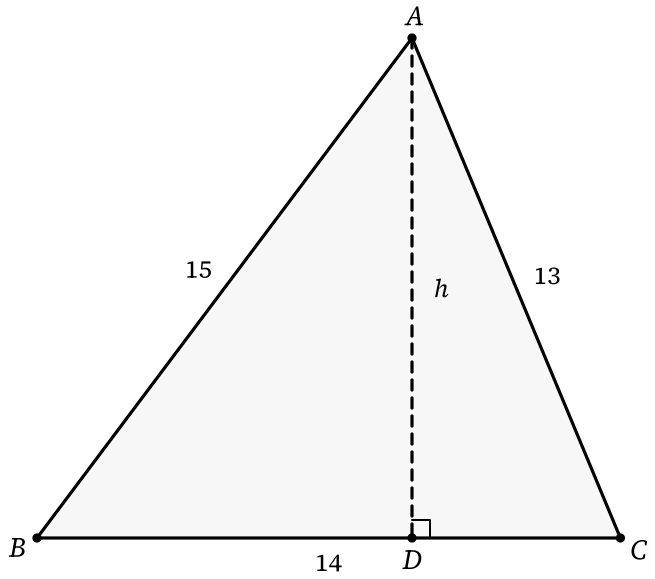
By the Pythagorean Theorem, $AB^2 = BC^2 + AC^2$. Since AC is the length of a segment, $AC > 0$, so $AC^2 > 0$. Therefore $AB^2 = BC^2 + AC^2 > BC^2$.

Both AB and BC are positive. For positive numbers, if $AB \le BC$ then $AB^2 \le AB \cdot BC \le BC^2$, which contradicts $AB^2 > BC^2$. So $AB > BC$.

Exchanging the roles of the two legs, $AB^2 > AC^2$ because $BC^2 > 0$, and the same reasoning gives $AB > AC$.

∎

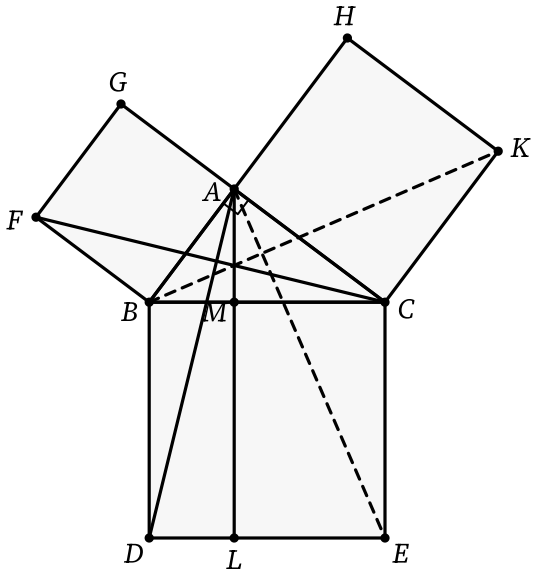
11. **Hard** In the figure, $\triangle ABC$ has $AB = 15$, $AC = 13$ and $BC = 14$, and $\overline{AD}$ is the altitude to $\overline{BC}$, with D between B and C. Find the altitude $h = AD$. (Hint: let $BD = x$, so $DC = 14 - x$, and write h^2 two ways.)



Show answer

$15^2 - x^2 = 13^2 - (14 - x)^2$ gives $225 = -27 + 28x$, so $x = 9$ and $h = \sqrt{225 - 81} = 12$.

12. Hard This is the key step of Euclid's proof. In the windmill figure, $\triangle ABC$ has a right angle at A, and $BCED$ and $ABFG$ are squares built outward on $\overline{BC}$ and $\overline{AB}$. Prove that $\triangle ABD \cong \triangle FBC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\triangle ABC$ with right angle $\angle BAC$; $BCED$ and $ABFG$ are squares built outward on $\overline{BC}$ and $\overline{AB}$

Prove: $\triangle ABD \cong \triangle FBC$

	STATEMENTS	REASONS
1	$BCED$ and $ABFG$ are squares	Given
2	$\overline{BD} \cong \overline{BC}$ and $\overline{BA} \cong \overline{BF}$	Definition of a square (all sides congruent)
3	$\angle CBD$ and $\angle ABF$ are right angles	Definition of a square (all angles right angles)
4	$m\angle CBD = m\angle ABF$	All right angles are congruent
5	$m\angle ABD = m\angle ABC + m\angle CBD$	Angle Addition Postulate
6	$m\angle FBC = m\angle FBA + m\angle ABC$	Angle Addition Postulate
7	$m\angle ABD = m\angle FBC$	Substitution Property of Equality (steps 4, 5, 6)

	STATEMENTS	REASONS
8	$\triangle ABD \cong \triangle FBC$	SAS (steps 2 and 7)

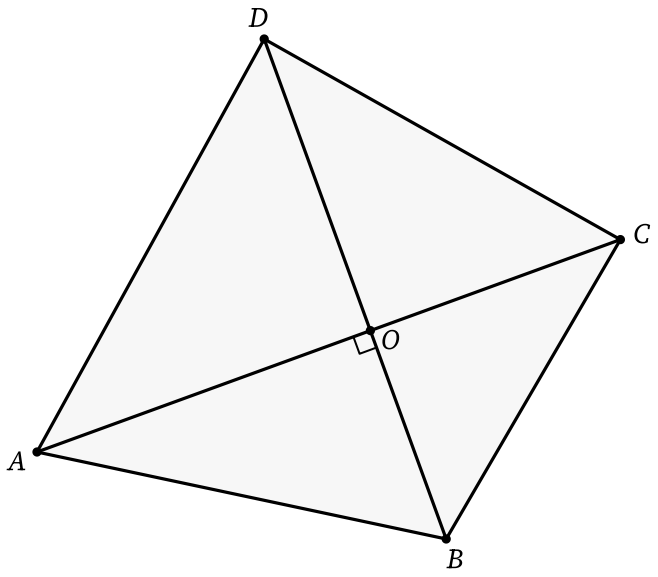
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13. **Challenge** Henry's tool chest is a box 12 inches long, 4 inches wide and 3 inches high (inside measurements). What is the longest thin straight rod that fits inside the closed chest, running from a bottom corner to the opposite top corner? (Hint: first find the diagonal of the bottom.)

Show answer

Bottom diagonal: $\sqrt{12^2 + 4^2} = \sqrt{160}$. Rod: $\sqrt{160 + 3^2} = \sqrt{169} = 13$ in.

14. **Challenge** In the figure, the diagonals $\overline{AC}$ and $\overline{BD}$ of quadrilateral $ABCD$ are perpendicular and meet at O. Prove that $AB^2 + CD^2 = BC^2 + DA^2$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Quadrilateral $ABCD$ with diagonals $\overline{AC} \perp \overline{BD}$ at O

Prove: $AB^2 + CD^2 = BC^2 + DA^2$

	STATEMENTS	REASONS
1	$\overline{AC} \perp \overline{BD}$ at O	Given
2	$\angle AOB$, $\angle BOC$, $\angle COD$ and $\angle DOA$ are right angles	Perpendicular lines form four right angles
3	$AB^2 = OA^2 + OB^2$ and $CD^2 = OC^2 + OD^2$	Pythagorean Theorem (right $\triangle AOB$ and $\triangle COD$)
4	$BC^2 = OB^2 + OC^2$ and $DA^2 = OD^2 + OA^2$	Pythagorean Theorem (right $\triangle BOC$ and $\triangle DOA$)
5	$AB^2 + CD^2 = OA^2 + OB^2 + OC^2 + OD^2$	Addition Property of Equality (step 3)
6	$BC^2 + DA^2 = OA^2 + OB^2 + OC^2 + OD^2$	Addition Property of Equality (step 4)
7	$AB^2 + CD^2 = BC^2 + DA^2$	Transitive Property of Equality

COMMON MISTAKES

- Putting the hypotenuse in the wrong place. c must be the side opposite the right angle. If you know the hypotenuse and a leg, subtract the squares ($a^2 = c^2 - b^2$); adding them gives a leg longer than the hypotenuse, which is impossible.
- Writing $\sqrt{a^2 + b^2} = a + b$. Square roots do not split over addition: $\sqrt{9 + 16} = \sqrt{25} = 5$, not $3 + 4 = 7$.
- Leaving a radical unsimplified. $\sqrt{80}$ should be written $4\sqrt{5}$: look for the largest perfect-square factor (here 16) and take its root outside.
- Using the theorem in a triangle that only looks right. Use it only when a right angle is given (in words or with a right-angle mark); Lesson 9.2 shows how to test a triangle for a right angle.

> **Make it a game · Cut and Rearrange.** Cut four congruent right triangles from paper, plus a square. Arrange them inside a big square of side a + b to form the square on the hypotenuse; rearrange them to form the two smaller squares. Explain why the areas match. Then teach the proof to your parent, who may ask any step.

9.2 The Converse of the Pythagorean Theorem

OBJECTIVES

- State and prove the Converse of the Pythagorean Theorem (Euclid I.48).
- Decide whether three lengths form a right triangle and recognize Pythagorean triples and their multiples.
- Classify a triangle as acute, right or obtuse by comparing c^2 with $a^2 + b^2$, and explain why the test works.

WATCH

Converse of Pythagoras' Theorem (1 of 2: What is a converse?)

Eddie Woo · 9:53

https://www.youtube.com/watch?v=XIM3UToQPZs

> **WHEN TO WATCH**
> Before the Converse of the Pythagorean Theorem: what a converse is.

Note: Eddie Woo teaches in Australia: he says 'Pythagoras' theorem'. Pythagorean triples are taught from the Core ideas.

BACKUP VIDEO

Pythagorean Inequality Theorem - Obtuse, Acute or Right?

Mario's Math Tutoring · 3:04

https://www.youtube.com/watch?v=ZYKTA-5aAeQ

Before the acute and obtuse tests: comparing c^2 with $a^2 + b^2$ to classify a triangle.

KEY TERMS

Converse of the Pythagorean Theorem
If the square of one side of a triangle equals the sum of the squares of the other two sides, then the angle opposite that side is a right angle.

Pythagorean triple
Three positive whole numbers a, b, c with $a^2 + b^2 = c^2$, such as 3, 4, 5 or 5, 12, 13.

Family of a triple

The whole-number multiples ka, kb, kc of a Pythagorean triple; for example 6, 8, 10 and 9, 12, 15 belong to the 3-4-5 family.

Acute triangle

A triangle whose three angles are all acute (less than 90°).

Obtuse triangle

A triangle with one obtuse angle (greater than 90° and less than 180°).

CORE IDEAS

In Lesson 1.3 you learned that the converse of a true statement need not be true. The Pythagorean Theorem says: if a triangle is right, then $a^2 + b^2 = c^2$. Its converse says: if $a^2 + b^2 = c^2$, then the triangle is right. This converse happens to be true, but it needs its own proof. Euclid gives it as the very last proposition of Book I (I.48): "If in a triangle the square on one of the sides be equal to the squares on the remaining two sides, the angle contained by the remaining two sides is right."

Why the converse is true. Suppose $\triangle ABC$ has $BC = a$, $AC = b$, $AB = c$ and $a^2 + b^2 = c^2$. Build a second triangle DEF with a right angle at F and legs $FE = a$, $FD = b$. By the Pythagorean Theorem its hypotenuse satisfies $DE^2 = a^2 + b^2 = c^2$, so $DE = c$. Now the two triangles have all three sides equal, so they are congruent by SSS, and the angle at C equals the right angle at F. Notice the clever move: the theorem is used to prove its own converse, by way of a helper triangle.

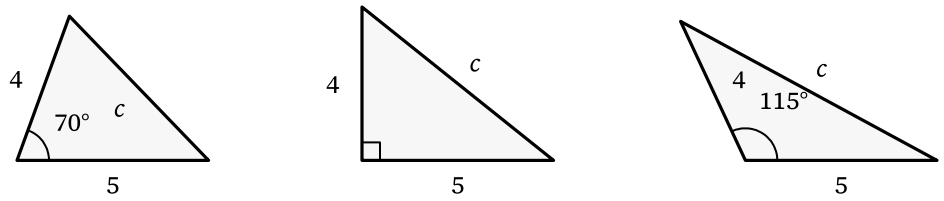
Pythagorean triples. Whole-number solutions of $a^2 + b^2 = c^2$ are called Pythagorean triples. Memorize the most common: 3-4-5, 5-12-13, 8-15-17 and 7-24-25. Multiplying a triple by any whole number k gives another one, because $(ka)^2 + (kb)^2 = k^2(a^2 + b^2) = k^2c^2 = (kc)^2$. So 6-8-10, 9-12-15 and 30-40-50 all belong to the 3-4-5 family.

Acute or obtuse? Picture two sides, a and b, hinged at C. As you open the hinge, the third side c grows (the Hinge Theorem, Lesson 5.6). At exactly 90°, $c^2 = a^2 + b^2$. So if the angle at C is smaller than 90°, then $c^2 < a^2 + b^2$; if it is larger, then $c^2 > a^2 + b^2$. Turning this around (with the converse of the Hinge Theorem) gives a test for any triangle.

The test, step by step. (1) Let c be the longest side. (2) Check the Triangle Inequality: $a + b > c$; if it fails, there is no triangle at all. (3) Compare: if $c^2 = a^2 + b^2$ the triangle

is right; if $c^2 < a^2 + b^2$ it is acute; if $c^2 > a^2 + b^2$ it is obtuse. You must use the longest side, because only the angle opposite the longest side can be right or obtuse; the largest angle is opposite the longest side (I.18).

Squaring a corner. Carpenters use the converse every day. To check a corner, measure 3 feet along one wall and 4 feet along the other; if the marks are exactly 5 feet apart, the corner is square. For more accuracy on a large job they use a multiple such as 6-8-10 or 9-12-15. If the diagonal is too short, the corner is acute; if too long, it is obtuse.

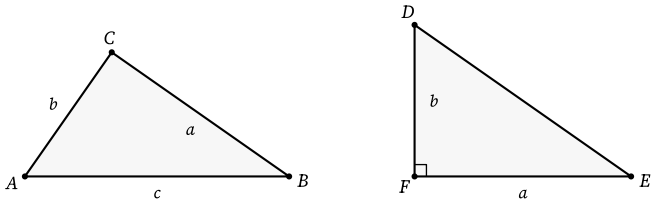


Opening the hinge: $c^2 < 41$ (acute), $c^2 = 41$ (right), $c^2 > 41$ (obtuse), where $41 = 4^2 + 5^2$.

POSTULATES AND THEOREMS

THEOREM **Converse of the Pythagorean Theorem** Euclid I.48

If the sides of $\triangle ABC$ satisfy $BC^2 + AC^2 = AB^2$, then $\angle C$ is a right angle.



TWO-COLUMN PROOF

Given: $\triangle ABC$ with $BC = a$, $AC = b$, $AB = c$, and $a^2 + b^2 = c^2$

Prove: $\angle C$ is a right angle

	STATEMENTS	REASONS
1	$BC = a$, $AC = b$, $AB = c$, and $a^2 + b^2 = c^2$	Given
2	Draw $\triangle DEF$ with right angle $\angle F$, $FE = a$ and $FD = b$	A perpendicular can be constructed (Euclid I.11) and segments copied (Euclid I.3)
3	$DE^2 = a^2 + b^2$	Pythagorean Theorem (right $\triangle DEF$)

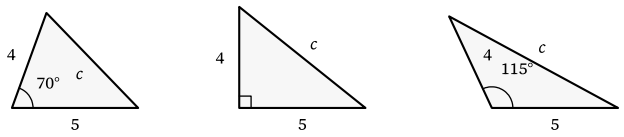
	STATEMENTS	REASONS
4	$DE^2 = c^2$	Substitution Property of Equality
5	$DE = c$	Lengths are positive, so take the positive square root
6	$\triangle ABC \cong \triangle DEF$	SSS ($AC = DF$, $BC = EF$, $AB = DE$)
7	$\angle C \cong \angle F$	CPCTC
8	$\angle C$ is a right angle	Definition of congruent angles ($m\angle C = m\angle F = 90°$)

∎

Euclid's proof of I.48 is the same: he builds a right angle, copies the sides, and uses I.47 and I.8 (SSS).

THEOREM **Acute Triangle Test**

If c is the longest side of $\triangle ABC$ and $c^2 < a^2 + b^2$, then $\triangle ABC$ is acute.



Opening the hinge: $c^2 < 41$ (acute), $c^2 = 41$ (right), $c^2 > 41$ (obtuse), where $41 = 4^2 + 5^2$.

PARAGRAPH PROOF

Given: $\triangle ABC$ with $BC = a$, $AC = b$, $AB = c$; c is the longest side and $c^2 < a^2 + b^2$

Prove: $\triangle ABC$ is acute

Draw right $\triangle DEF$ with right angle at F, $FE = a$ and $FD = b$. By the Pythagorean Theorem, $DE^2 = a^2 + b^2$. Since $c^2 < a^2 + b^2 = DE^2$ and both lengths are positive, $c < DE$.

Now $\triangle ABC$ and $\triangle DEF$ have two pairs of congruent sides ($CB = FE$ and $CA = FD$), and the third side AB is shorter than the third side DE. By the converse of the Hinge Theorem (SSS Inequality, Euclid I.25), the angle opposite the shorter third side is smaller: $m\angle C < m\angle F = 90°$. So $\angle C$ is acute.

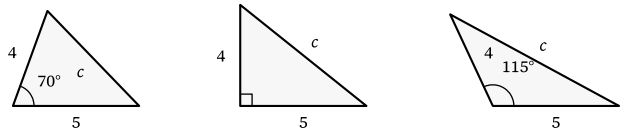
Finally, AB is the longest side, so $\angle C$ is the largest angle of the triangle (the larger side is opposite the larger angle, Euclid I.18, and equal sides are opposite equal

angles). So $m\angle A \le m\angle C < 90^\circ$ and $m\angle B \le m\angle C < 90^\circ$. All three angles are acute, so $\triangle ABC$ is acute.

∎

THEOREM **Obtuse Triangle Test**

If the sides of $\triangle ABC$ satisfy $c^2 > a^2 + b^2$, where $c = AB$, then $\angle C$ is obtuse and $\triangle ABC$ is obtuse.



Opening the hinge: $c^2 < 41$ (acute), $c^2 = 41$ (right), $c^2 > 41$ (obtuse), where $41 = 4^2 + 5^2$.

INDIRECT PROOF

Given: $\triangle ABC$ with $BC = a$, $AC = b$, $AB = c$, and $c^2 > a^2 + b^2$

Prove: $\angle C$ is obtuse

Assume, to the contrary, that $\angle C$ is not obtuse. Then $\angle C$ is either a right angle or an acute angle.

Case 1: $\angle C$ is a right angle. Then by the Pythagorean Theorem $c^2 = a^2 + b^2$, which contradicts the given $c^2 > a^2 + b^2$.

Case 2: $\angle C$ is acute. Draw right $\triangle DEF$ with right angle at F, $FE = a$ and $FD = b$, so $DE^2 = a^2 + b^2$. The two triangles have two pairs of congruent sides, and the included angle $\angle C$ is smaller than the included angle $\angle F = 90^\circ$. By the Hinge Theorem (SAS Inequality, Euclid I.24), $AB < DE$, so $c^2 < DE^2 = a^2 + b^2$. This also contradicts the given.

Both cases lead to a contradiction, so the assumption is false: $\angle C$ is obtuse, and a triangle with an obtuse angle is an obtuse triangle.

∎

KEY FORMULAS

$$c^2 = a^2 + b^2 \;\Rightarrow\; \text{right triangle}$$

$$c^2 < a^2 + b^2 \;\Rightarrow\; \text{acute triangle}$$

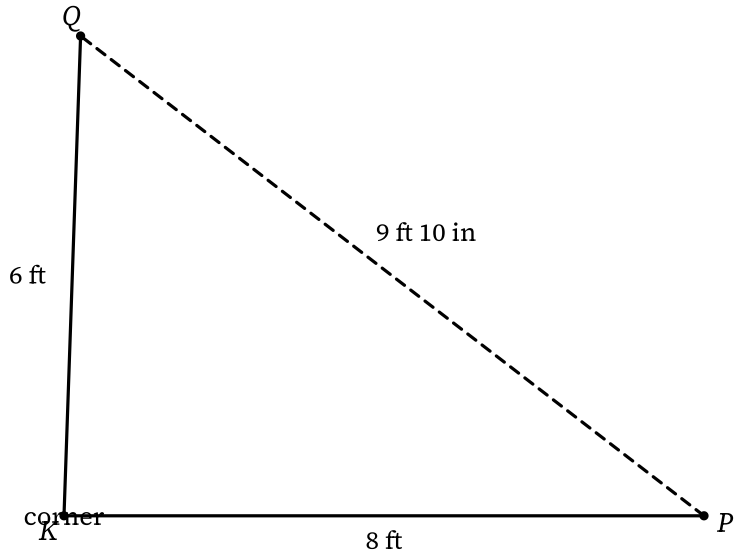
$$c^2 > a^2 + b^2 \;\Rightarrow\; \text{obtuse triangle}$$

From history. Builders in many times and places have laid out square corners with the 3-4-5 rule. A cord with 12 equally spaced knots, pulled tight around three stakes into sides of 3, 4 and 5 spaces, makes a right angle; master masons and carpenters used cords and rods this way. You may read that the surveyors of ancient Egypt, the "rope-stretchers," did exactly this; that is a modern guess, and no ancient Egyptian text confirms it, so it should be told with caution.

Euclid connection. Book I of the Elements ends with a theorem and its converse: I.47 (the Pythagorean Theorem) and I.48 (its converse). Euclid proves I.48 just as we do here, by building a right triangle with the same legs, using I.47 to find its hypotenuse, and applying SSS (I.8).

Practice habit. Before you use any converse, check that it has been proved. For this test, always list the sides from shortest to longest, check the Triangle Inequality, and only then compare c^2 with $a^2 + b^2$. Memorize the triples 3-4-5, 5-12-13, 8-15-17 and 7-24-25.

Example 1. Henry is framing a shed. To check a corner, he marks a point 6 feet from the corner along one wall and a point 8 feet from the corner along the other wall. The distance between the marks is 9 feet 10 inches. Is the angle at the corner acute, right or obtuse?

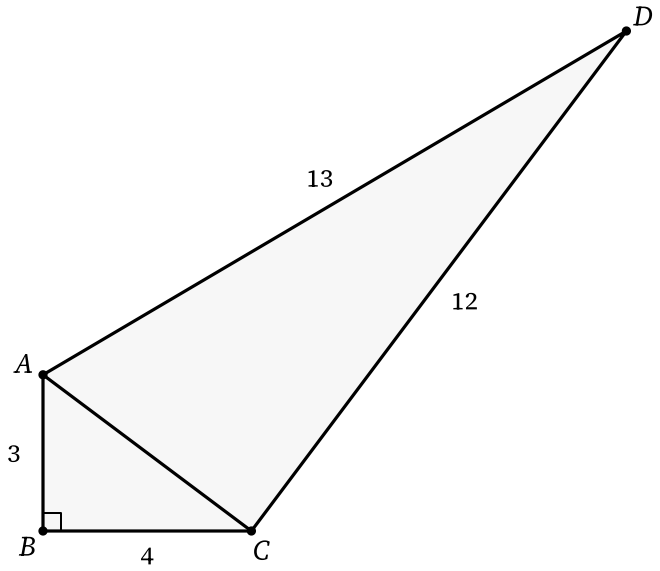


Show solution

1. The two wall lengths and the measured distance form a triangle with sides 6 ft, 8 ft and 9 ft 10 in. Use one unit: 9 ft 10 in $= 9\frac{10}{12} = \frac{59}{6}$ ft ≈ 9.833 ft. This is the longest side, and it is opposite the corner.
2. Check the Triangle Inequality: $6 + 8 = 14 > 9.833$, so the triangle exists.
3. Compare: $6^2 + 8^2 = 36 + 64 = 100$, while $\left(\frac{59}{6}\right)^2 = \frac{3481}{36} \approx 96.69$.
4. Since $96.69 < 100$, $c^2 < a^2 + b^2$, so the corner angle is acute: a little less than 90°. Henry should push the walls apart until the diagonal reads exactly 10 feet.

Answer: The corner is acute (slightly less than 90°), so it is not square.

Example 2. In quadrilateral $ABCD$ in the figure, $AB = 3$, $BC = 4$, $CD = 12$, $DA = 13$, and $\angle ABC$ is a right angle. Prove that $\angle ACD$ is a right angle.



Show solution

1. The diagonal $\overline{AC}$ is a side of both triangles. Find it from right $\triangle ABC$ with the Pythagorean Theorem.
2. Then test $\triangle ACD$ with the Converse of the Pythagorean Theorem: is $AC^2 + CD^2 = DA^2$?
3. Be careful to use the theorem in the first triangle (where the right angle is given) and the converse in the second (where it is to be proved).

TWO-COLUMN PROOF

Given: $AB = 3$, $BC = 4$, $CD = 12$, $DA = 13$; $\angle ABC$ is a right angle

Prove: $\angle ACD$ is a right angle

	STATEMENTS	REASONS
1	$AB = 3$, $BC = 4$, $CD = 12$, $DA = 13$; $\angle ABC$ is a right angle	Given
2	$AC^2 = AB^2 + BC^2$	Pythagorean Theorem (right $\triangle ABC$)
3	$AC^2 = 9 + 16 = 25$	Substitution Property of Equality

	STATEMENTS	REASONS
4	$AC^2 + CD^2 = 25 + 144 = 169$	Addition Property of Equality; Substitution
5	$DA^2 = 13^2 = 169$	Given; squaring
6	$AC^2 + CD^2 = DA^2$	Transitive Property of Equality
7	$\angle ACD$ is a right angle	Converse of the Pythagorean Theorem ($\overline{DA}$ is opposite $\angle ACD$)

∎

Answer: See the model proof.

PRACTICE

1. Easy Samuel nails three boards into a triangular frame with sides 9 ft, 12 ft and 15 ft. Is the frame a right triangle?

   Show answer

   Yes: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$.

2. Easy Classify the triangle with sides 6, 7 and 9 as acute, right or obtuse.

   Show answer

   Acute: $9^2 = 81 < 6^2 + 7^2 = 85$.

3. Easy Classify the triangle with sides 7, 24 and 25 as acute, right or obtuse.

   Show answer

   Right: $7^2 + 24^2 = 49 + 576 = 625 = 25^2$.

4. Easy Classify the triangle with sides 5, 8 and 10 as acute, right or obtuse.

   Show answer

   Obtuse: $10^2 = 100 > 5^2 + 8^2 = 89$.

5. Medium Classify the triangle with sides $\sqrt{11}$, 5 and 6 as acute, right or obtuse.

   Show answer

   Right: the longest side is 6 (since $\sqrt{11} \approx 3.3$), and $(\sqrt{11})^2 + 5^2 = 11 + 25 = 36 = 6^2$.

6. **Medium** A right triangle belongs to the 8-15-17 family and has hypotenuse 51. Find its two legs.

Show answer

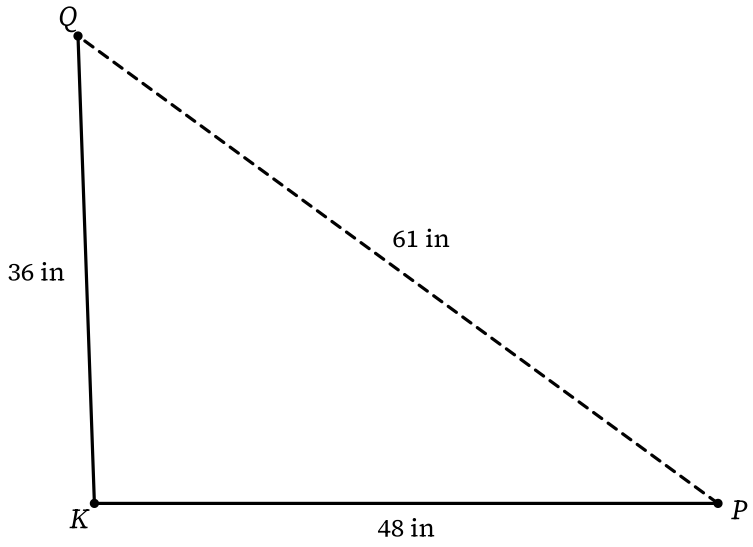
$51 = 3 \times 17$, so the legs are $3 \times 8 = 24$ and $3 \times 15 = 45$.

7. **Medium** Grace says the triangle with sides 4, 5 and 10 is obtuse because $10^2 > 4^2 + 5^2$. Is she right? Classify it as acute, right or obtuse, or say that no such triangle exists.

Show answer

No such triangle exists, because $4 + 5 = 9 < 10$ violates the Triangle Inequality. Always check the Triangle Inequality first.

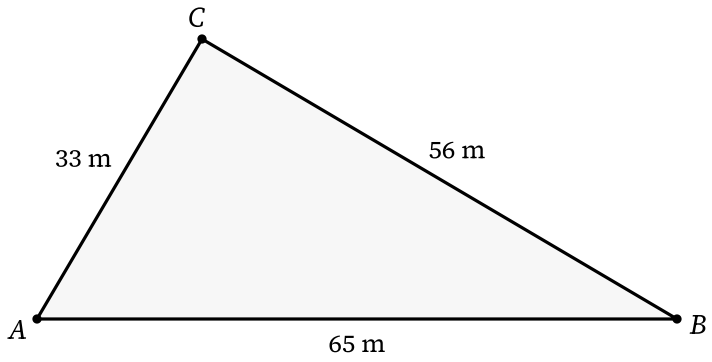
8. **Medium** Ruth checks the corner of a raised garden bed. She measures 36 inches along one side and 48 inches along the other, and the distance between the two marks is 61 inches. Is the corner angle acute, right or obtuse?



Show answer

Obtuse: $36^2 + 48^2 = 1296 + 2304 = 3600$, but $61^2 = 3721 > 3600$. The corner is a little more than 90°.

9. **Medium** A surveyor paces a triangular field ABC and finds $AB = 65$ m, $BC = 56$ m and $CA = 33$ m. Which corner of the field, if any, is a right angle?



Show answer

Corner C: the longest side $AB = 65$ is opposite C, and $33^2 + 56^2 = 1089 + 3136 = 4225 = 65^2$.

10. **Medium** Explain why the acute/right/obtuse test must compare the square of the LONGEST side with the sum of the squares of the other two. Use the triangle with sides 5, 12 and 13 to show what goes wrong otherwise.

Show answer

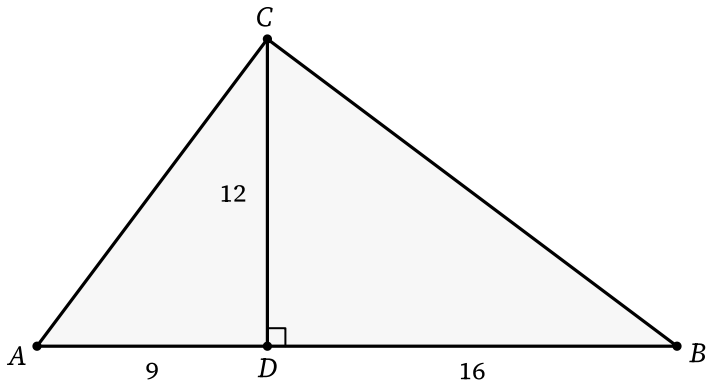
Only the largest angle of a triangle can be right or obtuse, and the largest angle is opposite the longest side. The test tells you about the angle opposite the side you call c. For 5, 12, 13, using $c = 13$ gives $169 = 25 + 144$, so the triangle is right. If you wrongly used $c = 12$, you would get $144 < 25 + 169$, which only tells you the angle opposite 12 is acute; it says nothing about the largest angle, and you might wrongly call the triangle acute.

11. **Hard** A triangle has sides 8, 15 and x, where x is the longest side. For which values of x is the triangle obtuse? Write an inequality.

Show answer

Obtuse needs $x^2 > 8^2 + 15^2 = 289$, so $x > 17$; a triangle needs $x < 8 + 15 = 23$. So $17 < x < 23$.

12. **Hard** In the figure, D is on $\overline{AB}$ and $\overline{CD} \perp \overline{AB}$, with $AD = 9$, $DB = 16$ and $CD = 12$. Prove that $\angle ACB$ is a right angle.



Show answer

See the model proof.

TWO-COLUMN PROOF

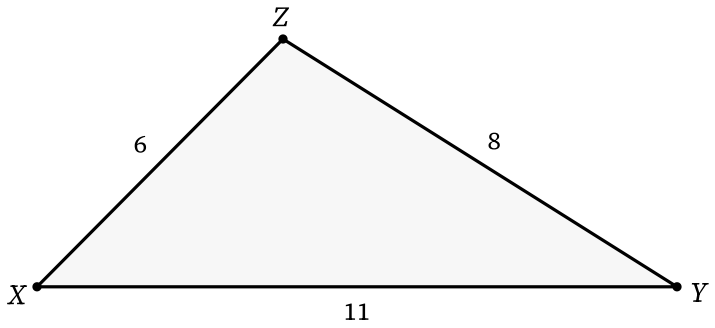
Given: D is on $\overline{AB}$ and $\overline{CD} \perp \overline{AB}$; $AD = 9$, $DB = 16$, $CD = 12$

Prove: $\angle ACB$ is a right angle

	STATEMENTS	REASONS
1	$\overline{CD} \perp \overline{AB}$; $AD = 9$, $DB = 16$, $CD = 12$	Given
2	$\angle CDA$ and $\angle CDB$ are right angles	Definition of perpendicular lines
3	$AC^2 = AD^2 + CD^2 = 81 + 144 = 225$	Pythagorean Theorem (right $\triangle ADC$); Substitution
4	$BC^2 = DB^2 + CD^2 = 256 + 144 = 400$	Pythagorean Theorem (right $\triangle BDC$); Substitution
5	$AB = AD + DB = 25$	Segment Addition Postulate
6	$AC^2 + BC^2 = 625 = AB^2$	Addition Property of Equality; Substitution ($25^2 = 625$)
7	$\angle ACB$ is a right angle	Converse of the Pythagorean Theorem

∎

13. **Hard** Write an indirect proof that the triangle XYZ with sides $XY = 11$, $YZ = 8$ and $XZ = 6$ has no right angle.



Show answer

See the model proof.

INDIRECT PROOF

Given: $\triangle XYZ$ with $XY = 11$, $YZ = 8$, $XZ = 6$

Prove: $\triangle XYZ$ has no right angle

Assume, to the contrary, that $\triangle XYZ$ has a right angle.

The side opposite the right angle is the hypotenuse, and the hypotenuse is longer than each leg (Lesson 9.1). The longest side is $\overline{XY}$, so the right angle would have to be $\angle Z$, with legs 6 and 8 and hypotenuse 11.

Then the Pythagorean Theorem would give $6^2 + 8^2 = 11^2$. But $6^2 + 8^2 = 100$ and $11^2 = 121$, and $100 \neq 121$. This is a contradiction.

So the assumption is false, and $\triangle XYZ$ has no right angle. (In fact $121 > 100$, so $\angle Z$ is obtuse.)

∎

14. **Challenge** Prove that for any whole numbers $m > n > 0$, the numbers $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$ are the sides of a right triangle. (A form of this recipe for triples appears in Euclid's Book X; $m = 2$, $n = 1$ gives 3, 4, 5.)

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Whole numbers $m > n > 0$; $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$
Prove: a, b, c are the sides of a right triangle
Since $m > n > 0$, all three numbers are positive. Expanding, $a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = c^2$.

Next, a triangle with these sides exists. Because $c^2 = a^2 + b^2$ is greater than a^2 and than b^2, c is the longest side. And $(a + b)^2 = a^2 + 2ab + b^2 = c^2 + 2ab > c^2$, so $a + b > c$. The other two parts of the Triangle Inequality hold because c is the longest side.

So there is a triangle with sides a, b, c, and since $a^2 + b^2 = c^2$, the Converse of the Pythagorean Theorem says the angle opposite c is a right angle.

∎

COMMON MISTAKES

- Testing with the wrong side. The side you call c must be the longest one; otherwise the comparison says nothing about the largest angle.
- Skipping the Triangle Inequality. Lengths like 4, 5, 10 satisfy $c^2 > a^2 + b^2$ but form no triangle at all. Check $a + b > c$ first.
- Calling a triangle right because $a^2 + b^2$ is close to c^2. In geometry the converse needs exact equality; $9^2 + 12^2 = 225$ but $15.1^2 = 228.01$, so a 9-12-15.1 triangle is obtuse, though barely.
- Mixing units. Change 9 ft 10 in to $\frac{59}{6}$ ft (or to 118 in) before squaring.

> **Make it a game · Rope Stretchers.** Mark a string into 12 equal parts and have your parent hold it as a 3-4-5 triangle. Check the corner against a book. Then parent calls three lengths and you say right, acute or obtuse in 10 seconds: 5, 12, 13; 6, 8, 11; 7, 8, 9. Name as many triples as you can in 90 seconds.

9.3 Similar Right Triangles and the Geometric Mean

OBJECTIVES

- Prove that the altitude to the hypotenuse of a right triangle forms two triangles similar to the original triangle and to each other (Euclid VI.8).
- Use the geometric mean (altitude) and geometric mean (leg) relationships to find lengths in right triangles.
- Give a second proof of the Pythagorean Theorem from the leg relationships, and solve practical problems with them.

WATCH

Geometric Mean Theorems Altitude and Leg

Mario's Math Tutoring · 12:39

https://www.youtube.com/watch?v=Wqhj1pesd_8

> **WHEN TO WATCH**
> After the Core ideas: the altitude to the hypotenuse makes three similar triangles, then the altitude and leg geometric-mean rules, with examples.

BACKUP VIDEO

Mystery Theorem (6 of 7: Folding paper - similar triangles)

Eddie Woo · 8:29

https://www.youtube.com/watch?v=3vlQyhKDQS8

As extra practice after the examples: two right triangles shown similar with angles, then used to find a length (part 6 of Eddie Woo's folded-paper problem).

KEY TERMS

Altitude to the hypotenuse

The segment from the vertex of the right angle perpendicular to the hypotenuse.

Segments of the hypotenuse

The two parts $\overline{AD}$ and $\overline{DB}$ into which the foot D of the altitude divides the hypotenuse $\overline{AB}$.

Geometric mean

For positive numbers a and b, the positive number x with $\frac{a}{x} = \frac{x}{b}$, that is, $x = \sqrt{ab}$.

Mean proportional

Euclid's name for the geometric mean: a length x with $a : x = x : b$.

Adjacent segment

For a leg of the right triangle, the segment of the hypotenuse that touches that leg (for leg $\overline{AC}$ it is $\overline{AD}$).

CORE IDEAS

Draw right $\triangle ABC$ with the right angle at C, and draw the altitude $\overline{CD}$ to the hypotenuse $\overline{AB}$. Because $\angle A$ and $\angle B$ are acute, the foot D lands between A and B. Now the figure holds three right triangles: the whole $\triangle ABC$ and the two small triangles $\triangle ACD$ and $\triangle CBD$.

Why all three are similar. $\triangle ACD$ has a right angle at D and shares $\angle A$ with $\triangle ABC$, so the two are similar by AA. In the same way $\triangle CBD$ shares $\angle B$ with $\triangle ABC$. Two triangles similar to the same triangle have the same angles, so the two small triangles are similar to each other too. Here is a quick check of the angles: $\angle ACD$ and $\angle B$ are both complementary to $\angle A$, so they are congruent.

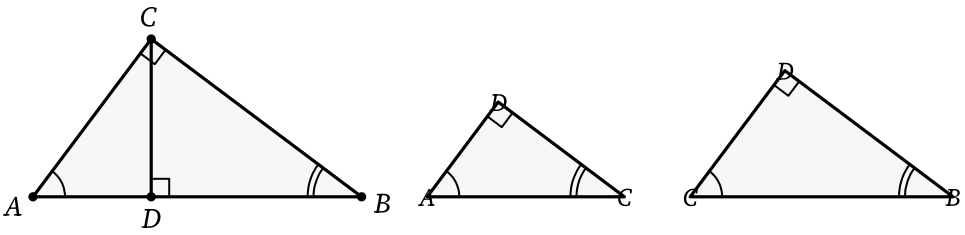
Match the vertices carefully. The safest way is to pull the three triangles apart and turn them so that matching angles sit in the same place: right angle with right angle, the angle equal to $\angle A$ with $\angle A$. Then $\triangle ACD \sim \triangle ABC \sim \triangle CBD$, with $A \leftrightarrow A \leftrightarrow C$, $C \leftrightarrow B \leftrightarrow B$ and $D \leftrightarrow C \leftrightarrow D$. Every proportion in this lesson comes from reading this statement letter by letter.

Geometric mean (altitude). From $\triangle ACD \sim \triangle CBD$, $\frac{AD}{CD} = \frac{CD}{DB}$, so $CD^2 = AD \cdot DB$. The altitude is the geometric mean of the two segments of the hypotenuse. Euclid notes this right after VI.8: the perpendicular is a mean proportional between the segments of the base.

Geometric mean (leg). From $\triangle ACD \sim \triangle ABC$, $\frac{AD}{AC} = \frac{AC}{AB}$, so $AC^2 = AB \cdot AD$. From $\triangle CBD \sim \triangle ABC$, $BC^2 = AB \cdot BD$. Each leg is the geometric mean of the whole hypotenuse and the segment of the hypotenuse next to that leg.

The Pythagorean Theorem again. Add the two leg relationships: $AC^2 + BC^2 = AB \cdot AD + AB \cdot BD = AB(AD + DB) = AB \cdot AB = AB^2$. This is the similar-triangle proof from Lesson 9.1, and now you can see where each step comes from.

A practical trick. Hold a carpenter's square at eye level so that you sight the top of a tree along one arm and its base along the other. The trunk is then the hypotenuse of a right triangle with the right angle at your eye, and your level line of sight to the trunk is the altitude. So (distance to the trunk)^2 = (eye height) $\times$ (height of the tree above eye level).

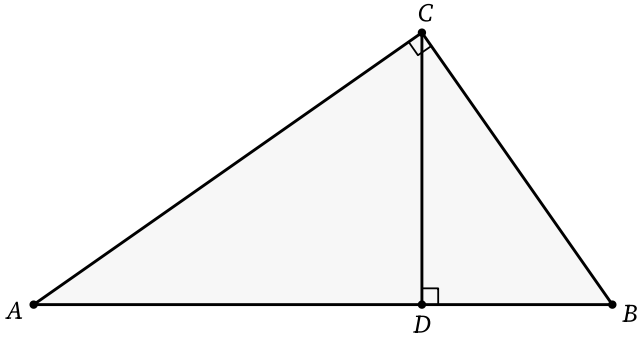


△ABC, △ACD and △CBD pulled apart and turned so that matching angles line up.

POSTULATES AND THEOREMS

THEOREM **Right Triangle Altitude Similarity Theorem** Euclid VI.8

If the altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the original triangle and to each other: $\triangle ACD \sim \triangle ABC \sim \triangle CBD$.



TWO-COLUMN PROOF

Given: $\triangle ABC$ with right angle $\angle ACB$; $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$

Prove: $\triangle ACD \sim \triangle ABC \sim \triangle CBD$

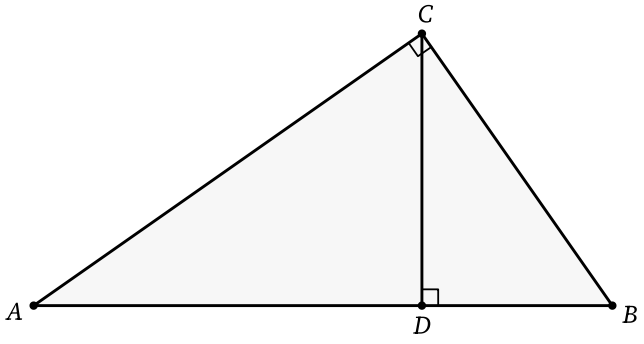
	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$	Given
2	$\angle ADC$ and $\angle CDB$ are right angles	Definition of perpendicular lines
3	$\angle ADC \cong \angle ACB$	All right angles are congruent

	STATEMENTS	REASONS
4	$\angle CAD \cong \angle BAC$	Reflexive Property of Congruence (same angle, since D is on $\overrightarrow{AB}$)
5	$\triangle ACD \sim \triangle ABC$	AA Similarity
6	$\angle CDB \cong \angle ACB$	All right angles are congruent
7	$\angle CBD \cong \angle ABC$	Reflexive Property of Congruence (same angle)
8	$\triangle CBD \sim \triangle ABC$	AA Similarity
9	$\angle BCD \cong \angle BAC$	Corresponding angles of similar triangles are congruent (step 8)
10	$\angle ADC \cong \angle CDB$	All right angles are congruent
11	$\triangle ACD \sim \triangle CBD$	AA Similarity ($\angle CAD \cong \angle BCD$ by steps 4 and 9 and the Transitive Property of Congruence; $\angle ADC \cong \angle CDB$ by step 10)
12	$\triangle ACD \sim \triangle ABC \sim \triangle CBD$	Steps 5, 8 and 11

∎

COROLLARY **Geometric Mean (Altitude) Corollary** Euclid VI.8

The altitude to the hypotenuse of a right triangle is the geometric mean of the two segments of the hypotenuse: $CD^2 = AD \cdot DB$, so $CD = \sqrt{AD \cdot DB}$.



PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle ACB$ and altitude $\overline{CD}$ to the hypotenuse

Prove: $CD^2 = AD \cdot DB$

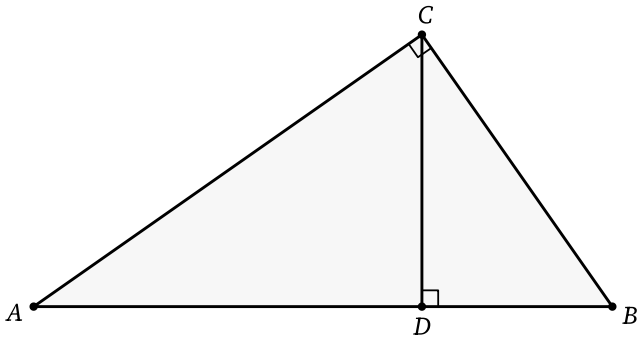
By the Right Triangle Altitude Similarity Theorem, $\triangle ACD \sim \triangle CBD$ with $A \leftrightarrow C$, $C \leftrightarrow B$ and $D \leftrightarrow D$. Corresponding sides of similar triangles are proportional, so $\frac{AD}{CD} = \frac{CD}{DB}$ (side $\overline{AD}$ matches $\overline{CD}$, and side $\overline{CD}$ matches $\overline{DB}$). By the Cross Products Property, $CD^2 = AD \cdot DB$, and since $CD > 0$, $CD = \sqrt{AD \cdot DB}$.

∎

Euclid states this as a porism (a corollary) of VI.8.

COROLLARY **Geometric Mean (Leg) Corollary**

Each leg of a right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to that leg: $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$.



FLOW PROOF

Given: Right $\triangle ABC$ with right angle $\angle ACB$ and altitude $\overline{CD}$ to the hypotenuse

Prove: $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$

(1) Right $\triangle ABC$ with altitude $\overline{CD}$ to the hypotenuse
Given

(2) $\triangle ACD \sim \triangle ABC$ and $\triangle CBD \sim \triangle ABC$
Right Triangle Altitude Similarity Theorem
from (1)

(3) $\frac{AD}{AC} = \frac{AC}{AB}$
Corresponding sides of similar triangles are proportional
from (2)

(4) $\frac{BD}{BC} = \frac{BC}{AB}$
Corresponding sides of similar triangles are proportional
from (2)

(5) $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$
Cross Products Property
from (3), (4)

∎

KEY FORMULAS

$$CD = \sqrt{AD \cdot DB}$$

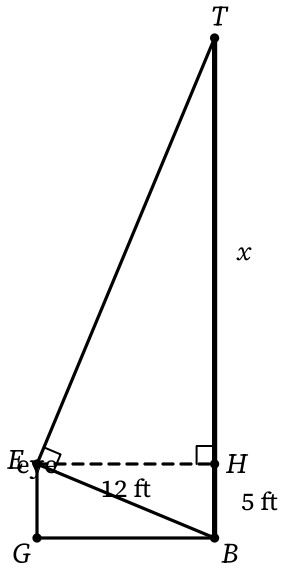
$$AC = \sqrt{AB \cdot AD}$$

$$BC = \sqrt{AB \cdot BD}$$

Euclid connection. Euclid VI.8 reads: “If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.” His porism adds that the perpendicular is a mean proportional between the segments of the base, and he uses exactly this figure, set inside a semicircle, to construct a mean proportional in VI.13 (Lesson 8.1).

Practice habit. Whenever you see a right triangle with the altitude to its hypotenuse, sketch the three triangles separately, turned so that matching angles line up, and write the similarity statement with the vertices in matching order before you write any proportion.

Example 1. Henry wants the height of an oak tree. He holds a carpenter's square at eye level, 5 feet above the ground, and backs up until he can sight the top of the tree along one arm and the base of the trunk along the other. He is then 12 feet from the trunk. How tall is the tree?

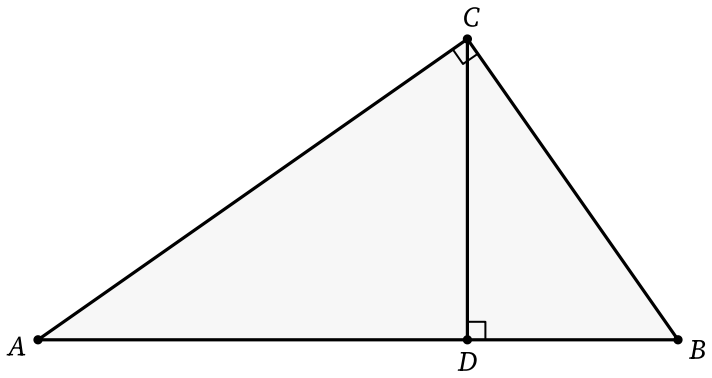


Show solution

1. The square makes a right angle at Henry's eye E, so $\triangle TEB$ is a right triangle with hypotenuse $\overline{TB}$ (the trunk). His level line of sight $\overline{EH}$ to the trunk is perpendicular to the trunk, so it is the altitude to the hypotenuse.
2. The altitude splits the trunk into $HB = 5$ ft (his eye height) and $TH = x$ ft (the part above eye level).
3. Geometric mean (altitude): $EH^2 = TH \cdot HB$, so $12^2 = 5x$, $144 = 5x$ and $x = 28.8$ ft.
4. Height of the tree $= TH + HB = 28.8 + 5 = 33.8$ ft.

Answer: 33.8 ft

Example 2. In right $\triangle ABC$ with right angle C, $\overline{CD}$ is the altitude to the hypotenuse. Prove that $CD \cdot AB = AC \cdot BC$.



Show solution

1. A product of lengths equal to a product of lengths usually comes from a proportion between similar triangles, by cross products.
2. The four lengths CD, AB, AC, BC appear in $\triangle ACD$ and $\triangle ABC$, which are similar by AA.
3. Write the similarity with matching vertices ($A \leftrightarrow A$, $C \leftrightarrow B$, $D \leftrightarrow C$), read off $\dfrac{CD}{BC} = \dfrac{AC}{AB}$, and cross-multiply. (Check: both products equal twice the area of $\triangle ABC$.)

TWO-COLUMN PROOF

Given: $\triangle ABC$ with right angle $\angle ACB$; $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$

Prove: $CD \cdot AB = AC \cdot BC$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$	Given
2	$\angle ADC$ is a right angle	Definition of perpendicular lines
3	$\angle ADC \cong \angle ACB$	All right angles are congruent
4	$\angle CAD \cong \angle BAC$	Reflexive Property of Congruence
5	$\triangle ACD \sim \triangle ABC$	AA Similarity
6	$\dfrac{CD}{BC} = \dfrac{AC}{AB}$	Corresponding sides of similar triangles are proportional

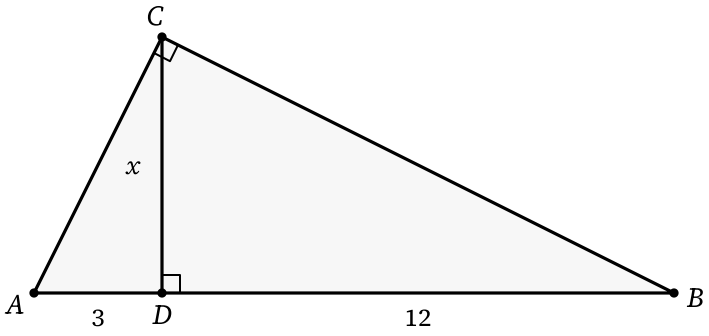
	STATEMENTS	REASONS
7	$CD \cdot AB = AC \cdot BC$	Cross Products Property

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Answer: See the model proof.

PRACTICE

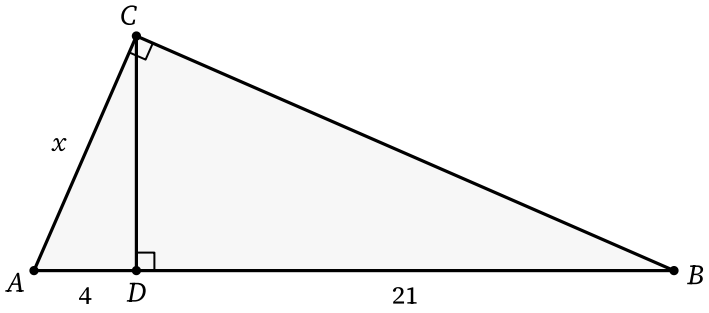
1. Easy In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $AD = 3$ and $DB = 12$. Find $x = CD$.



Show answer

$x = \sqrt{3 \cdot 12} = \sqrt{36} = 6$

2. Easy In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $AD = 4$ and $DB = 21$. Find $x = AC$.



Show answer

$AB = 25$, so $x = \sqrt{AB \cdot AD} = \sqrt{25 \cdot 4} = 10$

3. **Easy** Right $\triangle PQR$ has its right angle at R, and $\overline{RS}$ is the altitude to the hypotenuse $\overline{PQ}$. Complete the similarity statement with the vertices in matching order: $\triangle PQR \sim \triangle P??\sim \triangle R??$.

Show answer

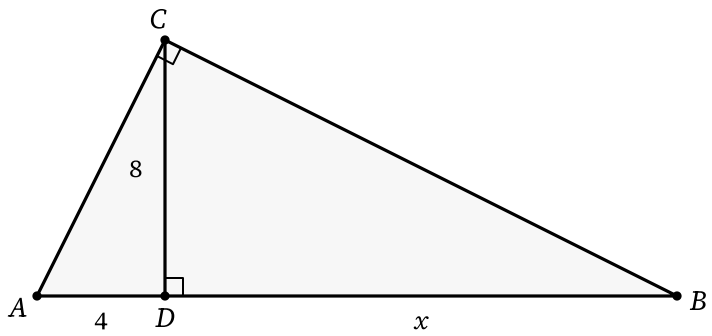
$\triangle PQR \sim \triangle PRS \sim \triangle RQS$ (right angle R matches right angle S; $\angle P$ matches $\angle P$ and $\angle QRS$).

4. **Easy** Find the geometric mean of 8 and 18.

Show answer

$\sqrt{8 \cdot 18} = \sqrt{144} = 12$

5. **Medium** In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $CD = 8$ and $AD = 4$. Find $x = DB$.



Show answer

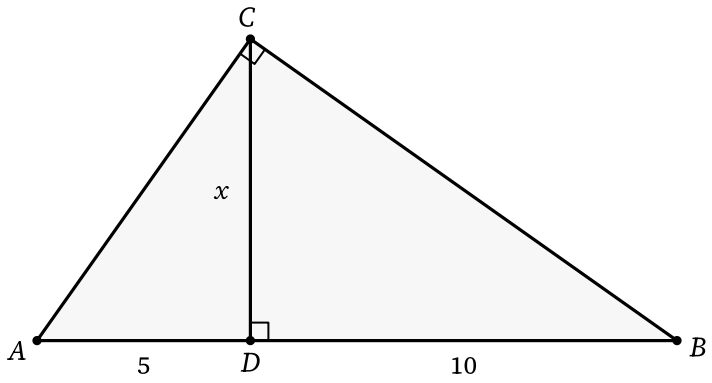
$8^2 = 4x$, so $x = 16$

6. **Medium** Surveyors lay out a right-triangle plot ABC with the right angle at corner C; side $\overline{AB}$ runs along the road. A footpath from C meets the road at D at a right angle. The fence $\overline{BC}$ is 120 m long, and D is 80 m from B. How long is the road frontage AB?

Show answer

Leg rule: $BC^2 = AB \cdot BD$, so $14400 = 80 \cdot AB$ and $AB = 180$ m.

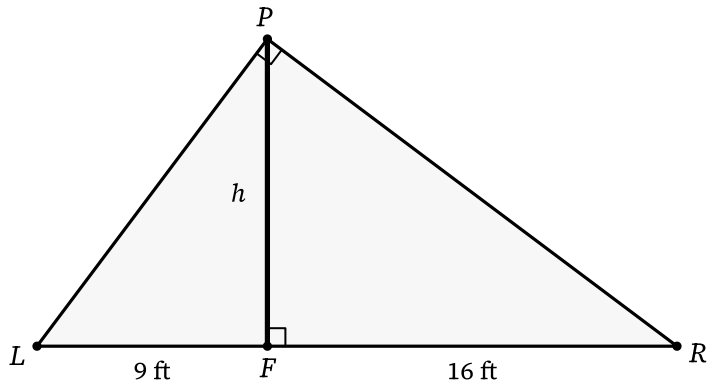
7. **Medium** In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $AD = 5$ and $DB = 10$. Find $x = CD$ in simplest radical form.



Show answer

$$x = \sqrt{5 \cdot 10} = \sqrt{50} = 5\sqrt{2}$$

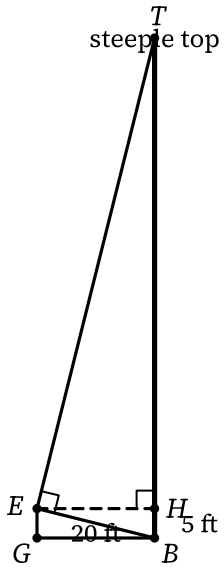
8. **Medium** The two rafters of a gable roof meet at a right angle at the peak P. A vertical post $\overline{PF}$ runs from the peak straight down to the floor beam $\overline{LR}$, meeting it 9 ft from one end and 16 ft from the other. How tall is the post?



Show answer

$$h = \sqrt{9 \cdot 16} = \sqrt{144} = 12 \text{ ft}$$

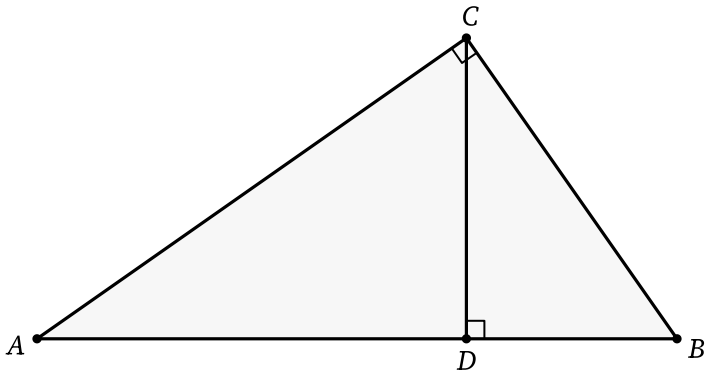
9. **Medium** Grace uses the carpenter's-square trick on a church steeple. With her eye 5 ft above level ground, she sights the top and the base of the steeple along the two arms when she stands 20 ft from it. How tall is the steeple?



Show answer

$20^2 = 5x$ gives $x = 80$ ft above eye level, so the steeple is $80 + 5 = 85$ ft tall.

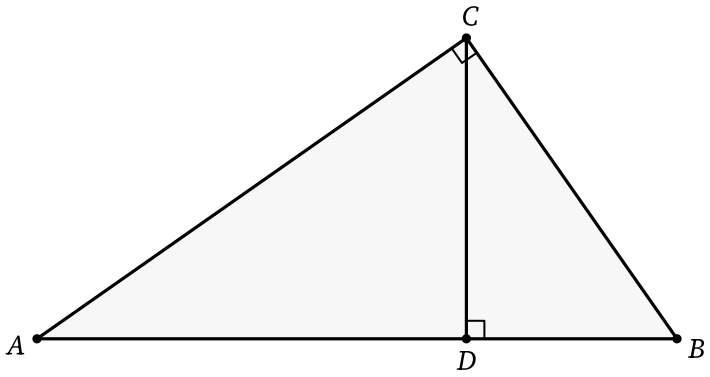
10. **Medium** In right $\triangle ABC$ with right angle C and altitude $\overline{CD}$, explain why $\angle ACD \cong \angle B$ without using similar triangles.



Show answer

In right $\triangle ACD$, $\angle ACD$ and $\angle A$ are the acute angles, so they are complementary. In right $\triangle ABC$, $\angle B$ and $\angle A$ are complementary. Two angles complementary to the same angle are congruent (Congruent Complements Theorem), so $\angle ACD \cong \angle B$.

11. Hard In right $\triangle ABC$ with right angle C and altitude $\overline{CD}$ to the hypotenuse, prove that $\dfrac{AC^2}{BC^2} = \dfrac{AD}{DB}$.



Show answer

See the model proof.

PARAGRAPH PROOF

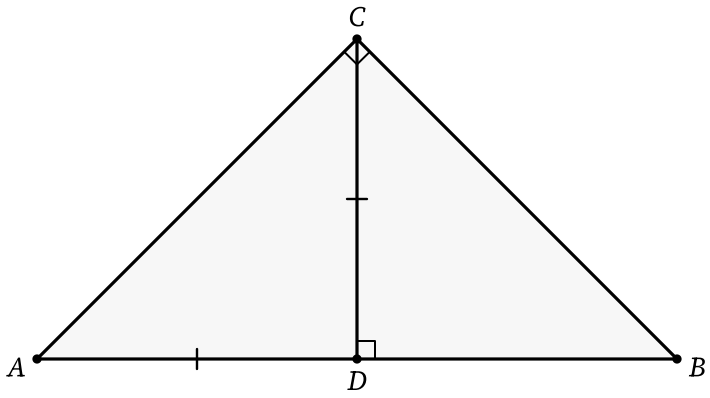
Given: Right $\triangle ABC$ with right angle $\angle ACB$ and altitude $\overline{CD}$ to the hypotenuse

Prove: $\dfrac{AC^2}{BC^2} = \dfrac{AD}{DB}$

By the Geometric Mean (Leg) Corollary, $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot DB$. Since $BC^2 \neq 0$, we may divide: $\dfrac{AC^2}{BC^2} = \dfrac{AB \cdot AD}{AB \cdot DB}$. The common factor AB is not zero, so it divides out, leaving $\dfrac{AC^2}{BC^2} = \dfrac{AD}{DB}$.

∎

12. **Hard** In the figure, $\triangle ABC$ has a right angle at C, $\overline{CD}$ is the altitude to the hypotenuse, and $CD = AD$. Prove that $AC = BC$.



Show answer

See the model proof.

TWO-COLUMN PROOF

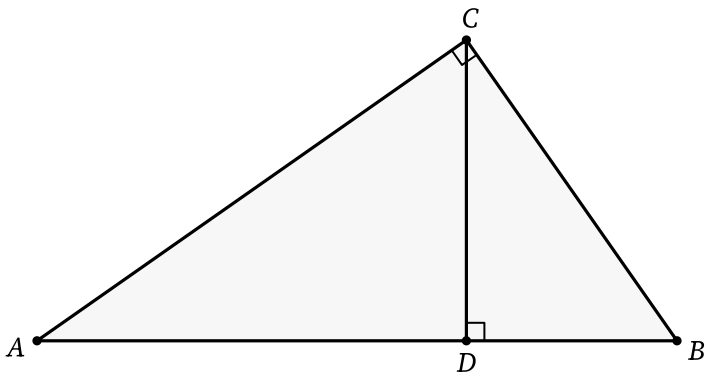
Given: $\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$ with D on $\overline{AB}$; $CD = AD$

Prove: $AC = BC$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$; $CD = AD$	Given
2	$CD^2 = AD \cdot DB$	Geometric Mean (Altitude) Corollary
3	$AD^2 = AD \cdot DB$	Substitution Property of Equality
4	$AD = DB$	Division Property of Equality (divide by $AD \neq 0$)
5	D is the midpoint of $\overline{AB}$	Definition of midpoint (D is between A and B)
6	$\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector (steps 1 and 5)
7	$AC = BC$	Perpendicular Bisector Theorem

∎

13. **Hard** Write a flow proof of the Pythagorean Theorem using the Geometric Mean (Leg) Corollary. Given: right $\triangle ABC$ with right angle C and altitude $\overline{CD}$ to the hypotenuse. Prove: $AC^2 + BC^2 = AB^2$.



Show answer

See the model proof.

FLOW PROOF

Given: Right $\triangle ABC$ with right angle $\angle ACB$ and altitude $\overline{CD}$ to the hypotenuse

Prove: $AC^2 + BC^2 = AB^2$

(1) Right $\triangle ABC$ with altitude $\overline{CD}$ to the hypotenuse
Given

(2) $AC^2 = AB \cdot AD$
Geometric Mean (Leg) Corollary
from (1)

(3) $BC^2 = AB \cdot BD$
Geometric Mean (Leg) Corollary
from (1)

(4) $AC^2 + BC^2 = AB \cdot AD + AB \cdot BD$
Addition Property of Equality
from (2), (3)

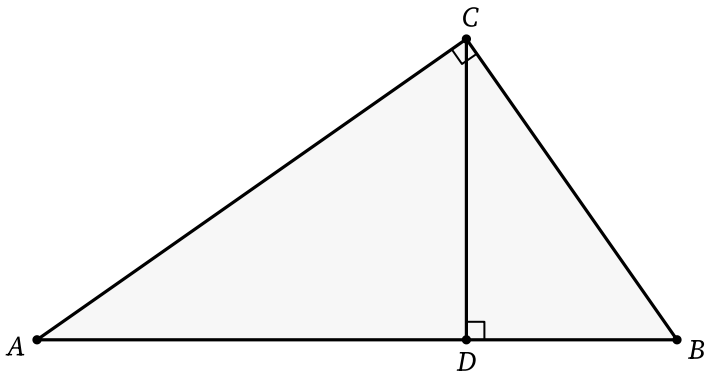
(5) $AC^2 + BC^2 = AB(AD + BD)$
Distributive Property
from (4)

(6) $AD + BD = AB$
Segment Addition Postulate (D is between A and B)

(7) $AC^2 + BC^2 = AB^2$
Substitution Property of Equality
from (5), (6)

∎

14. **Challenge** In right $\triangle ABC$ with right angle C and altitude $\overline{CD}$ to the hypotenuse, prove that $\dfrac{1}{CD^2} = \dfrac{1}{AC^2} + \dfrac{1}{BC^2}$. (You may use the result of Worked Example 2.) Check your result on a 6-8-10 triangle, where $CD = 4.8$.



Show answer

See the model proof. Check: $\dfrac{1}{4.8^2} = \dfrac{1}{23.04} \approx 0.043403$ and $\dfrac{1}{36} + \dfrac{1}{64} \approx 0.027778 + 0.015625 = 0.043403$.

TWO-COLUMN PROOF

Given: Right $\triangle ABC$ with right angle $\angle ACB$ and altitude $\overline{CD}$ to the hypotenuse

Prove: $\dfrac{1}{CD^2} = \dfrac{1}{AC^2} + \dfrac{1}{BC^2}$

	STATEMENTS	REASONS
1	Right $\triangle ABC$ with altitude $\overline{CD}$ to the hypotenuse	Given
2	$CD \cdot AB = AC \cdot BC$	Proved in Worked Example 2
3	$CD^2 \cdot AB^2 = AC^2 \cdot BC^2$	Multiplication Property of Equality (square both sides)
4	$AB^2 = AC^2 + BC^2$	Pythagorean Theorem
5	$CD^2(AC^2 + BC^2) = AC^2 \cdot BC^2$	Substitution Property of Equality
6	$\dfrac{1}{CD^2} = \dfrac{AC^2 + BC^2}{AC^2 \cdot BC^2}$	Division Property of Equality (divide by $CD^2 \cdot AC^2 \cdot BC^2 \neq 0$)
7	$\dfrac{1}{CD^2} = \dfrac{1}{AC^2} + \dfrac{1}{BC^2}$	Write the fraction as a sum: $\dfrac{AC^2}{AC^2 \cdot BC^2} + \dfrac{BC^2}{AC^2 \cdot BC^2}$, and

STATEMENTS	REASONS
	simplify

∎

COMMON MISTAKES

- Pairing a leg with the wrong segment. Leg $\overline{AC}$ goes with the segment it touches, $\overline{AD}$, and the whole hypotenuse: $AC^2 = AB \cdot AD$, not $AB \cdot DB$.
- Using the whole hypotenuse in the altitude rule. The altitude uses the two pieces: $CD^2 = AD \cdot DB$, not $AD \cdot AB$.
- Writing the similarity statement with vertices out of order, which produces a false proportion. Redraw the three triangles side by side, turned the same way, and match right angle to right angle first.
- Forgetting the last step in a word problem: in the carpenter's-square problem, x is only the part above eye level, so add the eye height.

> **Make it a game · Geometric Mean Machine.** Cut a paper right triangle, then cut it along the altitude from the right angle. Match angles to show all three triangles are similar. Next, parent names the hypotenuse's two segments, such as 4 and 9. Find the altitude and each leg within 20 seconds, and name the mean proportional. Ten rounds earns a trophy.

9.4 Special Right Triangles

OBJECTIVES

- Derive the side ratios $1 : 1 : \sqrt{2}$ of the $45°$-$45°$-$90°$ triangle (half a square) and $1 : \sqrt{3} : 2$ of the $30°$-$60°$-$90°$ triangle (half an equilateral triangle).
- Find exact side lengths of special right triangles in simplest radical form from any one side.
- Construct a $30°$-$60°$-$90°$ triangle with compass and straightedge and justify the construction.
- Solve practical problems with roof pitches, diagonal braces, square tiles and hexagonal tiles.

WATCH

Exact Values of Trigonometric Ratios (1 of 2: 30°)

Eddie Woo · 7:31

https://www.youtube.com/watch?v=SiA-WTnKNgs

> **WHEN TO WATCH**
> Before the 30°-60°-90° triangle: half of an equilateral triangle.

Note: He finds exact trig values from these triangles, which the course does in Lesson 9.6.

BACKUP VIDEO

Exact Values of Trigonometric Ratios (2 of 2: 60° and 45°)

Eddie Woo · 5:11

https://www.youtube.com/watch?v=VkgH1X2JBuM

Before the 45°-45°-90° triangle: half of a square.

KEY TERMS

$45°$-$45°$-$90°$ triangle

An isosceles right triangle; its acute angles are both $45°$ and its legs are congruent.

$30°$-$60°$-$90°$ triangle

A right triangle whose acute angles measure $30°$ and $60°$.

Short leg

In a $30°$-$60°$-$90°$ triangle, the leg opposite the $30°$ angle.

Long leg

In a $30°$-$60°$-$90°$ triangle, the leg opposite the $60°$ angle.

Exact answer

An answer written with radicals (such as $5\sqrt{3}$) rather than a rounded decimal (such as 8.66).

CORE IDEAS

Two right triangles appear so often in building, design and later mathematics that it pays to know their sides by heart. Each comes from cutting a familiar figure in half: the $45°$-$45°$-$90°$ triangle is half a square, and the $30°$-$60°$-$90°$ triangle is half an equilateral triangle. Knowing their shapes lets you find exact lengths without a calculator.

The $45°$-$45°$-$90°$ triangle. A diagonal cuts a square into two such triangles. Its two acute angles are congruent, so the sides opposite them, the legs, are congruent (converse of the Isosceles Triangle Theorem, Euclid I.6). Call each leg x. By the Pythagorean Theorem, the hypotenuse c satisfies $c^2 = x^2 + x^2 = 2x^2$, so $c = x\sqrt{2}$. The sides are in the ratio $1 : 1 : \sqrt{2}$. To go from the hypotenuse back to a leg, divide by $\sqrt{2}$: a hypotenuse of 10 gives legs $\dfrac{10}{\sqrt{2}} = \dfrac{10\sqrt{2}}{2} = 5\sqrt{2}$.

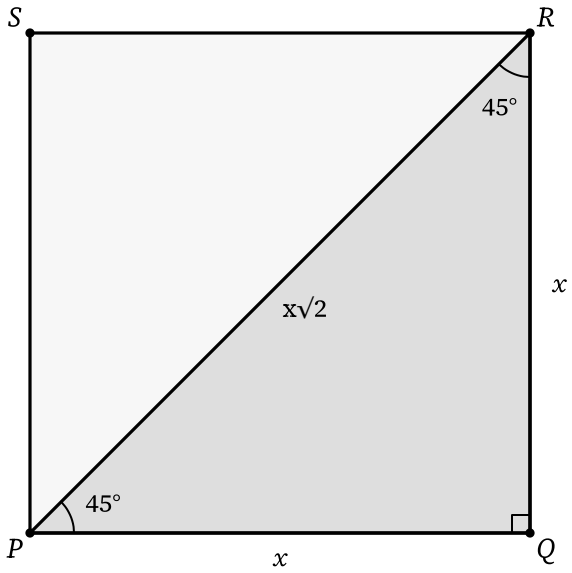
The $30°$-$60°$-$90°$ triangle. The altitude of an equilateral triangle with side $2x$ cuts it into two congruent right triangles. Each has angles $60°$ (a corner of the equilateral triangle), $90°$, and therefore $30°$. Its hypotenuse is a whole side, $2x$, and its short leg is half a side, x. So **the hypotenuse is twice the short leg**. By the Pythagorean Theorem the long leg is $\sqrt{(2x)^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$. The sides are in the ratio $1 : \sqrt{3} : 2$.

Keep the legs straight. The short leg is opposite the $30°$ angle and the long leg is opposite the $60°$ angle, because the larger angle is opposite the longer side (I.19). A quick check: the long leg is about 1.73 times the short leg, and the hypotenuse is exactly 2 times it, so the long leg is always between the other two.

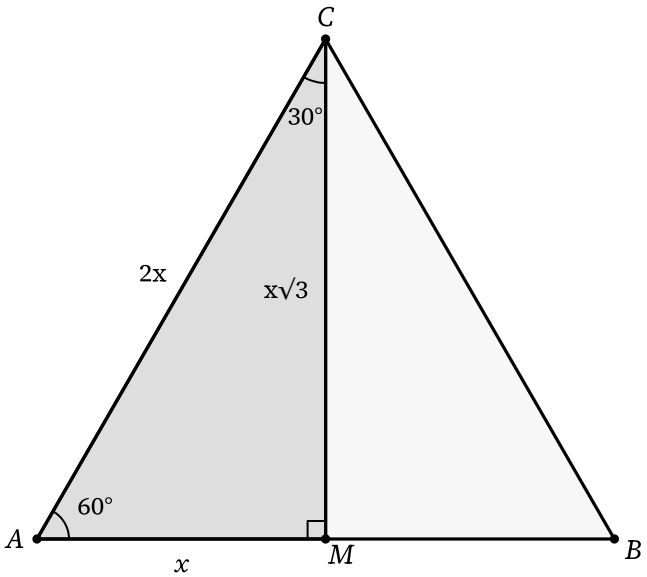
Strategy. In a $30°$-$60°$-$90°$ problem, find the short leg first, whatever side you are given: halve the hypotenuse, or divide the long leg by $\sqrt{3}$. Then multiply to get the

others. In a $45°$-$45°$-$90°$ problem, find a leg first.

Where they appear. A roof with a 12-in-12 pitch (rising 12 inches for every 12 inches of run) makes a $45°$ angle, so its rafters are $\sqrt{2}$ times the run. A diagonal brace across a square frame is $\sqrt{2}$ times the side. Hexagonal nuts, bolts and floor tiles split into equilateral triangles, so the distance across a hexagon's flat sides involves $\sqrt{3}$.



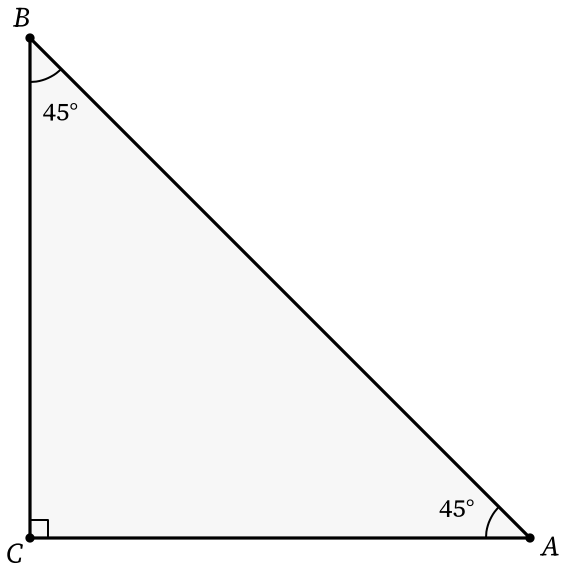
Half a square: legs x and x, hypotenuse x√2.



Half an equilateral triangle: short leg x, long leg x√3, hypotenuse 2x.

THEOREM $45°$-$45°$-$90°$ **Triangle Theorem**

In a $45°$-$45°$-$90°$ *triangle, the legs are congruent and the hypotenuse is* $\sqrt{2}$ *times as long as each leg.*



TWO-COLUMN PROOF

Given: $\triangle ABC$ with $m\angle C = 90°$ and $m\angle A = m\angle B = 45°$

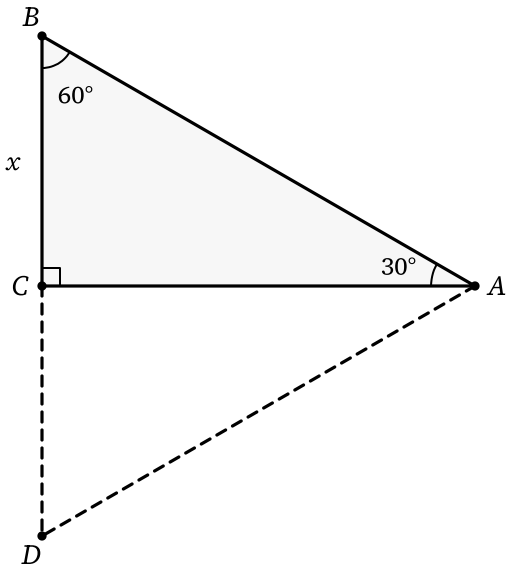
Prove: $AC = BC$ and $AB = AC\sqrt{2}$

	STATEMENTS	REASONS
1	$m\angle C = 90°$, $m\angle A = m\angle B = 45°$	Given
2	$\angle A \cong \angle B$	Definition of congruent angles
3	$AC = BC$; call this length x	Converse of the Isosceles Triangle Theorem (sides opposite congruent angles are congruent)
4	$AB^2 = AC^2 + BC^2$	Pythagorean Theorem
5	$AB^2 = x^2 + x^2 = 2x^2$	Substitution Property of Equality
6	$AB = x\sqrt{2}$	Take the positive square root ($\sqrt{2x^2} = x\sqrt{2}$ since $x > 0$)
7	$AC = BC$ and $AB = AC\sqrt{2}$	Steps 3 and 6

∎

THEOREM $30°$-$60°$-$90°$ **Triangle Theorem**

In a $30°$-$60°$-$90°$ *triangle, the hypotenuse is twice as long as the short leg, and the long leg is* $\sqrt{3}$ *times as long as the short leg.*



PARAGRAPH PROOF

Given: $\triangle ABC$ with $m\angle C = 90°$, $m\angle A = 30°$, $m\angle B = 60°$, and short leg $BC = x$

Prove: $AB = 2x$ and $AC = x\sqrt{3}$

Extend $\overline{BC}$ beyond C to a point D with $CD = BC = x$ (Ruler Postulate), and draw $\overline{AD}$. Since $\angle ACD$ forms a linear pair with the right angle $\angle ACB$, it is also a right angle. Then $\triangle ACB \cong \triangle ACD$ by SAS: $\overline{CB} \cong \overline{CD}$, $\angle ACB \cong \angle ACD$ (right angles), and $\overline{AC} \cong \overline{AC}$ (Reflexive Property).

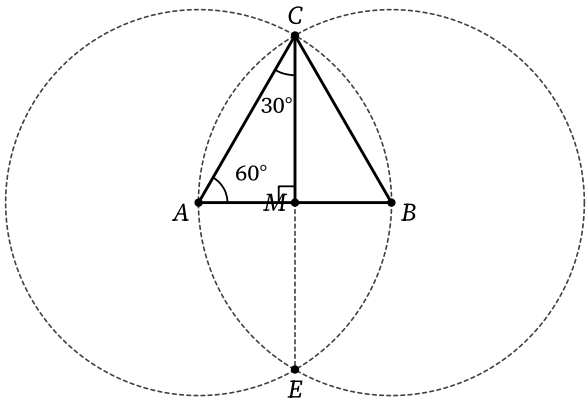
By CPCTC, $AD = AB$ and $m\angle D = m\angle B = 60°$. In $\triangle ABD$, the angles at B and D are $60°$, so by the Triangle Angle Sum Theorem the angle at A is $180° - 60° - 60° = 60°$. An equiangular triangle is equilateral (Lesson 4.6), so $AB = BD = BC + CD = 2x$ (Segment Addition Postulate).

Finally, by the Pythagorean Theorem in right $\triangle ACB$, $AC^2 = AB^2 - BC^2 = (2x)^2 - x^2 = 3x^2$, so $AC = x\sqrt{3}$. Thus $AB = 2x$ and $AC = x\sqrt{3}$.

∎

The proof shows the idea of the lesson: a $30°$-$60°$-$90°$ *triangle is exactly half of an equilateral triangle.*

CONSTRUCTION **Construct a $30°$-$60°$-$90°$ triangle** Euclid I.1, I.10



1. Draw a segment $\overline{AB}$ with the straightedge.
2. Set the compass to the length AB. Draw a circle centered at A through B, and a circle centered at B through A.
3. Label the two points where the circles cross C (above $\overline{AB}$) and E (below $\overline{AB}$).
4. Draw $\overline{AC}$ and $\overline{BC}$. Triangle ABC is equilateral (Euclid I.1).
5. Draw $\overline{CE}$; label the point where it crosses $\overline{AB}$ as M.
6. Triangle AMC is a $30°$-$60°$-$90°$ triangle with the right angle at M, the $60°$ angle at A, and the $30°$ angle at C.

Why it works. All the compass radii equal AB, so $CA = CB = AB$ and $EA = EB$. Triangle ABC is equilateral, so $m\angle CAB = 60°$. Since C and E are each equidistant from A and B, they both lie on the perpendicular bisector of $\overline{AB}$ (Converse of the Perpendicular Bisector Theorem), so $\overleftrightarrow{CE}$ is that bisector: $\angle AMC$ is a right angle and M is the midpoint of $\overline{AB}$

(this is Euclid's bisection of a segment, I.10). By the Triangle Angle Sum Theorem, $m\angle ACM = 180^\circ - 90^\circ - 60^\circ = 30^\circ$. As expected, the short leg $AM = \frac{1}{2}AB = \frac{1}{2}AC$ is half the hypotenuse.

KEY FORMULAS

$$45^\circ\text{-}45^\circ\text{-}90^\circ: \quad \text{leg} = x, \ \ \text{leg} = x, \ \ \text{hypotenuse} = x\sqrt{2}$$

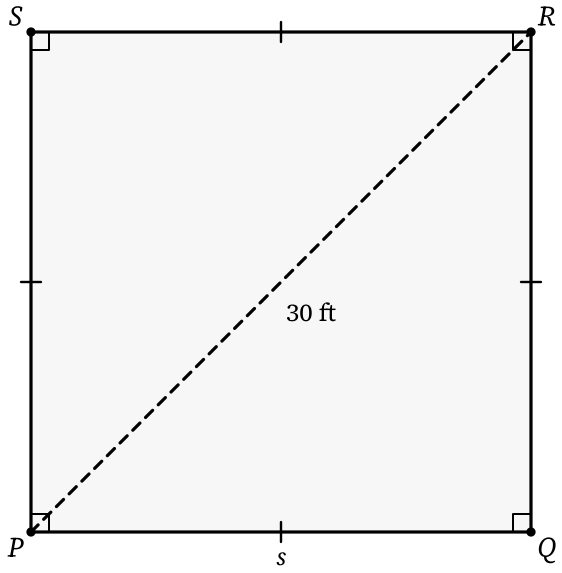
$$30^\circ\text{-}60^\circ\text{-}90^\circ: \quad \text{short leg} = x, \ \ \text{long leg} = x\sqrt{3}, \ \ \text{hypotenuse} = 2x$$

From history. Greek mathematicians discovered that the diagonal and the side of a square have no common measure: no unit, however small, fits a whole number of times into both. In our language, $\sqrt{2}$ is irrational. Later writers credit this discovery to the Pythagoreans; the dramatic stories told about it are legends. The discovery is one reason Euclid treats lengths as magnitudes rather than as numbers, and Book X of the Elements is devoted to such incommensurable lengths.

Euclid connection. Both special triangles come from Euclid's first constructions: the equilateral triangle of I.1 split by the bisector of I.10 gives the 30°-60°-90° triangle, and the square of I.46 split by its diagonal gives the 45°-45°-90° triangle. Euclid would not write $x\sqrt{2}$; he would say that the square on the diagonal is double the square on the side, which is I.47 applied to a square.

Practice habit. Sketch the two ratio triangles (x, x, $x\sqrt{2}$ and x, $x\sqrt{3}$, $2x$) at the top of your paper before you begin, and recite: “short leg opposite 30°, hypotenuse twice the short leg.” Always find the short leg (or a leg) first, and check that your long leg lies between the short leg and the hypotenuse.

Example 1. Ruth's square herb garden has a straight diagonal path 30 feet long from one corner to the opposite corner. How long is each side of the garden? Give the exact answer in simplest radical form and the answer to the nearest tenth of a foot.

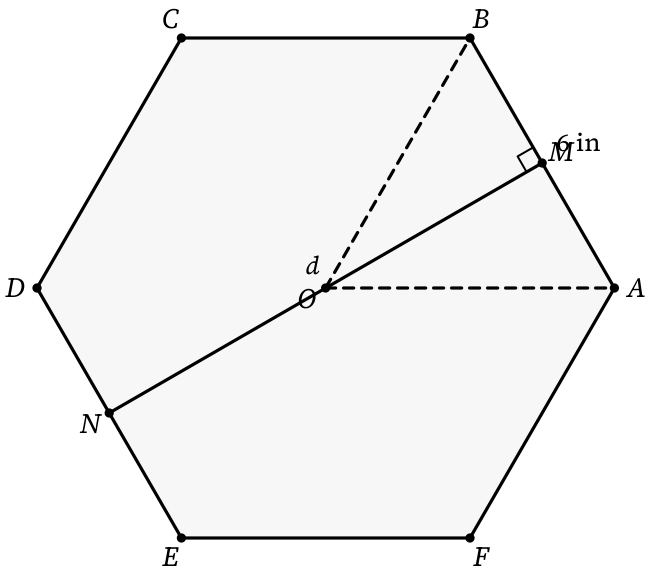


Show solution

1. The diagonal of a square cuts it into two $45°$-$45°$-$90°$ triangles; the path is the hypotenuse and the sides are the legs.
2. Hypotenuse $=$ leg $\cdot \sqrt{2}$, so $s\sqrt{2} = 30$ and $s = \dfrac{30}{\sqrt{2}}$.
3. Rationalize the denominator: $\dfrac{30}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{30\sqrt{2}}{2} = 15\sqrt{2}$.
4. With a calculator, $15\sqrt{2} \approx 21.2$ ft. Check: $21.2 < 30$, and $21.2 \times 1.414 \approx 30$.

Answer: $15\sqrt{2}$ ft ≈ 21.2 ft

Example 2. A floor tile is a regular hexagon with sides 6 inches long. Find the distance d across the tile between two opposite (parallel) sides. Give the exact answer and the answer to the nearest tenth of an inch.



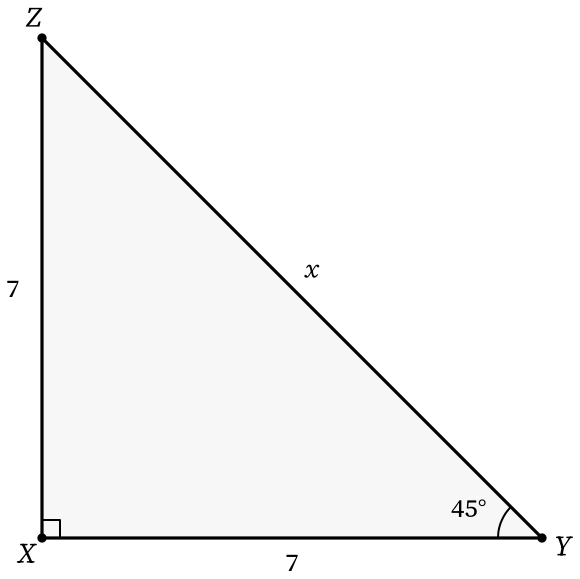
Show solution

1. Segments from the center O to the six vertices cut the regular hexagon into six congruent isosceles triangles, each with vertex angle $\dfrac{360^\circ}{6} = 60^\circ$. The base angles are then 60° too, so each triangle is equilateral with side 6 in.
2. The segment $\overline{OM}$ from the center perpendicular to side $\overline{AB}$ is the altitude of equilateral $\triangle OAB$. It cuts $\triangle OAB$ into two 30°-60°-90° triangles, such as $\triangle OMB$, with hypotenuse $OB = 6$ and short leg $MB = 3$.
3. The long leg is $OM = 3\sqrt{3}$.
4. The distance across the flats is twice this: $d = 2 \cdot 3\sqrt{3} = 6\sqrt{3} \approx 10.4$ in.

Answer: $d = 6\sqrt{3}$ in ≈ 10.4 in

PRACTICE

1. **Easy** In the figure, $\triangle XYZ$ is a $45°$-$45°$-$90°$ triangle with legs of length 7. Find the hypotenuse x.



Show answer

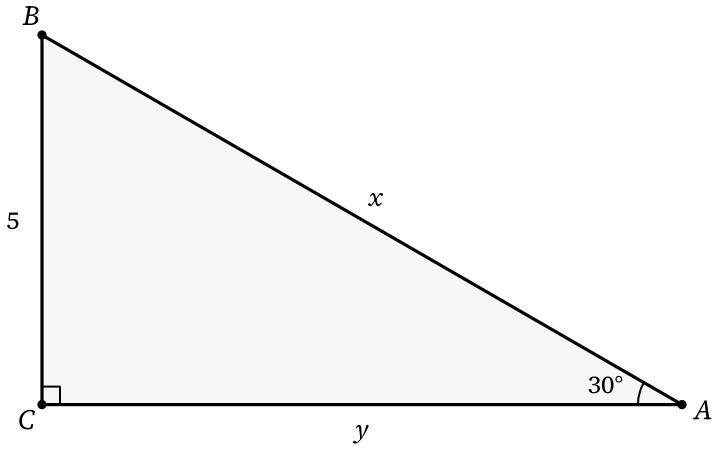
$x = 7\sqrt{2}$

2. **Easy** The hypotenuse of a $45°$-$45°$-$90°$ triangle is 10. Find the length of each leg in simplest radical form.

Show answer

$\frac{10}{\sqrt{2}} = 5\sqrt{2}$

3. **Easy** In the figure, $\triangle ABC$ has a right angle at C, $m\angle A = 30°$, and short leg $BC = 5$. Find the hypotenuse x and the long leg y.



Show answer

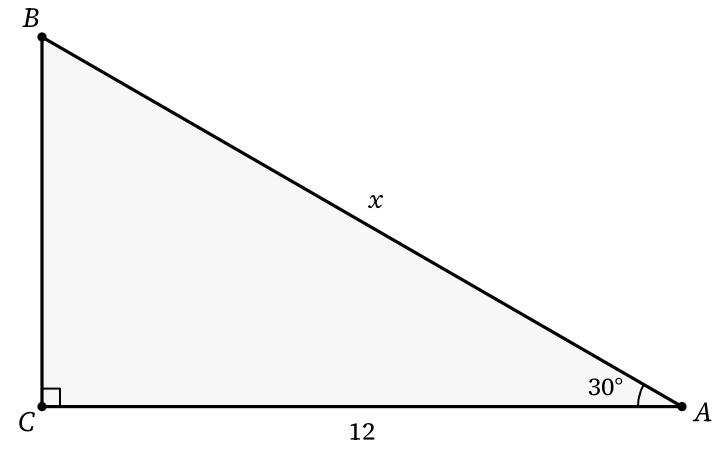
$x = 10,\ y = 5\sqrt{3}$

4. **Easy** The hypotenuse of a $30°$-$60°$-$90°$ triangle is 18. Find the long leg in simplest radical form.

Show answer

Short leg $= 9$, so long leg $= 9\sqrt{3}$.

5. **Medium** In the figure, $\triangle ABC$ has a right angle at C, $m\angle A = 30°$, and long leg $AC = 12$. Find the hypotenuse x in simplest radical form.



Show answer

Short leg $= \dfrac{12}{\sqrt{3}} = 4\sqrt{3}$, so $x = 2 \cdot 4\sqrt{3} = 8\sqrt{3}$.

6. **Medium** Samuel is cutting square quilt blocks 12 inches on a side into two triangles along a diagonal. How long is each diagonal cut? Give the exact answer.

Show answer

$12\sqrt{2}$ in (≈ 17.0 in)

7. **Medium** The front of an A-frame cabin is an equilateral triangle 14 feet on each side. How tall is the cabin at its peak? Round to the nearest tenth of a foot.

Show answer

The altitude splits the front into two $30°$-$60°$-$90°$ triangles with short leg 7 ft, so the height is $7\sqrt{3} \approx 12.1$ ft.

8. **Medium** A porch roof has a 12-in-12 pitch, so it makes a $45°$ angle with the horizontal. The horizontal run from the wall to the edge of the roof is 15 feet. How long is a rafter from the wall to the edge (no overhang)? Round to the nearest tenth of a foot.

Show answer

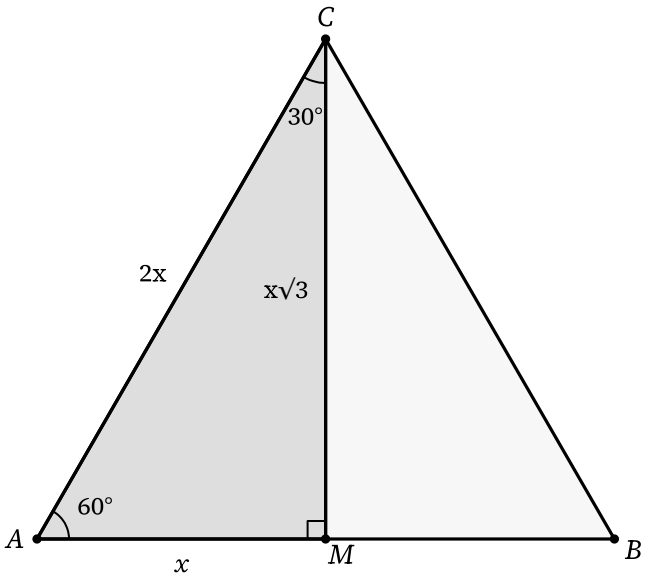
$15\sqrt{2} \approx 21.2$ ft

9. **Medium** Find the area of an equilateral triangle with sides 10 cm. Give the exact answer.

Show answer

The altitude is $5\sqrt{3}$ cm, so the area is $\frac{1}{2} \cdot 10 \cdot 5\sqrt{3} = 25\sqrt{3}$ square cm (≈ 43.3 square cm).

10. **Medium** Explain, using an equilateral triangle, why the hypotenuse of every $30°$-$60°$-$90°$ triangle is exactly twice its short leg.

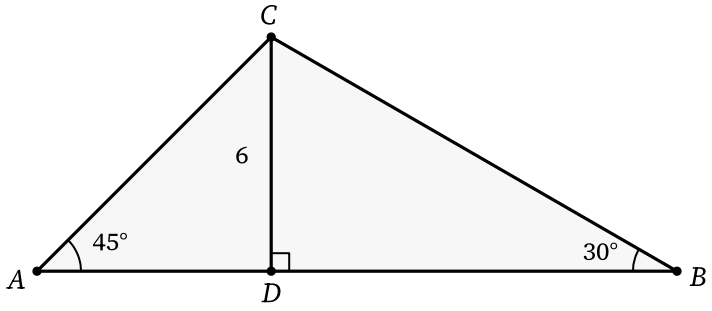


Half an equilateral triangle: short leg x, long leg x$\sqrt{3}$, hypotenuse 2x.

Show answer

Two copies of a $30°$-$60°$-$90°$ triangle placed together along the long leg form a triangle whose angles are all $60°$ (the two $30°$ angles join to make $60°$). An equiangular triangle is equilateral, so its sides are equal. One side is the hypotenuse; the opposite side is made of the two short legs. So the hypotenuse equals two short legs.

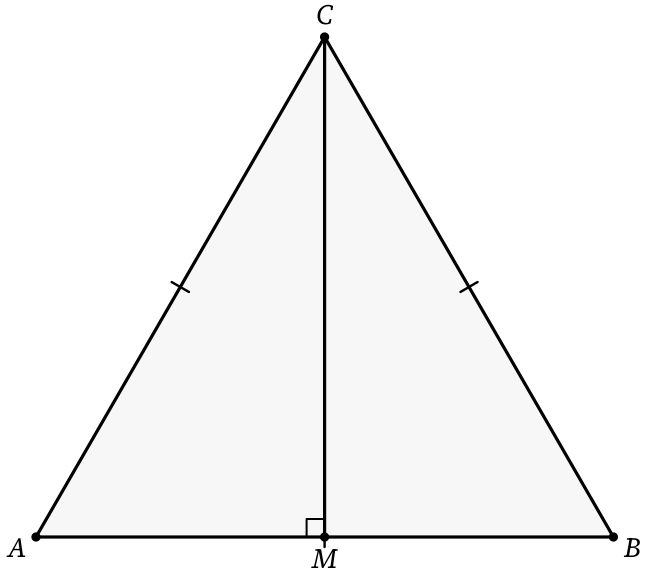
11. **Hard** In the figure, $\triangle ABC$ has $m\angle A = 45°$, $m\angle B = 30°$, and the altitude $\overline{CD}$ to $\overline{AB}$ has length 6. Find AB in simplest radical form.



Show answer

$\triangle ADC$ is $45°$-$45°$-$90°$, so $AD = 6$. $\triangle BDC$ is $30°$-$60°$-$90°$ with short leg $CD = 6$, so $DB = 6\sqrt{3}$. $AB = 6 + 6\sqrt{3}$ (≈ 16.4).

12. Hard Write a flow proof that the altitude of an equilateral triangle with side s has length $\frac{s\sqrt{3}}{2}$. Given: equilateral $\triangle ABC$ with side s and altitude $\overline{CM}$ to $\overline{AB}$. Prove: $CM = \frac{s\sqrt{3}}{2}$.



Show answer

See the model proof.

FLOW PROOF

Given: $\triangle ABC$ is equilateral with side s; $\overline{CM} \perp \overline{AB}$ with M on $\overline{AB}$

Prove: $CM = \frac{s\sqrt{3}}{2}$

(1) $\triangle ABC$ is equilateral with side s; $\overline{CM} \perp \overline{AB}$
Given

(2) $m\angle A = 60°$
An equilateral triangle is equiangular; Triangle Angle Sum Theorem
from (1)

(3) $\angle AMC$ is a right angle
Definition of perpendicular lines
from (1)

(4) $m\angle ACM = 30°$, so $\triangle AMC$ is a $30°$-$60°$-$90°$ triangle
Triangle Angle Sum Theorem
from (2), (3)

(5) $AM = \frac{1}{2}AC = \frac{s}{2}$

$30°$-$60°$-$90°$ Triangle Theorem ($\overline{AM}$ is the short leg, $\overline{AC}$ the hypotenuse)

from (1), (4)

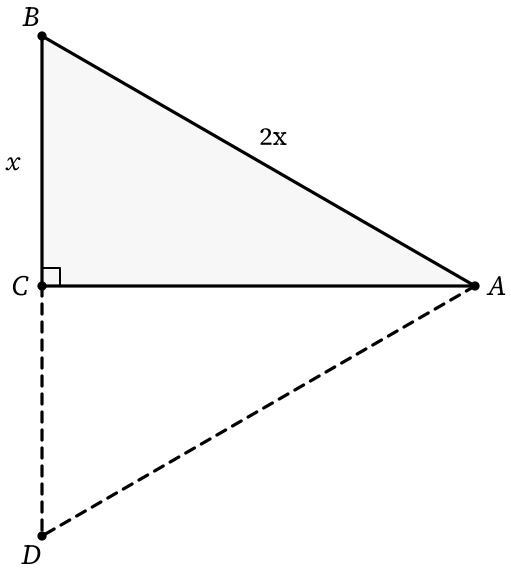
(6) $CM = \frac{s\sqrt{3}}{2}$

$30°$-$60°$-$90°$ Triangle Theorem (long leg $= \sqrt{3}\cdot$ short leg)

from (4), (5)

∎

13. **Hard** Prove the converse of part of the 30°-60°-90° Triangle Theorem. Given: $\triangle ABC$ with right angle $\angle ACB$ and $AB = 2 \cdot BC$. Prove: $m\angle BAC = 30°$. (Hint: extend $\overline{BC}$ to D with $CD = BC$, as in the figure.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\angle ACB$ is a right angle; $AB = 2 \cdot BC$

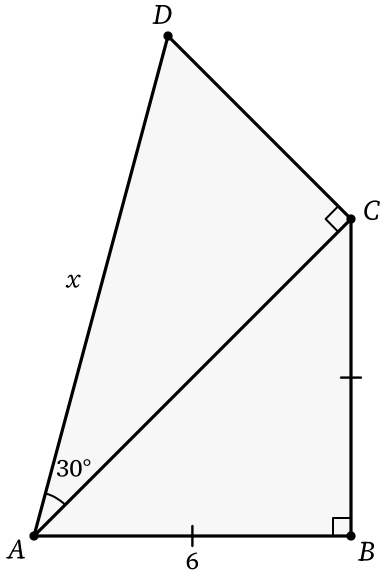
Prove: $m\angle BAC = 30°$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; $AB = 2 \cdot BC$	Given
2	Extend $\overline{BC}$ beyond C to D with $CD = BC$; draw $\overline{AD}$	Ruler Postulate (Euclid Post. 2 and I.3)
3	$\angle ACD$ is a right angle	Linear Pair Theorem (the supplement of a right angle is a right angle)
4	$\angle ACB \cong \angle ACD$	All right angles are congruent
5	$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
6	$\triangle ACB \cong \triangle ACD$	SAS (steps 2, 4, 5)
7	$AD = AB$ and $\angle BAC \cong \angle DAC$	CPCTC

	STATEMENTS	REASONS
8	$BD = BC + CD = 2 \cdot BC$	Segment Addition Postulate; Substitution
9	$BD = AB$	Substitution (steps 1 and 8)
10	$\triangle ABD$ is equilateral	Definition of equilateral triangle ($AB = AD = BD$)
11	$m\angle BAD = 60°$	An equilateral triangle is equiangular, so each angle is $60°$
12	$m\angle BAC + m\angle DAC = 60°$	Angle Addition Postulate
13	$m\angle BAC = 30°$	Substitution (step 7) and Division Property of Equality

∎

14. **Challenge** In the figure, $\triangle ABC$ is an isosceles right triangle with the right angle at B and $AB = 6$. Triangle ACD has a right angle at C and $m\angle CAD = 30°$. Find $x = AD$ in simplest radical form.



Show answer

$AC = 6\sqrt{2}$. In $\triangle ACD$, $\overline{AC}$ is the long leg, so $CD = \dfrac{6\sqrt{2}}{\sqrt{3}} = 2\sqrt{6}$ and $x = 2 \cdot 2\sqrt{6} = 4\sqrt{6}$ (≈ 9.80).

COMMON MISTAKES

- Putting $\sqrt{3}$ on the hypotenuse. In a 30°-60°-90° triangle the hypotenuse is $2x$ and the long leg is $x\sqrt{3}$; remember that $2 > \sqrt{3} \approx 1.73$.
- Matching the long leg with the 30° angle. The short leg is opposite the 30° angle; the long leg is opposite the 60° angle.
- Leaving a radical in the denominator. $\dfrac{10}{\sqrt{2}}$ should be written $5\sqrt{2}$, and $\dfrac{12}{\sqrt{3}}$ should be written $4\sqrt{3}$.
- Using the special ratios in a right triangle whose angles are not given as 45° or 30° and 60°. Check the angles first; otherwise use the Pythagorean Theorem or the trigonometry of Lessons 9.5 and 9.6.

> **Make it a game · Radical Duel.** Fold a square and an equilateral triangle to rediscover the ratios 1 : 1 : square root 2 and 1 : square root 3 : 2. Then play: parent calls a triangle and one side, such as 45-45-90 with leg 7, or 30-60-90 with hypotenuse 10. Give the other sides in simplest radical form within 15 seconds. First to five correct wins.

9.5 The Tangent Ratio

OBJECTIVES

- Explain, using AA similarity, why the ratio of the legs of a right triangle depends only on the size of an acute angle.
- Identify the opposite leg, adjacent leg and hypotenuse relative to an acute angle, and define $\tan A$.
- Use the tangent ratio and a calculator in degree mode to find a missing leg of a right triangle.
- Connect the slope of a line to the tangent of the angle it makes with the horizontal.

WATCH

Trigonometry (1 of 2: Exploring the connection between ratio of sides on different triangles)

Eddie Woo · 7:53

https://www.youtube.com/watch?v=0AGBkR7q53k

> **WHEN TO WATCH**
> Before the Core ideas: why the ratio of sides depends only on the angle.

Note: He introduces all three ratios together; this lesson uses only the tangent.

BACKUP VIDEO

Identifying Sides in a Right-Angled Triangle

Eddie Woo · 4:59

https://www.youtube.com/watch?v=8DqNp_p-nJo

Before Example 1: naming the opposite side, the adjacent side and the hypotenuse.

KEY TERMS

Opposite leg

For an acute angle of a right triangle, the leg that does not touch the angle.

Adjacent leg

For an acute angle of a right triangle, the leg that forms one side of the angle (the side of the angle that is not the hypotenuse).

Tangent

For an acute angle A of a right triangle, $\tan A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$.

Trigonometric ratio

A ratio of two sides of a right triangle that depends only on the measure of an acute angle. The word trigonometry means "triangle measurement."

Degree mode

The calculator setting in which angles are measured in degrees. Check it by computing $\tan 45^\circ$, which must equal 1.

CORE IDEAS

Draw two right triangles of different sizes, each with a 35° angle. They have a right angle and a 35° angle in common, so they are similar by AA (Euclid VI.4). Corresponding sides of similar triangles are proportional, so the ratio of the leg across from the 35° angle to the leg beside it is the same in both triangles, about 0.700, no matter how large the triangle is. That ratio belongs to the angle itself. This is the single idea behind all of trigonometry.

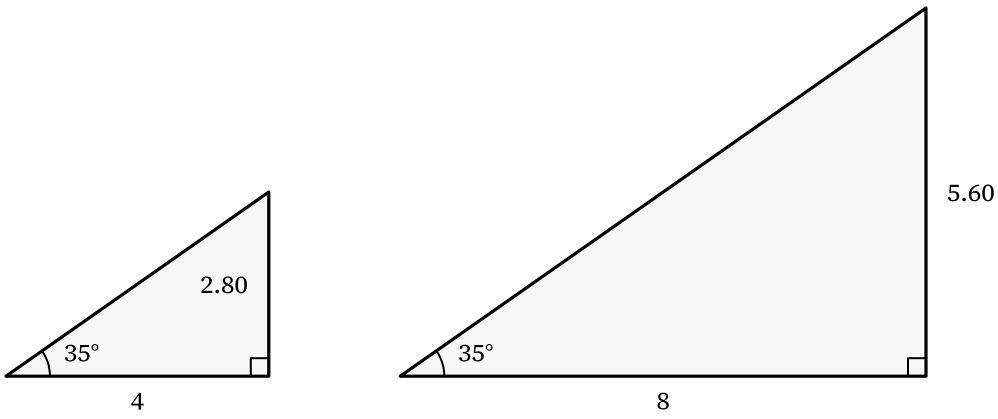
Naming the sides. In right $\triangle ABC$ with the right angle at C, the **hypotenuse** $\overline{AB}$ is opposite the right angle. For the acute angle A, the **opposite leg** is $\overline{BC}$ (it does not touch A) and the **adjacent leg** is $\overline{AC}$ (it is a side of $\angle A$). If you change to angle B, the two legs trade names: $\overline{AC}$ is opposite B and $\overline{BC}$ is adjacent to B. Always name the sides from the point of view of the angle you are using.

The tangent ratio. For an acute angle A of a right triangle, $\tan A = \dfrac{\text{opposite}}{\text{adjacent}}$. You already know three exact values from Lesson 9.4: $\tan 45^\circ = \dfrac{x}{x} = 1$, $\tan 30^\circ = \dfrac{x}{x\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ and $\tan 60^\circ = \dfrac{x\sqrt{3}}{x} = \sqrt{3}$. As the angle grows from near 0° toward 90°, the opposite leg grows compared with the adjacent leg, so $\tan A$ grows from near 0, passes 1 at 45°, and becomes very large near 90°.

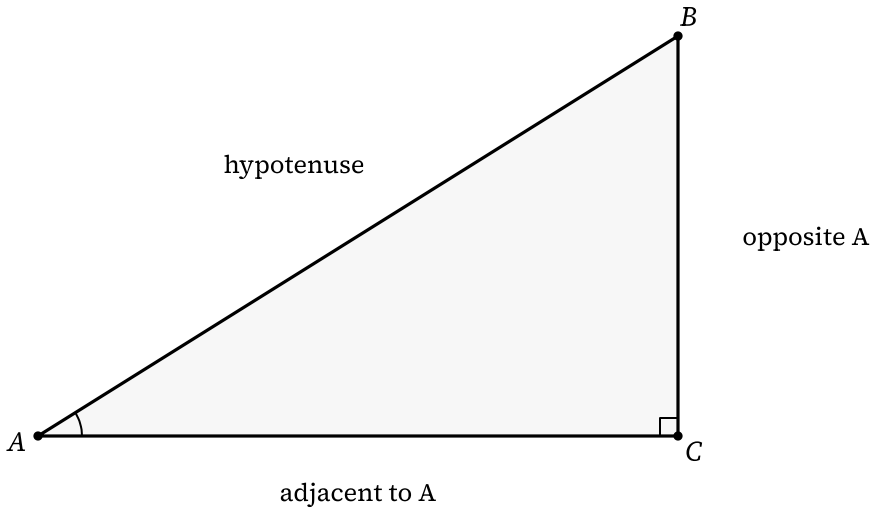
Using a calculator. For other angles, a scientific calculator gives the tangent. First make sure it is in **degree mode**: $\tan 45^\circ$ must show 1. (In radian mode, $\tan 40$ gives about -1.12, a sure sign of the wrong mode.) Keep all the digits until the final answer, and round only at the end.

Finding a side. Write the ratio as an equation and solve it. If the unknown is the opposite leg: $\tan 35^\circ = \dfrac{x}{12}$, so $x = 12 \tan 35^\circ \approx 8.4$. If the unknown is the adjacent leg: $\tan 28^\circ = \dfrac{9}{x}$, so $x \tan 28^\circ = 9$ and $x = \dfrac{9}{\tan 28^\circ} \approx 16.9$. A quick check: when the angle is less than 45°, the opposite leg must be shorter than the adjacent leg.

Slope and tangent. A line rising at an angle θ above the horizontal goes up $\tan\theta$ units for every 1 unit it runs, so its slope is $\dfrac{\text{rise}}{\text{run}} = \tan\theta$. A roof rising 6 inches in 12 inches of run, a ramp, and a road climbing a hill are all described by a slope and therefore by a tangent. In a later Trigonometry course you will study $\tan\theta$ as a function of the angle and extend it beyond 90°.

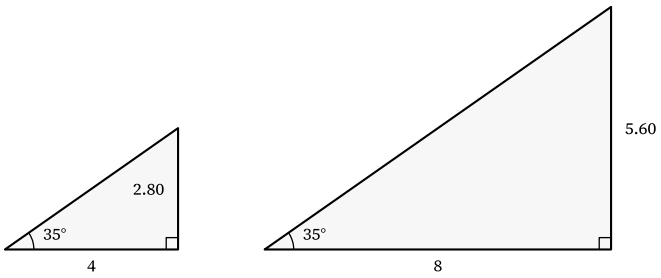


2.80/4 = 5.60/8 ≈ 0.700 = tan 35°



THEOREM **The tangent depends only on the angle** Euclid VI.4

If acute angles A and D of right triangles ABC and DEF (right angles at C and F) are congruent, then $\dfrac{BC}{AC} = \dfrac{EF}{DF}$; that is, $\tan A = \tan D$.



2.80/4 = 5.60/8 ≈ 0.700 = tan 35°

TWO-COLUMN PROOF

Given: Right $\triangle ABC$ and right $\triangle DEF$ with right angles $\angle C$ and $\angle F$; $\angle A \cong \angle D$

Prove: $\dfrac{BC}{AC} = \dfrac{EF}{DF}$

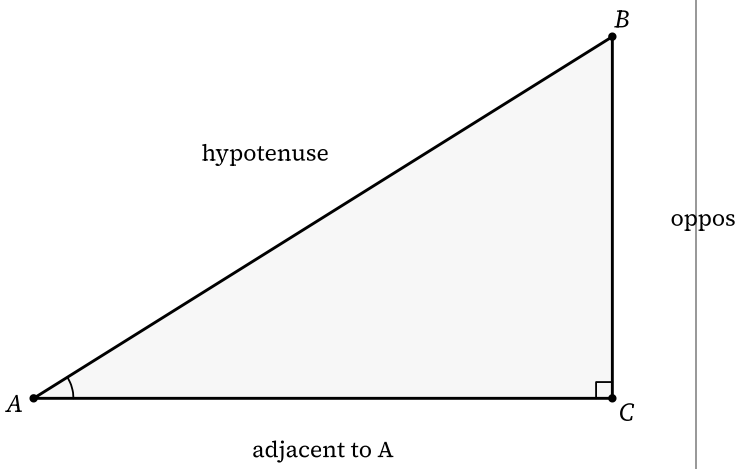
	STATEMENTS	REASONS
1	$\angle C$ and $\angle F$ are right angles; $\angle A \cong \angle D$	Given
2	$\angle C \cong \angle F$	All right angles are congruent
3	$\triangle ABC \sim \triangle DEF$	AA Similarity
4	$\dfrac{BC}{EF} = \dfrac{AC}{DF}$	Corresponding sides of similar triangles are proportional
5	$BC \cdot DF = EF \cdot AC$	Cross Products Property
6	$\dfrac{BC}{AC} = \dfrac{EF}{DF}$	Division Property of Equality (divide both sides by $AC \cdot DF$)

∎

This theorem is what makes $\tan A$ *well defined: its value does not depend on which right triangle you draw.*

COROLLARY **Tangents of complementary angles**

If A and B are the acute angles of a right triangle, then $\tan A \cdot \tan B = 1$*; equivalently,* $\tan(90^\circ - A) = \dfrac{1}{\tan A}$.



PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$

Prove: $\tan A \cdot \tan B = 1$

For $\angle A$ the opposite leg is $\overline{BC}$ and the adjacent leg is $\overline{AC}$, so $\tan A = \dfrac{BC}{AC}$. For $\angle B$ the legs trade roles, so $\tan B = \dfrac{AC}{BC}$. Multiplying, $\tan A \cdot \tan B = \dfrac{BC}{AC} \cdot \dfrac{AC}{BC} = 1$. Since the acute angles of a right triangle are complementary, $m\angle B = 90^\circ - m\angle A$, so $\tan(90^\circ - A) = \dfrac{1}{\tan A}$.

∎

KEY FORMULAS

$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 30^\circ = \frac{\sqrt{3}}{3}, \qquad \tan 45^\circ = 1, \qquad \tan 60^\circ = \sqrt{3}$$

$$\text{slope} = \tan\theta$$

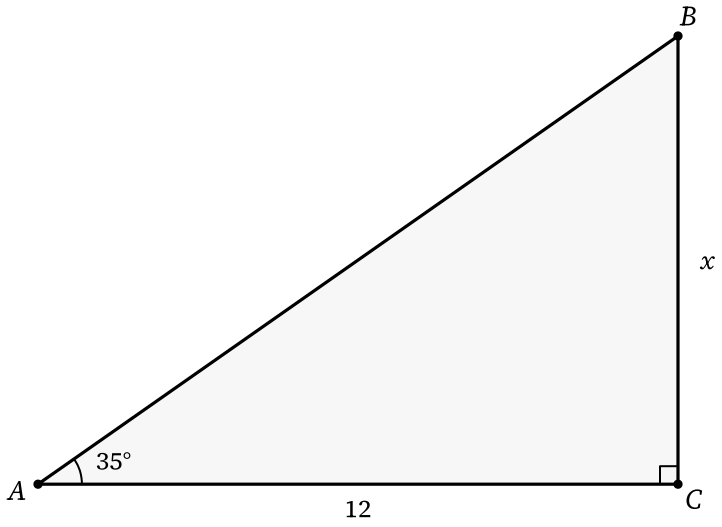
From history. Greek astronomers needed to compute lengths from angles long before the word “tangent” existed. Later writers credit Hipparchus of Nicaea (second century BC) with the first table of chords in a circle, and Ptolemy's Almagest (second century AD) contains a full chord table that astronomers used for centuries. Ratios like our tangent were later tabulated by astronomers who measured the shadow of a vertical stick, called a gnomon, at different heights of the sun: the length of the shadow compared with the stick depends only on the sun's angle.

Euclid connection. Euclid has no trigonometry, but he proves the fact that makes it possible. VI.4 says that in equiangular triangles the sides about the equal angles are proportional; so all right triangles with the same acute angle have the same ratio of legs, and that ratio can be put in a table once and for all.

Practice habit. Before writing any ratio, mark the angle you are using and label the three sides "opp," "adj" and "hyp" from that angle's point of view. Check degree mode with $\tan 45^\circ = 1$, keep full calculator digits, and round only the final answer.

WORKED EXAMPLES

Example 1. In the figure, right $\triangle ABC$ has its right angle at C, $m\angle A = 35^\circ$ and $AC = 12$. Find $x = BC$ to the nearest tenth.

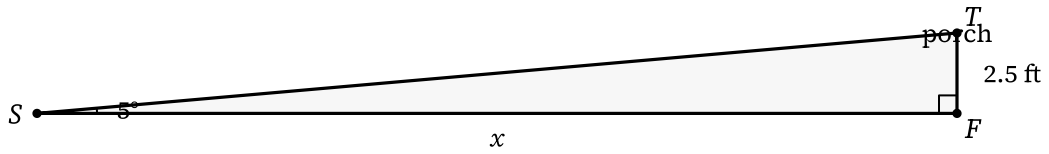


Show solution

1. From $\angle A$: $\overline{BC}$ is the opposite leg (unknown) and $\overline{AC}$ is the adjacent leg (12). The ratio that uses opposite and adjacent is the tangent.
2. $\tan 35^\circ = \dfrac{x}{12}$.
3. Multiply both sides by 12: $x = 12\tan 35^\circ \approx 12(0.700208) \approx 8.4025$.
4. Rounded, $x \approx 8.4$. Check: $35^\circ < 45^\circ$, so the opposite leg should be shorter than the adjacent leg, and $8.4 < 12$.

Answer: $x \approx 8.4$

Example 2. The Miller family is building a wheelchair ramp to a porch that is 2.5 feet above the ground. For safety the ramp should make an angle of no more than 5° with the level ground. At exactly 5°, how far from the porch (measured horizontally) must the ramp begin? Round to the nearest tenth of a foot.



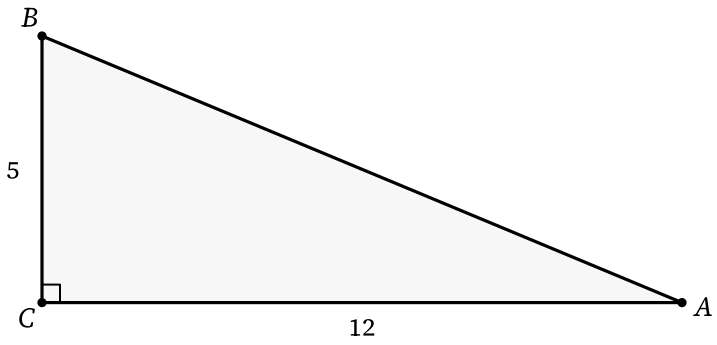
Show solution

1. The ramp, the ground and the porch's height form a right triangle. From the 5° angle at the start S of the ramp, the rise $FT = 2.5$ ft is the opposite leg and the horizontal distance $SF = x$ is the adjacent leg.
2. $\tan 5^\circ = \dfrac{2.5}{x}$.
3. Multiply both sides by x and divide by $\tan 5^\circ$: $x = \dfrac{2.5}{\tan 5^\circ} \approx \dfrac{2.5}{0.0874887} \approx 28.575$.
4. So the ramp must begin about 28.6 ft from the porch. Check: a 5° slope is very gentle (slope about 0.087), so the run should be more than 10 times the rise, and it is.

Answer: about 28.6 ft

PRACTICE

1. **Easy** In the figure, right $\triangle ABC$ has its right angle at C, $BC = 5$ and $AC = 12$. Find $\tan A$ and $\tan B$ as fractions.



Show answer

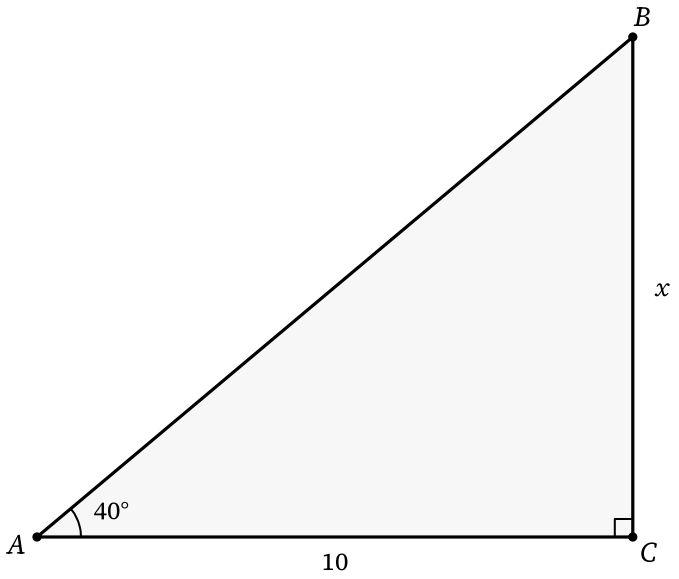
$$\tan A = \frac{5}{12}, \tan B = \frac{12}{5}$$

2. **Easy** Use a calculator in degree mode to find $\tan 40^\circ$ to four decimal places.

Show answer

$$\tan 40^\circ \approx 0.8391$$

3. **Easy** In the figure, right $\triangle ABC$ has its right angle at C, $m\angle A = 40^\circ$ and $AC = 10$. Find $x = BC$ to the nearest tenth.



Show answer

$$x = 10 \tan 40^\circ \approx 8.4$$

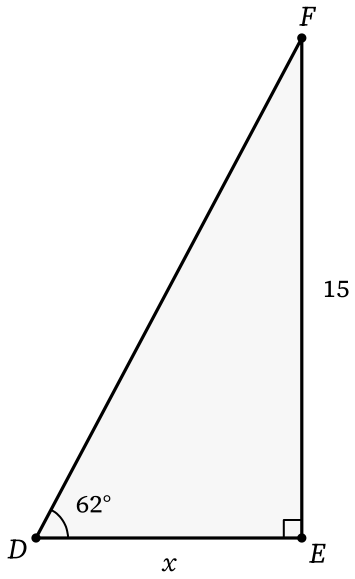
4. **Easy** In the figure, right $\triangle PQR$ has its right angle at Q, $m\angle P = 25^\circ$ and $PQ = 20$. Find $x = QR$ to the nearest tenth.



Show answer

$x = 20 \tan 25^\circ \approx 9.3$

5. **Medium** In the figure, right $\triangle DEF$ has its right angle at E, $m\angle D = 62^\circ$ and $EF = 15$. Find $x = DE$ to the nearest tenth.



Show answer

$\tan 62^\circ = \dfrac{15}{x}$, so $x = \dfrac{15}{\tan 62^\circ} \approx 8.0$

6. **Medium** Use a $30°$-$60°$-$90°$ triangle with short leg x to find the exact value of $\tan 30°$ in simplest radical form.

Show answer

The leg opposite $30°$ is x and the adjacent leg is $x\sqrt{3}$, so $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.

7. **Medium** Henry's shed roof makes an $18°$ angle with the horizontal. The roof runs 10 feet horizontally from the low wall to the high wall. How much higher is the high wall? Round to the nearest tenth of a foot.

Show answer

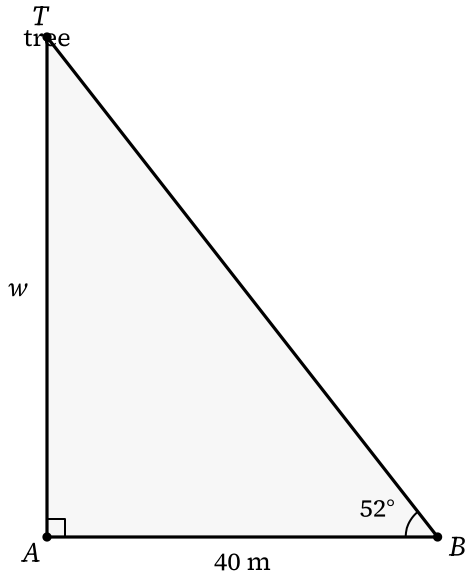
$10 \tan 18° \approx 3.2$ ft

8. **Medium** A line rises from left to right and makes a $35°$ angle with the positive x-axis. What is its slope, to the nearest hundredth?

Show answer

$\text{slope} = \tan 35° \approx 0.70$

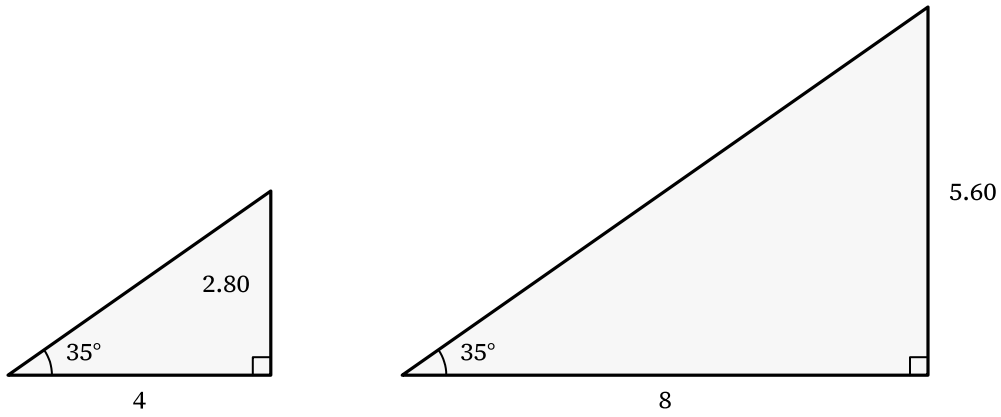
9. **Medium** Samuel stands at point A on the bank of a river, directly across from a tree T on the far bank. He walks 40 m along the straight bank to point B, so that $\angle TAB$ is a right angle, and measures $m\angle ABT = 52°$. How wide is the river from A to T? Round to the nearest tenth of a meter.



Show answer

$w = 40 \tan 52° \approx 51.2$ m

10. **Medium** Grace measured a small right triangle with a 35° angle and found the ratio of opposite leg to adjacent leg to be about 0.70. Ruth drew a much larger right triangle with a 35° angle. Explain why Ruth's ratio must also be about 0.70.

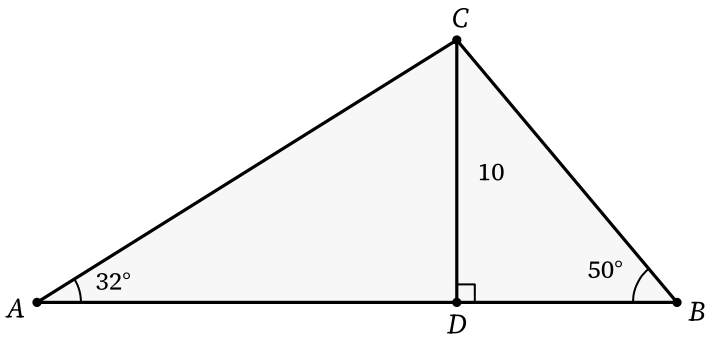


2.80/4 = 5.60/8 ≈ 0.700 = tan 35°

Show answer

Both triangles have a right angle and a 35° angle, so they are similar by AA. In similar triangles corresponding sides are proportional, so the ratio of the two legs is the same in both triangles. The size of the triangle does not matter; only the angle does.

11. **Hard** In the figure, $\overline{CD}$ is the altitude from C to $\overline{AB}$ in $\triangle ABC$, with $CD = 10$, $m\angle A = 32^\circ$ and $m\angle B = 50^\circ$. Find AB to the nearest tenth.



Show answer

$AD = \dfrac{10}{\tan 32^\circ} \approx 16.003$ and $DB = \dfrac{10}{\tan 50^\circ} \approx 8.391$, so $AB \approx 24.4$.

12. Hard Prove that if $\angle A$ is an acute angle of right $\triangle ABC$ (right angle at C) and $\tan A = 1$, then $m\angle A = 45°$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$; $\tan A = 1$

Prove: $m\angle A = 45°$

By the definition of tangent, $\tan A = \dfrac{BC}{AC}$, so $\dfrac{BC}{AC} = 1$ and $BC = AC$ (Multiplication Property of Equality). Then $\triangle ABC$ is isosceles with $\overline{BC} \cong \overline{AC}$, so the angles opposite these sides are congruent by the Isosceles Triangle Theorem: $\angle A \cong \angle B$.

The acute angles of a right triangle are complementary, so $m\angle A + m\angle B = 90°$. Substituting $m\angle B = m\angle A$ gives $2\,m\angle A = 90°$, so $m\angle A = 45°$.

∎

13. **Hard** Prove the converse of this lesson's theorem: if right triangles ABC and DEF have right angles at C and F and $\tan A = \tan D$, then $\angle A \cong \angle D$.

Show answer

See the model proof.

TWO-COLUMN PROOF

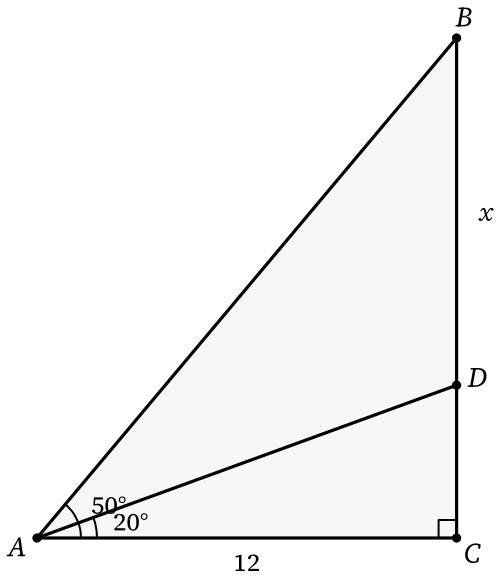
Given: Right $\triangle ABC$ and right $\triangle DEF$ with right angles $\angle C$ and $\angle F$; $\tan A = \tan D$

Prove: $\angle A \cong \angle D$

	STATEMENTS	REASONS
1	$\angle C$ and $\angle F$ are right angles; $\tan A = \tan D$	Given
2	$\dfrac{BC}{AC} = \dfrac{EF}{DF}$	Definition of tangent; Substitution
3	$BC \cdot DF = EF \cdot AC$	Cross Products Property
4	$\dfrac{CB}{FE} = \dfrac{CA}{FD}$	Division Property of Equality (divide both sides by $EF \cdot DF$)
5	$\angle C \cong \angle F$	All right angles are congruent
6	$\triangle ABC \sim \triangle DEF$	SAS Similarity (steps 4 and 5)
7	$\angle A \cong \angle D$	Corresponding angles of similar triangles are congruent

∎

14. **Challenge** In the figure, right $\triangle ABC$ has its right angle at C and $AC = 12$. Point D is on $\overline{CB}$ with $m\angle CAD = 20°$, and $m\angle CAB = 50°$. Find $x = DB$ to the nearest tenth.



Show answer

$CB = 12\tan 50° \approx 14.301$ and $CD = 12\tan 20° \approx 4.368$, so $x \approx 14.301 - 4.368 \approx 9.9$.

COMMON MISTAKES

- Leaving the calculator in radian mode. Test it: $\tan 45°$ must equal 1. If you get about 1.62, switch to degree mode.
- Swapping the legs. The opposite leg is the one that does not touch the angle. When you move to the other acute angle, the opposite and adjacent legs trade places.
- Multiplying when you should divide. If the unknown is in the denominator, as in $\tan 62° = \dfrac{15}{x}$, then $x = \dfrac{15}{\tan 62°}$, not $15\tan 62°$.
- Rounding $\tan A$ to two places before multiplying. Use the full calculator value and round only the final answer.

> **Make it a game · Same Angle, Same Ratio.** With a protractor, draw a 35-degree angle. Draw three perpendicular lines across one ray at different distances, making three right triangles. Guess the ratio of opposite to adjacent, then measure each triangle. Are they equal? Explain why with AA similarity, and name that ratio the tangent. Parent challenges you to repeat at 60 degrees.

9.6 The Sine and Cosine Ratios

OBJECTIVES

- Define $\sin A$ and $\cos A$ and use SOH-CAH-TOA to choose the right ratio for a problem.
- Prove and use the cofunction relationship $\sin A = \cos(90^\circ - A)$.
- Give exact values of sine, cosine and tangent for 30°, 45° and 60° from the special right triangles.
- Use inverse ratios ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$) to find angles and solve right triangles.

WATCH

Trigonometry (2 of 2: Defining the ratios found as sine, cosine and tangent)

Eddie Woo · 9:15

https://www.youtube.com/watch?v=zpG0tLvhVoQ

> **WHEN TO WATCH**
> Before Example 1: defining sine, cosine and tangent.

BACKUP VIDEO

Trigonometry in Right Triangles (3 of 5: Finding an angle)

Eddie Woo · 8:47

https://www.youtube.com/watch?v=ok2r4Px6Xa0

Before the inverse-ratio examples: finding an angle.

KEY TERMS

Sine

For an acute angle A of a right triangle, $\sin A = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$.

Cosine

For an acute angle A of a right triangle, $\cos A = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$.

SOH-CAH-TOA

A memory aid: Sine = Opposite/Hypotenuse, Cosine = Adjacent/Hypotenuse, Tangent = Opposite/Adjacent.

Cofunctions

Sine and cosine are cofunctions: the sine of an angle equals the cosine of its complement. “Cosine” means “complement's sine.”

Inverse ratio

$\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ give the acute angle whose sine, cosine or tangent is x. For example, $\sin^{-1}(0.5) = 30^\circ$.

Solve a right triangle

Find all three side lengths and all three angle measures from the given information.

CORE IDEAS

The tangent compares the two legs. When a problem involves the hypotenuse, use two more ratios: $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\cos A = \dfrac{\text{adjacent}}{\text{hypotenuse}}$. For the same reason as in Lesson 9.5, these depend only on the angle: all right triangles with the same acute angle are similar by AA, so the ratios of their corresponding sides are equal.

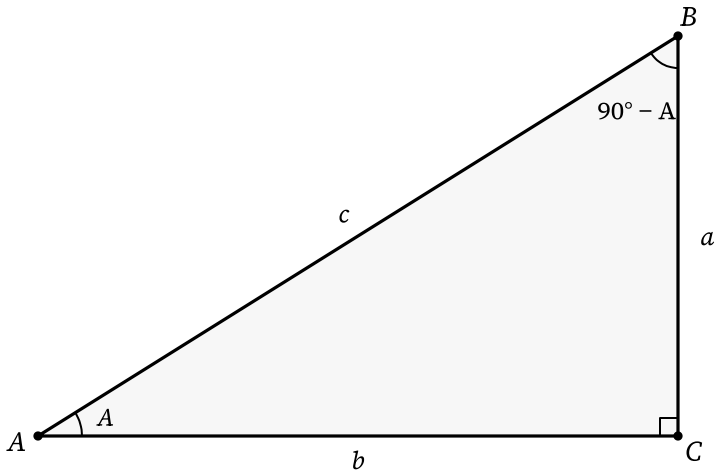
Choosing the ratio. Remember SOH-CAH-TOA. Look at the angle you know (or want), then circle the side you know and the side you want. Opposite and hypotenuse: use sine. Adjacent and hypotenuse: use cosine. Opposite and adjacent: use tangent. Because the hypotenuse is the longest side, the sine and cosine of an acute angle are always between 0 and 1; a calculator answer of 1.3 for a sine means a mistake.

Cofunctions. In right $\triangle ABC$ with the right angle at C, the leg opposite $\angle A$ is the leg adjacent to $\angle B$. So $\sin A = \dfrac{BC}{AB} = \cos B$. The acute angles are complementary, so $B = 90^\circ - A$, and $\sin A = \cos(90^\circ - A)$. This is where the name comes from: the cosine of A is the sine of A's complement. For example, $\sin 20^\circ = \cos 70^\circ \approx 0.342$.

Exact values. The special right triangles of Lesson 9.4 give exact values. From the 45°-45°-90° triangle with sides $1, 1, \sqrt{2}$: $\sin 45^\circ = \cos 45^\circ = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. From the 30°-60°-90° triangle with sides $1, \sqrt{3}, 2$: $\sin 30^\circ = \cos 60^\circ = \dfrac{1}{2}$ and $\sin 60^\circ = \cos 30^\circ = \dfrac{\sqrt{3}}{2}$. Sketch the triangles rather than memorizing a table; then you can always rebuild the values.

Finding an angle. If you know two sides, find their ratio and then use an inverse key. If $\sin A = \dfrac{5}{13}$, then $A = \sin^{-1}\left(\dfrac{5}{13}\right) \approx 22.6^\circ$. Read $\sin^{-1}$ as "the angle whose sine is"; the -1 does NOT mean $\dfrac{1}{\sin}$.

Solving a right triangle. To solve a right triangle, find every missing side and angle. Use the Pythagorean Theorem for a third side, the trigonometric ratios for sides from an angle, the inverse ratios for angles from sides, and the fact that the acute angles add to 90°. Whenever you can, compute from the given numbers rather than from your rounded answers, so that rounding errors do not pile up.

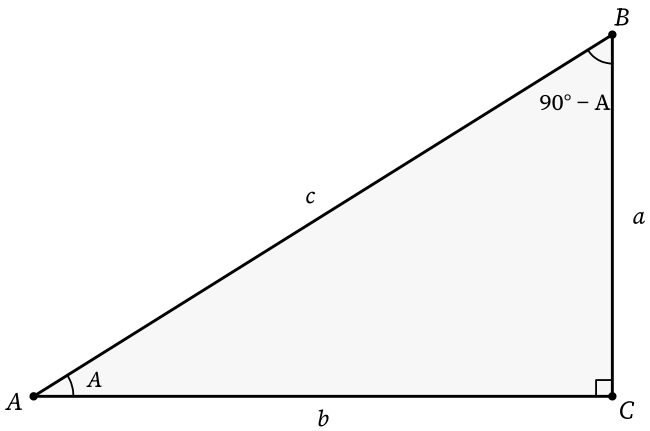


sin A = a/c = cos(90° − A), cos A = b/c = sin(90° − A), tan A = a/b

POSTULATES AND THEOREMS

THEOREM **Cofunction Theorem**

If A is an acute angle, then $\sin A = \cos(90^\circ - A)$ *and* $\cos A = \sin(90^\circ - A)$.



sin A = a/c = cos(90° − A), cos A = b/c = sin(90° − A), tan A = a/b

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$ and acute angle $\angle A$

Prove: $\sin A = \cos(90^\circ - A)$ and $\cos A = \sin(90^\circ - A)$

The acute angles of a right triangle are complementary, so $m\angle B = 90^\circ - m\angle A$.

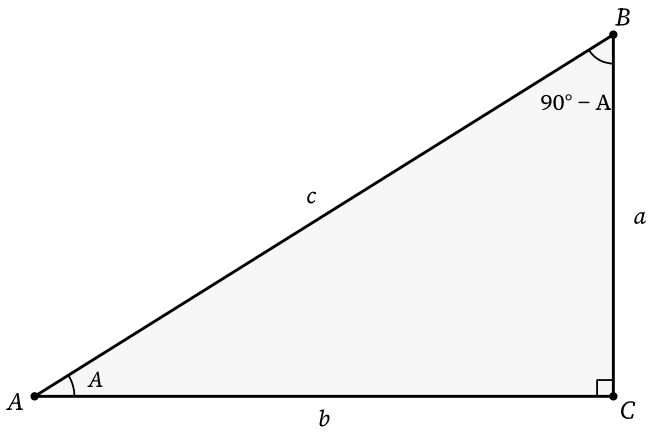
By the definition of sine, $\sin A = \dfrac{BC}{AB}$, since $\overline{BC}$ is opposite $\angle A$. But $\overline{BC}$ is a side of $\angle B$, so it is the leg adjacent to $\angle B$, and by the definition of cosine $\cos B = \dfrac{BC}{AB}$. Therefore $\sin A = \cos B = \cos(90^\circ - A)$ by substitution.

In the same way, $\cos A = \dfrac{AC}{AB}$, and $\overline{AC}$ is opposite $\angle B$, so $\sin B = \dfrac{AC}{AB}$. Therefore $\cos A = \sin B = \sin(90^\circ - A)$.

∎

THEOREM **Pythagorean Identity** Euclid I.47

For every acute angle A, $\sin^2 A + \cos^2 A = 1$. *(Here $\sin^2 A$ means $(\sin A)^2$.)*



sin A = a/c = cos(90° − A), cos A = b/c = sin(90° − A), tan A = a/b

TWO-COLUMN PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$

Prove: $\sin^2 A + \cos^2 A = 1$

	STATEMENTS	REASONS
1	$\angle C$ is a right angle	Given
2	$\sin A = \dfrac{BC}{AB}$ and $\cos A = \dfrac{AC}{AB}$	Definitions of sine and cosine
3	$\sin^2 A + \cos^2 A = \dfrac{BC^2}{AB^2} + \dfrac{AC^2}{AB^2}$	Substitution Property of Equality

	STATEMENTS	REASONS
4	$\sin^2 A + \cos^2 A = \dfrac{BC^2 + AC^2}{AB^2}$	Adding fractions with a common denominator
5	$BC^2 + AC^2 = AB^2$	Pythagorean Theorem
6	$\sin^2 A + \cos^2 A = \dfrac{AB^2}{AB^2}$	Substitution Property of Equality
7	$\sin^2 A + \cos^2 A = 1$	Simplify ($AB \neq 0$)

∎

This identity is the Pythagorean Theorem rewritten for a right triangle whose hypotenuse is 1.

KEY FORMULAS

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin A = \cos(90^\circ - A), \qquad \cos A = \sin(90^\circ - A)$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}, \qquad \sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}, \qquad \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

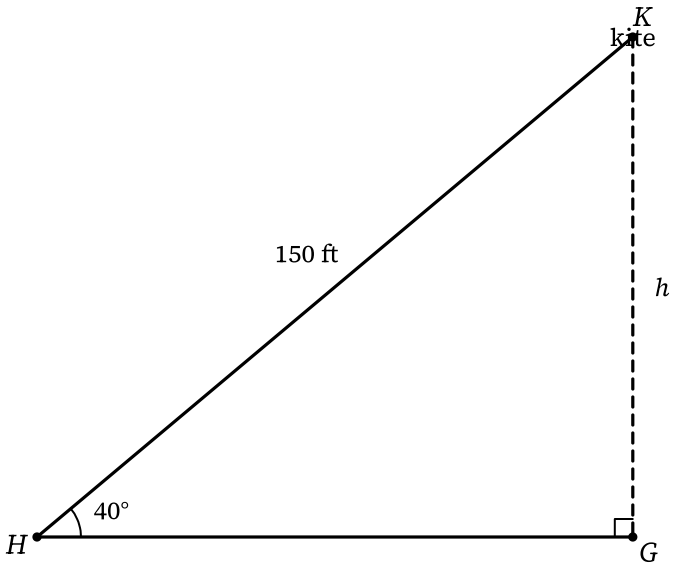
$$\sin^2 A + \cos^2 A = 1$$

From history. Astronomers in India, such as Aryabhata (about AD 500), worked with the half-chord of a circle, which is our sine. Their word for it passed into Arabic and then, through a translation into Latin, became “sinus,” from which we get “sine.” The word “cosine” came later, as a short form of “the sine of the complement.”

Euclid connection. Euclid has no sine or cosine, but in Book II he extends I.47 to triangles that are not right: II.12 (obtuse) and II.13 (acute) say the square on the side opposite the angle exceeds or falls short of the squares on the other two sides by twice a certain rectangle. In the Trigonometry course you will write that rectangle with a cosine, as the Law of Cosines.

Practice habit. Write SOH-CAH-TOA at the top of your paper. For each problem, mark the angle, label the sides “opp,” “adj” and “hyp” from that angle, circle the side you know and the side you want, and choose the one ratio that uses exactly those two. Check that every acute-angle sine and cosine is between 0 and 1 and that the two acute angles add to 90°.

Example 1. Ruth flies a kite on 150 feet of string, which she holds at ground level. The string is straight and makes a 40° angle with the level ground. How high is the kite? Round to the nearest tenth of a foot.

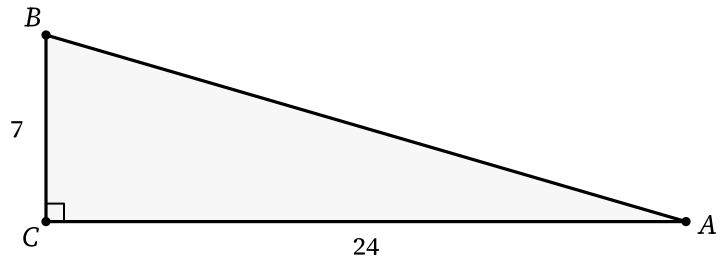


Show solution

1. The string is the hypotenuse (150 ft). The height h of the kite is the leg opposite the 40° angle.
2. Opposite and hypotenuse: use sine. $\sin 40^\circ = \dfrac{h}{150}$.
3. $h = 150 \sin 40^\circ \approx 150(0.642788) \approx 96.42$.
4. The kite is about 96.4 ft high. Check: the height must be less than the 150-ft string, and since $40^\circ < 45^\circ$, less than the horizontal distance $150 \cos 40^\circ \approx 114.9$ ft.

Answer: about 96.4 ft

Example 2. Solve right $\triangle ABC$ with the right angle at C, $BC = 7$ and $AC = 24$: find AB, $m\angle A$ and $m\angle B$. Round angles to the nearest tenth of a degree.



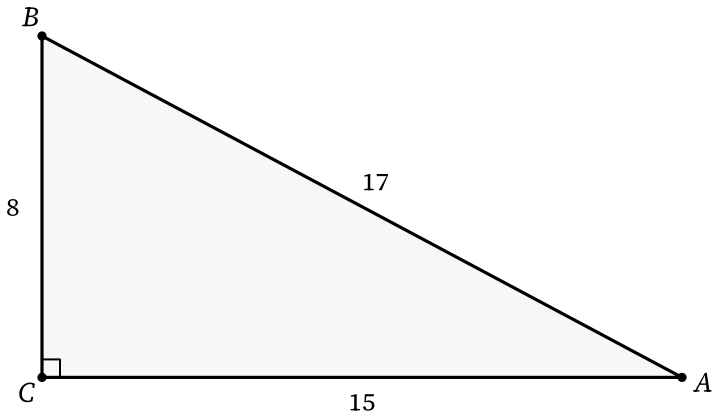
Show solution

1. Third side: $AB = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$.
2. For $\angle A$, the given legs are opposite (7) and adjacent (24), so use tangent: $\tan A = \dfrac{7}{24}$ and $m\angle A = \tan^{-1}\left(\dfrac{7}{24}\right) \approx 16.26^\circ \approx 16.3^\circ$.
3. The acute angles are complementary: $m\angle B = 90^\circ - 16.26^\circ \approx 73.7^\circ$.
4. Check with a different ratio from the given numbers: $\sin B = \dfrac{24}{25} = 0.96$, and $\sin^{-1}(0.96) \approx 73.74^\circ$. It agrees.

Answer: $AB = 25$, $m\angle A \approx 16.3^\circ$, $m\angle B \approx 73.7^\circ$

PRACTICE

1. **Easy** In the figure, right $\triangle ABC$ has its right angle at C, $BC = 8$, $AC = 15$ and $AB = 17$. Find $\sin A$, $\cos A$ and $\tan A$ as fractions.



Show answer

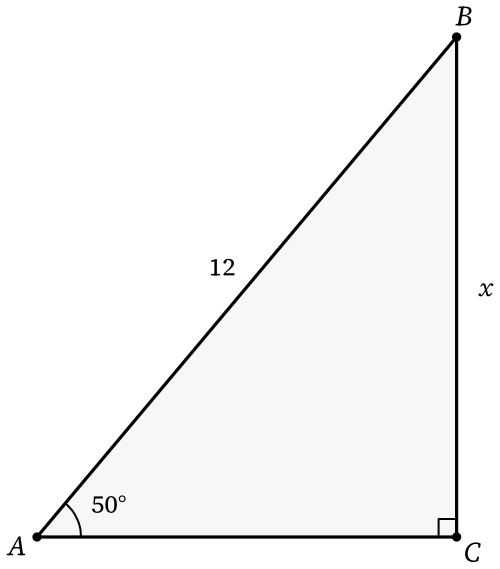
$$\sin A = \frac{8}{17}, \cos A = \frac{15}{17}, \tan A = \frac{8}{15}$$

2. **Easy** Use a $30°$-$60°$-$90°$ triangle to find the exact value of $\cos 30°$.

Show answer

With sides $1, \sqrt{3}, 2$, the leg adjacent to $30°$ is $\sqrt{3}$, so $\cos 30° = \frac{\sqrt{3}}{2}$.

3. **Easy** In the figure, right $\triangle ABC$ has its right angle at C, hypotenuse $AB = 12$ and $m\angle A = 50°$. Find $x = BC$ to the nearest tenth.



Show answer

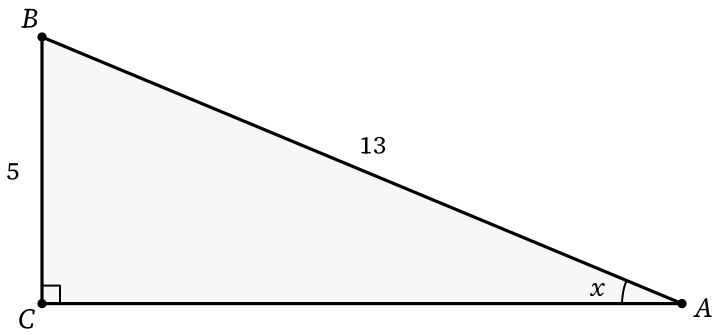
$x = 12 \sin 50° \approx 9.2$

4. **Easy** Find x if $\sin 38° = \cos x°$ and $0 < x < 90$.

Show answer

$x = 90 - 38 = 52$

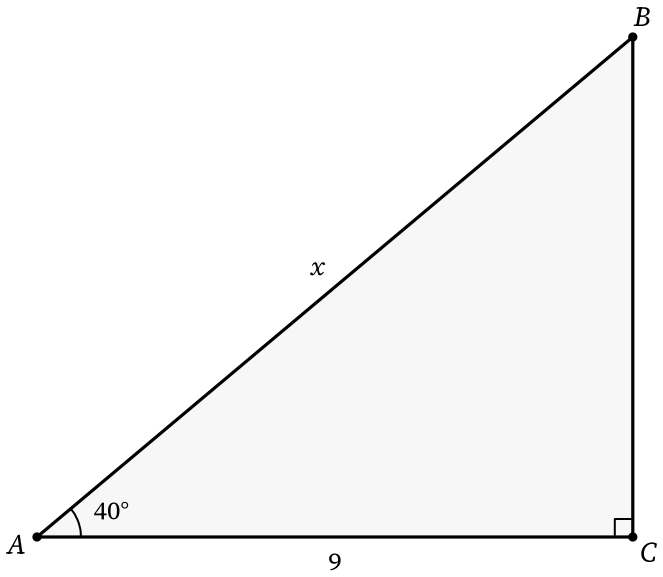
5. **Medium** In the figure, right $\triangle ABC$ has its right angle at C, $BC = 5$ and hypotenuse $AB = 13$. Find $x = m\angle A$ to the nearest tenth of a degree.



Show answer

$\sin A = \dfrac{5}{13}$, so $x = \sin^{-1}\left(\dfrac{5}{13}\right) \approx 22.6°$

6. **Medium** In the figure, right $\triangle ABC$ has its right angle at C, $AC = 9$ and $m\angle A = 40°$. Find the hypotenuse $x = AB$ to the nearest tenth.



Show answer

$\cos 40° = \dfrac{9}{x}$, so $x = \dfrac{9}{\cos 40°} \approx 11.7$

7. **Medium** A straight ramp 20 feet long rises 1.5 feet from the ground to a church doorway. What angle does the ramp make with the level ground? Round to the nearest tenth of a degree.

Show answer

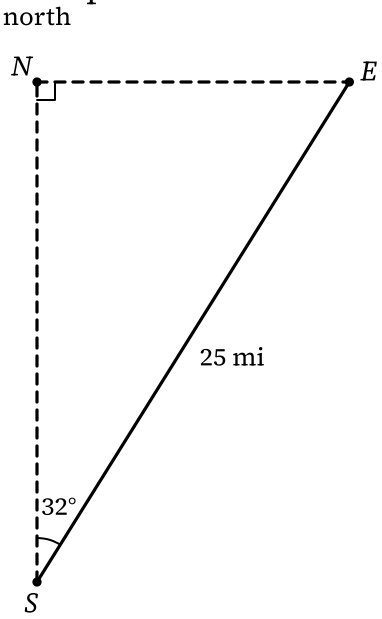
The ramp is the hypotenuse and the rise is opposite the angle: $\sin^{-1}\left(\dfrac{1.5}{20}\right) = \sin^{-1}(0.075) \approx 4.3°$

8. **Medium** A guy wire runs in a straight line from the top of a 40-foot farm windmill tower to a stake in the level ground, making a $58°$ angle with the ground. How long is the wire? Round to the nearest tenth of a foot.

Show answer

$\sin 58° = \dfrac{40}{L}$, so $L = \dfrac{40}{\sin 58°} \approx 47.2$ ft

9. **Medium** A ship leaves harbor S and sails 25 miles in a straight line on a course 32° east of due north, to point E. How far north and how far east of the harbor is the ship? Round each to the nearest tenth of a mile.



Show answer

North: $25\cos 32^\circ \approx 21.2$ mi; east: $25\sin 32^\circ \approx 13.2$ mi

10. **Medium** Henry's calculator work gives $\sin A = 1.25$ for an acute angle A of a right triangle. Without finding his error, explain why this answer must be wrong.

Show answer

$\sin A$ is the opposite leg divided by the hypotenuse. The hypotenuse is the longest side of a right triangle, so the fraction has a smaller numerator than denominator and must be less than 1 (and greater than 0). A sine of 1.25 is impossible; he probably divided the hypotenuse by the leg.

11. **Hard** Solve right $\triangle ABC$ with the right angle at C, hypotenuse $AB = 16$ and $m\angle A = 34^\circ$: find $m\angle B$, $a = BC$ and $b = AC$. Round sides to the nearest tenth.

Show answer

$m\angle B = 56^\circ$; $a = 16\sin 34^\circ \approx 8.9$; $b = 16\cos 34^\circ \approx 13.3$

12. Hard Prove that for an acute angle A of right $\triangle ABC$ (right angle at C), $\tan A = \dfrac{\sin A}{\cos A}$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$

Prove: $\tan A = \dfrac{\sin A}{\cos A}$

	STATEMENTS	REASONS
1	$\angle C$ is a right angle	Given
2	$\sin A = \dfrac{BC}{AB}$ and $\cos A = \dfrac{AC}{AB}$	Definitions of sine and cosine
3	$\dfrac{\sin A}{\cos A} = \dfrac{BC}{AB} \div \dfrac{AC}{AB}$	Substitution Property of Equality
4	$\dfrac{\sin A}{\cos A} = \dfrac{BC}{AB} \cdot \dfrac{AB}{AC} = \dfrac{BC}{AC}$	To divide by a fraction, multiply by its reciprocal; simplify
5	$\tan A = \dfrac{BC}{AC}$	Definition of tangent
6	$\tan A = \dfrac{\sin A}{\cos A}$	Substitution Property of Equality (steps 4 and 5)

∎

13. Hard Prove that for every acute angle A, $\sin A < \tan A$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$ and acute angle $\angle A$

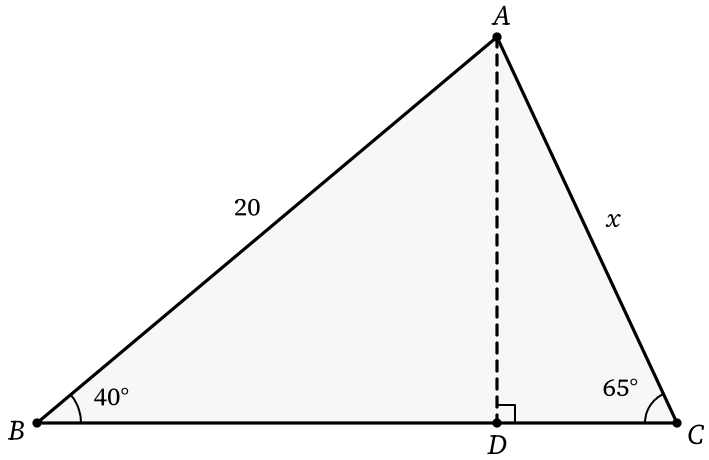
Prove: $\sin A < \tan A$

By definition, $\sin A = \dfrac{BC}{AB}$ and $\tan A = \dfrac{BC}{AC}$. Both fractions have the same positive numerator BC.

The hypotenuse is longer than each leg (Lesson 9.1, problem 10), so $AB > AC > 0$. Dividing the same positive number by a larger positive number gives a smaller result, so $\dfrac{BC}{AB} < \dfrac{BC}{AC}$, that is, $\sin A < \tan A$.

∎

14. Challenge In the figure, $\triangle ABC$ has $m\angle B = 40°$, $m\angle C = 65°$ and $AB = 20$. $\overline{AD}$ is the altitude to $\overline{BC}$. Find $x = AC$ to the nearest tenth. (Hint: find AD first, keeping all its digits.)



Show answer

$AD = 20 \sin 40° \approx 12.856$; then $\sin 65° = \dfrac{AD}{x}$, so $x = \dfrac{20 \sin 40°}{\sin 65°} \approx 14.2$

COMMON MISTAKES

- Reading $\sin^{-1}$ as a reciprocal. $\sin^{-1}(0.5)$ means the angle whose sine is 0.5, which is $30°$; it does not mean $\dfrac{1}{\sin 0.5}$.

- Choosing the wrong ratio. With the adjacent leg and the hypotenuse, use cosine, not sine. Label opp, adj and hyp from the angle you are using before you choose.
- Getting an error message from $\sin^{-1}$ or $\cos^{-1}$. The input must be between 0 and 1 for an acute angle; if it is larger, you have divided the hypotenuse by a leg instead of a leg by the hypotenuse.
- Building on rounded answers. When you solve a triangle, compute each new part from the given numbers when you can, or keep full calculator values.

> **Make it a game · Swap Your Viewpoint.** Draw a right triangle with sides 5, 12, 13. Write sin, cos and tan for angle A. Then switch your viewpoint to angle B and rewrite them, noticing that opposite and adjacent swap. Parent flashes angle choices and you shout the ratio within 5 seconds. Finish by proving sin A = cos(90 - A) from the swap.

9.7 Angles of Elevation and Depression

OBJECTIVES

- Identify angles of elevation and depression, and prove that the angle of elevation from A to B is congruent to the angle of depression from B to A.
- Use trigonometric ratios to find heights and distances of steeples, trees, cliffs, ships and lighthouses.
- Solve two-triangle problems in which a height is measured from two points.

WATCH

Finding an Angle of Elevation

Eddie Woo · 5:27

https://www.youtube.com/watch?v=0hv6Kz8sxSw

> **WHEN TO WATCH**
> Before Example 1: a worked angle-of-elevation problem.

Note: The backup also covers bearings, which this course does not use.

BACKUP VIDEO

Angles for Position (Elevation/Depression & Bearings)

Eddie Woo · 11:43

https://www.youtube.com/watch?v=83t9aU7HFDo

Before the practice set: angles of elevation and depression.

KEY TERMS

Line of sight

The straight line from an observer's eye to the object being observed.

Angle of elevation

The angle between the horizontal and the line of sight when an observer looks UP at an object.

Angle of depression

The angle between the horizontal and the line of sight when an observer looks DOWN at an object.

Theodolite

A surveyor's instrument: a telescope mounted so that it can measure horizontal angles and angles of elevation and depression.

Two-observation method

Finding a height whose base cannot be reached by measuring angles of elevation from two points a known distance apart in line with the object.

CORE IDEAS

Surveyors, builders and sailors often need a height or a distance that they cannot measure with a tape: the height of a steeple, the width of a river, the distance to a ship. They can measure an **angle** and one length, and right-triangle trigonometry does the rest. Every such problem is measured from a **horizontal** line through the observer's eye.

Elevation and depression. When you look up at an object, the angle between the horizontal and your line of sight is the **angle of elevation**. When you look down, it is the **angle of depression**. Both are measured from the horizontal, never from the vertical. A common error is to place the angle of depression at the object below instead of at the observer above.

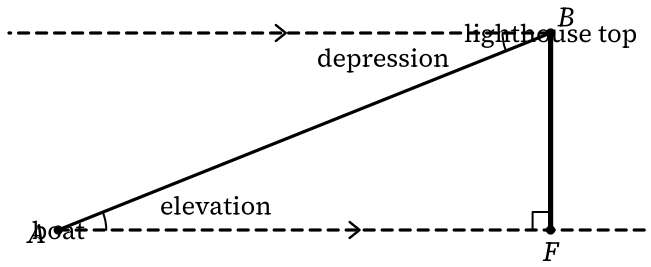
Why they are congruent. Suppose a keeper at the top B of a lighthouse looks down at a boat A, and a sailor in the boat looks up at B. The keeper's horizontal line and the sailor's horizontal line are parallel (both are level), and the line of sight is a transversal. The angle of depression at B and the angle of elevation at A are alternate interior angles, so they are congruent (Alternate Interior Angles Theorem, Euclid I.29). This is very useful: an angle of depression measured from above can be placed inside the right triangle at the bottom.

A plan for every problem. (1) Draw a large sketch with a horizontal line through the observer's eye. (2) Mark the angle, the known length, and the unknown. (3) Find a right triangle that contains the angle. (4) Choose sine, cosine or tangent with SOH-

CAH-TOA. (5) Solve, and add or subtract the observer's eye height if the triangle starts at eye level rather than at the ground.

Two observations. Often you cannot walk to the foot of a cliff or tower. Then measure the angle of elevation α from a point A, walk a known distance d straight toward the object to point B, and measure the larger angle β. With x the unknown distance from B to the foot and h the height: $h = x\tan\beta$ and $h = (x+d)\tan\alpha$. Two equations, two unknowns: set them equal, solve for x, then find h.

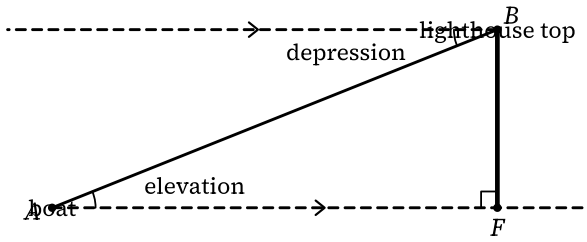
Slopes and pitches. A ramp or roof that rises r over a horizontal run s makes an angle θ with $\tan\theta = \dfrac{r}{s}$, so its angle of elevation is $\theta = \tan^{-1}\left(\dfrac{r}{s}\right)$. A roof described as "9 in 12" rises 9 inches for every 12 inches of run.



POSTULATES AND THEOREMS

THEOREM **Elevation-Depression Theorem** Euclid I.29

The angle of elevation from point A to point B is congruent to the angle of depression from B to A.



TWO-COLUMN PROOF

Given: $\overrightarrow{AH}$ and $\overrightarrow{BK}$ are horizontal, pointing in opposite directions, with H and K on opposite sides of $\overleftrightarrow{AB}$; $\overleftrightarrow{BF}$ is vertical, with F on $\overleftrightarrow{AH}$; $\angle HAB$ is the angle of elevation from A to B; $\angle KBA$ is the angle of depression from B to A

Prove: $\angle HAB \cong \angle KBA$

	STATEMENTS	REASONS
1	$\overrightarrow{AH}$ and $\overrightarrow{BK}$ are horizontal; $\overleftrightarrow{BF}$ is vertical	Given

	STATEMENTS	REASONS
2	$\overleftrightarrow{AH} \perp \overleftrightarrow{BF}$ and $\overleftrightarrow{BK} \perp \overleftrightarrow{BF}$	A horizontal line meets a vertical line at a right angle
3	$\overleftrightarrow{AH} \parallel \overleftrightarrow{BK}$	In a plane, two lines perpendicular to the same line are parallel
4	$\overleftrightarrow{AB}$ is a transversal of $\overleftrightarrow{AH}$ and $\overleftrightarrow{BK}$	Definition of transversal
5	$\angle HAB$ and $\angle KBA$ are alternate interior angles	Definition of alternate interior angles (H and K are on opposite sides of the transversal)
6	$\angle HAB \cong \angle KBA$	Alternate Interior Angles Theorem

∎

In the figure, F is the foot of the lighthouse, directly below B on the horizontal line through A.

KEY FORMULAS

$$\text{height} = (\text{horizontal distance}) \cdot \tan(\text{angle of elevation})$$

$$\text{horizontal distance} = \frac{\text{height}}{\tan(\text{angle of depression})}$$

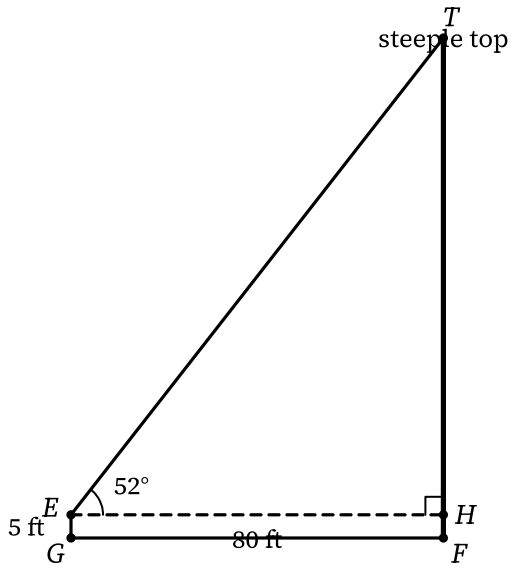
$$h = \frac{d}{\dfrac{1}{\tan\alpha} - \dfrac{1}{\tan\beta}} \quad (\text{two observations, } \alpha < \beta)$$

From history. Surveyors have long measured land with chains for distance and instruments for angles; the theodolite combines a telescope with circles marked in degrees and minutes, so a surveyor can measure both horizontal angles and angles of elevation. George Washington worked as a licensed land surveyor as a young man, beginning in 1749, laying out boundaries in the Virginia wilderness with chain and compass. Today surveyors use electronic instruments, but the right-triangle reasoning is the same.

Euclid connection. The congruence of the angles of elevation and depression is Euclid's I.29: a straight line falling on parallel straight lines makes the alternate angles equal. Euclid's own I.26 (ASA and AAS) is the reason a single acute angle and a single distance fix a whole right triangle, so that its other sides can be computed.

Practice habit. Draw a large sketch with a horizontal line at the observer's eye, mark the angle at the observer measured from that horizontal, and write which right triangle you will use before choosing a ratio. Estimate first (a 45° angle means height equals distance) and compare with your answer.

Example 1. Grace stands on level ground 80 feet from the base of a church steeple. Her eyes are 5 feet above the ground, and the angle of elevation from her eyes to the top of the steeple is 52°. How tall is the steeple? Round to the nearest tenth of a foot.

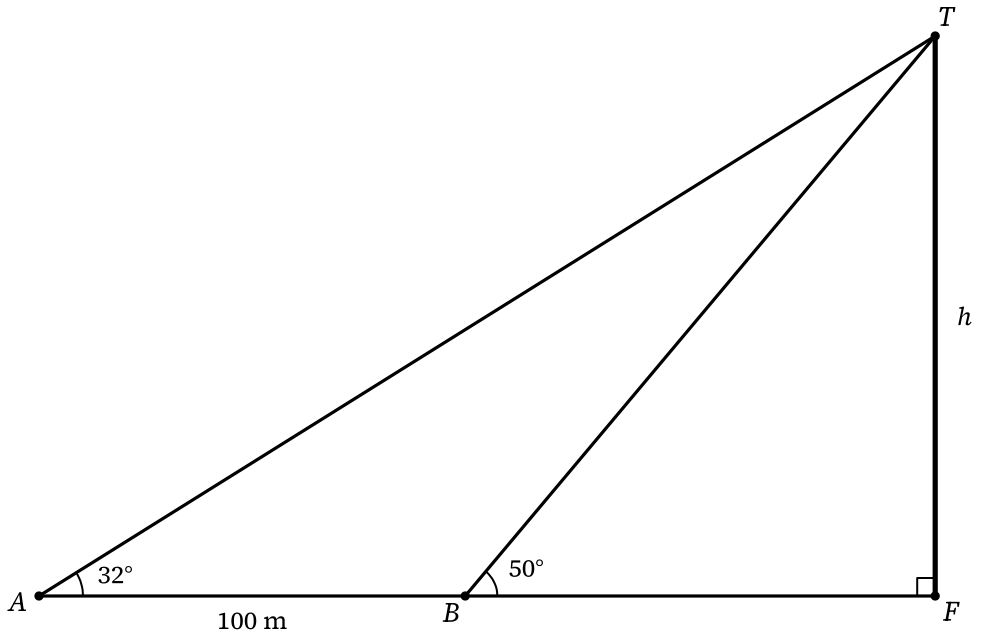


Show solution

1. Draw the horizontal line $\overline{EH}$ from Grace's eye E to the steeple; it is 80 ft long and 5 ft above the ground. The right triangle is $\triangle EHT$, with the right angle at H and the 52° angle at E.
2. From E, $\overline{HT}$ is opposite and $\overline{EH}$ is adjacent, so use tangent: $\tan 52^\circ = \dfrac{HT}{80}$.
3. $HT = 80 \tan 52^\circ \approx 80(1.279942) \approx 102.40$ ft.
4. Add the eye height: the steeple is $102.40 + 5 \approx 107.4$ ft tall. Estimate: 52° is a little more than 45°, so HT should be a little more than 80 ft. It is.

Answer: about 107.4 ft

Example 2. From point A on level ground, the angle of elevation to the top T of a sea cliff is 32°. Samuel walks 100 m straight toward the cliff to point B, where the angle of elevation is 50°. How high is the cliff? Round to the nearest tenth of a meter.



Show solution

1. Let F be the foot of the cliff, $h = TF$ and $x = BF$. Two right triangles share the side $\overline{TF}$: $\triangle TBF$ and $\triangle TAF$, with $AF = x + 100$.
2. From $\triangle TBF$: $h = x \tan 50^\circ$. From $\triangle TAF$: $h = (x + 100)\tan 32^\circ$.
3. Set them equal: $x \tan 50^\circ = x \tan 32^\circ + 100 \tan 32^\circ$, so $x(\tan 50^\circ - \tan 32^\circ) = 100 \tan 32^\circ$ and $x = \dfrac{100 \tan 32^\circ}{\tan 50^\circ - \tan 32^\circ} \approx \dfrac{62.4869}{0.566885} \approx 110.23$ m.
4. Then $h = x \tan 50^\circ \approx 110.23(1.191754) \approx 131.4$ m. Check with the other triangle: $(210.23)\tan 32^\circ \approx 131.4$ m.

Answer: about 131.4 m

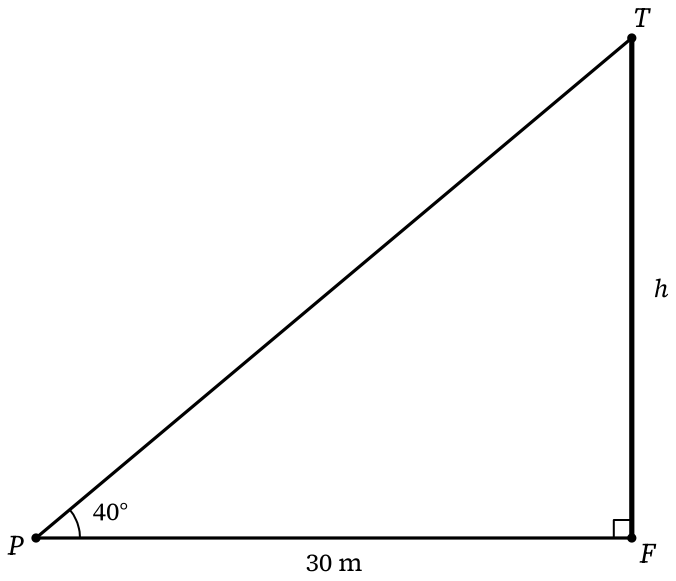
PRACTICE

1. **Easy** From the top of a lighthouse, the keeper sees a fishing boat at an angle of depression of 9°. What is the angle of elevation from the boat to the top of the lighthouse? Give a reason.

   Show answer

   9°, because the angle of elevation and the angle of depression are alternate interior angles formed by parallel horizontal lines.

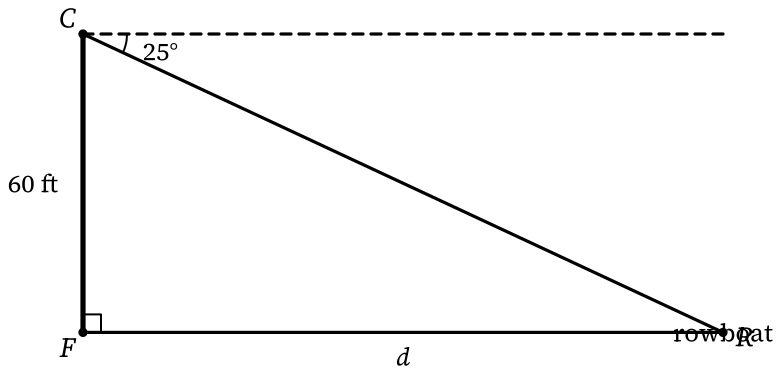
2. **Easy** Henry stands 30 m from the base of a tall pine tree on level ground. The angle of elevation from the ground where he stands to the top of the tree is 40°. How tall is the tree? Round to the nearest tenth of a meter.



   Show answer

   $h = 30 \tan 40^\circ \approx 25.2 \text{ m}$

3. **Easy** From the top of a 60-foot cliff, the angle of depression to a rowboat on the water is 25°. How far is the rowboat from the base of the cliff? Round to the nearest tenth of a foot.



Show answer

The 25° angle also appears at the boat (alternate interior angles), so $\tan 25^\circ = \dfrac{60}{d}$ and $d = \dfrac{60}{\tan 25^\circ} \approx 128.7$ ft.

4. **Easy** On a sunny afternoon, a fence post 6 feet tall casts a shadow 4 feet long on level ground. What is the angle of elevation of the sun? Round to the nearest tenth of a degree.

Show answer

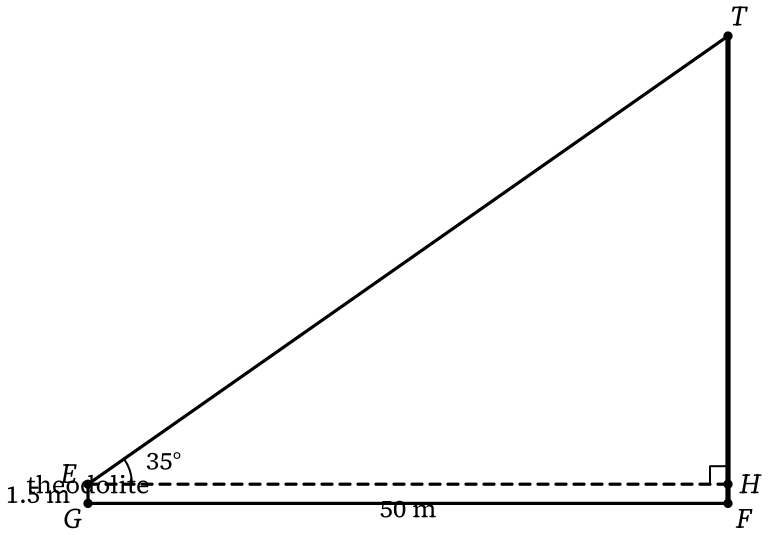
$$\tan^{-1}\left(\frac{6}{4}\right) = \tan^{-1}(1.5) \approx 56.3^\circ$$

5. **Medium** The lamp of a lighthouse is 30 m above sea level. The keeper sees a ship at an angle of depression of 4°. How far is the ship from the base of the lighthouse? Round to the nearest meter.

Show answer

$$d = \frac{30}{\tan 4^\circ} \approx 429 \text{ m}$$

6. **Medium** A surveyor sets up a theodolite whose telescope is 1.5 m above level ground, 50 m from the base of a bell tower. The angle of elevation to the top of the tower is 35°. How tall is the tower? Round to the nearest tenth of a meter.



Show answer

$$50 \tan 35^\circ + 1.5 \approx 35.0 + 1.5 \approx 36.5 \text{ m}$$

7. **Medium** A barn roof has a "9 in 12" pitch: it rises 9 inches for every 12 inches of horizontal run. What angle does the roof make with the horizontal? Round to the nearest tenth of a degree.

Show answer

$$\tan^{-1}\left(\frac{9}{12}\right) = \tan^{-1}(0.75) \approx 36.9^\circ$$

8. **Medium** A ramp to a schoolhouse door must rise 2 feet, and its angle of elevation may be at most 4.8°. What is the shortest horizontal run the ramp can have? Round to the nearest tenth of a foot.

Show answer

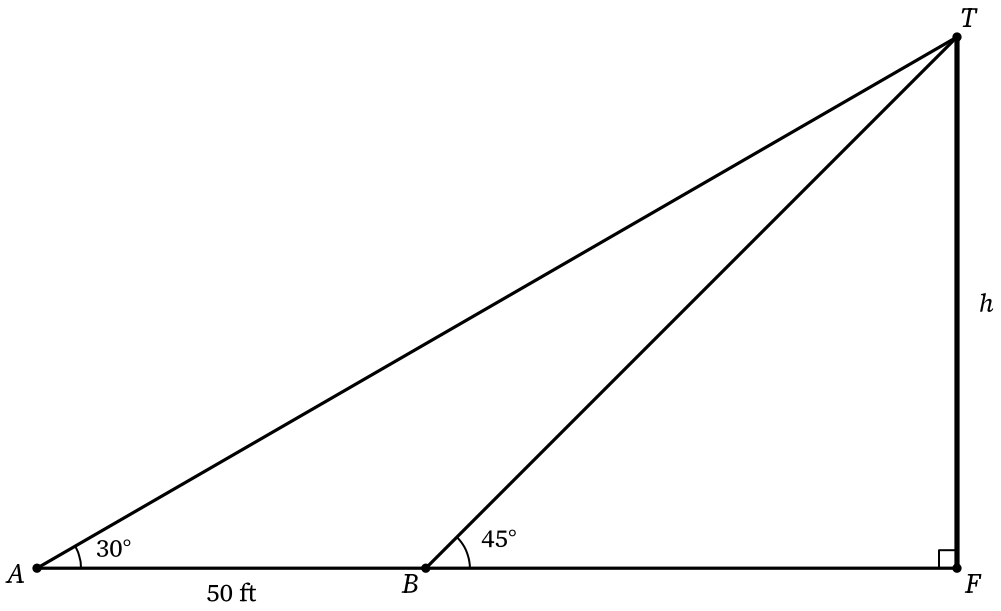
$$\frac{2}{\tan 4.8^\circ} \approx 23.8 \text{ ft}$$

9. **Medium** Ruth says: “From the top of the tower the angle of depression to the well is 30°, so in my right triangle the angle at the top of the tower is 30°.” Explain her mistake and where the 30° angle belongs in the triangle formed by the tower, the ground and the line of sight.

Show answer

The angle of depression is measured from the HORIZONTAL at the top of the tower, not from the tower itself, which is vertical. Inside the triangle, the angle at the top between the tower and the line of sight is $90^\circ - 30^\circ = 60^\circ$. By alternate interior angles, the 30° angle appears at the well, between the ground and the line of sight.

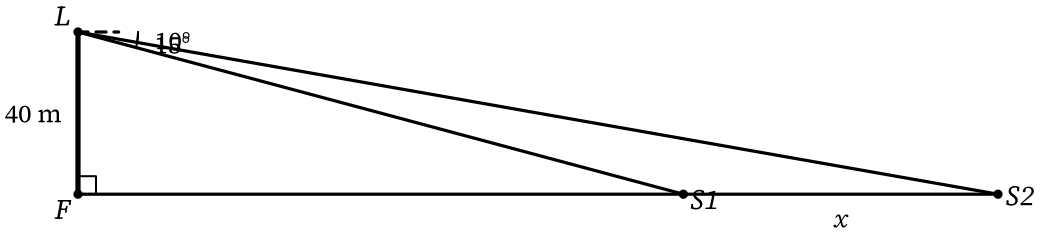
10. **Hard** From a point A, the angle of elevation to the top of a grain elevator is 30°. Walking 50 ft straight toward it to B, the angle of elevation becomes 45°. Find the height h of the elevator exactly (in simplest radical form) and to the nearest tenth of a foot. (Use the exact values of $\tan 30^\circ$ and $\tan 45^\circ$.)



Show answer

From B: $h = x$ (since $\tan 45^\circ = 1$). From A: $h = (x + 50)\dfrac{\sqrt{3}}{3}$. So $h\sqrt{3} = h + 50$, $h = \dfrac{50}{\sqrt{3} - 1} = 25(\sqrt{3} + 1) \approx 68.3$ ft.

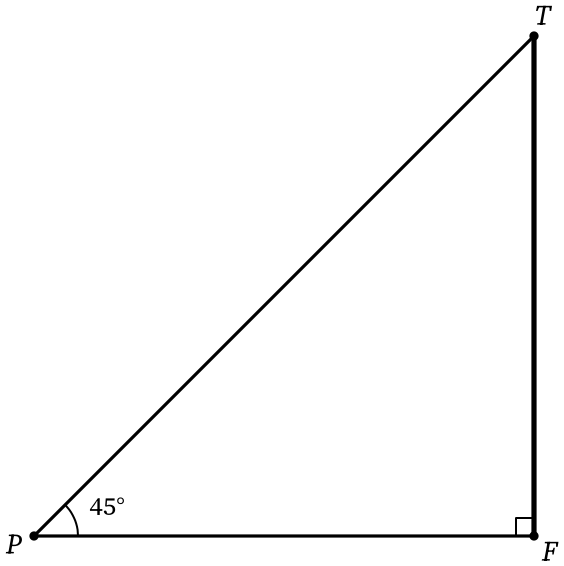
11. **Hard** From the top of a lighthouse 40 m above the sea, the angles of depression to two boats directly in line with the lighthouse are 10° and 15°. How far apart are the boats? Round to the nearest tenth of a meter.



Show answer

$$\frac{40}{\tan 10^\circ} - \frac{40}{\tan 15^\circ} \approx 226.85 - 149.28 \approx 77.6 \text{ m}$$

12. Hard In the figure, the angle of elevation from point P on level ground to the top T of a tower $\overline{TF}$ is $45°$, and the tower is perpendicular to the ground. Prove that the height of the tower equals its distance from P: $TF = PF$.



Show answer

See the model proof.

TWO-COLUMN PROOF

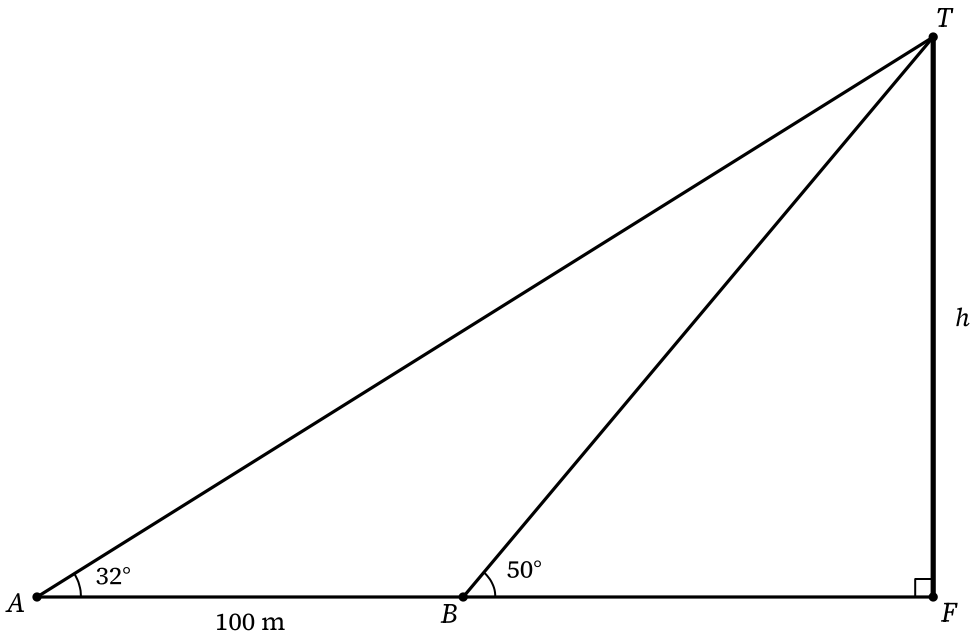
Given: $\overline{TF} \perp \overline{PF}$; $m\angle TPF = 45°$

Prove: $TF = PF$

	STATEMENTS	REASONS
1	$\overline{TF} \perp \overline{PF}$; $m\angle TPF = 45°$	Given
2	$\angle TFP$ is a right angle	Definition of perpendicular lines
3	$m\angle TPF + m\angle PTF = 90°$	Acute angles of a right triangle are complementary
4	$m\angle PTF = 45°$	Substitution and Subtraction Property of Equality
5	$\angle TPF \cong \angle PTF$	Definition of congruent angles
6	$TF = PF$	Converse of the Isosceles Triangle Theorem (Euclid I.6)

∎

13. Hard Prove the two-observation formula. A tower of height h stands on level ground. From point A the angle of elevation of its top is α; from point B, which is d units closer on the same line, it is β, with $\alpha < \beta$. Prove that $h = \dfrac{d \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overline{TF}$ is a vertical tower of height h with foot F; A, B, F are collinear on level ground with B between A and F and $AB = d$; $m\angle TAF = \alpha$, $m\angle TBF = \beta$, $\alpha < \beta$

Prove: $h = \dfrac{d \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$

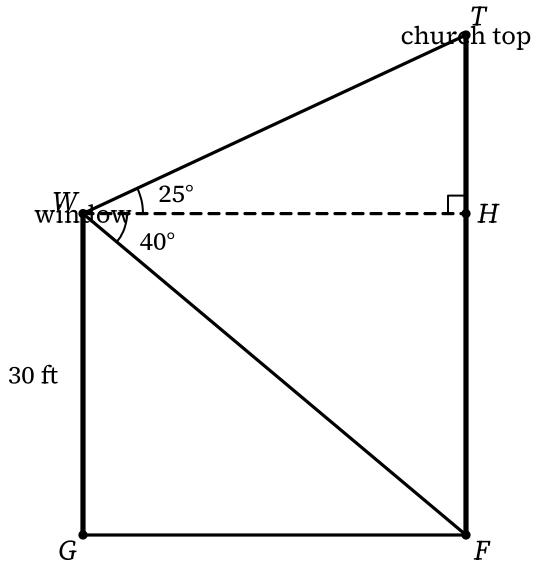
Let $x = BF$. Then $AF = x + d$ by the Segment Addition Postulate. Triangles TBF and TAF are right triangles with the right angle at F. By the definition of tangent, $\tan\beta = \dfrac{h}{x}$ and $\tan\alpha = \dfrac{h}{x+d}$, so $h = x\tan\beta$ and $h = (x+d)\tan\alpha$.

Setting the two expressions equal (Transitive Property): $x\tan\beta = x\tan\alpha + d\tan\alpha$, so $x(\tan\beta - \tan\alpha) = d\tan\alpha$. Since $\alpha < \beta$, $\tan\alpha < \tan\beta$ (the tangent grows as the acute angle grows), so $\tan\beta - \tan\alpha \neq 0$ and we may divide: $x = \dfrac{d\tan\alpha}{\tan\beta - \tan\alpha}$.

Substituting into $h = x\tan\beta$ gives $h = \dfrac{d\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}$.

∎

14. **Challenge** From a window 30 ft above the street, Samuel sees the top of the church across the town square at an angle of elevation of 25° and the base of the church at an angle of depression of 40°. The ground is level. How tall is the church? Round to the nearest tenth of a foot.



Show answer

$$\text{Horizontal distance} = \frac{30}{\tan 40^\circ} \approx 35.753 \text{ ft; height above the window} = 35.753 \tan 25^\circ \approx 16.672 \text{ ft; church} \approx 30 + 16.672 \approx 46.7 \text{ ft.}$$

COMMON MISTAKES

- Measuring the angle of depression from the vertical. Both angles of elevation and depression are measured from the horizontal line through the observer.
- Placing the angle of depression at the top of the triangle. Use alternate interior angles to move it to the bottom, where it is an angle of the right triangle.
- Forgetting the observer's eye height or instrument height. If the horizontal line starts at eye level, add that height at the end.
- Rounding the first triangle's answer before using it in the second. In two-triangle problems keep full calculator values until the end.

> **Make it a game · Clinometer Challenge.** Tape a straw to a protractor with a weighted string hanging from its center. Sight the top of a tree, wall or lamp post and read the angle. Measure your distance in steps and your eye height. Guess the object's height first, then compute with tangent. Closest guess to the computed height scores.

UNIT 9

Right Triangles and Trigonometry

This unit is about the most useful triangle of all: the right triangle. It begins with the Pythagorean Theorem, the great proposition that closes Book I of Euclid's Elements (I.47), and its converse (I.48). Students study three different proofs: Euclid's own windmill proof by areas, a rearrangement proof and a proof with similar triangles. They then use the theorem and its converse to find lengths, test for right angles and classify triangles as acute, right or obtuse.

Next, similar triangles (Unit 8) are put to work: the altitude to the hypotenuse creates three similar right triangles (Euclid VI.8), which leads to the geometric-mean relationships. Then come the two special right triangles, the $45°$-$45°$-$90°$ triangle (half a square) and the $30°$-$60°$-$90°$ triangle (half an equilateral triangle).

The unit ends with right-triangle trigonometry. Because all right triangles with the same acute angle are similar, the ratios of their sides depend only on the angle; these ratios are the tangent, sine and cosine. Students learn to use a scientific calculator (allowed from this unit on) to find sides and angles, and they apply trigonometry the way surveyors, builders and navigators do: to find the heights of steeples and cliffs, the widths of rivers and the distances to ships.

PREREQUISITES

- Using the Pythagorean Theorem numerically, simplifying square roots and rationalizing a one-term denominator (Algebra I)
- Similar triangles: AA, SSS and SAS Similarity and proportions from similar triangles (Lessons 8.3–8.4)
- The Triangle Inequality and the Hinge Theorem and its converse (Lesson 5.6)
- The Isosceles Triangle Theorem and its converse; equilateral triangles are equiangular (Lesson 4.6)
- Alternate Interior Angles Theorem for parallel lines (Lesson 3.2)
- Writing two-column, paragraph, flow and indirect proofs (Units 2–8)

MASTERY CHECKLIST

- ☐ I can state the Pythagorean Theorem in symbols and in Euclid's words, and explain Euclid's proof, a rearrangement proof and a similar-triangles proof.
- ☐ I can find a missing side of a right triangle and give the answer in simplest radical form or rounded as asked.
- ☐ I can prove the Converse of the Pythagorean Theorem and use it to decide whether a triangle is right.
- ☐ I can classify a triangle as acute, right or obtuse by comparing c^2 with $a^2 + b^2$, after checking the Triangle Inequality.
- ☐ I can prove that the altitude to the hypotenuse forms three similar right triangles and use the geometric-mean relationships for the altitude and the legs.
- ☐ I can derive and use the side ratios of $45°$-$45°$-$90°$ and $30°$-$60°$-$90°$ triangles, and construct these triangles with compass and straightedge.

- ☐ I can explain why the trigonometric ratios depend only on the angle, and define $\sin A$, $\cos A$, and $\tan A$.
- ☐ I can use SOH-CAH-TOA and a calculator in degree mode to find missing sides, and inverse ratios to find missing angles.
- ☐ I can use the cofunction relationship $\sin A = \cos(90^\circ - A)$ and the exact values for 30°, 45°, and 60°.
- ☐ I can prove that angles of elevation and depression are congruent and solve height-and-distance problems, including two-observation problems.

LESSONS IN THIS UNIT

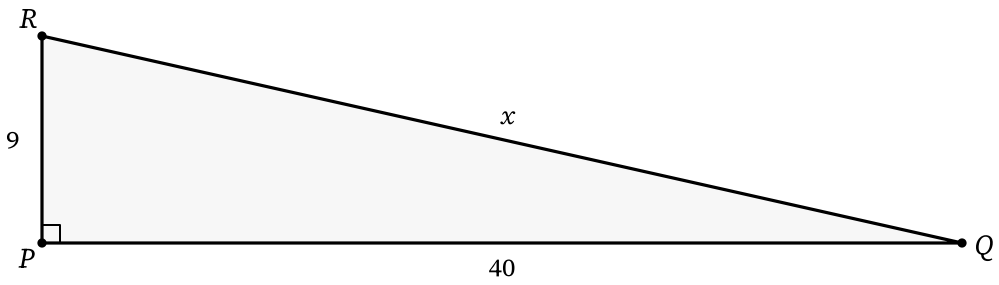
9.1	**The Pythagorean Theorem and Its Proofs**	2 sessions of 45 minutes
9.2	**The Converse of the Pythagorean Theorem**	45 minutes
9.3	**Similar Right Triangles and the Geometric Mean**	2 sessions of 45 minutes
9.4	**Special Right Triangles**	2 sessions of 45 minutes
9.5	**The Tangent Ratio**	2 sessions of 45 minutes
9.6	**The Sine and Cosine Ratios**	2 sessions of 45 minutes

Unit 9 test

You have 60 minutes. A scientific calculator is allowed; make sure it is in degree mode. You will need a compass and straightedge for problem 13. Show all your work. Give exact answers in simplest radical form when a problem asks for them; otherwise round as each problem directs, using full calculator values until the final step. In every proof, give a reason for every step. Figures are drawn to scale, but use only the information given in words and marked on the figure.

1. **5 pts**

   In the figure, right $\triangle PQR$ has legs $PQ = 40$ and $PR = 9$. Find the hypotenuse x.



2. **5 pts**

   A right triangle has a hypotenuse of 14 and one leg of 6. Find the other leg in simplest radical form.

3. **6 pts**

   Classify the triangle with sides 11, 13 and 17 as acute, right or obtuse. Show the comparison you use.

4. **6 pts**

   Samuel is building a deck. To check a corner he marks 12 ft along one side and 16 ft along the other, and the distance between the marks is 20 ft 3 in. Is the corner angle acute, right or obtuse?

5. **6 pts**

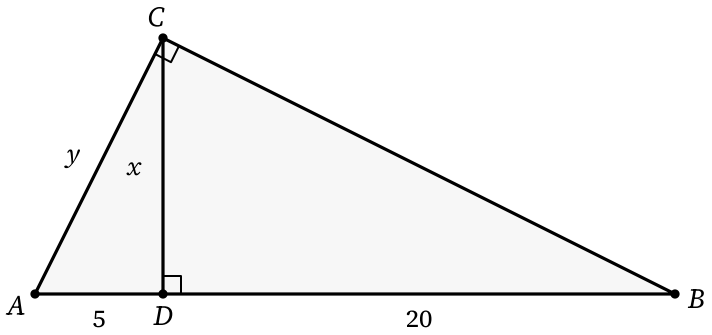
The diagonal of a square is 14 cm. Find the length of a side in simplest radical form.

6. **6 pts**

A rafter makes a 30° angle with the horizontal ceiling joist, and the peak of the roof is 6 ft above the joist. How long is the horizontal run from the end of the rafter to the point directly below the peak? Give the exact answer in simplest radical form.

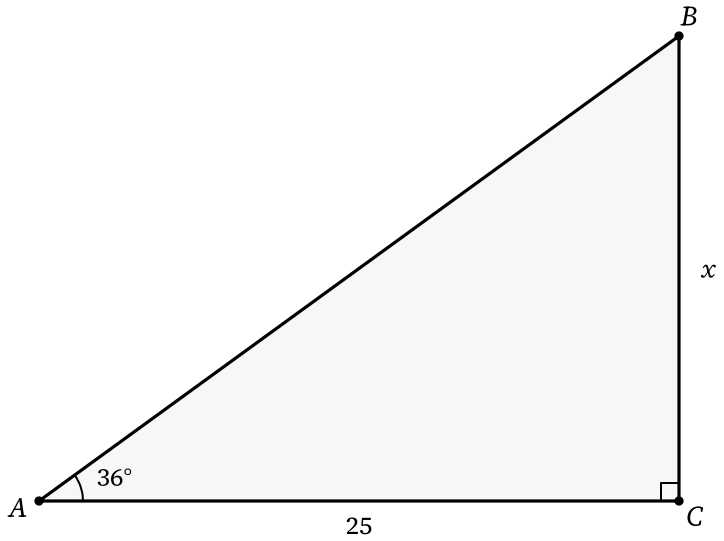
7. **7 pts**

In the figure, $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, with $AD = 5$ and $DB = 20$. Find $x = CD$ and $y = AC$. Give y in simplest radical form.



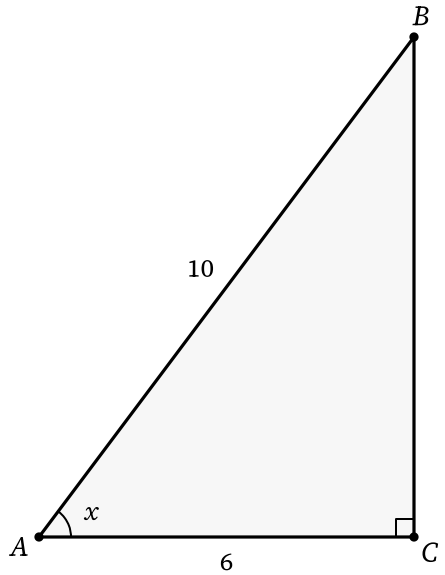
8. **6 pts**

In the figure, right $\triangle ABC$ has its right angle at C, $m\angle A = 36°$ and $AC = 25$. Find $x = BC$ to the nearest tenth.



9. **6 pts**

In the figure, right $\triangle ABC$ has its right angle at C, $AC = 6$ and hypotenuse $AB = 10$. Find $x = m\angle A$ to the nearest tenth of a degree.



10. **5 pts**

A 14-foot ladder leans against the wall of a barn and makes a 72° angle with the level ground. How high up the wall does it reach? Round to the nearest tenth of a foot.

11. **5 pts**

Explain why a carpenter who measures 6 ft and 8 ft along two walls from a corner and finds exactly 10 ft between the marks knows the corner is square. Then explain why, on a large foundation, a carpenter would rather use 6-8-10 (or 9-12-15) than 3-4-5.

12. **7 pts**

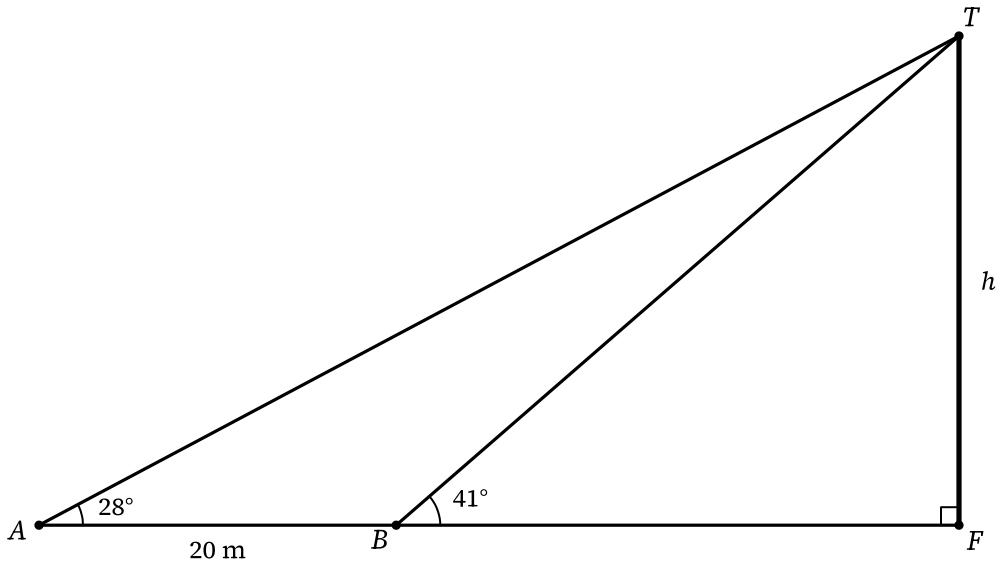
From a fishing boat, the angle of elevation to the top of a lighthouse that stands 35 m above sea level is 6°. How far is the boat from the base of the lighthouse? Round to the nearest meter.

13. **6 pts**

You are given a segment $\overline{AB}$. Using only a compass and straightedge, construct an isosceles right triangle (a 45°-45°-90° triangle) with $\overline{AB}$ as one leg. List your steps and explain why the triangle has angles of 45°, 45° and 90°.

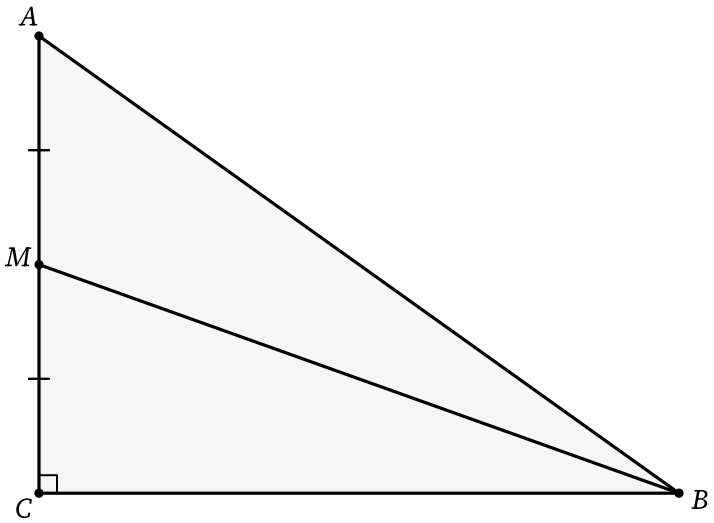
14. **8 pts**

From point A on level ground, the angle of elevation to the top T of a church tower is 28°. Grace walks 20 m straight toward the tower to point B, where the angle of elevation is 41°. Find the height h of the tower to the nearest tenth of a meter.



15. **8 pts**

In the figure, $\triangle ABC$ has a right angle at C, and M is the midpoint of $\overline{AC}$. Prove that $AB^2 - BM^2 = 3\,CM^2$.



16. **8 pts**

In right $\triangle ABC$ with the right angle at C, prove that $\sin A \cdot \tan B = \cos A$.

ANSWER KEY

1. $x = \sqrt{81 + 1600} = \sqrt{1681} = 41$
2. $\sqrt{196 - 36} = \sqrt{160} = 4\sqrt{10}$
3. Acute: $11 + 13 > 17$, so the triangle exists, and $17^2 = 289 < 11^2 + 13^2 = 290$.
4. Obtuse: 20 ft 3 in $= 20.25$ ft, and $20.25^2 = 410.0625 > 12^2 + 16^2 = 400$.
5. $\dfrac{14}{\sqrt{2}} = 7\sqrt{2}$ cm
6. The rise of 6 ft is the short leg (opposite 30°), so the run is the long leg: $6\sqrt{3}$ ft ≈ 10.4 ft.
7. $x = \sqrt{5 \cdot 20} = 10$; $y = \sqrt{AB \cdot AD} = \sqrt{25 \cdot 5} = 5\sqrt{5}$
8. $x = 25 \tan 36^\circ \approx 18.2$
9. $\cos A = \dfrac{6}{10}$, so $x = \cos^{-1}(0.6) \approx 53.1^\circ$
10. $14 \sin 72^\circ \approx 13.3$ ft
11. The three measurements form a triangle with $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, so by the Converse of the Pythagorean Theorem the angle opposite the 10-ft side, the corner, is a right angle. Longer measurements are better because a small error in reading the tape is a smaller fraction of a long length, and the marks lie farther along the walls, so an angle that is slightly off makes a larger, easier-to-see difference in the diagonal.
12. $d = \dfrac{35}{\tan 6^\circ} \approx 333$ m
13. Steps: (1) Extend $\overline{BA}$ beyond A. With center A, draw an arc crossing the line at two points X and Y on either side of A. (2) With centers X and Y and a larger equal radius, draw arcs that cross at P; draw $\overleftrightarrow{AP}$, which is perpendicular to $\overline{AB}$ at A (Euclid I.11). (3) With center A and radius AB, draw an arc crossing $\overleftrightarrow{AP}$ at C. (4) Draw $\overline{BC}$. Why: $\angle BAC$ is a right angle by construction, and $AC = AB$ because they are radii of the same circle, so $\triangle BAC$ is isosceles. Its base angles $\angle B$ and $\angle C$ are congruent (Isosceles Triangle Theorem, Euclid I.5) and complementary (acute angles of a right triangle), so each is 45°.
14. Let $x = BF$. Then $h = x \tan 41^\circ = (x + 20) \tan 28^\circ$, so $x = \dfrac{20 \tan 28^\circ}{\tan 41^\circ - \tan 28^\circ} \approx 31.50$ m and $h \approx 31.50 \tan 41^\circ \approx 27.4$ m.
15. See the model proof.

    TWO-COLUMN PROOF

    Given: $\angle ACB$ is a right angle; M is the midpoint of $\overline{AC}$

Prove: $AB^2 - BM^2 = 3\,CM^2$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; M is the midpoint of $\overline{AC}$	Given
2	$AB^2 = AC^2 + BC^2$	Pythagorean Theorem (right $\triangle ACB$)
3	$\angle MCB$ is a right angle	$\angle MCB$ and $\angle ACB$ are the same angle, since M lies on $\overrightarrow{CA}$
4	$BM^2 = CM^2 + BC^2$	Pythagorean Theorem (right $\triangle MCB$)
5	$AB^2 - BM^2 = AC^2 - CM^2$	Subtraction Property of Equality (steps 2 and 4)
6	$AM = MC$, so $AC = AM + MC = 2\,CM$	Definition of midpoint; Segment Addition Postulate; Substitution
7	$AC^2 = 4\,CM^2$	Multiplication Property of Equality (square both sides)
8	$AB^2 - BM^2 = 3\,CM^2$	Substitution ($4\,CM^2 - CM^2 = 3\,CM^2$)

∎

16. See the model proof.

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with right angle $\angle C$

Prove: $\sin A \cdot \tan B = \cos A$

By the definition of sine, $\sin A = \dfrac{BC}{AB}$, because $\overline{BC}$ is the leg opposite $\angle A$ and $\overline{AB}$ is the hypotenuse. For $\angle B$, the opposite leg is $\overline{AC}$ and the adjacent leg is $\overline{BC}$, so by the definition of tangent $\tan B = \dfrac{AC}{BC}$.

Multiplying, $\sin A \cdot \tan B = \dfrac{BC}{AB} \cdot \dfrac{AC}{BC} = \dfrac{AC}{AB}$, since $BC \neq 0$. By the definition of cosine, $\cos A = \dfrac{AC}{AB}$, because $\overline{AC}$ is the leg adjacent to $\angle A$. Therefore, by substitution, $\sin A \cdot \tan B = \cos A$.

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UNIT 10

Circles

This unit is a study of the circle, the figure Euclid gives all of Book III of the *Elements* to. Students learn the vocabulary of circles (chords, secants, tangents, central angles and arcs), then prove the great circle theorems one after another: a diameter perpendicular to a chord bisects it (III.3), a tangent is perpendicular to the radius (III.18), an inscribed angle is half its arc (III.20), the angle in a semicircle is right (III.31), and the products of chord and secant pieces are equal (III.35 and III.36). Each theorem is proved from congruent or similar triangles that students already know, so the unit is a rich review of Units 4, 8 and 9 as well.

Compass-and-straightedge work continues: finding the center of a circle (III.1), completing a circle from a broken piece (III.25), and constructing tangents from an outside point (III.17). Practical problems include the radius of a stone arch, belts around pulleys, a carpenter's square on a round tabletop, what a satellite can see, and how far the horizon is from a lighthouse. The unit ends where geometry meets algebra, with the equation of a circle in the coordinate plane.

Parents: the most important habits in this unit are drawing the radius to every point of tangency, tracing the intercepted arc before computing an angle, and naming the correct product rule before computing a length. A scientific calculator is allowed for square roots, π and decimal answers.

PREREQUISITES

- Prove triangles congruent with SSS, SAS, ASA, AAS and HL, and use CPCTC (Unit 4).
- Use the Isosceles Triangle Theorem and the Exterior Angle Theorem (Lessons 4.1 and 4.6).
- Use the Perpendicular Bisector Theorem and its converse (Lesson 5.1).
- Prove triangles similar by AA and solve proportions (Unit 8).
- Use the Pythagorean Theorem and its converse (Lessons 9.1 and 9.2).
- Use the distance and midpoint formulas, slopes of perpendicular lines, and solve quadratic equations (Algebra I).

MASTERY CHECKLIST

- ☐ I can name the parts of a circle and find the measures of central angles, minor arcs, major arcs and semicircles.
- ☐ I can find the center of a circle, and complete a circle from an arc, with compass and straightedge.
- ☐ I can use congruent chords, arcs and central angles, and the perpendicular from the center to a chord, to find lengths such as the radius of an arch.
- ☐ I can prove that a tangent is perpendicular to the radius at the point of tangency, use congruent tangent segments, and construct a tangent from an outside point.
- ☐ I can prove and use the Inscribed Angle Theorem, Thales' Theorem and the Inscribed Quadrilateral Theorem.
- ☐ I can find angles formed by chords, secants and tangents with the vertex on, inside or outside a circle.
- ☐ I can use the Intersecting Chords, Secant-Secant and Tangent-Secant Theorems, including finding the distance to the horizon.
- ☐ I can write and read the equation of a circle, complete the square, and decide whether a point is inside, on or outside a circle.
- ☐ I can write two-column, paragraph, flow and indirect

proofs about circles with a correct reason for every step.

- ☐ I can name the propositions of Euclid's Book III behind the main circle theorems: III.1, III.3, III.18, III.20, III.31, III.32, III.35 and III.36.

LESSONS IN THIS UNIT

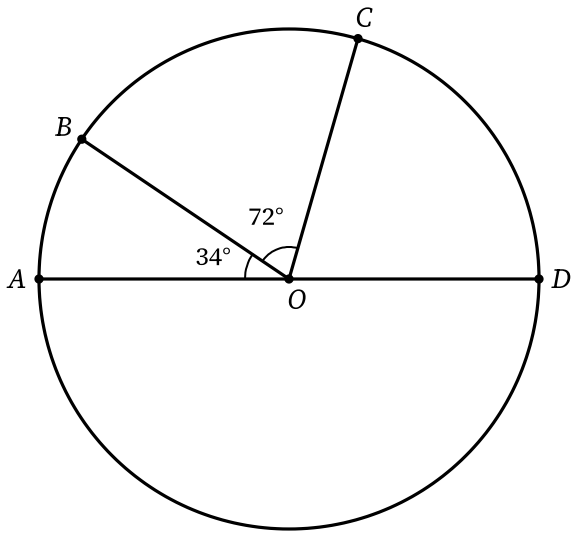
10.1	**Circles, Arcs and Central Angles**	45 minutes
10.2	**Chords**	2 sessions of 45 minutes
10.3	**Tangents**	2 sessions of 45 minutes
10.4	**Inscribed Angles**	2 sessions of 45 minutes
10.5	**Angles Formed by Chords, Secants and Tangents**	2 sessions of 45 minutes
10.6	**Segment Lengths in Circles**	2 sessions of 45 minutes
10.7	**Circles in the Coordinate Plane**	2 sessions of 45 minutes

Unit 10 test

You have 50 minutes. A scientific calculator is allowed for square roots, π and decimal answers, and you will need a compass and straightedge for Problem 13. Show your work on every problem. In each proof, give a reason for every statement. Give exact answers unless a problem tells you how to round, and include units. The test is worth 100 points.

1. **5 pts**

In circle O in the figure, $\overline{AD}$ is a diameter, $m\angle AOB = 34°$ and $m\angle BOC = 72°$, with B and C on the same semicircle. Find (a) $m\overset{\frown}{CD}$ and (b) $m\overset{\frown}{BDA}$.

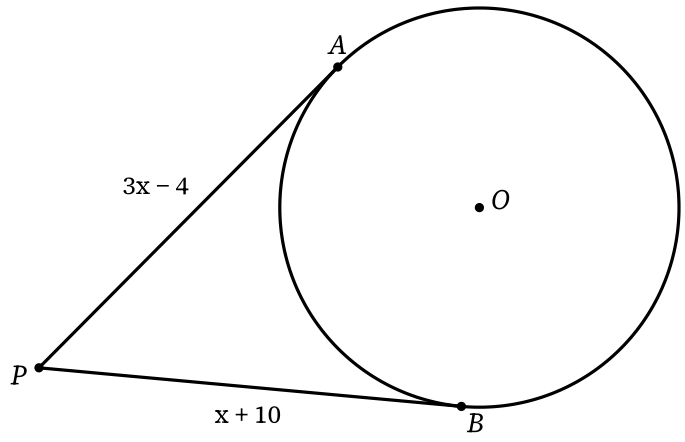


2. **6 pts**

A round wooden tabletop has a diameter of 34 inches. A straight inlaid strip of maple runs across it as a chord 30 inches long. How far is the strip from the center of the tabletop?

3. **5 pts**

In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B, with $PA = 3x - 4$ and $PB = x + 10$. Find x and PA.

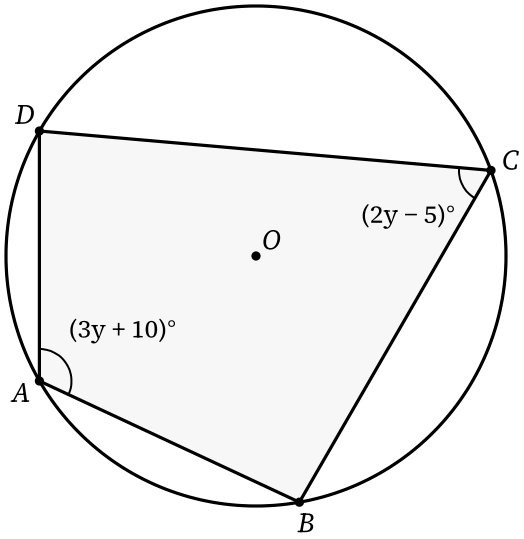


4. **5 pts**

A circle has radius 9 cm. Point P is 41 cm from the center, and $\overline{PT}$ is tangent to the circle at T. Find PT.

5. **6 pts**

Quadrilateral $ABCD$ is inscribed in a circle, with $m\angle DAB = (3y + 10)^\circ$ and $m\angle BCD = (2y - 5)^\circ$, as in the figure. Find y and $m\angle DAB$.

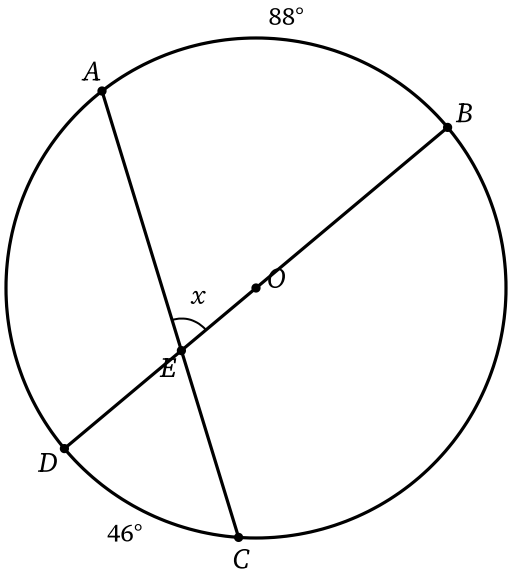


6. **5 pts**

$\overline{AB}$ is a diameter of a circle with $AB = 20$ cm, and C is a point on the circle with $AC = 12$ cm. Find BC.

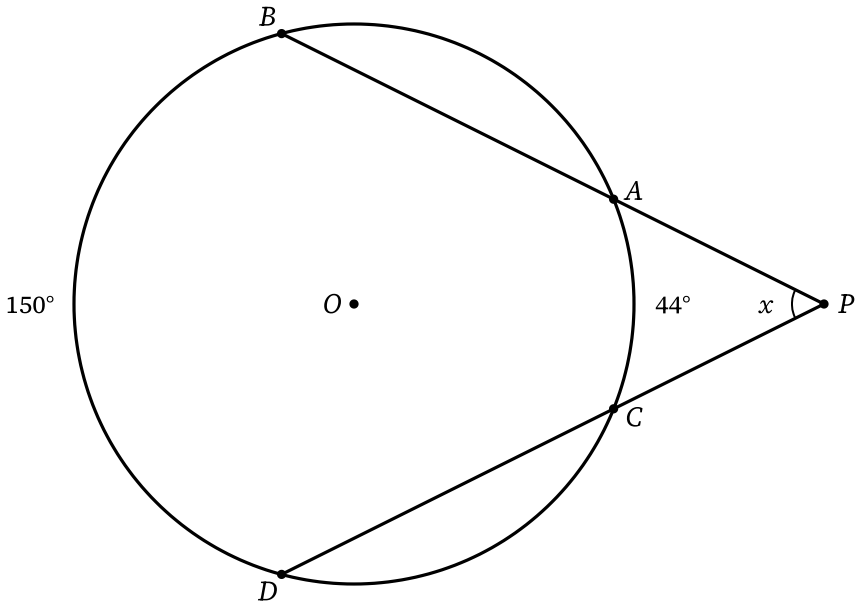
7. **5 pts**

Two straight lead strips in a round stained-glass window are chords that cross at E, as in the figure. The arc inside the angle where they cross measures $88°$, and the arc inside its vertical angle measures $46°$. Find x, the measure of the angle between the strips.



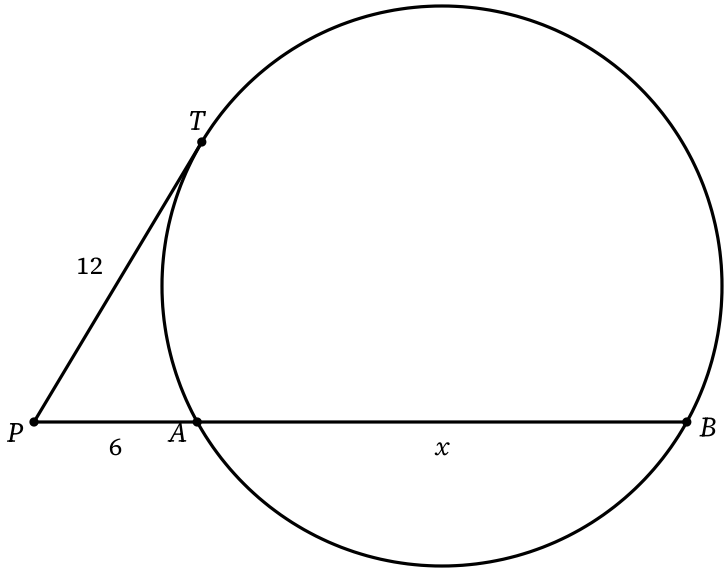
8. **5 pts**

In the figure, two secants meet outside the circle at P. The far arc $\overset{\frown}{BD}$ measures $150°$ and the near arc $\overset{\frown}{AC}$ measures $44°$. Find $x = m\angle APC$.



9. **6 pts**

In the figure, $\overline{PT}$ is tangent to the circle at T with $PT = 12$, and a secant from P meets the circle at A and B with $PA = 6$. Find $x = AB$.



10. **7 pts**

A fire lookout stands on a mountain top 2640 feet (half a mile) above the surrounding plain. Treating the earth as a sphere of radius 3960 miles, how far away is the horizon? Give the answer in miles, to the nearest tenth.

11. **7 pts**

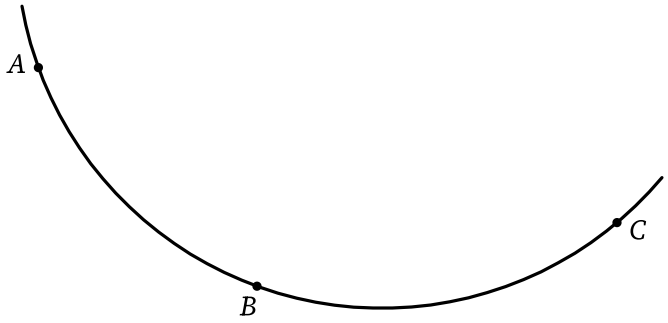
Find the center and radius of the circle $x^2 + y^2 - 4x + 10y + 4 = 0$.

12. **6 pts**

Write an equation of the circle that has a diameter with endpoints $(-3, 1)$ and $(5, 7)$.

13. **8 pts**

Ruth found a piece of a broken round plate; its edge is the arc in the figure, with points A, B and C marked on it. Describe, step by step, how to find the center of the plate with compass and straightedge, and explain why your construction works.

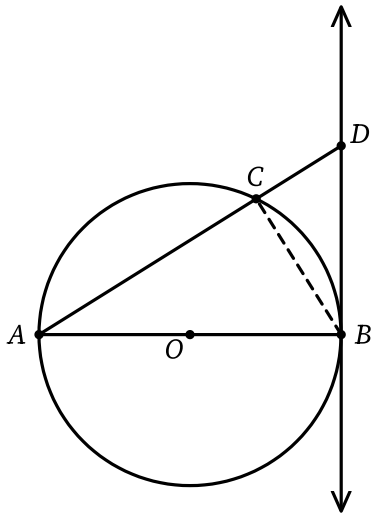


14. **6 pts**

Grace claims that a trapezoid whose legs are not congruent can be inscribed in a circle. Is she right? Explain.

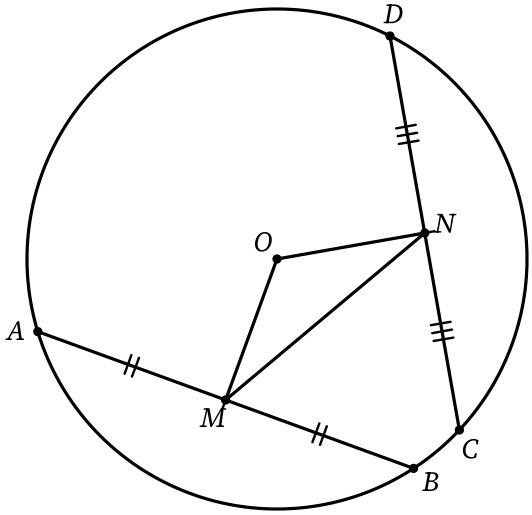
15. **9 pts**

Given: $\overline{AB}$ is a diameter of $\odot O$, $\overleftrightarrow{BD}$ is tangent to $\odot O$ at B, and $\overline{AD}$ meets the circle again at C, as in the figure. Prove: $AB^2 = AC \cdot AD$. Write a two-column proof.



16. **9 pts**

Given: In $\odot O$, chords $\overline{AB} \cong \overline{CD}$, neither of them a diameter; M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$; and O, M, N are not collinear, as in the figure. Prove: $\angle OMN \cong \angle ONM$. Write a paragraph proof.



ANSWER KEY

1. (a) $180^\circ - 34^\circ - 72^\circ = 74^\circ$; (b) $B\overset{\frown}{D}A$ is the major arc with endpoints B and A, so it measures $360^\circ - 34^\circ = 326^\circ$.
2. The radius is 17 in, and the perpendicular from the center bisects the chord, so the distance is $\sqrt{17^2 - 15^2} = \sqrt{64} = 8$ inches.
3. Tangent segments from one point are congruent: $3x - 4 = x + 10$, so $x = 7$ and $PA = 17$.
4. The radius $\overline{OT}$ is perpendicular to the tangent, so $PT = \sqrt{41^2 - 9^2} = \sqrt{1600} = 40$ cm.
5. Opposite angles of an inscribed quadrilateral are supplementary: $(3y + 10) + (2y - 5) = 180$, so $5y = 175$, $y = 35$, and $m\angle DAB = 115^\circ$.
6. $\angle ACB$ is inscribed in a semicircle, so it is a right angle, and $BC = \sqrt{20^2 - 12^2} = \sqrt{256} = 16$ cm.
7. $x = \dfrac{1}{2}(88^\circ + 46^\circ) = 67^\circ$
8. $x = \dfrac{1}{2}(150^\circ - 44^\circ) = 53^\circ$
9. $12^2 = 6 \cdot PB$, so $PB = 24$ and $x = 24 - 6 = 18$.
10. The line of sight to the horizon is tangent, and the line through the earth's center is a secant with external part $h = 0.5$ mi and whole $h + 7920$ mi. By the Tangent-Secant Theorem, $d^2 = 0.5 \times 7920.5 = 3960.25$, so $d \approx 62.9$ miles.
11. $(x^2 - 4x + 4) + (y^2 + 10y + 25) = -4 + 4 + 25$, so $(x - 2)^2 + (y + 5)^2 = 25$: center $(2, -5)$, radius 5.
12. Center $(\dfrac{-3 + 5}{2}, \dfrac{1 + 7}{2}) = (1, 4)$; the diameter is $\sqrt{8^2 + 6^2} = 10$, so $r = 5$ and $(x - 1)^2 + (y - 4)^2 = 25$.
13. Draw chords $\overline{AB}$ and $\overline{BC}$. Construct the perpendicular bisector of each: with one compass width greater than half the chord, draw arcs from both endpoints that cross on both sides, and draw the line through the crossings. The point O where the two bisectors meet is the center, and OA is the radius. Why: the center is equidistant from the endpoints of any chord (radii), so by the Converse of the Perpendicular Bisector Theorem it lies on the perpendicular bisector of every chord. The chords $\overline{AB}$ and $\overline{BC}$ are not parallel, so their bisectors are not parallel and meet in exactly one point, which must be the center (Euclid III.25).
14. No. Suppose trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$ is inscribed in a circle. With transversal $\overline{BD}$, the alternate interior angles $\angle ABD$ and $\angle BDC$ are congruent.

These inscribed angles intercept $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$, so those arcs are congruent, and congruent arcs have congruent chords: $\overline{AD} \cong \overline{BC}$. So the legs of an inscribed trapezoid are always congruent.

15. See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB}$ is a diameter of $\odot O$; $\overleftrightarrow{BD}$ is tangent to $\odot O$ at B; $\overline{AD}$ meets $\odot O$ again at C

Prove: $AB^2 = AC \cdot AD$

	STATEMENTS	REASONS
1	$\overline{AB}$ is a diameter; $\overleftrightarrow{BD}$ is tangent at B; C is on $\odot O$ and on $\overline{AD}$	Given
2	Draw $\overline{BC}$	Through any two points there is exactly one line
3	$\angle ACB$ is a right angle	An angle inscribed in a semicircle is a right angle (III.31)
4	$\overline{AB} \perp \overleftrightarrow{BD}$	Tangent Theorem (the radius $\overline{OB}$ lies along $\overline{AB}$)
5	$\angle ABD$ is a right angle	Definition of perpendicular lines
6	$\angle ACB \cong \angle ABD$	All right angles are congruent
7	$\angle BAC \cong \angle DAB$	Reflexive Property (C is on $\overline{AD}$, so these are the same angle)
8	$\triangle ACB \sim \triangle ABD$	AA Similarity Postulate
9	$\frac{AC}{AB} = \frac{AB}{AD}$	Corresponding sides of similar triangles are proportional
10	$AB^2 = AC \cdot AD$	Cross Products Property

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16. See the model proof.

PARAGRAPH PROOF

Given: In $\odot O$, chords $\overline{AB} \cong \overline{CD}$, neither a diameter; M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$; O, M, N are not collinear

Prove: $\angle OMN \cong \angle ONM$

M is the midpoint of $\overline{AB}$, so $MA = MB$; and $OA = OB$ because all radii are congruent. So M and O are two different points (since $\overline{AB}$ is not a diameter, O is not on it), each equidistant from A and B. By the Converse of the Perpendicular Bisector Theorem, both lie on the perpendicular bisector of $\overline{AB}$, and since two points determine a line, $\overleftrightarrow{OM}$ is that bisector. Thus $\overline{OM} \perp \overline{AB}$, and OM is the distance from the center to $\overline{AB}$. In the same way, $\overline{ON} \perp \overline{CD}$ and ON is the distance from the center to $\overline{CD}$.

Congruent chords are equidistant from the center (Equidistant Chords Theorem), so $OM = ON$. Since O, M, N are not collinear, they form $\triangle OMN$ with $\overline{OM} \cong \overline{ON}$, an isosceles triangle. By the Isosceles Triangle Theorem, the angles opposite these sides are congruent: $\angle ONM \cong \angle OMN$.

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10.1 Circles, Arcs and Central Angles

OBJECTIVES

- Identify the center, radii, diameters, chords, secants and tangents of a circle.
- Find the measures of central angles, minor arcs, major arcs and semicircles, using the Arc Addition Postulate.
- Explain why congruent arcs must lie in the same circle or in congruent circles.
- Construct the center of a given circle with compass and straightedge (Euclid III.1).

WATCH

Components of a Circle

Eddie Woo · 10:30

https://www.youtube.com/watch?v=gWgMXIJ5mKM

> **WHEN TO WATCH**
> Before the Core ideas: the parts of a circle.

Note: Eddie Woo teaches in Australia: he says 'interval' for segment. Arc measure and the Arc Addition Postulate are taught from the Core ideas.

BACKUP VIDEO

Naming Parts of a Circle | Naming Circles, Radii, Diameters, and Chords | Math with Mr. J

Math with Mr. J · 5:28

https://www.youtube.com/watch?v=cnSThy0AdhA

For review: radii, diameters and chords.

KEY TERMS

Circle

The set of all points in a plane that are a given distance (the radius) from a given point (the center). A circle is named by its center: circle O or $\odot O$.

Radius

A segment from the center to a point on the circle, or the length of such a segment. All radii of a circle are congruent; this comes straight from the definition of a circle.

Chord and diameter

A chord is a segment whose endpoints both lie on the circle. A diameter is a chord that passes through the center; its length is twice the radius.

Secant and tangent

A secant is a line that meets a circle in two points. A tangent is a line in the plane of the circle that meets the circle in exactly one point, called the point of tangency.

Central angle

An angle whose vertex is the center of a circle.

Minor arc, major arc, semicircle

A minor arc is the part of the circle inside a central angle; its measure is the measure of the central angle. The major arc is the rest of the circle, with measure 360° minus the minor arc. If the endpoints are the ends of a diameter, each arc is a semicircle and measures 180°.

Congruent circles, concentric circles, congruent arcs

Congruent circles have congruent radii. Concentric circles lie in the same plane and have the same center. Congruent arcs are arcs of the same circle or of congruent circles that have equal measures.

CORE IDEAS

Euclid defines a circle as a plane figure bounded by one line such that all the straight lines drawn to it from one point inside it are equal (Book I, Definition 15). In modern words, a **circle** is the set of all points in a plane at a fixed distance r from a fixed point, the **center**. Notice what the definition hands you for free: every radius has length r, so **all radii of a circle are congruent**. That single fact will be the reason for many steps in this unit, because any two radii together with a chord form an isosceles triangle.

Segments and lines can meet a circle in different ways. A **chord** joins two points of the circle. A **diameter** is a chord through the center, so it is made of two radii and

has length $d = 2r$. A **secant** is a whole line through two points of the circle, so every secant contains a chord. A **tangent** touches the circle at exactly one point. Watch the words: a chord is a segment, a secant is a line.

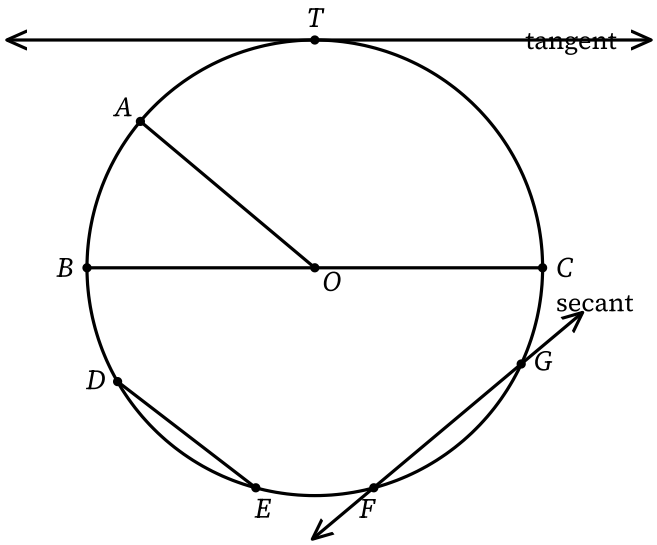
A **central angle** has its vertex at the center. Its sides cut the circle into two pieces called arcs. The arc inside the angle is a **minor arc**, named by its endpoints: $\overset{\frown}{AB}$. We *define* the measure of a minor arc to be the measure of its central angle: $m\overset{\frown}{AB} = m\angle AOB$. Why is this a good definition? One full turn about the center is 360°, so measuring each arc by how much of that turn it fills at the center makes the whole circle 360°, and equal turns give equal arcs.

The rest of the circle is the **major arc**. It measures 360° minus the minor arc, and it is named with three letters, such as $\overset{\frown}{ACB}$, where C is a point on it. Three letters are needed because the two letters A and B alone cannot tell you which of the two arcs is meant. When $\overline{AB}$ is a diameter the central angle is a straight angle, both arcs measure 180°, and each is a **semicircle** (also named with three letters).

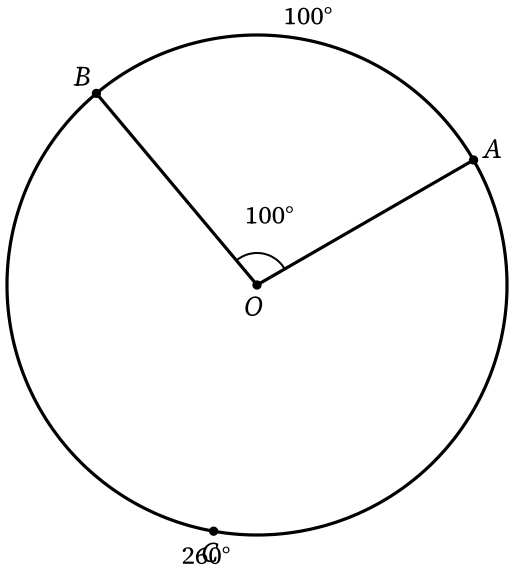
Arcs add the way segments and angles add. If two arcs of the same circle share exactly one endpoint, the arc they make together has a measure equal to their sum: this is the **Arc Addition Postulate**. It works only when the arcs do not overlap, just as the Angle Addition Postulate works only for adjacent angles.

An arc's measure tells how much of a turn it is, not how long it is. A 60° arc of a dinner plate and a 60° arc of a wagon wheel have equal measures but are not the same size. That is why **congruent arcs** must lie in the same circle or in congruent circles. **Concentric circles**, like the rings of an archery target, share a center; arcs between the same two rays from the center have equal measures but different sizes.

If you are handed a circle without its center marked, you can still find the center with compass and straightedge. The key is that the center is equally far from any two points of the circle, so it must lie on the perpendicular bisector of every chord (the Converse of the Perpendicular Bisector Theorem, Lesson 5.1). This is the first proposition of Euclid's Book III.



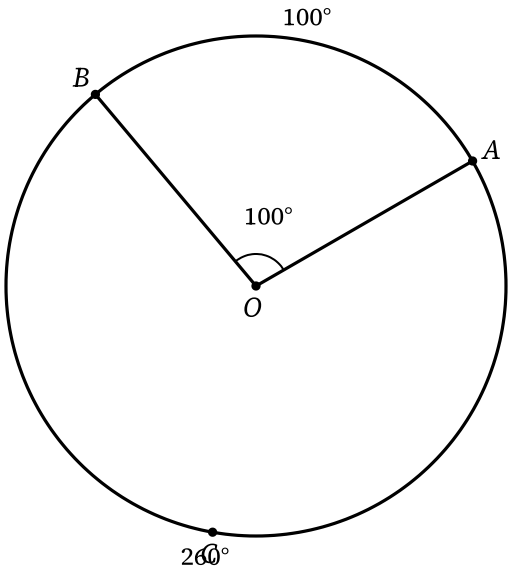
Radius OA, diameter BC, chord DE, secant FG and a tangent at T.



Central angle AOB, minor arc AB and major arc ACB.

POSTULATE **Arc Addition Postulate**

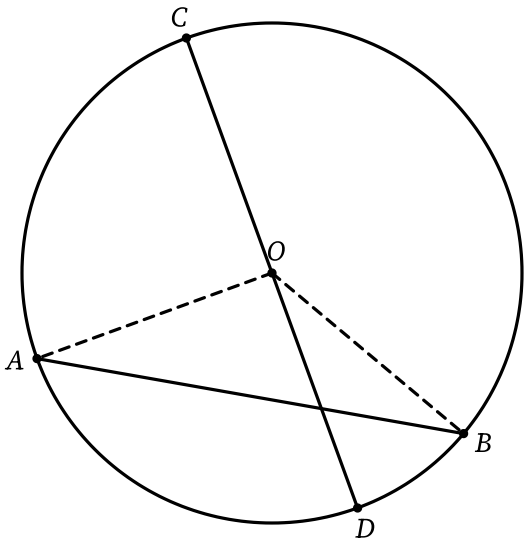
The measure of an arc formed by two adjacent arcs (arcs of the same circle that share exactly one endpoint) is the sum of the measures of the two arcs. If B lies on $\overset{\frown}{ABC}$, then $m\overset{\frown}{ABC} = m\overset{\frown}{AB} + m\overset{\frown}{BC}$.



Central angle AOB, minor arc AB and major arc ACB.

THEOREM **The Diameter Is the Longest Chord** Euclid III.15

A diameter of a circle is longer than any chord of that circle that is not a diameter.



PARAGRAPH PROOF

Given: $\overline{AB}$ is a chord of $\odot O$, which has radius r; $\overline{AB}$ does not pass through O; $\overline{CD}$ is a diameter of $\odot O$

Prove: $AB < CD$

Draw radii $\overline{OA}$ and $\overline{OB}$. Because $\overline{AB}$ does not pass through O, the points A, O and B are not collinear, so they are the vertices of $\triangle AOB$.

By the Triangle Inequality Theorem (Euclid I.20), $AB < OA + OB$. All radii of a circle are congruent, so $OA = OB = r$, and by substitution $AB < r + r = 2r$.

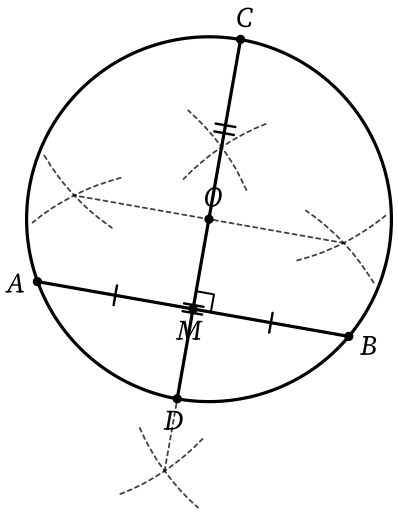
The diameter $\overline{CD}$ passes through the center, so O is between C and D, and by the Segment Addition Postulate $CD = CO + OD = r + r = 2r$. Substituting, $AB < CD$. So every chord that is not a diameter is shorter than a diameter.

∎

Euclid III.15: 'Of straight lines in a circle the diameter is greatest.'

CONSTRUCTIONS

CONSTRUCTION **Find the center of a given circle** Euclid III.1



1. Draw any chord $\overline{AB}$ of the circle.
2. Open the compass to more than half of AB. Draw arcs centered at A and at B, both above and below $\overline{AB}$, without changing the compass width.
3. Draw the line through the two points where the arcs cross. It is the perpendicular bisector of $\overline{AB}$ and crosses $\overline{AB}$ at its midpoint M. Label the points C and D where this line meets the circle.
4. Bisect $\overline{CD}$ the same way. Its midpoint O is the center of the circle.

Why it works. The center is the same distance from A and B, since both distances are radii. By the Converse of the Perpendicular Bisector Theorem, the center lies on the perpendicular bisector of $\overline{AB}$, which is line CD. The center is also the same distance from C and D, and the only point of line CD equidistant from C and D is the midpoint of $\overline{CD}$. So

that midpoint is the center. (Euclid's own proof of III.1 is indirect: he supposes some other point is the center and reaches a contradiction.)

KEY FORMULAS

$$d = 2r$$

$$m\overset{\frown}{AB} = m\angle AOB \quad \text{(minor arc)}$$

$$m\overset{\frown}{ACB} = 360^\circ - m\overset{\frown}{AB} \quad \text{(major arc)}$$

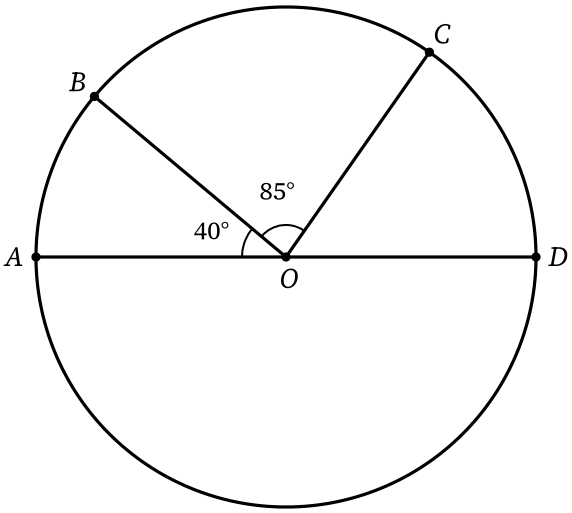
$$m\overset{\frown}{ABC} = m\overset{\frown}{AB} + m\overset{\frown}{BC}$$

From history. Euclid gives all of Book III of the *Elements* to circles: chords, tangents, and the angles in and around a circle, which is the subject of this whole unit. About a century after Euclid, Apollonius of Perga (about 200 BC) wrote the *Conics*, the great ancient work on the curves made by slicing a cone with a plane: the circle, the ellipse, the parabola and the hyperbola. Later writers called him 'the Great Geometer.'

Euclid connection. Definitions 15–17 of Book I define the circle, its center and its diameter. Book III opens with Proposition 1, 'To find the centre of a given circle,' which is the construction in this lesson, and Proposition III.15 proves that the diameter is the greatest straight line in a circle.

Practice habit. Recite the definitions of chord, secant, tangent and arc measure until they are automatic. Always name a major arc or a semicircle with three letters, and whenever you see two radii in a figure, mark them congruent right away.

Example 1. In circle O, $\overline{AD}$ is a diameter, $m\angle AOB = 40^\circ$ and $m\angle BOC = 85^\circ$, with B and C on the same semicircle as in the figure. Find (a) $m\overset{\frown}{CD}$ and (b) $m\overset{\frown}{BAD}$.

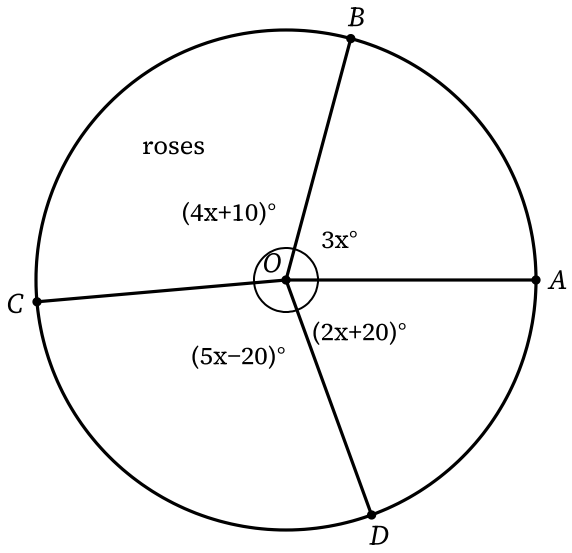


Show solution

1. A minor arc has the same measure as its central angle, so $m\overset{\frown}{AB} = 40^\circ$ and $m\overset{\frown}{BC} = 85^\circ$.
2. Because $\overline{AD}$ is a diameter, the arc from A through B and C to D is a semicircle of 180°. By the Arc Addition Postulate, $40^\circ + 85^\circ + m\overset{\frown}{CD} = 180^\circ$, so $m\overset{\frown}{CD} = 55^\circ$.
3. $\overset{\frown}{BAD}$ starts at B, passes through A, and continues around the other semicircle to D. By the Arc Addition Postulate, $m\overset{\frown}{BAD} = m\overset{\frown}{BA} + 180^\circ = 40^\circ + 180^\circ = 220^\circ$.
4. Check: $\overset{\frown}{BAD}$ and $\overset{\frown}{BCD}$ together make the whole circle, and $220^\circ + (85^\circ + 55^\circ) = 360^\circ$.

Answer: (a) $m\overset{\frown}{CD} = 55^\circ$; (b) $m\overset{\frown}{BAD} = 220^\circ$

Example 2. The Miller family's round garden is divided into four flower beds by straight paths that run from the center to the edge, as in the figure. The central angles of the beds measure $3x^\circ$, $(4x + 10)^\circ$, $(5x - 20)^\circ$ and $(2x + 20)^\circ$. Find x, the measure of the arc of the garden's edge that borders the rose bed (the $(4x + 10)^\circ$ bed), and the measure of the major arc that borders the other three beds.



Show solution

1. The four central angles fill the whole turn around the center without overlapping, so by the Arc Addition Postulate their four arcs make up the whole circle, and their measures add to 360°.
2. $3x + (4x + 10) + (5x - 20) + (2x + 20) = 360$, so $14x + 10 = 360$, $14x = 350$ and $x = 25$.
3. The rose bed's arc measures $4(25) + 10 = 110^\circ$.
4. The major arc measures $360^\circ - 110^\circ = 250^\circ$. Check with the other three beds: $3(25) = 75$, $5(25) - 20 = 105$ and $2(25) + 20 = 70$, and $75^\circ + 105^\circ + 70^\circ = 250^\circ$.

Answer: $x = 25$; the rose bed's arc measures 110°; the major arc measures 250°.

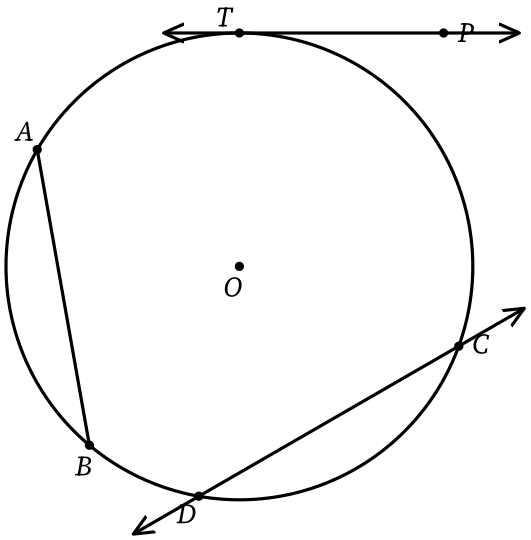
PRACTICE

1. **Easy** A round clock face has a radius of 7.5 inches. What is its diameter?

   Show answer

   15 inches

2. **Easy** In the figure, $\overleftrightarrow{PT}$ meets circle O only at T, and $\overleftrightarrow{CD}$ meets it at C and D. Classify each as a chord, secant or tangent: (a) $\overline{AB}$, (b) $\overleftrightarrow{CD}$, (c) $\overleftrightarrow{PT}$.



   Show answer

   (a) chord; (b) secant; (c) tangent

3. **Easy** In circle P, $m\angle APB = 64°$, and C is a point on the circle outside $\angle APB$. Find $m\overset{\frown}{AB}$ and $m\overset{\frown}{ACB}$.

   Show answer

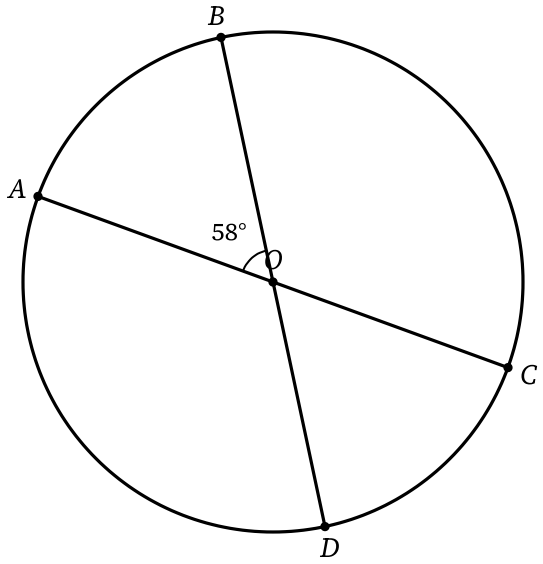
   $m\overset{\frown}{AB} = 64°$ and $m\overset{\frown}{ACB} = 360° - 64° = 296°$

4. **Easy** $\overline{AC}$ is a diameter of circle O, and B is a point on the circle with $m\overset{\frown}{AB} = 48°$. Find the measure of the minor arc $\overset{\frown}{BC}$.

   Show answer

   $\overset{\frown}{ABC}$ is a semicircle, so $m\overset{\frown}{BC} = 180° - 48° = 132°$.

5. **Medium** In circle O, $\overline{AC}$ and $\overline{BD}$ are diameters and $m\angle AOB = 58°$, as in the figure. Find (a) $m\overset{\frown}{BC}$, (b) $m\overset{\frown}{CD}$ and (c) $m\overset{\frown}{BCA}$.



Show answer

(a) $\angle BOC$ and $\angle AOB$ form a linear pair, so $m\overset{\frown}{BC} = 180° - 58° = 122°$. (b) $\angle COD$ and $\angle AOB$ are vertical angles, so $m\overset{\frown}{CD} = 58°$. (c) $\overset{\frown}{BCA}$ is the major arc with endpoints B and A, so it measures $360° - 58° = 302°$.

6. **Medium** In circle O, $\overline{AC}$ is a diameter and B is a point on the circle. The two arcs that make up the semicircle $\overset{\frown}{ABC}$ measure $m\overset{\frown}{AB} = (4x + 6)°$ and $m\overset{\frown}{BC} = (6x - 16)°$. Find x and $m\overset{\frown}{AB}$.

Show answer

$(4x + 6) + (6x - 16) = 180$, so $10x - 10 = 180$ and $x = 19$; $m\overset{\frown}{AB} = 4(19) + 6 = 82°$ (and $m\overset{\frown}{BC} = 98°$).

7. **Medium** The face of a church tower clock is a circle with the twelve numbers equally spaced around its rim. Find the measure of the minor arc of the rim from the 12 to the 5, and the measure of the major arc from the 12 to the 5.

Show answer

Each of the 12 equal arcs measures $360° \div 12 = 30°$. From 12 to 5 there are 5 of them: $150°$. The major arc is $360° - 150° = 210°$.

8. **Medium** A wagon wheel has 16 spokes equally spaced around its hub, each running from the center to the rim. Find the measure of the minor arc of the rim between two spokes that have four other spokes between them.

Show answer

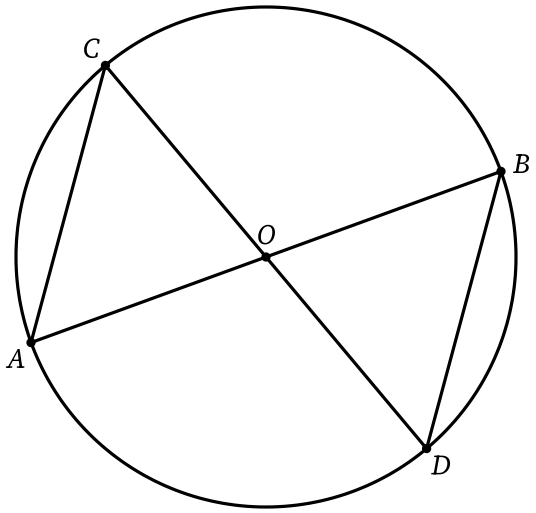
Adjacent spokes cut off arcs of $360^\circ \div 16 = 22.5^\circ$. With four spokes between them, the two spokes are 5 gaps apart: $5 \times 22.5^\circ = 112.5^\circ$.

9. **Medium** Ruth says that a 60° arc of a circle with radius 2 inches is congruent to a 60° arc of a circle with radius 5 inches, because the arcs have equal measures. Is she right? Explain.

Show answer

No. Arc measure tells what fraction of a full turn the arc is, not its size. The arc on the 5-inch circle is much longer than the arc on the 2-inch circle, so the two arcs cannot be made to fit on each other. Arcs with equal measures are congruent only when they lie in the same circle or in congruent circles.

10. **Hard** Given: $\overline{AB}$ and $\overline{CD}$ are diameters of $\odot O$, as in the figure. Prove: $\overline{AC} \cong \overline{BD}$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

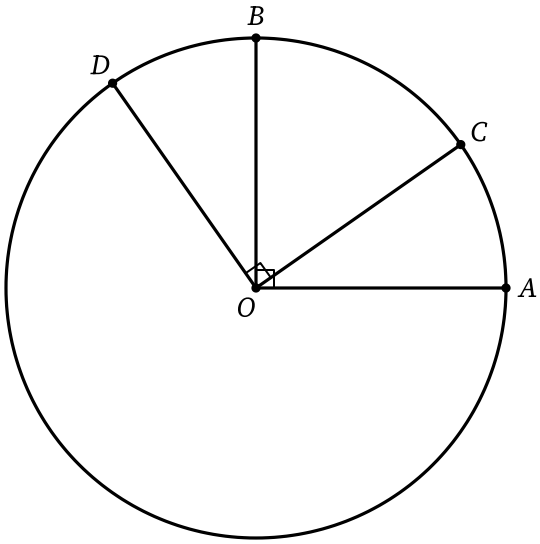
Given: $\overline{AB}$ and $\overline{CD}$ are diameters of $\odot O$

Prove: $\overline{AC} \cong \overline{BD}$

	STATEMENTS	REASONS
1	$\overline{AB}$ and $\overline{CD}$ are diameters of $\odot O$	Given
2	O is between A and B, and O is between C and D	Definition of diameter (a chord through the center)
3	$\overline{OA} \cong \overline{OB}$ and $\overline{OC} \cong \overline{OD}$	All radii of a circle are congruent
4	$\angle AOC \cong \angle BOD$	Vertical Angles Theorem
5	$\triangle AOC \cong \triangle BOD$	SAS
6	$\overline{AC} \cong \overline{BD}$	CPCTC

∎

11. Hard Given: In $\odot O$, $\overline{OA} \perp \overline{OB}$ and $\overline{OC} \perp \overline{OD}$, with C in the interior of $\angle AOB$ and B in the interior of $\angle COD$, as in the figure. Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: In $\odot O$, $\overline{OA} \perp \overline{OB}$ and $\overline{OC} \perp \overline{OD}$; C is in the interior of $\angle AOB$ and B is in the interior of $\angle COD$

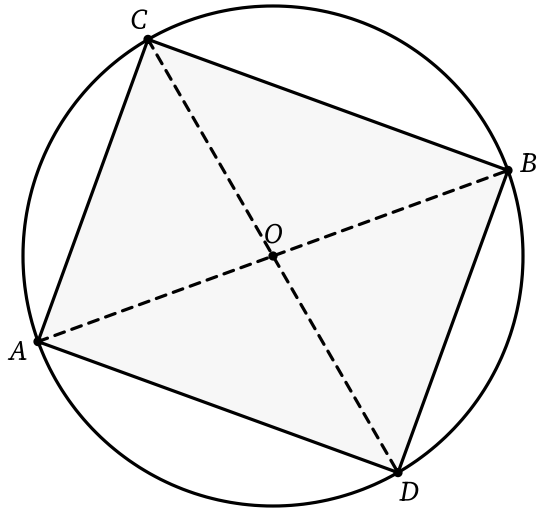
Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$

	STATEMENTS	REASONS
1	$\overline{OA} \perp \overline{OB}$ and $\overline{OC} \perp \overline{OD}$	Given
2	$m\angle AOB = 90°$ and $m\angle COD = 90°$	Definition of perpendicular lines
3	$m\angle AOC + m\angle COB = m\angle AOB$ and $m\angle COB + m\angle BOD = m\angle COD$	Angle Addition Postulate
4	$m\angle AOC + m\angle COB = 90°$ and $m\angle COB + m\angle BOD = 90°$	Substitution Property of Equality
5	$\angle AOC$ and $\angle COB$ are complementary; $\angle COB$ and $\angle BOD$ are complementary	Definition of complementary angles

	STATEMENTS	REASONS
6	$\angle AOC \cong \angle BOD$	Congruent Complements Theorem
7	$m\overset{\frown}{AC} = m\angle AOC$ and $m\overset{\frown}{BD} = m\angle BOD$	Definition of the measure of a minor arc
8	$m\overset{\frown}{AC} = m\overset{\frown}{BD}$	Substitution Property of Equality
9	$\overset{\frown}{AC} \cong \overset{\frown}{BD}$	Definition of congruent arcs (same circle, equal measures)

∎

12. Hard Given: $\overline{AB}$ and $\overline{CD}$ are two different diameters of $\odot O$, as in the figure. Prove: quadrilateral $ACBD$ is a rectangle. Write a paragraph proof. (You may use the tests for parallelograms and rectangles from Unit 6.)



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overline{AB}$ and $\overline{CD}$ are two different diameters of $\odot O$

Prove: $ACBD$ is a rectangle

A diameter is a chord through the center, so O lies on both $\overline{AB}$ and $\overline{CD}$. All radii of a circle are congruent, so $OA = OB$ and $OC = OD$; therefore O is the midpoint of $\overline{AB}$ and also the midpoint of $\overline{CD}$.

The segments $\overline{AB}$ and $\overline{CD}$ are the diagonals of quadrilateral $ACBD$. Since they share the midpoint O, the diagonals bisect each other, so $ACBD$ is a parallelogram (if the diagonals of a quadrilateral bisect each other, it is a parallelogram).

Each diameter has length $2r$, where r is the radius, so $AB = CD$: the diagonals of parallelogram $ACBD$ are congruent. A parallelogram with congruent diagonals is a rectangle. Therefore $ACBD$ is a rectangle.

∎

13. **Challenge** Imagine both hands of a clock extended to the rim. At exactly 3:40, what is the measure of the minor arc of the rim between the hour hand and the minute hand? (Careful: the hour hand moves too.)

Show answer

The minute hand moves $360^\circ \div 60 = 6^\circ$ per minute, so it is $40 \times 6^\circ = 240^\circ$ past the 12. The hour hand moves 30° per hour, so it is $3 \times 30^\circ + \frac{40}{60} \times 30^\circ = 90^\circ + 20^\circ = 110^\circ$ past the 12. The arc between them measures $240^\circ - 110^\circ = 130^\circ$, which is less than 180°, so it is the minor arc: 130°.

COMMON MISTAKES

- Naming a major arc with two letters. $\overset{\frown}{AB}$ always means the minor arc; for the major arc or a semicircle, put a third point of the arc in the middle, as in $\overset{\frown}{ACB}$.
- Mixing up chord and secant. A chord is a segment with both endpoints on the circle; a secant is a whole line through two points of the circle. Every secant contains a chord.
- Calling arcs with equal measures congruent when they lie in circles of different sizes. First check that the arcs are in the same circle or in circles with equal radii.
- Adding arcs that overlap. The Arc Addition Postulate is for arcs that share only an endpoint; if you add overlapping arcs, the overlap is counted twice. Sketch the arcs and check that together they make one arc.

> **Make it a game · Plate Draw.** Trace a plate on paper. Parent calls a part and you draw it in a different color within 5 seconds: radius, diameter, chord, secant, tangent, central angle, minor arc, major arc. Then parent gives a central angle of 70 degrees; you state the minor arc, and the major arc as 360 minus 70. Aim for no errors.

10.2 Chords

OBJECTIVES

- Prove and use the relationships among congruent central angles, chords and arcs.
- Prove that a diameter perpendicular to a chord bisects the chord and its arc, and that the perpendicular bisector of a chord passes through the center.
- Use the right triangle formed by a radius, half a chord and the distance to the center to find lengths.
- Find the radius of a wheel or an arch from a chord and its height, and construct the center of an arc.

WATCH

Chord Properties (Circle Geometry)

Eddie Woo · 13:21

https://www.youtube.com/watch?v=twUwM_In_GU

> **WHEN TO WATCH**
> After the chord theorems: the chord properties and why they are true.

Note: Eddie Woo teaches in Australia: he spells it 'centre'.

BACKUP VIDEO

Finding the Centre of a Circle Passing Through 3 Points

Eddie Woo · 3:57

https://www.youtube.com/watch?v=eDwHrzrbKrg

Before the broken-wheel construction: finding the centre of a circle through three points.

KEY TERMS

Arc of a chord

The minor arc whose endpoints are the endpoints of the chord. The chord and its arc are said to subtend (stand on) each other, and they share the same central angle.

Distance from the center to a chord

The length of the perpendicular segment from the center to the chord.

Perpendicular bisector of a chord

The line that is perpendicular to the chord at its midpoint.

Rise (height) of an arc

The distance from the midpoint of a chord to the arc, measured along the perpendicular bisector of the chord. Builders call it the rise of an arch.

CORE IDEAS

Every chord $\overline{AB}$ that is not a diameter has a central angle $\angle AOB$ and a minor arc $\overset{\frown}{AB}$. The two radii $\overline{OA}$ and $\overline{OB}$ together with the chord form an isosceles triangle, and that triangle is the key to everything in this lesson.

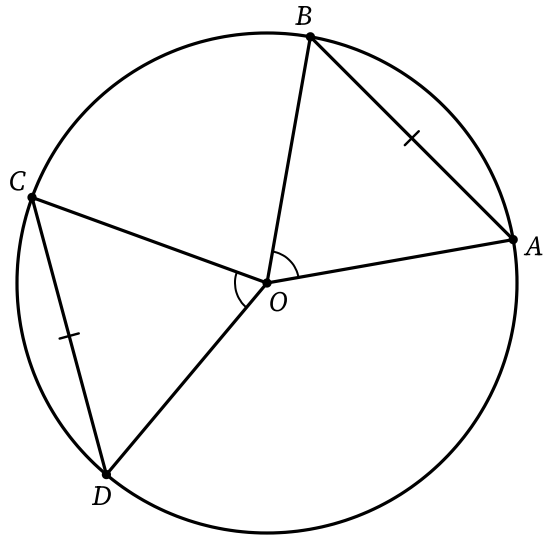
Why do congruent central angles give congruent chords? If $\angle AOB \cong \angle COD$ in the same circle, triangles AOB and COD have two sides that are radii and the included angle congruent, so they are congruent by SAS and the chords match. Going the other way, congruent chords make the triangles congruent by SSS, so the central angles match. Since an arc is measured by its central angle, central angles, chords and arcs are congruent together or not at all, as long as we stay in one circle or in congruent circles.

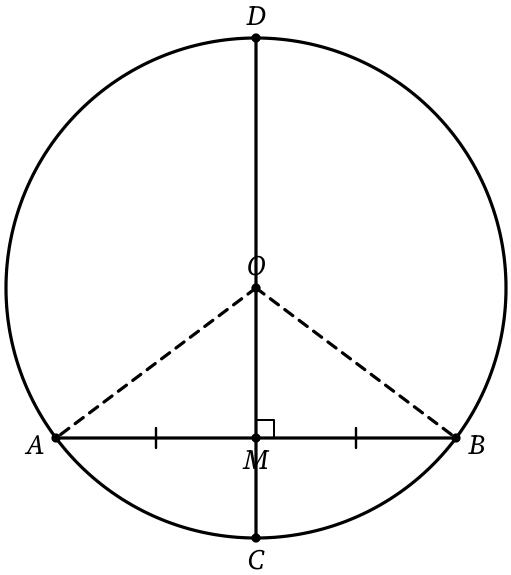
Now draw a line through the center perpendicular to a chord. It cuts the isosceles triangle into two right triangles with the same hypotenuse length (a radius) and a shared leg, so they are congruent by HL. Therefore the line **bisects the chord and its arc**. Read the other way, the **perpendicular bisector of any chord passes through the center**, because the center is equidistant from the chord's endpoints. That is how you find the center of an arc.

The **distance** from the center to a chord means the length of the perpendicular segment (Lesson 3.4). The half-chord, that distance and a radius form a right triangle, so if a chord of length c is at distance d from the center of a circle of radius r, then $r^2 = (\frac{c}{2})^2 + d^2$. For example, in a circle of radius 10, a chord of length 16 is $\sqrt{10^2 - 8^2} = 6$ units from the center.

Because one right triangle ties the chord, the distance and the radius together, **chords at equal distances from the center are congruent, and congruent chords are at equal distances**. Solving for the chord, $c = 2\sqrt{r^2 - d^2}$, which gets smaller as d gets larger: chords nearer the center are longer, and the diameter, at distance 0, is the longest of all.

A practical use: a piece of a broken wheel or the curve of a stone arch is an arc. Measure a chord c across it and the rise h at its middle. The center lies on the perpendicular bisector of the chord, $r - h$ beyond the chord's midpoint, so $(r - h)^2 + (\frac{c}{2})^2 = r^2$. Expanding, the r^2 terms cancel, leaving a simple linear equation for r.



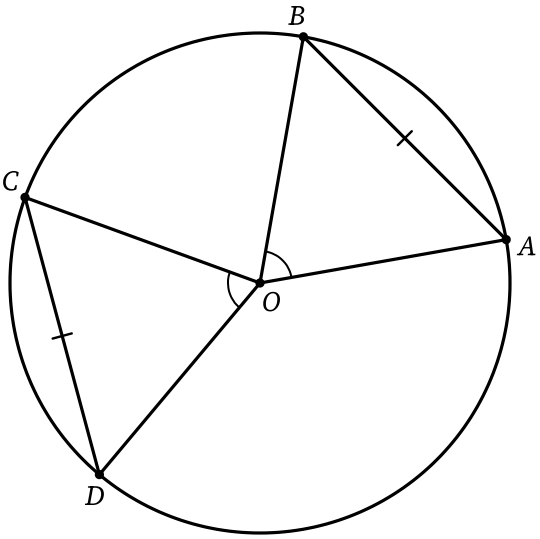


POSTULATES AND THEOREMS

THEOREM **Congruent Central Angles, Chords and Arcs Theorem**

Euclid III.26–III.29

In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent, if and only if their chords are congruent.



PARAGRAPH PROOF

Given: $\overline{AB}$ and $\overline{CD}$ are chords of $\odot O$ that are not diameters, with central angles $\angle AOB$ and $\angle COD$

Prove: $\angle AOB \cong \angle COD$ exactly when $\overline{AB} \cong \overline{CD}$, exactly when $\overset{\frown}{AB} \cong \overset{\frown}{CD}$
All radii of a circle are congruent, so $\overline{OA} \cong \overline{OC}$ and $\overline{OB} \cong \overline{OD}$.

Suppose $\angle AOB \cong \angle COD$. Then $\triangle AOB \cong \triangle COD$ by SAS, so $\overline{AB} \cong \overline{CD}$ by CPCTC. Conversely, suppose $\overline{AB} \cong \overline{CD}$. Then $\triangle AOB \cong \triangle COD$ by SSS, so $\angle AOB \cong \angle COD$ by CPCTC.

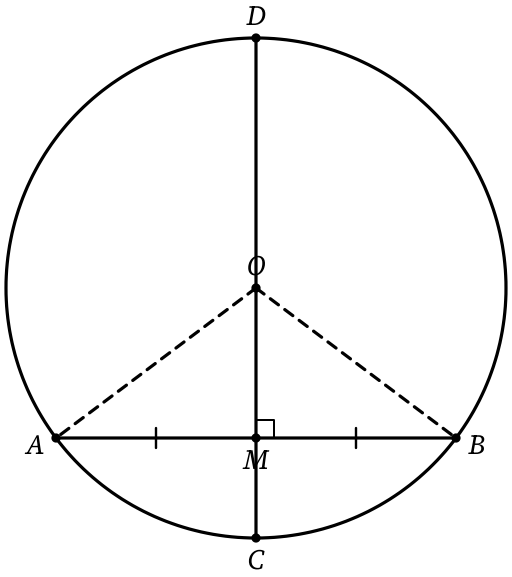
Finally, the measure of a minor arc is defined to be the measure of its central angle, so $m\overset{\frown}{AB} = m\overset{\frown}{CD}$ exactly when $m\angle AOB = m\angle COD$. The arcs lie in the same circle, so they are congruent exactly when their central angles are. In two congruent circles every radius has the same length, so the same argument works word for word.

∎

If both chords are diameters, both central angles are straight angles and both arcs are semicircles, so the statement is immediate.

THEOREM **Perpendicular Diameter Theorem** Euclid III.3

If a diameter of a circle (or any line through the center) is perpendicular to a chord, then it bisects the chord and its arc.



TWO-COLUMN PROOF

Given: $\overline{CD}$ is a diameter of $\odot O$; $\overline{CD} \perp \overline{AB}$ at M, with M between O and C
Prove: $\overline{AM} \cong \overline{BM}$ and $\overset{\frown}{AC} \cong \overset{\frown}{BC}$

	STATEMENTS	REASONS
1	$\overline{CD} \perp \overline{AB}$ at M	Given

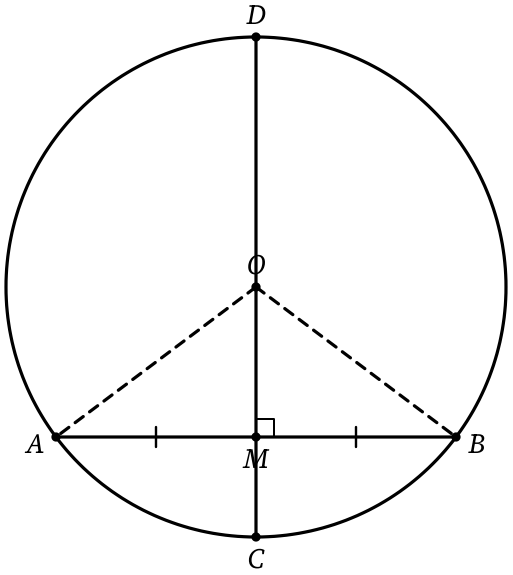
	STATEMENTS	REASONS
2	$\angle OMA$ and $\angle OMB$ are right angles	Definition of perpendicular lines
3	$\overline{OA} \cong \overline{OB}$	All radii of a circle are congruent
4	$\overline{OM} \cong \overline{OM}$	Reflexive Property of Congruence
5	$\triangle OMA \cong \triangle OMB$	HL
6	$\overline{AM} \cong \overline{BM}$	CPCTC
7	$\angle AOM \cong \angle BOM$	CPCTC
8	$\angle AOC \cong \angle BOC$	Step 7, since $\overrightarrow{OM}$ and $\overrightarrow{OC}$ are the same ray
9	$\overset{\frown}{AC} \cong \overset{\frown}{BC}$	Congruent Central Angles, Chords and Arcs Theorem
10	$\overline{AM} \cong \overline{BM}$ and $\overset{\frown}{AC} \cong \overset{\frown}{BC}$	Steps 6 and 9

∎

If $\overline{AB}$ is itself a diameter, then $M = O$, $AM = BM$ because both are radii, and two perpendicular diameters cut the circle into four $90°$ arcs. Euclid III.3 also proves the converse: a line through the center that bisects a chord which is not a diameter is perpendicular to it.

COROLLARY **The Perpendicular Bisector of a Chord Contains the Center** Euclid III.1

The perpendicular bisector of a chord of a circle passes through the center of the circle.



PARAGRAPH PROOF

Given: ℓ is the perpendicular bisector of chord $\overline{AB}$ of $\odot O$

Prove: O lies on ℓ

All radii of a circle are congruent, so $OA = OB$: the center is equidistant from the endpoints of the chord. By the Converse of the Perpendicular Bisector Theorem (Lesson 5.1), a point equidistant from A and B lies on the perpendicular bisector of $\overline{AB}$. Therefore O lies on ℓ.

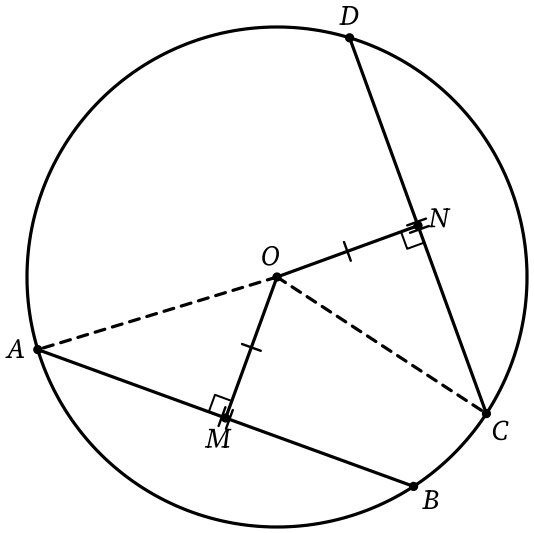
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Euclid states this as a corollary (porism) of III.1.

THEOREM **Equidistant Chords**

Theorem Euclid III.14

In the same circle or in congruent circles, two chords are congruent if and only if they are equidistant from the center.



FLOW PROOF

Given: In $\odot O$, $\overline{OM} \perp \overline{AB}$ at M and $\overline{ON} \perp \overline{CD}$ at N; $\overline{OM} \cong \overline{ON}$

Prove: $\overline{AB} \cong \overline{CD}$

(1) $\overline{OM} \perp \overline{AB}$ and $\overline{ON} \perp \overline{CD}$
Given

(2) $\overline{OM} \cong \overline{ON}$
Given

(3) $\angle OMA$ and $\angle ONC$ are right angles
Definition of perpendicular lines
from (1)

(4) $\overline{OA} \cong \overline{OC}$

All radii of a circle are congruent

(5) $\triangle OMA \cong \triangle ONC$
HL
from (2), (3), (4)

(6) $AM = CN$
CPCTC
from (5)

(7) M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$
Perpendicular Diameter Theorem
from (1)

(8) $AB = 2 \cdot AM$ and $CD = 2 \cdot CN$
Definition of midpoint and Segment Addition Postulate
from (7)

(9) $AB = CD$
Substitution Property of Equality
from (6), (8)

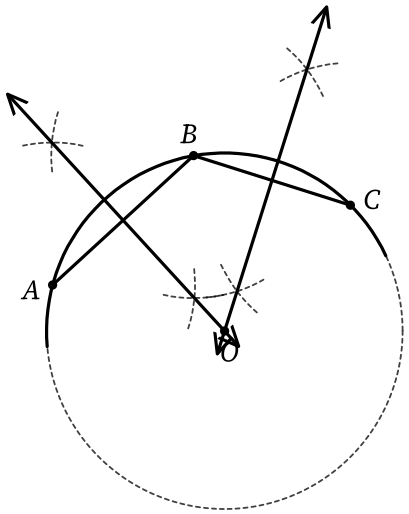
(10) $\overline{AB} \cong \overline{CD}$
Definition of congruent segments
from (9)

∎

This proves the 'if' direction. The converse (congruent chords are equidistant from the center) is Practice Problem 11. Euclid III.14: 'In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.'

CONSTRUCTION **Complete a circle from an arc (restore a broken wheel)** Euclid III.25

1. Mark three points A, B and C on the arc, spread well apart.
2. Draw the chords $\overline{AB}$ and $\overline{BC}$.
3. Construct the perpendicular bisector of $\overline{AB}$: with one compass width (more than half of AB), draw arcs centered at A and at B that cross on both sides of the chord, and draw the line through the crossings. Do the same for $\overline{BC}$.
4. Label the point where the two perpendicular bisectors cross O. Set the compass to OA and draw the whole circle.



Why it works. The perpendicular bisector of a chord passes through the center, so the center lies on both bisectors. The chords $\overline{AB}$ and $\overline{BC}$ meet at B and do not lie on one line, so they are not parallel; then their perpendicular bisectors are not parallel either (lines perpendicular to two parallel lines would themselves be parallel), and they cross at exactly one point. That point is the center, and $OA = OB = OC$ is the radius.

KEY FORMULAS

$$r^2 = (\frac{c}{2})^2 + d^2$$

$$c = 2\sqrt{r^2 - d^2}$$

$$r = \frac{(c/2)^2 + h^2}{2h} \quad (\text{chord } c,\ \text{rise } h)$$

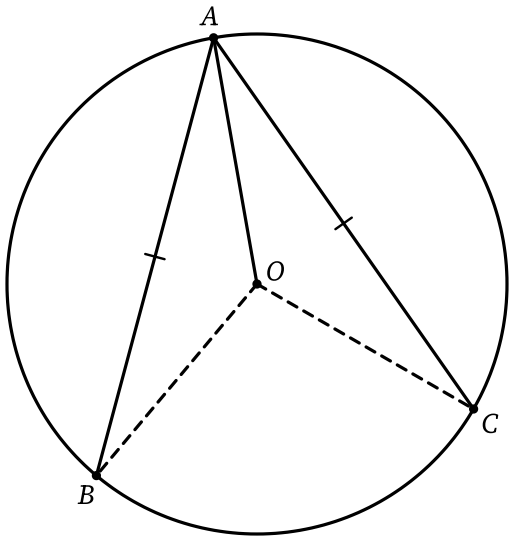
From history. Wheelwrights, coopers and stonemasons of earlier centuries used these facts every day. A mason replacing a damaged stone in a round arch, or a wheelwright rebuilding a broken wheel, could recover the whole circle from one surviving piece by bisecting two chords, which is exactly the method Euclid gives in Book III, Proposition 25.

Euclid connection. Euclid III.3: 'If in a circle a straight line through the centre bisects a straight line not through the centre, it also cuts it at right angles; and if it cuts it at right angles, it also bisects it.' III.14 is the Equidistant Chords Theorem, III.26–III.29 connect equal angles, arcs and chords in equal circles, and III.25 completes a circle from a piece of it.

Practice habit. Whenever a problem shows a chord and the center, draw the perpendicular from the center to the chord and a radius to one endpoint. Label the right triangle with half the chord, the distance and the radius before you compute anything.

WORKED EXAMPLES

Example 1. Given: In $\odot O$, chords $\overline{AB} \cong \overline{AC}$, as in the figure. Prove: $\overrightarrow{AO}$ bisects $\angle BAC$.



Show solution

1. Mark the given congruent chords. Draw the radii $\overline{OB}$ and $\overline{OC}$; they are congruent because all radii are.
2. Look for two triangles that contain the angles $\angle BAO$ and $\angle CAO$: triangles ABO and ACO. They share the side $\overline{AO}$.
3. Three pairs of congruent sides give SSS; then CPCTC gives the congruent angles.

TWO-COLUMN PROOF

Given: In $\odot O$, $\overline{AB} \cong \overline{AC}$

Prove: $\overrightarrow{AO}$ bisects $\angle BAC$

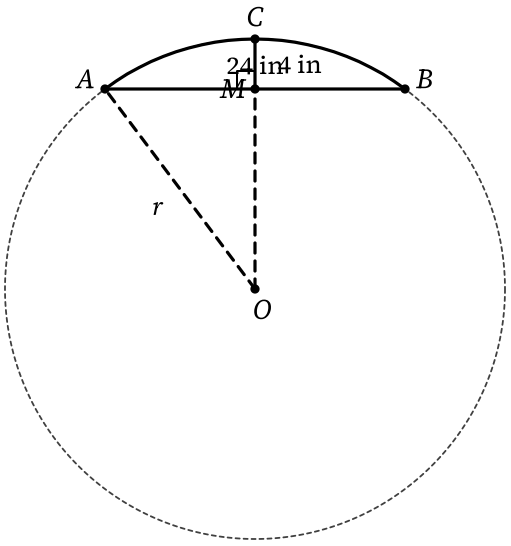
	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$\overline{OB} \cong \overline{OC}$	All radii of a circle are congruent
3	$\overline{AO} \cong \overline{AO}$	Reflexive Property of Congruence

	STATEMENTS	REASONS
4	$\triangle ABO \cong \triangle ACO$	SSS
5	$\angle BAO \cong \angle CAO$	CPCTC
6	$\overrightarrow{AO}$ bisects $\angle BAC$	Definition of angle bisector

∎

Answer: See the proof.

Example 2. Henry finds a piece of a broken wagon wheel and wants to build a new wheel of the same size. He measures a chord across the piece, $AB = 24$ inches. At the middle of the chord, the rim rises $MC = 4$ inches above the chord, measured along the perpendicular bisector of the chord. What is the radius of the wheel?



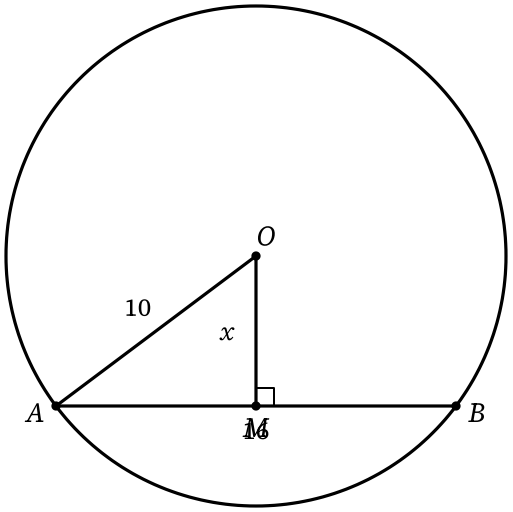
Show solution

1. The perpendicular bisector of a chord passes through the center, so the center O lies on line MC, $\overline{OM} \perp \overline{AB}$, and $AM = 24 \div 2 = 12$ inches.
2. Let r be the radius. Then $OA = OC = r$, so $OM = r - 4$.
3. In right triangle OMA, the Pythagorean Theorem gives $(r-4)^2 + 12^2 = r^2$. Expanding, $r^2 - 8r + 16 + 144 = r^2$, so $-8r + 160 = 0$ and $r = 20$.
4. Check: $OM = 20 - 4 = 16$, and $16^2 + 12^2 = 256 + 144 = 400 = 20^2$. The new wheel needs a radius of 20 inches, a diameter of 40 inches.

Answer: The radius is 20 inches (diameter 40 inches).

PRACTICE

1. **Easy** In circle O in the figure, the radius is $OA = 10$, chord $AB = 16$, and $\overline{OM} \perp \overline{AB}$ at M. Find OM, the distance from the center to the chord.



Show answer

OM bisects the chord, so $AM = 8$ and $OM = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$.

2. **Easy** In $\odot O$, chords $\overline{AB} \cong \overline{CD}$ and $m\angle AOB = 72°$. Find $m\overset{\frown}{CD}$.

Show answer

Congruent chords have congruent central angles and arcs, so $m\overset{\frown}{CD} = 72°$.

3. **Easy** In $\odot O$, diameter $\overline{CD}$ is perpendicular to chord $\overline{AB}$ at M, and $AB = 18$ cm. Find AM.

Show answer

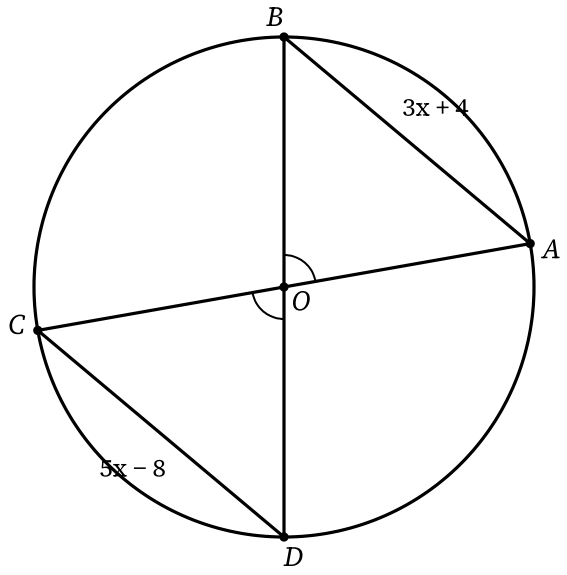
A diameter perpendicular to a chord bisects it, so $AM = 9$ cm.

4. **Easy** A chord of length 10 is 12 units from the center of a circle. Find the radius of the circle.

Show answer

$r = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

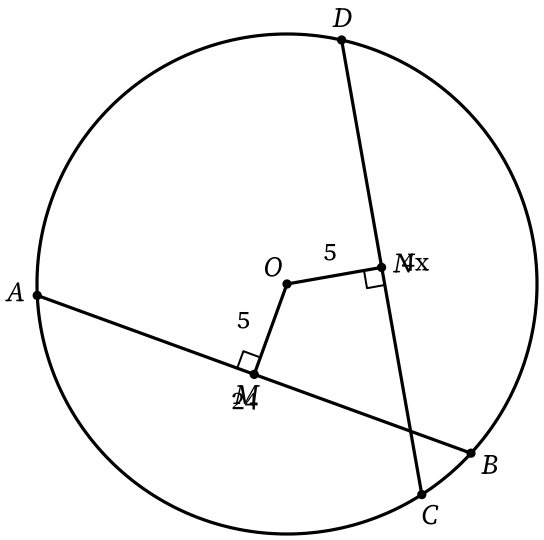
5. **Medium** In $\odot O$ in the figure, $\angle AOB \cong \angle COD$, $AB = 3x + 4$ and $CD = 5x - 8$. Find x and AB.



Show answer

Congruent central angles have congruent chords: $3x + 4 = 5x - 8$, so $x = 6$ and $AB = 22$.

6. **Medium** In $\odot O$ in the figure, chords $\overline{AB}$ and $\overline{CD}$ are each 5 cm from the center: $\overline{OM} \perp \overline{AB}$, $\overline{ON} \perp \overline{CD}$ and $OM = ON = 5$. If $AB = 24$ cm and $CD = 4x$ cm, find x and the radius of the circle.



Show answer

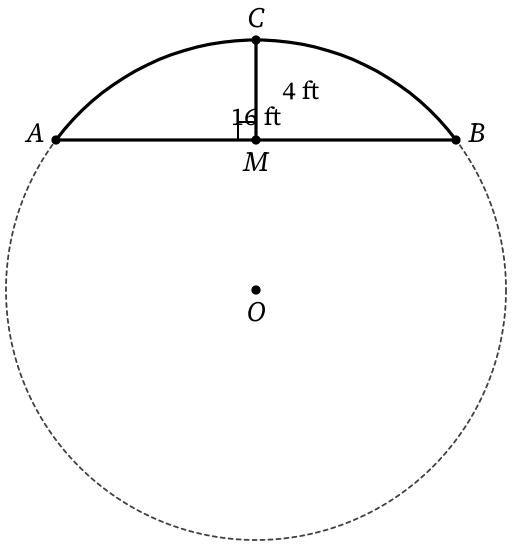
Chords equidistant from the center are congruent: $4x = 24$, so $x = 6$. $\overline{OM}$ bisects $\overline{AB}$, so $AM = 12$ and $r = \sqrt{12^2 + 5^2} = 13$ cm.

7. **Medium** In $\odot P$, chords $\overline{AB} \cong \overline{BC}$. The minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ share only the point B, with $m\overset{\frown}{AB} = (2x + 20)^\circ$ and $m\overset{\frown}{BC} = (5x - 16)^\circ$. Find $m\overset{\frown}{ABC}$.

Show answer

Congruent chords have congruent arcs: $2x + 20 = 5x - 16$, so $x = 12$ and each arc measures 44°. By Arc Addition, $m\overset{\frown}{ABC} = 88^\circ$.

8. **Medium** The top of a stone church doorway is a circular arch, as in the figure. The opening is $AB = 16$ feet wide, and the arch rises $MC = 4$ feet above the middle of $\overline{AB}$. What is the radius of the circle the mason used?



Show answer

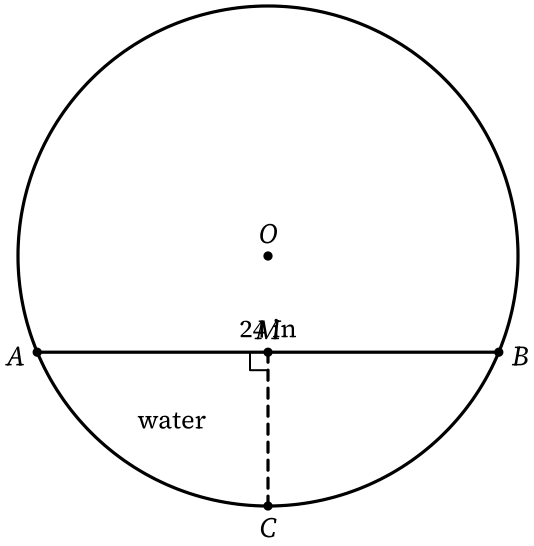
With $OM = r - 4$ and $AM = 8$: $(r-4)^2 + 8^2 = r^2$, so $-8r + 80 = 0$ and $r = 10$ feet.

9. **Medium** Samuel says that any line that bisects a chord of a circle must pass through the center. Is he right? Explain.

Show answer

No. Every line through the midpoint of a chord bisects it, and there are infinitely many such lines. What is true is that the *perpendicular* bisector passes through the center, because the center is equidistant from the chord's endpoints (Converse of the Perpendicular Bisector Theorem). A bisector that is not perpendicular generally misses the center.

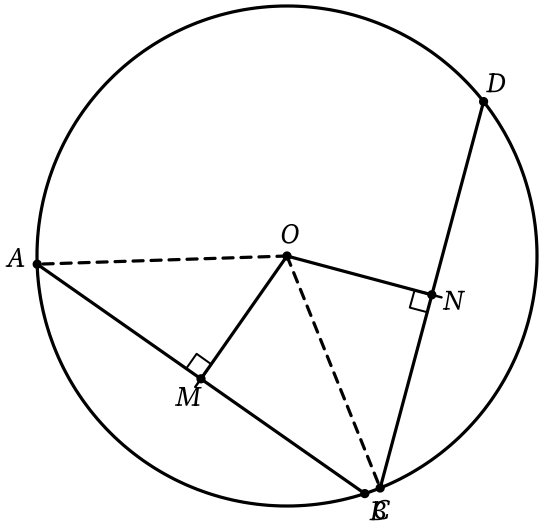
10. Hard A round drainage culvert under a farm road has an inside diameter of 26 inches. After a rain, the water surface across it is $AB = 24$ inches wide, and the culvert is less than half full, as in the figure. How deep is the water at its deepest point C?



Show answer

The radius is 13 in, and the perpendicular from the center bisects the surface chord, so the center is $\sqrt{13^2 - 12^2} = 5$ in above the surface. Less than half full means the surface is below the center, so the depth is $CM = 13 - 5 = 8$ inches.

11. **Hard** Prove the converse half of the Equidistant Chords Theorem. Given: In $\odot O$, $\overline{AB} \cong \overline{CD}$, $\overline{OM} \perp \overline{AB}$ at M and $\overline{ON} \perp \overline{CD}$ at N, as in the figure. Prove: $\overline{OM} \cong \overline{ON}$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: In $\odot O$, $\overline{AB} \cong \overline{CD}$; $\overline{OM} \perp \overline{AB}$ at M and $\overline{ON} \perp \overline{CD}$ at N

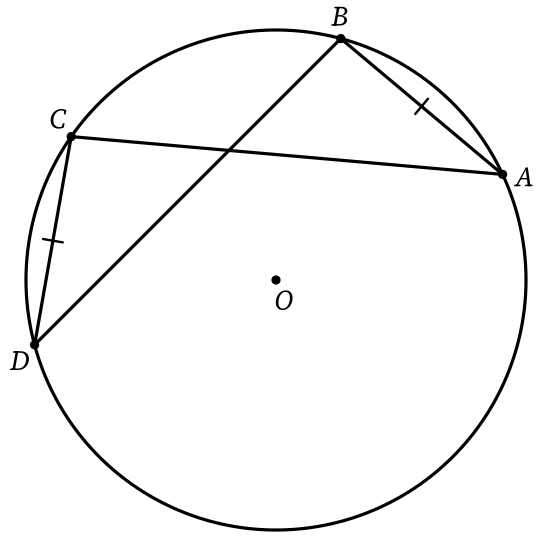
Prove: $\overline{OM} \cong \overline{ON}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{CD}$; $\overline{OM} \perp \overline{AB}$ and $\overline{ON} \perp \overline{CD}$	Given
2	M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$	Perpendicular Diameter Theorem
3	$AM = \frac{1}{2}AB$ and $CN = \frac{1}{2}CD$	Definition of midpoint
4	$AB = CD$	Definition of congruent segments
5	$AM = CN$	Substitution Property of Equality
6	$\angle OMA$ and $\angle ONC$ are right angles	Definition of perpendicular lines

	STATEMENTS	REASONS
7	$\overline{OA} \cong \overline{OC}$	All radii of a circle are congruent
8	$\triangle OMA \cong \triangle ONC$	HL
9	$\overline{OM} \cong \overline{ON}$	CPCTC

∎

12. Hard Given: Points A, B, C and D lie in that order on a minor arc of $\odot O$ (an arc less than a semicircle), and chords $\overline{AB} \cong \overline{CD}$, as in the figure. Prove: $\overline{AC} \cong \overline{BD}$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

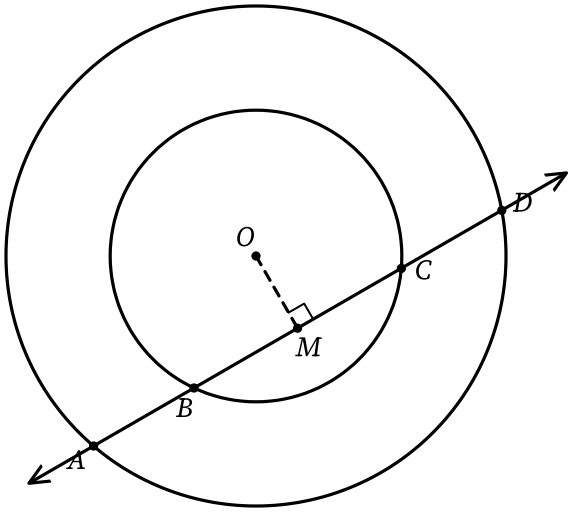
Given: A, B, C, D lie in that order on a minor arc of $\odot O$; $\overline{AB} \cong \overline{CD}$
Prove: $\overline{AC} \cong \overline{BD}$
Because the chords $\overline{AB}$ and $\overline{CD}$ are congruent, their minor arcs are congruent by the Congruent Central Angles, Chords and Arcs Theorem, so $m\overset{\frown}{AB} = m\overset{\frown}{CD}$.

All four points lie in order on an arc smaller than a semicircle, so $A\overset{\frown}{BC}$ and $B\overset{\frown}{CD}$ are minor arcs. By the Arc Addition Postulate, $mA\overset{\frown}{BC} = m\overset{\frown}{AB} + mB\overset{\frown}{C}$ and $mB\overset{\frown}{CD} = mB\overset{\frown}{C} + m\overset{\frown}{CD}$. Substituting $m\overset{\frown}{CD} = m\overset{\frown}{AB}$ shows $mA\overset{\frown}{BC} = mB\overset{\frown}{CD}$.

So the minor arc of chord $\overline{AC}$ is congruent to the minor arc of chord $\overline{BD}$, and by the same theorem the chords are congruent: $\overline{AC} \cong \overline{BD}$.

∎

13. Hard Given: Two concentric circles have center O. A line that does not pass through O meets the larger circle at A and D and the smaller circle at B and C, in the order A, B, C, D, as in the figure. Prove: $\overline{AB} \cong \overline{CD}$. Write a two-column proof. (Hint: draw the perpendicular from O to the line.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Two concentric circles with center O; A line not through O meets the larger circle at A and D and the smaller circle at B and C, in the order A, B, C, D
Prove: $\overline{AB} \cong \overline{CD}$

	STATEMENTS	REASONS
1	The line meets the larger circle at A and D and the smaller circle at B and C	Given
2	Draw $\overline{OM} \perp \overline{AD}$ with M on $\overline{AD}$	Construction of a perpendicular from a point to a line (Euclid I.12)
3	$AM = MD$	Perpendicular Diameter Theorem (chord $\overline{AD}$ of the larger circle)
4	$BM = MC$	Perpendicular Diameter Theorem (chord $\overline{BC}$ of the

	STATEMENTS	REASONS
		smaller circle)
5	$AM = AB + BM$ and $MD = MC + CD$	Segment Addition Postulate
6	$AB + BM = MC + CD$	Substitution Property of Equality (steps 3 and 5)
7	$AB + BM = BM + CD$	Substitution Property of Equality (step 4)
8	$AB = CD$	Subtraction Property of Equality
9	$\overline{AB} \cong \overline{CD}$	Definition of congruent segments

∎

14. **Challenge** In a circle of radius 25 cm, two parallel chords have lengths 48 cm and 14 cm, and they lie on opposite sides of the center. How far apart are the chords?

Show answer

The 48-cm chord is $\sqrt{25^2 - 24^2} = 7$ cm from the center and the 14-cm chord is $\sqrt{25^2 - 7^2} = 24$ cm from the center. Both distances lie along the same line through the center, perpendicular to both chords, so on opposite sides the chords are $7 + 24 = 31$ cm apart.

COMMON MISTAKES

- Using the whole chord instead of half of it in the right triangle. The perpendicular from the center bisects the chord, so the leg of the right triangle is $\frac{c}{2}$, not c.
- Measuring the distance from the center to a chord along a slanted segment. Distance always means the perpendicular segment; mark the right angle.
- Assuming that any line through the midpoint of a chord passes through the center. Only the perpendicular bisector must; check for the right angle.
- Using the chord theorems for circles of different sizes. Congruent chords give congruent arcs only in the same circle or in congruent circles.

> **Make it a game · Lost Center.** Parent traces a plate circle on paper and marks the center on the back. You must find the center without it: draw two chords, then fold each so its endpoints meet to make the perpendicular bisector. Where the folds cross is the center. Closest to the secret mark wins. Explain why the method works.

10.3 Tangents

OBJECTIVES

- Prove that a tangent is perpendicular to the radius at the point of tangency, and prove the converse.
- Prove and use the fact that two tangent segments from an outside point are congruent.
- Solve problems with tangents, circumscribed polygons and belts around pulleys using right triangles.
- Construct a tangent to a circle from an outside point (Euclid III.17).

WATCH

Tangent Properties - Circle Geometry (Tangent is perpendicular to radius)

Eddie Woo · 9:21

https://www.youtube.com/watch?v=ZGs8hGAcAYQ

> **WHEN TO WATCH**
> After the Tangent Theorem: why a tangent is perpendicular to the radius.

Note: Eddie Woo teaches in Australia: he says 'external point' for outside point.

BACKUP VIDEO

Tangent Properties - Circle Geometry (Tangents from an external point are equal)

Eddie Woo · 9:21

https://www.youtube.com/watch?v=l-YLSXA3NLw

Before the Tangent Segments Theorem: tangents from an outside point are equal.

KEY TERMS

Point of tangency

The single point where a tangent line meets a circle.

Tangent segment

A segment of a tangent line that has one endpoint at the point of tangency, such as $\overline{PT}$ from an outside point P to the point of tangency T.

Circumscribed polygon

A polygon whose sides are all tangent to a circle. The circle is then inscribed in the polygon.

Common tangent

A line that is tangent to two circles, such as the straight part of a belt running around two pulleys.

CORE IDEAS

A **tangent** line touches a circle at exactly one point, the **point of tangency**. A **tangent segment** runs along a tangent line from an outside point to the point of tangency. A belt on a pulley, a rope pulled tight around a post, and a line of sight that just grazes a round silo all follow tangents.

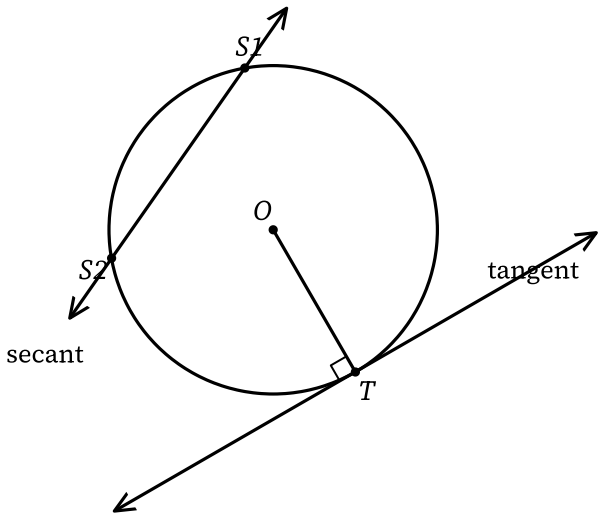
Here is the central fact: **a tangent is perpendicular to the radius drawn to the point of tangency**. Why must this be? Suppose the tangent at T were tilted, not perpendicular to $\overline{OT}$. Then the perpendicular from O would land at some other point F of the line, and the point G just as far beyond F as T is on the near side would also be at distance r from O, giving a second point of the line on the circle. A tangent has only one such point, so the tilt is impossible. This is an indirect proof, and it is Euclid's method in III.18.

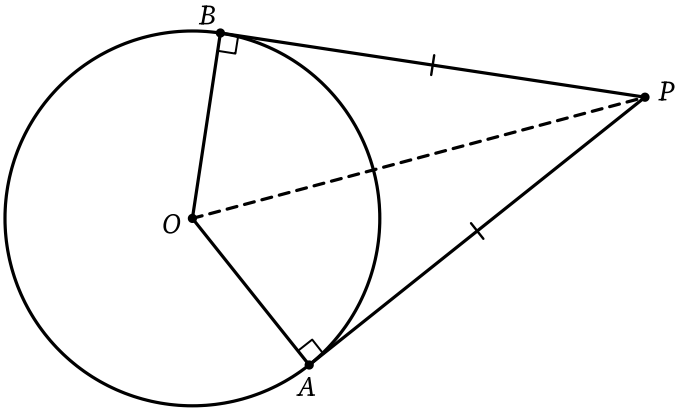
The converse is true too: a line perpendicular to a radius at its outer endpoint is a tangent. Every other point P of that line is the end of the hypotenuse $\overline{OP}$ of a right triangle, so $OP > OT = r$ and P lies outside the circle. This gives a quick test: if $OT^2 + PT^2 = OP^2$, then $\angle OTP$ is a right angle and $\overline{PT}$ is tangent.

Because of that right angle, every tangent problem hides a right triangle whose legs are the radius r and the tangent segment PT, and whose hypotenuse is $\overline{OP}$: $OP^2 = r^2 + PT^2$. Draw the radius to the point of tangency first; the rest is the Pythagorean Theorem.

From a point outside a circle you can draw two tangents (the construction below shows how). Their **tangent segments are congruent**: the right triangles OAP and OBP share the hypotenuse $\overline{OP}$ and have legs $OA = OB$, so they are congruent by HL. As a bonus, $\overrightarrow{PO}$ bisects the angle between the two tangents.

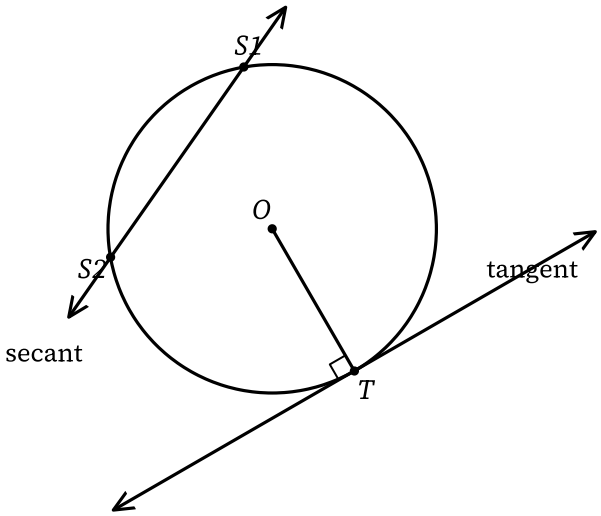
A polygon is **circumscribed** about a circle when every side is tangent to it; the circle is then inscribed in the polygon, like the incircle of Lesson 5.2. At each vertex the two tangent segments are equal, so each side splits into two tangent pieces, and missing sides can be found with algebra.


S1
O
S2
tangent
secant
T


B
P
O
A

THEOREM **Tangent Theorem** Euclid III.18

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.



INDIRECT PROOF

Given: Line ℓ is tangent to $\odot O$ at T

Prove: $\ell \perp \overline{OT}$

Assume, for the sake of contradiction, that ℓ is not perpendicular to $\overline{OT}$. The center O is not on ℓ (a line through the center contains a diameter and meets the circle twice), so let F be the foot of the perpendicular from O to ℓ. Then $F \neq T$, because $\overline{OT}$ is not perpendicular to ℓ.

On ℓ, mark the point G on the opposite side of F from T with $FG = FT$ (Ruler Postulate). In triangles OFT and OFG: $\overline{OF} \cong \overline{OF}$ (Reflexive Property), $\angle OFT \cong \angle OFG$ (both are right angles), and $\overline{FT} \cong \overline{FG}$. So $\triangle OFT \cong \triangle OFG$ by SAS, and $OG = OT$ by CPCTC.

Since OT is the radius, OG equals the radius too, so G lies on $\odot O$. But $G \neq T$, since they lie on opposite sides of F. Then ℓ meets the circle at two points, T and G, which contradicts the fact that a tangent meets the circle at exactly one point.

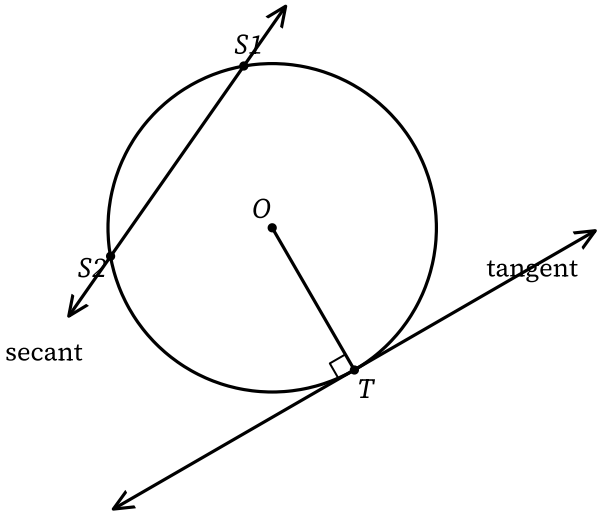
Therefore the assumption is false, and $\ell \perp \overline{OT}$.

∎

Euclid III.18: 'If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.' Euclid's proof is also indirect.

THEOREM **Converse of the Tangent Theorem** Euclid III.16

If a line in the plane of a circle is perpendicular to a radius at its endpoint on the circle, then the line is tangent to the circle.



PARAGRAPH PROOF

Given: T is on $\odot O$; line $\ell \perp \overline{OT}$ at T

Prove: ℓ is tangent to $\odot O$

Let P be any point of ℓ other than T. Then $\triangle OTP$ has a right angle at T, and its other two angles are acute, because the angles of a triangle add to $180°$. The largest angle of a triangle is opposite the longest side (Euclid I.19), so the hypotenuse $\overline{OP}$ is longer than the leg $\overline{OT}$: $OP > OT = r$.

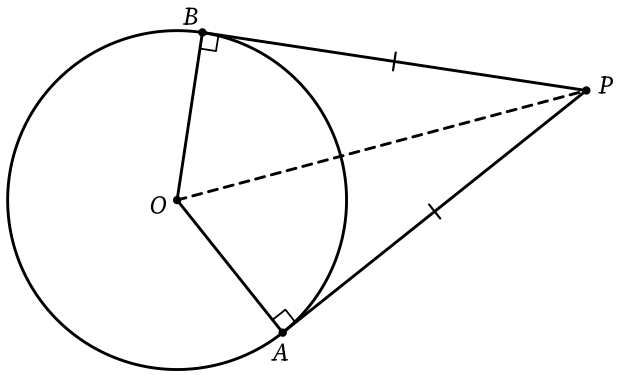
So every point of ℓ except T is farther than r from O and lies outside the circle. The line meets the circle only at T, so it is tangent.

∎

Euclid states this as the corollary (porism) of III.16: the straight line drawn at right angles to the diameter at its end touches the circle.

THEOREM **Tangent Segments Theorem**

If two segments from the same outside point are tangent to a circle, then they are congruent.



TWO-COLUMN PROOF

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B

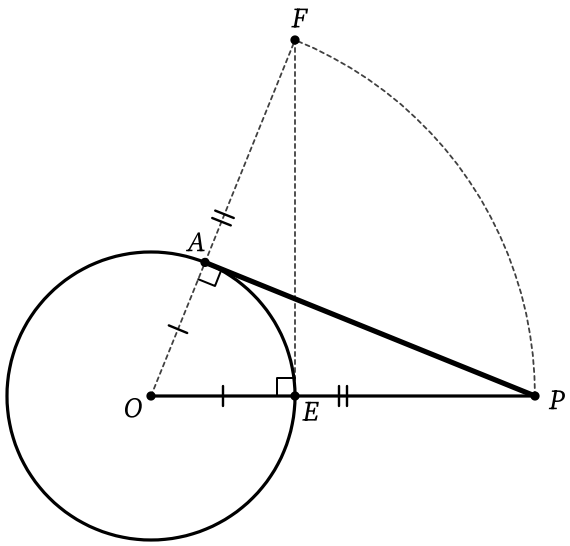
Prove: $\overline{PA} \cong \overline{PB}$

	STATEMENTS	REASONS
1	$\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B	Given
2	$\overline{OA} \perp \overline{PA}$ and $\overline{OB} \perp \overline{PB}$	Tangent Theorem
3	$\angle OAP$ and $\angle OBP$ are right angles	Definition of perpendicular lines
4	$\overline{OA} \cong \overline{OB}$	All radii of a circle are congruent
5	$\overline{OP} \cong \overline{OP}$	Reflexive Property of Congruence
6	$\triangle OAP \cong \triangle OBP$	HL
7	$\overline{PA} \cong \overline{PB}$	CPCTC

∎

CONSTRUCTION **Construct a tangent from an outside point (Euclid's method)** Euclid III.17

1. Draw $\overline{OP}$ from the center O to the outside point P, and label the point E where it crosses the given circle.
2. With center O and radius OP, draw an arc, part of a larger circle with the same center.
3. At E, construct the line perpendicular to $\overline{OP}$ (Euclid I.11). Label the point F where it meets the large arc.
4. Draw $\overline{OF}$ and label the point A where it crosses the given circle.
5. Draw $\overline{PA}$. It is tangent to the circle at A. Using the perpendicular on the other side of $\overline{OP}$ gives the second tangent.



Why it works. $OA = OE$ (radii of the given circle), $OP = OF$ (radii of the large circle), and $\angle AOP$ and $\angle EOF$ are the same angle. So $\triangle OAP \cong \triangle OEF$ by SAS, and $\angle OAP \cong \angle OEF$, which is a right angle by construction. Then $\overline{PA}$ is perpendicular to the radius $\overline{OA}$ at A, so by the Converse of the Tangent Theorem it is tangent. (Lesson 10.4 gives a quicker construction that uses a semicircle.)

KEY FORMULAS

$$OP^2 = r^2 + PT^2$$

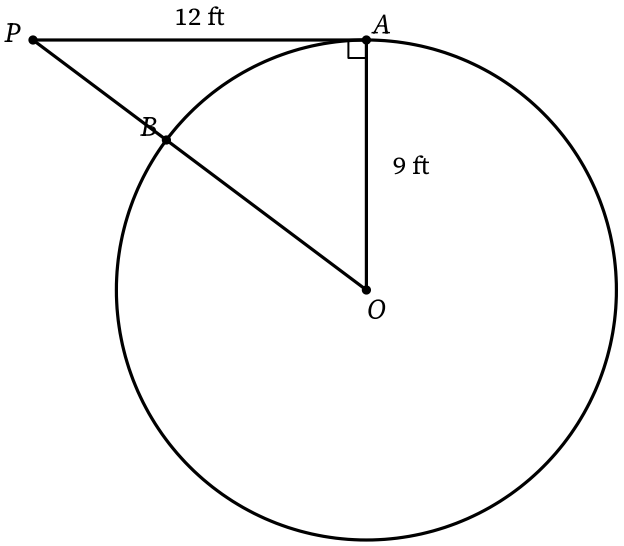
$$PA = PB \quad \text{(tangent segments from the same point } P\text{)}$$

From history. In III.16 Euclid also proves that no other straight line can be squeezed between a circle and its tangent at the point of contact. The thin 'horn angle' between a circle and its tangent, smaller than every angle made by two straight lines and yet not nothing, puzzled later commentators for many centuries.

Euclid connection. III.17 constructs a tangent from a given point, III.18 proves that the tangent is perpendicular to the radius, and III.19 adds that the line perpendicular to a tangent at its point of contact passes through the center. Euclid proves III.18 indirectly, just as we do.

Practice habit. The moment you see a tangent, draw the radius to the point of tangency and mark the right angle. Recite the Tangent Theorem and the Tangent Segments Theorem until you can state them without looking.

Example 1. A round water tank on the Miller farm has a radius of 9 feet. A straight fence touches the tank at only one point, A, so the fence is tangent to the tank. Henry stands on the fence at P, 12 feet from A. How far is Henry from the center O of the tank, and how far is he from the nearest point B of the tank wall?

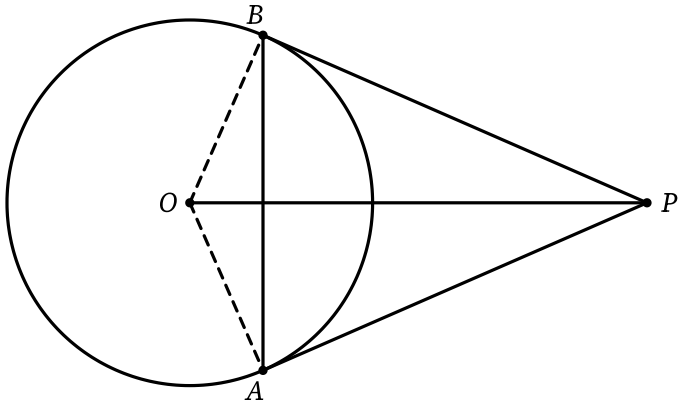


Show solution

1. The fence is tangent at A, so $\overline{OA} \perp \overline{AP}$ by the Tangent Theorem: $\triangle OAP$ is a right triangle with legs $OA = 9$ and $AP = 12$.
2. By the Pythagorean Theorem, $OP = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$ feet.
3. For any point X of the tank wall, the Triangle Inequality gives $PX + XO \geq PO$, so $PX \geq 15 - 9 = 6$; the shortest distance occurs at B, where $\overline{OP}$ crosses the circle. So $PB = OP - OB = 15 - 9 = 6$ feet.

Answer: Henry is 15 feet from the center and 6 feet from the tank wall.

Example 2. Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B, as in the figure. Prove: $\overleftrightarrow{OP}$ is the perpendicular bisector of $\overline{AB}$.



Show solution

1. Work backward: a line is the perpendicular bisector of $\overline{AB}$ if two of its points are each equidistant from A and B.
2. P is equidistant from A and B by the Tangent Segments Theorem; O is equidistant because $\overline{OA}$ and $\overline{OB}$ are radii.
3. Use the Converse of the Perpendicular Bisector Theorem for each point, then 'two points determine a line.'

PARAGRAPH PROOF

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B

Prove: $\overleftrightarrow{OP}$ is the perpendicular bisector of $\overline{AB}$

By the Tangent Segments Theorem, $PA = PB$. All radii of a circle are congruent, so $OA = OB$. Thus P and O are each equidistant from A and B.

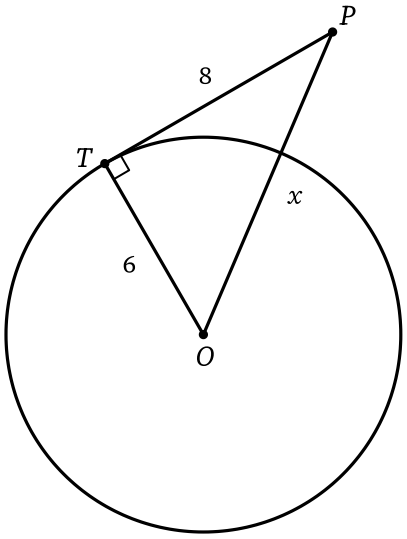
By the Converse of the Perpendicular Bisector Theorem, every point equidistant from A and B lies on the perpendicular bisector of $\overline{AB}$, so both P and O lie on it. The points P and O are different (P is outside the circle and O is inside), and through two points there is exactly one line. Therefore $\overleftrightarrow{OP}$ is the perpendicular bisector of $\overline{AB}$.

∎

Answer: See the proof.

PRACTICE

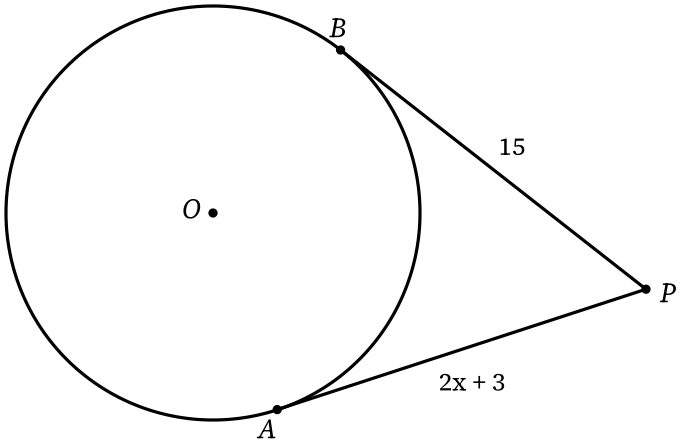
1. **Easy** In the figure, $\overline{PT}$ is tangent to $\odot O$ at T, $OT = 6$ and $PT = 8$. Find OP.



Show answer

$\overline{OT} \perp \overline{PT}$, so $OP = \sqrt{6^2 + 8^2} = 10$.

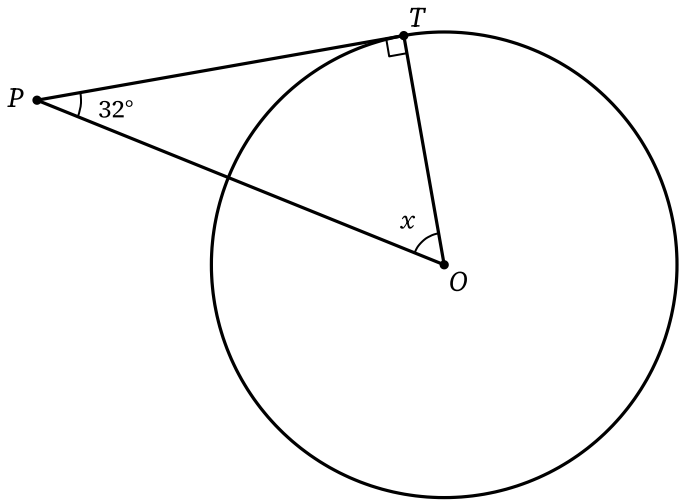
2. **Easy** In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B. If $PA = 2x + 3$ and $PB = 15$, find x.



Show answer

Tangent segments from P are congruent: $2x + 3 = 15$, so $x = 6$.

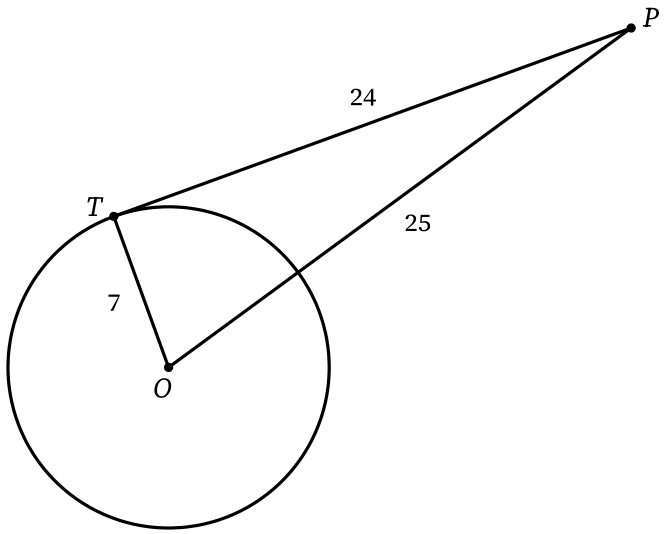
3. **Easy** In the figure, $\overline{PT}$ is tangent to $\odot O$ at T and $m\angle OPT = 32°$. Find $m\angle POT$.



Show answer

$m\angle OTP = 90°$ by the Tangent Theorem, so $m\angle POT = 180° - 90° - 32° = 58°$.

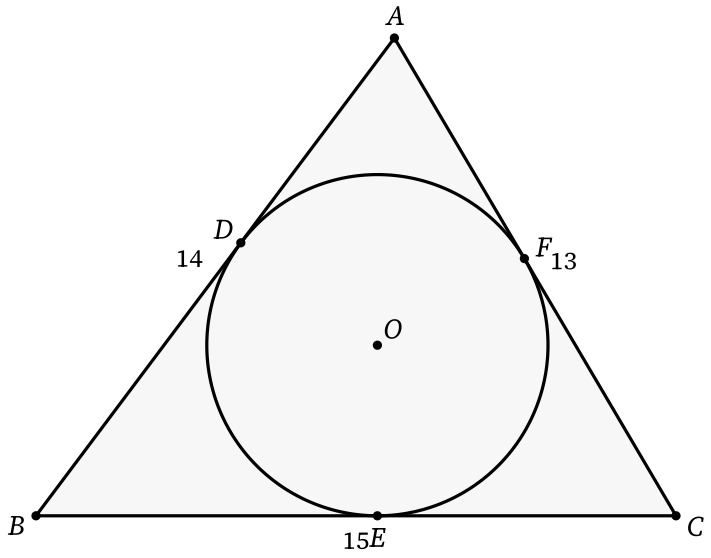
4. **Easy** In the figure, T is on $\odot O$, $OT = 7$, $TP = 24$ and $OP = 25$. Is $\overline{TP}$ tangent to $\odot O$ at T? Explain.



Show answer

Yes. $7^2 + 24^2 = 49 + 576 = 625 = 25^2$, so by the Converse of the Pythagorean Theorem $\angle OTP$ is a right angle. $\overline{TP}$ is perpendicular to the radius $\overline{OT}$ at T, so it is tangent by the Converse of the Tangent Theorem.

5. **Medium** In the figure, the circle with center O is inscribed in $\triangle ABC$ and touches $\overline{AB}$ at D, $\overline{BC}$ at E and $\overline{CA}$ at F. If $AB = 14$, $BC = 15$ and $CA = 13$, find AD.



Show answer

Let $AD = AF = x$, $BD = BE = y$ and $CE = CF = z$ (tangent segments). Then $x + y = 14$, $y + z = 15$ and $x + z = 13$. Adding the first and third and subtracting the second: $2x = 14 + 13 - 15 = 12$, so $AD = 6$.

6. **Medium** Quadrilateral $ABCD$ is circumscribed about a circle, so each of its sides is tangent to the circle. If $AB = 10$, $BC = 13$ and $CD = 12$, find DA.

Show answer

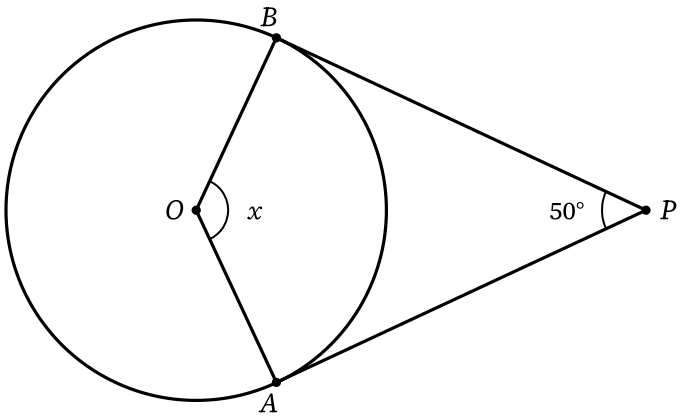
At each vertex the two tangent segments are equal, and each side is the sum of two tangent segments. $AB + CD$ and $BC + DA$ each contain one tangent segment from every vertex, so $AB + CD = BC + DA$: $10 + 12 = 13 + DA$, and $DA = 9$.

7. **Medium** A round grain silo has a radius of 10 feet. Grace stands 26 feet from the center of the silo and looks along a line of sight that just grazes the side of the silo. How far is it from Grace to the point where her line of sight touches the silo?

Show answer

The line of sight is tangent, so it is perpendicular to the radius at the point of tangency: $\sqrt{26^2 - 10^2} = \sqrt{676 - 100} = \sqrt{576} = 24$ feet.

8. **Medium** In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B, and $m\angle APB = 50°$. Find $m\angle AOB$.



Show answer

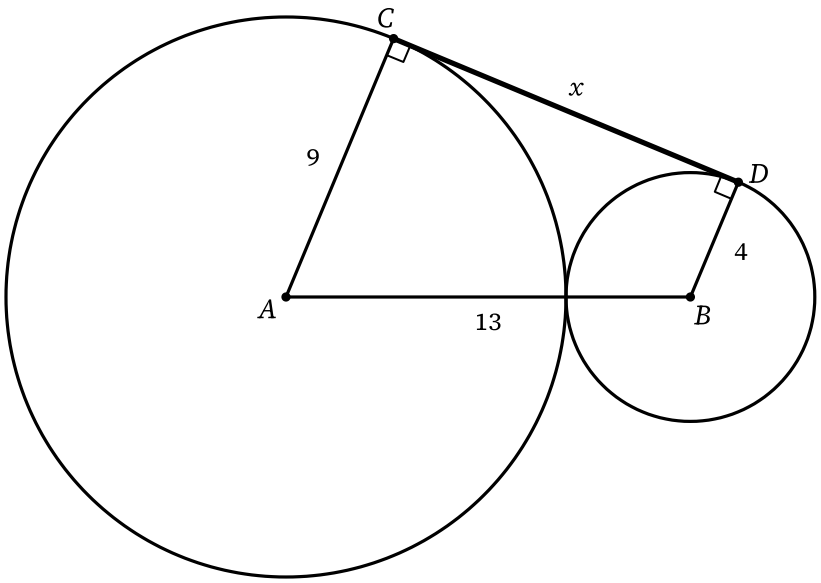
The angles at A and B are right angles (Tangent Theorem), and the angles of quadrilateral $PAOB$ add to $360°$: $m\angle AOB = 360° - 90° - 90° - 50° = 130°$.

9. **Medium** Are the tangent lines at the two endpoints of a diameter parallel? Explain.

Show answer

Yes. Each tangent is perpendicular to the diameter at its endpoint, by the Tangent Theorem. In a plane, two lines perpendicular to the same line are parallel (Lesson 3.3).

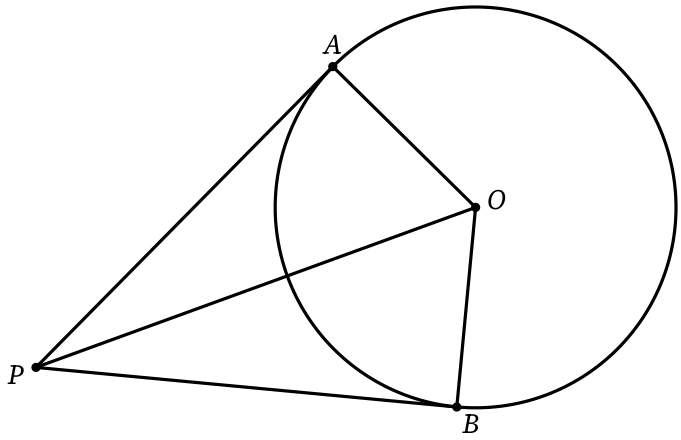
10. Hard A belt runs around two pulleys with radii 9 inches and 4 inches whose centers A and B are 13 inches apart. A straight part of the belt, $\overline{CD}$, is tangent to both pulleys, as in the figure. Find the length CD.



Show answer

$\overline{AC}$ and $\overline{BD}$ are both perpendicular to $\overline{CD}$, so they are parallel. Draw the segment from B parallel to $\overline{CD}$ meeting $\overline{AC}$ at E; then $EBDC$ is a rectangle, so $EC = 4$, $AE = 9 - 4 = 5$ and $BE = CD$. In right triangle AEB, $BE = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. So $CD = 12$ inches.

11. Hard Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B, as in the figure. Prove: $\overrightarrow{PO}$ bisects $\angle APB$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B

Prove: $\overrightarrow{PO}$ bisects $\angle APB$

(1) $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B
Given

(2) $\overline{OA} \perp \overline{PA}$ and $\overline{OB} \perp \overline{PB}$
Tangent Theorem
from (1)

(3) $\angle OAP$ and $\angle OBP$ are right angles
Definition of perpendicular lines
from (2)

(4) $\overline{OA} \cong \overline{OB}$
All radii of a circle are congruent

(5) $\overline{OP} \cong \overline{OP}$
Reflexive Property of Congruence

(6) $\triangle OAP \cong \triangle OBP$
HL
from (3), (4), (5)

(7) $\angle APO \cong \angle BPO$

CPCTC

from (6)

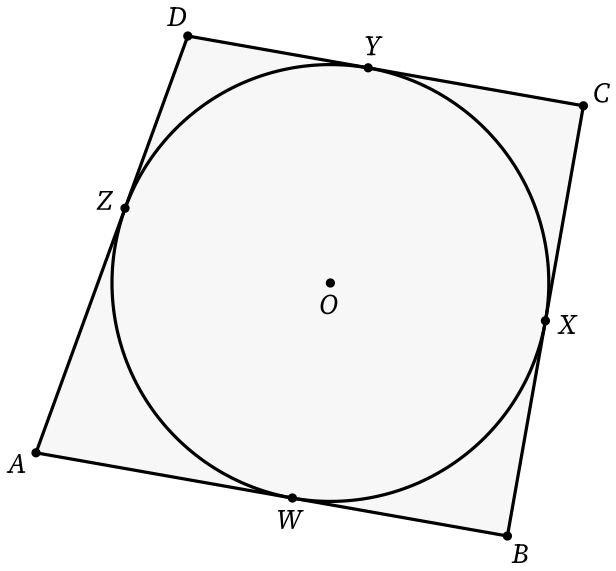
(8) $\overrightarrow{PO}$ bisects $\angle APB$

Definition of angle bisector

from (7)

∎

12. Hard Given: The sides of quadrilateral $ABCD$ are tangent to $\odot O$ at W (on $\overline{AB}$), X (on $\overline{BC}$), Y (on $\overline{CD}$) and Z (on $\overline{DA}$), as in the figure. Prove: $AB + CD = BC + DA$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: The sides of $ABCD$ are tangent to $\odot O$ at W, X, Y and Z on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$

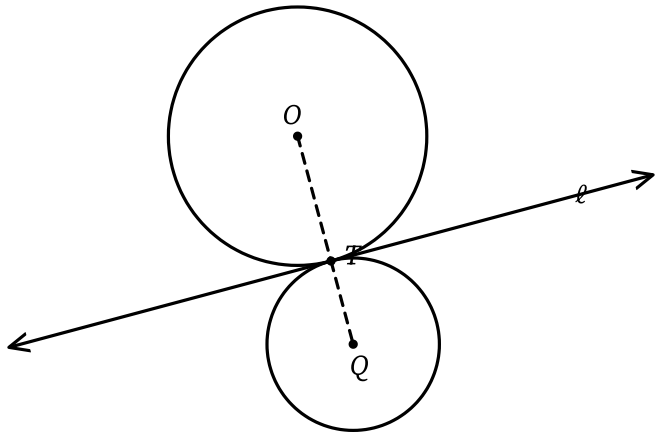
Prove: $AB + CD = BC + DA$

	STATEMENTS	REASONS
1	The sides of $ABCD$ are tangent to $\odot O$ at W, X, Y and Z	Given
2	$AW = AZ$, $BW = BX$, $CX = CY$ and $DY = DZ$	Tangent Segments Theorem
3	$AB = AW + WB$ and $CD = CY + YD$	Segment Addition Postulate
4	$BC = BX + XC$ and $DA = DZ + ZA$	Segment Addition Postulate
5	$AB + CD = AW + WB + CY + YD$	Addition Property of Equality (step 3)
6	$AB + CD = AZ + BX + CX + DZ$	Substitution Property of Equality (step 2)

	STATEMENTS	REASONS
7	$AB + CD = (BX + XC) + (DZ + ZA)$	Commutative and Associative Properties of Addition
8	$AB + CD = BC + DA$	Substitution Property of Equality (step 4)

∎

13. Hard Given: $\odot O$ and $\odot Q$ are both tangent to line ℓ at the same point T, as in the figure. Prove: O, T and Q are collinear. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\odot O$ and $\odot Q$ are tangent to line ℓ at T

Prove: O, T and Q are collinear

By the Tangent Theorem, $\overline{OT} \perp \ell$ and $\overline{QT} \perp \ell$, so $\overrightarrow{TO}$ and $\overrightarrow{TQ}$ each make right angles with ℓ.

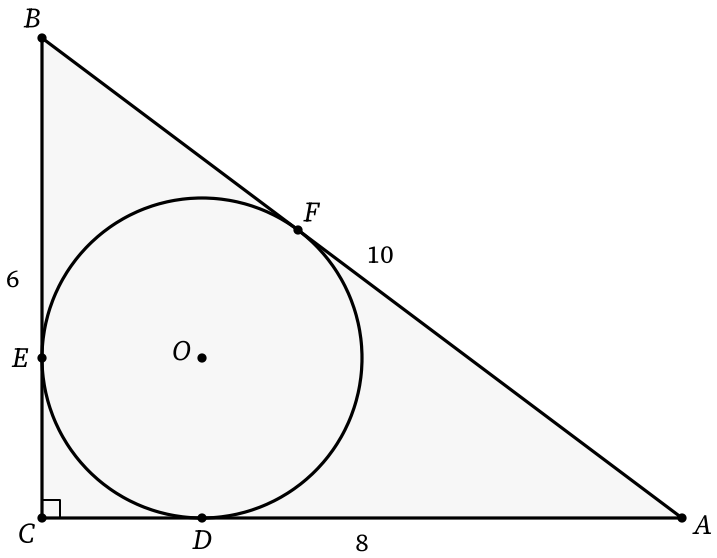
If O and Q are on the same side of ℓ, then $\overrightarrow{TO}$ and $\overrightarrow{TQ}$ make 90° angles with the same ray of ℓ on the same side of ℓ, so by the Protractor Postulate they are the same ray, and O, T, Q lie on one line.

If O and Q are on opposite sides of ℓ, as in the figure, let $\overrightarrow{TR}$ be the ray opposite $\overrightarrow{TQ}$. It lies on O's side of ℓ and makes a 90° angle with ℓ, because it forms a linear pair with a right angle. By the Protractor Postulate, $\overrightarrow{TR}$ is the same ray as $\overrightarrow{TO}$, so O lies on line TQ.

In either case O, T and Q are collinear. (This is Euclid III.19: the perpendicular to a tangent at its point of contact passes through the center.)

∎

14. **Challenge** In the figure, right triangle ABC has legs $CA = 8$ and $CB = 6$ and hypotenuse $AB = 10$. A circle with center O is inscribed in the triangle. Find the radius of the circle.



Show answer

Let the circle touch $\overline{CA}$ at D, $\overline{CB}$ at E and $\overline{AB}$ at F, with radius r. Quadrilateral $ODCE$ has right angles at D, C and E, so it is a rectangle, and since $OD = OE = r$ it is a square: $CD = CE = r$. Then $AF = AD = 8 - r$ and $BF = BE = 6 - r$, so $AB = (8 - r) + (6 - r) = 14 - 2r = 10$, and $r = 2$.

COMMON MISTAKES

- Assuming a line is tangent because it looks as if it just touches. Prove it: show a right angle with the radius, for example with the Converse of the Pythagorean Theorem.
- Putting the tangent segment in the hypotenuse's place. In the right triangle, the hypotenuse is always $\overline{OP}$, from the center to the outside point; the radius and the tangent segment are the legs.
- Setting tangent segments equal when they come from different outside points. Only the two tangent segments from the same point are congruent.
- Forgetting to draw the radius to the point of tangency. The right angle you need is invisible until you draw it.

Make it a game · Teach and Catch. Take a plate and a ruler and teach your parent four facts about tangents: tangent is perpendicular to the radius, the converse, two tangent segments from outside are congruent, and the point of tangency is a single point. Parent sneaks in one false fact. If you catch it and fix it with a drawing, you win.

10.4 Inscribed Angles

OBJECTIVES

- Prove the Inscribed Angle Theorem in all three cases (Euclid III.20).
- Use the facts that inscribed angles on the same arc are congruent, that an angle in a semicircle is right, and that opposite angles of an inscribed quadrilateral are supplementary.
- Write proofs that combine inscribed angles with congruent triangles and parallel lines.
- Find the center of a circle with a carpenter's square and construct tangents with a semicircle.

WATCH

Angle Properties - Circle Geometry (Angle at the centre/circumference)

Eddie Woo · 7:19

https://www.youtube.com/watch?v=S0ajCv9yRIc

> **WHEN TO WATCH**
> Before the Inscribed Angle Theorem: the angle at the centre is twice the angle at the circumference.

Note: Eddie Woo teaches in Australia: he says 'angle at the circumference' for inscribed angle and 'cyclic quadrilateral' for inscribed quadrilateral.

BACKUP VIDEO

Angle Properties - Circle Geometry (Angle in a semicircle)

Eddie Woo · 9:40

https://www.youtube.com/watch?v=S16GCythNfg

Before Thales' theorem: the angle in a semicircle.

KEY TERMS

Inscribed angle

An angle whose vertex is on a circle and whose sides contain chords of the circle.

Intercepted arc

The arc that lies in the interior of an angle and has its endpoints on the sides of the angle. An inscribed angle $\angle ABC$ intercepts the arc $\overset{\frown}{AC}$ that does not contain B.

Inscribed polygon and circumscribed circle

A polygon is inscribed in a circle when all of its vertices lie on the circle; the circle is then circumscribed about the polygon.

CORE IDEAS

An **inscribed angle** has its vertex on the circle and chords for sides. Like a central angle, it cuts off an arc, its **intercepted arc**, lying inside the angle between its sides. The question of this lesson: if a central angle and an inscribed angle intercept the same arc, how are they related?

Start with the simplest case, where one side of the inscribed angle $\angle ABC$ is a diameter $\overline{BC}$. Draw the radius $\overline{OA}$. Because $OA = OB$, triangle AOB is isosceles and its base angles are equal, say x each. The central angle $\angle AOC$ is an exterior angle of that triangle, so it equals $x + x = 2x$. The central angle is twice the inscribed angle, so **the inscribed angle is half its intercepted arc.**

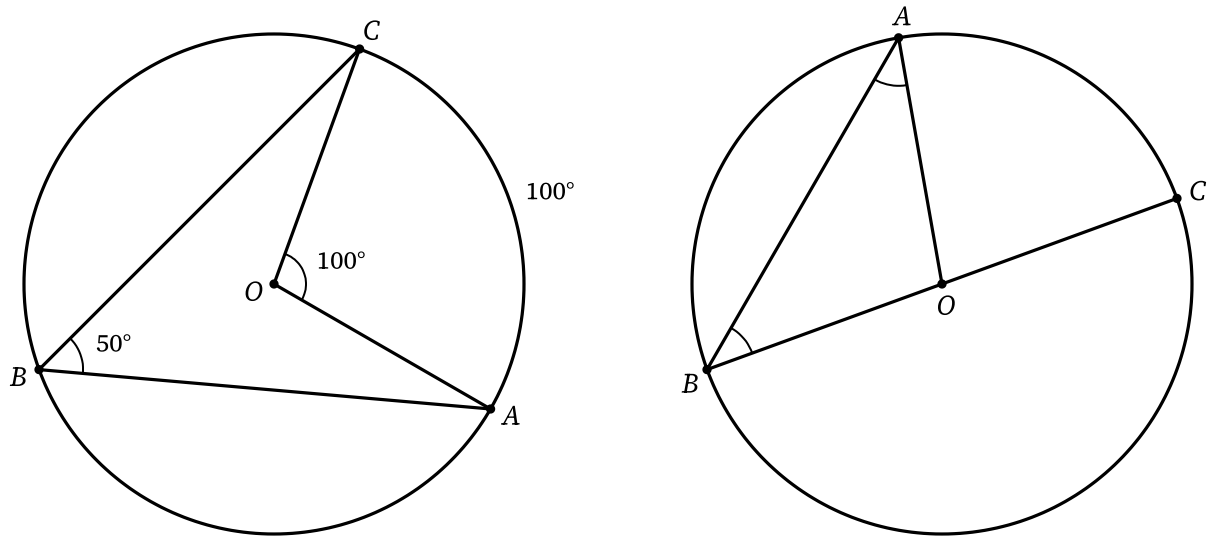
If the center lies inside or outside the inscribed angle, draw the diameter from the vertex. It splits the problem into two angles of the simple kind, which you add (center inside) or subtract (center outside). In every case $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$. This is Euclid III.20, and it is the most useful single fact about circles.

Three consequences follow at once. **Inscribed angles that intercept the same arc are congruent** (III.21), because each is half the same arc: the vertex can slide anywhere along the rest of the circle and the angle does not change. **An angle inscribed in a semicircle is a right angle** (III.31), because it is half of 180°. Conversely, a right

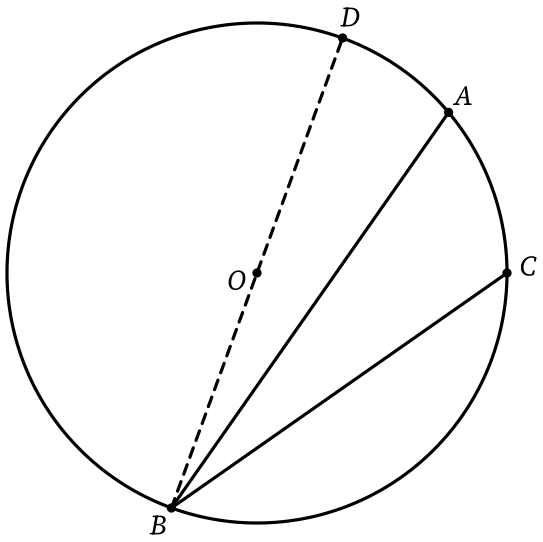
inscribed angle stands on a semicircle, so its chord is a diameter; that is why the corner of a carpenter's square laid on a circle marks the ends of a diameter.

In a quadrilateral $ABCD$ inscribed in a circle, the opposite angles $\angle A$ and $\angle C$ intercept the two arcs $\overset{\frown}{BCD}$ and $\overset{\frown}{DAB}$, which together make the whole circle. So $m\angle A + m\angle C = \frac{1}{2}(360^\circ) = 180^\circ$: **opposite angles of an inscribed quadrilateral are supplementary** (III.22). This is a strong test: for example, a parallelogram that is not a rectangle can never be inscribed in a circle.

To use these facts well, always identify the intercepted arc first. Trace from one side of the angle to the other along the arc that lies inside the angle, away from the vertex. Then halve its measure.



Case 1: the center lies on side BC.

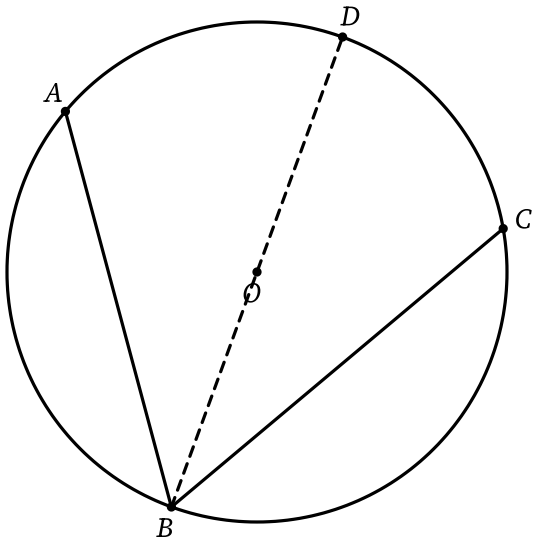


Case 3: the center lies outside the angle.

POSTULATES AND THEOREMS

THEOREM **Inscribed Angle Theorem** Euclid III.20

The measure of an inscribed angle is half the measure of its intercepted arc.



Case 2: the center lies inside the angle.

PARAGRAPH PROOF

Given: $\angle ABC$ is inscribed in $\odot O$ and intercepts $\overset{\frown}{AC}$

Prove: $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$

Case 1: O lies on a side of the angle, say on $\overline{BC}$ (figure L10.4-case1). Draw radius $\overline{OA}$. All radii are congruent, so $OA = OB$, and by the Isosceles Triangle Theorem $m\angle OAB = m\angle OBA$; call this measure x. Because B, O, C are collinear, $\angle AOC$ is

an exterior angle of $\triangle AOB$, so by the Exterior Angle Theorem $m\angle AOC = x + x = 2x$. By the definition of arc measure, $m\overset{\frown}{AC} = m\angle AOC = 2x$. Since $m\angle ABC = m\angle OBA = x$, we get $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$.

Case 2: O lies in the interior of the angle (figure L10.4-case2). Draw the diameter $\overline{BD}$. Its endpoint D lies on $\overset{\frown}{AC}$ and $\overrightarrow{BD}$ lies between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, so by the Angle Addition Postulate $m\angle ABC = m\angle ABD + m\angle DBC$. Each of these two angles has the center on a side, so by Case 1 $m\angle ABD = \frac{1}{2} m\overset{\frown}{AD}$ and $m\angle DBC = \frac{1}{2} m\overset{\frown}{DC}$. Adding, $m\angle ABC = \frac{1}{2}(m\overset{\frown}{AD} + m\overset{\frown}{DC}) = \frac{1}{2} m\overset{\frown}{AC}$ by the Arc Addition Postulate.

Case 3: O lies in the exterior of the angle (figure L10.4-case3). Again draw the diameter $\overline{BD}$. Now both sides lie on the same side of $\overline{BD}$; name the points so that A lies on $\overset{\frown}{DC}$ and $\overrightarrow{BA}$ is between $\overrightarrow{BD}$ and $\overrightarrow{BC}$. By the Angle Addition Postulate, $m\angle ABC = m\angle DBC - m\angle DBA$. By Case 1, $m\angle DBC = \frac{1}{2} m\overset{\frown}{DC}$ and $m\angle DBA = \frac{1}{2} m\overset{\frown}{DA}$. So $m\angle ABC = \frac{1}{2}(m\overset{\frown}{DC} - m\overset{\frown}{DA}) = \frac{1}{2} m\overset{\frown}{AC}$, because $m\overset{\frown}{DA} + m\overset{\frown}{AC} = m\overset{\frown}{DC}$ by the Arc Addition Postulate.

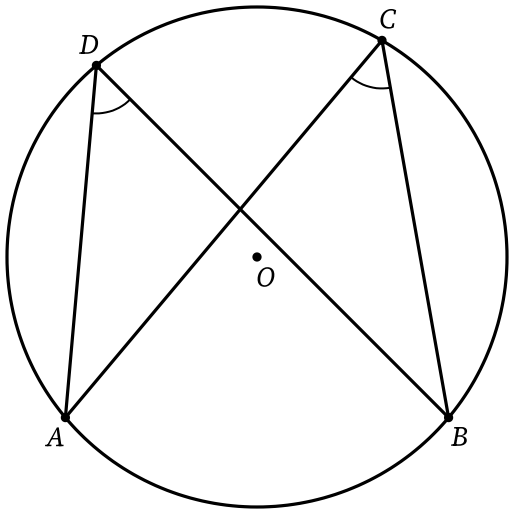
The three cases cover every position of the center, so the theorem is proved.

∎

Euclid III.20: 'In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.'

COROLLARY **Inscribed Angles on the Same Arc** Euclid III.21

If two inscribed angles intercept the same arc (or congruent arcs), then the angles are congruent.



PARAGRAPH PROOF

Given: $\angle ACB$ and $\angle ADB$ are inscribed in $\odot O$ and both intercept $\overset{\frown}{AB}$

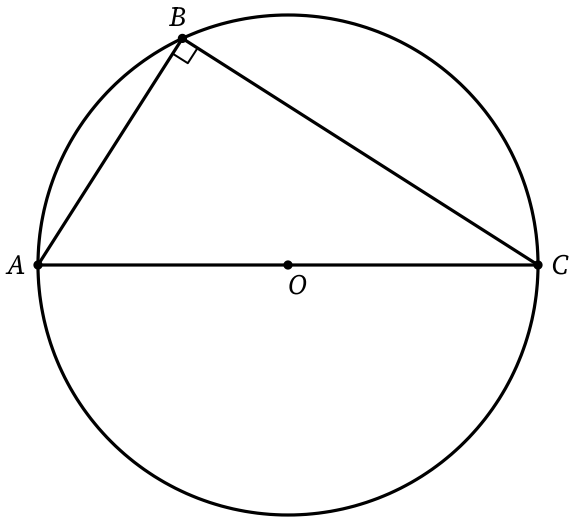
Prove: $\angle ACB \cong \angle ADB$

By the Inscribed Angle Theorem, $m\angle ACB = \frac{1}{2}m\overset{\frown}{AB}$ and $m\angle ADB = \frac{1}{2}m\overset{\frown}{AB}$. By the Substitution Property, $m\angle ACB = m\angle ADB$, so the angles are congruent.

∎

COROLLARY **Angle in a Semicircle (Thales' Theorem)** Euclid III.31

An angle inscribed in a semicircle is a right angle. Conversely, if an inscribed angle is a right angle, then its intercepted arc is a semicircle and its chord is a diameter.



PARAGRAPH PROOF

Given: $\angle ABC$ is inscribed in $\odot O$

Prove: $\overline{AC}$ is a diameter exactly when $m\angle ABC = 90°$

If $\overline{AC}$ is a diameter, the arc $\overset{\frown}{AC}$ that does not contain B is a semicircle, so by the Inscribed Angle Theorem $m\angle ABC = \frac{1}{2}(180°) = 90°$.

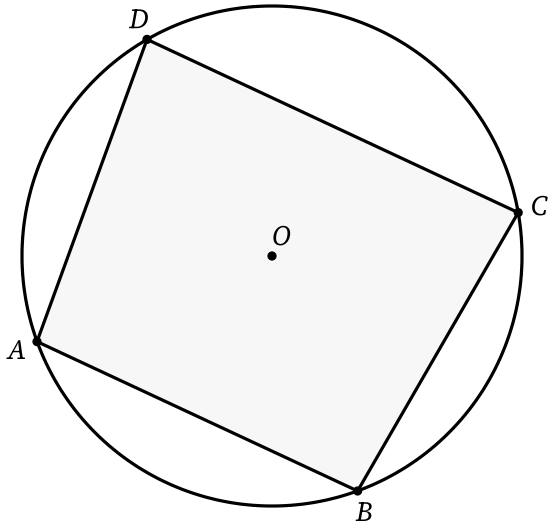
Conversely, if $m\angle ABC = 90°$, then by the Inscribed Angle Theorem the intercepted arc measures $180°$. Its central angle $\angle AOC$ is then a straight angle, so A, O and C are collinear with O between A and C. A chord through the center is a diameter, so $\overline{AC}$ is a diameter.

∎

Euclid III.31: 'In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle.'

THEOREM **Inscribed Quadrilateral Theorem** Euclid III.22

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.



TWO-COLUMN PROOF

Given: Quadrilateral $ABCD$ is inscribed in $\odot O$

Prove: $\angle DAB$ and $\angle BCD$ are supplementary

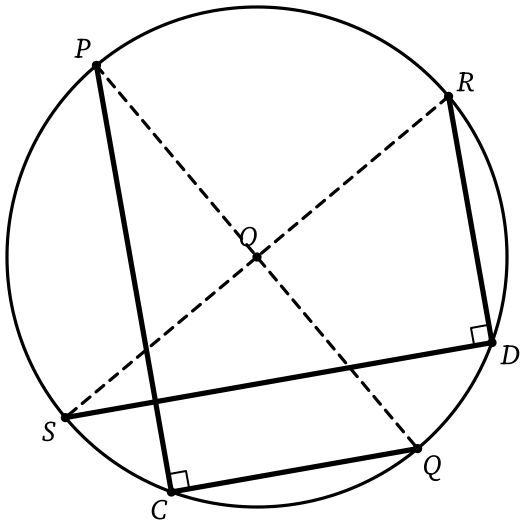
	STATEMENTS	REASONS
1	$ABCD$ is inscribed in $\odot O$	Given
2	$m\angle DAB = \frac{1}{2} m\overset{\frown}{BCD}$	Inscribed Angle Theorem
3	$m\angle BCD = \frac{1}{2} m\overset{\frown}{DAB}$	Inscribed Angle Theorem
4	$m\angle DAB + m\angle BCD = \frac{1}{2}(m\overset{\frown}{BCD} + m\overset{\frown}{DAB})$	Addition Property of Equality and Distributive Property
5	$m\overset{\frown}{BCD} + m\overset{\frown}{DAB} = 360^\circ$	Arc Addition Postulate (the two arcs make up the whole circle)
6	$m\angle DAB + m\angle BCD = \frac{1}{2}(360^\circ) = 180^\circ$	Substitution Property of Equality
7	$\angle DAB$ and $\angle BCD$ are supplementary	Definition of supplementary angles

∎

The same argument, with arcs $C\overset{\frown}{DA}$ and $A\overset{\frown}{BC}$, shows that $\angle ABC$ and $\angle CDA$ are supplementary.

CONSTRUCTION **Find the center of a circle with a carpenter's square** Euclid III.31

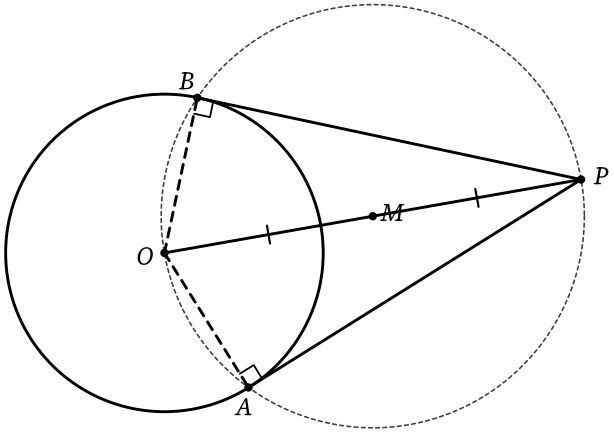
1. Set the corner of a carpenter's square (a right angle) on the edge of the circle at a point C.
2. Mark the points P and Q where the two edges of the square cross the circle, and draw $\overline{PQ}$.
3. Move the corner to another point D of the circle, turned so that its edges do not pass through P and Q. Mark the crossing points R and S, and draw $\overline{RS}$. (Check that $\overline{RS}$ is a different line from $\overline{PQ}$.)
4. The point where $\overline{PQ}$ and $\overline{RS}$ cross is the center O.



Why it works. $\angle PCQ$ is a right angle inscribed in the circle, so by the converse of Thales' Theorem (III.31) its chord $\overline{PQ}$ is a diameter. In the same way $\overline{RS}$ is a diameter. Every diameter passes through the center. Because $\overline{RS}$ was chosen to be a different line from $\overline{PQ}$, and two different lines meet in at most one point, the two diameters cross exactly at the center.

CONSTRUCTION **Construct the tangents from an outside point (semicircle method)**

1. Draw $\overline{OP}$ from the center O of the given circle to the outside point P.
2. Construct the perpendicular bisector of $\overline{OP}$ to find its midpoint M.
3. With center M and radius MO, draw a circle. It crosses the given circle at two points, A and B.
4. Draw $\overline{PA}$ and $\overline{PB}$. Both are tangent to the given circle.



Why it works. $\overline{OP}$ is a diameter of the circle centered at M, and A lies on that circle, so $\angle OAP$ is inscribed in a semicircle and is a right angle (III.31). Then $\overline{PA}$ is perpendicular to the radius $\overline{OA}$ at A, so it is tangent by the Converse of the Tangent Theorem (Lesson 10.3). The same reasoning works for B.

KEY FORMULAS

$$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$$

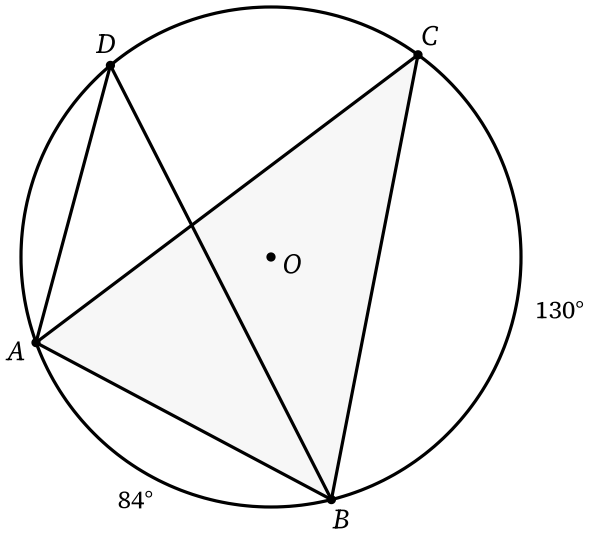
$$m\angle A + m\angle C = 180^\circ, \quad m\angle B + m\angle D = 180^\circ \quad (ABCD \text{ inscribed})$$

From history. Later Greek writers credit Thales of Miletus (about 600 BC) with discovering that an angle inscribed in a semicircle is a right angle, which is why the fact is often called Thales' Theorem. One ancient report adds that he sacrificed an ox to celebrate, but that story should be treated as legend. Euclid proves the fact as Book III, Proposition 31, using the angle sum of a triangle.

Euclid connection. III.20 ('the angle at the centre is double of the angle at the circumference') is the Inscribed Angle Theorem; Euclid draws the diameter through the vertex and handles the center inside the angle by adding and outside the angle by subtracting, just as in Cases 2 and 3. III.21 gives equal angles in the same segment, III.22 the inscribed quadrilateral, and III.31 the angle in a semicircle.

Practice habit. Recite the Inscribed Angle Theorem and its three consequences (same arc, semicircle, inscribed quadrilateral) until they are automatic. For every inscribed angle, trace its intercepted arc with your pencil before you write any number.

Example 1. In $\odot O$ in the figure, $\triangle ABC$ is inscribed and D is another point of the circle. The arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ that contain none of the other labeled points measure 84° and 130°. Find (a) $m\angle ADB$, (b) $m\angle BAC$ and (c) $m\angle ABC$.

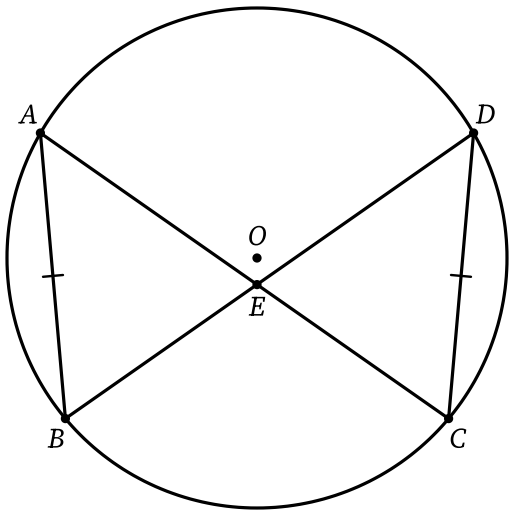


Show solution

1. (a) $\angle ADB$ is inscribed and intercepts $\overset{\frown}{AB}$, so $m\angle ADB = \frac{1}{2}(84^\circ) = 42^\circ$.
2. (b) $\angle BAC$ intercepts $\overset{\frown}{BC}$, so $m\angle BAC = \frac{1}{2}(130^\circ) = 65^\circ$.
3. (c) $\angle ABC$ intercepts the arc from C to A that does not contain B. By the Arc Addition Postulate it measures $360^\circ - 84^\circ - 130^\circ = 146^\circ$, so $m\angle ABC = 73^\circ$.
4. Check: $\angle ACB$ intercepts $\overset{\frown}{AB}$, so it measures 42°, and $65^\circ + 73^\circ + 42^\circ = 180^\circ$, as the Triangle Angle Sum Theorem requires.

Answer: (a) 42°; (b) 65°; (c) 73°

Example 2. Given: A, B, C and D lie in that order on $\odot O$, chords $\overline{AB} \cong \overline{DC}$, and chords $\overline{AC}$ and $\overline{BD}$ intersect at E, as in the figure. Prove: $\triangle ABE \cong \triangle DCE$.



Show solution

1. Mark the given congruent chords. The triangles share no sides, so look for congruent angles.
2. $\angle BAE$ and $\angle CDE$ are inscribed angles that both intercept $\overset{\frown}{BC}$; $\angle ABE$ and $\angle DCE$ both intercept $\overset{\frown}{AD}$.
3. Two angles and the included side give ASA.

TWO-COLUMN PROOF

Given: A, B, C, D lie in that order on $\odot O$; $\overline{AB} \cong \overline{DC}$; $\overline{AC}$ and $\overline{BD}$ intersect at E
Prove: $\triangle ABE \cong \triangle DCE$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{DC}$	Given
2	$\angle BAE \cong \angle CDE$	Inscribed angles that intercept the same arc ($\overset{\frown}{BC}$) are congruent; E is on $\overline{AC}$ and on $\overline{BD}$
3	$\angle ABE \cong \angle DCE$	Inscribed angles that intercept the same arc ($\overset{\frown}{AD}$) are congruent

	STATEMENTS	REASONS
4	$\triangle ABE \cong \triangle DCE$	ASA

∎

Answer: See the proof.

PRACTICE

1. Easy An inscribed angle intercepts an arc of $96°$. Find the measure of the angle.

   Show answer

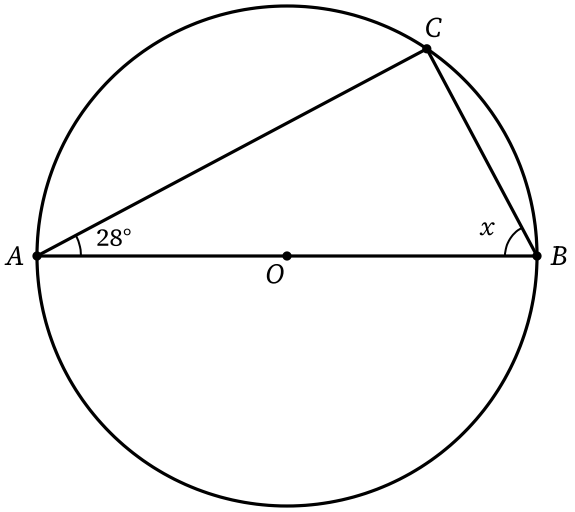
   $$\frac{1}{2}(96°) = 48°$$

2. Easy An inscribed angle measures $37°$. Find the measure of its intercepted arc.

   Show answer

   $$2(37°) = 74°$$

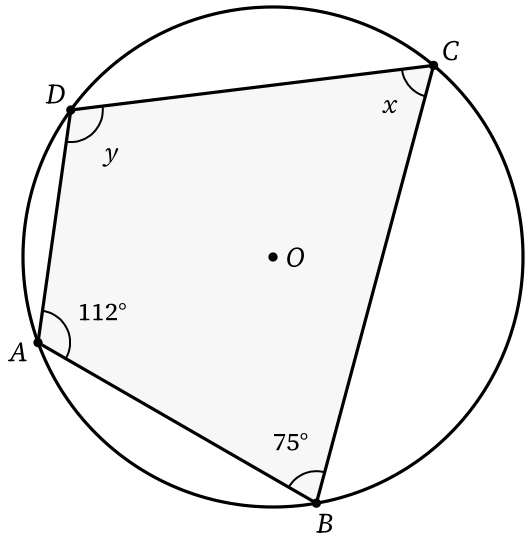
3. Easy In the figure, $\overline{AB}$ is a diameter of $\odot O$, C is on the circle and $m\angle CAB = 28°$. Find $m\angle CBA$.



   Show answer

   $\angle ACB$ is inscribed in a semicircle, so it is a right angle, and $m\angle CBA = 180° - 90° - 28° = 62°$.

4. **Easy** Quadrilateral $ABCD$ is inscribed in a circle, with $m\angle DAB = 112^\circ$ and $m\angle ABC = 75^\circ$, as in the figure. Find $x = m\angle BCD$ and $y = m\angle CDA$.



Show answer

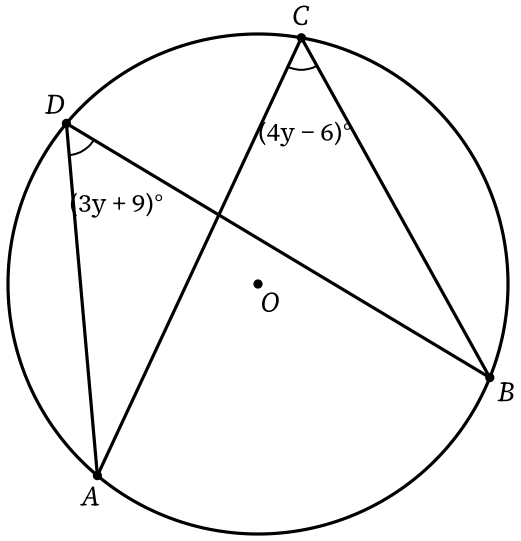
Opposite angles are supplementary: $x = 180^\circ - 112^\circ = 68^\circ$ and $y = 180^\circ - 75^\circ = 105^\circ$.

5. **Medium** An inscribed angle measures $(2x + 4)^\circ$ and its intercepted arc measures $(5x - 10)^\circ$. Find x and the measure of the angle.

Show answer

$2(2x + 4) = 5x - 10$, so $4x + 8 = 5x - 10$ and $x = 18$. The angle measures $2(18) + 4 = 40^\circ$ (and the arc 80°).

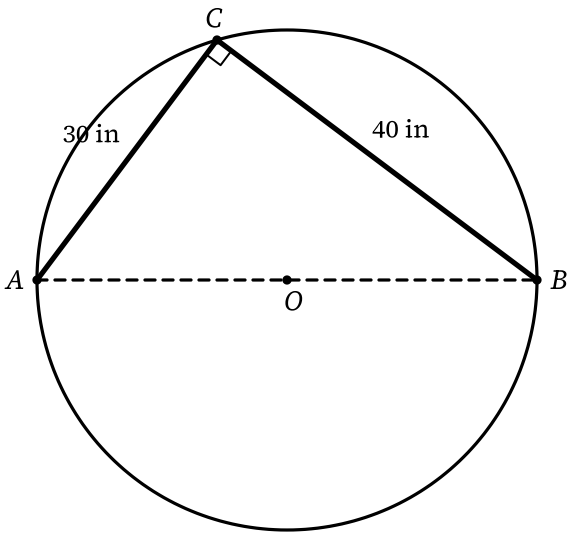
6. **Medium** In the figure, $\angle ACB$ and $\angle ADB$ are inscribed in the circle and both intercept $\overset{\frown}{AB}$. If $m\angle ACB = (4y - 6)^\circ$ and $m\angle ADB = (3y + 9)^\circ$, find y and $m\overset{\frown}{AB}$.



Show answer

The angles are congruent: $4y - 6 = 3y + 9$, so $y = 15$. Each angle measures 54°, so $m\overset{\frown}{AB} = 108^\circ$.

7. **Medium** Henry sets the corner C of a carpenter's square on the edge of a round tabletop. The edges of the square cross the rim at A and B, with $CA = 30$ inches and $CB = 40$ inches, as in the figure. What is the diameter of the tabletop?



Show answer

$\angle ACB$ is a right inscribed angle, so $\overline{AB}$ is a diameter. By the Pythagorean Theorem, $AB = \sqrt{30^2 + 40^2} = 50$ inches.

8. **Medium** A triangle is inscribed in a circle so that its vertices divide the circle into three arcs whose measures are in the ratio $3 : 4 : 5$. Find the three angles of the triangle.

Show answer

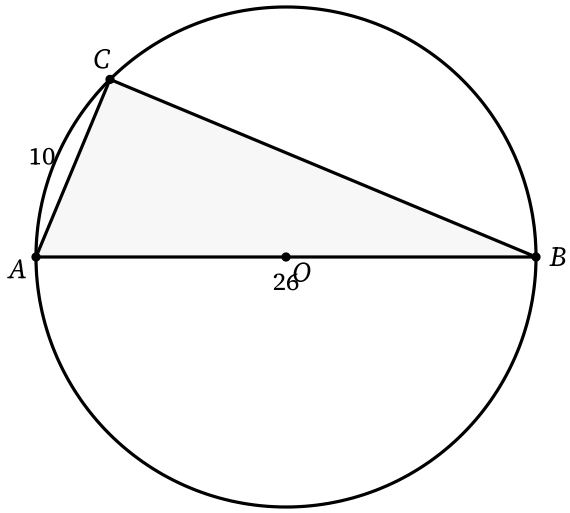
The arcs are $3k + 4k + 5k = 360$, so $k = 30$: 90°, 120° and 150°. Each angle of the triangle intercepts the arc opposite it, so the angles are 45°, 60° and 75°.

9. **Medium** Explain why the center of the circle that passes through all three vertices of a right triangle is the midpoint of the hypotenuse.

Show answer

Let $\triangle ABC$ have its right angle at C, and consider the circle through A, B and C. Then $\angle ACB$ is a right inscribed angle, so by the converse of Thales' Theorem its chord $\overline{AB}$ is a diameter. The center of a circle is the midpoint of every diameter, so the center is the midpoint of the hypotenuse $\overline{AB}$. (This agrees with Lesson 5.2: the circumcenter of a right triangle lies on its hypotenuse.)

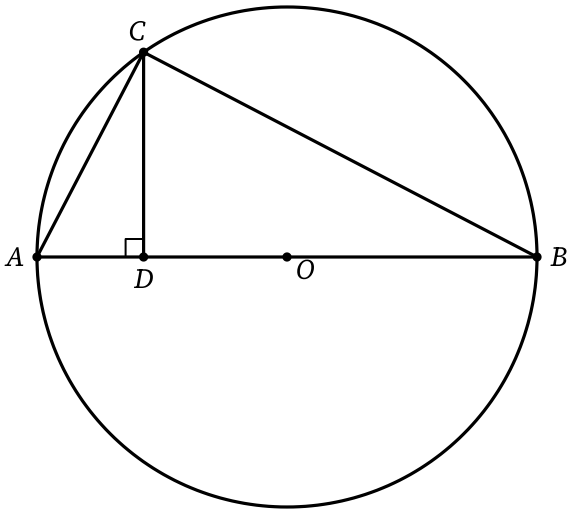
10. **Hard** In the figure, $\overline{AB}$ is a diameter of $\odot O$ with $AB = 26$, and C is a point on the circle with $AC = 10$. Find BC, and find $m\angle ABC$ to the nearest tenth of a degree.



Show answer

$\angle ACB = 90^\circ$ (angle in a semicircle), so $BC = \sqrt{26^2 - 10^2} = \sqrt{576} = 24$. Then $\sin\angle ABC = \dfrac{AC}{AB} = \dfrac{10}{26}$, so $m\angle ABC = \sin^{-1}\left(\dfrac{10}{26}\right) \approx 22.6^\circ$.

11. Hard Given: $\overline{AB}$ is a diameter of $\odot O$, C is on the circle, and $\overline{CD} \perp \overline{AB}$ at D, as in the figure. Prove: $\angle ACD \cong \angle ABC$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB}$ is a diameter of $\odot O$ and C is on $\odot O$; $\overline{CD} \perp \overline{AB}$ at D

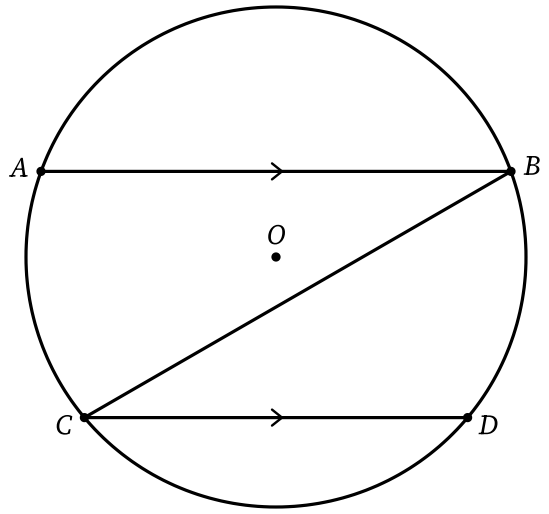
Prove: $\angle ACD \cong \angle ABC$

	STATEMENTS	REASONS
1	$\overline{AB}$ is a diameter of $\odot O$ and C is on $\odot O$	Given
2	$\angle ACB$ is a right angle	An angle inscribed in a semicircle is a right angle (III.31)
3	$\overline{CD} \perp \overline{AB}$	Given
4	$\angle ADC$ is a right angle	Definition of perpendicular lines
5	$\angle CAB$ and $\angle ABC$ are complementary	The acute angles of a right triangle are complementary ($\triangle ACB$)

	STATEMENTS	REASONS
6	$\angle CAD$ and $\angle ACD$ are complementary	The acute angles of a right triangle are complementary ($\triangle ADC$)
7	$\angle CAD$ is the same angle as $\angle CAB$	D lies on $\overline{AB}$, so $\overrightarrow{AD}$ and $\overrightarrow{AB}$ are the same ray
8	$\angle ACD \cong \angle ABC$	Congruent Complements Theorem (both are complements of $\angle CAB$)

∎

12. Hard Given: In $\odot O$, chords $\overline{AB} \parallel \overline{CD}$, as in the figure. Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$. Write a flow proof. (Use the transversal $\overline{BC}$.)



Show answer

See the model proof.

FLOW PROOF

Given: In $\odot O$, chords $\overline{AB} \parallel \overline{CD}$

Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$

(1) $\overline{AB} \parallel \overline{CD}$
Given

(2) $\angle ABC \cong \angle DCB$
Alternate Interior Angles Theorem (transversal $\overline{BC}$)
from (1)

(3) $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$
Inscribed Angle Theorem

(4) $m\angle DCB = \frac{1}{2} m\overset{\frown}{BD}$
Inscribed Angle Theorem

(5) $\frac{1}{2} m\overset{\frown}{AC} = \frac{1}{2} m\overset{\frown}{BD}$
Substitution Property of Equality
from (2), (3), (4)

(6) $m\overset{\frown}{AC} = m\overset{\frown}{BD}$
Multiplication Property of Equality
from (5)

(7) $\overset{\frown}{AC} \cong \overset{\frown}{BD}$
Definition of congruent arcs
from (6)

∎

13. **Hard** Prove that if a rhombus is inscribed in a circle, then it is a square. Write a paragraph proof.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: Rhombus $ABCD$ is inscribed in a circle

Prove: $ABCD$ is a square

Every rhombus is a parallelogram, and opposite angles of a parallelogram are congruent, so $m\angle A = m\angle C$. Because $ABCD$ is inscribed in a circle, its opposite angles are supplementary (Inscribed Quadrilateral Theorem): $m\angle A + m\angle C = 180^\circ$.

Substituting, $2m\angle A = 180^\circ$, so $m\angle A = 90^\circ$. Consecutive angles of a parallelogram are supplementary, so each of the other angles is also 90°, and $ABCD$ is a rectangle.

A quadrilateral that is both a rhombus and a rectangle is a square. Therefore $ABCD$ is a square.

∎

14. **Challenge** Triangle ABC is inscribed in a circle with center O and radius 10 cm, and $m\angle BAC = 30^\circ$. Find BC.

Show answer

$\angle BAC$ intercepts $\overset{\frown}{BC}$, so $m\overset{\frown}{BC} = 60^\circ$ and the central angle $\angle BOC$ measures 60°. In $\triangle BOC$, $OB = OC = 10$, so the base angles are equal and each is $(180^\circ - 60^\circ) \div 2 = 60^\circ$. The triangle is equiangular, hence equilateral, so $BC = 10$ cm.

COMMON MISTAKES

- Forgetting to halve. A central angle equals its arc, but an inscribed angle is half its arc. Check: is the vertex at the center or on the circle?
- Using the wrong arc. The intercepted arc lies inside the angle, between its sides, on the far side from the vertex. Trace it before you compute.
- Making adjacent angles of an inscribed quadrilateral supplementary. Only opposite angles are supplementary (unless the quadrilateral happens to have parallel sides).
- Using the Inscribed Quadrilateral Theorem when not all four vertices are on the circle. Check that every vertex lies on the circle first.

> **Make it a game · Corner on the Circle.** Trace a plate and draw a diameter. Place the corner of a book so its two edges pass through the diameter's endpoints, and mark where the corner lies. Slide to several spots. All land on the circle, since an angle in a semicircle is right. Then parent gives an arc of 100 degrees; give the inscribed angle.

10.5 Angles Formed by Chords, Secants and Tangents

OBJECTIVES

- Prove that an angle formed by a tangent and a chord is half its intercepted arc (Euclid III.32).
- Prove and use the rules for angles whose vertex is inside a circle (half the sum of the arcs) and outside a circle (half the difference of the arcs).
- Choose the correct rule by locating the vertex: on, inside or outside the circle.
- Solve practical problems, such as how much of the earth a satellite can see.

WATCH

Angle in the Alternate Segment Theorem

Eddie Woo · 10:52

https://www.youtube.com/watch?v=mum10EF2z1A

> **WHEN TO WATCH**
> Before the Tangent-Chord Angle Theorem.

Note: Eddie Woo teaches in Australia: he calls the tangent-chord theorem the 'alternate segment theorem'. The proofs of the inside and outside angle rules are taught from the Core ideas.

BACKUP VIDEO

Circle Theorems involving Angle Measures, Inscribed Angles, Intersecting Chords, Secants & Tangents

Mario's Math Tutoring · 5:30

https://www.youtube.com/watch?v=XckhcRlr4w8

Before the inside and outside angle theorems: where the vertex is (center, inside, on or outside the circle) decides the rule, with examples.

KEY TERMS

Tangent-chord angle

An angle formed by a tangent and a chord that meet at the point of tangency. Its intercepted arc is the arc that lies inside the angle.

Angle formed by two chords

An angle whose vertex is the point where two chords cross inside the circle. It intercepts one arc, and its vertical angle intercepts another.

Near arc and far arc

For an angle whose vertex is outside a circle, the two intercepted arcs: the near arc is the one closer to the vertex, the far arc the one farther away.

CORE IDEAS

Lesson 10.4 measured angles with the vertex on the circle. Now the vertex may be on the circle, inside it or outside it, and the sides may be chords, secants or tangents. One pattern organizes everything: **vertex on the circle, half the arc; vertex inside, half the sum of the arcs; vertex outside, half the difference of the arcs.**

A tangent and a chord meeting at the point of tangency behave exactly like an inscribed angle: the angle is **half its intercepted arc** (Euclid III.32). Why? Draw the diameter from the point of tangency. The tangent is perpendicular to it, so that angle is 90°, exactly half of a semicircle. The tangent-chord angle is that 90° angle minus (or plus) an inscribed angle, and half-arcs add and subtract along with it.

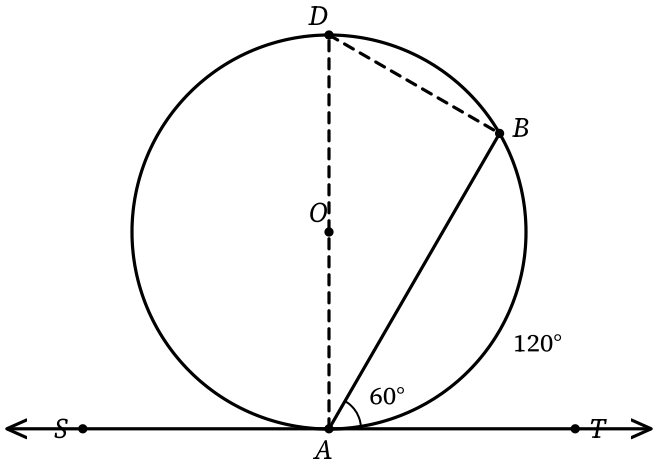
When two chords cross at a point E inside the circle, the angle at E is **half the sum** of the arc it intercepts and the arc intercepted by its vertical angle. Why? Draw one more chord to make a triangle with a vertex at E. The angle at E is then an *exterior* angle of that triangle, so it equals the sum of two inscribed angles, each of which is half an arc.

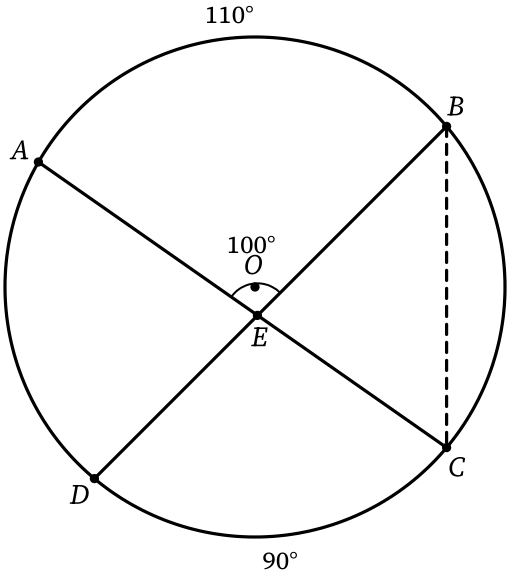
When the vertex P is outside the circle (two secants, a secant and a tangent, or two tangents), the angle is **half the difference** of the far arc and the near arc. Why? Again draw a chord to make a triangle, but now it is an inscribed angle on the far arc that is the exterior angle, and the angle at P is one of its remote interior angles. So the angle at P is the far half-arc *minus* the near half-arc.

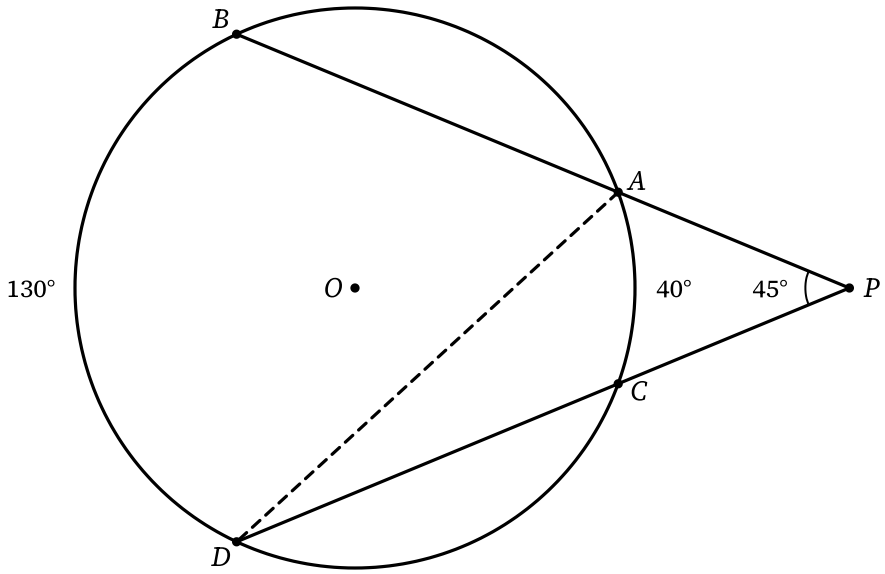
A good check: an angle with its vertex inside the circle is larger than an inscribed angle on the same arc, and an angle with its vertex outside is smaller. For two tangents, the formula simplifies: if the near arc is x, the far arc is $360^\circ - x$, and the angle is $\frac{1}{2}((360^\circ - x) - x) = 180^\circ - x$. The angle between the tangents and the near arc are supplementary.

This last fact answers a practical question. A satellite far above the earth sees the part of the earth between its two tangent lines of sight. The angle between those

lines tells exactly how big an arc of the earth it can see.



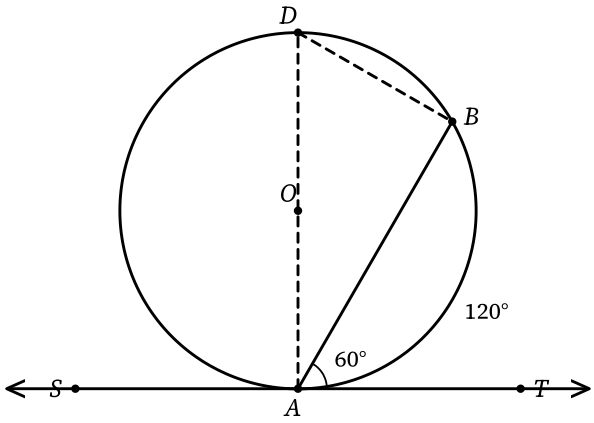




POSTULATES AND THEOREMS

THEOREM **Tangent-Chord Angle Theorem** Euclid III.32

If a tangent and a chord meet at the point of tangency, then the measure of the angle they form is half the measure of its intercepted arc.



PARAGRAPH PROOF

Given: Line ST is tangent to $\odot O$ at A, with A between S and T; $\overline{AB}$ is a chord

Prove: $m\angle BAT = \frac{1}{2} m\overset{\frown}{AB}$, where $\overset{\frown}{AB}$ is the arc inside $\angle BAT$

Draw the diameter $\overline{AD}$. By the Tangent Theorem, $\overline{AD} \perp \overleftrightarrow{ST}$, so $m\angle DAT = 90°$, and the semicircle from A to D on the side of T lies inside $\angle DAT$.

If $\overline{AB}$ is the diameter ($B = D$), then $m\angle BAT = 90° = \frac{1}{2}(180°)$, and the statement holds.

If $\angle BAT$ is acute, as in the figure, B lies on that semicircle and $\overrightarrow{AB}$ is between $\overrightarrow{AT}$ and $\overrightarrow{AD}$. By the Angle Addition Postulate and the Inscribed Angle Theorem, $m\angle BAT = m\angle DAT - m\angle DAB = 90^\circ - \frac{1}{2}m\overset{\frown}{BD}$. By the Arc Addition Postulate, $m\overset{\frown}{AB} + m\overset{\frown}{BD} = 180^\circ$, so $\frac{1}{2}m\overset{\frown}{AB} = 90^\circ - \frac{1}{2}m\overset{\frown}{BD}$. Therefore $m\angle BAT = \frac{1}{2}m\overset{\frown}{AB}$.

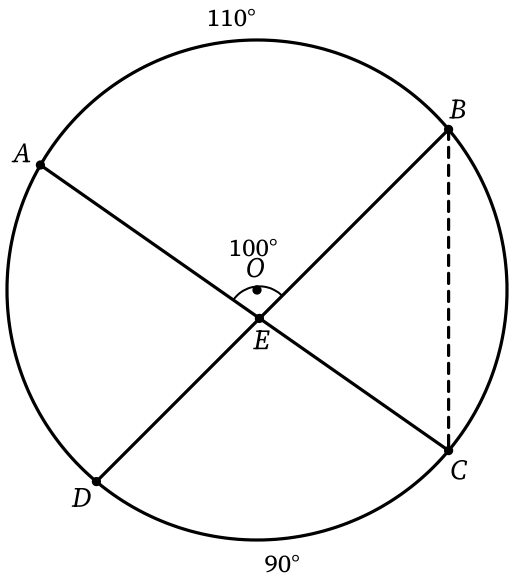
If $\angle BAT$ is obtuse, then $\overrightarrow{AD}$ is between $\overrightarrow{AT}$ and $\overrightarrow{AB}$, and $m\angle BAT = m\angle DAT + m\angle DAB = 90^\circ + \frac{1}{2}m\overset{\frown}{DB} = \frac{1}{2}(180^\circ + m\overset{\frown}{DB})$. The intercepted arc is the semicircle from A to D followed by $\overset{\frown}{DB}$, so again the angle is half its intercepted arc.

■

Euclid III.32: 'If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.'

THEOREM **Two Chords Angle Theorem**

If two chords intersect inside a circle, then the measure of each angle formed is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.



TWO-COLUMN PROOF

Given: Chords $\overline{AC}$ and $\overline{BD}$ of $\odot O$ intersect at E

Prove: $m\angle AEB = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$

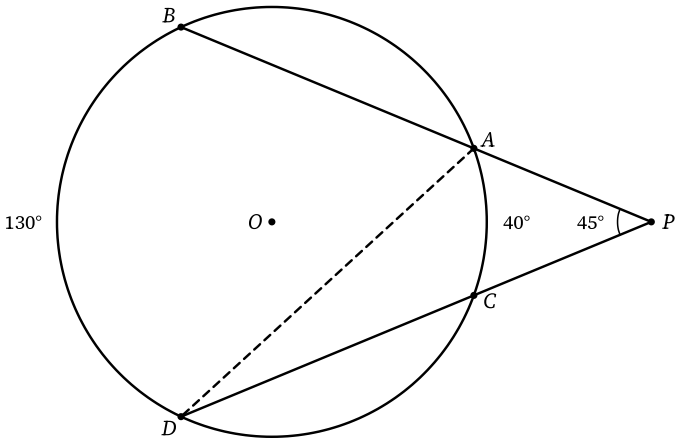
	STATEMENTS	REASONS
1	Chords $\overline{AC}$ and $\overline{BD}$ intersect at E	Given
2	Draw $\overline{BC}$	Through any two points there is exactly one line

	STATEMENTS	REASONS
3	$m\angle AEB = m\angle EBC + m\angle ECB$	Exterior Angle Theorem ($\angle AEB$ is an exterior angle of $\triangle BEC$, since A, E, C are collinear)
4	$m\angle EBC = \frac{1}{2}m\overset{\frown}{CD}$	Inscribed Angle Theorem ($\angle EBC$ is $\angle DBC$, which intercepts $\overset{\frown}{CD}$)
5	$m\angle ECB = \frac{1}{2}m\overset{\frown}{AB}$	Inscribed Angle Theorem ($\angle ECB$ is $\angle ACB$, which intercepts $\overset{\frown}{AB}$)
6	$m\angle AEB = \frac{1}{2}m\overset{\frown}{CD} + \frac{1}{2}m\overset{\frown}{AB}$	Substitution Property of Equality
7	$m\angle AEB = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$	Distributive Property

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THEOREM **Outside Angle Theorem**

If two secants, a secant and a tangent, or two tangents meet at a point outside a circle, then the measure of the angle they form is half the difference of the measures of the intercepted arcs (far arc minus near arc).



PARAGRAPH PROOF

Given: Two lines from an outside point P meet $\odot O$ as described in each case

Prove: $m\angle P = \frac{1}{2}(\text{far arc} - \text{near arc})$

Two secants. Let one secant meet the circle at A and B and the other at C and D, with A between P and B and C between P and D, as in the figure. Draw $\overline{AD}$. Because P, A, B are collinear, $\angle BAD$ is an exterior angle of $\triangle PAD$, so by the Exterior Angle Theorem $m\angle BAD = m\angle P + m\angle ADP$. By the Inscribed Angle

Theorem, $m\angle BAD = \frac{1}{2} m\overset{\frown}{BD}$; and $\angle ADP$ is $\angle ADC$ (since C is on $\overline{DP}$), so $m\angle ADP = \frac{1}{2} m\overset{\frown}{AC}$. Subtracting, $m\angle P = \frac{1}{2}(m\overset{\frown}{BD} - m\overset{\frown}{AC})$.

A tangent and a secant. Let $\overline{PA}$ be tangent at A and let the secant meet the circle at C and D, with C between P and D. Draw $\overline{AC}$. Then $\angle ACD$ is an exterior angle of $\triangle PAC$, so $m\angle ACD = m\angle P + m\angle PAC$. By the Inscribed Angle Theorem $m\angle ACD = \frac{1}{2} m\overset{\frown}{AD}$ (the far arc), and by the Tangent-Chord Angle Theorem $m\angle PAC = \frac{1}{2} m\overset{\frown}{AC}$ (the near arc). Subtracting, $m\angle P = \frac{1}{2}(m\overset{\frown}{AD} - m\overset{\frown}{AC})$.

Two tangents. This case is proved in Practice Problem 13.

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The two-tangent case is Practice Problem 13.

KEY FORMULAS

$$m\angle 1 = \frac{1}{2} m\overset{\frown}{AB} \quad \text{(vertex on the circle)}$$

$$m\angle 1 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD}) \quad \text{(vertex inside)}$$

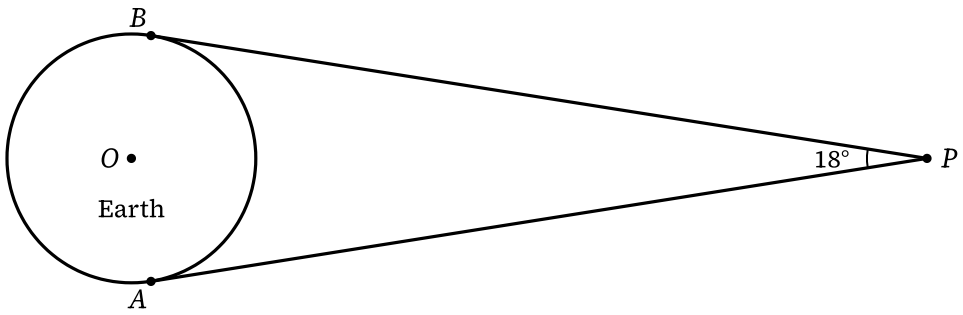
$$m\angle P = \frac{1}{2}(\text{far arc} - \text{near arc}) \quad \text{(vertex outside)}$$

$$m\angle P = 180^\circ - (\text{near arc}) \quad \text{(two tangents)}$$

Euclid connection. III.32 is Euclid's tangent-chord theorem, stated as 'the angles it makes with the tangent are equal to the angles in the alternate segments.' The inside and outside rules are not propositions of the *Elements*, but they follow directly from Euclid's I.32 (the exterior angle of a triangle) and III.20 (the inscribed angle), exactly as our proofs show.

Practice habit. Before computing, decide where the vertex is: on, inside or outside the circle. Say the rule out loud ('half the arc,' 'half the sum,' 'half the difference'), then estimate the angle from the figure as a check.

Example 1. A weather satellite P is far above the equator. Its two lines of sight that just graze the earth (tangent lines to the equator) meet at the satellite at an angle of 18°, as in the figure. What is the measure of the arc of the equator that the satellite can see?

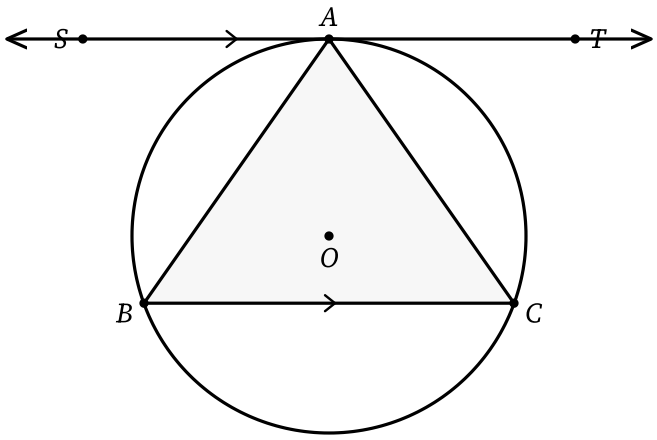


Show solution

1. The two lines of sight are tangents from the outside point P. The satellite sees the near arc between the points of tangency; call its measure x. The far arc is the rest of the equator, $360^\circ - x$.
2. Outside Angle Theorem: $18 = \frac{1}{2}((360 - x) - x) = 180 - x$.
3. So $x = 162$. The satellite sees an arc of 162°, a little less than half of the equator.

Answer: 162°

Example 2. Given: $\overleftrightarrow{ST}$ is tangent to $\odot O$ at A, $\triangle ABC$ is inscribed in $\odot O$, and $\overleftrightarrow{ST} \parallel \overline{BC}$, as in the figure. Prove: $\overline{AB} \cong \overline{AC}$.



Show solution

1. Work backward: to prove two sides congruent, prove the angles opposite them, $\angle ACB$ and $\angle ABC$, congruent and use the Converse of the Isosceles Triangle Theorem.
2. $\angle SAB$ (tangent and chord) and $\angle ACB$ (inscribed) are both half of $\overset{\frown}{AB}$.
3. The parallel lines give alternate interior angles $\angle SAB \cong \angle ABC$. Chain the congruences together.

FLOW PROOF

Given: $\overleftrightarrow{ST}$ is tangent to $\odot O$ at A; $\triangle ABC$ is inscribed in $\odot O$; $\overleftrightarrow{ST} \parallel \overline{BC}$

Prove: $\overline{AB} \cong \overline{AC}$

(1) $\overleftrightarrow{ST}$ is tangent to $\odot O$ at A
Given

(2) $\overleftrightarrow{ST} \parallel \overline{BC}$
Given

(3) $m\angle SAB = \frac{1}{2} m\overset{\frown}{AB}$
Tangent-Chord Angle Theorem
from (1)

(4) $m\angle ACB = \frac{1}{2} m\overset{\frown}{AB}$

Inscribed Angle Theorem

(5) $\angle SAB \cong \angle ACB$
Substitution Property of Equality
from (3), (4)

(6) $\angle SAB \cong \angle ABC$
Alternate Interior Angles Theorem (transversal $\overline{AB}$)
from (2)

(7) $\angle ABC \cong \angle ACB$
Transitive Property of Congruence
from (5), (6)

(8) $\overline{AB} \cong \overline{AC}$
Converse of the Isosceles Triangle Theorem
from (7)

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Answer: See the proof.

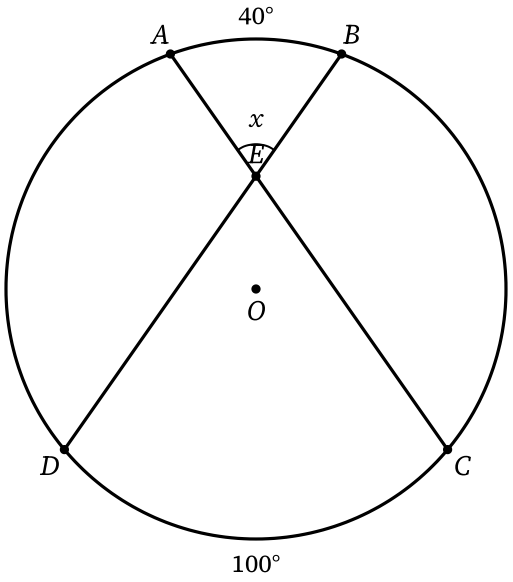
PRACTICE

1. Easy A tangent and a chord of a circle meet at the point of tangency, and the arc inside the angle they form measures $128°$. Find the measure of the angle.

   Show answer

   $$\frac{1}{2}(128°) = 64°$$

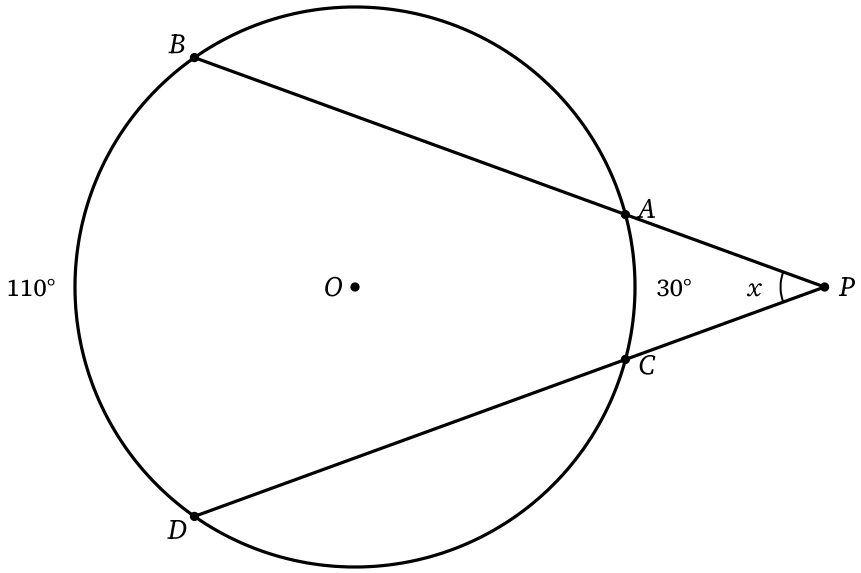
2. **Easy** In the figure, chords $\overline{AC}$ and $\overline{BD}$ intersect at E, $m\overset{\frown}{AB} = 40^\circ$ and $m\overset{\frown}{CD} = 100^\circ$. Find $x = m\angle AEB$.



Show answer

$$x = \frac{1}{2}(40^\circ + 100^\circ) = 70^\circ$$

3. **Easy** In the figure, two secants meet at the outside point P. The far arc $\overset{\frown}{BD}$ measures $110°$ and the near arc $\overset{\frown}{AC}$ measures $30°$. Find $x = m\angle APC$.



Show answer

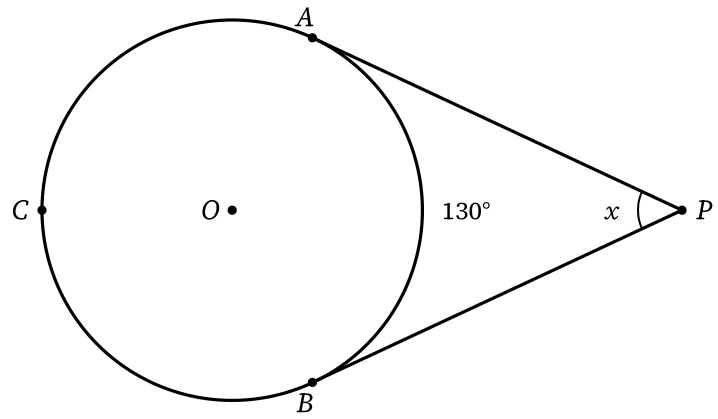
$$x = \frac{1}{2}(110° - 30°) = 40°$$

4. **Easy** The angle between a tangent and a chord at the point of tangency measures $73°$. Find the measure of the intercepted arc.

Show answer

$$2(73°) = 146°$$

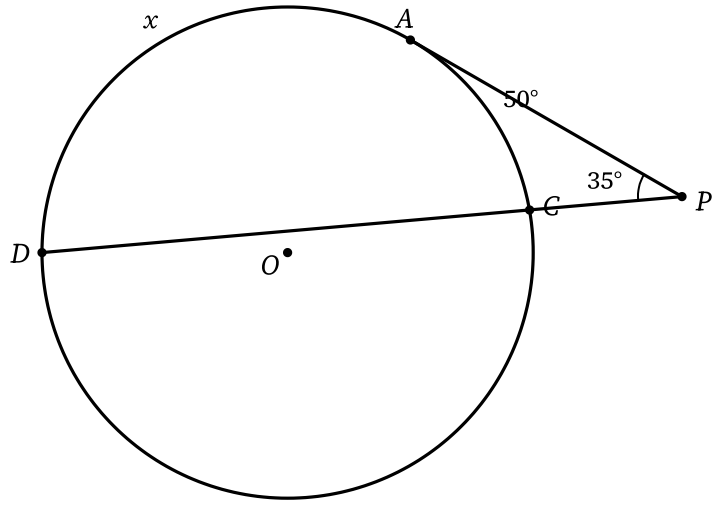
5. **Medium** In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B, and the minor arc $\overset{\frown}{AB}$ measures 130°. Find $x = m\angle APB$.



Show answer

The far arc is $360^\circ - 130^\circ = 230^\circ$, so $x = \dfrac{1}{2}(230^\circ - 130^\circ) = 50^\circ$.

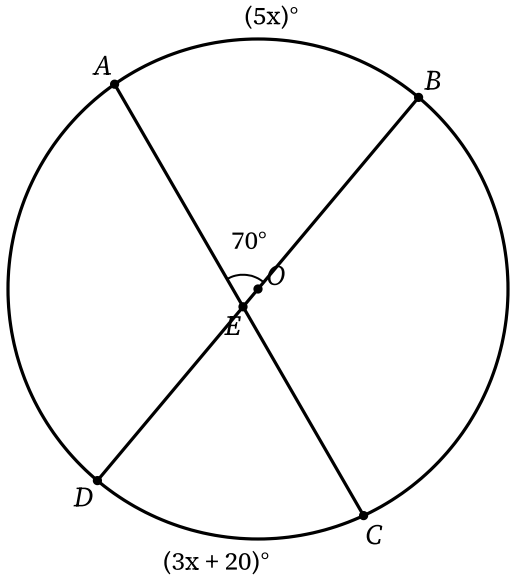
6. **Medium** In the figure, $\overline{PA}$ is tangent to the circle at A, and a secant from P meets the circle at C and D. If $m\angle APC = 35^\circ$ and the near arc $\overset{\frown}{AC}$ measures 50°, find x, the measure of the far arc $\overset{\frown}{AD}$.



Show answer

$35 = \dfrac{1}{2}(x - 50)$, so $x - 50 = 70$ and $x = 120^\circ$.

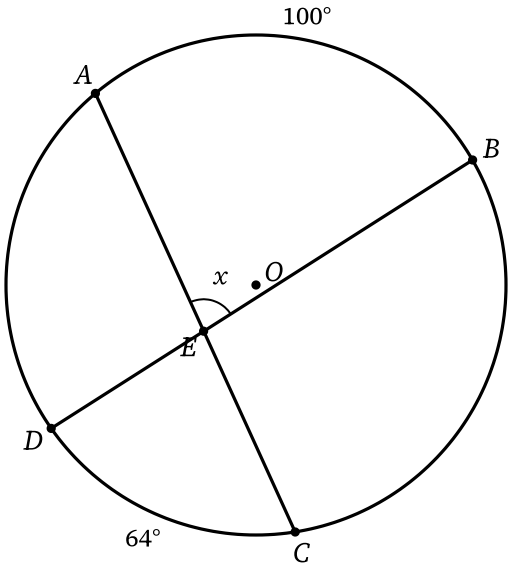
7. **Medium** In the figure, chords $\overline{AC}$ and $\overline{BD}$ intersect at E, $m\angle AEB = 70^\circ$, $m\overset{\frown}{AB} = (5x)^\circ$ and $m\overset{\frown}{CD} = (3x + 20)^\circ$. Find x.



Show answer

$70 = \frac{1}{2}(5x + 3x + 20)$, so $8x + 20 = 140$ and $x = 15$ (the arcs measure 75° and 65°).

8. **Medium** In a round rose window, two straight lead strips are chords that cross inside the window, as in the figure. The arcs cut off by the angle where they cross and by its vertical angle measure 100° and 64°. Find x, the measure of the angle between the strips.



Show answer

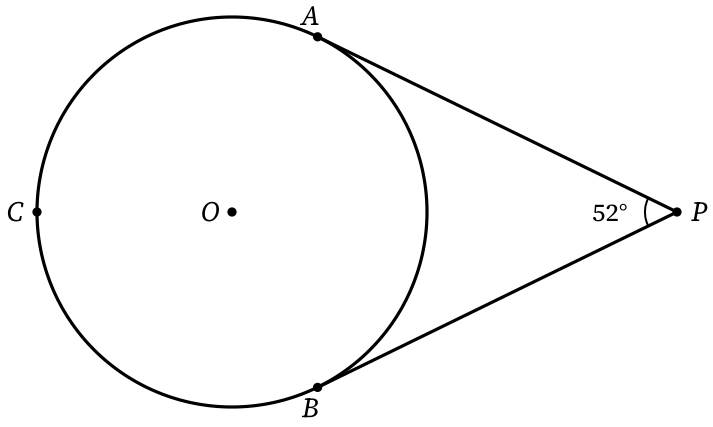
$$x = \frac{1}{2}(100^\circ + 64^\circ) = 82^\circ$$

9. **Medium** Two secants from an outside point P meet a circle at A, B and at C, D, with A and C nearer to P. Explain why $\angle P$ must be smaller than the inscribed angle $\angle BAD$ that intercepts the far arc.

Show answer

Because P, A, B are collinear, $\angle BAD$ is an exterior angle of $\triangle PAD$. By the Exterior Angle Theorem, $m\angle BAD = m\angle P + m\angle ADP$, and $m\angle ADP > 0$, so $m\angle P < m\angle BAD$. That is why the outside angle is half the far arc *minus* half the near arc.

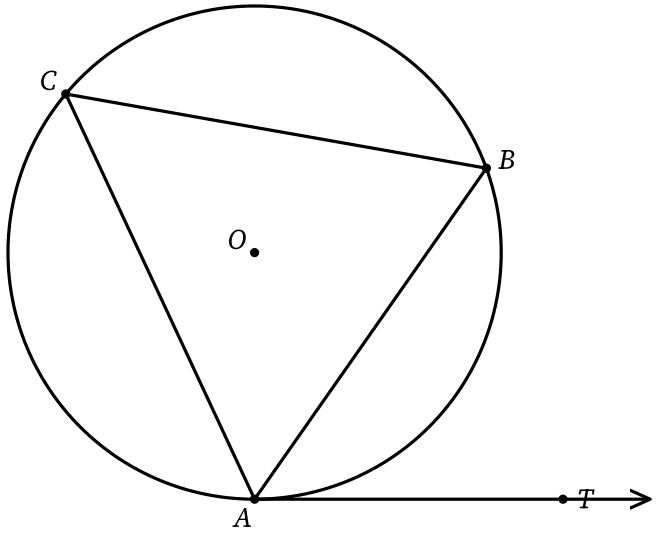
10. Hard In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B, C is on the major arc, and $m\angle APB = 52^\circ$. Find $m\overset{\frown}{AB}$ and $m\overset{\frown}{ACB}$.



Show answer

Let $m\overset{\frown}{AB} = x$. Then $52 = \frac{1}{2}((360 - x) - x) = 180 - x$, so $m\overset{\frown}{AB} = 128^\circ$ and $m\overset{\frown}{ACB} = 232^\circ$.

11. Hard Given: $\overrightarrow{AT}$ is tangent to $\odot O$ at A; B and C are on $\odot O$, with C not on the arc $\overset{\frown}{AB}$ that lies inside $\angle TAB$, as in the figure. Prove: $\angle TAB \cong \angle ACB$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\overrightarrow{AT}$ is tangent to $\odot O$ at A; B and C are on $\odot O$, and C is not on the arc $\overset{\frown}{AB}$ inside $\angle TAB$

Prove: $\angle TAB \cong \angle ACB$

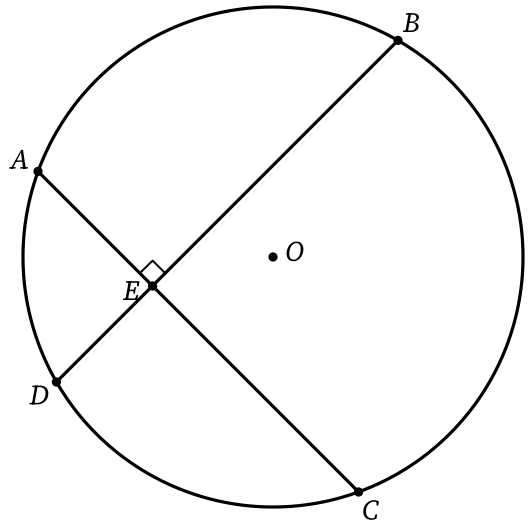
By the Tangent-Chord Angle Theorem, $m\angle TAB = \frac{1}{2} m\overset{\frown}{AB}$, where $\overset{\frown}{AB}$ is the arc inside $\angle TAB$.

The vertex C of the inscribed angle $\angle ACB$ is not on that arc, so $\angle ACB$ intercepts the same arc $\overset{\frown}{AB}$, and by the Inscribed Angle Theorem $m\angle ACB = \frac{1}{2} m\overset{\frown}{AB}$.

By substitution, $m\angle TAB = m\angle ACB$, so $\angle TAB \cong \angle ACB$. This is Euclid's own form of III.32: the angle between a tangent and a chord equals the angle in the alternate segment.

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12. **Hard** Given: In $\odot O$, chords $\overline{AC}$ and $\overline{BD}$ are perpendicular at E, as in the figure. Prove: $m\overset{\frown}{AB} + m\overset{\frown}{CD} = 180°$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: In $\odot O$, chords $\overline{AC} \perp \overline{BD}$ at E

Prove: $m\overset{\frown}{AB} + m\overset{\frown}{CD} = 180°$

(1) $\overline{AC} \perp \overline{BD}$ at E
Given

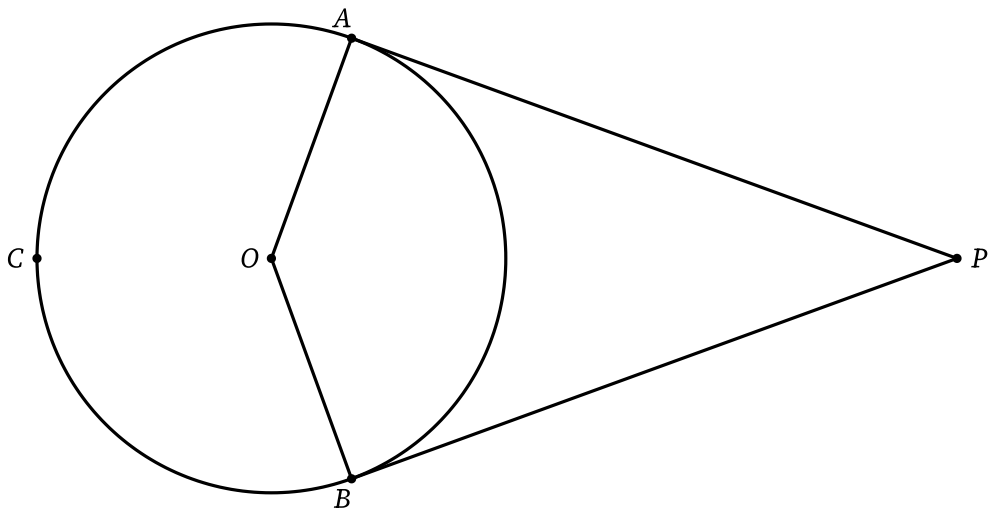
(2) $m\angle AEB = 90°$
Definition of perpendicular lines
from (1)

(3) $m\angle AEB = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$
Two Chords Angle Theorem

(4) $\frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD}) = 90°$
Substitution Property of Equality
from (2), (3)

(5) $m\overset{\frown}{AB} + m\overset{\frown}{CD} = 180°$
Multiplication Property of Equality
from (4)

13. **Hard** Prove the two-tangent case of the Outside Angle Theorem. Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B, and C is on the major arc, as in the figure. Prove: $m\angle APB = \frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB})$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B; C is on the major arc $\overset{\frown}{ACB}$

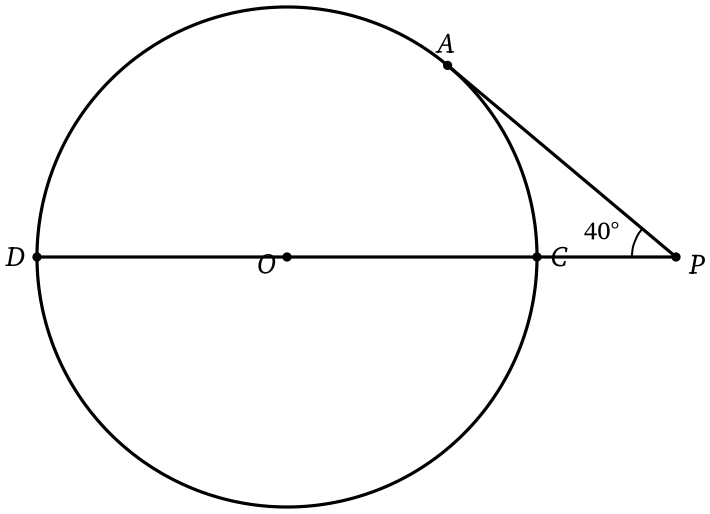
Prove: $m\angle APB = \frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB})$

	STATEMENTS	REASONS
1	$\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$ at A and B	Given
2	$\overline{OA} \perp \overline{PA}$ and $\overline{OB} \perp \overline{PB}$	Tangent Theorem
3	$m\angle PAO = 90°$ and $m\angle OBP = 90°$	Definition of perpendicular lines
4	$m\angle APB + m\angle PAO + m\angle AOB + m\angle OBP = 360°$	Polygon Interior Angle Sum Theorem (quadrilateral $PAOB$)
5	$m\angle APB + m\angle AOB = 180°$	Substitution and Subtraction Properties of Equality
6	$m\angle AOB = m\overset{\frown}{AB}$	Definition of the measure of a minor arc

	STATEMENTS	REASONS
7	$m\angle APB = 180^\circ - m\overset{\frown}{AB}$	Substitution and Subtraction Properties of Equality
8	$m\overset{\frown}{ACB} = 360^\circ - m\overset{\frown}{AB}$	Arc Addition Postulate
9	$\frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB}) = \frac{1}{2}(360^\circ - 2m\overset{\frown}{AB}) = 180^\circ - m\overset{\frown}{AB}$	Substitution Property of Equality and Distributive Property
10	$m\angle APB = \frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB})$	Transitive Property of Equality (steps 7 and 9)

∎

14. **Challenge** In the figure, $\overline{PA}$ is tangent to $\odot O$ at A, and a secant from P passes through the center, meeting the circle at C (nearer to P) and D. If $m\angle APO = 40^\circ$, find $m\overset{\frown}{AC}$. Solve it with the Outside Angle Theorem, then check with $\triangle OAP$.



Show answer

Let $m\overset{\frown}{AC} = x$. Since $\overline{CD}$ is a diameter, the far arc $\overset{\frown}{AD}$ measures $180^\circ - x$. Then $40 = \frac{1}{2}((180 - x) - x) = 90 - x$, so $x = 50^\circ$. Check: in $\triangle OAP$, $m\angle OAP = 90^\circ$ (Tangent Theorem), so $m\angle AOP = 180^\circ - 90^\circ - 40^\circ = 50^\circ$, and the central angle $\angle AOC$ equals $m\overset{\frown}{AC}$.

COMMON MISTAKES

- Using half the sum for an angle whose vertex is outside the circle. Outside angles use half the *difference*, far arc minus near arc.
- Using the wrong pair of arcs for two chords. Use the arc inside the angle and the arc inside its vertical angle, not the two arcs beside it.
- Taking the whole arc for a tangent-chord angle. Like an inscribed angle, it is half its arc.
- Subtracting in the wrong order and getting a negative angle. Always subtract the near arc from the far arc.

> **Make it a game · Circle Angle Cards.** Make nine cards, each showing a circle with two arcs and a vertex that is inside, on or outside the circle. Draw one, name the rule (half the sum, half the arc, or half the difference), and compute within 15 seconds. Complete the deck, record your time, then shuffle and beat it.

10.6 Segment Lengths in Circles

OBJECTIVES

- Prove the Intersecting Chords, Secant-Secant and Tangent-Secant Theorems with similar triangles.
- Find unknown segment lengths in circles, including by solving quadratic equations.
- Choose the correct product rule for chords, secants and tangents.
- Find the distance to the horizon from a tower, a ship's mast or a mountain.

WATCH

Intercept Properties - Circle Geometry (1 of 3: Intersecting chords)

Eddie Woo · 5:59

https://www.youtube.com/watch?v=O1hCcaEjnBE

> **WHEN TO WATCH**
> Before the Intersecting Chords Theorem.

Note: Eddie Woo teaches in Australia: he calls these 'intercept properties'. The proof of the Tangent-Secant Theorem is taught from the lesson.

BACKUP VIDEO

Circle Theorems involving Secants, Chords, and Tangents

Mario's Math Tutoring · 3:46

https://www.youtube.com/watch?v=ASZl3SnKwI8

Before the Secant-Secant and Tangent-Secant Theorems: all three segment-length rules, with an example of each.

KEY TERMS

Segments of a chord

When two chords cross at a point E, each chord is divided into two segments; for chord $\overline{AC}$ they are $\overline{AE}$ and $\overline{EC}$.

Secant segment

A segment of a secant line from a point outside the circle to the farther point where the line meets the circle, such as $\overline{PB}$. It is the 'whole' in the product rules.

External secant segment

The part of a secant segment that lies outside the circle, from the outside point to the nearer point where the line meets the circle, such as $\overline{PA}$.

CORE IDEAS

Lesson 10.5 measured the angles made by chords, secants and tangents; this lesson measures the lengths of their pieces. In every case, inscribed angles give a pair of similar triangles (Unit 8), and the proportion between their sides turns into a simple rule about products.

Two chords. If chords $\overline{AC}$ and $\overline{BD}$ cross at E, then $AE \cdot EC = BE \cdot ED$. Why? Draw $\overline{AB}$ and $\overline{DC}$. Triangles AEB and DEC have vertical angles at E, and $\angle BAC$ and $\angle BDC$ are inscribed angles on the same arc, so the triangles are similar by AA. Cross-multiplying their proportion gives the products. For example, if $AE = 4$, $EC = 6$ and $BE = 3$, then $3 \cdot ED = 24$ and $ED = 8$.

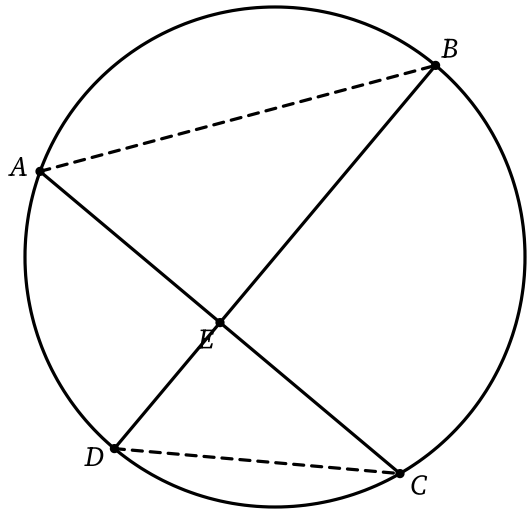
Two secants. From an outside point P, a secant meets the circle first at A and then at B. The whole secant segment is $\overline{PB}$ and its external part is $\overline{PA}$. For two secants from P, **external part times whole is the same**: $PA \cdot PB = PC \cdot PD$. The reason is again a pair of similar triangles, PBC and PDA, which share $\angle P$ and have inscribed angles on the same arc.

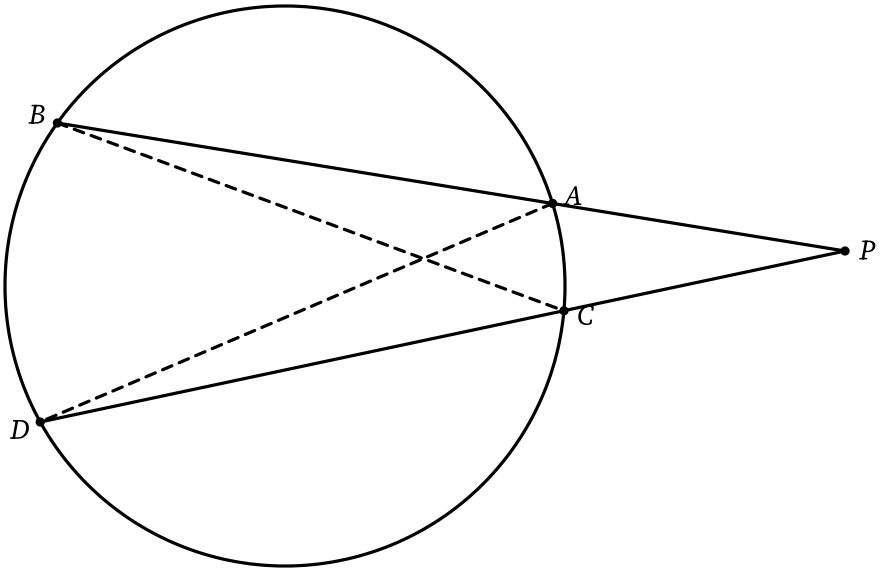
A tangent and a secant. Think of a tangent as a secant whose two meeting points have slid together into one point T: its external part and its whole are both PT. So the rule becomes $PT^2 = PA \cdot PB$. The proof uses the tangent-chord angle of Lesson 10.5 to get the similar triangles.

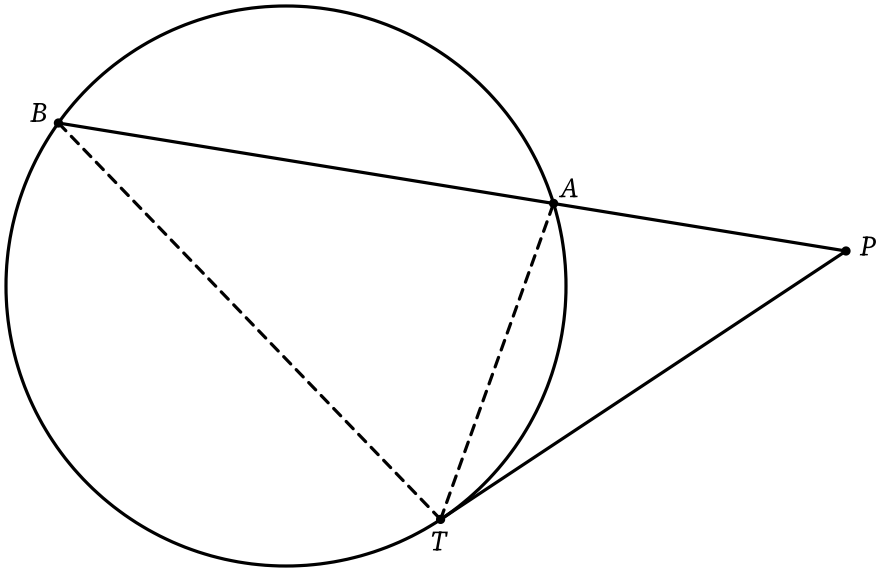
Notice the pattern. For a fixed point P and a fixed circle, every line through P that meets the circle gives the same product: the two pieces of a chord if P is inside, or external part times whole if P is outside. (Later mathematicians named this number the *power* of the point.)

How far is the horizon? Stand at height h above the earth, whose radius is R. The line from your eye through the earth's center is a secant: its external part is h and its whole is $h + 2R$. Your line of sight to the horizon just grazes the earth, so it is a

tangent of length d, and $d^2 = h(h + 2R)$. (This is the Pythagorean Theorem in disguise: $d^2 + R^2 = (R + h)^2$.)



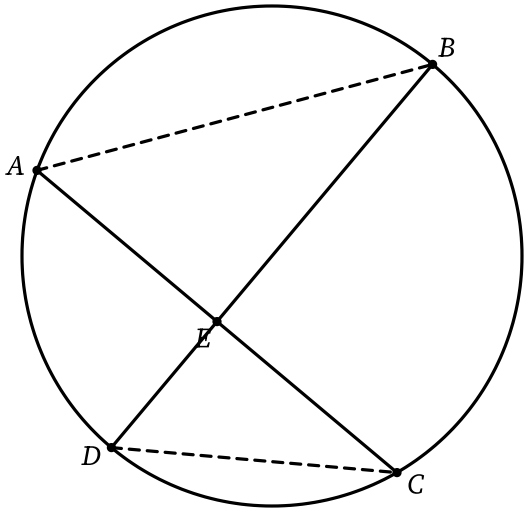




POSTULATES AND THEOREMS

THEOREM **Intersecting Chords Theorem** Euclid III.35

If two chords intersect inside a circle, then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other.



TWO-COLUMN PROOF

Given: Chords $\overline{AC}$ and $\overline{BD}$ of a circle intersect at E

Prove: $AE \cdot EC = BE \cdot ED$

	STATEMENTS	REASONS
1	Chords $\overline{AC}$ and $\overline{BD}$ intersect at E	Given

	STATEMENTS	REASONS
2	Draw $\overline{AB}$ and $\overline{DC}$	Through any two points there is exactly one line
3	$\angle AEB \cong \angle DEC$	Vertical Angles Theorem
4	$\angle BAE \cong \angle CDE$	Inscribed angles that intercept the same arc ($\overset{\frown}{BC}$) are congruent
5	$\triangle AEB \sim \triangle DEC$	AA Similarity Postulate
6	$\frac{AE}{DE} = \frac{EB}{EC}$	Corresponding sides of similar triangles are proportional
7	$AE \cdot EC = DE \cdot EB$	Cross Products Property
8	$AE \cdot EC = BE \cdot ED$	Commutative Property of Multiplication

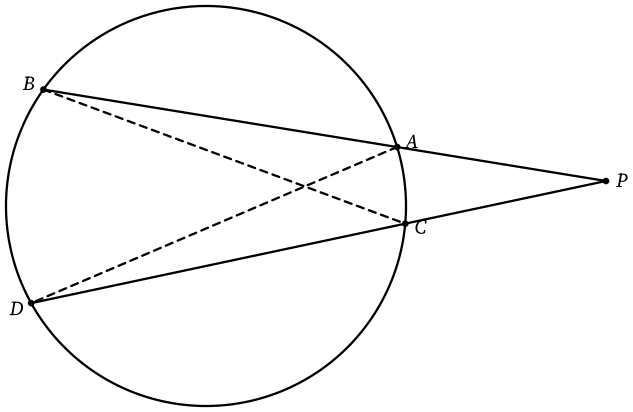
∎

Euclid III.35: 'If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.'

THEOREM **Secant-Secant Theorem**

Euclid III.36 (corollary)

If two secants are drawn to a circle from an outside point, then the product of the lengths of one secant segment and its external part equals the product of the lengths of the other secant segment and its external part.



PARAGRAPH PROOF

Given: From outside point P, one secant meets the circle at A and B and another at C and D, with A between P and B and C between P and D

Prove: $PA \cdot PB = PC \cdot PD$

Draw $\overline{BC}$ and $\overline{AD}$. In $\triangle PBC$ and $\triangle PDA$, the angle at P is the same angle, so $\angle BPC \cong \angle DPA$ (Reflexive Property).

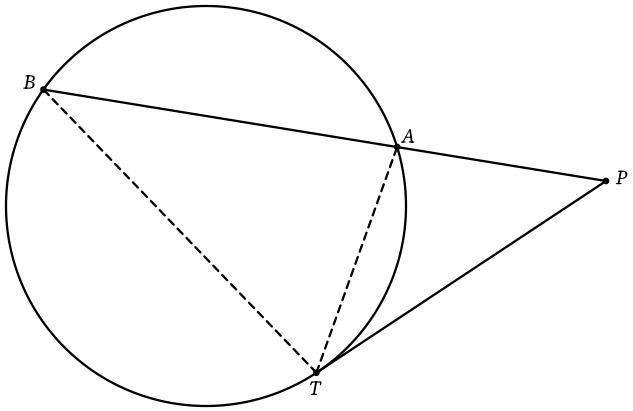
$\angle PBC$ is the inscribed angle $\angle ABC$, and $\angle PDA$ is the inscribed angle $\angle CDA$. Both intercept $\overset{\frown}{AC}$, so they are congruent. Therefore $\triangle PBC \sim \triangle PDA$ by the AA Similarity Postulate.

Corresponding sides of similar triangles are proportional, so $\frac{PB}{PD} = \frac{PC}{PA}$. By the Cross Products Property, $PA \cdot PB = PC \cdot PD$.

∎

THEOREM **Tangent-Secant Theorem** Euclid III.36

If a tangent and a secant are drawn to a circle from an outside point, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external part.



FLOW PROOF

Given: $\overline{PT}$ is tangent to the circle at T; A secant from P meets the circle at A and B, with A between P and B

Prove: $PT^2 = PA \cdot PB$

(1) $\overline{PT}$ is tangent at T; the secant meets the circle at A and B
Given

(2) Draw $\overline{TA}$ and $\overline{TB}$
Through any two points there is exactly one line

(3) $\angle TPA \cong \angle BPT$
Reflexive Property (A is on $\overline{PB}$, so these are the same angle)
from (1)

(4) $m\angle PTA = \frac{1}{2} m\overset{\frown}{TA}$
Tangent-Chord Angle Theorem
from (1), (2)

(5) $m\angle PBT = \frac{1}{2} m\overset{\frown}{TA}$
Inscribed Angle Theorem
from (2)

(6) $\angle PTA \cong \angle PBT$
Substitution Property of Equality
from (4), (5)

(7) $\triangle PTA \sim \triangle PBT$
AA Similarity Postulate
from (3), (6)

(8) $\frac{PT}{PB} = \frac{PA}{PT}$
Corresponding sides of similar triangles are proportional
from (7)

(9) $PT^2 = PA \cdot PB$
Cross Products Property
from (8)

∎

Euclid III.36: the rectangle contained by the whole secant and its part outside the circle 'will be equal to the square on the tangent.'

KEY FORMULAS

$$AE \cdot EC = BE \cdot ED \quad \text{(two chords)}$$

$$PA \cdot PB = PC \cdot PD \quad \text{(two secants: outside part} \times \text{whole)}$$

$$PT^2 = PA \cdot PB \quad \text{(tangent and secant)}$$

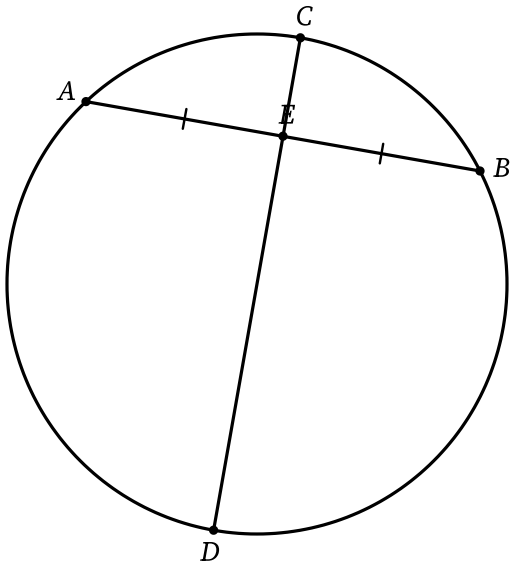
$$d^2 = h(h + 2R) \quad \text{(distance to the horizon)}$$

Euclid connection. III.35 and III.36 are these theorems. Euclid speaks of 'the rectangle contained by' two segments and 'the square on' a segment, meaning areas, where we speak of products of lengths. He proves them with the Pythagorean Theorem (I.47) and the rectangle facts of Book II rather than with similar triangles, which he does not develop until Book VI; the similar-triangle proofs here are a shorter modern route to the same truths.

> **Practice habit.** Say the product rule in words before you put in numbers: 'piece times piece' for chords, 'outside part times whole' for secants, 'tangent squared' for a tangent. Label every segment on the diagram, and write the whole secant length beside the outside part.

WORKED EXAMPLES

Example 1. Given: Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at E, and E is the midpoint of $\overline{AB}$, as in the figure. Prove: $AE^2 = CE \cdot ED$.



Show solution

1. The prove statement is a product of chord pieces, so start from the Intersecting Chords Theorem: $AE \cdot EB = CE \cdot ED$.
2. The midpoint gives $EB = AE$; substitute.
3. Notice what the result says: AE is the geometric mean of CE and ED (Lesson 8.1).

TWO-COLUMN PROOF

Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect at E; E is the midpoint of $\overline{AB}$

Prove: $AE^2 = CE \cdot ED$

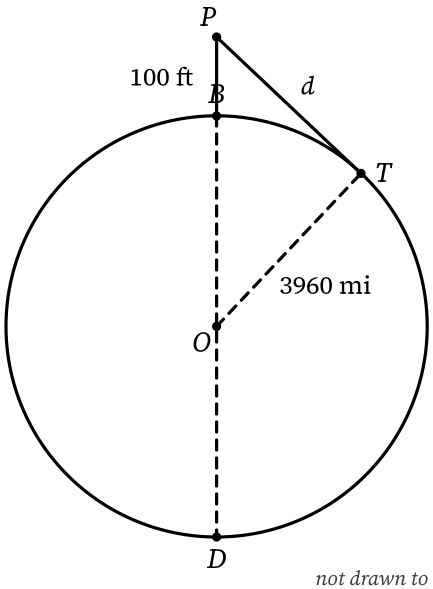
	STATEMENTS	REASONS
1	Chords $\overline{AB}$ and $\overline{CD}$ intersect at E	Given
2	$AE \cdot EB = CE \cdot ED$	Intersecting Chords Theorem
3	E is the midpoint of $\overline{AB}$	Given
4	$AE = EB$	Definition of midpoint
5	$AE \cdot AE = CE \cdot ED$	Substitution Property of Equality

	STATEMENTS	REASONS
6	$AE^2 = CE \cdot ED$	Definition of a square ($x \cdot x = x^2$)

∎

Answer: See the proof.

Example 2. How far is the horizon from the top of a lighthouse 100 feet above the sea? Treat the earth as a sphere of radius 3960 miles; the line of sight to the horizon is tangent to the earth. Give the distance in miles, to the nearest tenth.



not drawn to scale

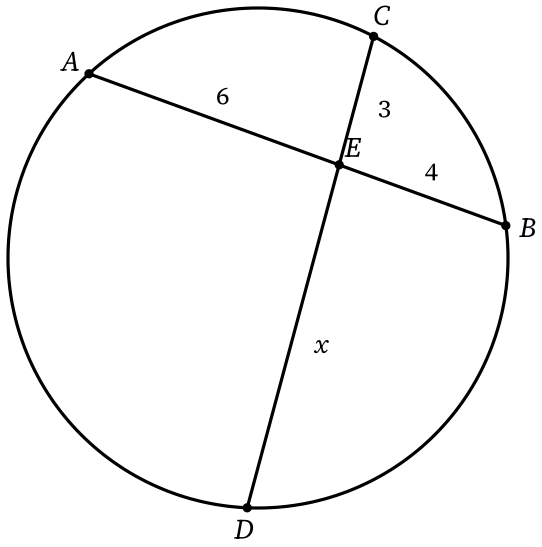
Show solution

1. Use one unit throughout: $h = 100 \text{ ft} = \dfrac{100}{5280} \text{ mi} \approx 0.018939 \text{ mi}$.
2. The line from the top P through the earth's center is a secant. Its external part is $PB = h$ and its whole is $PD = h + 2R = h + 7920$ miles.
3. Tangent-Secant Theorem: $d^2 = PB \cdot PD = h(h + 7920) = \dfrac{100}{5280} \times 7920 + h^2 = 150 + 0.000359 \approx 150.0004$.
4. $d \approx \sqrt{150.0004} \approx 12.247$, so the horizon is about 12.2 miles away.

Answer: about 12.2 miles

PRACTICE

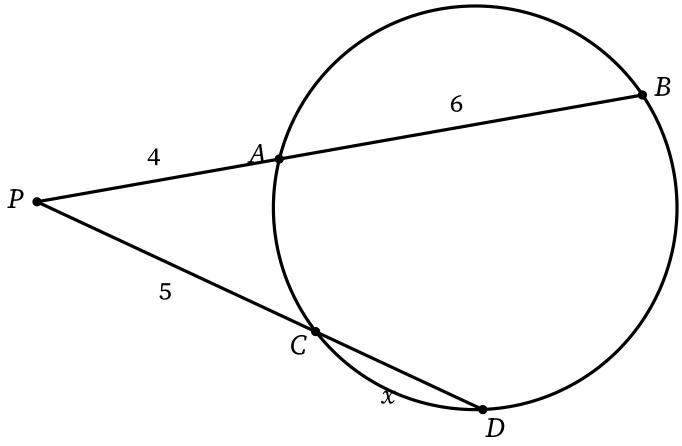
1. **Easy** In the figure, chords $\overline{AB}$ and $\overline{CD}$ intersect at E, with $AE = 6$, $EB = 4$ and $CE = 3$. Find $x = ED$.



Show answer

$6 \cdot 4 = 3x$, so $x = 8$.

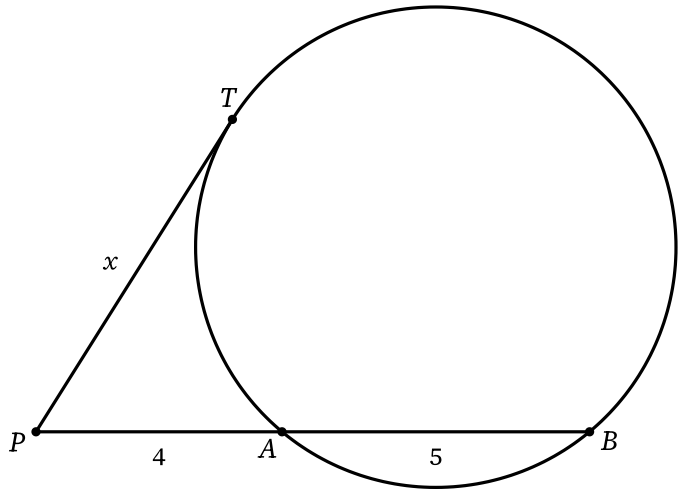
2. **Easy** In the figure, two secants are drawn from P. $PA = 4$, $AB = 6$ and $PC = 5$. Find $x = CD$.



Show answer

$PB = 10$, so $4 \cdot 10 = 5 \cdot PD$ and $PD = 8$. Then $x = CD = 8 - 5 = 3$.

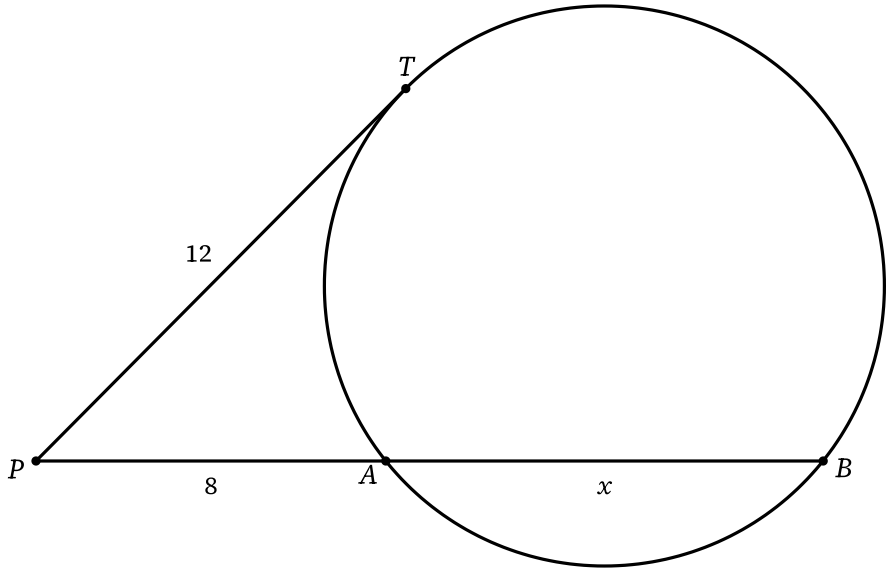
3. **Easy** In the figure, $\overline{PT}$ is tangent to the circle at T, and a secant from P meets the circle at A and B with $PA = 4$ and $AB = 5$. Find $x = PT$.



Show answer

$x^2 = PA \cdot PB = 4 \cdot 9 = 36$, so $x = 6$.

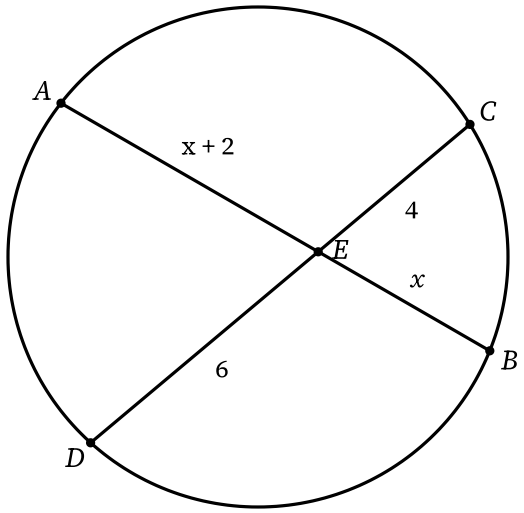
4. **Easy** In the figure, $\overline{PT}$ is tangent to the circle at T with $PT = 12$, and a secant from P meets the circle at A and B with $PA = 8$. Find $x = AB$.



Show answer

$12^2 = 8 \cdot PB$, so $PB = 18$ and $x = 18 - 8 = 10$.

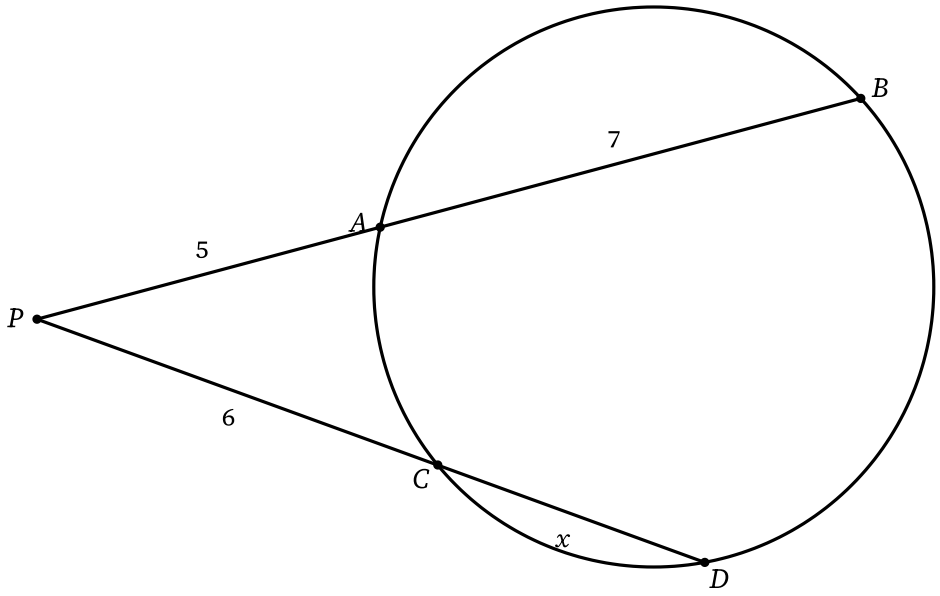
5. **Medium** In the figure, chords $\overline{AB}$ and $\overline{CD}$ intersect at E, with $AE = x + 2$, $EB = x$, $CE = 4$ and $ED = 6$. Find x.



Show answer

$x(x + 2) = 24$, so $x^2 + 2x - 24 = 0$ and $(x + 6)(x - 4) = 0$. A length is positive, so $x = 4$.

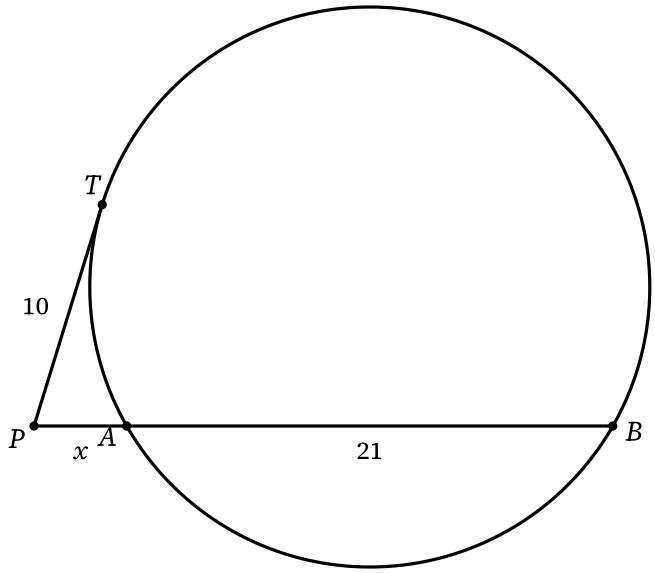
6. **Medium** In the figure, two secants are drawn from P, with $PA = 5$, $AB = 7$, $PC = 6$ and $CD = x$. Find x.



Show answer

$5 \cdot 12 = 6(6 + x)$, so $60 = 36 + 6x$ and $x = 4$.

7. **Medium** In the figure, $\overline{PT}$ is tangent to the circle at T with $PT = 10$, and a secant from P meets the circle at A and B with $PA = x$ and $AB = 21$. Find x.



Show answer

$10^2 = x(x + 21)$, so $x^2 + 21x - 100 = 0$ and $(x + 25)(x - 4) = 0$. A length is positive, so $x = 4$.

8. **Medium** A lookout stands in the crow's nest of a ship, 66 feet above the water. Treating the earth as a sphere of radius 3960 miles, how far away is the horizon? Give the answer in miles, to the nearest tenth.

Show answer

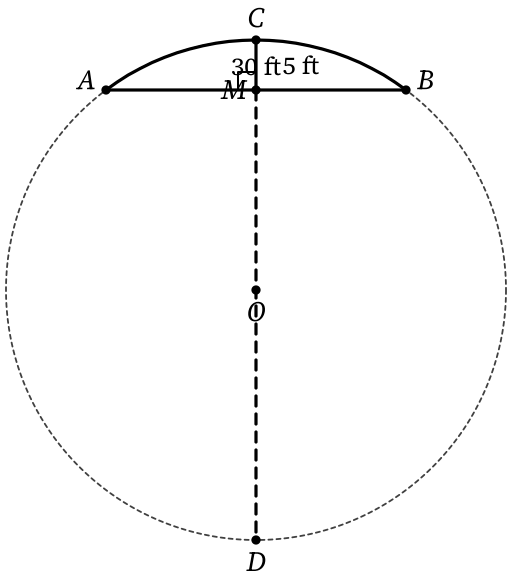
$h = \dfrac{66}{5280} = 0.0125$ mi, so $d^2 = h(h + 7920) = 0.0125 \times 7920.0125 \approx 99.0002$ and $d \approx 9.9$ miles.

9. **Medium** Use the Tangent-Secant Theorem to explain why the two tangent segments $\overline{PT}$ and $\overline{PS}$ from an outside point P to a circle are congruent.

Show answer

Draw any secant from P, meeting the circle at A and B. By the Tangent-Secant Theorem, $PT^2 = PA \cdot PB$ and also $PS^2 = PA \cdot PB$. So $PT^2 = PS^2$, and since lengths are positive, $PT = PS$.

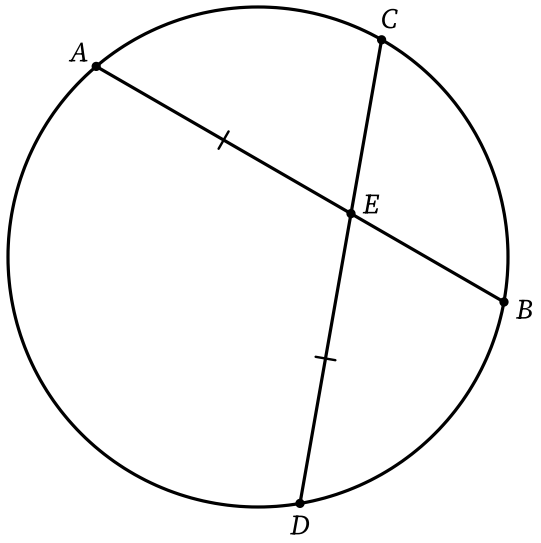
10. **Hard** A stone bridge has a circular arch. The span is $AB = 30$ feet and the arch rises $MC = 5$ feet above the midpoint M of $\overline{AB}$, as in the figure. Use the Intersecting Chords Theorem to find the radius of the arch.



Show answer

The perpendicular bisector of chord $\overline{AB}$ passes through the center, so extending $\overline{CM}$ gives a diameter $\overline{CD}$. By the Intersecting Chords Theorem, $AM \cdot MB = CM \cdot MD$: $15 \cdot 15 = 5 \cdot MD$, so $MD = 45$. The diameter is $5 + 45 = 50$ feet and the radius is 25 feet.

11. Hard Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect at E, and $\overline{AE} \cong \overline{DE}$, as in the figure. Prove: $\overline{CE} \cong \overline{BE}$. Write a two-column proof.



Show answer

See the model proof.

TWO-COLUMN PROOF

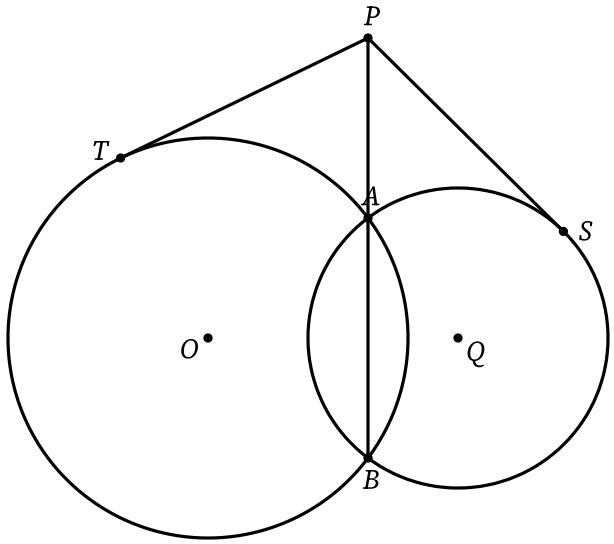
Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect at E; $\overline{AE} \cong \overline{DE}$

Prove: $\overline{CE} \cong \overline{BE}$

	STATEMENTS	REASONS
1	Chords $\overline{AB}$ and $\overline{CD}$ intersect at E	Given
2	$AE \cdot EB = CE \cdot ED$	Intersecting Chords Theorem
3	$\overline{AE} \cong \overline{DE}$, so $AE = DE$	Given; definition of congruent segments
4	$AE \cdot EB = CE \cdot AE$	Substitution Property of Equality
5	$EB = CE$	Division Property of Equality ($AE \neq 0$)
6	$\overline{CE} \cong \overline{BE}$	Definition of congruent segments

∎

12. Hard Given: Circles O and Q intersect at A and B. Point P lies on line AB outside both circles, with A between P and B. $\overline{PT}$ is tangent to $\odot O$ at T and $\overline{PS}$ is tangent to $\odot Q$ at S, as in the figure. Prove: $PT = PS$. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $\odot O$ and $\odot Q$ intersect at A and B; P is on $\overleftrightarrow{AB}$, outside both circles, with A between P and B; $\overline{PT}$ is tangent to $\odot O$ at T; $\overline{PS}$ is tangent to $\odot Q$ at S

Prove: $PT = PS$

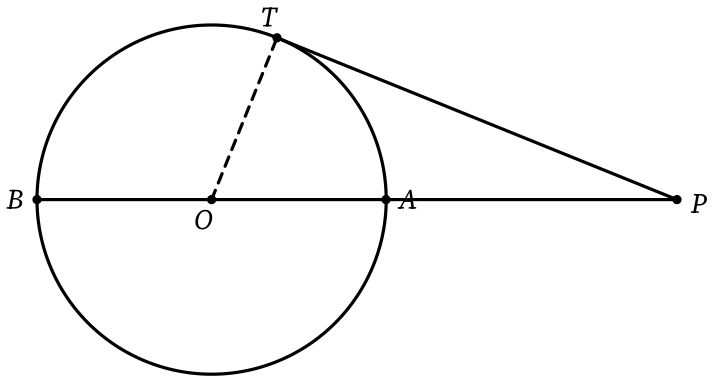
Line PAB meets $\odot O$ at A and B, so it is a secant of $\odot O$ from P, with external part PA and whole PB. By the Tangent-Secant Theorem in $\odot O$, $PT^2 = PA \cdot PB$.

The same line also meets $\odot Q$ at A and B, so by the Tangent-Secant Theorem in $\odot Q$, $PS^2 = PA \cdot PB$.

By the Transitive Property of Equality, $PT^2 = PS^2$. Both lengths are positive, so $PT = PS$.

∎

13. **Hard** Given: $\overline{PT}$ is tangent to $\odot O$ at T; the secant from P through the center O meets the circle at A and B, with A between P and O; the radius is r, as in the figure. Prove: $PT^2 = PO^2 - r^2$. Write a flow proof.



Show answer

See the model proof.

FLOW PROOF

Given: $\overline{PT}$ is tangent to $\odot O$ at T; Secant $\overline{PB}$ passes through O and meets the circle at A and B, with A between P and O; The radius of $\odot O$ is r

Prove: $PT^2 = PO^2 - r^2$

(1) $\overline{PT}$ is tangent at T; secant $\overline{PB}$ through O meets the circle at A and B
Given

(2) $PT^2 = PA \cdot PB$
Tangent-Secant Theorem
from (1)

(3) $OA = OB = r$
All radii of a circle are congruent

(4) $PA = PO - r$
Segment Addition Postulate (A is between P and O)
from (3)

(5) $PB = PO + r$
Segment Addition Postulate (O is between P and B)
from (3)

(6) $PT^2 = (PO - r)(PO + r)$
Substitution Property of Equality

from (2), (4), (5)

(7) $PT^2 = PO^2 - r^2$
Distributive Property (multiplying the binomials)
from (6)

∎

14. **Challenge** Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at E. If $AB = 10$, $CE = 2$, $ED = 12$ and $AE < EB$, find AE and EB.

Show answer

Let $AE = x$, so $EB = 10 - x$. Then $x(10 - x) = 2 \cdot 12 = 24$, so $x^2 - 10x + 24 = 0$ and $(x - 4)(x - 6) = 0$. Since $AE < EB$, $AE = 4$ and $EB = 6$.

COMMON MISTAKES

- Multiplying the external part by the inside part ($PA \cdot AB$) for a secant. The rule is external part times the *whole* secant segment, $PA \cdot PB$.
- For a tangent and a secant, multiplying PT by the secant instead of squaring PT. The tangent plays both roles, so it is squared.
- Keeping a negative solution of the quadratic equation. Lengths are positive; reject the negative root.
- Mixing feet and miles in horizon problems. Convert everything to one unit before you multiply.

> **Make it a game · Predict the Fourth Piece.** Parent traces a plate, draws two crossing chords, measures all four pieces secretly, and tells you three. Predict the fourth using the intersecting chords product, then measure to check; within 2 millimeters earns a point. Next, try one secant and a tangent from an outside point. Which theorem gives the tangent length?

10.7 Circles in the Coordinate Plane

OBJECTIVES

- Derive the equation $(x-h)^2+(y-k)^2=r^2$ of a circle from the distance formula.
- Write the equation of a circle from its center and radius, from its center and a point, or from the endpoints of a diameter.
- Complete the square to find the center and radius of a circle written in general form.
- Decide whether a point is inside, on or outside a circle, and use coordinates to check circle theorems in specific cases.

WATCH

Circle Equations (2 of 3: Generalising an equation)

Eddie Woo · 8:54

https://www.youtube.com/watch?v=5aKhvfNBn4w

> **WHEN TO WATCH**
> Before Example 1: building the equation of a circle from the distance formula.

Note: Eddie Woo teaches in Australia: he says 'centre' and works on the 'Cartesian plane'.

BACKUP VIDEO

Applications of Completing the Square (2 of 2: Finding a circle's centre)

Eddie Woo · 6:13

https://www.youtube.com/watch?v=wWPC8EeH6vw

Before the completing-the-square examples: finding the centre and radius.

KEY TERMS

Standard form of the equation of a circle

$(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

General form

The equation of a circle multiplied out and collected on one side, $x^2+y^2+Dx+Ey+F=0$.

Completing the square

Adding $(\frac{b}{2})^2$ to $x^2 + bx$ to make the perfect square $(x + \frac{b}{2})^2$; the same amount must be added to the other side of the equation.

CORE IDEAS

Euclid defined a circle as all the points at one distance from a center. On a coordinate plane, 'distance' has a formula, so the definition turns straight into an equation. This joining of geometry and algebra is the gift of Descartes.

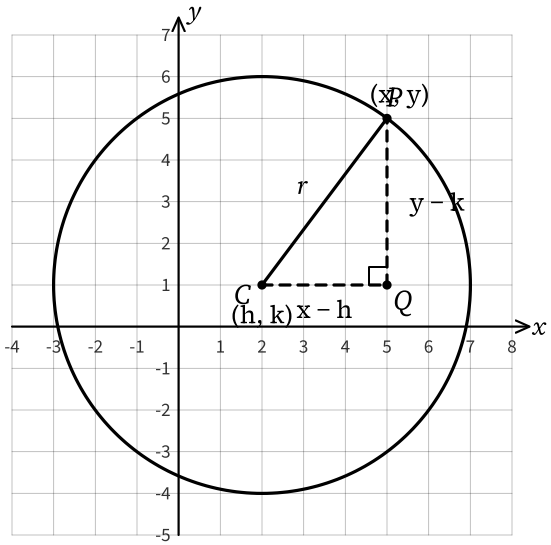
Let the center be $C(h, k)$ and the radius r. A point $P(x, y)$ is on the circle exactly when $CP = r$. The horizontal leg from C to P is $x - h$ and the vertical leg is $y - k$, so by the distance formula (the Pythagorean Theorem on the grid) $\sqrt{(x-h)^2 + (y-k)^2} = r$. Squaring both sides gives the **standard form** $(x-h)^2 + (y-k)^2 = r^2$. When the center is the origin, it is simply $x^2 + y^2 = r^2$.

To read the center, watch the signs: the equation has *minus* h and *minus* k. So $(x + 4)^2 + (y - 1)^2 = 49$ is $(x - (-4))^2 + (y - 1)^2 = 7^2$: center $(-4, 1)$ and radius 7 (not 49).

To write an equation you need the center and the radius. If you are given the center and a point on the circle, the radius is the distance between them. If you are given the endpoints of a diameter, the center is their midpoint and the radius is half their distance apart.

If the equation has been multiplied out, as in $x^2 + y^2 - 6x + 4y - 12 = 0$, **complete the square** in x and in y to return to standard form. Group the terms: $(x^2 - 6x) + (y^2 + 4y) = 12$. Add $(\frac{-6}{2})^2 = 9$ and $(\frac{4}{2})^2 = 4$ to both sides: $(x - 3)^2 + (y + 2)^2 = 25$. If the right side comes out zero or negative, the equation is not a circle (a single point, or no points at all).

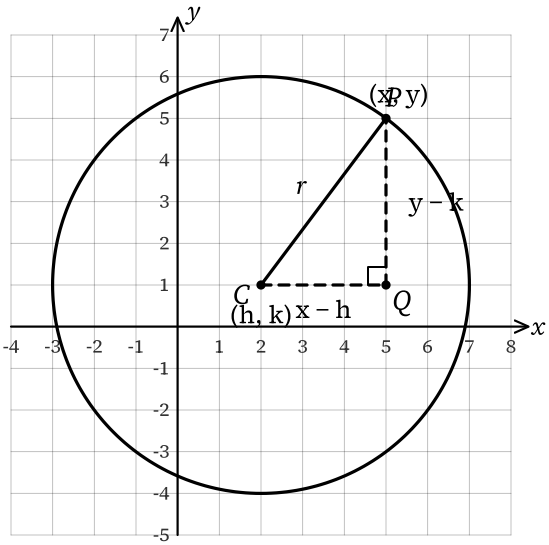
A point is **inside** the circle when its distance from the center is less than r, **on** it when the distance equals r, and **outside** when it is greater. In the equation, compare $(x-h)^2 + (y-k)^2$ with r^2: less, equal or greater. Squaring does not change the order, because both quantities are positive.



POSTULATES AND THEOREMS

THEOREM **Equation of a Circle**

The circle with center (h, k) and radius r is the graph of $(x - h)^2 + (y - k)^2 = r^2$.



PARAGRAPH PROOF

Given: A circle has center $C(h, k)$ and radius $r > 0$

Prove: A point $P(x, y)$ is on the circle exactly when $(x - h)^2 + (y - k)^2 = r^2$

By the definition of a circle, $P(x, y)$ is on the circle exactly when $CP = r$. By the distance formula, $CP = \sqrt{(x - h)^2 + (y - k)^2}$, so P is on the circle exactly when $\sqrt{(x - h)^2 + (y - k)^2} = r$.

Both sides of this equation are nonnegative, and two nonnegative numbers are equal exactly when their squares are equal. So the equation is equivalent to $(x -$

$h)^2 + (y - k)^2 = r^2$, and the points on the circle are exactly the points whose coordinates satisfy it.

∎

COROLLARY **Inside, On or Outside**

A point (x_1, y_1) is inside, on or outside the circle $(x - h)^2 + (y - k)^2 = r^2$ according as $(x_1 - h)^2 + (y_1 - k)^2$ is less than, equal to or greater than r^2.

PARAGRAPH PROOF

Given: A circle with center $C(h, k)$ and radius r, and a point $Q(x_1, y_1)$

Prove: $CQ < r$, $CQ = r$ or $CQ > r$ exactly when $(x_1 - h)^2 + (y_1 - k)^2$ is less than, equal to or greater than r^2

By the distance formula, $CQ^2 = (x_1 - h)^2 + (y_1 - k)^2$. For nonnegative numbers, $CQ < r$ exactly when $CQ^2 < r^2$, and likewise for $=$ and $>$. A point is inside the circle when it is closer to the center than r, on it when the distance is r, and outside when the distance is greater than r.

∎

KEY FORMULAS

$$(x - h)^2 + (y - k)^2 = r^2$$

$$x^2 + y^2 = r^2 \quad \text{(center at the origin)}$$

$$\text{center} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right), \quad r = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{(diameter endpoints)}$$

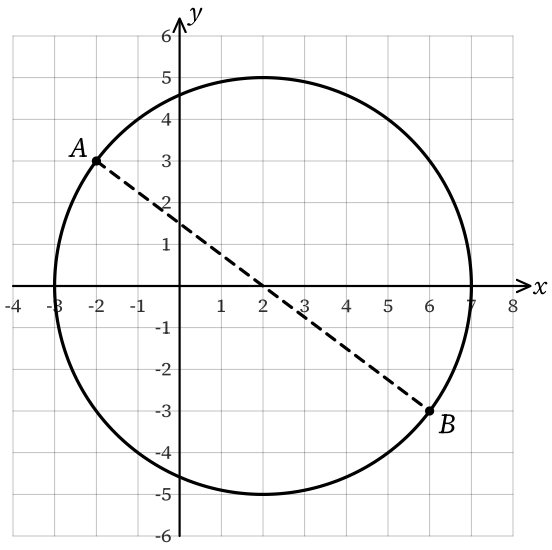
From history. René Descartes published *La Géométrie* in 1637 as an appendix to his *Discourse on Method*. In it he showed how curves can be described by equations and how geometry problems can be solved with algebra, which is why the coordinate plane is often called the Cartesian plane. The equation of a circle in this lesson is one of the simplest fruits of his idea.

Euclid connection. The equation of a circle is Euclid's definition of a circle (Book I, Definition 15: all points at an equal distance from the center) combined with Euclid's I.47 (the Pythagorean Theorem, which is what the distance formula is), written in Descartes' language of coordinates.

Practice habit. Before you write an equation, write down the center (h, k) and the radius r separately; afterward, check the equation by substituting a point you know is on the circle. Always square the radius on the right side.

WORKED EXAMPLES

Example 1. Write an equation of the circle that has a diameter with endpoints $A(-2, 3)$ and $B(6, -3)$, as in the figure.

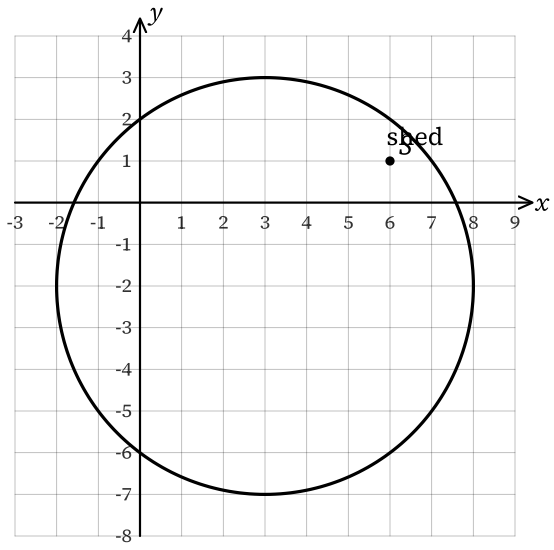


Show solution

1. The center is the midpoint of the diameter: $\left(\frac{-2+6}{2}, \frac{3+(-3)}{2}\right) = (2, 0)$.
2. The radius is the distance from the center to A: $r = \sqrt{(-2-2)^2 + (3-0)^2} = \sqrt{16+9} = 5$. (Check: $AB = \sqrt{8^2 + 6^2} = 10$, and half of 10 is 5.)
3. Substitute $h = 2$, $k = 0$, $r = 5$: $(x-2)^2 + y^2 = 25$.
4. Check with B: $(6-2)^2 + (-3)^2 = 16 + 9 = 25$.

Answer: $(x-2)^2 + y^2 = 25$

Example 2. A center-pivot sprinkler on the Miller farm waters a circular field. On the farm map, where 1 unit is 10 meters, the edge of the watered circle has the equation $x^2 + y^2 - 6x + 4y - 12 = 0$. Find the center and radius of the circle, and decide whether the tool shed at $S(6, 1)$ is inside the watered circle.



Show solution

1. Group the x-terms and the y-terms and move the constant: $(x^2 - 6x) + (y^2 + 4y) = 12$.
2. Complete each square by adding $(\frac{-6}{2})^2 = 9$ and $(\frac{4}{2})^2 = 4$ to both sides: $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$, so $(x - 3)^2 + (y + 2)^2 = 25$.
3. The center is $(3, -2)$ and the radius is $\sqrt{25} = 5$ units, which is 50 meters.
4. For the shed, $(6 - 3)^2 + (1 + 2)^2 = 9 + 9 = 18$, and $18 < 25$, so the shed is inside the circle. It gets watered, so it is wise to keep its door closed.

Answer: Center $(3, -2)$, radius 5 units (50 meters); yes, the shed is inside the watered circle.

PRACTICE

1. Easy Find the center and radius of the circle $(x + 4)^2 + (y - 1)^2 = 49$.

   Show answer

   Center $(-4, 1)$, radius 7

2. **Easy** Write an equation of the circle with center $(2, -5)$ and radius 3.

Show answer

$(x-2)^2 + (y+5)^2 = 9$

3. **Easy** A round church window is drawn on a grid marked in feet, with its center at the origin. The rim of the window passes through the point $(6, 8)$. Write an equation of the rim.

Show answer

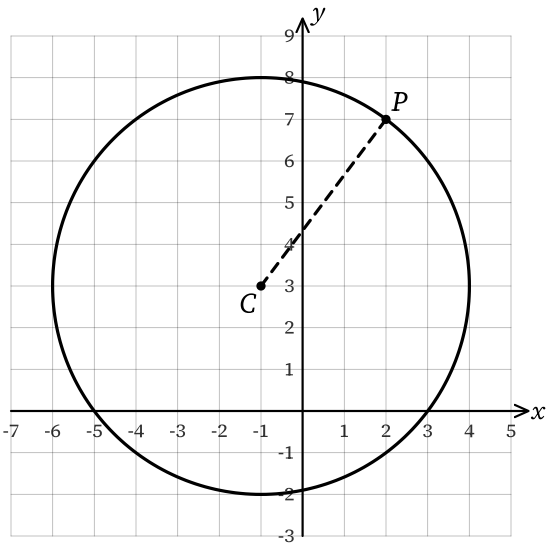
$r = \sqrt{6^2 + 8^2} = 10$, so $x^2 + y^2 = 100$.

4. **Easy** Is the point $(1, 2)$ inside, on or outside the circle $x^2 + y^2 = 4$?

Show answer

$1^2 + 2^2 = 5 > 4$, so the point is outside the circle.

5. **Medium** Write an equation of the circle with center $C(-1, 3)$ that passes through $P(2, 7)$, as in the figure.



Show answer

$r^2 = (2 - (-1))^2 + (7 - 3)^2 = 9 + 16 = 25$, so $(x+1)^2 + (y-3)^2 = 25$.

6. **Medium** Write an equation of the circle that has a diameter with endpoints $(1, -2)$ and $(7, 6)$.

Show answer

Center $(\frac{1+7}{2}, \frac{-2+6}{2}) = (4, 2)$; the diameter is $\sqrt{6^2 + 8^2} = 10$, so $r = 5$ and $(x-4)^2 + (y-2)^2 = 25$.

7. **Medium** Find the center and radius of the circle $x^2 + y^2 + 10x - 2y + 1 = 0$.

Show answer

$(x^2 + 10x + 25) + (y^2 - 2y + 1) = -1 + 25 + 1$, so $(x+5)^2 + (y-1)^2 = 25$: center $(-5, 1)$, radius 5.

8. **Medium** On a sea chart marked in miles, a lighthouse stands at $(4, 6)$ and its light can be seen up to 10 miles away. A fishing boat is at $(-3, -1)$. Can the crew see the light?

Show answer

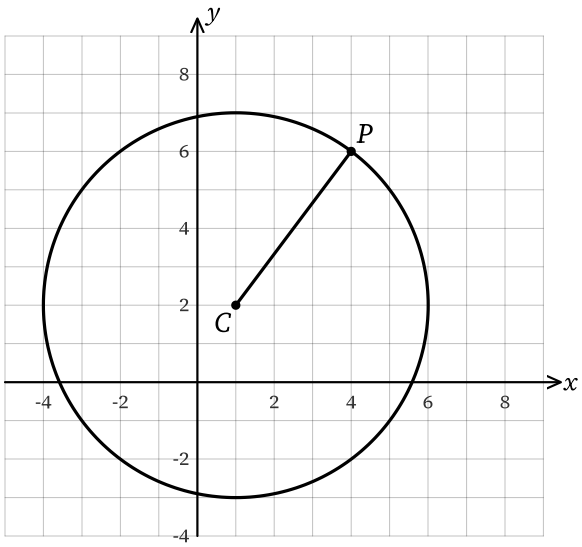
Yes. $(-3-4)^2 + (-1-6)^2 = 49 + 49 = 98 < 100 = 10^2$, so the boat is inside the circle of radius 10 around the lighthouse (about 9.9 miles away).

9. **Medium** Is the graph of $x^2 + y^2 + 4x + 6y + 20 = 0$ a circle? Explain.

Show answer

No. Completing the square gives $(x+2)^2 + (y+3)^2 = -20 + 4 + 9 = -7$. A sum of two squares can never be negative, so no point satisfies the equation; there is no circle.

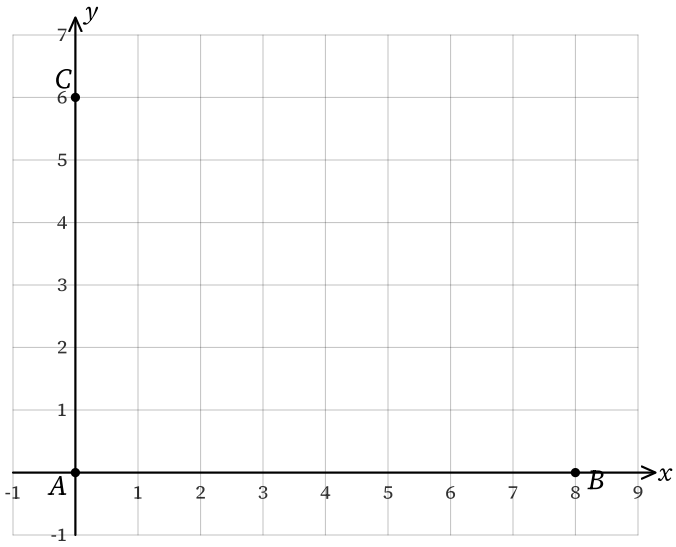
10. Hard The point $P(4, 6)$ lies on the circle $(x-1)^2 + (y-2)^2 = 25$, as in the figure. Write an equation of the line tangent to the circle at P, in slope-intercept form.



Show answer

The radius from $C(1, 2)$ to $P(4, 6)$ has slope $\frac{4}{3}$. The tangent is perpendicular to the radius at P (Tangent Theorem), so its slope is $-\frac{3}{4}$: $y - 6 = -\frac{3}{4}(x - 4)$, which is $y = -\frac{3}{4}x + 9$.

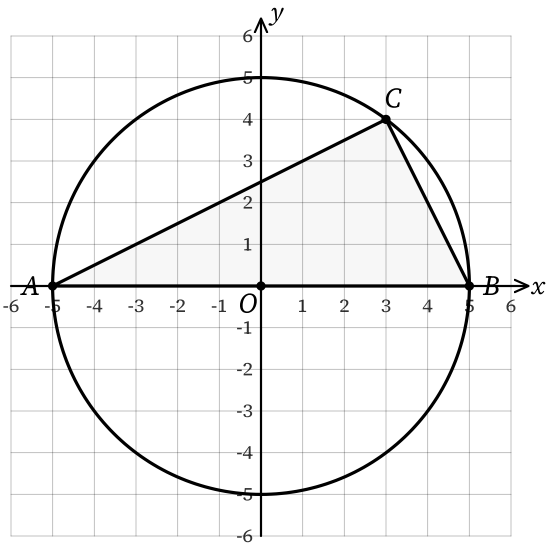
11. Hard Three farmhouses stand at $A(0,0)$, $B(8,0)$ and $C(0,6)$ on a map marked in hundreds of feet, as in the figure. The families want to dig a well the same distance from all three houses. Write an equation of the circle through the three houses, and give the location of the well.



Show answer

The axes are perpendicular, so $\angle BAC$ is a right angle; by the converse of Thales' Theorem, $\overline{BC}$ is a diameter of the circle through A, B and C. The center (the well) is the midpoint of $\overline{BC}$, $(4,3)$, and the radius is $\frac{1}{2}\sqrt{8^2+6^2}=5$. The circle is $(x-4)^2+(y-3)^2=25$, and the well goes at $(4,3)$, 500 feet from each house.

12. Hard Given: $A(-5,0)$, $B(5,0)$ and $C(3,4)$, as in the figure. Prove: C lies on the circle $x^2+y^2=25$, which has diameter $\overline{AB}$, and $\angle ACB$ is a right angle. Write a paragraph proof using coordinates.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $A(-5,0)$, $B(5,0)$, $C(3,4)$

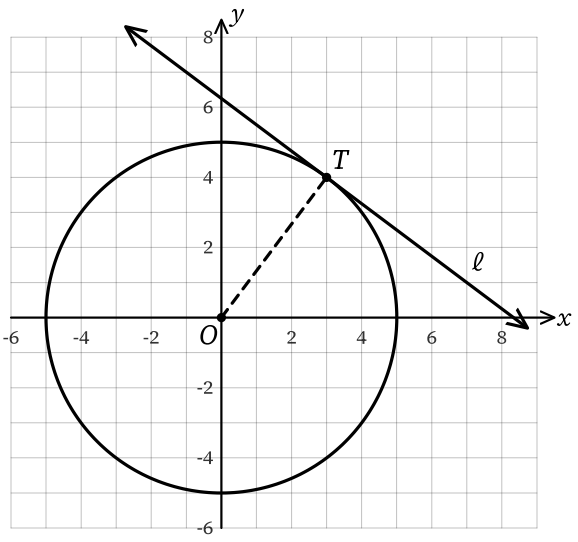
Prove: C lies on the circle $x^2+y^2=25$ with diameter $\overline{AB}$, and $\angle ACB$ is a right angle

$3^2+4^2=9+16=25$, so C satisfies the equation and lies on the circle with center $O(0,0)$ and radius 5. A and B satisfy it too ($(\pm 5)^2+0^2=25$), and $O(0,0)$ is the midpoint of $\overline{AB}$, so $\overline{AB}$ is a chord through the center: a diameter.

The slope of $\overline{CA}$ is $\dfrac{4-0}{3-(-5)}=\dfrac{1}{2}$ and the slope of $\overline{CB}$ is $\dfrac{4-0}{3-5}=-2$. Their product is $\dfrac{1}{2}\cdot(-2)=-1$, so $\overline{CA}\perp\overline{CB}$ (lines with slopes whose product is -1 are perpendicular), and $\angle ACB$ is a right angle. This agrees with Thales' Theorem (III.31): an angle inscribed in a semicircle is right.

∎

13. **Hard** Given: the circle $x^2 + y^2 = 25$, the point $T(3, 4)$ and the line ℓ: $3x + 4y = 25$, as in the figure. Prove: ℓ is tangent to the circle at T. Write a paragraph proof.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: The circle $x^2 + y^2 = 25$ with center $O(0, 0)$; $T(3, 4)$; Line ℓ: $3x + 4y = 25$
Prove: ℓ is tangent to the circle at T
T is on the circle, because $3^2 + 4^2 = 25$, and T is on ℓ, because $3(3) + 4(4) = 9 + 16 = 25$. So $\overline{OT}$ is a radius and ℓ passes through its endpoint T.

The slope of $\overline{OT}$ is $\frac{4}{3}$. Solving $3x + 4y = 25$ for y gives $y = -\frac{3}{4}x + \frac{25}{4}$, so ℓ has slope $-\frac{3}{4}$. The product of the slopes is $\frac{4}{3} \cdot (-\frac{3}{4}) = -1$, so $\ell \perp \overline{OT}$.

A line perpendicular to a radius at its endpoint on the circle is tangent (Converse of the Tangent Theorem). Therefore ℓ is tangent to the circle at T.

∎

14. **Challenge** A circle has its center on the x-axis and passes through the points $(1, 4)$ and $(7, 2)$. Write an equation of the circle.

Show answer

Let the center be $(h, 0)$. Both points are the same distance from it: $(1 - h)^2 + 4^2 = (7 - h)^2 + 2^2$. Expanding, $h^2 - 2h + 17 = h^2 - 14h + 53$, so $12h = 36$ and $h = 3$. Then $r^2 = (1 - 3)^2 + 16 = 20$, and the circle is $(x - 3)^2 + y^2 = 20$. (Check: $(7 - 3)^2 + 2^2 = 20$.)

COMMON MISTAKES

- Flipping the signs of the center wrongly. $(x - 3)^2 + (y + 2)^2 = 25$ has center $(3, -2)$: the center's coordinates are the numbers that make each square zero.
- Forgetting to square the radius, or reading r^2 as r. The circle $x^2 + y^2 = 49$ has radius 7, not 49.
- Completing the square on one side only. Whatever you add to the left side must also be added to the right side.
- Using the diameter as the radius when given the endpoints of a diameter. Find the midpoint for the center and take half the distance for the radius.

> **Make it a game · Circle Battleship.** On graph paper, parent secretly hides a button on a point of the circle (x - 3)^2 + (y + 2)^2 = 25, telling you only the equation. Guess points. Parent answers inside, on or outside the circle by substituting into the equation. Find the button in the fewest guesses. Then write the circle equation from a center and radius.

UNIT 10

Circles

This unit is a study of the circle, the figure Euclid gives all of Book III of the *Elements* to. Students learn the vocabulary of circles (chords, secants, tangents, central angles and arcs), then prove the great circle theorems one after another: a diameter perpendicular to a chord bisects it (III.3), a tangent is perpendicular to the radius (III.18), an inscribed angle is half its arc (III.20), the angle in a semicircle is right (III.31), and the products of chord and secant pieces are equal (III.35 and III.36). Each theorem is proved from congruent or similar triangles that students already know, so the unit is a rich review of Units 4, 8 and 9 as well.

Compass-and-straightedge work continues: finding the center of a circle (III.1), completing a circle from a broken piece (III.25), and constructing tangents from an outside point (III.17). Practical problems include the radius of a stone arch, belts around pulleys, a carpenter's square on a round tabletop, what a satellite can see, and how far the horizon is from a lighthouse. The unit ends where geometry meets algebra, with the equation of a circle in the coordinate plane.

Parents: the most important habits in this unit are drawing the radius to every point of tangency, tracing the intercepted arc before computing an angle, and naming the correct product rule before computing a length. A scientific calculator is allowed for square roots, π and decimal answers.

PREREQUISITES

- Prove triangles congruent with SSS, SAS, ASA, AAS and HL, and use CPCTC (Unit 4).
- Use the Isosceles Triangle Theorem and the Exterior Angle Theorem (Lessons 4.1 and 4.6).
- Use the Perpendicular Bisector Theorem and its converse (Lesson 5.1).
- Prove triangles similar by AA and solve proportions (Unit 8).
- Use the Pythagorean Theorem and its converse (Lessons 9.1 and 9.2).
- Use the distance and midpoint formulas, slopes of perpendicular lines, and solve quadratic equations (Algebra I).

MASTERY CHECKLIST

- ☐ I can name the parts of a circle and find the measures of central angles, minor arcs, major arcs and semicircles.
- ☐ I can find the center of a circle, and complete a circle from an arc, with compass and straightedge.
- ☐ I can use congruent chords, arcs and central angles, and the perpendicular from the center to a chord, to find lengths such as the radius of an arch.
- ☐ I can prove that a tangent is perpendicular to the radius at the point of tangency, use congruent tangent segments, and construct a tangent from an outside point.
- ☐ I can prove and use the Inscribed Angle Theorem, Thales' Theorem and the Inscribed Quadrilateral Theorem.
- ☐ I can find angles formed by chords, secants and tangents with the vertex on, inside or outside a circle.
- ☐ I can use the Intersecting Chords, Secant-Secant and Tangent-Secant Theorems, including finding the distance to the horizon.
- ☐ I can write and read the equation of a circle, complete the square, and decide whether a point is inside, on or outside a circle.
- ☐ I can write two-column, paragraph, flow and indirect

proofs about circles with a correct reason for every step.

- ☐ I can name the propositions of Euclid's Book III behind the main circle theorems: III.1, III.3, III.18, III.20, III.31, III.32, III.35 and III.36.

LESSONS IN THIS UNIT

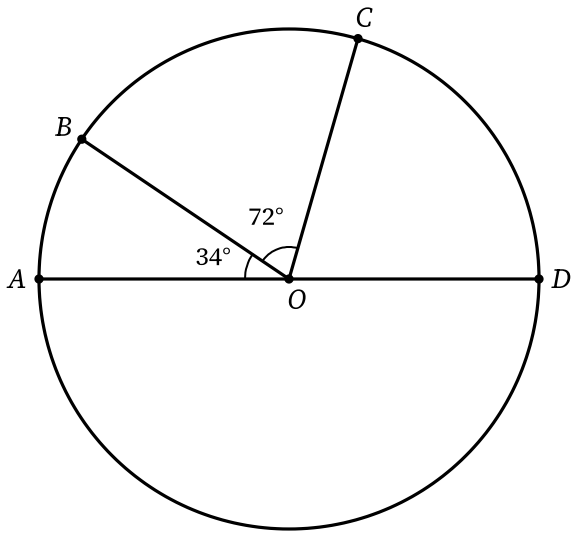
10.1	**Circles, Arcs and Central Angles**	45 minutes
10.2	**Chords**	2 sessions of 45 minutes
10.3	**Tangents**	2 sessions of 45 minutes
10.4	**Inscribed Angles**	2 sessions of 45 minutes
10.5	**Angles Formed by Chords, Secants and Tangents**	2 sessions of 45 minutes
10.6	**Segment Lengths in Circles**	2 sessions of 45 minutes
10.7	**Circles in the Coordinate Plane**	2 sessions of 45 minutes

Unit 10 test

You have 50 minutes. A scientific calculator is allowed for square roots, π and decimal answers, and you will need a compass and straightedge for Problem 13. Show your work on every problem. In each proof, give a reason for every statement. Give exact answers unless a problem tells you how to round, and include units. The test is worth 100 points.

1. **5 pts**

In circle O in the figure, $\overline{AD}$ is a diameter, $m\angle AOB = 34°$ and $m\angle BOC = 72°$, with B and C on the same semicircle. Find (a) $m\overset{\frown}{CD}$ and (b) $m\overset{\frown}{BDA}$.

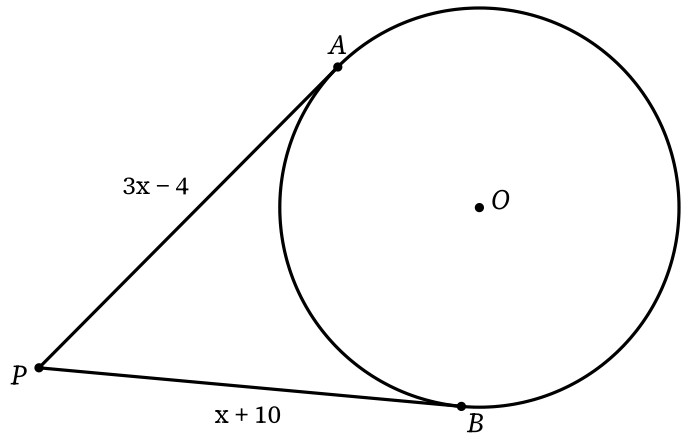


2. **6 pts**

A round wooden tabletop has a diameter of 34 inches. A straight inlaid strip of maple runs across it as a chord 30 inches long. How far is the strip from the center of the tabletop?

3. **5 pts**

In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B, with $PA = 3x - 4$ and $PB = x + 10$. Find x and PA.

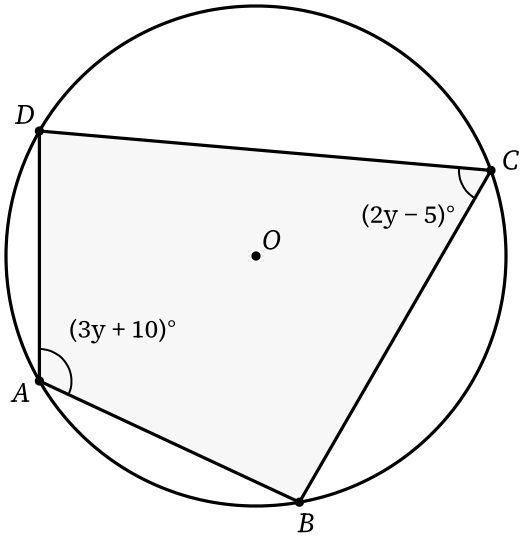


4. **5 pts**

A circle has radius 9 cm. Point P is 41 cm from the center, and $\overline{PT}$ is tangent to the circle at T. Find PT.

5. **6 pts**

Quadrilateral $ABCD$ is inscribed in a circle, with $m\angle DAB = (3y + 10)^\circ$ and $m\angle BCD = (2y - 5)^\circ$, as in the figure. Find y and $m\angle DAB$.

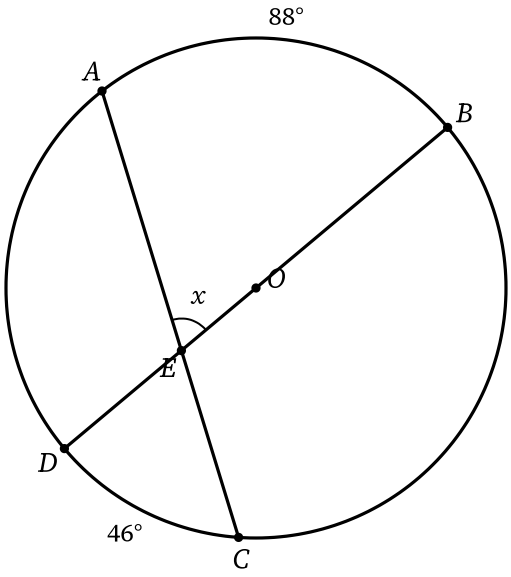


6. **5 pts**

$\overline{AB}$ is a diameter of a circle with $AB = 20$ cm, and C is a point on the circle with $AC = 12$ cm. Find BC.

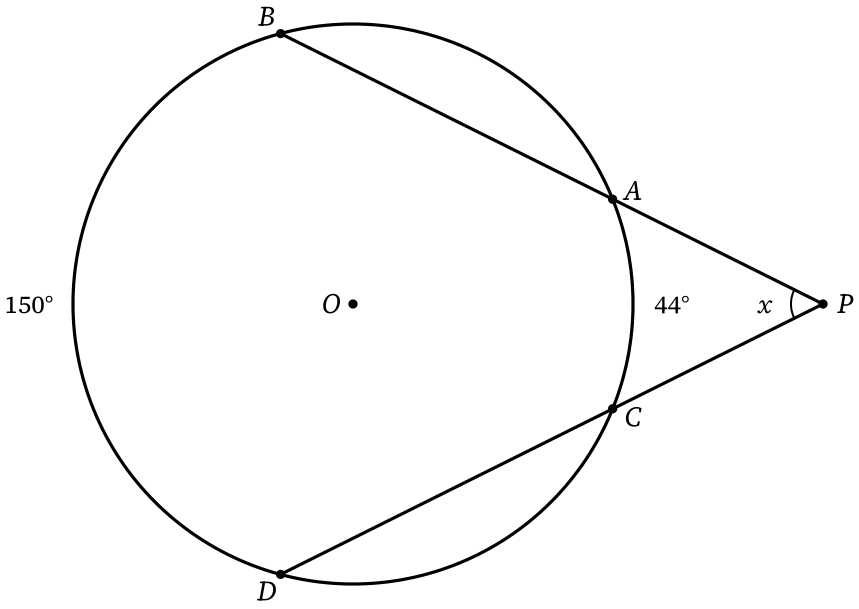
7. **5 pts**

Two straight lead strips in a round stained-glass window are chords that cross at E, as in the figure. The arc inside the angle where they cross measures $88°$, and the arc inside its vertical angle measures $46°$. Find x, the measure of the angle between the strips.



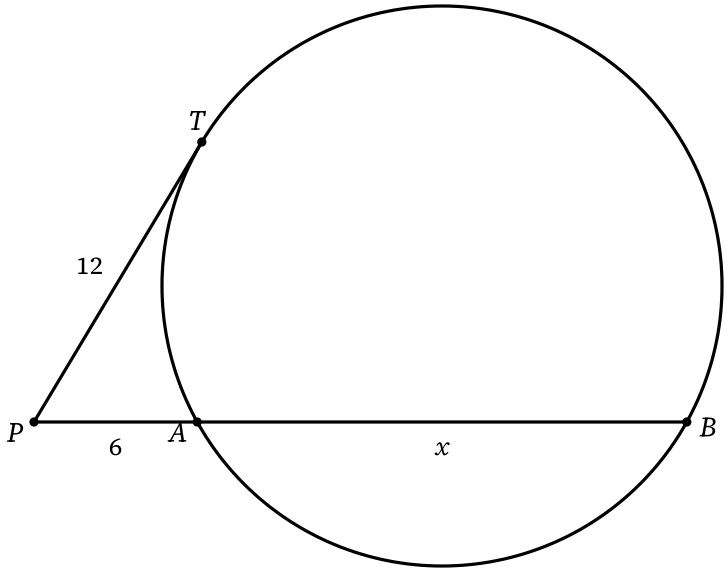
8. **5 pts**

In the figure, two secants meet outside the circle at P. The far arc $\overset{\frown}{BD}$ measures $150°$ and the near arc $\overset{\frown}{AC}$ measures $44°$. Find $x = m\angle APC$.



9. **6 pts**

In the figure, $\overline{PT}$ is tangent to the circle at T with $PT = 12$, and a secant from P meets the circle at A and B with $PA = 6$. Find $x = AB$.



10. **7 pts**

A fire lookout stands on a mountain top 2640 feet (half a mile) above the surrounding plain. Treating the earth as a sphere of radius 3960 miles, how far away is the horizon? Give the answer in miles, to the nearest tenth.

11. **7 pts**

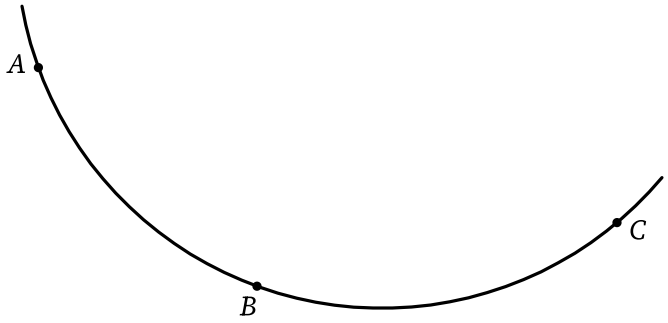
Find the center and radius of the circle $x^2 + y^2 - 4x + 10y + 4 = 0$.

12. **6 pts**

Write an equation of the circle that has a diameter with endpoints $(-3, 1)$ and $(5, 7)$.

13. **8 pts**

Ruth found a piece of a broken round plate; its edge is the arc in the figure, with points A, B and C marked on it. Describe, step by step, how to find the center of the plate with compass and straightedge, and explain why your construction works.

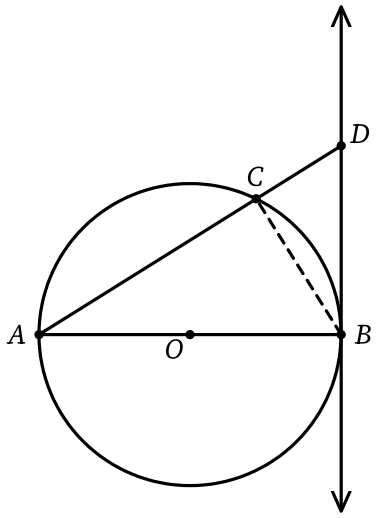


14. **6 pts**

Grace claims that a trapezoid whose legs are not congruent can be inscribed in a circle. Is she right? Explain.

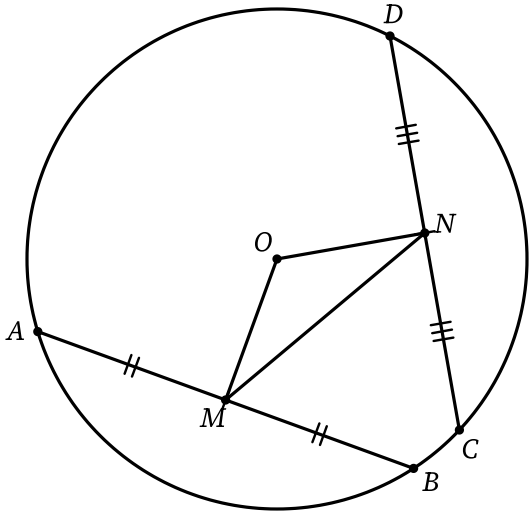
15. **9 pts**

Given: $\overline{AB}$ is a diameter of $\odot O$, $\overleftrightarrow{BD}$ is tangent to $\odot O$ at B, and $\overline{AD}$ meets the circle again at C, as in the figure. Prove: $AB^2 = AC \cdot AD$. Write a two-column proof.



16. **9 pts**

Given: In $\odot O$, chords $\overline{AB} \cong \overline{CD}$, neither of them a diameter; M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$; and O, M, N are not collinear, as in the figure. Prove: $\angle OMN \cong \angle ONM$. Write a paragraph proof.



ANSWER KEY

1. (a) $180^\circ - 34^\circ - 72^\circ = 74^\circ$; (b) $\overset{\frown}{BDA}$ is the major arc with endpoints B and A, so it measures $360^\circ - 34^\circ = 326^\circ$.
2. The radius is 17 in, and the perpendicular from the center bisects the chord, so the distance is $\sqrt{17^2 - 15^2} = \sqrt{64} = 8$ inches.
3. Tangent segments from one point are congruent: $3x - 4 = x + 10$, so $x = 7$ and $PA = 17$.
4. The radius $\overline{OT}$ is perpendicular to the tangent, so $PT = \sqrt{41^2 - 9^2} = \sqrt{1600} = 40$ cm.
5. Opposite angles of an inscribed quadrilateral are supplementary: $(3y + 10) + (2y - 5) = 180$, so $5y = 175$, $y = 35$, and $m\angle DAB = 115^\circ$.
6. $\angle ACB$ is inscribed in a semicircle, so it is a right angle, and $BC = \sqrt{20^2 - 12^2} = \sqrt{256} = 16$ cm.
7. $x = \dfrac{1}{2}(88^\circ + 46^\circ) = 67^\circ$
8. $x = \dfrac{1}{2}(150^\circ - 44^\circ) = 53^\circ$
9. $12^2 = 6 \cdot PB$, so $PB = 24$ and $x = 24 - 6 = 18$.
10. The line of sight to the horizon is tangent, and the line through the earth's center is a secant with external part $h = 0.5$ mi and whole $h + 7920$ mi. By the Tangent-Secant Theorem, $d^2 = 0.5 \times 7920.5 = 3960.25$, so $d \approx 62.9$ miles.
11. $(x^2 - 4x + 4) + (y^2 + 10y + 25) = -4 + 4 + 25$, so $(x - 2)^2 + (y + 5)^2 = 25$: center $(2, -5)$, radius 5.
12. Center $(\dfrac{-3 + 5}{2}, \dfrac{1 + 7}{2}) = (1, 4)$; the diameter is $\sqrt{8^2 + 6^2} = 10$, so $r = 5$ and $(x - 1)^2 + (y - 4)^2 = 25$.
13. Draw chords $\overline{AB}$ and $\overline{BC}$. Construct the perpendicular bisector of each: with one compass width greater than half the chord, draw arcs from both endpoints that cross on both sides, and draw the line through the crossings. The point O where the two bisectors meet is the center, and OA is the radius. Why: the center is equidistant from the endpoints of any chord (radii), so by the Converse of the Perpendicular Bisector Theorem it lies on the perpendicular bisector of every chord. The chords $\overline{AB}$ and $\overline{BC}$ are not parallel, so their bisectors are not parallel and meet in exactly one point, which must be the center (Euclid III.25).
14. No. Suppose trapezoid $ABCD$ with $\overline{AB} \parallel \overline{DC}$ is inscribed in a circle. With transversal $\overline{BD}$, the alternate interior angles $\angle ABD$ and $\angle BDC$ are congruent.

These inscribed angles intercept $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$, so those arcs are congruent, and congruent arcs have congruent chords: $\overline{AD} \cong \overline{BC}$. So the legs of an inscribed trapezoid are always congruent.

15. See the model proof.

TWO-COLUMN PROOF

Given: $\overline{AB}$ is a diameter of $\odot O$; $\overleftrightarrow{BD}$ is tangent to $\odot O$ at B; $\overline{AD}$ meets $\odot O$ again at C

Prove: $AB^2 = AC \cdot AD$

	STATEMENTS	REASONS
1	$\overline{AB}$ is a diameter; $\overleftrightarrow{BD}$ is tangent at B; C is on $\odot O$ and on $\overline{AD}$	Given
2	Draw $\overline{BC}$	Through any two points there is exactly one line
3	$\angle ACB$ is a right angle	An angle inscribed in a semicircle is a right angle (III.31)
4	$\overline{AB} \perp \overleftrightarrow{BD}$	Tangent Theorem (the radius $\overline{OB}$ lies along $\overline{AB}$)
5	$\angle ABD$ is a right angle	Definition of perpendicular lines
6	$\angle ACB \cong \angle ABD$	All right angles are congruent
7	$\angle BAC \cong \angle DAB$	Reflexive Property (C is on $\overline{AD}$, so these are the same angle)
8	$\triangle ACB \sim \triangle ABD$	AA Similarity Postulate
9	$\dfrac{AC}{AB} = \dfrac{AB}{AD}$	Corresponding sides of similar triangles are proportional
10	$AB^2 = AC \cdot AD$	Cross Products Property

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16. See the model proof.

PARAGRAPH PROOF

Given: In $\odot O$, chords $\overline{AB} \cong \overline{CD}$, neither a diameter; M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$; O, M, N are not collinear

Prove: $\angle OMN \cong \angle ONM$

M is the midpoint of $\overline{AB}$, so $MA = MB$; and $OA = OB$ because all radii are congruent. So M and O are two different points (since $\overline{AB}$ is not a diameter, O is not on it), each equidistant from A and B. By the Converse of the Perpendicular Bisector Theorem, both lie on the perpendicular bisector of $\overline{AB}$, and since two points determine a line, $\overleftrightarrow{OM}$ is that bisector. Thus $\overline{OM} \perp \overline{AB}$, and OM is the distance from the center to $\overline{AB}$. In the same way, $\overline{ON} \perp \overline{CD}$ and ON is the distance from the center to $\overline{CD}$.

Congruent chords are equidistant from the center (Equidistant Chords Theorem), so $OM = ON$. Since O, M, N are not collinear, they form $\triangle OMN$ with $\overline{OM} \cong \overline{ON}$, an isosceles triangle. By the Isosceles Triangle Theorem, the angles opposite these sides are congruent: $\angle ONM \cong \angle OMN$.

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UNIT 11

Area and Perimeter

This unit asks a simple question: how much surface does a figure cover? Starting from three area postulates (a rectangle's area is base times height, congruent figures have equal areas, and areas of non-overlapping pieces add), you will derive the area formulas for parallelograms, triangles, trapezoids, kites, rhombuses and regular polygons by cutting and rearranging, just as Euclid showed that figures "on the same base and between the same parallels" are equal (I.35–I.41).

You will then turn to circles: why π is the same for every circle, how to find circumference, arc length, and the areas of circles, sectors, segments and rings, and how Eratosthenes and Archimedes used these ideas. The unit ends with the beautiful rule that similar figures have areas in the ratio of the squares of their scale factor (Euclid VI.19 and XII.2), with composite regions and geometric probability.

Parents: expect practical problems (gardens, roofs, stained glass, sprinklers, blueprints) alongside proofs that explain each formula. A scientific calculator is used for π, trigonometry and decimal answers; students should give exact answers (in terms of π or in radical form) whenever asked.

PREREQUISITES

- Properties of parallelograms, rhombuses, trapezoids and kites (Lessons 6.2–6.5)
- Similar polygons and the AA, SAS and SSS similarity tests (Lessons 8.2–8.4)
- The Pythagorean Theorem and special right triangles (Lessons 9.1 and 9.4)
- Tangent, sine and cosine ratios with a calculator (Lessons 9.5–9.6)
- Central angles, arcs, chords and tangents of circles (Lessons 10.1–10.3)
- Simplifying square roots and solving linear and simple quadratic equations (Algebra I)

MASTERY CHECKLIST

- ☐ I can state the three area postulates and derive the area formulas for parallelograms and triangles from them.
- ☐ I can explain Euclid's propositions on figures with the same base between the same parallels (I.35–I.38, I.41).
- ☐ I can find areas and perimeters of figures on the coordinate plane, including by enclosing a figure in a rectangle.
- ☐ I can derive and use the area formulas for trapezoids, rhombuses and kites, and find areas of composite figures.
- ☐ I can find the central angle and apothem of a regular polygon and use

$A = \frac{1}{2}ap.$

- ☐ I can inscribe a regular hexagon in a circle with compass and straightedge and explain why it works.
- ☐ I can explain why

π is the same for every circle and find circumferences and arc lengths.

- ☐ I can find the areas of circles, sectors, segments and rings, exactly and rounded.
- ☐ I can use the fact that similar figures with scale factor

- a : b have areas in the ratio a^2 : b^2, and find a scale factor from areas.
- ☐ I can find geometric probabilities as ratios of areas.
- ☐ I can write two-column, paragraph, flow and indirect proofs about areas.

LESSONS IN THIS UNIT

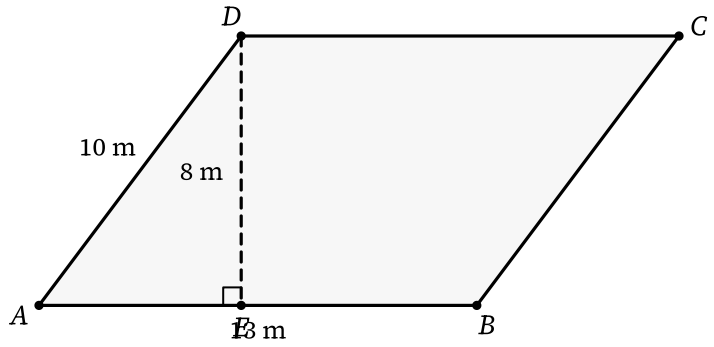
11.1	**Areas of Rectangles, Parallelograms and Triangles**	45 minutes
11.2	**Areas of Trapezoids, Rhombuses and Kites**	2 sessions of 45 minutes
11.3	**Areas of Regular Polygons**	2 sessions of 45 minutes
11.4	**Circumference and Arc Length**	2 sessions of 45 minutes
11.5	**Areas of Circles and Sectors**	2 sessions of 45 minutes
11.6	**Areas of Similar Figures and Composite Regions**	2 sessions of 45 minutes

Unit 11 Test

You have 55 minutes. A scientific calculator is allowed for π, trigonometry and decimal approximations. Show your work for every problem. Give exact answers (in terms of π or in simplest radical form) when a problem asks for them; otherwise round as the problem says. Include units in your answers. For Problem 14 you need a compass and straightedge; leave all construction marks visible. In proofs, give a reason for every step. Total: 100 points.

1. **5 pts**

   In the figure, parallelogram $ABCD$ has $AB = 13$ m, $AD = 10$ m, and height $DE = 8$ m to base $\overline{AB}$. Find its area.



2. **6 pts**

   Find the area of $\triangle ABC$ with vertices $A(0, 1)$, $B(7, 0)$ and $C(3, 5)$.

3. **6 pts**

   One side of a barn roof is a trapezoid. Its parallel edges are 30 ft and 18 ft long, and the perpendicular distance between them is 12 ft. How many square feet of shingles are needed to cover it?

4. **6 pts**

   A rhombus has sides of 17 cm and one diagonal of 16 cm. Find the length of the other diagonal and the area of the rhombus.

5. **6 pts**

   Find the area of a regular hexagon with side 8 in. Give an exact answer in simplest radical form.

6. **7 pts**

   Grace's herb garden is a regular pentagon with sides of 6 ft. Find its area to the nearest tenth of a square foot.

7. **6 pts**

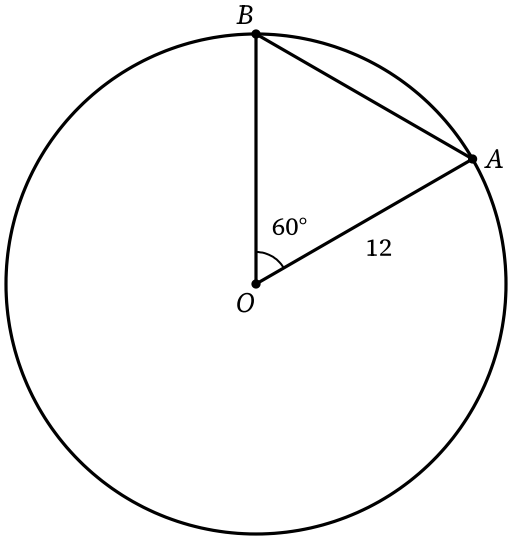
   A wagon wheel has a radius of 14 in. How many feet does the wagon travel when the wheel makes 50 full turns? Round to the nearest tenth of a foot.

8. **6 pts**

Find the length of an arc with central angle 140° in a circle of radius 9 cm. Give an exact answer in terms of π.

9. **6 pts**

In the figure, circle O has radius 12 and $m\angle AOB = 60^\circ$. Find the area of the segment bounded by chord $\overline{AB}$ and minor arc $\overset{\frown}{AB}$, exactly and to the nearest tenth.



10. **6 pts**

A circular fountain with radius 5 ft is surrounded by a brick walkway 4 ft wide. Find the area of the walkway to the nearest tenth of a square foot.

11. **6 pts**

Two similar quilt blocks have areas 27 in^2 and 75 in^2. A side of the smaller block is 9 in. Find the corresponding side of the larger block.

12. **6 pts**

A square target is 12 in on each side and has a circle of radius 3 in painted in its center. A dart hits the target at a random point. Find the probability that it lands inside the circle, exactly and to the nearest hundredth.

13. **6 pts**

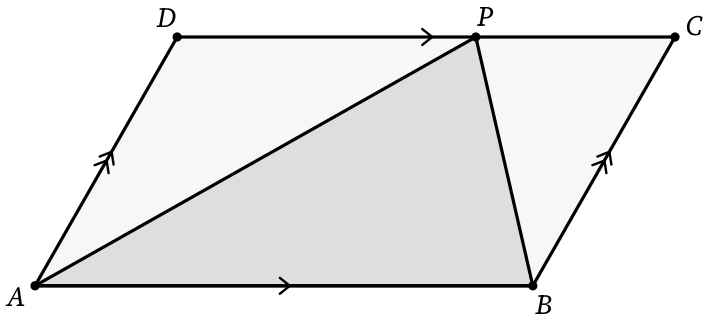
Explain why a circle with diameter 20 in has twice the circumference but four times the area of a circle with diameter 10 in.

14. **6 pts**

Using a compass and straightedge, inscribe a regular hexagon in a circle of radius 2 in. List your steps, explain why each side of the hexagon equals the radius, and find the exact area of the hexagon.

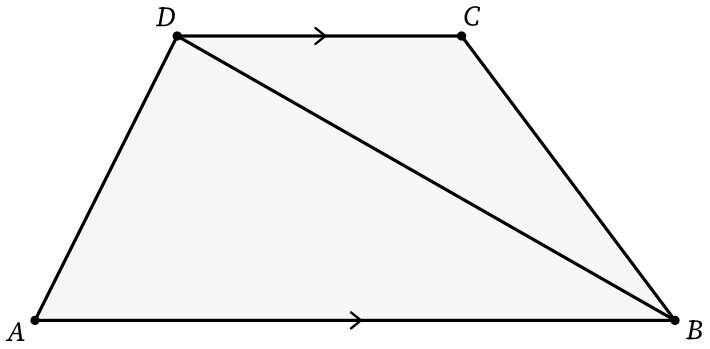
15. **8 pts**

In the figure, $ABCD$ is a parallelogram and P is any point on side $\overline{DC}$. Write a two-column proof that $\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$ (Euclid I.41).



16. **8 pts**

In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and $\overline{BD}$ is a diagonal. Write a paragraph proof that $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{AB}{DC}$.



ANSWER KEY

1. $A = bh = 13 \times 8 = 104$ square meters
2. Enclosing rectangle from $(0, 0)$ to $(7, 5)$: area 35. Corner triangles: $\frac{1}{2}(7)(1) = 3.5$, $\frac{1}{2}(4)(5) = 10$, $\frac{1}{2}(3)(4) = 6$. Area $= 35 - 19.5 = 15.5$ square units
3. $\frac{1}{2}(12)(30 + 18) = 288$ square feet
4. The diagonals are perpendicular bisectors of each other: half-diagonals 8 and $\sqrt{17^2 - 8^2} = 15$, so the other diagonal is 30 cm. Area $= \frac{1}{2}(16)(30) = 240$ cm^2
5. Apothem $= 4\sqrt{3}$, perimeter $= 48$, $A = \frac{1}{2}(4\sqrt{3})(48) = 96\sqrt{3}$ square inches
6. Central angle 72°; half of it is 36°, so $a = \frac{3}{\tan 36^\circ} \approx 4.129$ ft. $p = 30$ ft, $A \approx \frac{1}{2}(4.129)(30) \approx 61.9$ square feet
7. One turn: $2\pi(14) = 28\pi$ in. Fifty turns: $1400\pi \approx 4398.2$ in ≈ 366.5 ft
8. $\frac{140}{360} \cdot 18\pi = 7\pi$ cm
9. Sector: $\frac{60}{360} \cdot 144\pi = 24\pi$. $\triangle AOB$ is equilateral (isosceles with a 60° vertex angle) with side 12, area $\frac{\sqrt{3}}{4}(144) = 36\sqrt{3}$. Segment $= 24\pi - 36\sqrt{3} \approx 13.0$ square units
10. $\pi(9^2 - 5^2) = 56\pi \approx 175.9$ square feet
11. Area ratio $\frac{75}{27} = \frac{25}{9}$, so the scale factor is $\frac{5}{3}$ and the side is $9 \cdot \frac{5}{3} = 15$ in
12. $\frac{9\pi}{144} = \frac{\pi}{16} \approx 0.20$
13. All circles are similar, and here the scale factor is 2. Circumference is a length, so it is multiplied by 2 (10π to 20π). Area is two-dimensional, so it is multiplied by $2^2 = 4$ (25π to 100π).
14. Steps: draw the circle with center O and mark A on it; with the compass still set to the radius, step around the circle from A to mark B, C, D, E, F; join the six points. Each triangle such as $\triangle OAB$ has $OA = OB = AB = 2$, so it is equilateral and its central angle is 60°; six such angles fill 360°, so the marks close up (Euclid IV.15). Area: six equilateral triangles of side 2, each $\frac{\sqrt{3}}{4}(4) = \sqrt{3}$, total $6\sqrt{3}$ square inches.
15. See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; P lies on $\overline{DC}$

Prove: $\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; P lies on $\overline{DC}$	Given
2	$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$	Definition of parallelogram
3	Let h be the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$	The distance between two parallel lines is the same at every point
4	The height of $\triangle ABP$ to base $\overline{AB}$ is h	Definition of height (P lies on $\overleftrightarrow{DC}$)
5	$\text{Area}(\triangle ABP) = \frac{1}{2}AB \cdot h$	Triangle Area Theorem
6	$\text{Area}(ABCD) = AB \cdot h$	Parallelogram Area Theorem (the height to base $\overline{AB}$ is h)
7	$\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$	Substitution Property of Equality

∎

16. See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; $\overline{BD}$ is a diagonal

Prove: $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{AB}{DC}$

Let h be the distance between the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$; it is the same at every point.

In $\triangle ABD$, take $\overline{AB}$ as the base. The vertex D lies on $\overleftrightarrow{DC}$, so the height to that base is h, and by the Triangle Area Theorem $\text{Area}(\triangle ABD) = \frac{1}{2}AB \cdot h$. In $\triangle CDB$, take $\overline{DC}$ as the base. The vertex B lies on $\overleftrightarrow{AB}$, so the height is again h, and $\text{Area}(\triangle CDB) = \frac{1}{2}DC \cdot h$.

Dividing, $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{\frac{1}{2}AB \cdot h}{\frac{1}{2}DC \cdot h} = \dfrac{AB}{DC}$, since the common nonzero factor $\frac{1}{2}h$ divides out.

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11.1 Areas of Rectangles, Parallelograms and Triangles

OBJECTIVES

- State the area postulates and use them to derive the area formulas for parallelograms and triangles.
- Find the areas and perimeters of rectangles, parallelograms and triangles, including figures on the coordinate plane.
- Explain Euclid's propositions about figures on the same base and between the same parallels (I.35–I.38, I.41).

WATCH

Area (2 of 4: Formulas for area of Rectangles, Parallelograms and Triangles and how they are formed)

Eddie Woo · 8:34

https://www.youtube.com/watch?v=8NSwGvpbr2k

> **WHEN TO WATCH**
> Before the Core ideas: where the area formulas for rectangles, parallelograms and triangles come from.

Note: Eddie Woo teaches in Australia: he says 'perpendicular height' for height.

BACKUP VIDEO

Parallel Lines (1 of 5: Relationship between parallel lines, the triangles formed and the area)

Eddie Woo · 11:18

https://www.youtube.com/watch?v=0pV-zUIHcLA

After the same-base, same-parallels propositions: triangles between parallel lines.

KEY TERMS

area

The number of square units needed to cover a region of the plane. It is measured in square units such as square inches (in^2) or square meters (m^2).

base and height of a parallelogram

Any side may be chosen as the base; the height is the perpendicular distance between the base and the opposite side (the line containing it).

base and height of a triangle

Any side may be chosen as the base; the height (altitude) is the perpendicular distance from the opposite vertex to the line containing the base.

Area Congruence Postulate

Congruent figures have equal areas.

Area Addition Postulate

If a region is split into non-overlapping parts, its area is the sum of the areas of the parts.

Rectangle Area Postulate

The area of a rectangle is the product of its base and its height: $A = bh$. (A square of side s has area s^2.)

between the same parallels

Euclid's phrase for figures whose bases lie on one line and whose opposite sides or vertices lie on a second line parallel to it, so that all of them have the same height.

CORE IDEAS

Length tells how far; **area** tells how much surface a region covers. We measure area by counting unit squares: a rectangle 5 units long and 3 units high holds 3 rows of 5 unit squares, or 15 square units. For lengths that are not whole numbers the counting picture still guides us, so we take $A = bh$ for a rectangle as a postulate, the starting point for every other area formula.

Two more postulates do the rest of the work. The **Area Congruence Postulate** says congruent figures have equal areas: if one shape fits exactly on another, they cover the same amount of surface. The **Area Addition Postulate** says that when a region is

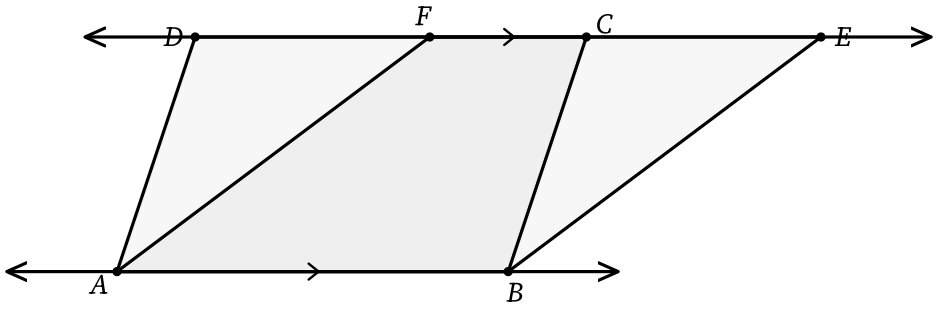
cut into pieces that do not overlap, the whole area is the sum of the pieces. This is Euclid's Common Notion 2 ("if equals are added to equals, the wholes are equal") applied to regions.

Why a parallelogram has area bh. Cut a right triangle off one end of a parallelogram by dropping a perpendicular from a top vertex to the base, and slide that triangle to the other end. The pieces now form a rectangle with the same base b and the same height h. Nothing was added or lost, so the areas are equal. Notice that the height is the *perpendicular* distance, not the slanted side: leaning a parallelogram over (like pushing a stack of cards) does not change its area as long as the base and height stay the same.

Why a triangle has area $\frac{1}{2}bh$. Two copies of any triangle fit together to make a parallelogram with the same base and height. The diagonal splits that parallelogram into two congruent triangles, so each triangle is exactly half of bh. Any side of a triangle can serve as the base, as long as you use the height drawn to *that* side.

Euclid states these facts without formulas. In I.35 he proves that parallelograms on the same base and between the same parallels are "equal," meaning equal in area even though they are not congruent; I.36 extends this to equal bases, I.37 and I.38 do the same for triangles, and I.41 says a parallelogram is double a triangle on the same base between the same parallels. Our formulas say the same thing: same base and same height give the same bh.

On the coordinate plane, horizontal and vertical segments are easy to measure, so look for a horizontal or vertical base. If no side is horizontal or vertical, surround the figure with a rectangle whose sides follow the grid lines, and subtract the right triangles in the corners (Area Addition Postulate again). Perimeter is the distance around, found by adding side lengths with the distance formula.



Euclid I.35: same base, same parallels, equal areas

POSTULATES AND THEOREMS

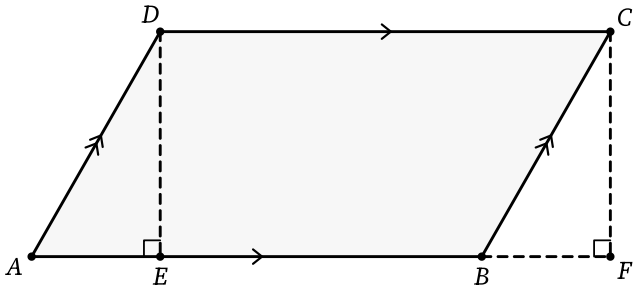
POSTULATE **Area Postulates**

(1) Rectangle Area Postulate: a rectangle with base b and height h has area bh. (2) Area Congruence Postulate: congruent figures have equal areas. (3) Area Addition Postulate: the area of a region is the sum of the areas of its non-overlapping parts.

Postulate (2) restates Euclid's Common Notion 4 ("things which coincide with one another are equal") for areas, and (3) restates Common Notion 2.

THEOREM **Parallelogram Area Theorem** Euclid I.35

The area of a parallelogram is the product of a base and its corresponding height: $A = bh$.



TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram with base $AB = b$; $\overline{DE} \perp \overleftrightarrow{AB}$ at E and $\overline{CF} \perp \overleftrightarrow{AB}$ at F; $DE = h$

Prove: $\text{Area}(ABCD) = bh$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; $\overline{DE} \perp \overleftrightarrow{AB}$, $\overline{CF} \perp \overleftrightarrow{AB}$	Given
2	$\overline{DE} \parallel \overline{CF}$	Two lines perpendicular to the same line are parallel

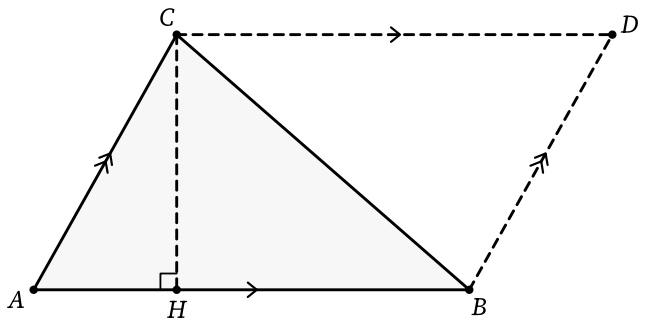
	STATEMENTS	REASONS
3	$\overline{DC} \parallel \overline{EF}$	Definition of parallelogram ($\overline{DC} \parallel \overline{AB}$, and E, F lie on $\overleftrightarrow{AB}$)
4	$DEFC$ is a parallelogram	Definition of parallelogram
5	$DEFC$ is a rectangle	A parallelogram with one right angle is a rectangle (consecutive angles are supplementary)
6	$\overline{DE} \cong \overline{CF}$ and $EF = DC$	Opposite sides of a parallelogram are congruent
7	$\overline{AD} \cong \overline{BC}$ and $DC = AB = b$	Opposite sides of a parallelogram are congruent
8	$\angle AED$ and $\angle BFC$ are right angles	Definition of perpendicular
9	$\triangle AED \cong \triangle BFC$	HL
10	$\text{Area}(\triangle AED) = \text{Area}(\triangle BFC)$	Area Congruence Postulate
11	$\text{Area}(ABCD) = \text{Area}(\triangle AED) + \text{Area}(EBCD)$	Area Addition Postulate
12	$\text{Area}(EFCD) = \text{Area}(EBCD) + \text{Area}(\triangle BFC)$	Area Addition Postulate
13	$\text{Area}(ABCD) = \text{Area}(EFCD)$	Substitution Property of Equality (rows 10, 11, 12)
14	$\text{Area}(EFCD) = EF \cdot DE = bh$	Rectangle Area Postulate; Substitution ($EF = DC = b$, $DE = h$)
15	$\text{Area}(ABCD) = bh$	Substitution Property of Equality

∎

The figure shows the usual case, where the foot E falls on the base. If the parallelogram leans so far that E lies outside $\overline{AB}$, the same congruent triangles are subtracted instead of added, which is exactly how Euclid argues in I.35.

THEOREM **Triangle Area Theorem** Euclid I.41

The area of a triangle is half the product of a base and its corresponding height: $A = \frac{1}{2}bh$.



PARAGRAPH PROOF

Given: $\triangle ABC$ with base $AB = b$ and height $CH = h$, where $\overline{CH} \perp \overleftrightarrow{AB}$

Prove: $\text{Area}(\triangle ABC) = \frac{1}{2}bh$

Through C draw the line parallel to $\overleftrightarrow{AB}$, and through B draw the line parallel to $\overleftrightarrow{AC}$ (each exists and is unique by the Parallel Postulate; Euclid constructs them in I.31). These two lines meet, because they are parallel to the intersecting lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$; call the meeting point D. Both pairs of opposite sides of $ABDC$ are parallel, so $ABDC$ is a parallelogram by definition.

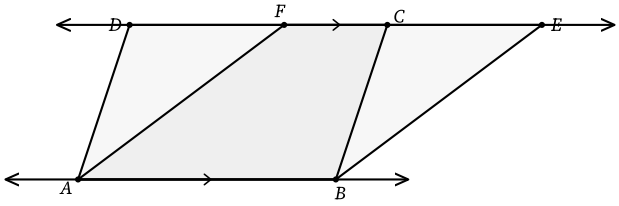
Opposite sides of a parallelogram are congruent, so $\overline{AB} \cong \overline{DC}$ and $\overline{AC} \cong \overline{DB}$, and $\overline{BC} \cong \overline{CB}$ by the Reflexive Property. Therefore $\triangle ABC \cong \triangle DCB$ by SSS, and by the Area Congruence Postulate the two triangles have equal areas. The diagonal $\overline{BC}$ splits the parallelogram into these two triangles, so by the Area Addition Postulate $\text{Area}(ABDC) = 2 \cdot \text{Area}(\triangle ABC)$.

The parallelogram has base $AB = b$, and its height is the perpendicular distance from line CD to line AB, which is $CH = h$. By the Parallelogram Area Theorem, $\text{Area}(ABDC) = bh$. So $2 \cdot \text{Area}(\triangle ABC) = bh$, and dividing by 2 gives $\text{Area}(\triangle ABC) = \frac{1}{2}bh$.

∎

COROLLARY **Same Base, Same Parallels** Euclid I.37

Triangles on the same base (or on equal bases) whose third vertices lie on one line parallel to the base have



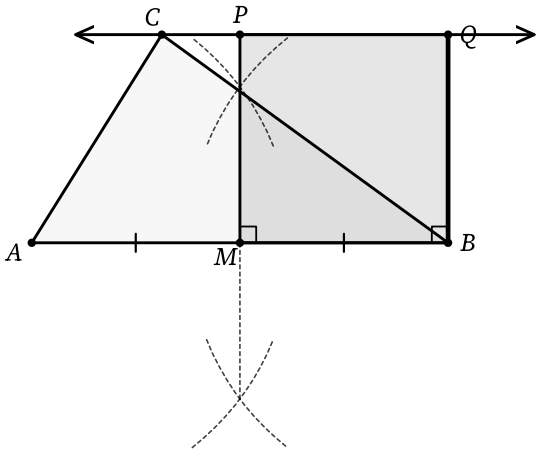
Euclid I.35: same base, same parallels, equal areas

equal areas. The same holds for parallelograms.

This follows at once from the area theorems: the bases are equal, and the heights are equal because the distance between two parallel lines is the same everywhere (Lesson 3.4). Euclid proves these facts as I.35–I.38. Practice Problem 9 asks you to write the proof.

CONSTRUCTIONS

CONSTRUCTION **Construct a rectangle equal in area to a given triangle** Euclid I.42



1. Start with $\triangle ABC$ and its base $\overline{AB}$.
2. Bisect $\overline{AB}$ (Euclid I.10): with the compass set wider than half of AB, draw arcs centered at A and at B above and below the segment, and draw the line through the two crossing points. It meets $\overline{AB}$ at its midpoint M and is perpendicular to $\overline{AB}$.
3. Through C construct the line parallel to $\overline{AB}$ (Euclid I.31, by copying an angle). Call the point where the perpendicular through M meets this parallel line P.
4. At B construct the perpendicular to $\overline{AB}$ (Euclid I.11) and let it meet the parallel line at Q.
5. Draw rectangle $MBQP$. It has the same area as $\triangle ABC$.

Why it works. $\overline{MP}$ and $\overline{BQ}$ are perpendicular to $\overline{AB}$, so they are parallel, and $\overline{PQ} \parallel \overline{MB}$ by construction; so $MBQP$ is a parallelogram with a right angle, a rectangle. Its base is $MB = \frac{1}{2}AB$ and its height is the distance between the parallel lines, which equals the height h of the triangle. So its area is $\frac{1}{2}AB \cdot h$, the area of the triangle. Euclid's I.42 does this for a parallelogram with any given angle; using a right angle gives a rectangle.

KEY FORMULAS

$$A_{\text{rectangle}} = bh$$

$$A_{\text{parallelogram}} = bh$$

$$A_{\text{triangle}} = \frac{1}{2}bh$$

From history. Heron of Alexandria, usually dated to the first century AD, wrote practical books on measurement for surveyors and builders. In his *Metrica* he gives a way to find a triangle's area from its three sides alone: if s is half the perimeter, then $A = \sqrt{s(s-a)(s-b)(s-c)}$. For a triangle with sides 13, 14 and 15, $s = 21$ and $A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84$. The formula is now called Heron's formula; you do not need it in this course, but it is a fine example of how far area reasoning can go.

Euclid connection. Euclid I.35 reads: "Parallelograms which are on the same base and in the same parallels are equal to one another." Euclid never multiplies lengths; he shows that figures are "equal" by cutting and adding congruent pieces, just as our proof of $A = bh$ does. I.41 (a parallelogram is double a triangle on the same base between the same parallels) is the ancestor of $A = \frac{1}{2}bh$.

Practice habit. Recite the three area postulates and the two area theorems until they are automatic. Before using any area formula, find the height and mark it with a right-angle symbol on your figure; if you cannot draw the right angle, you have not found the height.

WORKED EXAMPLES

Example 1. The Miller family's vegetable garden is shaped like a parallelogram. Its long sides are 24 ft, its slanted sides are 15 ft, and the perpendicular distance between the long sides is 12 ft, as in the figure. Find the area of the garden and the length of fence needed to go around it.

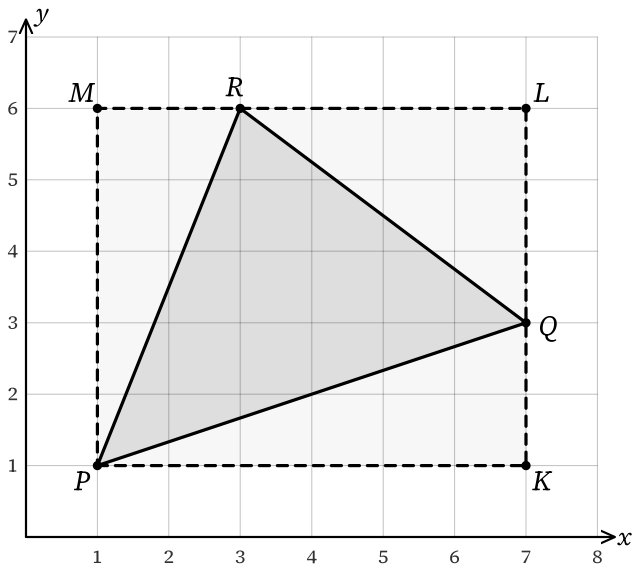


Show solution

1. Choose the 24-ft side as the base. Its height is the perpendicular distance to the opposite side, 12 ft (not the slanted 15-ft side).
2. Area: $A = bh = 24 \times 12 = 288$ square feet.
3. Perimeter: opposite sides of a parallelogram are congruent, so $P = 2(24 + 15) = 2(39) = 78$ feet.
4. Check: the slanted side 15 ft is longer than the height 12 ft, as it must be (the perpendicular is the shortest segment from a point to a line).

Answer: Area $= 288$ square feet; fence $= 78$ feet

Example 2. Find the area and the perimeter of $\triangle PQR$ with vertices $P(1, 1)$, $Q(7, 3)$ and $R(3, 6)$. Give the perimeter to the nearest tenth.



Show solution

1. No side is horizontal or vertical, so enclose the triangle in the rectangle $PKLM$ with $K(7, 1)$, $L(7, 6)$, $M(1, 6)$. Its base is $7 - 1 = 6$ and its height is $6 - 1 = 5$, so its area is $6 \times 5 = 30$.
2. The rectangle is made of $\triangle PQR$ and three right triangles. $\triangle PKQ$ has legs $PK = 6$ and $KQ = 2$, area $\frac{1}{2}(6)(2) = 6$. $\triangle QLR$ has legs $QL = 3$ and $LR = 4$, area $\frac{1}{2}(3)(4) = 6$. $\triangle RMP$ has legs $RM = 2$ and $MP = 5$, area $\frac{1}{2}(2)(5) = 5$.
3. By the Area Addition Postulate, $\text{Area}(\triangle PQR) = 30 - (6 + 6 + 5) = 13$ square units.
4. Perimeter with the distance formula: $PQ = \sqrt{6^2 + 2^2} = \sqrt{40} \approx 6.325$, $QR = \sqrt{4^2 + 3^2} = 5$, $RP = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.385$. Sum ≈ 16.710, about 16.7 units.

Answer: Area $= 13$ square units; perimeter ≈ 16.7 units

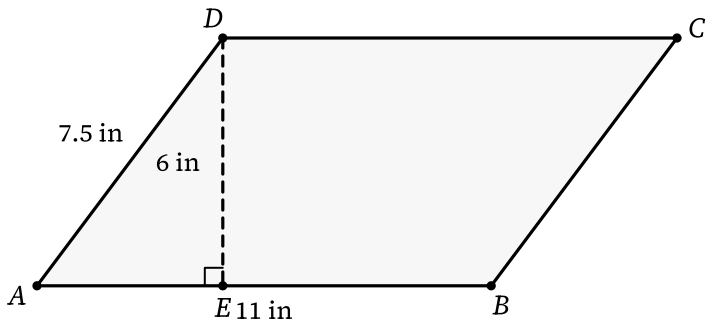
PRACTICE

1. **Easy** Ruth is planting a rectangular flower bed 14 ft long and 9 ft wide. Find (a) its area, to know how much mulch to buy, and (b) its perimeter, to know how much edging to buy.

   Show answer

   (a) 126 square feet; (b) 46 feet

2. **Easy** In the figure, parallelogram $ABCD$ has $AB = 11$ in, $AD = 7.5$ in, and height $DE = 6$ in to base $\overline{AB}$. Find its area.



   Show answer

   $A = bh = 11 \times 6 = 66$ square inches (the 7.5-in side is not the height)

3. **Easy** The triangular gable end of a shed is 18 ft wide at the bottom and rises 7 ft from the bottom edge to the peak. How many square feet of siding cover the gable?

   Show answer

   $A = \frac{1}{2}(18)(7) = 63$ square feet

4. **Easy** A triangle has area 54 square meters and a base of 12 meters. Find the height to that base.

   Show answer

   $54 = \frac{1}{2}(12)h$, so $h = 9$ meters

5. **Medium** A right triangle has legs 8 cm and 15 cm. Find (a) its area and (b) its perimeter.

   Show answer

   (a) $\frac{1}{2}(8)(15) = 60$ cm^2; (b) the hypotenuse is $\sqrt{64 + 225} = 17$, so the perimeter is $8 + 15 + 17 = 40$ cm

6. **Medium** Find the area of the parallelogram with vertices $A(1, 2)$, $B(8, 2)$, $C(10, 6)$ and $D(3, 6)$.

   Show answer

   Base $AB = 7$ (horizontal), height $= 6 - 2 = 4$, so $A = 28$ square units

7. **Medium** Triangle JKL has vertices $J(-2, 0)$, $K(6, 0)$ and $L(4, 5)$. Find (a) its area and (b) its perimeter to the nearest tenth.

   Show answer

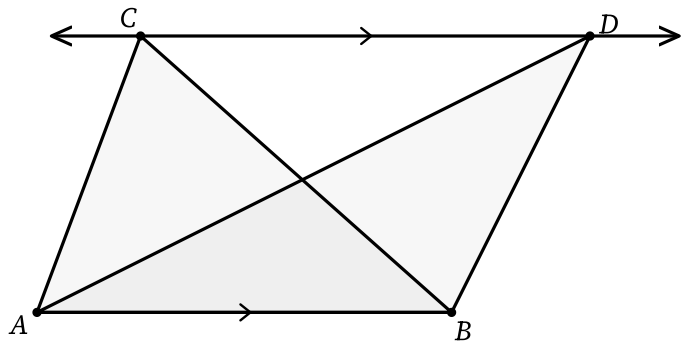
   (a) base $JK = 8$, height 5, area $= 20$ square units; (b) $8 + \sqrt{29} + \sqrt{61} \approx 8 + 5.385 + 7.810 \approx 21.2$ units

8. **Medium** Henry is painting one wall of a room. The wall is 16 ft wide and 9 ft high. It has a doorway 3 ft wide and 7 ft high and a window 4 ft wide and 3 ft high, which will not be painted. How many square feet will he paint?

   Show answer

   $16 \times 9 - 3 \times 7 - 4 \times 3 = 144 - 21 - 12 = 111$ square feet

9. **Medium** Explain why, and prove: in the figure, $\triangle ABC$ and $\triangle ABD$ share the base $\overline{AB}$, and C and D lie on a line parallel to $\overleftrightarrow{AB}$. Prove that $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD)$ (Euclid I.37), even though the triangles have different shapes.



Show answer

See the model proof. Same base and equal heights give equal areas.

PARAGRAPH PROOF

Given: $\triangle ABC$ and $\triangle ABD$ have base $\overline{AB}$; $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$

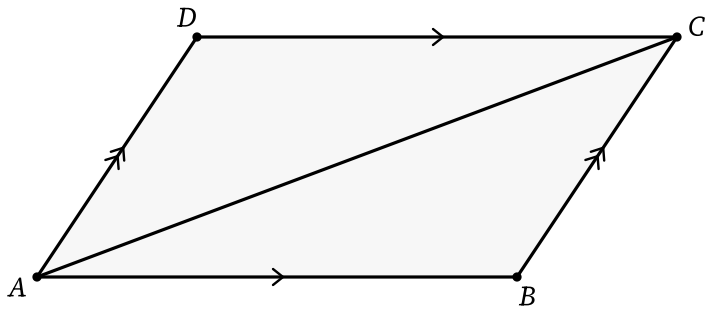
Prove: $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD)$

Draw $\overline{CH} \perp \overleftrightarrow{AB}$ and $\overline{DK} \perp \overleftrightarrow{AB}$ with H and K on $\overleftrightarrow{AB}$. Two lines perpendicular to the same line are parallel, so $\overline{CH} \parallel \overline{DK}$; also $\overline{CD} \parallel \overline{HK}$ because $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$. So $CHKD$ is a parallelogram, and since its angle at H is a right angle it is a rectangle. Opposite sides of a parallelogram are congruent, so $CH = DK$; call this common length h.

CH is the height of $\triangle ABC$ to base $\overline{AB}$ and DK is the height of $\triangle ABD$ to the same base. By the Triangle Area Theorem, $\text{Area}(\triangle ABC) = \frac{1}{2} AB \cdot h$ and $\text{Area}(\triangle ABD) = \frac{1}{2} AB \cdot h$. By the Transitive Property of Equality the two areas are equal. The shapes differ, but area depends only on the base and the height.

∎

10. **Hard** In the figure, $ABCD$ is a parallelogram with diagonal $\overline{AC}$. Write a two-column proof that $\text{Area}(\triangle ABC) = \frac{1}{2}\,\text{Area}(ABCD)$ (Euclid I.34: the diagonal bisects the parallelogram).



Show answer

See the model proof.

TWO-COLUMN PROOF

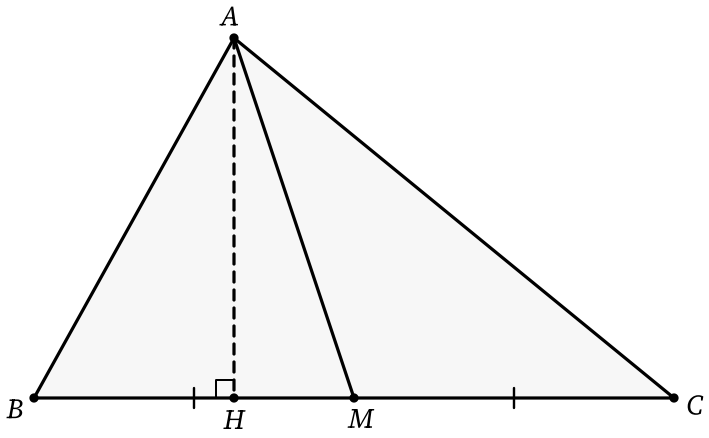
Given: $ABCD$ is a parallelogram

Prove: $\text{Area}(\triangle ABC) = \frac{1}{2}\,\text{Area}(ABCD)$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram	Given
2	$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$	Opposite sides of a parallelogram are congruent
3	$\overline{AC} \cong \overline{CA}$	Reflexive Property of Congruence
4	$\triangle ABC \cong \triangle CDA$	SSS
5	$\text{Area}(\triangle ABC) = \text{Area}(\triangle CDA)$	Area Congruence Postulate
6	$\text{Area}(ABCD) = \text{Area}(\triangle ABC) + \text{Area}(\triangle CDA)$	Area Addition Postulate
7	$\text{Area}(ABCD) = 2 \cdot \text{Area}(\triangle ABC)$	Substitution Property of Equality
8	$\text{Area}(\triangle ABC) = \frac{1}{2}\,\text{Area}(ABCD)$	Division Property of Equality

∎

11. **Hard** In the figure, M is the midpoint of side $\overline{BC}$ of $\triangle ABC$, so $\overline{AM}$ is a median. Write a flow proof that the median divides the triangle into two triangles of equal area: $\text{Area}(\triangle ABM) = \text{Area}(\triangle ACM)$.



Show answer

See the model proof.

FLOW PROOF

Given: M is the midpoint of $\overline{BC}$

Prove: $\text{Area}(\triangle ABM) = \text{Area}(\triangle ACM)$

(1) M is the midpoint of $\overline{BC}$
Given

(2) $BM = MC$
Definition of midpoint
from (1)

(3) Draw $\overline{AH} \perp \overleftrightarrow{BC}$ with H on $\overleftrightarrow{BC}$; let $AH = h$
Through a point not on a line there is exactly one perpendicular to the line (Euclid I.12)

(4) h is the height of $\triangle ABM$ to base $\overline{BM}$ and of $\triangle ACM$ to base $\overline{MC}$
Definition of height of a triangle (B, M, C lie on one line)
from (3)

(5) $\text{Area}(\triangle ABM) = \frac{1}{2} \cdot BM \cdot h$
Triangle Area Theorem
from (4)

(6) $\text{Area}(\triangle ACM) = \frac{1}{2} \cdot MC \cdot h$
Triangle Area Theorem

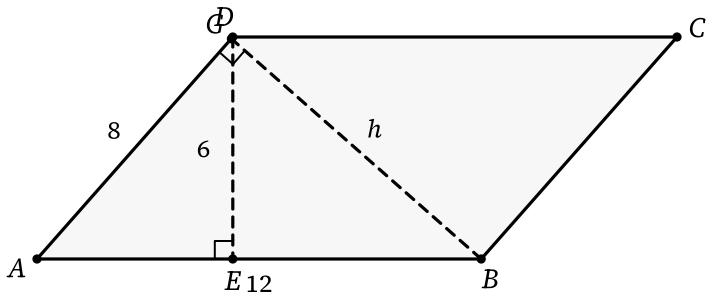
from (4)

(7) $\text{Area}(\triangle ABM) = \dfrac{1}{2} \cdot MC \cdot h$
Substitution Property of Equality
from (2), (5)

(8) $\text{Area}(\triangle ABM) = \text{Area}(\triangle ACM)$
Transitive Property of Equality
from (6), (7)

∎

12. **Hard** In the figure, parallelogram $ABCD$ has sides $AB = 12$ and $AD = 8$. The height to base $\overline{AB}$ is $DE = 6$. Find the height $h = BG$ to base $\overline{AD}$.



Show answer

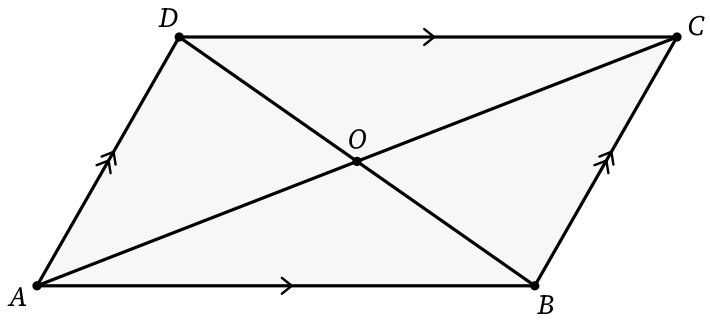
Area $= 12 \times 6 = 72$; using base AD: $8h = 72$, so $h = 9$

13. **Challenge** Triangle ABC has vertices $A(0, 0)$, $B(8, 0)$ and $C(5, 6)$. Find the length of the altitude from A to $\overline{BC}$. Give an exact answer in simplest radical form.

Show answer

Area $= \dfrac{1}{2}(8)(6) = 24$. $BC = \sqrt{3^2 + 6^2} = 3\sqrt{5}$. Then $\dfrac{1}{2}(3\sqrt{5})h = 24$, so $h = \dfrac{16}{\sqrt{5}} = \dfrac{16\sqrt{5}}{5}$ units

14. **Challenge** In the figure, the diagonals of parallelogram $ABCD$ meet at O. Prove that the diagonals divide the parallelogram into four triangles of equal area.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a parallelogram; diagonals $\overline{AC}$ and $\overline{BD}$ meet at O
Prove: $\triangle AOB$, $\triangle BOC$, $\triangle COD$ and $\triangle DOA$ have equal areas
The diagonals of a parallelogram bisect each other, so $AO = OC$ and $BO = OD$.

Compare $\triangle AOB$ and $\triangle COB$. Their bases $\overline{AO}$ and $\overline{OC}$ lie on line AC and are equal, and both have the same height: the perpendicular distance from B to $\overleftrightarrow{AC}$. By the Triangle Area Theorem their areas are equal. In the same way, $\triangle BOC$ and $\triangle DOC$ have equal bases $BO = OD$ on line BD and the same height from C to $\overleftrightarrow{BD}$, so their areas are equal; and $\triangle COD$ and $\triangle AOD$ have equal bases $CO = OA$ and the same height from D to $\overleftrightarrow{AC}$, so their areas are equal.

By the Transitive Property of Equality, all four triangles have the same area, and each is one fourth of the parallelogram by the Area Addition Postulate.

∎

COMMON MISTAKES

- Using the slanted side of a parallelogram or triangle as the height. The height must be perpendicular to the base; mark the right angle before you multiply.
- Forgetting the $\frac{1}{2}$ in the triangle formula. Remember why it is there: a triangle is half of a parallelogram with the same base and height.
- Pairing a base with the wrong height. Each height belongs to one particular base, the side (or line) it is perpendicular to.

- Mixing up area and perimeter, or their units. Perimeter is a length (feet); area is in square units (square feet).

> **Make it a game · Graph Paper Duel.** Parent draws a triangle, a parallelogram and a rectangle on graph paper with named vertices. You and your parent each find area and perimeter separately, then compare. If answers differ, settle it by counting squares. The person with fewer errors wins. Finish by cutting a parallelogram and sliding a triangle to form a rectangle.

11.2 Areas of Trapezoids, Rhombuses and Kites

OBJECTIVES

- Derive the trapezoid area formula in two ways and use it to solve problems.
- Prove that the area of a rhombus or kite is half the product of its diagonals, and apply the formula.
- Find the areas of composite figures by adding and subtracting simpler regions.
- Write proofs about areas using the area postulates and earlier area theorems.

WATCH

Area (3 of 4: Formula for area of Kites, Rhombuses and Trapeziums and how they are formed)

Eddie Woo · 8:57

https://www.youtube.com/watch?v=NLvpJyttq0Y

> **WHEN TO WATCH**
> Before Example 1: the area formulas for kites, rhombuses and trapezoids and where they come from.

Note: Eddie Woo teaches in Australia and says 'trapezium'; Mr. J says both words.

BACKUP VIDEO

Area of a Trapezoid (Trapezium) | Math with Mr. J

Math with Mr. J · 6:14

https://www.youtube.com/watch?v=-_SIZw5H4dA

Before the trapezoid examples: a step-by-step trapezoid area.

KEY TERMS

bases of a trapezoid

The two parallel sides of a trapezoid, often called b_1 and b_2. The other two sides are the legs.

height of a trapezoid

The perpendicular distance between the lines containing the two bases.

rhombus

A parallelogram with four congruent sides. Its diagonals are perpendicular and bisect each other (Lesson 6.4).

kite

A quadrilateral with two pairs of consecutive congruent sides and no opposite sides congruent. Its diagonals are perpendicular (Lesson 6.5).

composite figure

A figure made of simpler figures, such as rectangles, triangles and trapezoids. Its area is found by adding the areas of non-overlapping pieces or subtracting the areas of cut-out pieces.

CORE IDEAS

A trapezoid has two parallel sides, its **bases** b_1 and b_2, and its **height** h is the distance between them. Because the bases are parallel, the distance between them is the same everywhere (Lesson 3.4), and that single number h is what makes a neat formula possible.

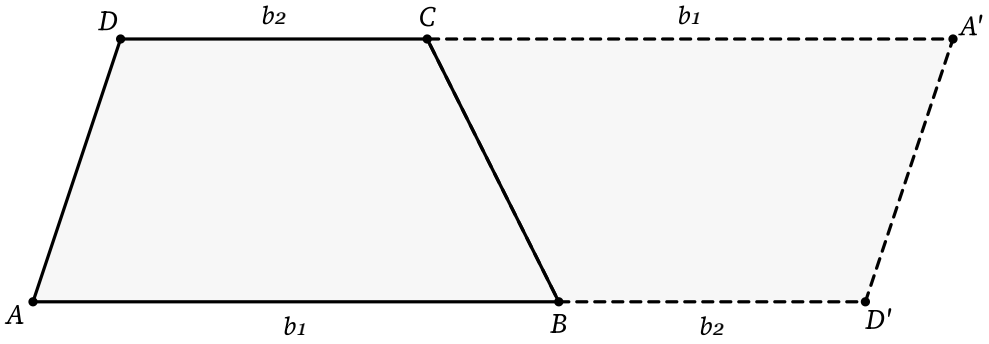
First way: cut along a diagonal. A diagonal splits the trapezoid into two triangles. One has base b_1 and the other has base b_2, and both have height h, because each triangle's top vertex lies on the other parallel line. Adding, $A = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1 + b_2)$.

Second way: double it. Make a copy of the trapezoid, turn it upside down (a rotation of 180° about the midpoint of a leg), and fit it beside the original. Together they form a parallelogram whose base is $b_1 + b_2$ and whose height is still h. The trapezoid is half of it, so again $A = \frac{1}{2}h(b_1 + b_2)$. You can also read the formula as *(average of the bases) times (height)*, or, since the midsegment of a trapezoid is $\frac{1}{2}(b_1 + b_2)$ (Lesson 6.5), as *(midsegment) times (height)*.

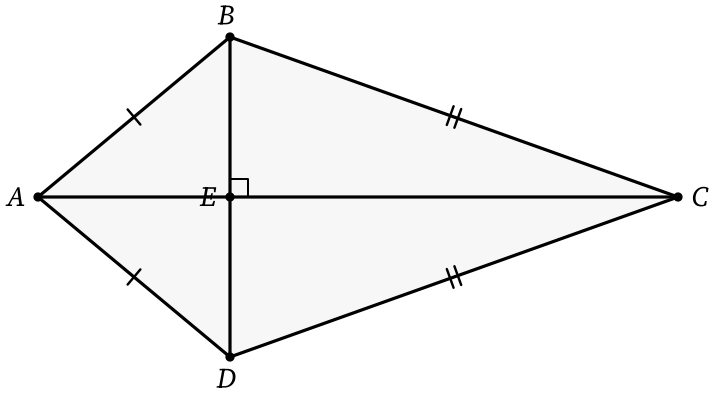
Kites and rhombuses. Their diagonals are perpendicular. Cut along one diagonal d_1: you get two triangles whose common base is d_1 and whose heights are the two pieces of the other diagonal, which add up to d_2. So $A = \frac{1}{2}d_1 \cdot (\text{piece}) + \frac{1}{2}d_1 \cdot$

(other piece) $= \frac{1}{2}d_1 d_2$. Another way to see it: the kite fills exactly half of the rectangle that is d_1 by d_2 drawn around it.

For a **composite figure**, break it into pieces you know (rectangles, triangles, trapezoids), find each area, and add; or find the area of a larger simple figure and subtract the parts that are missing. Both are the Area Addition Postulate at work. Always check that the pieces do not overlap and leave nothing out.



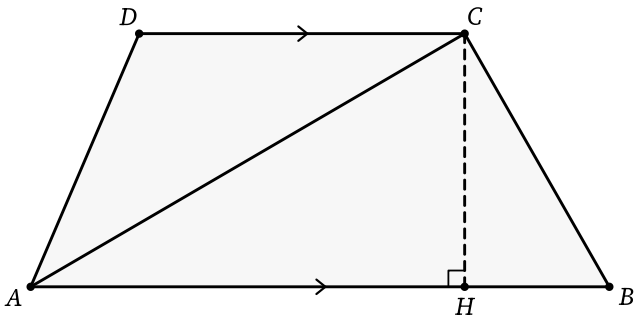
Two copies of a trapezoid make a parallelogram with base $b_1 + b_2$



POSTULATES AND THEOREMS

THEOREM **Trapezoid Area Theorem**

The area of a trapezoid with bases b_1 and b_2 and height h is $A = \frac{1}{2}h(b_1 + b_2)$.



TWO-COLUMN PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; $AB = b_1$, $DC = b_2$, and the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ is h

Prove: $\text{Area}(ABCD) = \dfrac{1}{2}h(b_1 + b_2)$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$; $AB = b_1$, $DC = b_2$; the distance between the parallel lines is h	Given
2	Draw diagonal $\overline{AC}$	Two points determine a line
3	$\text{Area}(ABCD) = \text{Area}(\triangle ABC) + \text{Area}(\triangle ACD)$	Area Addition Postulate
4	The height of $\triangle ABC$ to base $\overline{AB}$ is h, and the height of $\triangle ACD$ to base $\overline{DC}$ is h	C lies on $\overleftrightarrow{DC}$ and A lies on $\overleftrightarrow{AB}$; the distance between parallel lines is constant
5	$\text{Area}(\triangle ABC) = \dfrac{1}{2}b_1 h$ and $\text{Area}(\triangle ACD) = \dfrac{1}{2}b_2 h$	Triangle Area Theorem
6	$\text{Area}(ABCD) = \dfrac{1}{2}b_1 h + \dfrac{1}{2}b_2 h$	Substitution Property of Equality
7	$\text{Area}(ABCD) = \dfrac{1}{2}h(b_1 + b_2)$	Distributive Property

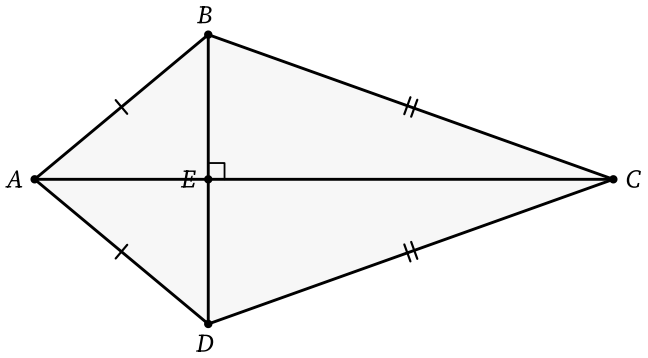
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A second derivation (two copies form a parallelogram with base $b_1 + b_2$) is shown in the core ideas and the first figure.

THEOREM

Kite and Rhombus Area Theorem

If the diagonals of a convex quadrilateral are perpendicular, its area is half the product of the diagonals: $A = \dfrac{1}{2}d_1 d_2$. In particular this holds for every rhombus and every convex kite



(and, by subtracting instead of adding, for a concave kite as well).

PARAGRAPH PROOF

Given: Convex quadrilateral $ABCD$ with diagonals $\overline{AC} \perp \overline{BD}$ meeting at E; $AC = d_1$, $BD = d_2$

Prove: $\text{Area}(ABCD) = \frac{1}{2}d_1 d_2$

Diagonal $\overline{AC}$ divides $ABCD$ into $\triangle ABC$ and $\triangle ADC$, so by the Area Addition Postulate $\text{Area}(ABCD) = \text{Area}(\triangle ABC) + \text{Area}(\triangle ADC)$.

Because $\overline{BD} \perp \overline{AC}$ at E, the segment $\overline{BE}$ is the height of $\triangle ABC$ to base $\overline{AC}$, and $\overline{DE}$ is the height of $\triangle ADC$ to the same base. By the Triangle Area Theorem, $\text{Area}(\triangle ABC) = \frac{1}{2}AC \cdot BE$ and $\text{Area}(\triangle ADC) = \frac{1}{2}AC \cdot DE$.

Adding and using the Distributive Property, $\text{Area}(ABCD) = \frac{1}{2}AC(BE + DE)$. Since E lies between B and D, the Segment Addition Postulate gives $BE + DE = BD$. So $\text{Area}(ABCD) = \frac{1}{2}AC \cdot BD = \frac{1}{2}d_1 d_2$.

The diagonals of a rhombus are perpendicular (Lesson 6.4), and so are the diagonals of a kite (Lesson 6.5), so the formula applies to every rhombus and every convex kite. (For a concave kite $ABCD$ with $AB = AD$, $CB = CD$ and the reflex angle at C, the line $\overleftrightarrow{AC}$ still meets $\overline{BD}$ at right angles at E, with C between A and E. Then $\text{Area}(ABCD) = \text{Area}(\triangle ABD) - \text{Area}(\triangle CBD) = \frac{1}{2}BD \cdot AE - \frac{1}{2}BD \cdot CE = \frac{1}{2}BD \cdot AC$.)

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KEY FORMULAS

$$A_{\text{trapezoid}} = \frac{1}{2}h(b_1 + b_2)$$

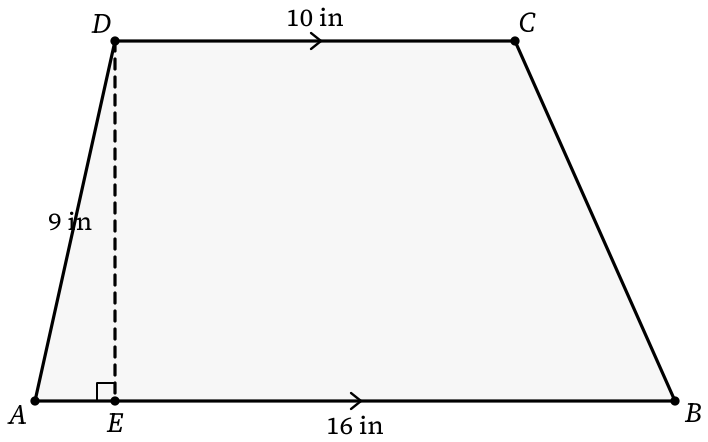
$$A_{\text{rhombus or kite}} = \frac{1}{2}d_1 d_2$$

From history. Surveyors have long measured irregular fields by splitting them into triangles and trapezoids. In the method of offsets, a surveyor stretches a straight baseline (a chain or tape) along the field and measures perpendicular distances, called offsets, from the baseline out to the boundary at regular intervals. Each strip between two offsets is a trapezoid whose parallel sides are the offsets, so the field's area is a sum of trapezoid areas.

Euclid connection. Euclid's Book I definitions call every four-sided figure other than the square, oblong, rhombus and rhomboid a "trapezium," and he gives no trapezoid formula. But his method is ours: I.34 shows a diagonal bisects a parallelogram, and I.37 and I.39 show that triangles on the same base are equal in area exactly when their vertices lie on a parallel to the base. Practice Problem 14 proves I.39 indirectly, as Euclid does.

Practice habit. Recite the trapezoid and kite area formulas together with the reason for each (two triangles of height h; two triangles whose heights add to d_2). For composite figures, sketch the pieces separately and label each piece's base and height before computing.

Example 1. A stained-glass pane for a church window is a trapezoid. Its parallel sides are 16 in and 10 in, and the perpendicular distance between them is 9 in, as in the figure. (a) Find the area of the pane. (b) If the colored glass costs \$0.40 per square inch, what does the glass for one pane cost?

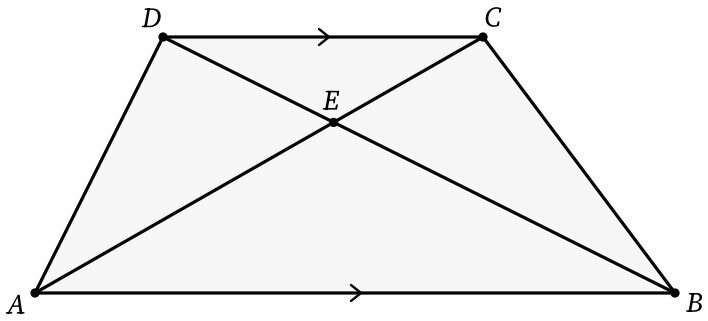


Show solution

1. The bases are the parallel sides, $b_1 = 16$ and $b_2 = 10$; the height is $h = 9$.
2. $A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(9)(16 + 10) = \frac{1}{2}(9)(26) = 117$ square inches.
3. Cost: $117 \times 0.40 = 46.80$, so \$46.80.
4. Check with the average of the bases: $\frac{16 + 10}{2} = 13$ and $13 \times 9 = 117$.

Answer: (a) 117 square inches; (b) \$46.80

Example 2. In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and its diagonals meet at E. Prove that $\text{Area}(\triangle AED) = \text{Area}(\triangle BEC)$.



Show solution

1. The two small triangles are not congruent, so look for larger triangles with equal areas.
2. $\triangle ABD$ and $\triangle ABC$ share base $\overline{AB}$, and their third vertices D and C lie on a line parallel to $\overline{AB}$, so they have equal areas (Lesson 11.1, Euclid I.37).
3. Both of those triangles contain $\triangle ABE$. Subtract it from each.

TWO-COLUMN PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; diagonals $\overline{AC}$ and $\overline{BD}$ meet at E

Prove: $\text{Area}(\triangle AED) = \text{Area}(\triangle BEC)$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$; $\overline{AC}$ and $\overline{BD}$ meet at E	Given
2	$\text{Area}(\triangle ABD) = \text{Area}(\triangle ABC)$	Triangles on the same base whose vertices lie on a line parallel to the base have equal areas (Euclid I.37)
3	$\text{Area}(\triangle ABD) = \text{Area}(\triangle ABE) + \text{Area}(\triangle AED)$	Area Addition Postulate (E is on $\overline{BD}$)
4	$\text{Area}(\triangle ABC) = \text{Area}(\triangle ABE) + \text{Area}(\triangle BEC)$	Area Addition Postulate (E is on $\overline{AC}$)
5	$\text{Area}(\triangle ABE) + \text{Area}(\triangle AED) = \text{Area}(\triangle ABE) + \text{Area}(\triangle BEC)$	Substitution Property of Equality
6	$\text{Area}(\triangle AED) = \text{Area}(\triangle BEC)$	Subtraction Property of Equality

∎

Answer: See the proof.

PRACTICE

1. Easy A trapezoid has bases 7 cm and 13 cm and height 6 cm. Find its area.

   Show answer

   $\frac{1}{2}(6)(7+13) = 60 \text{ cm}^2$

2. Easy Samuel is building a kite. Its two cross sticks, which are the diagonals of the kite, are 30 in and 20 in long and are perpendicular. How many square inches of fabric does the face of the kite need?

   Show answer

   $\frac{1}{2}(30)(20) = 300$ square inches

3. Easy A rhombus has diagonals 10 cm and 16 cm. Find its area.

   Show answer

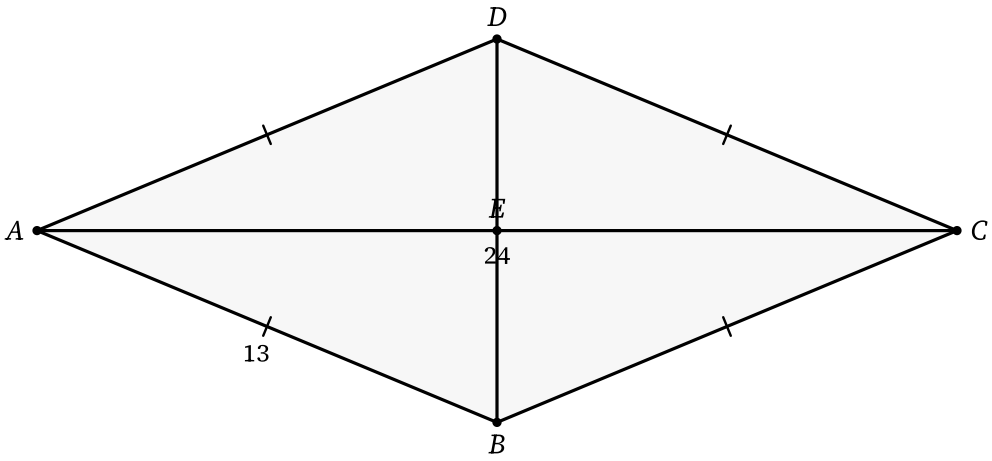
   $\frac{1}{2}(10)(16) = 80 \text{ cm}^2$

4. Easy A trapezoid has area 90 square feet and bases 8 ft and 12 ft. Find its height.

   Show answer

   $90 = \frac{1}{2}h(20) = 10h$, so $h = 9$ feet

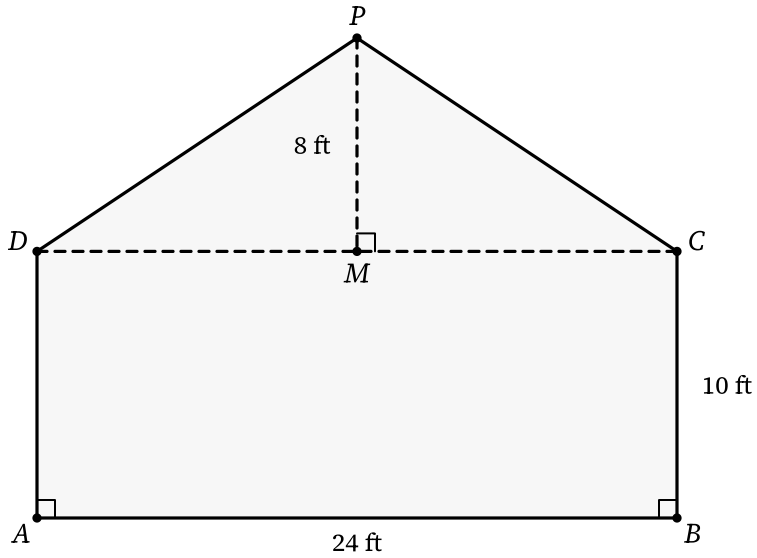
5. **Medium** In the figure, rhombus $ABCD$ has side 13 and diagonal $AC = 24$. Find (a) the length of diagonal $\overline{BD}$ and (b) the area of the rhombus.



Show answer

(a) The diagonals bisect each other at right angles, so $AE = 12$ and $DE = \sqrt{13^2 - 12^2} = 5$; $BD = 10$. (b) $\frac{1}{2}(24)(10) = 120$ square units

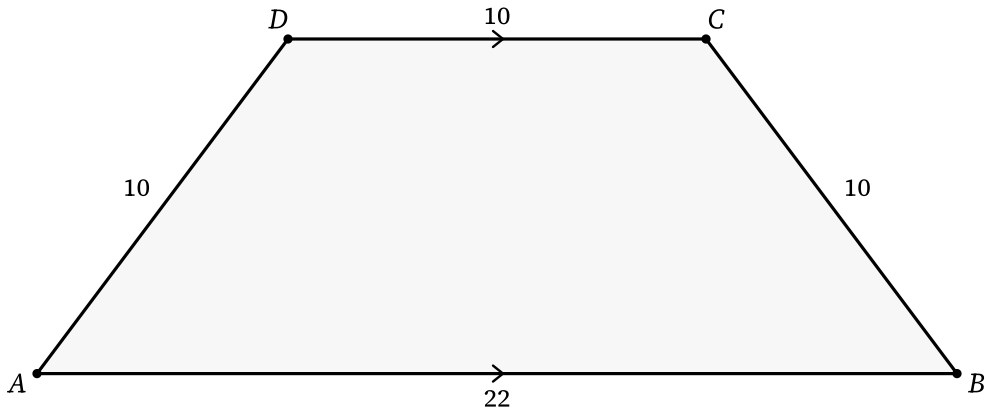
6. **Medium** The front wall of the Miller family's barn is a rectangle 24 ft wide and 10 ft high with a triangular gable on top. The peak is 8 ft above the top of the rectangle, as in the figure. How many square feet of boards cover the whole front wall?



Show answer

$24 \times 10 + \frac{1}{2}(24)(8) = 240 + 96 = 336$ square feet

7. **Medium** In the figure, isosceles trapezoid $ABCD$ has bases $AB = 22$ and $DC = 10$ and legs $AD = BC = 10$. Find its area.



Show answer

Dropping perpendiculars from D and C cuts off $\dfrac{22 - 10}{2} = 6$ at each end, so $h = \sqrt{10^2 - 6^2} = 8$. Area $= \dfrac{1}{2}(8)(22 + 10) = 128$ square units

8. **Medium** A building lot lies between two parallel streets that are 150 ft apart. The lot has 120 ft of frontage on one street and 80 ft on the other, and its side lines run straight from street to street. Find the area of the lot.

Show answer

The lot is a trapezoid: $\dfrac{1}{2}(150)(120 + 80) = 15{,}000$ square feet

9. **Medium** Write a flow proof: if $ABCD$ is a square with diagonal $AC = d$, then $\text{Area}(ABCD) = \frac{1}{2}d^2$.

Show answer

See the model proof.

FLOW PROOF

Given: $ABCD$ is a square; $AC = d$

Prove: $\text{Area}(ABCD) = \frac{1}{2}d^2$

(1) $ABCD$ is a square; $AC = d$
Given

(2) $ABCD$ is a rhombus
A square has four congruent sides (definition of square)
from (1)

(3) $ABCD$ is a rectangle
A square has four right angles (definition of square)
from (1)

(4) $\overline{AC} \perp \overline{BD}$
The diagonals of a rhombus are perpendicular
from (2)

(5) $BD = AC = d$
The diagonals of a rectangle are congruent
from (1), (3)

(6) $\text{Area}(ABCD) = \frac{1}{2}AC \cdot BD$
Kite and Rhombus Area Theorem
from (4)

(7) $\text{Area}(ABCD) = \frac{1}{2}d \cdot d = \frac{1}{2}d^2$
Substitution Property of Equality
from (5), (6)

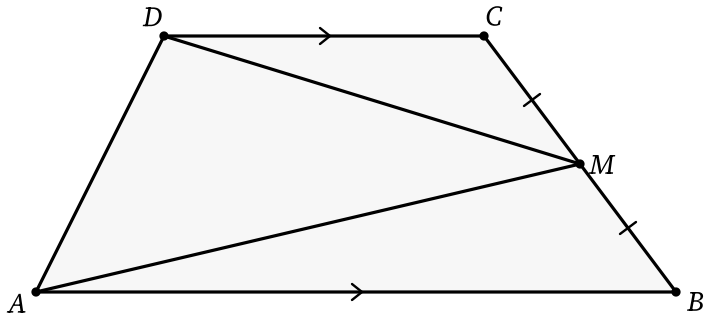
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10. **Medium** Explain why the trapezoid formula $A = \frac{1}{2}h(b_1 + b_2)$ gives the parallelogram formula when $b_2 = b_1$, and the triangle formula when b_2 shrinks to 0.

Show answer

If $b_2 = b_1 = b$, the figure has a pair of parallel, congruent opposite sides, so it is a parallelogram (Euclid I.33), and the formula gives $\frac{1}{2}h(2b) = bh$. If the top base shrinks to a point, the figure becomes a triangle with base b_1 and height h, and the formula gives $\frac{1}{2}h(b_1 + 0) = \frac{1}{2}b_1 h$. One formula covers all three shapes because each is built from triangles of height h.

11. **Hard** In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and M is the midpoint of leg $\overline{BC}$. Write a two-column proof that $\text{Area}(\triangle AMD) = \frac{1}{2}\,\text{Area}(ABCD)$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and height h; M is the midpoint of $\overline{BC}$

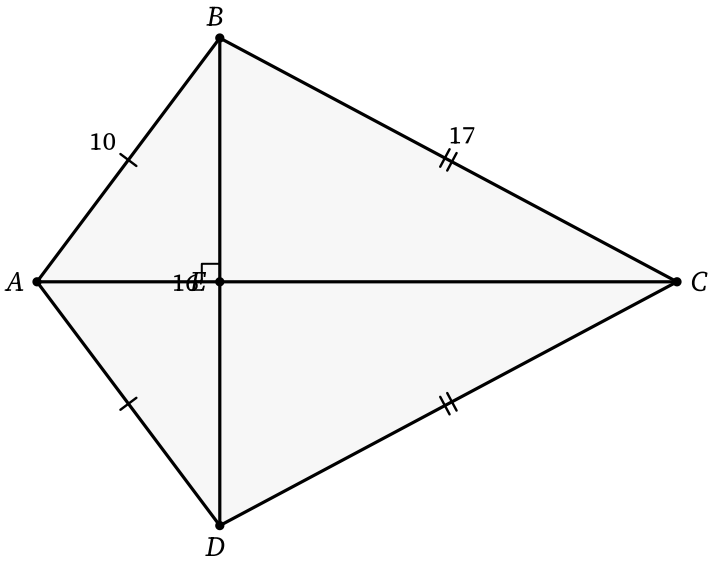
Prove: $\text{Area}(\triangle AMD) = \frac{1}{2}\,\text{Area}(ABCD)$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{DC}$ with height h; M is the midpoint of $\overline{BC}$	Given
2	$BM = MC$	Definition of midpoint
3	$\text{Area}(\triangle ABM) = \text{Area}(\triangle ACM)$	Triangle Area Theorem: equal bases $BM = MC$ on $\overleftrightarrow{BC}$ and the same height from A
4	$\text{Area}(\triangle ABC) = \text{Area}(\triangle ABM) + \text{Area}(\triangle ACM) = 2 \cdot \text{Area}(\triangle ABM)$	Area Addition Postulate; Substitution
5	$\text{Area}(\triangle DCM) = \text{Area}(\triangle DBM)$, so $\text{Area}(\triangle DCB) = 2 \cdot \text{Area}(\triangle DCM)$	Same reasoning as rows 3–4, using vertex D
6	$\text{Area}(\triangle ABC) = \frac{1}{2}AB \cdot h$ and $\text{Area}(\triangle DCB) = \frac{1}{2}DC \cdot h$	Triangle Area Theorem (each height is the distance between the parallel lines)
7	$\text{Area}(\triangle ABM) + \text{Area}(\triangle DCM) = \frac{1}{4}h(AB + DC)$	Division and Substitution Properties; Distributive Property

	STATEMENTS	REASONS
8	$\text{Area}(ABCD) = \frac{1}{2}h(AB + DC)$	Trapezoid Area Theorem
9	$\text{Area}(ABCD) = \text{Area}(\triangle ABM) + \text{Area}(\triangle AMD) + \text{Area}(\triangle DCM)$	Area Addition Postulate
10	$\text{Area}(\triangle AMD) = \frac{1}{2}h(AB + DC) - \frac{1}{4}h(AB + DC) = \frac{1}{4}h(AB + DC)$	Subtraction and Substitution Properties of Equality
11	$\text{Area}(\triangle AMD) = \frac{1}{2}\,\text{Area}(ABCD)$	Substitution Property of Equality (row 8)

∎

12. **Hard** In the figure, kite $ABCD$ has $AB = AD = 10$, $CB = CD = 17$, and diagonal $BD = 16$. Find (a) the length of diagonal $\overline{AC}$ and (b) the area of the kite.



Show answer

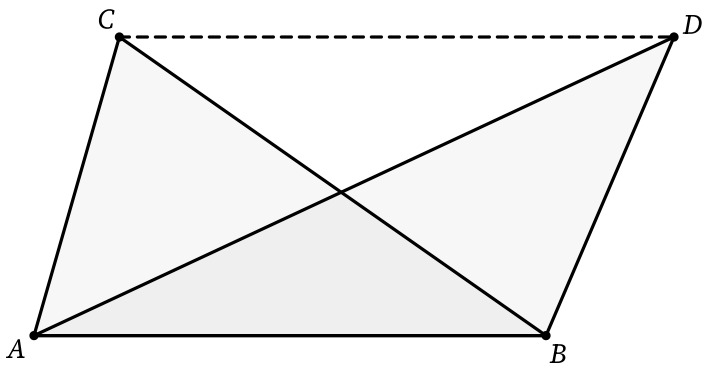
(a) Diagonal $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$, so $BE = 8$. Then $AE = \sqrt{10^2 - 8^2} = 6$ and $CE = \sqrt{17^2 - 8^2} = 15$, so $AC = 21$. (b) $\frac{1}{2}(21)(16) = 168$ square units

13. **Challenge** A rhombus has perimeter 40 inches, and its diagonals are in the ratio $3:4$. Find its area.

Show answer

Each side is 10 in. The half-diagonals are $3k$ and $4k$ and form a right triangle with a side: $(3k)^2 + (4k)^2 = 10^2$, so $k = 2$. The diagonals are 12 in and 16 in, and the area is $\frac{1}{2}(12)(16) = 96$ square inches

14. **Challenge** Prove Euclid I.39 by an indirect proof: if $\triangle ABC$ and $\triangle ABD$ (with $C \neq D$) lie on the same side of $\overline{AB}$ and have equal areas, then $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$.



Show answer

See the model proof.

INDIRECT PROOF

Given: $\triangle ABC$ and $\triangle ABD$ lie on the same side of $\overline{AB}$, $C \neq D$;
$\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD)$
Prove: $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$

Assume, for the sake of contradiction, that $\overleftrightarrow{CD}$ is not parallel to $\overleftrightarrow{AB}$.

Through C draw the line ℓ parallel to $\overleftrightarrow{AB}$ (Playfair's Axiom; Euclid I.31). $\overleftrightarrow{BD}$ is not parallel to ℓ: it passes through B, and the only line through B parallel to ℓ is $\overleftrightarrow{AB}$, which is a different line because D is not on $\overleftrightarrow{AB}$. So $\overleftrightarrow{BD}$ meets ℓ at a point E. D is not on ℓ, since otherwise $\overleftrightarrow{CD}$ would be ℓ, which is parallel to $\overleftrightarrow{AB}$, against our assumption. So $E \neq D$. Every point of ℓ lies on the same side of $\overleftrightarrow{AB}$ as C, which is also D's side, so E lies on ray $\overrightarrow{BD}$.

$\triangle ABC$ and $\triangle ABE$ share base $\overline{AB}$, and their third vertices C and E lie on $\ell \parallel \overleftrightarrow{AB}$. So $\text{Area}(\triangle ABE) = \text{Area}(\triangle ABC)$ (Euclid I.37, Lesson 11.1), and by the given and the Transitive Property, $\text{Area}(\triangle ABE) = \text{Area}(\triangle ABD)$.

If E is between B and D, then $\overline{AE}$ splits $\triangle ABD$ into $\triangle ABE$ and $\triangle AED$, so $\text{Area}(\triangle ABD) = \text{Area}(\triangle ABE) + \text{Area}(\triangle AED) > \text{Area}(\triangle ABE)$. If D is between B and E, then in the same way $\text{Area}(\triangle ABE) = \text{Area}(\triangle ABD) + \text{Area}(\triangle ADE) > \text{Area}(\triangle ABD)$. Either way the whole is greater than the part (Common Notion 5), which contradicts $\text{Area}(\triangle ABE) = \text{Area}(\triangle ABD)$.

Therefore the assumption is false, and $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$.

COMMON MISTAKES

- Using a slanted leg of a trapezoid as its height. The height is perpendicular to both bases; if it is not given, find it with a right triangle as in Problem 7.
- Forgetting to add the bases before multiplying, as in $\frac{1}{2}h \cdot b_1 + b_2$. Write the parentheses: $\frac{1}{2}h(b_1 + b_2)$.
- Using the sides of a rhombus or kite in $\frac{1}{2}d_1 d_2$. The formula needs the full diagonals; if you know half a diagonal, double it first.
- Counting a region twice in a composite figure. Sketch the pieces separately and make sure they do not overlap.

> **Make it a game · Mystery Figure Riddles.** Parent reads riddles: 'I am a rhombus with diagonals 10 and 24. What is my area? What is my side?' 'I am a trapezoid with bases 7 and 13 and height 5.' Answer each in 30 seconds. Then draw a house-shaped composite figure and find its area by splitting it. Write two riddles for your parent.

11.3 Areas of Regular Polygons

OBJECTIVES

- Identify the center, radius, apothem and central angle of a regular polygon, and find the central angle as $360^\circ/n$.
- Derive the formula $A = \frac{1}{2}ap$ by dividing a regular polygon into congruent isosceles triangles.
- Find apothems and areas of regular polygons using special right triangles and trigonometry.
- Inscribe a regular hexagon in a circle with compass and straightedge (Euclid IV.15).

WATCH

Math Antics - Polygons

mathantics · 7:27

https://www.youtube.com/watch?v=IaoZhhx_I9s

> **WHEN TO WATCH**
> Before the Core ideas: regular polygons and their parts.

BACKUP VIDEO

Find Area of Regular Polygon Given Apothem

Mario's Math Tutoring · 2:36

https://www.youtube.com/watch?v=CAXJh2avnu8

After the area derivation: an octagon split into triangles, then A = ½ap. It uses the tangent ratio (Lesson 9.5) to find the side.

KEY TERMS

regular polygon

A polygon that is both equilateral (all sides congruent) and equiangular (all angles congruent).

center of a regular polygon

The point that is the same distance from every vertex; it is the center of the circle through all the vertices.

radius of a regular polygon

A segment from the center to a vertex, or its length r.

apothem

The perpendicular segment from the center to a side, or its length a. It is the same length for every side.

central angle of a regular polygon

An angle whose vertex is the center and whose sides pass through two consecutive vertices. A regular n-gon has n of them, each $\dfrac{360^\circ}{n}$.

CORE IDEAS

A regular polygon has a **center**: one point equally far from all the vertices, so a circle drawn around it passes through every vertex. (We accept that every regular polygon has a center; Euclid constructs the circle around regular polygons in Book IV.) The distance from the center to a vertex is the **radius** r, and the perpendicular distance from the center to a side is the **apothem** a.

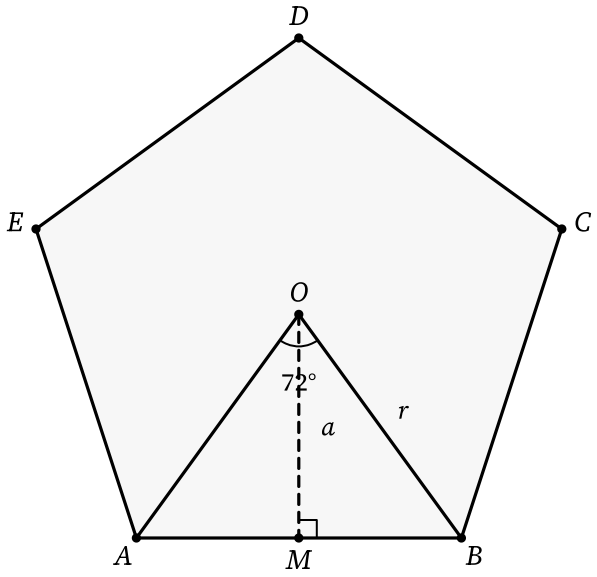
Draw a radius to every vertex. This cuts a regular n-gon into n triangles, each with two sides equal to r and a third side equal to the side s of the polygon. By SSS all n triangles are congruent, so their angles at the center are congruent. Together these **central angles** go once around the center, so each one is $\dfrac{360^\circ}{n}$. A hexagon's central angle is 60°, a square's is 90°, an octagon's is 45°.

Why $A = \dfrac{1}{2}ap$. Each of the n congruent triangles has base s and height a (the apothem is exactly the height from the center to that side), so each has area $\dfrac{1}{2}sa$. Adding all n of them, $A = n \cdot \dfrac{1}{2}sa = \dfrac{1}{2}a(ns)$, and ns is the perimeter p. So $A = \dfrac{1}{2}ap$. You can picture this as the n triangles unrolled into a row: their bases add up to the perimeter and they all have height a.

Finding the apothem. The apothem cuts one of the isosceles triangles into two congruent right triangles (by HL), so it bisects the side and bisects the central angle. In each right triangle the legs are a and $\dfrac{s}{2}$, the hypotenuse is r, and the angle at the center is $\dfrac{1}{2} \cdot \dfrac{360^\circ}{n} = \dfrac{180^\circ}{n}$. For triangles, squares and hexagons this angle is 60°, 45°

or 30°, so special right triangles give exact answers. For other polygons use trigonometry: $\tan\left(\frac{180^\circ}{n}\right) = \frac{s/2}{a}$.

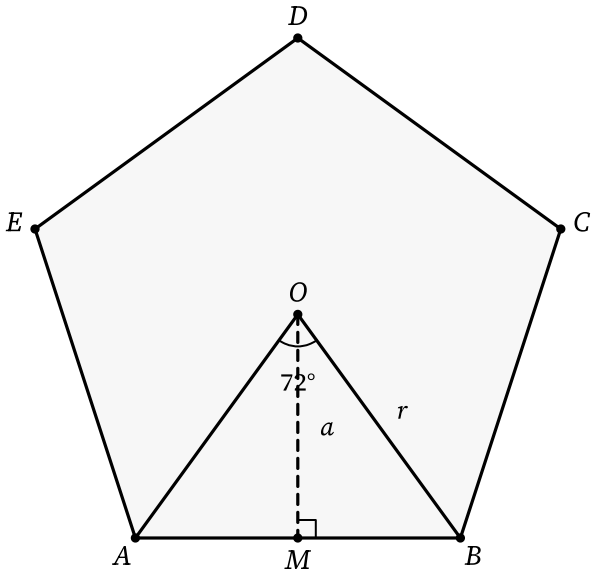
The hexagon is special: its central angle is 60°, and each central triangle has two sides equal to r, so its base angles are also 60° and it is equilateral. That means **the side of a regular hexagon equals its radius**, which is why you can step a compass around a circle exactly six times.



POSTULATES AND THEOREMS

THEOREM **Central Angle Theorem for Regular Polygons**

The central angles of a regular n-gon are congruent, and each measures $\frac{360^\circ}{n}$.



PARAGRAPH PROOF

Given: A regular n-gon with center O

Prove: Each central angle measures $\dfrac{360^\circ}{n}$

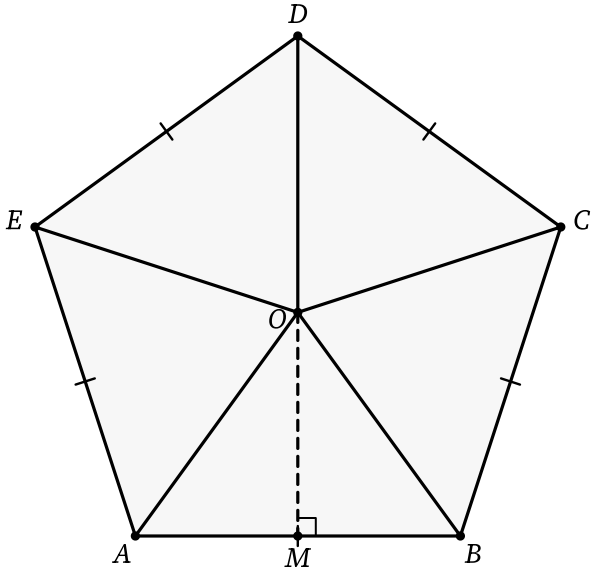
Let A, B, C be any three consecutive vertices. By the definition of the center, $OA = OB = OC$, and by the definition of a regular polygon, $AB = BC$. So $\triangle AOB \cong \triangle BOC$ by SSS, and $\angle AOB \cong \angle BOC$ by CPCTC. Repeating this all the way around shows that all n central angles are congruent; call the measure of each one x.

The n central angles do not overlap and together fill the whole angle around O, which is 360° (Angle Addition Postulate). So $nx = 360^\circ$, and by the Division Property of Equality, $x = \dfrac{360^\circ}{n}$.

∎

THEOREM **Regular Polygon Area Theorem**

The area of a regular polygon is half the product of its apothem and its perimeter: $A = \dfrac{1}{2}ap$.



TWO-COLUMN PROOF

Given: A regular n-gon with center O, side s, apothem a and perimeter p

Prove: $A = \dfrac{1}{2}ap$

	STATEMENTS	REASONS
1	A regular n-gon has center O, side s, apothem a, perimeter p	Given

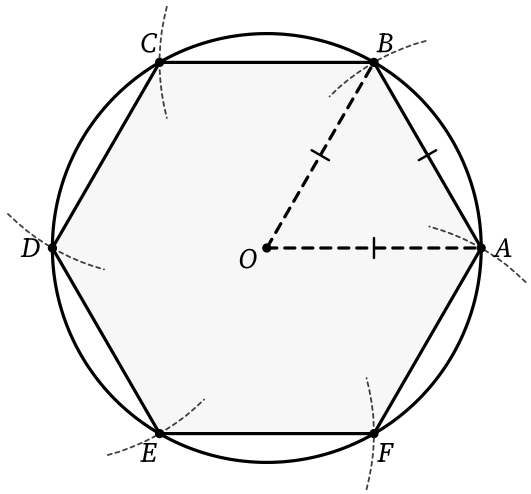
	STATEMENTS	REASONS
2	Radii from O to the vertices divide the polygon into n triangles	Two points determine a line
3	The n triangles are congruent	SSS (all radii are congruent; all sides are congruent)
4	Each triangle has base s and height a	Definition of apothem (the perpendicular distance from O to each side is a)
5	The area of each triangle is $\frac{1}{2}sa$	Triangle Area Theorem
6	$A = n \cdot \frac{1}{2}sa$	Area Addition Postulate
7	$A = \frac{1}{2}a(ns)$	Commutative and Associative Properties of Multiplication
8	$ns = p$	Definition of perimeter (n sides, each of length s)
9	$A = \frac{1}{2}ap$	Substitution Property of Equality

∎

CONSTRUCTIONS

CONSTRUCTION **Inscribe a regular hexagon in a circle** Euclid IV.15

1. Draw a circle with center O and mark any point A on it.
2. Keep the compass set to the radius of the circle. Put the point on A and draw an arc crossing the circle at B.
3. Without changing the compass, put the point on B and mark C on the circle; then



from C mark D, from D mark E, and from E mark F.

4. Check: the arc from F should land exactly on A. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$.

Why it works. In each triangle such as $\triangle OAB$, $OA = OB$ (radii) and $AB = OA$ (the compass width), so the triangle is equilateral and $m\angle AOB = 60^\circ$. Six such angles make $6 \times 60^\circ = 360^\circ$, so the sixth arc closes up at A. The six sides are congruent (all equal to the radius), and the six angles of the hexagon are congruent (each is made of two 60° angles of equilateral triangles, 120°), so the hexagon is regular.

KEY FORMULAS

$$\text{central angle} = \frac{360^\circ}{n}$$

$$A = \frac{1}{2}ap$$

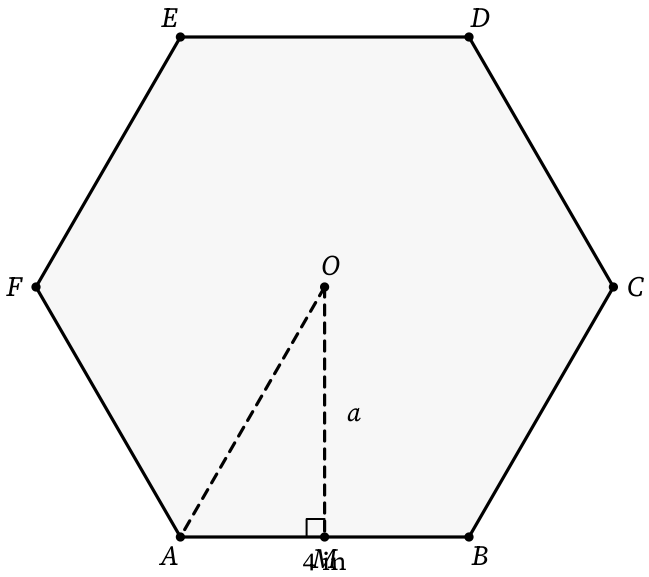
$$\tan\left(\frac{180^\circ}{n}\right) = \frac{s/2}{a}$$

From history. Only three regular polygons tile a floor with no gaps: the equilateral triangle, the square and the regular hexagon. Pappus of Alexandria, writing around AD 300, admired the bees for building their honeycombs from hexagons, since of these three shapes the hexagon encloses the most area for the same perimeter, so the bees store the most honey for the least wax. Hexagonal floor tiles in old churches and kitchens use the same fact.

Euclid connection. Book IV of the Elements is about polygons and circles. Euclid inscribes a regular pentagon (IV.11), a regular hexagon (IV.15) and a regular fifteen-sided polygon (IV.16) in a circle. After IV.15 he notes that "the side of the hexagon is equal to the radius of the circle," the fact behind our construction.

Practice habit. Recite the four parts of a regular polygon (center, radius, apothem, central angle) and the formula $A = \frac{1}{2}ap$. Then always draw one central triangle and its right-triangle half, labeled with r, a, $\frac{s}{2}$ and the angle $\frac{180^\circ}{n}$, before you compute anything.

Example 1. Ruth is tiling a kitchen floor with regular hexagonal tiles that are 4 in on each side, as in the figure. Find the area of one tile, exactly and to the nearest tenth of a square inch.

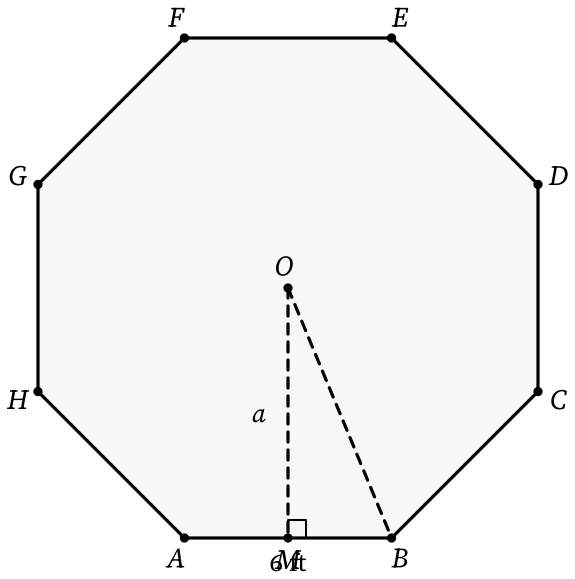


Show solution

1. The central angle is $\dfrac{360^\circ}{6} = 60^\circ$; the apothem $\overline{OM}$ splits it into two 30° angles and bisects the side, so $AM = 2$.
2. In right $\triangle OMA$, the angle at O is 30°, so the side opposite it is $AM = 2$ and the apothem is the longer leg: $a = 2\sqrt{3}$ in (30°-60°-90° triangle).
3. Perimeter: $p = 6 \times 4 = 24$ in.
4. $A = \dfrac{1}{2}ap = \dfrac{1}{2}(2\sqrt{3})(24) = 24\sqrt{3} \approx 41.6$ square inches.

Answer: $24\sqrt{3} \approx 41.6$ square inches

Example 2. Henry is building a gazebo whose floor is a regular octagon with sides of 6 ft, as in the figure. How many square feet of decking does the floor need? Round to the nearest tenth.



Show solution

1. Central angle: $\dfrac{360^\circ}{8} = 45^\circ$. The apothem $\overline{OM}$ bisects it, so $m\angle MOB = 22.5^\circ$, and it bisects side $\overline{AB}$, so $MB = 3$ ft.
2. In right $\triangle OMB$: $\tan 22.5^\circ = \dfrac{MB}{OM} = \dfrac{3}{a}$, so $a = \dfrac{3}{\tan 22.5^\circ} \approx \dfrac{3}{0.41421} \approx 7.243$ ft.
3. Perimeter: $p = 8 \times 6 = 48$ ft.
4. $A = \dfrac{1}{2}ap \approx \dfrac{1}{2}(7.243)(48) \approx 173.8$ square feet. (Keep the unrounded apothem in your calculator until the last step.)

Answer: about 173.8 square feet

PRACTICE

1. Easy Find the measure of each central angle of a regular decagon (10 sides).

   Show answer

   $\dfrac{360^\circ}{10} = 36^\circ$

2. **Easy** Each central angle of a regular polygon measures 24°. How many sides does it have?

   Show answer

   $n = \dfrac{360}{24} = 15$ sides

3. **Easy** A regular pentagon has sides of 8 cm and an apothem of about 5.5 cm. Use $A = \dfrac{1}{2}ap$ to estimate its area.

   Show answer

   $p = 40$, so $A \approx \dfrac{1}{2}(5.5)(40) = 110$ cm^2

4. **Easy** A square has an apothem of 7 in. Find its side length and its area, and check that $A = \dfrac{1}{2}ap$ agrees with $A = s^2$.

   Show answer

   The apothem is half a side, so $s = 14$ in and $A = 196$ square inches; also $\dfrac{1}{2}(7)(56) = 196$

5. **Medium** Find the area of an equilateral triangle with side 12 by finding its apothem and using $A = \dfrac{1}{2}ap$. Give an exact answer.

   Show answer

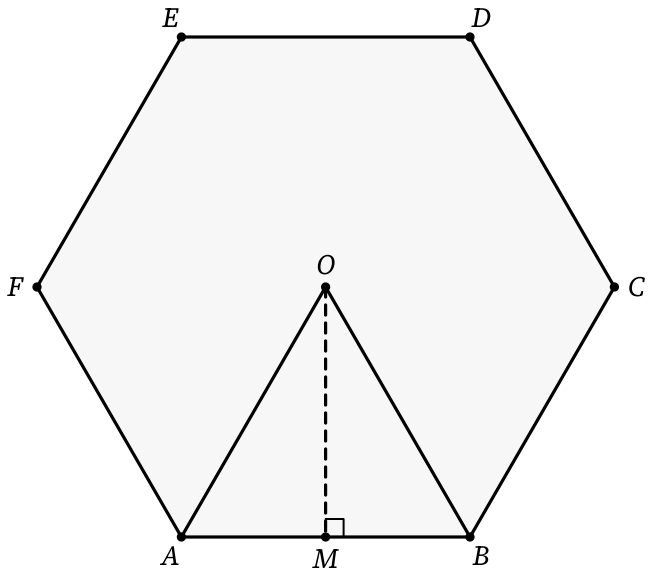
   Half-side 6, angle at the center 60°, so $a = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$; $p = 36$; $A = \dfrac{1}{2}(2\sqrt{3})(36) = 36\sqrt{3}$ square units

6. **Medium** A regular hexagon has radius 10 cm. Find its area in simplest radical form.

   Show answer

   Side $= 10$, apothem $= 5\sqrt{3}$, $p = 60$, so $A = \dfrac{1}{2}(5\sqrt{3})(60) = 150\sqrt{3}$ cm^2

7. **Medium** In the figure, $ABCDEF$ is a regular hexagon with center O, and $\overline{OM}$ is the apothem to side $\overline{AB}$. Find (a) $m\angle AOB$ and (b) $m\angle AOM$.



Show answer

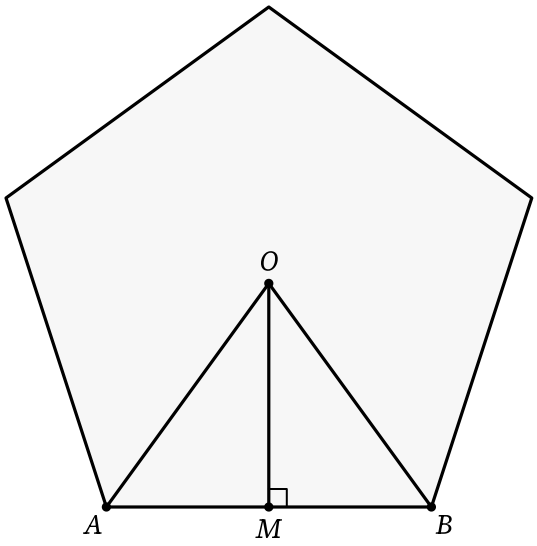
(a) $\dfrac{360^\circ}{6} = 60^\circ$; (b) the apothem bisects the central angle, so 30°

8. **Medium** A garden stepping stone is a regular hexagon with an apothem of 9 in. Find its area to the nearest tenth of a square inch.

Show answer

In the 30°-60°-90° triangle the apothem is the longer leg, so the half-side is $\dfrac{9}{\sqrt{3}} = 3\sqrt{3}$ and $s = 6\sqrt{3}$. Then $p = 36\sqrt{3}$ and $A = \dfrac{1}{2}(9)(36\sqrt{3}) = 162\sqrt{3} \approx 280.6$ square inches

9. **Medium** In the figure, O is the center of a regular polygon, $\overline{AB}$ is a side, and $\overline{OM} \perp \overline{AB}$ at M. Write a two-column proof that the apothem bisects the side and the central angle: $\overline{AM} \cong \overline{BM}$ and $\angle AOM \cong \angle BOM$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: O is the center of a regular polygon with side $\overline{AB}$; $\overline{OM} \perp \overline{AB}$ at M

Prove: $\overline{AM} \cong \overline{BM}$ and $\angle AOM \cong \angle BOM$

	STATEMENTS	REASONS
1	O is the center; $\overline{OM} \perp \overline{AB}$	Given
2	$\overline{OA} \cong \overline{OB}$	Definition of the center of a regular polygon
3	$\angle OMA$ and $\angle OMB$ are right angles	Definition of perpendicular
4	$\overline{OM} \cong \overline{OM}$	Reflexive Property of Congruence
5	$\triangle OMA \cong \triangle OMB$	HL
6	$\overline{AM} \cong \overline{BM}$ and $\angle AOM \cong \angle BOM$	CPCTC

∎

10. **Medium** Explain why no regular polygon has a central angle of 25°. Write your explanation as an indirect proof.

Show answer

See the model proof.

INDIRECT PROOF

Prove: No regular polygon has a central angle of 25°

Assume, for the sake of contradiction, that some regular n-gon has central angles of 25°.

By the Central Angle Theorem, each central angle measures $\frac{360^\circ}{n}$, so $\frac{360}{n} = 25$ and $n = \frac{360}{25} = 14.4$.

But the number of sides of a polygon must be a whole number. This is a contradiction, so the assumption is false: no regular polygon has a central angle of 25°.

∎

11. **Hard** A rose window in a chapel is a regular octagon whose radius (center to vertex) is 5 ft. Find the area of the glass to the nearest tenth of a square foot.

Show answer

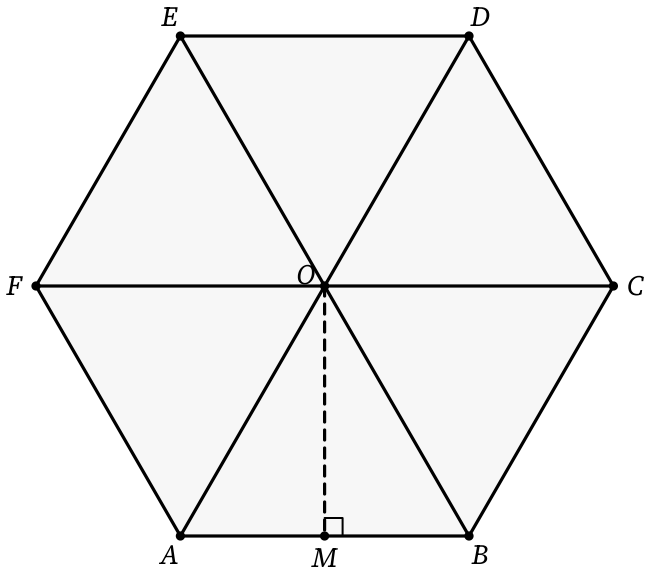
Half the central angle is 22.5°, so $a = 5\cos 22.5^\circ \approx 4.619$ ft and $s = 2(5\sin 22.5^\circ) \approx 3.827$ ft. Then $p \approx 30.615$ ft and $A = \frac{1}{2}ap \approx 70.7$ square feet

12. **Hard** Find the area of a regular pentagon with sides of 10 cm, to the nearest tenth of a square centimeter.

Show answer

Half the central angle is 36°, so $a = \frac{5}{\tan 36^\circ} \approx 6.882$ cm; $p = 50$ cm; $A \approx \frac{1}{2}(6.882)(50) \approx 172.0$ cm^2

13. **Hard** Prove that a regular hexagon with side s has area $\dfrac{3\sqrt{3}}{2}s^2$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: Regular hexagon $ABCDEF$ with center O and side s

Prove: $\text{Area}(ABCDEF) = \dfrac{3\sqrt{3}}{2}s^2$

The radii $\overline{OA}, \ldots, \overline{OF}$ divide the hexagon into six congruent triangles, and by the Central Angle Theorem each central angle measures $\dfrac{360^\circ}{6} = 60^\circ$. In $\triangle AOB$, $OA = OB$, so by the Isosceles Triangle Theorem $\angle OAB \cong \angle OBA$. The angles of the triangle sum to 180°, so each base angle is $\dfrac{180^\circ - 60^\circ}{2} = 60^\circ$. An equiangular triangle is equilateral, so $OA = OB = AB = s$.

Let M be the foot of the apothem on $\overline{AB}$. The apothem bisects the side, so $AM = \dfrac{s}{2}$, and $\triangle OMA$ is a 30°-60°-90° triangle with hypotenuse s. Its longer leg is $a = \dfrac{s}{2}\sqrt{3} = \dfrac{\sqrt{3}}{2}s$.

The perimeter is $p = 6s$. By the Regular Polygon Area Theorem, $A = \dfrac{1}{2}ap = \dfrac{1}{2} \cdot \dfrac{\sqrt{3}}{2}s \cdot 6s = \dfrac{3\sqrt{3}}{2}s^2$.

∎

14. **Challenge** A regular polygon has side 8 and apothem $4\sqrt{3}$. (a) How many sides does it have? (b) Find its area in simplest radical form.

Show answer

(a) In the right triangle formed by the apothem and half a side, $\tan\theta = \dfrac{4}{4\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so the half central angle is $\theta = 30^\circ$ and the central angle is 60°; $n = \dfrac{360}{60} = 6$. (b) $p = 48$, so $A = \dfrac{1}{2}(4\sqrt{3})(48) = 96\sqrt{3}$ square units

COMMON MISTAKES

- Confusing the radius with the apothem. The radius goes to a vertex; the apothem goes to the midpoint of a side and meets it at a right angle, so it is the shorter of the two.
- Using the whole central angle in the right triangle. The right triangle has the HALF angle $\dfrac{180^\circ}{n}$ at the center and the HALF side $\dfrac{s}{2}$ opposite it.
- Using a side instead of the perimeter in $A = \dfrac{1}{2}ap$. Multiply the side by the number of sides first.
- Rounding the apothem early. Keep full calculator accuracy and round only the final area.

> **Make it a game · Fold the Plate.** Fold a paper circle in half three times for eight equal sectors, mark where creases meet the edge, and connect to form a regular octagon. Find the central angle (360/8), draw an apothem, measure the apothem and a side, and compute area as half the apothem times the perimeter. Parent challenges you with a hexagon.

11.4 Circumference and Arc Length

OBJECTIVES

- Explain why the ratio of circumference to diameter is the same number π for every circle.
- Use $C = \pi d = 2\pi r$ to find circumferences, radii and diameters, exactly and rounded.
- Find the length of an arc as a fraction of the circumference, and use arc length to find a radius or a central angle.
- Describe how Eratosthenes measured the earth's circumference with an arc-length proportion.

WATCH

Math Antics - Circles, What Is PI?

mathantics · 7:58

https://www.youtube.com/watch?v=cC0fZ_lkFpQ

> **WHEN TO WATCH**
> Before the Core ideas: what pi is and why C = πd.

Note: Eratosthenes' measurement is taught from the lesson.

BACKUP VIDEO

Arc Length - Math Antics Extras

mathantics · 3:40

https://www.youtube.com/watch?v=oL_Vf9fYCpg

Before the arc-length examples.

KEY TERMS

circumference
The distance around a circle; its perimeter.

π (pi)
The ratio of the circumference of any circle to its diameter, $\pi = \dfrac{C}{d} \approx 3.14159$. It is an irrational number, so its decimal never ends or repeats.

arc length

The distance along an arc, measured in units of length (inches, feet). It is different from the arc's degree measure, which is measured in degrees.

radian

A unit of angle measure: a central angle of 1 radian cuts off an arc whose length equals the radius. A full circle is 2π radians, about 6.28. (A preview; radians are used in the Trigonometry course.)

CORE IDEAS

Wrap a string once around a jar lid and lay it next to the lid's diameter: the string is a little more than three diameters long. Do the same with a bicycle wheel or a round table, and you get the same answer, a little more than three. This ratio, circumference divided by diameter, is called π, and it is about 3.14159.

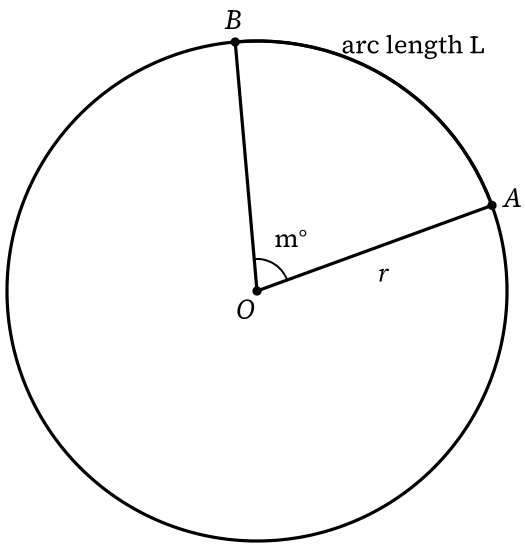
Why is the ratio the same for every circle? All circles are similar: a dilation with center at the center of one circle and scale factor $k = \dfrac{r_2}{r_1}$ maps it onto any other circle with radius r_2. A dilation multiplies every length by k, so it multiplies both the diameter and the circumference by k, and their ratio does not change. You can see the same thing with inscribed regular polygons: in a circle twice as large, every inscribed n-gon is twice as large (its central triangles are similar by SAS Similarity), so its perimeter is also twice as large, and the circumference is the number those perimeters approach as n grows.

Since $\dfrac{C}{d} = \pi$, we get $C = \pi d$, and because $d = 2r$, also $C = 2\pi r$. For an exact answer leave π in it ("10π inches"); for a measurement use the π key on your calculator and round at the end. The old approximations 3.14 and $\dfrac{22}{7}$ are close but not exact.

An **arc** is part of a circle, so its length is the same fraction of the circumference that its central angle is of $360°$. An arc of $90°$ is a quarter of the circle, so its length is a quarter of the circumference. In general, an arc of $m°$ has length $\dfrac{m}{360} \cdot 2\pi r$. Arc length is a length (feet); arc measure is an angle (degrees). Two arcs can have the same degree measure and different lengths if they lie on circles of different sizes.

The arc-length proportion also works backward. If you know the length of an arc and its central angle, you can find the circumference of the whole circle: that is exactly how Eratosthenes measured the earth. And it suggests a new way to measure

angles: the **radian** measures a central angle by the ratio $\dfrac{\text{arc length}}{\text{radius}}$, so a full turn is $\dfrac{2\pi r}{r} = 2\pi$ radians. You will use radians in later courses.



POSTULATES AND THEOREMS

THEOREM **Circumference Formula**

The circumference of a circle with diameter d and radius r is $C = \pi d = 2\pi r$.

PARAGRAPH PROOF

Given: A circle with diameter d, radius r and circumference C

Prove: $C = \pi d = 2\pi r$

By the definition of π, $\dfrac{C}{d} = \pi$ for every circle. (This definition makes sense because the ratio is the same for all circles, as explained in the core ideas: all circles are similar.)

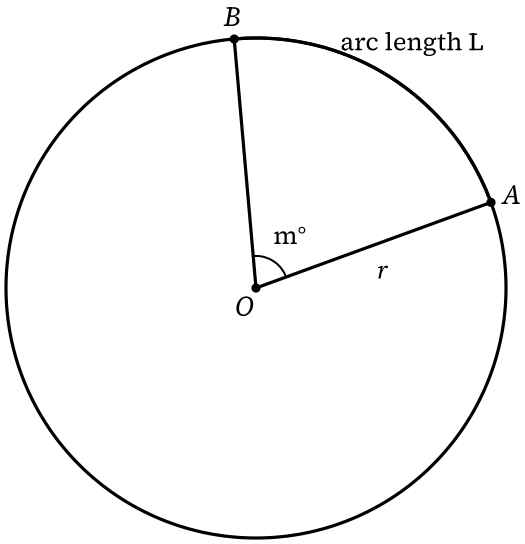
Multiplying both sides by d gives $C = \pi d$ (Multiplication Property of Equality). A diameter is made of two radii, so $d = 2r$, and substituting gives $C = \pi(2r) = 2\pi r$.

∎

The fact that all circles have the same ratio $\dfrac{C}{d}$ rests on similarity together with a limit (the perimeters of inscribed polygons approach the circumference); we accept that limit step without proof in this course, as Archimedes made it rigorous with his method of exhaustion.

THEOREM **Arc Length Formula**

In a circle of radius r, the length of an arc whose central angle measures m° is $L = \dfrac{m}{360} \cdot 2\pi r$.



PARAGRAPH PROOF

Given: Circle O with radius r; arc $\overset{\frown}{AB}$ with central angle m°

Prove: Length of $\overset{\frown}{AB} = \dfrac{m}{360} \cdot 2\pi r$

In the same circle, congruent central angles cut off congruent arcs (Lesson 10.2), so arcs with equal central angles have equal lengths. When two adjacent arcs make up a larger arc, the length of the whole is the sum of the lengths of the parts, and by the Arc Addition Postulate its degree measure is the sum of their measures. So arc length is proportional to the central angle: an arc of 1° is $\dfrac{1}{360}$ of the circle, and an arc of m° is $\dfrac{m}{360}$ of the circle.

The whole circle has length $2\pi r$, so the arc has length $\dfrac{m}{360} \cdot 2\pi r$.

∎

For whole-number degree measures the proportion follows directly by adding 1° arcs; for other measures we accept it as the natural extension.

KEY FORMULAS

$$\pi = \frac{C}{d}$$

$$C = \pi d = 2\pi r$$

$$\text{arc length} = \frac{m}{360} \cdot 2\pi r$$

From history. Eratosthenes, the librarian at Alexandria (about 240 BC), is reported by later writers to have measured the earth this way. At noon on midsummer day the sun shone straight down a well at Syene, so it was directly overhead there, while at Alexandria a vertical rod cast a shadow showing the sun about 7.2° from overhead, one-fiftieth of a full circle. The sun's rays are so far away that they arrive parallel, so by alternate interior angles the arc of the earth from Syene to Alexandria also measures 7.2°. He took the distance between the cities as 5,000 stadia, so the whole circumference was $50 \times 5{,}000 = 250{,}000$ stadia, a result remarkably close to the true size, though we do not know the exact length of his stadion.

Euclid connection. Euclid compares areas of circles (XII.2) but never measures the length of a curve; the Elements has no circumference formula. Archimedes, in *Measurement of a Circle*, bounded the circumference between the perimeters of inscribed and circumscribed regular polygons, using the same picture of polygons approaching a circle that we use here.

Practice habit. Recite $C = \pi d = 2\pi r$ and the arc-length proportion $\dfrac{L}{2\pi r} = \dfrac{m}{360}$. Before computing, check whether you were given a radius or a diameter and circle it, and label units on every answer: arc length in units of length, arc measure in degrees.

WORKED EXAMPLES

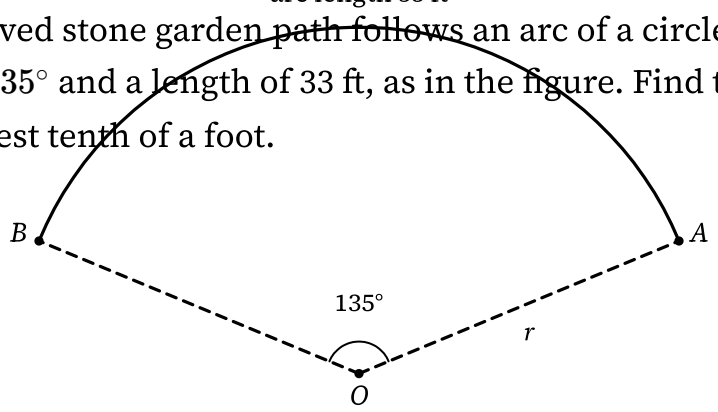
Example 1. Grace's bicycle wheel has a diameter of 26 in. (a) Find the circumference of the wheel, exactly and to the nearest tenth of an inch. (b) About how many full turns does the wheel make when she rides 1 mile (5,280 ft)?

Show solution

1. (a) $C = \pi d = 26\pi \approx 81.68$, so about 81.7 inches.
2. (b) Change the mile to inches so the units match: $5{,}280 \times 12 = 63{,}360$ in.
3. Each turn rolls the bicycle forward one circumference, so the number of turns is $\dfrac{63{,}360}{26\pi} \approx \dfrac{63{,}360}{81.681} \approx 775.7$.
4. So the wheel makes about 775 full turns (and part of another).

Answer: (a) $26\pi \approx 81.7$ in; (b) about 775 full turns

Example 2. A curved stone garden path follows an arc of a circle. The arc has a central angle of 135° and a length of 33 ft, as in the figure. Find the radius of the circle to the nearest tenth of a foot.



Show solution

1. Set up the arc-length proportion: $\dfrac{33}{2\pi r} = \dfrac{135}{360}$.
2. Simplify the right side: $\dfrac{135}{360} = \dfrac{3}{8}$, so $33 = \dfrac{3}{8} \cdot 2\pi r = \dfrac{3\pi r}{4}$.
3. Solve: $r = \dfrac{4 \times 33}{3\pi} = \dfrac{44}{\pi} \approx 14.006$.
4. Check: $\dfrac{135}{360} \cdot 2\pi(14.006) \approx 33.0$. The radius is about 14.0 feet.

Answer: $r = \dfrac{44}{\pi} \approx 14.0$ feet

PRACTICE

1. Easy Find the circumference of a circle with radius 9 cm. Give an exact answer in terms of π.

   Show answer

   $C = 2\pi(9) = 18\pi$ cm

2. Easy A round dining table has a diameter of 48 in. Henry wants to put a strip of trim around its edge. How long must the strip be, to the nearest inch?

   Show answer

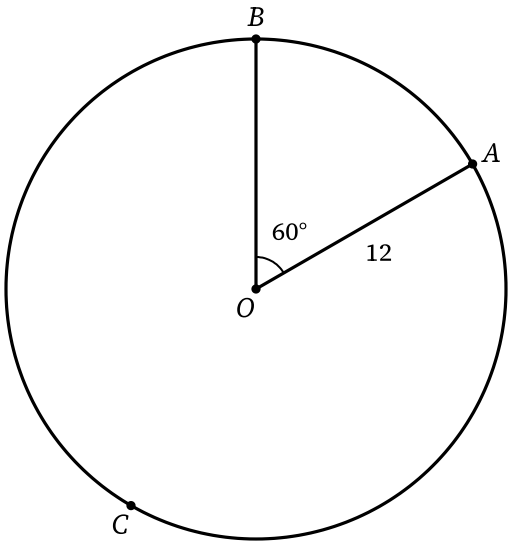
   $C = 48\pi \approx 150.8$, so about 151 inches

3. Easy A circle has circumference 30π m. Find its radius.

   Show answer

   $2\pi r = 30\pi$, so $r = 15$ m

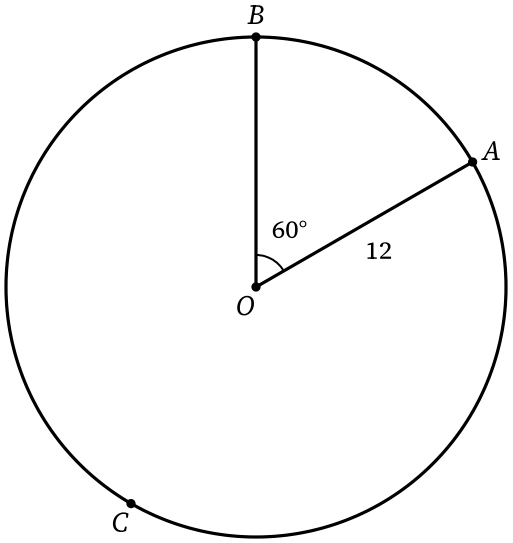
4. **Easy** In the figure, circle O has radius 12 and $m\angle AOB = 60°$. Find the length of minor arc $\overset{\frown}{AB}$ in terms of π.



Show answer

$$\frac{60}{360} \cdot 2\pi(12) = 4\pi$$

5. **Medium** In the same circle as Problem 4 (radius 12, $m\angle AOB = 60°$), find the length of the major arc $\overset{\frown}{ACB}$ in terms of π.



Show answer

The major arc measures $360° - 60° = 300°$, so its length is $\frac{300}{360} \cdot 24\pi = 20\pi$

6. **Medium** A circular running track around a sports field has an inside lane of radius 30 m and an outside lane of radius 31.2 m. How much farther does a runner in the outside lane travel in one full lap? Round to the nearest tenth of a meter.

   Show answer

   $2\pi(31.2) - 2\pi(30) = 2\pi(1.2) = 2.4\pi \approx 7.5$ m

7. **Medium** An arc of a circle with radius 10 in has length 5π in. Find the measure of its central angle.

   Show answer

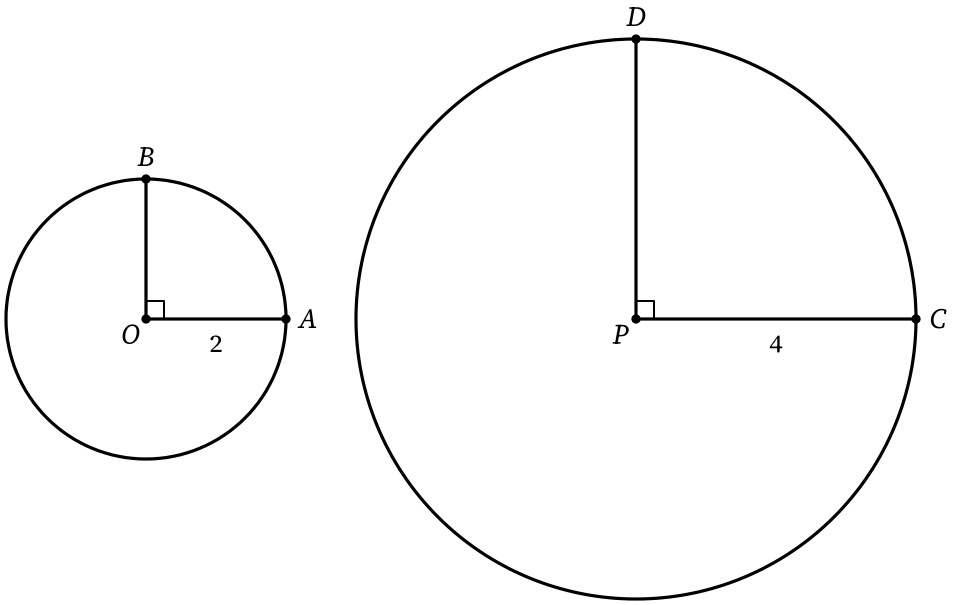
   $\frac{m}{360} \cdot 20\pi = 5\pi$, so $\frac{m}{360} = \frac{1}{4}$ and $m = 90^\circ$

8. **Medium** The top of a church window is a semicircle on a horizontal sill 4 ft wide, so the semicircle has diameter 4 ft. A craftsman bends a strip of lead along the curved edge. How long is the curved edge, to the nearest tenth of a foot?

   Show answer

   Half the circumference: $\frac{1}{2}\pi(4) = 2\pi \approx 6.3$ ft

9. **Medium** Explain why two arcs can have the same degree measure but different lengths. Give an example with numbers.



Show answer

Degree measure depends only on the central angle, but length also depends on the size of the circle. For example, a 90° arc of a circle with radius 2 has length $\frac{1}{4}(4\pi) = \pi$, while a 90° arc of a circle with radius 4 has length $\frac{1}{4}(8\pi) = 2\pi$. The arcs are both a quarter of their circles, but the larger circle is twice as long around.

10. Hard Prove that in any circle, the ratio of an arc's length to the radius depends only on the central angle: $\frac{L}{r} = \frac{\pi m}{180}$ for an arc of m°. (This ratio is the radian measure of the angle.)

Show answer

See the model proof.

TWO-COLUMN PROOF

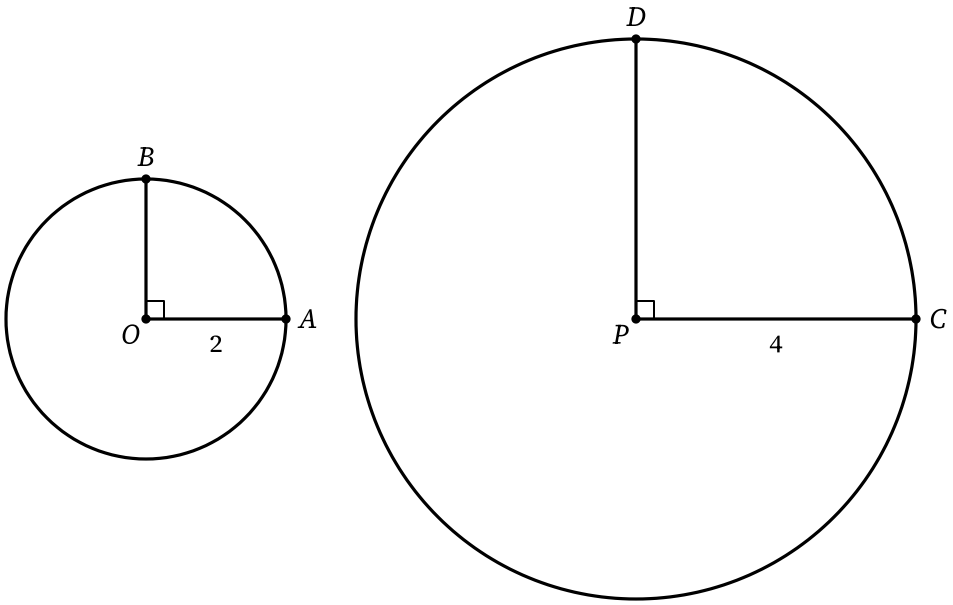
Given: A circle of radius r; an arc of length L with central angle m°

Prove: $\frac{L}{r} = \frac{\pi m}{180}$

	STATEMENTS	REASONS
1	The circle has radius r; the arc has length L and central angle m°	Given
2	$L = \frac{m}{360} \cdot 2\pi r$	Arc Length Formula
3	$L = \frac{\pi m r}{180}$	Simplifying ($\frac{2}{360} = \frac{1}{180}$); Commutative Property
4	$\frac{L}{r} = \frac{\pi m}{180}$	Division Property of Equality ($r > 0$)

∎

11. **Hard** Write a paragraph proof: if two circles have radii r_1 and r_2, then arcs of the two circles with the same central angle m° have lengths in the ratio $r_1 : r_2$.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: Circles with radii r_1 and r_2; an arc of each circle with central angle m°, of lengths L_1 and L_2

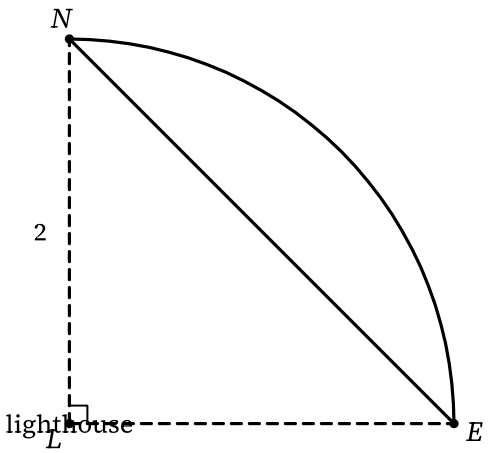
Prove: $\dfrac{L_1}{L_2} = \dfrac{r_1}{r_2}$

By the Arc Length Formula, $L_1 = \dfrac{m}{360} \cdot 2\pi r_1$ and $L_2 = \dfrac{m}{360} \cdot 2\pi r_2$.

Dividing the first equation by the second (Division Property of Equality; $L_2 \neq 0$), $\dfrac{L_1}{L_2} = \dfrac{\frac{m}{360} \cdot 2\pi r_1}{\frac{m}{360} \cdot 2\pi r_2}$. The common nonzero factor $\dfrac{m}{360} \cdot 2\pi$ divides out, leaving $\dfrac{L_1}{L_2} = \dfrac{r_1}{r_2}$. So arcs with equal central angles have lengths proportional to the radii, which is another way of saying that all circles are similar.

∎

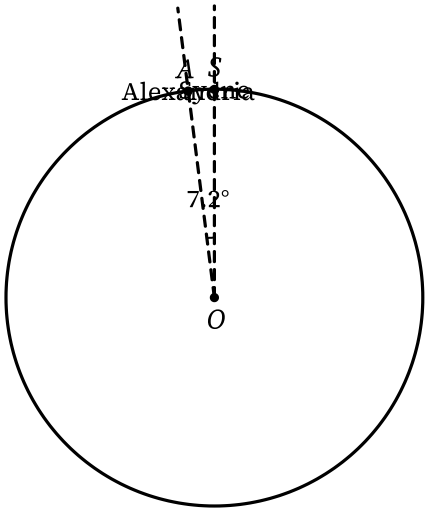
12. Hard A ship sails along a circular route around a lighthouse, staying 2 nautical miles from it. It sails from due north of the lighthouse to due east, the shorter way. Then it sails straight back to its starting point. Find the total distance sailed, to the nearest tenth of a nautical mile.



Show answer

The arc is $90°$: $\frac{1}{4} \cdot 2\pi(2) = \pi \approx 3.142$. The straight return is the hypotenuse of a right isosceles triangle with legs 2: $2\sqrt{2} \approx 2.828$. Total ≈ 5.970, about 6.0 nautical miles

13. **Hard** Eratosthenes found that the arc of the earth from Syene to Alexandria measures 7.2°, and he took the distance between the cities as 5,000 stadia. (a) What fraction of a full circle is 7.2°? (b) What circumference did he find?

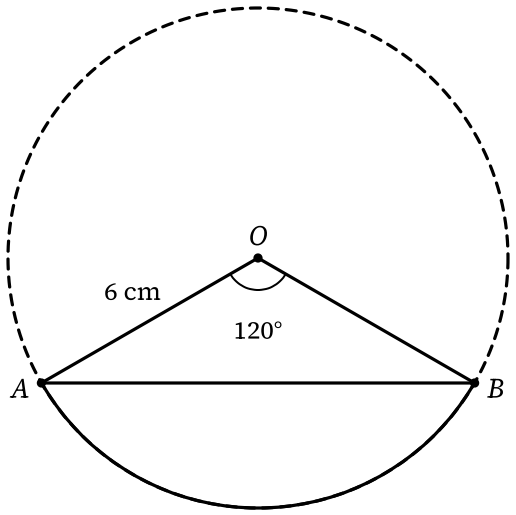


Center of the earth O; the arc from Syene to Alexandria

Show answer

(a) $\dfrac{7.2}{360} = \dfrac{1}{50}$; (b) $50 \times 5{,}000 = 250{,}000$ stadia

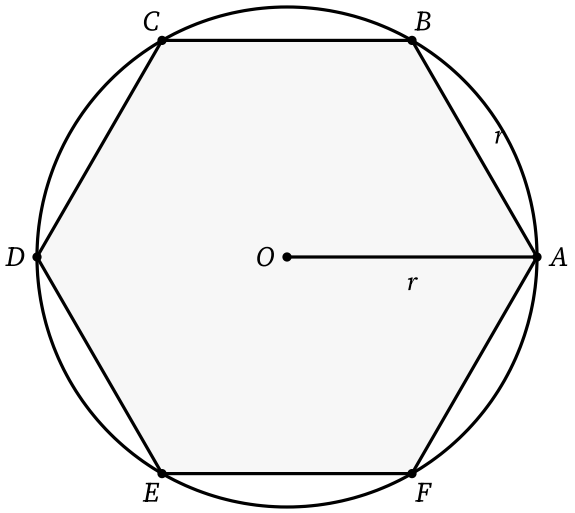
14. **Challenge** In the figure, $\overline{AB}$ is a chord of circle O with radius 6 cm and $m\angle AOB = 120°$. Find the perimeter of the region bounded by chord $\overline{AB}$ and minor arc $\overset{\frown}{AB}$. Give an exact answer and an answer to the nearest tenth.



Show answer

Arc: $\frac{120}{360} \cdot 12\pi = 4\pi$. Chord: $\triangle AOB$ is isosceles with vertex angle $120°$; the perpendicular from O bisects $\overline{AB}$ and makes $30°$-$60°$-$90°$ triangles with hypotenuse 6, so half the chord is $3\sqrt{3}$ and $AB = 6\sqrt{3}$. Perimeter $= 4\pi + 6\sqrt{3} \approx 12.566 + 10.392 \approx 23.0$ cm

15. **Challenge** Use an indirect proof to show that $\pi > 3$. (Hint: a regular hexagon inscribed in a circle has perimeter $6r$, and a chord is shorter than its arc because a segment is the shortest path between two points.)



Show answer

See the model proof.

INDIRECT PROOF

Given: A circle with radius r and an inscribed regular hexagon

Prove: $\pi > 3$

Assume, for the sake of contradiction, that $\pi \le 3$. Then the circumference is $C = 2\pi r \le 6r$.

Inscribe a regular hexagon in the circle (Euclid IV.15). Its side equals the radius, so its perimeter is $6r$. The six vertices divide the circle into six arcs, and each side of the hexagon is a chord joining the ends of one arc. A segment is the shortest path between two points, so each side is shorter than its arc. Adding the six inequalities, $6r < C$.

Combining, $6r < C \le 6r$, so $6r < 6r$, which is impossible. Therefore the assumption is false, and $\pi > 3$.

∎

COMMON MISTAKES

- Using the diameter in $C = 2\pi r$ (or the radius in $C = \pi d$). Circle which one you were given before you start.

- Confusing arc length with arc measure. A 60° arc can be 4π inches long or 4π miles long; write degrees for measure and units of length for length.
- Rounding π to 3.14 in the middle of a calculation. Use the π key and round only the final answer, or leave the answer in terms of π when asked for an exact value.
- Using the minor arc when the major arc is asked for. The major arc's central angle is 360° minus the minor arc's.

> **Make it a game · Pi Hunt.** Gather three round objects: a lid, a cup, a can. For each, wrap a string around it, then measure the string and the diameter. Divide circumference by diameter. How close to 3.14? Guess first, and the closest guess scores. Then find the arc length of a quarter of the lid's edge.

11.5 Areas of Circles and Sectors

OBJECTIVES

- Explain why the area of a circle is πr^2 by cutting it into thin sectors and rearranging them into a near-rectangle.
- Find the areas of circles, sectors, segments and rings (annuli), exactly and rounded.
- Solve practical problems involving sprinklers, windows, paths and grazing areas.
- Prove simple facts about circle and sector areas, including a ring area found from a tangent chord.

WATCH

Quick Visual Proof: Area of a Circle

Eddie Woo · 6:47

https://www.youtube.com/watch?v=Sx-5YJbszuI

> **WHEN TO WATCH**
> After the Core ideas: the circle cut into sectors and rearranged.

Note: Segments and rings are taught from the lesson.

BACKUP VIDEO

Area & Perimeter of Sectors

Eddie Woo · 12:49

https://www.youtube.com/watch?v=RHyvVt8T1Eo

Before the sector examples: sector area and perimeter.

KEY TERMS

sector

The region bounded by two radii of a circle and the arc between them, shaped like a slice of pie.

segment of a circle

The region bounded by a chord and the arc it cuts off.

annulus (ring)

The region between two concentric circles.

concentric circles

Circles in the same plane with the same center and different radii.

CORE IDEAS

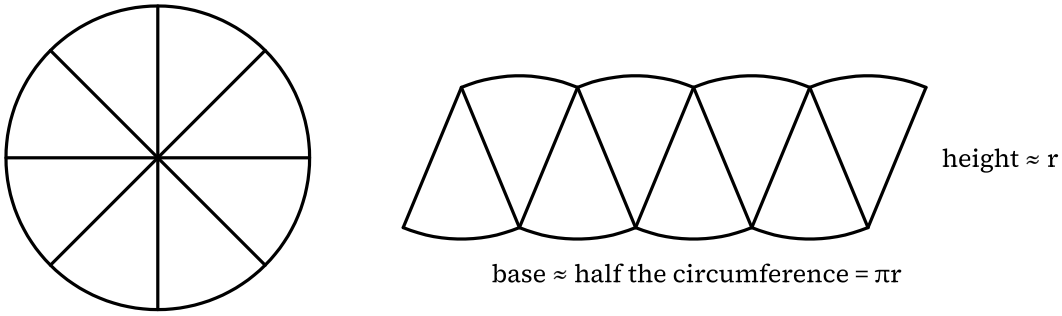
Why $A = \pi r^2$. Cut a circle into many equal sectors, like a pie, and lay them side by side, alternately point-up and point-down. The pieces form a shape that looks almost like a parallelogram. Its height is about r (the length of each slice), and its base is made of the curved edges of half the slices, so it is about half the circumference, $\frac{1}{2}(2\pi r) = \pi r$. With more and thinner slices the shape gets closer and closer to a rectangle with base πr and height r, whose area is $\pi r \cdot r = \pi r^2$.

Archimedes said the same thing another way: a circle has the same area as a right triangle whose legs are the radius r and the circumference C. That triangle's area is $\frac{1}{2} r C = \frac{1}{2} r (2\pi r) = \pi r^2$. Both pictures show that the area of a circle depends on the square of the radius, so doubling the radius makes the area four times as large, not twice.

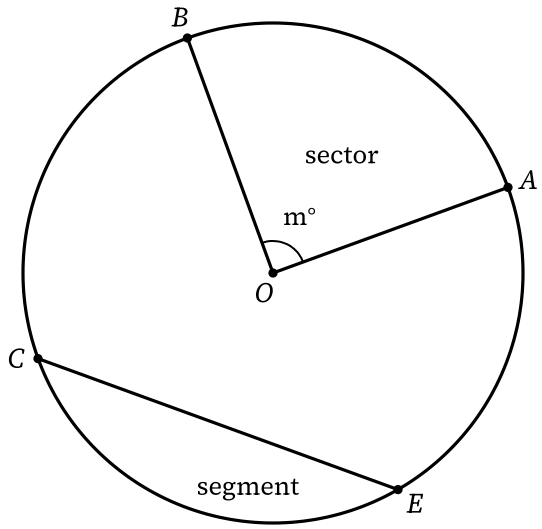
A **sector** is a fraction of the circle, exactly as an arc is a fraction of the circumference. A sector with central angle m° has area $\dfrac{m}{360} \cdot \pi r^2$. A sector of 90° is a quarter of the circle.

A **segment** cut off by a minor arc is the part of a sector that is left when you remove the triangle formed by the two radii and the chord: $\text{segment} = \text{sector} - \text{triangle}$. For central angles of 60°, 90° and 120° you can find the triangle's area with special right triangles; otherwise use trigonometry for its height.

A **ring** (annulus) between concentric circles of radii R and r is the big circle minus the small one: $A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. Subtract the areas, not the radii: a ring from radius 15 to radius 12 is not $\pi \cdot 3^2$.



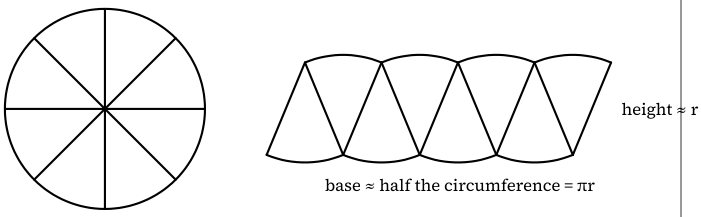
Eight slices of a circle rearranged into a near-parallelogram



POSTULATES AND THEOREMS

THEOREM **Circle Area Formula**

The area of a circle with radius r is $A = \pi r^2$.



Eight slices of a circle rearranged into a near-parallelogram

PARAGRAPH PROOF

Given: A circle with radius r and circumference $C = 2\pi r$

Prove: $A = \pi r^2$

Divide the circle into n congruent sectors and rearrange them, alternately pointing up and down, into a row. By the Area Addition Postulate the row has the same area as the circle. Its height is the radius r (to within the small bulge of each arc), and its bottom edge is made of $\frac{n}{2}$ arcs, half the circumference, πr.

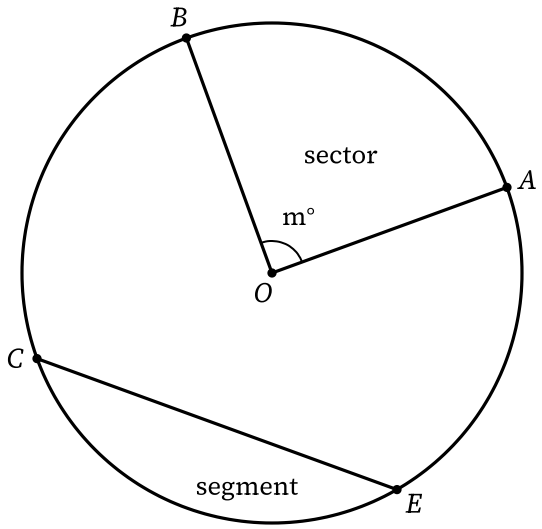
As n grows, the slices become thinner, the bulges flatten, and the slanted ends straighten, so the row becomes as close as we like to a rectangle with base πr and height r. The circle's area therefore equals that rectangle's area, $\pi r \cdot r = \pi r^2$ (Rectangle Area Postulate).

∎

The step "as close as we like" is a limit. We accept it without a full proof in this course; Archimedes made it rigorous by showing that the circle's area can be neither more nor less than the right triangle with legs r and C, a double proof by contradiction.

THEOREM **Sector Area Formula**

The area of a sector of a circle with radius r and central angle m° is $A = \frac{m}{360} \cdot \pi r^2$.



PARAGRAPH PROOF

Given: A circle of radius r; a sector with central angle m°

Prove: Area of sector $= \frac{m}{360} \cdot \pi r^2$

Sectors of the same circle with congruent central angles are congruent (one can be rotated onto the other about the center), so by the Area Congruence Postulate they have equal areas. The 360 sectors of 1° fill the circle without overlapping, so by the Area Addition Postulate each has area $\frac{1}{360}\pi r^2$.

A sector of m° is made of m such 1° sectors, so its area is $m \cdot \dfrac{1}{360}\pi r^2 = \dfrac{m}{360}\cdot \pi r^2$. (For measures that are not whole numbers we accept the same proportion.)

∎

COROLLARY **Segment and Ring Areas**

The area of a segment cut off by a minor arc is the area of its sector minus the area of the triangle formed by the chord and the two radii. (For a major arc, add the triangle instead.) The area of a ring between concentric circles of radii $R > r$ is $\pi(R^2 - r^2)$.

Both follow from the Area Addition Postulate: the sector is made of the segment and the triangle, and the large circle is made of the small circle and the ring.

KEY FORMULAS

$$A_{\text{circle}} = \pi r^2$$

$$A_{\text{sector}} = \frac{m}{360}\cdot \pi r^2$$

$$A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$$

$$A_{\text{ring}} = \pi(R^2 - r^2)$$

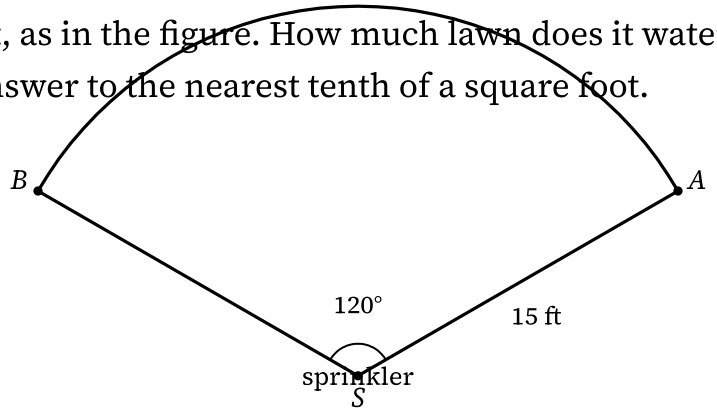
From history. Archimedes of Syracuse (about 250 BC), in his short book *Measurement of a Circle*, proved that a circle's area equals that of a right triangle whose legs are the radius and the circumference. He then trapped the circle between regular polygons inside and outside it, doubling the number of sides from 6 up to 96, and showed that $3\dfrac{10}{71} < \pi < 3\dfrac{1}{7}$, that is, roughly $3.1408 < \pi < 3.1429$. The familiar estimate $\dfrac{22}{7}$ is his upper bound.

Euclid connection. Euclid's XII.2 proves that "circles are to one another as the squares on the diameters," using Eudoxus's method of exhaustion (polygons squeezed against the circle, with proof by contradiction). That is exactly the statement that every circle's area is the same constant times r^2; Archimedes later identified the constant with the ratio $\dfrac{C}{d}$, the number we call π. Lesson 11.6 uses XII.2 for similar figures.

Practice habit. Recite the formulas for circle, sector, segment and ring area together with the picture behind each (pie slices, fraction of the circle, sector minus triangle, big minus small). Write the exact answer in terms of π first, and round only at the very end.

WORKED EXAMPLES

Example 1. A lawn sprinkler turns back and forth through an angle of 120° and sprays water 15 ft, as in the figure. How much lawn does it water? Give an exact answer and an answer to the nearest tenth of a square foot.

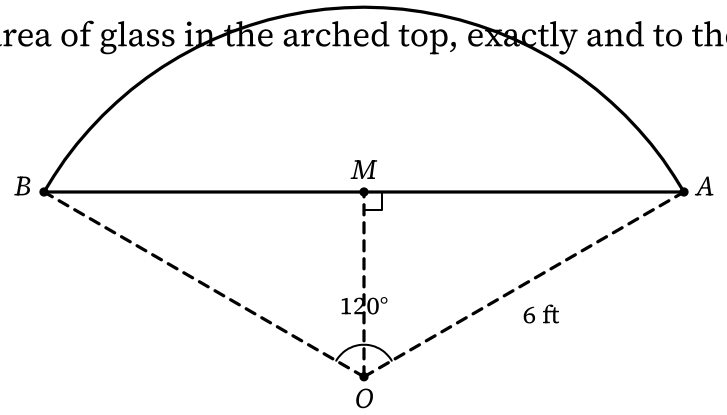


Show solution

1. The watered region is a sector with $r = 15$ ft and central angle 120°.
2. $\dfrac{120}{360} = \dfrac{1}{3}$, so the sector is one third of the circle.
3. $A = \dfrac{1}{3}\pi(15)^2 = \dfrac{225\pi}{3} = 75\pi \approx 235.6$ square feet.

Answer: $75\pi \approx 235.6$ square feet

Example 2. The top of an arched church window is a segment of a circle with radius 6 ft. The chord across the bottom of the arch cuts off a 120° arc, as in the figure. Find the area of glass in the arched top, exactly and to the nearest tenth of a square foot.



Show solution

1. Sector: $\dfrac{120}{360} \cdot \pi(6)^2 = \dfrac{1}{3}(36\pi) = 12\pi$.
2. Triangle AOB: draw $\overline{OM} \perp \overline{AB}$. Since $OA = OB$, $\overline{OM}$ bisects the 120° angle and the chord, so $\triangle OMA$ is a 30°-60°-90° triangle with hypotenuse 6: $OM = 3$ and $AM = 3\sqrt{3}$, so $AB = 6\sqrt{3}$.
3. $\text{Area}(\triangle AOB) = \dfrac{1}{2}(6\sqrt{3})(3) = 9\sqrt{3}$.
4. Segment $=$ sector $-$ triangle $= 12\pi - 9\sqrt{3} \approx 37.699 - 15.588 \approx 22.1$ square feet.

Answer: $12\pi - 9\sqrt{3} \approx 22.1$ square feet

PRACTICE

1. Easy Find the area of a circle with radius 7 cm, in terms of π.

   Show answer

   49π cm^2

2. Easy Find the area of a circle with diameter 10 m, to the nearest tenth of a square meter.

   Show answer

   $r = 5$, so $A = 25\pi \approx 78.5$ m^2

3. **Easy** The Miller family orders a 14-inch (diameter) pizza cut into 8 equal slices. Find the area of one slice to the nearest tenth of a square inch.

   Show answer

   $\frac{1}{8}\pi(7)^2 = \frac{49\pi}{8} \approx 19.2$ square inches

4. **Easy** A circle has area 64π square feet. Find its radius.

   Show answer

   $\pi r^2 = 64\pi$, so $r = 8$ feet

5. **Medium** Find the area of a sector with radius 10 in and central angle 72°, in terms of π.

   Show answer

   $\frac{72}{360} \cdot 100\pi = 20\pi$ square inches

6. **Medium** A circular flower bed of radius 12 ft is surrounded by a gravel path 3 ft wide. Find the area of the path to the nearest tenth of a square foot.

   Show answer

   Outer radius 15 ft: $\pi(15^2 - 12^2) = 81\pi \approx 254.5$ square feet

7. **Medium** A sector of a circle with radius 6 cm has area 15π cm². Find its central angle.

   Show answer

   $\frac{m}{360} \cdot 36\pi = 15\pi$, so $m = \frac{15 \times 360}{36} = 150^\circ$

8. **Medium** A 12-inch pizza costs \$10.00 and a 16-inch pizza costs \$16.00 (both are diameters). Which is the better buy per square inch? Explain your reasoning with numbers.

   Show answer

   The 16-inch pizza. Areas: $36\pi \approx 113.1$ in² and $64\pi \approx 201.1$ in². Cost per square inch: $\frac{10}{113.1} \approx \$0.088$ and $\frac{16}{201.1} \approx \$0.080$. The larger pizza has $\frac{64}{36} \approx 1.78$ times the area for only 1.6 times the price.

9. **Medium** Write a paragraph proof: a sector with radius r and arc length L has area $\frac{1}{2}rL$. (This is like a triangle with base L and height r, just as Archimedes pictured the whole circle.)

Show answer

See the model proof.

PARAGRAPH PROOF

Given: A sector with radius r, central angle m° and arc length L

Prove: Area of sector $= \frac{1}{2}rL$

By the Arc Length Formula, $L = \frac{m}{360} \cdot 2\pi r$. By the Sector Area Formula, the sector's area is $\frac{m}{360} \cdot \pi r^2$.

Rewrite the area: $\frac{m}{360} \cdot \pi r^2 = \frac{1}{2}r \cdot \left(\frac{m}{360} \cdot 2\pi r\right)$, since $\frac{1}{2}r \cdot 2\pi r = \pi r^2$ (Commutative and Associative Properties). The quantity in parentheses is L, so by Substitution the area is $\frac{1}{2}rL$.

∎

10. **Medium** Write an indirect proof: if two circles have equal areas, then they have equal radii.

Show answer

See the model proof.

INDIRECT PROOF

Given: Circles with radii r_1 and r_2 have equal areas

Prove: $r_1 = r_2$

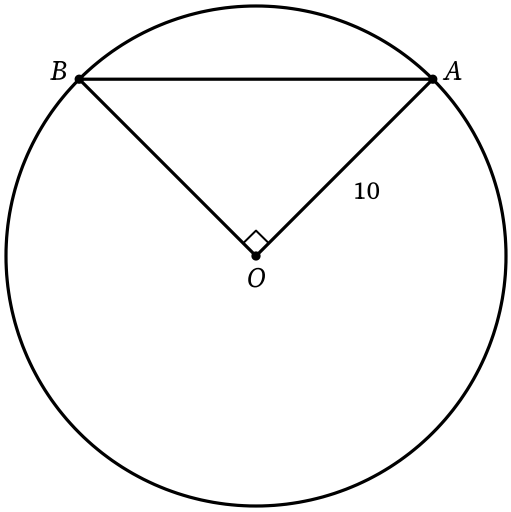
Assume, for the sake of contradiction, that $r_1 \neq r_2$. Then one radius is smaller; say $r_1 < r_2$.

Both radii are positive, so multiplying the inequality $r_1 < r_2$ by r_1 and by r_2 gives $r_1^2 < r_1 r_2 < r_2^2$. Multiplying by the positive number π, $\pi r_1^2 < \pi r_2^2$, so the first circle has a smaller area than the second.

This contradicts the given fact that the areas are equal. Therefore the assumption is false, and $r_1 = r_2$.

∎

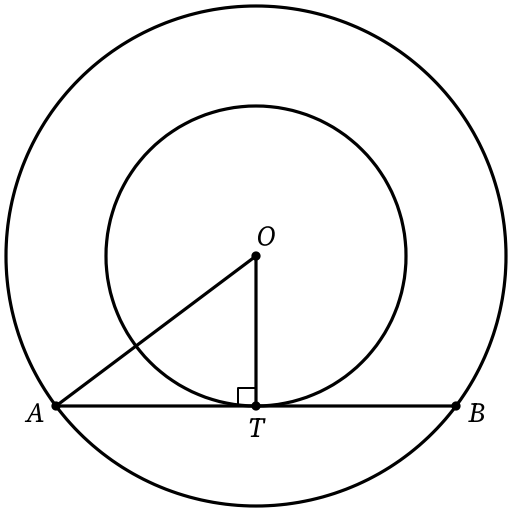
11. Hard In the figure, circle O has radius 10 and $m\angle AOB = 90^\circ$. Find the area of the segment bounded by chord $\overline{AB}$ and minor arc $\overset{\frown}{AB}$, exactly and to the nearest tenth.



Show answer

Sector $\frac{1}{4} \cdot 100\pi = 25\pi$; triangle $\frac{1}{2}(10)(10) = 50$; segment $= 25\pi - 50 \approx 28.5$ square units

12. Hard In the figure, two circles have center O. Chord $\overline{AB}$ of the larger circle is tangent to the smaller circle at T, and $AT = t$. Write a two-column proof that the area of the ring between the circles is πt^2. (So the ring's area can be found from one chord length, without knowing either radius.)



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Concentric circles with center O, radii R (larger) and r; chord $\overline{AB}$ of the larger circle is tangent to the smaller circle at T; $AT = t$

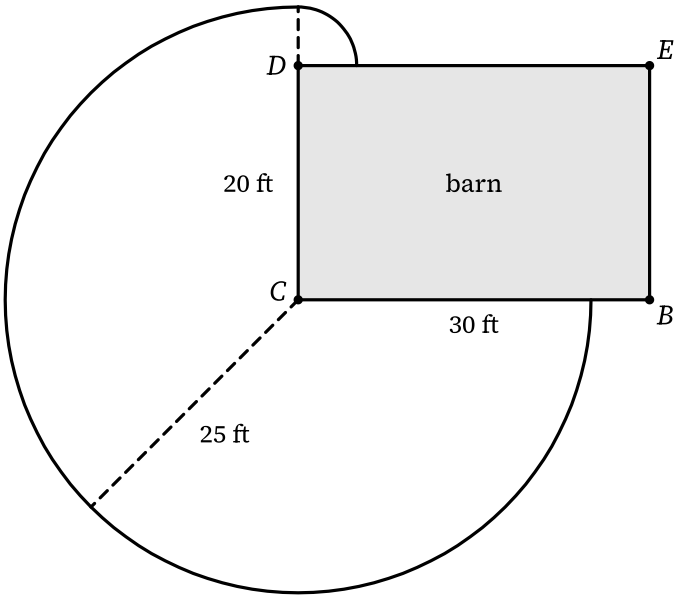
Prove: Area of ring $= \pi t^2$

	STATEMENTS	REASONS
1	$\overline{AB}$ is tangent to the smaller circle at T; $OA = R$, $OT = r$, $AT = t$	Given; definition of radius
2	$\overline{OT} \perp \overline{AB}$	A tangent is perpendicular to the radius at the point of tangency (Euclid III.18)
3	$\triangle OTA$ is a right triangle with hypotenuse $\overline{OA}$	Definition of right triangle
4	$OT^2 + AT^2 = OA^2$, so $r^2 + t^2 = R^2$	Pythagorean Theorem; Substitution

	STATEMENTS	REASONS
5	$R^2 - r^2 = t^2$	Subtraction Property of Equality
6	Area of ring $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$	Area Addition Postulate (ring area); Circle Area Formula; Distributive Property
7	Area of ring $= \pi t^2$	Substitution Property of Equality

∎

13. **Challenge** A calf is tied to the outside corner C of a barn that is 30 ft long and 20 ft wide, with a 25-ft rope, as in the figure. The calf cannot go inside the barn, but the rope can bend around a corner. Find the area of the grass the calf can reach, exactly and to the nearest tenth of a square foot.



Show answer

Outside the barn the rope sweeps $\frac{3}{4}$ of a circle of radius 25: $\frac{3}{4} \cdot 625\pi = 468.75\pi$. Along the 20-ft wall the rope reaches corner D with 5 ft left, sweeping a quarter circle of radius 5: $\frac{1}{4} \cdot 25\pi = 6.25\pi$. Along the 30-ft wall the rope is too short to reach the next corner. Total $= 475\pi \approx 1492.3$ square feet

14. **Challenge** A sector of a circle with radius 6 cm has perimeter $(12 + 4\pi)$ cm (two radii and the arc). Find the area of the sector in terms of π.

Show answer

Arc length $= 4\pi$, so $\frac{m}{360} \cdot 12\pi = 4\pi$ and $m = 120°$. Area $= \frac{1}{3} \cdot 36\pi = 12\pi$ cm^2 (or $\frac{1}{2}rL = \frac{1}{2}(6)(4\pi) = 12\pi$)

COMMON MISTAKES

- Using the diameter in πr^2. A pizza "14 inches" across has radius 7, and its area is 49π, not 196π.
- Subtracting radii for a ring: $\pi(15 - 12)^2$ is wrong. Subtract the areas: $\pi(15^2 - 12^2)$.
- Forgetting to subtract the triangle when finding a segment, or subtracting a triangle whose height was guessed. Find the triangle's height with a right triangle.
- Thinking that doubling the radius doubles the area. Area grows with the square of the radius, so it becomes four times as large.

> **Make it a game · Slice and Solve.** Cut a paper circle with radius 6 and fold it into 12 equal sectors. Parent calls a number of slices, such as 5 or 3, and you pick them up and state the sector's area in terms of pi. Then rearrange all 12 slices into a near-rectangle and explain why the area is pi r squared. Find a ring area too.

11.6 Areas of Similar Figures and Composite Regions

OBJECTIVES

- Prove that if two similar figures have scale factor $a : b$, the ratio of their areas is $a^2 : b^2$.
- Find areas, scale factors and lengths of similar figures from given areas.
- Find the areas of composite regions made of circles and polygons.
- Compute geometric probabilities as ratios of areas.

WATCH

Areas of Similar Figures (1 of 2: Explanation)

Eddie Woo · 9:42

https://www.youtube.com/watch?v=eoR-XpI4W4o

> **WHEN TO WATCH**
> After the Core ideas: why areas scale by the square of the scale factor.

Note: Eddie Woo teaches in Australia: he says 'enlargement factor' for scale factor. Midway, the backup has a short plug for the presenter's ACT and SAT courses.

BACKUP VIDEO

Geometric Probability

Mario's Math Tutoring · 5:34

https://www.youtube.com/watch?v=8P6HGcwQYZ0

Before the geometric-probability examples: probability as a ratio of lengths, areas and volumes.

KEY TERMS

scale factor

The ratio of corresponding lengths in two similar figures (Lesson 8.2).

ratio of areas

For similar figures with scale factor $a : b$, the ratio of their areas, which is $a^2 : b^2$.

composite region

A region built from simpler shapes (polygons, circles, sectors) by joining them or cutting some away.

geometric probability

When a point is chosen at random so that it is equally likely to land anywhere in a region, the probability that it lands in a smaller part of the region is $\dfrac{\text{area of the part}}{\text{area of the whole region}}$.

CORE IDEAS

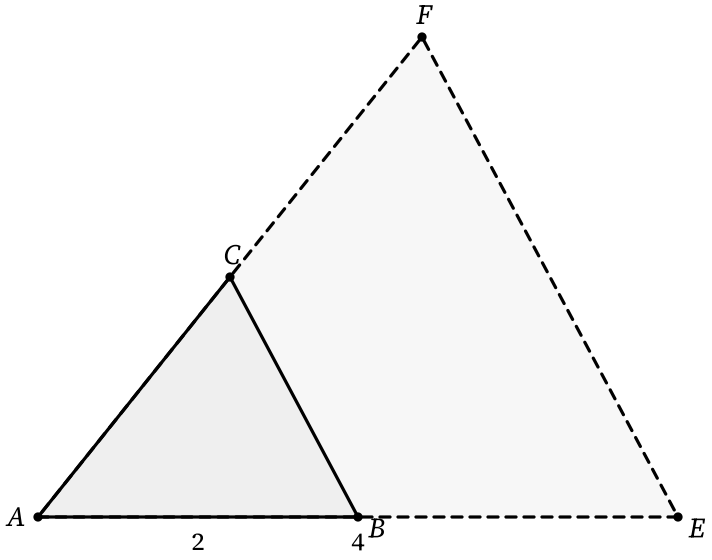
If you double every length of a square, you do not get twice the area; you get four times: a 2×2 square holds four 1×1 squares. Area is two-dimensional, so a scale factor k acts twice, once on each dimension: lengths are multiplied by k and areas by k^2.

Why it is true for triangles. If $\triangle DEF \sim \triangle ABC$ with scale factor k, then the base of $\triangle DEF$ is k times the base of $\triangle ABC$, and its height is also k times as long (the heights are corresponding sides of two smaller similar right triangles). So its area $\frac{1}{2}(k \cdot b)(k \cdot h) = k^2 \cdot \frac{1}{2}bh$. Any polygon can be cut into triangles by diagonals, and similar polygons are cut into similar triangles with the same scale factor, so the rule k^2 holds for every pair of similar polygons. All circles are similar, and their areas πr^2 are in the ratio of the squares of the radii.

The rule also works backward. If two similar figures have areas in the ratio $9 : 4$, their scale factor is $\sqrt{9} : \sqrt{4} = 3 : 2$. So from areas you can find lengths: take the square root of the area ratio to get the length ratio. Perimeters, like all lengths, are in the ratio k, not k^2.

A **composite region** is found by adding the areas of pieces that do not overlap, or by subtracting the areas of pieces cut away (the Area Addition Postulate). With circles, the usual moves are: a square minus a circle, a circle minus a polygon, a rectangle plus two semicircles (a running track), or a sector minus a triangle (a segment).

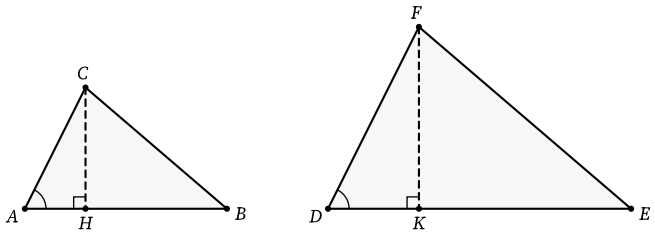
Geometric probability. If a dart is equally likely to land anywhere on a board, the chance it lands in a certain region is the fraction of the board's area that region covers. A target that covers one quarter of the board has probability $\dfrac{1}{4}$. This is why a bull's-eye with one-sixth the radius of the board is hit only $\dfrac{1}{36}$ of the time by a random throw: the area ratio is the square of the length ratio.



Scale factor 2: lengths double, area becomes 4 times as large

POSTULATES AND THEOREMS

THEOREM Areas of Similar Triangles Euclid VI.19



If two triangles are similar with scale factor k, the ratio of their areas is k^2. (Euclid VI.19: similar triangles are to one another in the duplicate ratio of their corresponding sides.)

TWO-COLUMN PROOF

Given: $\triangle DEF \sim \triangle ABC$ with $\dfrac{DE}{AB} = \dfrac{EF}{BC} = \dfrac{DF}{AC} = k$, labeled so that $\angle A$ is acute (every triangle has at least two acute angles); $\overline{CH} \perp \overleftrightarrow{AB}$ at H; $\overline{FK} \perp \overleftrightarrow{DE}$ at K

Prove: $\dfrac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABC)} = k^2$

	STATEMENTS	REASONS
1	$\triangle DEF \sim \triangle ABC$ with scale factor k; $\overline{CH} \perp \overleftrightarrow{AB}$, $\overline{FK} \perp \overleftrightarrow{DE}$	Given
2	$\angle D \cong \angle A$	Definition of similar polygons (corresponding angles are congruent)

	STATEMENTS	REASONS
3	$\angle CHA$ and $\angle FKD$ are right angles, so $\angle CHA \cong \angle FKD$	Definition of perpendicular; all right angles are congruent
4	$\triangle FKD \sim \triangle CHA$	AA Similarity (H lies on ray AB and K on ray DE because $\angle A$ and $\angle D$ are acute)
5	$\frac{FK}{CH} = \frac{DF}{AC} = k$	Definition of similar polygons (corresponding sides are proportional); Given
6	$FK = k \cdot CH$ and $DE = k \cdot AB$	Multiplication Property of Equality
7	$\text{Area}(\triangle ABC) = \frac{1}{2} AB \cdot CH$ and $\text{Area}(\triangle DEF) = \frac{1}{2} DE \cdot FK$	Triangle Area Theorem
8	$\text{Area}(\triangle DEF) = \frac{1}{2}(k \cdot AB)(k \cdot CH) = k^2 \cdot \frac{1}{2} AB \cdot CH$	Substitution; Commutative and Associative Properties
9	$\text{Area}(\triangle DEF) = k^2 \cdot \text{Area}(\triangle ABC)$	Substitution Property of Equality
10	$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABC)} = k^2$	Division Property of Equality

∎

THEOREM **Areas of Similar Polygons** Euclid VI.20

If two polygons are similar with scale factor $a : b$, the ratio of their areas is $a^2 : b^2$.

PARAGRAPH PROOF

Given: Convex polygons $P = ABCD\ldots$ and $P' = A'B'C'D'\ldots$ are similar with scale factor $k = \frac{a}{b}$ (each side of P is k times the corresponding side of P')

Prove: $\frac{\text{Area}(P)}{\text{Area}(P')} = k^2 = \frac{a^2}{b^2}$

Draw all the diagonals from A in P and from A' in P', cutting each polygon into triangles. In $\triangle ABC$ and $\triangle A'B'C'$, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = k$ and $\angle B \cong \angle B'$, so $\triangle ABC \sim \triangle A'B'C'$ by SAS Similarity, and therefore $\frac{AC}{A'C'} = k$ and $\angle BCA \cong \angle B'C'A'$.

Next, $\angle ACD$ is what remains of $\angle BCD$ after removing $\angle BCA$, and likewise in P'; since $\angle BCD \cong \angle B'C'D'$ and $\angle BCA \cong \angle B'C'A'$, the Angle Addition Postulate and subtraction give $\angle ACD \cong \angle A'C'D'$. Also $\dfrac{AC}{A'C'} = \dfrac{CD}{C'D'} = k$, so $\triangle ACD \sim \triangle A'C'D'$ by SAS Similarity. Continuing around the polygon, each triangle of P is similar to the corresponding triangle of P' with the same scale factor k.

By the Areas of Similar Triangles theorem, each triangle of P has k^2 times the area of its partner. Adding (Area Addition Postulate and the Distributive Property), $\text{Area}(P) = k^2 \cdot \text{Area}(P')$, so the ratio of the areas is $k^2 = \dfrac{a^2}{b^2}$.

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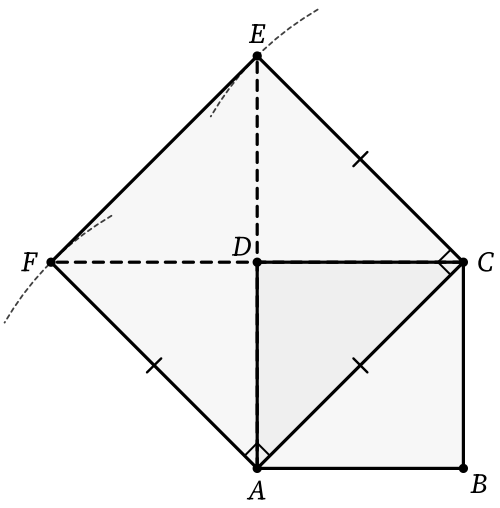
COROLLARY **Areas of Circles** Euclid XII.2

The areas of two circles are in the ratio of the squares of their radii (or of their diameters).

From the Circle Area Formula, $\dfrac{\pi r_1^2}{\pi r_2^2} = \dfrac{r_1^2}{r_2^2}$. Euclid proves this as XII.2 without knowing the value of π, by the method of exhaustion.

CONSTRUCTIONS

CONSTRUCTION **Construct a square with twice the area of a given square** Euclid I.46



1. Start with square $ABCD$ and draw its diagonal $\overline{AC}$.
2. At C construct the perpendicular to $\overline{AC}$ (Euclid I.11). Set the compass to length AC and, with the point at C, mark E on the perpendicular on the same side of $\overline{AC}$ as D.
3. At A construct the perpendicular to $\overline{AC}$ and mark F on it, on the same side, with $AF = AC$.

4. Draw $\overline{EF}$. Square $ACEF$ has twice the area of square $ABCD$.

Why it works. $\overline{AF}$ and $\overline{CE}$ are both perpendicular to $\overline{AC}$, so they are parallel; they are also equal and lie on the same side of $\overline{AC}$, so $ACEF$ is a parallelogram (Congruent and Parallel Sides Theorem, Lesson 6.3; Euclid I.33). It has a right angle at C, so it is a rectangle, and $EF = AC$ because opposite sides of a parallelogram are congruent. All four sides equal AC and all four angles are right, so $ACEF$ is the square on $\overline{AC}$ (Euclid I.46). If the side of $ABCD$ is s, the Pythagorean Theorem gives $AC^2 = s^2 + s^2 = 2s^2$, and AC^2 is the area of $ACEF$. In the language of this lesson, the new square is similar to the old one with scale factor $\sqrt{2}$, so its area is $(\sqrt{2})^2 = 2$ times as large. The dashed diagonals show it directly: $ACEF$ is made of four triangles congruent to $\triangle ACD$, while $ABCD$ is made of two.

KEY FORMULAS

$$\text{scale factor } \frac{a}{b} \implies \frac{\text{perimeter}_1}{\text{perimeter}_2} = \frac{a}{b}, \quad \frac{\text{area}_1}{\text{area}_2} = \frac{a^2}{b^2}$$

$$P(\text{landing in a region}) = \frac{\text{area of the region}}{\text{area of the whole figure}}$$

From history. In Plato's dialogue *Meno*, Socrates asks a young servant how to draw a square with twice the area of a given square. The boy first guesses that the side should be doubled, then sees from a drawing in the sand that doubling the side makes four times the area. Guided by questions, he finally finds the answer: build the new square on the diagonal of the old one, exactly the construction in this lesson.

Euclid connection. Euclid states VI.19 as "similar triangles are to one another in the duplicate ratio of the corresponding sides" ("duplicate ratio" means the ratio squared) and extends it to polygons in VI.20 and to circles in XII.2. In VI.31 he uses it to generalize the Pythagorean Theorem: any similar figures drawn on the sides of a right triangle, not just squares, have the figure on the hypotenuse equal to the other two together (Practice Problem 12).

Practice habit. Recite the rule "lengths k, areas k^2" and say why (two dimensions, each multiplied by k). When given areas, write the area ratio first, then take its square root to get the scale factor before finding any length.

WORKED EXAMPLES

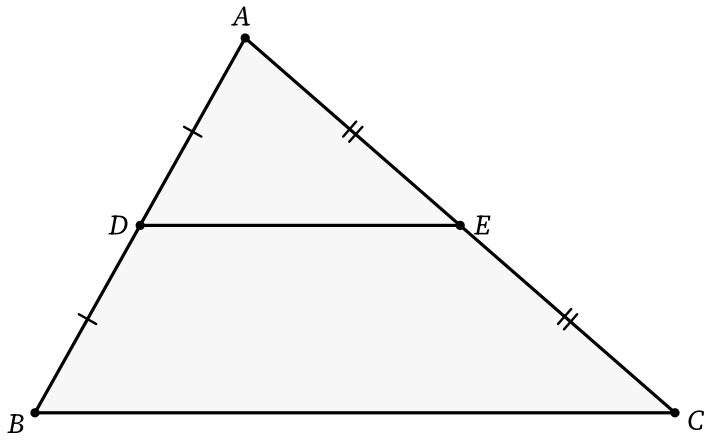
Example 1. On the blueprint for the Millers' new house, 1 inch represents 8 feet. The family room is a rectangle measuring 2.5 in by 1.5 in on the blueprint. (a) What is its area on the blueprint? (b) What is the actual floor area of the room?

Show solution

1. (a) Blueprint area: $2.5 \times 1.5 = 3.75$ square inches.
2. (b) The scale factor from blueprint to house is 8 feet per inch, so each square inch on the blueprint stands for $8^2 = 64$ square feet.
3. Actual area: $3.75 \times 64 = 240$ square feet.
4. Check by lengths: the room is $2.5 \times 8 = 20$ ft by $1.5 \times 8 = 12$ ft, and $20 \times 12 = 240$ square feet.

Answer: (a) 3.75 square inches; (b) 240 square feet

Example 2. In the figure, D and E are the midpoints of sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$. Prove that $\text{Area}(\triangle ADE) = \frac{1}{4}\,\text{Area}(\triangle ABC)$.



Show solution

1. The small triangle and the large triangle share $\angle A$, and the sides around it are halved, so look for SAS Similarity.
2. The scale factor is $\frac{1}{2}$, so the area ratio is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.
3. Notice what this means: the trapezoid $DBCE$ that is left over has three fourths of the area.

TWO-COLUMN PROOF

Given: D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{AC}$

Prove: $\text{Area}(\triangle ADE) = \frac{1}{4}\,\text{Area}(\triangle ABC)$

	STATEMENTS	REASONS
1	D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{AC}$	Given
2	$AD = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$	Definition of midpoint
3	$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$	Division Property of Equality
4	$\angle A \cong \angle A$	Reflexive Property of Congruence
5	$\triangle ADE \sim \triangle ABC$ with scale factor $\frac{1}{2}$	SAS Similarity

	STATEMENTS	REASONS
6	$\dfrac{\text{Area}(\triangle ADE)}{\text{Area}(\triangle ABC)} = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$	Areas of Similar Triangles (Euclid VI.19)
7	$\text{Area}(\triangle ADE) = \dfrac{1}{4}\,\text{Area}(\triangle ABC)$	Multiplication Property of Equality

∎

Answer: See the proof.

PRACTICE

1. Easy Two similar triangles have scale factor $2:3$. What is the ratio of their areas (smaller to larger)?

   Show answer

   $2^2 : 3^2 = 4 : 9$

2. Easy A rectangle has area 10 square inches. A similar rectangle has sides 3 times as long. Find the area of the larger rectangle.

   Show answer

   $10 \times 3^2 = 90$ square inches

3. Easy Two circles have radii 3 cm and 5 cm. What is the ratio of their areas (smaller to larger)?

   Show answer

   $9\pi : 25\pi = 9 : 25$

4. Easy A point is chosen at random inside a square that is 10 cm on each side. Inside the square is a shaded rectangle 4 cm by 5 cm. What is the probability that the point lands in the shaded rectangle?

   Show answer

   $\dfrac{20}{100} = \dfrac{1}{5} = 0.2$

5. **Medium** Two similar pentagons have areas 50 cm^2 and 72 cm^2. Find the ratio of their perimeters, smaller to larger.

Show answer

Area ratio $\frac{50}{72} = \frac{25}{36}$, so the scale factor is $\frac{5}{6}$, and the perimeters are in the ratio $5:6$

6. **Medium** On a map, 1 cm represents 2 km. A lake covers 3.5 cm^2 on the map. What is the actual area of the lake?

Show answer

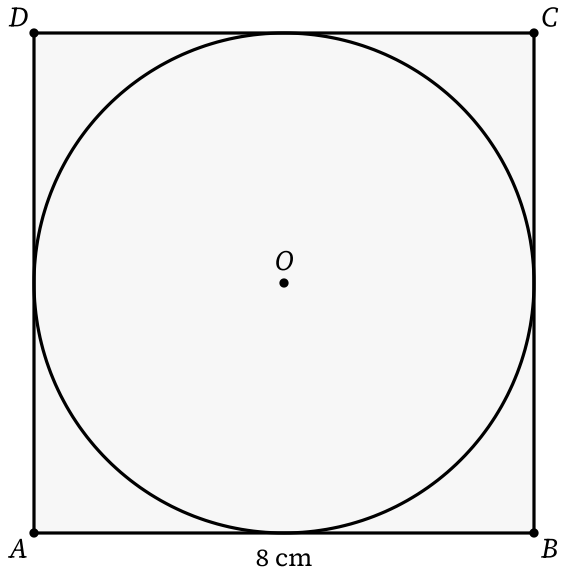
Each cm^2 represents $2^2 = 4$ km^2, so $3.5 \times 4 = 14$ km^2

7. **Medium** A triangle with a side of 6 in has area 18 in^2. A similar triangle has a corresponding side of 15 in. Find its area.

Show answer

$k = \frac{15}{6} = 2.5$, so the area is $18 \times 2.5^2 = 18 \times 6.25 = 112.5$ in^2

8. **Medium** In the figure, a circle is inscribed in square $ABCD$ with side 8 cm, touching all four sides. Find the total area of the four corner regions inside the square but outside the circle, exactly and to the nearest tenth.



Show answer

The circle's radius is 4, so the area is $64 - 16\pi \approx 13.7$ cm^2

9. **Medium** Samuel's circular dartboard has radius 9 in, and the bull's-eye at its center has radius 1.5 in. If a dart hits the board at a random point, what is the probability it hits the bull's-eye?

Show answer

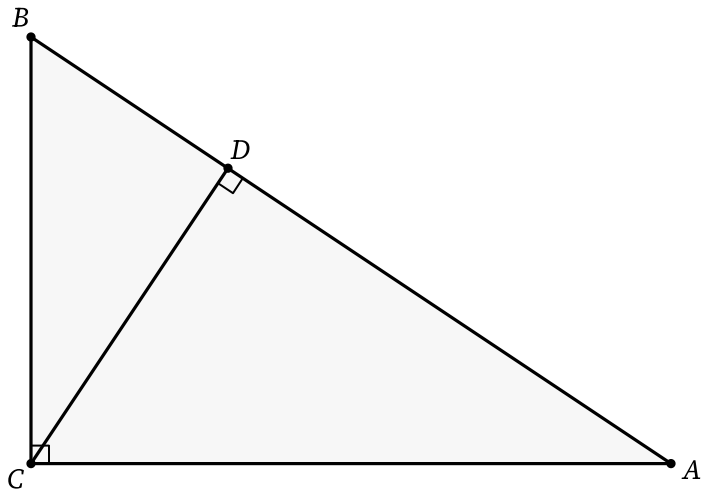
$$\frac{\pi(1.5)^2}{\pi(9)^2} = \left(\frac{1.5}{9}\right)^2 = \frac{1}{36} \approx 0.028$$

10. **Medium** Ruth's family doubles the length and the width of their rectangular vegetable garden. Ruth says they will need twice as much fertilizer. Explain why she is wrong, and say how much they will need.

Show answer

Fertilizer depends on area. Doubling both dimensions makes a similar rectangle with scale factor 2, so the area is multiplied by $2^2 = 4$: for example, a 10×6 garden (60 ft^2) becomes 20×12 (240 ft^2). They need four times as much fertilizer.

11. **Hard** In the figure, $\triangle ABC$ has a right angle at C, and $\overline{CD}$ is the altitude to the hypotenuse. Write a two-column proof that $\dfrac{\text{Area}(\triangle ACD)}{\text{Area}(\triangle CBD)} = \left(\dfrac{AC}{CB}\right)^2$.



Show answer

See the model proof.

TWO-COLUMN PROOF

Given: $\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$ at D

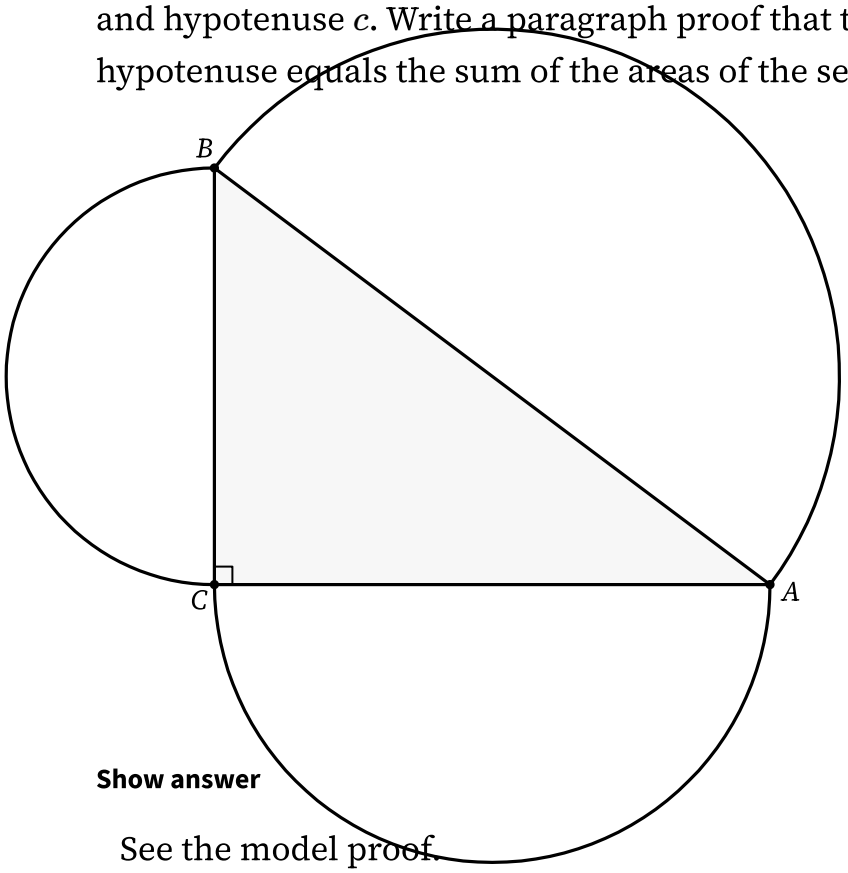
Prove: $\dfrac{\text{Area}(\triangle ACD)}{\text{Area}(\triangle CBD)} = \left(\dfrac{AC}{CB}\right)^2$

	STATEMENTS	REASONS
1	$\angle ACB$ is a right angle; $\overline{CD} \perp \overline{AB}$	Given
2	$\angle ADC$ and $\angle CDB$ are right angles, so $\angle ADC \cong \angle CDB$	Definition of perpendicular; all right angles are congruent
3	$m\angle A + m\angle ACD = 90°$	The acute angles of a right triangle are complementary ($\triangle ACD$)
4	$m\angle ACD + m\angle DCB = 90°$	Angle Addition Postulate; definition of right angle
5	$\angle A \cong \angle DCB$	Congruent Complements Theorem
6	$\triangle ACD \sim \triangle CBD$	AA Similarity

	STATEMENTS	REASONS
7	The scale factor from $\triangle CBD$ to $\triangle ACD$ is $\dfrac{AC}{CB}$	Corresponding sides of similar triangles (the hypotenuses)
8	$\dfrac{\text{Area}(\triangle ACD)}{\text{Area}(\triangle CBD)} = \left(\dfrac{AC}{CB}\right)^2$	Areas of Similar Triangles (Euclid VI.19)

∎

12. **Hard** Euclid VI.31 generalizes the Pythagorean Theorem. In the figure, semicircles are drawn outward on the three sides of right $\triangle ABC$, with legs a, b and hypotenuse c. Write a paragraph proof that the area of the semicircle on the hypotenuse equals the sum of the areas of the semicircles on the legs.



Show answer

See the model proof.

PARAGRAPH PROOF

Given: Right $\triangle ABC$ with legs a, b and hypotenuse c; a semicircle is drawn on each side as a diameter

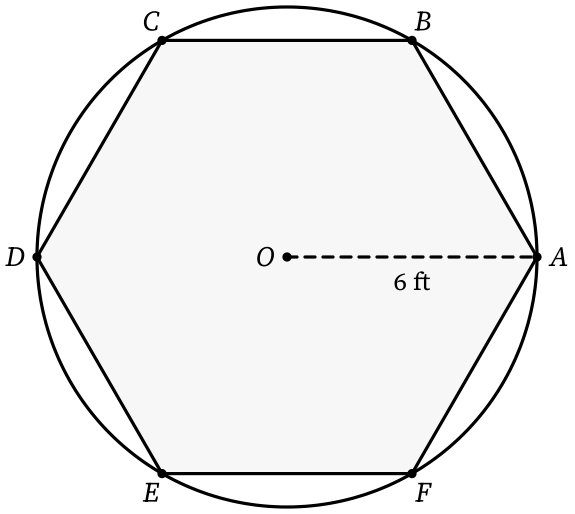
Prove: The semicircle on the hypotenuse has area equal to the sum of the areas of the other two

A semicircle with diameter d has radius $\frac{d}{2}$, so by the Circle Area Formula its area is $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{8}d^2$. So the three semicircles have areas $\frac{\pi}{8}a^2$, $\frac{\pi}{8}b^2$ and $\frac{\pi}{8}c^2$.

By the Pythagorean Theorem, $a^2 + b^2 = c^2$. Multiplying both sides by $\frac{\pi}{8}$ (Multiplication Property of Equality) and using the Distributive Property, $\frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 = \frac{\pi}{8}c^2$. That is, the semicircle on the hypotenuse equals the two semicircles on the legs together.

The same argument works for any three similar figures on the sides, because similar figures have areas equal to a fixed constant times the square of a corresponding length; this is Euclid's VI.31.

13. **Hard** A round church window has radius 6 ft. A regular hexagon of clear glass is inscribed in it, and the six pieces between the hexagon and the circle are colored glass. Find the area of the colored glass, exactly and to the nearest tenth of a square foot.



Show answer

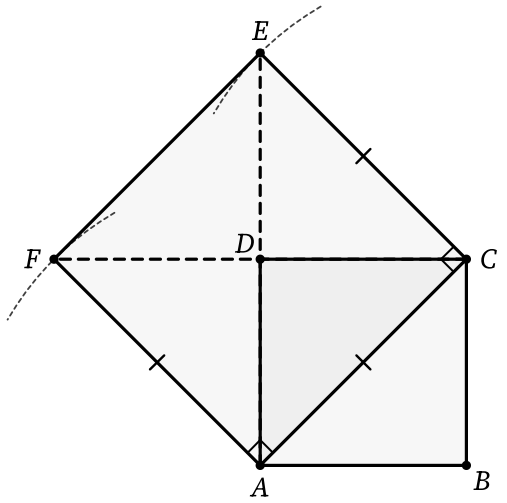
Circle: 36π. Hexagon: its side equals the radius, 6, so its area is $\frac{3\sqrt{3}}{2}(6)^2 = 54\sqrt{3}$. Colored glass $= 36\pi - 54\sqrt{3} \approx 113.10 - 93.53 \approx 19.6$ square feet

14. **Challenge** Two triangular gardens are similar. The larger has 2.25 times the area of the smaller, and its perimeter is 12 ft longer. Find the perimeters of both gardens.

Show answer

Scale factor $\sqrt{2.25} = 1.5$, so $P_{\text{large}} = 1.5P_{\text{small}}$. Then $1.5P - P = 12$, so $P = 24$. The perimeters are 24 ft and 36 ft

15. **Challenge** The construction in this lesson doubles a square of side 1 by building a square on its diagonal, whose side d satisfies $d^2 = 2$. Write an indirect proof that d cannot be written as a ratio $\frac{p}{q}$ of two whole numbers. (Use: the square of an odd number is odd.)



Show answer

See the model proof.

INDIRECT PROOF

Given: $d > 0$ and $d^2 = 2$

Prove: d is not a ratio of two whole numbers

Assume, for the sake of contradiction, that $d = \frac{p}{q}$ for whole numbers p and q, with the fraction in lowest terms (so p and q are not both even).

Then $\frac{p^2}{q^2} = 2$, so $p^2 = 2q^2$, and p^2 is even. If p were odd, p^2 would be odd, so p is even; write $p = 2m$. Then $4m^2 = 2q^2$, so $q^2 = 2m^2$ is even, and by the same reasoning q is even.

So p and q are both even, which contradicts the fraction being in lowest terms. Therefore the assumption is false, and d (that is, $\sqrt{2}$) is not a ratio of whole numbers. The Greeks knew this: the side and diagonal of a square have no common unit of measure.

∎

COMMON MISTAKES

- Using the scale factor for areas. If lengths are tripled, areas are multiplied by 9, not 3.
- Using the area ratio for lengths. If the areas are in the ratio $25 : 16$, the sides are in the ratio $5 : 4$; take the square root first.
- Applying the k^2 rule to figures that are not similar. Two rectangles with the same perimeter, or a square and a rectangle, need not have areas in any squared ratio.
- In geometric probability, dividing by the wrong whole. The denominator is the area of the entire region where the point can land.

> **Make it a game · Bean Target.** Draw a target with two circles of radii 2 and 4 centimeters, one inside the other. Predict the fraction of toss-in beans that will land in the inner circle (one quarter, since areas scale by the square). Toss twenty beans from a short distance and count. Compare. Then predict: a scale factor of 3 makes how many times the area?

UNIT 11

Area and Perimeter

This unit asks a simple question: how much surface does a figure cover? Starting from three area postulates (a rectangle's area is base times height, congruent figures have equal areas, and areas of non-overlapping pieces add), you will derive the area formulas for parallelograms, triangles, trapezoids, kites, rhombuses and regular polygons by cutting and rearranging, just as Euclid showed that figures "on the same base and between the same parallels" are equal (I.35–I.41).

You will then turn to circles: why π is the same for every circle, how to find circumference, arc length, and the areas of circles, sectors, segments and rings, and how Eratosthenes and Archimedes used these ideas. The unit ends with the beautiful rule that similar figures have areas in the ratio of the squares of their scale factor (Euclid VI.19 and XII.2), with composite regions and geometric probability.

Parents: expect practical problems (gardens, roofs, stained glass, sprinklers, blueprints) alongside proofs that explain each formula. A scientific calculator is used for π, trigonometry and decimal answers; students should give exact answers (in terms of π or in radical form) whenever asked.

PREREQUISITES

- Properties of parallelograms, rhombuses, trapezoids and kites (Lessons 6.2–6.5)
- Similar polygons and the AA, SAS and SSS similarity tests (Lessons 8.2–8.4)
- The Pythagorean Theorem and special right triangles (Lessons 9.1 and 9.4)
- Tangent, sine and cosine ratios with a calculator (Lessons 9.5–9.6)
- Central angles, arcs, chords and tangents of circles (Lessons 10.1–10.3)
- Simplifying square roots and solving linear and simple quadratic equations (Algebra I)

MASTERY CHECKLIST

- ☐ I can state the three area postulates and derive the area formulas for parallelograms and triangles from them.
- ☐ I can explain Euclid's propositions on figures with the same base between the same parallels (I.35–I.38, I.41).
- ☐ I can find areas and perimeters of figures on the coordinate plane, including by enclosing a figure in a rectangle.
- ☐ I can derive and use the area formulas for trapezoids, rhombuses and kites, and find areas of composite figures.
- ☐ I can find the central angle and apothem of a regular polygon and use $A = \frac{1}{2}ap$.
- ☐ I can inscribe a regular hexagon in a circle with compass and straightedge and explain why it works.
- ☐ I can explain why π is the same for every circle and find circumferences and arc lengths.
- ☐ I can find the areas of circles, sectors, segments and rings, exactly and rounded.
- ☐ I can use the fact that similar figures with scale factor

a : b have areas in the ratio a^2 : b^2, and find a scale factor from areas.

- ☐ I can find geometric probabilities as ratios of areas.
- ☐ I can write two-column, paragraph, flow and indirect proofs about areas.

LESSONS IN THIS UNIT

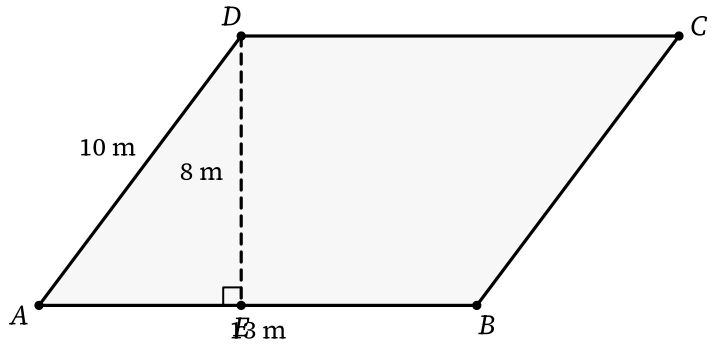
11.1	**Areas of Rectangles, Parallelograms and Triangles**	45 minutes
11.2	**Areas of Trapezoids, Rhombuses and Kites**	2 sessions of 45 minutes
11.3	**Areas of Regular Polygons**	2 sessions of 45 minutes
11.4	**Circumference and Arc Length**	2 sessions of 45 minutes
11.5	**Areas of Circles and Sectors**	2 sessions of 45 minutes
11.6	**Areas of Similar Figures and Composite Regions**	2 sessions of 45 minutes

Unit 11 Test

You have 55 minutes. A scientific calculator is allowed for π, trigonometry and decimal approximations. Show your work for every problem. Give exact answers (in terms of π or in simplest radical form) when a problem asks for them; otherwise round as the problem says. Include units in your answers. For Problem 14 you need a compass and straightedge; leave all construction marks visible. In proofs, give a reason for every step. Total: 100 points.

1. **5 pts**

   In the figure, parallelogram $ABCD$ has $AB = 13$ m, $AD = 10$ m, and height $DE = 8$ m to base $\overline{AB}$. Find its area.



2. **6 pts**

   Find the area of $\triangle ABC$ with vertices $A(0, 1)$, $B(7, 0)$ and $C(3, 5)$.

3. **6 pts**

   One side of a barn roof is a trapezoid. Its parallel edges are 30 ft and 18 ft long, and the perpendicular distance between them is 12 ft. How many square feet of shingles are needed to cover it?

4. **6 pts**

   A rhombus has sides of 17 cm and one diagonal of 16 cm. Find the length of the other diagonal and the area of the rhombus.

5. **6 pts**

   Find the area of a regular hexagon with side 8 in. Give an exact answer in simplest radical form.

6. **7 pts**

   Grace's herb garden is a regular pentagon with sides of 6 ft. Find its area to the nearest tenth of a square foot.

7. **6 pts**

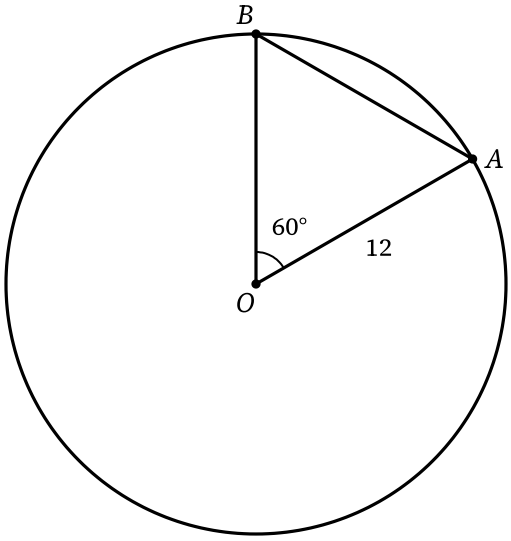
   A wagon wheel has a radius of 14 in. How many feet does the wagon travel when the wheel makes 50 full turns? Round to the nearest tenth of a foot.

8. **6 pts**

   Find the length of an arc with central angle 140° in a circle of radius 9 cm. Give an exact answer in terms of π.

9. **6 pts**

   In the figure, circle O has radius 12 and $m\angle AOB = 60^\circ$. Find the area of the segment bounded by chord $\overline{AB}$ and minor arc $\overset{\frown}{AB}$, exactly and to the nearest tenth.



10. **6 pts**

    A circular fountain with radius 5 ft is surrounded by a brick walkway 4 ft wide. Find the area of the walkway to the nearest tenth of a square foot.

11. **6 pts**

    Two similar quilt blocks have areas 27 in^2 and 75 in^2. A side of the smaller block is 9 in. Find the corresponding side of the larger block.

12. **6 pts**

    A square target is 12 in on each side and has a circle of radius 3 in painted in its center. A dart hits the target at a random point. Find the probability that it lands inside the circle, exactly and to the nearest hundredth.

13. **6 pts**

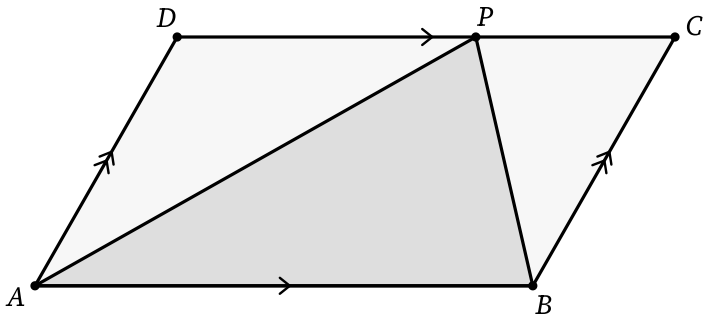
Explain why a circle with diameter 20 in has twice the circumference but four times the area of a circle with diameter 10 in.

14. **6 pts**

Using a compass and straightedge, inscribe a regular hexagon in a circle of radius 2 in. List your steps, explain why each side of the hexagon equals the radius, and find the exact area of the hexagon.

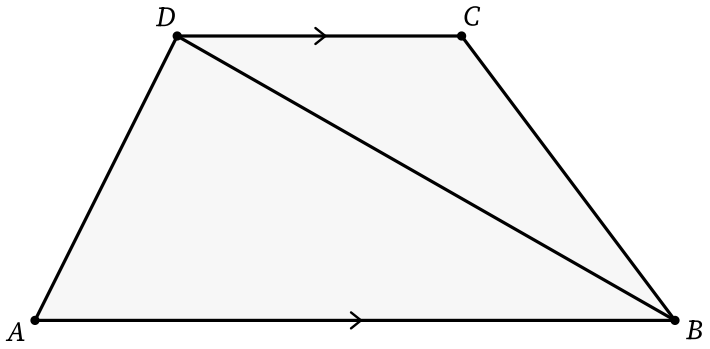
15. **8 pts**

In the figure, $ABCD$ is a parallelogram and P is any point on side $\overline{DC}$. Write a two-column proof that $\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$ (Euclid I.41).



16. **8 pts**

In the figure, $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$, and $\overline{BD}$ is a diagonal. Write a paragraph proof that $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{AB}{DC}$.



ANSWER KEY

1. $A = bh = 13 \times 8 = 104$ square meters
2. Enclosing rectangle from $(0, 0)$ to $(7, 5)$: area 35. Corner triangles: $\frac{1}{2}(7)(1) = 3.5$, $\frac{1}{2}(4)(5) = 10$, $\frac{1}{2}(3)(4) = 6$. Area $= 35 - 19.5 = 15.5$ square units
3. $\frac{1}{2}(12)(30 + 18) = 288$ square feet
4. The diagonals are perpendicular bisectors of each other: half-diagonals 8 and $\sqrt{17^2 - 8^2} = 15$, so the other diagonal is 30 cm. Area $= \frac{1}{2}(16)(30) = 240$ cm^2
5. Apothem $= 4\sqrt{3}$, perimeter $= 48$, $A = \frac{1}{2}(4\sqrt{3})(48) = 96\sqrt{3}$ square inches
6. Central angle $72°$; half of it is $36°$, so $a = \frac{3}{\tan 36°} \approx 4.129$ ft. $p = 30$ ft, $A \approx \frac{1}{2}(4.129)(30) \approx 61.9$ square feet
7. One turn: $2\pi(14) = 28\pi$ in. Fifty turns: $1400\pi \approx 4398.2$ in ≈ 366.5 ft
8. $\frac{140}{360} \cdot 18\pi = 7\pi$ cm
9. Sector: $\frac{60}{360} \cdot 144\pi = 24\pi$. $\triangle AOB$ is equilateral (isosceles with a $60°$ vertex angle) with side 12, area $\frac{\sqrt{3}}{4}(144) = 36\sqrt{3}$. Segment $= 24\pi - 36\sqrt{3} \approx 13.0$ square units
10. $\pi(9^2 - 5^2) = 56\pi \approx 175.9$ square feet
11. Area ratio $\frac{75}{27} = \frac{25}{9}$, so the scale factor is $\frac{5}{3}$ and the side is $9 \cdot \frac{5}{3} = 15$ in
12. $\frac{9\pi}{144} = \frac{\pi}{16} \approx 0.20$
13. All circles are similar, and here the scale factor is 2. Circumference is a length, so it is multiplied by 2 (10π to 20π). Area is two-dimensional, so it is multiplied by $2^2 = 4$ (25π to 100π).
14. Steps: draw the circle with center O and mark A on it; with the compass still set to the radius, step around the circle from A to mark B, C, D, E, F; join the six points. Each triangle such as $\triangle OAB$ has $OA = OB = AB = 2$, so it is equilateral and its central angle is $60°$; six such angles fill $360°$, so the marks close up (Euclid IV.15). Area: six equilateral triangles of side 2, each $\frac{\sqrt{3}}{4}(4) = \sqrt{3}$, total $6\sqrt{3}$ square inches.
15. See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; P lies on $\overline{DC}$

Prove: $\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram; P lies on $\overline{DC}$	Given
2	$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$	Definition of parallelogram
3	Let h be the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$	The distance between two parallel lines is the same at every point
4	The height of $\triangle ABP$ to base $\overline{AB}$ is h	Definition of height (P lies on $\overleftrightarrow{DC}$)
5	$\text{Area}(\triangle ABP) = \frac{1}{2}AB \cdot h$	Triangle Area Theorem
6	$\text{Area}(ABCD) = AB \cdot h$	Parallelogram Area Theorem (the height to base $\overline{AB}$ is h)
7	$\text{Area}(\triangle ABP) = \frac{1}{2}\,\text{Area}(ABCD)$	Substitution Property of Equality

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16. See the model proof.

PARAGRAPH PROOF

Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$; $\overline{BD}$ is a diagonal

Prove: $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{AB}{DC}$

Let h be the distance between the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$; it is the same at every point.

In $\triangle ABD$, take $\overline{AB}$ as the base. The vertex D lies on $\overleftrightarrow{DC}$, so the height to that base is h, and by the Triangle Area Theorem $\text{Area}(\triangle ABD) = \frac{1}{2}AB \cdot h$. In $\triangle CDB$, take $\overline{DC}$ as the base. The vertex B lies on $\overleftrightarrow{AB}$, so the height is again h, and $\text{Area}(\triangle CDB) = \frac{1}{2}DC \cdot h$.

Dividing, $\dfrac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle CDB)} = \dfrac{\frac{1}{2}AB \cdot h}{\frac{1}{2}DC \cdot h} = \dfrac{AB}{DC}$, since the common nonzero factor $\frac{1}{2}h$ divides out.

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UNIT 12

Surface Area and Volume

In this unit geometry steps off the flat page and into space. You will name and sketch prisms, pyramids, cylinders, cones and spheres, unfold them into nets, slice them into cross sections, and meet Euler's formula and the five Platonic solids with which Euclid closes the Elements. Then you will learn to measure solids in two ways: surface area, the amount of material that covers a solid, and volume, the amount of space it fills.

As always in this course, the formulas are explained, not just memorized. Lateral areas come from unfolding nets; volumes come from stacking layers and from Cavalieri's Principle, which compares solids slice by slice; the one-third in the pyramid and cone formulas is Euclid's (XII.7 and XII.10); and the sphere formulas come from Archimedes, who asked for a sphere inside a cylinder to be carved on his tomb. The unit ends with similar solids, where areas grow by the square of the scale factor and volumes by the cube.

Expect many practical problems: painting a barn, shingling a steeple, ordering concrete by the cubic yard, filling a grain bin. A scientific calculator is allowed for π and decimal answers. Parents: the most common errors in this unit are using a diameter for a radius, a height for a slant height, and dividing by 3 instead of 27 to get cubic yards. A quick check of these three habits catches most mistakes.

PREREQUISITES

- Areas of rectangles, triangles, parallelograms, trapezoids and regular polygons (Lessons 11.1 to 11.3)
- Circumference, arc length and the areas of circles and sectors (Lessons 11.4 and 11.5)
- The ratio of areas of similar figures (Lesson 11.6)
- The Pythagorean theorem and special right triangles (Lessons 9.1 and 9.4)
- Similar triangles and dilations (Lessons 7.5, 8.2 and 8.3)
- Solving equations, including ones with squares and cubes, and rounding answers (Algebra I)

MASTERY CHECKLIST

- ☐ I can name prisms, pyramids, cylinders, cones and spheres and count the faces, edges and vertices of a polyhedron.
- ☐ I can use Euler's formula $F + V = E + 2$ and explain why there are exactly five Platonic solids.
- ☐ I can match a solid with its net, describe cross sections and solids of revolution, and construct the net of a regular tetrahedron with compass and straightedge.
- ☐ I can find the lateral area and surface area of right prisms and cylinders and explain why $L = Ph$ and $L = 2\pi rh$.
- ☐ I can find the slant height of a regular pyramid or a right cone with the Pythagorean theorem.
- ☐ I can find the lateral area and surface area of regular pyramids and right cones and explain where $L = \frac{1}{2}P\ell$ and $L = \pi r\ell$ come from.
- ☐ I can state Cavalieri's Principle and use it to explain why $V = Bh$ for right and oblique prisms and cylinders.

- ☐ I can find volumes of pyramids and cones with $V = \frac{1}{3}Bh$ and explain why the factor is one-third.
- ☐ I can find the surface area and volume of spheres and hemispheres and explain Archimedes' sphere-and-cylinder discovery.
- ☐ I can use the ratios $a^2 : b^2$ and $a^3 : b^3$ for the surface areas and volumes of similar solids.
- ☐ I can solve composite-solid and practical problems, converting units such as cubic feet to cubic yards correctly.

LESSONS IN THIS UNIT

12.1	**Solids, Nets and Cross Sections**	45 minutes
12.2	**Surface Area of Prisms and Cylinders**	2 sessions of 45 minutes
12.3	**Surface Area of Pyramids and Cones**	2 sessions of 45 minutes
12.4	**Volume of Prisms and Cylinders**	2 sessions of 45 minutes
12.5	**Volume of Pyramids and Cones**	2 sessions of 45 minutes
12.6	**Spheres and Similar Solids**	2 sessions of 45 minutes

Unit 12 Test

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem: write the formula you use, substitute, and then compute. Give exact

answers in terms of π or in simplest radical form when a problem asks for them; otherwise round as the problem directs and include units (square units for area, cubic units for volume). In the proof, give a reason for every step. For Problem 15 you need a compass, a straightedge and a centimeter ruler.

1. **5 pts**

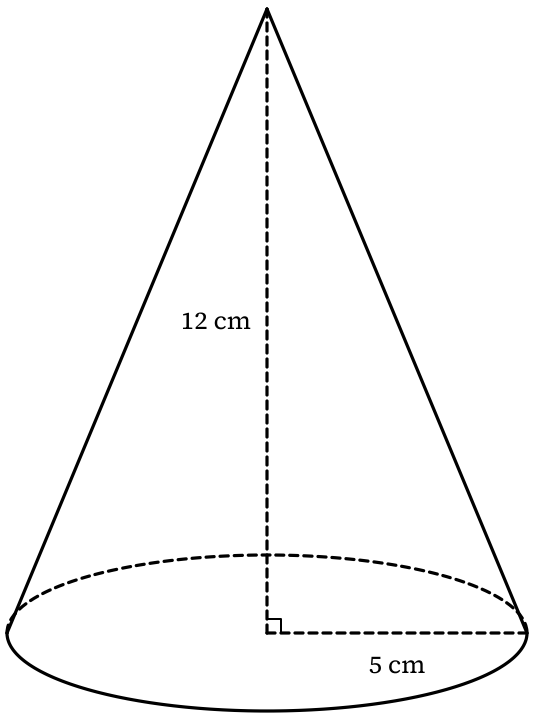
   A pyramid has a pentagon for its base. Find its number of faces, vertices and edges, in that order.

2. **5 pts**

   A convex polyhedron has 9 faces and 16 edges. How many vertices does it have?

3. **6 pts**

   The right cone in the figure has radius 5 cm and height 12 cm. It is sliced by a vertical plane through its apex and its axis. Name the shape of the cross section and find its area.

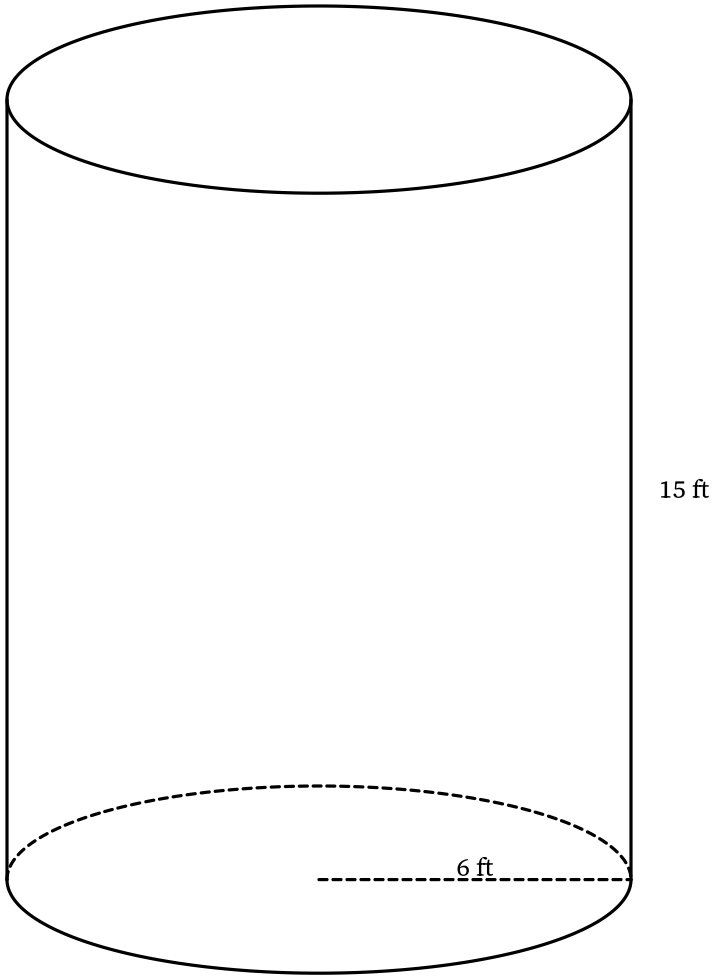


4. **6 pts**

   Find the total surface area of a rectangular box 9 cm long, 7 cm wide and 2 cm high.

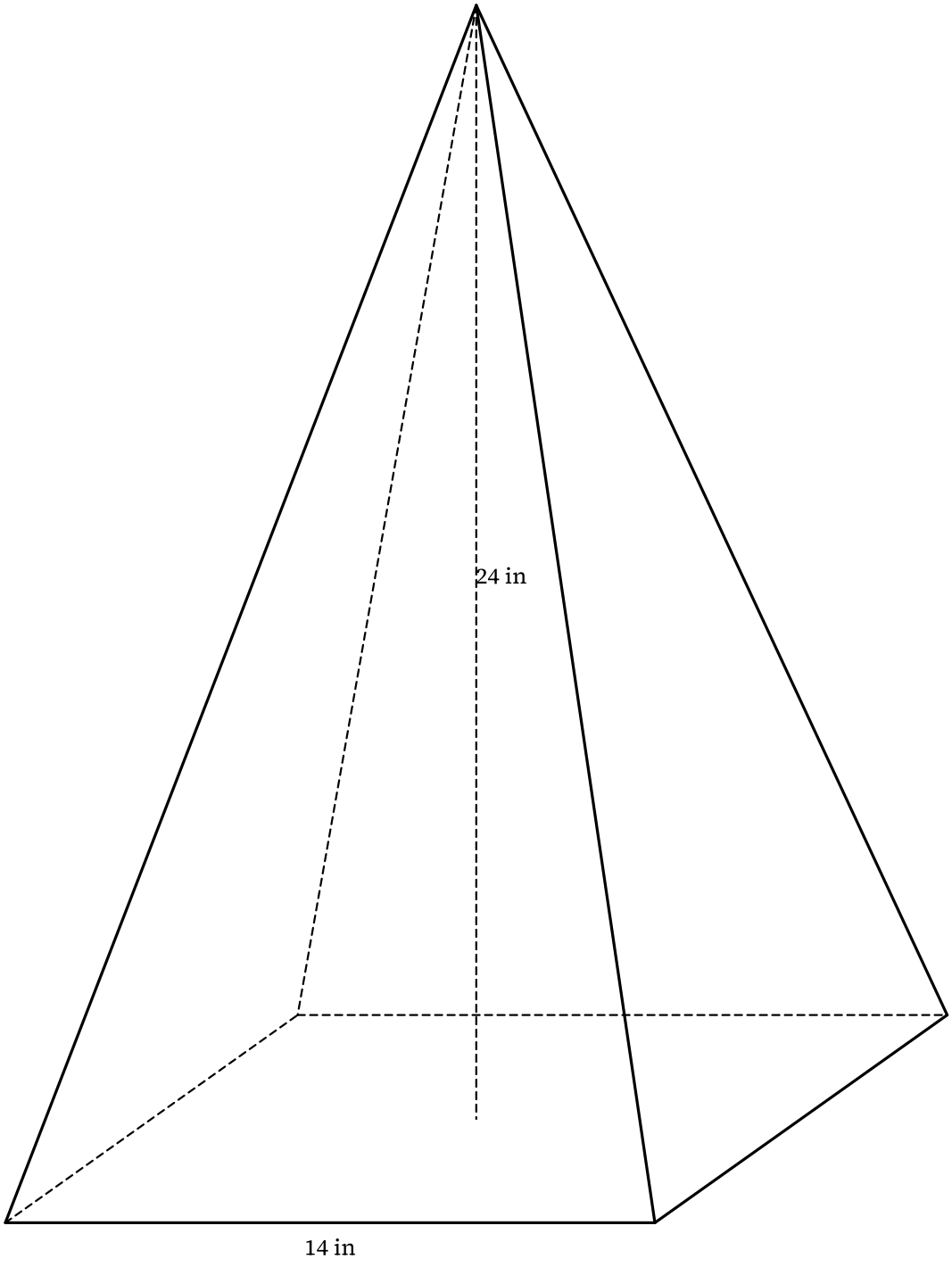
5. **7 pts**

   A cylindrical water tank with radius 6 ft and height 15 ft stands on the ground, as in the figure. Its curved side and its flat top will be painted, but not its bottom. Find the area to be painted to the nearest tenth of a square foot. If one gallon of paint covers 300 square feet, how many whole gallons are needed?



6. **7 pts**

A regular square pyramid has base edges 14 in and height 24 in. Find its slant height and its total surface area.



7. **7 pts**

   A right cone has radius 8 cm and height 15 cm. Find its total surface area in terms of π.

8. **7 pts**

   The Miller family is pouring a concrete driveway 40 ft long, 12 ft wide and 4 inches thick. How many cubic yards of concrete does it take, to the nearest tenth? If they order 6 cubic yards at \$150 per cubic yard, what is the cost?

9. **7 pts**

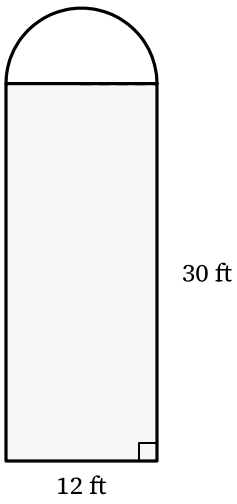
   Grace stacks 30 identical coins, each 2.4 cm in diameter and 0.2 cm thick, into a column, then nudges the column so it leans. Find the volume of the leaning stack to the nearest tenth of a cubic centimeter, and explain why leaning does not change the volume.

10. **7 pts**

    The attic of a gazebo is the space under a square pyramid roof with base edges 12 ft and height 5 ft. Find the volume of the attic.

11. **8 pts**

    A grain silo is a cylinder 12 ft in diameter and 30 ft tall, topped by a hemispherical dome with the same diameter, as in the figure. Find its total volume to the nearest cubic foot.



Side view of the silo

12. **7 pts**

A sphere has volume 288π cubic meters. Find its radius and then its surface area in terms of π.

13. **8 pts**

Ruth makes two similar clay pitchers whose heights are in the ratio $3:4$. The larger pitcher holds 64 ounces. The smaller pitcher needs 18 square inches of glaze on its outside. How many ounces does the smaller pitcher hold, and how much glaze does the larger one need?

14. **6 pts**

Prove that a sphere of radius r and a cone with base radius r and height $4r$ have equal volumes.

15. **7 pts**

With compass and straightedge (and a ruler only to measure the first segment), construct a net for a regular tetrahedron whose edges are 3 cm long. List your steps, and explain why the net folds into a solid whose four faces are congruent equilateral triangles.

ANSWER KEY

1. $F = 6$, $V = 6$, $E = 10$ (and $6 + 6 = 10 + 2$).
2. $9 + V = 16 + 2$, so $V = 9$ vertices.
3. An isosceles triangle with base $2 \times 5 = 10$ cm and height 12 cm; area $\frac{1}{2}(10)(12) = 60$ square centimeters.
4. $2(9 \cdot 7 + 9 \cdot 2 + 7 \cdot 2) = 2(63 + 18 + 14) = 190$ square centimeters
5. Side: $2\pi(6)(15) = 180\pi$; top: $\pi(6)^2 = 36\pi$. Total $216\pi \approx 678.6$ square feet. $678.6 \div 300 \approx 2.26$, so 3 gallons.
6. $\ell = \sqrt{24^2 + 7^2} = \sqrt{625} = 25$ in. $S = \frac{1}{2}(56)(25) + 14^2 = 700 + 196 = 896$ square inches.
7. $\ell = \sqrt{64 + 225} = 17$. $S = \pi(8)(17) + \pi(8)^2 = 136\pi + 64\pi = 200\pi$ square centimeters.
8. $V = 40 \times 12 \times \frac{4}{12} = 160$ cubic feet $= 160 \div 27 \approx 5.9$ cubic yards. Cost: $6 \times 150 =$ \$900.
9. The stack is $30 \times 0.2 = 6$ cm tall with radius 1.2 cm: $V = \pi(1.2)^2(6) = 8.64\pi \approx 27.1$ cubic centimeters. The leaning stack has the same height as the straight one, and at every level its cross section is the same circle, of the same area; by Cavalieri's Principle the volumes are equal.
10. $V = \frac{1}{3}(144)(5) = 240$ cubic feet
11. $r = 6$. Cylinder: $\pi(36)(30) = 1080\pi$. Hemisphere: $\frac{2}{3}\pi(216) = 144\pi$. Total $1224\pi \approx 3845$ cubic feet.
12. $\frac{4}{3}\pi r^3 = 288\pi$, so $r^3 = 216$ and $r = 6$ m. $S = 4\pi(36) = 144\pi$ square meters.
13. Volumes are in the ratio $27 : 64$, so the smaller holds $64 \times \frac{27}{64} = 27$ ounces. Areas are in the ratio $9 : 16$, so the larger needs $18 \times \frac{16}{9} = 32$ square inches of glaze.
14. See the model proof.

    TWO-COLUMN PROOF

    Given: A sphere with radius r; A cone with base radius r and height $4r$

    Prove: $V_{\text{sphere}} = V_{\text{cone}}$

	STATEMENTS	REASONS
1	$V_{\text{sphere}} = \frac{4}{3}\pi r^3$	Volume of a Sphere
2	$V_{\text{cone}} = \frac{1}{3}\pi r^2(4r)$	Volume of a Cone

	STATEMENTS	REASONS
3	$V_{\text{cone}} = \frac{4}{3}\pi r^3$	Properties of multiplication and exponents
4	$V_{\text{sphere}} = V_{\text{cone}}$	Transitive and Symmetric Properties of Equality (rows 1 and 3)

∎

15. Draw $\overline{AB}$ 6 cm long. Draw circles centered at A and at B with radius AB; they meet at C; draw $\overline{AC}$ and $\overline{BC}$ (Euclid I.1). Bisect each side with the perpendicular bisector construction (I.10) to find midpoints D, E, F of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, and draw $\overline{DE}$, $\overline{EF}$, $\overline{FD}$. Why: $AC = AB = BC$ as radii of the two circles, so $\triangle ABC$ is equilateral with sides 6 cm. The half-sides are 3 cm, and by the Triangle Midsegment Theorem each midsegment is $\frac{1}{2}(6) = 3$ cm. So the four small triangles each have three 3-cm sides: they are congruent (SSS) equilateral triangles, and folding along $\overline{DE}$, $\overline{EF}$, $\overline{FD}$ brings A, B and C together to form a regular tetrahedron.

12.1 Solids, Nets and Cross Sections

OBJECTIVES

- Identify prisms, pyramids, cylinders, cones and spheres and count the faces, edges and vertices of a polyhedron.
- Use Euler's formula $F + V = E + 2$ to find a missing count for a polyhedron.
- Match solids with their nets, and describe cross sections and solids of revolution.
- Explain why there are exactly five Platonic solids, and construct the net of a regular tetrahedron with compass and straightedge.

WATCH

Euler's Formula for Polyhedra | Math with Mr. J

Math with Mr. J · 3:08

https://www.youtube.com/watch?v=VygN0FxfzMg

> **WHEN TO WATCH**
> With Euler's formula: F + V − E = 2 checked on a prism and a pyramid.

Note: Polyhedron vocabulary, solids of revolution and the Platonic solids are taught from the lesson.

BACKUP VIDEO

Slicing three-dimensional figures (7th grade math)

Math Mammoth · 9:04

https://www.youtube.com/watch?v=PLmcnCZHmHE

With cross sections: a computer tool slices pyramids, prisms, cylinders and cones, straight and tilted.

KEY TERMS

polyhedron

A closed solid whose surface is made entirely of polygons. The polygons are its **faces**; a segment where two faces meet is an **edge**; a point where three or more edges meet is a **vertex** (plural: vertices).

prism

A polyhedron with two congruent, parallel faces called **bases**, joined by parallelograms called **lateral faces**. A prism is named by the shape of its bases (triangular prism, hexagonal prism).

pyramid

A polyhedron with one polygon **base** and triangular **lateral faces** that all meet at one point, the **apex**. A pyramid is named by its base (square pyramid, hexagonal pyramid).

cylinder, cone and sphere

Solids with curved surfaces. A **cylinder** has two congruent parallel circular bases joined by a curved surface; a **cone** has one circular base and an apex; a **sphere** is the set of all points in space at a given distance (the radius) from a given point (the center).

net

A flat pattern of polygons (and, for curved solids, circles and other shapes) that can be folded up to cover the surface of a solid exactly, with no gaps or overlaps.

cross section

The flat shape you get where a plane slices through a solid.

Platonic solid (regular polyhedron)

A convex polyhedron whose faces are all congruent regular polygons, with the same number of faces meeting at every vertex.

CORE IDEAS

Until now nearly everything in this course has lived in a flat plane. In this unit we step into space, where figures have length, width and height. Euclid begins his solid geometry (Book XI) with the definition: 'A solid is that which has length, breadth, and depth.' The simplest solids to study are **polyhedra** (singular: polyhedron), solids whose surfaces are made only of flat polygons, such as a brick, a cereal box or the roof of a steeple.

Two families of polyhedra will occupy us all unit. A **prism** has two congruent, parallel bases joined by lateral faces that are parallelograms (rectangles when the prism is a **right prism**, standing straight up). A **pyramid** has one base, and every lateral face is a triangle reaching up to the apex. In both cases the name comes from the base: a hexagonal prism has hexagons for bases; a square pyramid has a square for its base.

Some solids have curved surfaces: the **cylinder** (a can), the **cone** (a funnel or a church spire) and the **sphere** (a ball). Euclid defined each one as a **solid of revolution**, the solid swept out when a flat figure spins around a fixed line called the axis: a rectangle turning about one side sweeps out a cylinder, a right triangle turning about one leg sweeps out a cone (see the figure), and a semicircle turning about its diameter sweeps out a sphere. This is exactly how a woodworker's lathe shapes a table leg.

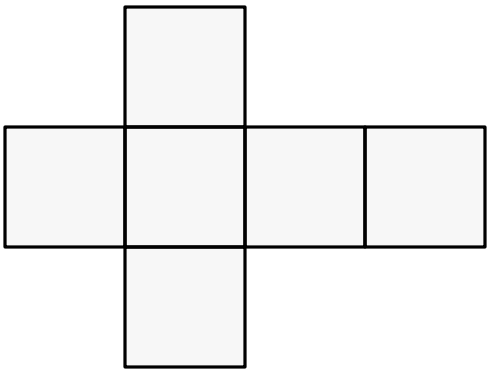
A **net** is a solid unfolded flat, like a cardboard box cut along some edges and pressed flat. The figure shows one of the eleven different nets of a cube: six squares that fold up so each square becomes one face. Nets matter because the area of a net is the total area of the solid's surface, which is the subject of the next two lessons. To check a net, imagine folding it: each face must land in its own place, with no two faces overlapping and no gap left open.

Count the faces F, vertices V and edges E of a few polyhedra and something remarkable appears. A cube has $F = 6$, $V = 8$, $E = 12$, and $6 + 8 = 12 + 2$. A square pyramid has $F = 5$, $V = 5$, $E = 8$, and $5 + 5 = 8 + 2$. Leonhard Euler noticed that $F + V = E + 2$ for every convex polyhedron. Checking examples is inductive reasoning and does not prove the rule for all polyhedra (Lesson 1.2), so we state Euler's formula as a theorem accepted without proof in this course. It is still very useful: if you know two of the counts, it gives you the third.

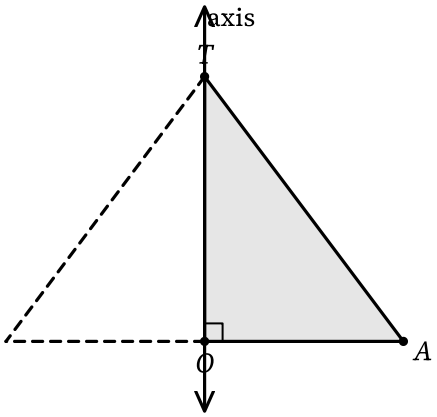
A **Platonic solid** is the most perfect kind of polyhedron: every face is the same regular polygon and every vertex looks the same. Why are there only five? At each vertex at least three faces must meet, and their angles must add to **less** than 360°; if they added to exactly 360° the faces would lie flat, like tiles on a floor, and no corner would form. Equilateral triangles (60°) can meet 3, 4 or 5 at a vertex; squares (90°) only 3; regular pentagons (108°) only 3; hexagons (120°) cannot work at all, since $3 \times 120^\circ = 360^\circ$. That gives exactly five possibilities, and the proof below makes the argument complete.

A **cross section** is the shape a plane cuts out of a solid, like the face you see when you slice a loaf of bread. The shape depends on the direction of the slice. A horizontal slice of a cylinder is a circle congruent to its base; a vertical slice through its axis is a rectangle. A slice of a square pyramid parallel to the base is a smaller square; every

plane that cuts through a sphere slices it in a circle. Cross sections will return in Lesson 12.4, where they are the key to Cavalieri's Principle for volume.



A net of a cube

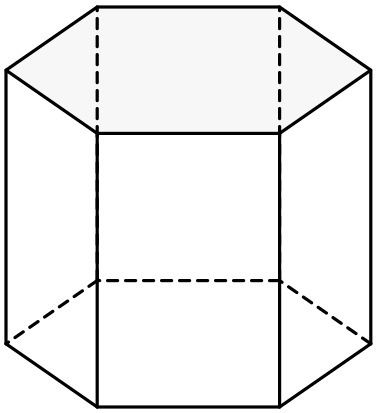


A right triangle turned about leg OT sweeps out a cone.

POSTULATES AND THEOREMS

THEOREM **Euler's Formula for Polyhedra**

For every convex polyhedron with F faces, V vertices and E edges, $F + V = E + 2$.



Hexagonal prism (hidden edges dashed)

Accepted without proof in this course: a complete proof needs ideas from topology, a later branch of mathematics. Check it on every polyhedron you meet.

THEOREM **There Are Exactly Five Platonic Solids** Euclid XIII.18

There are exactly five regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

PARAGRAPH PROOF

Given: A regular polyhedron: all faces are congruent regular polygons with n sides, and the same number k of faces meet at every vertex

Prove: The polyhedron is one of only five possible types.

At each vertex of a polyhedron at least three faces meet, since two faces alone meet only along an edge and cannot close up a corner; so $k \geq 3$. The face angles meeting at a vertex add to less than $360°$ (Euclid XI.21: any solid angle is contained by plane angles together less than four right angles). Each angle of a regular n-gon measures $\frac{(n-2)\cdot 180°}{n}$ (Polygon Interior Angle Sum Theorem, Lesson 6.1), so the angle is $60°$ for $n = 3$, $90°$ for $n = 4$, $108°$ for $n = 5$, $120°$ for $n = 6$, and larger than $120°$ for $n > 6$.

If $n \geq 6$, then k angles of at least $120°$ add to at least $3 \times 120° = 360°$, which is not less than $360°$; so no regular polyhedron has faces with six or more sides. If $n = 5$, then $3 \times 108° = 324° < 360°$ but $4 \times 108° = 432° > 360°$, so $k = 3$ (the dodecahedron). If $n = 4$, then $3 \times 90° = 270° < 360°$ but $4 \times 90° = 360°$, so $k = 3$ (the cube). If $n = 3$, then $3 \times 60°$, $4 \times 60°$ and $5 \times 60°$ are all less than $360°$ but $6 \times 60° = 360°$, so $k = 3$, 4 or 5 (the tetrahedron, octahedron and icosahedron).

So there are at most five possibilities. Euclid constructs a solid of each of the five kinds in Propositions XIII.13 to XIII.17, so all five really exist. Therefore there are exactly five regular polyhedra.

∎

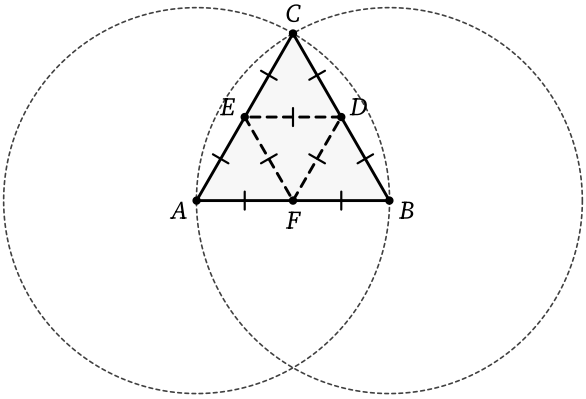
Euclid places this argument in the remark that closes Book XIII, right after Proposition 18: 'no other figure, besides the said five figures, can be constructed.'

CONSTRUCTIONS

CONSTRUCTION **Construct the net of a regular tetrahedron**

Euclid I.1 and I.10

1. Draw a segment $\overline{AB}$ twice as long as you want each edge of the tetrahedron to be.
2. With the compass point on A and width AB, draw a circle. Keeping the same width, draw a circle with center B. Label



one point where the circles cross C, and draw $\overline{AC}$ and $\overline{BC}$. (This is Euclid I.1: $\triangle ABC$ is equilateral.)

3. Bisect each side (Euclid I.10, using the perpendicular bisector construction from Lesson 2.5). Label the midpoints D on $\overline{BC}$, E on $\overline{CA}$ and F on $\overline{AB}$.
4. Draw $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$. Cut out $\triangle ABC$ and fold along the three inner segments until A, B and C meet at one point. The folded paper is a regular tetrahedron.

Why it works. $AC = AB$ and $BC = BA$ because each pair are radii of the same circle, so $\triangle ABC$ is equilateral. Each midpoint cuts a side into halves of length $\frac{1}{2}AB$, and by the Triangle Midsegment Theorem (Lesson 5.4) each of $\overline{DE}$, $\overline{EF}$, $\overline{FD}$ is also half a side, so all nine small segments have length $\frac{1}{2}AB$. The four small triangles are therefore congruent equilateral triangles (SSS), and when folded they become the four faces of a regular tetrahedron.

KEY FORMULAS

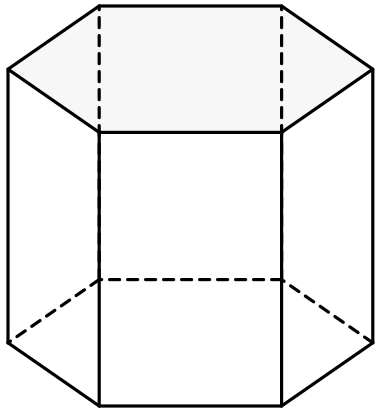
$$F + V = E + 2$$

From history. The five regular solids are often called the Platonic solids because the philosopher Plato discusses them in his dialogue Timaeus. Euclid's final book, Book XIII, constructs each of the five inside a sphere and then shows that no sixth regular solid can exist, a fitting summit for the whole Elements. About two thousand years later, in the 1750s, the Swiss mathematician Leonhard Euler published the relation $F + V = E + 2$, one of the first results in the branch of mathematics now called topology.

Euclid connection. Euclid defines the sphere, the cone and the cylinder as solids of revolution (Book XI, Definitions 14, 18 and 21): a semicircle turned about its diameter, a right triangle turned about one leg, and a rectangle turned about one side. The Elements ends in Book XIII with the construction of the five regular solids and the proof that there are no others.

Practice habit. Recite Euler's formula $F + V = E + 2$ and the names of the five Platonic solids until they are automatic. When you count parts of a polyhedron, draw a large sketch with hidden edges dashed and count in an organized way: bases first, then the lateral faces and edges.

Example 1. Count the faces, vertices and edges of the hexagonal prism in the figure, and check that Euler's formula holds.



Hexagonal prism (hidden edges dashed)

Show solution

1. Faces: 2 hexagonal bases plus 6 rectangular lateral faces (one for each side of the hexagon), so $F = 8$.
2. Vertices: each hexagonal base has 6 vertices, so $V = 6 + 6 = 12$.
3. Edges: 6 edges around the top base, 6 around the bottom base, and 6 vertical edges joining them, so $E = 18$.
4. Check: $F + V = 8 + 12 = 20$ and $E + 2 = 18 + 2 = 20$. Euler's formula holds.

Answer: $F = 8$, $V = 12$, $E = 18$; $8 + 12 = 20 = 18 + 2$.

Example 2. A traditional soccer ball is stitched from 12 regular pentagons and 20 regular hexagons, and exactly 3 patches meet at every corner. Treating the ball as a polyhedron, find the number of edges (seams) and the number of vertices, and check Euler's formula.

Show solution

1. Faces: $F = 12 + 20 = 32$.
2. Edges: the pentagons have $12 \times 5 = 60$ sides and the hexagons $20 \times 6 = 120$ sides, 180 sides in all. Every seam is shared by exactly two patches, so each edge was counted twice: $E = 180 \div 2 = 90$.
3. Vertices: the 180 patch corners are shared 3 at a time, so $V = 180 \div 3 = 60$.
4. Check: $F + V = 32 + 60 = 92$ and $E + 2 = 90 + 2 = 92$. Euler's formula holds.

Answer: 90 edges and 60 vertices; $32 + 60 = 92 = 90 + 2$.

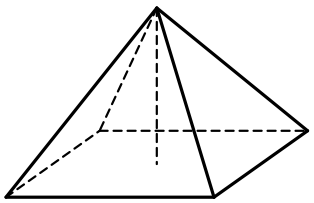
PRACTICE

1. Easy A solid has two congruent parallel triangular bases and three rectangular lateral faces. Name the solid.

   Show answer

   A triangular prism.

2. Easy Find the number of faces, vertices and edges of the square pyramid in the figure, in that order.



   Show answer

   $F = 5$, $V = 5$, $E = 8$ (and $5 + 5 = 8 + 2$).

3. **Easy** A convex polyhedron has 6 faces and 8 vertices. Use Euler's formula to find the number of edges.

   Show answer

   $6 + 8 = E + 2$, so $E = 12$ edges.

4. **Easy** A net is made of two congruent circles and one rectangle. What solid does it fold into?

   Show answer

   A cylinder.

5. **Medium** A convex polyhedron has 12 vertices and 30 edges. How many faces does it have?

   Show answer

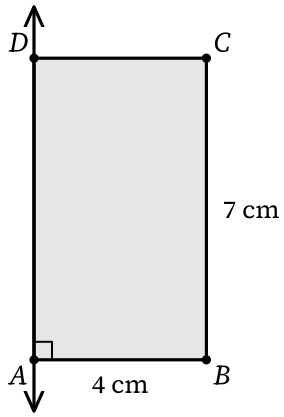
   $F + 12 = 30 + 2$, so $F = 20$ faces (this is the icosahedron).

6. **Medium** Samuel saws a cylindrical log with radius 3 inches and length 24 inches lengthwise, straight through its central axis. What are the dimensions of the flat rectangular cross section he exposes?

   Show answer

   The width is the diameter, $2 \times 3 = 6$ in, and the length is 24 in: a rectangle 6 in by 24 in.

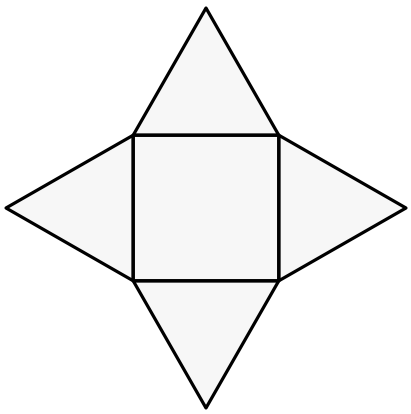
7. **Medium** In the figure, rectangle $ABCD$ has $AB = 4$ cm and $BC = 7$ cm. It is spun around the line through side $\overline{AD}$, as on a woodworker's lathe. Name the solid it sweeps out and give its radius and height.



Show answer

A cylinder with radius $AB = 4$ cm and height $AD = BC = 7$ cm.

8. **Medium** The net in the figure is a square with a congruent isosceles triangle on each side. What solid does it fold into?



Show answer

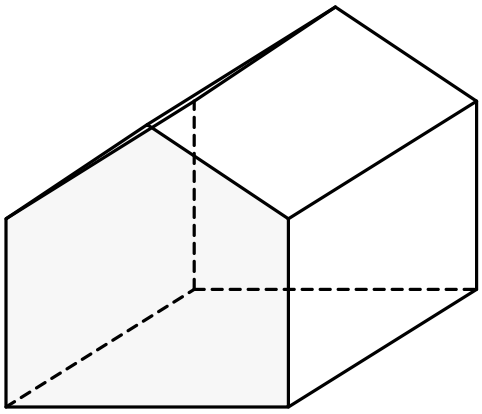
A square pyramid: the square is the base and the four triangles fold up to meet at the apex.

9. Medium Explain why no Platonic solid can have regular hexagons for its faces.

Show answer

Each angle of a regular hexagon is 120°. At least three faces must meet at each vertex, and three hexagon angles add to $3 \times 120^\circ = 360^\circ$. Face angles at a corner of a polyhedron must add to less than 360°; at exactly 360° the hexagons lie flat, like a tiled floor, and no corner forms. More hexagons would add to even more.

10. Hard The Miller family builds a small greenhouse frame shaped like a pentagonal prism (a house shape: a rectangle-walled room with a triangular gable at each end), as in the figure. Each edge is one wooden strut and each vertex is one metal connector. How many struts and how many connectors do they need? Check your counts with Euler's formula.



Greenhouse frame (hidden edges dashed)

Show answer

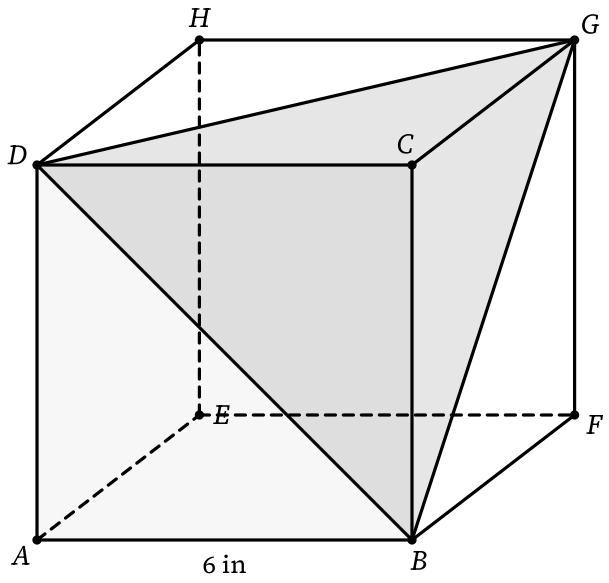
15 struts and 10 connectors. The prism has $F = 7$ faces, and $7 + 10 = 17 = 15 + 2$.

11. Hard A convex polyhedron has 20 faces, all of them triangles. How many edges does it have? How many vertices?

Show answer

The faces have $20 \times 3 = 60$ sides, and each edge is shared by two faces, so $E = 30$. Then $20 + V = 30 + 2$ gives $V = 12$.

12. **Hard** In the figure, cube $ABCDEFGH$ has edges of length 6 inches. A plane slices through vertices B, D and G, the three vertices next to corner C. Find the side length of the triangular cross section $\triangle BDG$ in simplest radical form.



Front face ABCD, back face EFGH

Show answer

Each side of $\triangle BDG$ is a diagonal of a square face of the cube ($\overline{BD}$ on the front face, $\overline{DG}$ on the top, $\overline{BG}$ on the right side). Each is the hypotenuse of a right triangle with legs 6 and 6, so each has length $\sqrt{36+36} = 6\sqrt{2}$ in. All three sides are equal, so the triangle is equilateral.

13. **Challenge** Prove that Euler's formula holds for every prism: if the bases of a prism are n-gons, then $F + V = E + 2$.

Show answer

See the model proof.

PARAGRAPH PROOF

Given: A prism whose two bases are n-gons

Prove: $F + V = E + 2$

Faces: the prism has 2 bases and one lateral face for each of the n sides of a base, so $F = n + 2$. Vertices: each base has n vertices and the prism has no others, so $V = 2n$. Edges: each base has n edges, and one lateral edge joins each vertex of the bottom base to the matching vertex of the top base, so $E = n + n + n = 3n$.

Then $F + V = (n + 2) + 2n = 3n + 2$, and $E + 2 = 3n + 2$. By the Transitive Property of Equality, $F + V = E + 2$.

∎

14. **Challenge** Could a pyramid have exactly 35 edges? Explain.

Show answer

No. A pyramid with an n-gon base has n base edges and n lateral edges (one from each base vertex to the apex), so it has $2n$ edges, an even number. 35 is odd.

COMMON MISTAKES

- Adding up the sides of every face and calling that the number of edges. Each edge is shared by two faces, so divide the total number of face sides by 2.
- Mixing up prisms and pyramids. A prism has two congruent parallel bases; a pyramid has one base and an apex. Find the base or bases first, then name the solid.
- Assuming any six squares joined together form a net of a cube. Imagine folding: if two squares would land on the same face (for example, five squares in a row), it is not a net.
- Using Euler's formula on a cylinder, cone or sphere. The formula is about polyhedra, whose faces are flat polygons.

Make it a game · Euler Hunt. Find four solids in the house: a tissue box, a can, a cereal box, a funnel. Count faces, vertices and edges for each polyhedron and check F + V = E + 2. Parent names a solid and you give the counts from memory. Finally, cut open a box into a net and name the shape it makes.

12.2 Surface Area of Prisms and Cylinders

OBJECTIVES

- Find the lateral area and total surface area of a right prism from its net, and explain why $L = Ph$.
- Find the lateral area and total surface area of a right cylinder, and explain why $L = 2\pi rh$.
- Solve practical problems about painting, wrapping and covering, deciding which surfaces the problem actually needs.

WATCH

Surface Area of a Prism (1 of 2: Introduction)

Eddie Woo · 5:53

https://www.youtube.com/watch?v=tFuBZvoBqE0

> **WHEN TO WATCH**
> Before Example 1: surface area of a prism from its net.

BACKUP VIDEO

Surface Area of a Cylinder

Eddie Woo · 9:56

https://www.youtube.com/watch?v=q7ZxLwPUSTs

Before the cylinder examples.

KEY TERMS

lateral faces and lateral edges

In a prism, the faces that are not bases are the lateral faces; the edges where two lateral faces meet are the lateral edges.

height (altitude) of a prism or cylinder

The perpendicular distance between the planes of the two bases.

right prism

A prism whose lateral edges are perpendicular to its bases, so every lateral face is a rectangle. A prism that leans is an **oblique prism**.

right cylinder

A cylinder whose axis (the segment joining the centers of the bases) is perpendicular to the bases.

lateral area

The total area of the lateral faces of a prism, or of the curved surface of a cylinder (not counting the bases).

surface area

The total area of all the surfaces of a solid: the lateral area plus the areas of the bases. It equals the area of the solid's net and is measured in square units.

CORE IDEAS

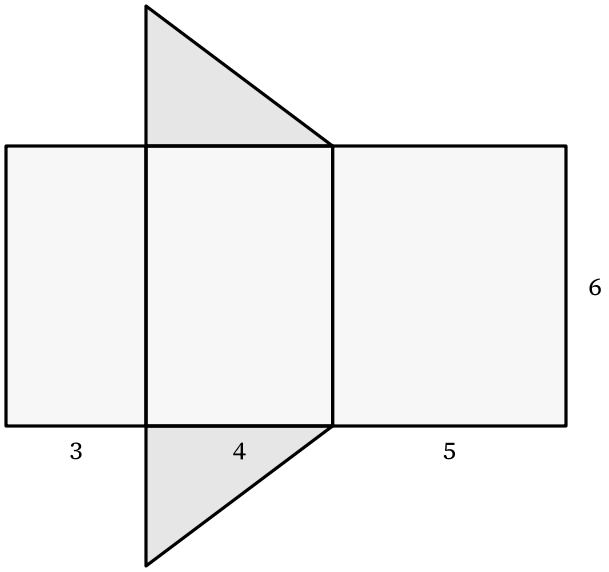
The **surface area** of a solid is the total area of its outside, the amount of paint, paper or sheet metal it takes to cover it exactly once. Because a net is the surface laid flat, the surface area of a solid equals the area of its net. Surface area is measured in square units (square inches, square feet), just like the area of any flat figure, because it *is* the area of flat pieces.

A prism has two kinds of faces: the two **bases** and the **lateral faces** around the sides. In a **right prism** the lateral edges stand perpendicular to the bases, so every lateral face is a rectangle whose height is the height h of the prism. In this lesson every prism and cylinder is a right one unless a problem says otherwise.

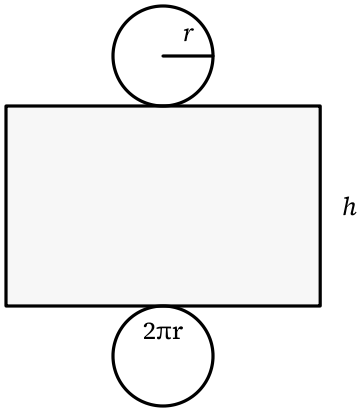
Why is the lateral area of a right prism equal to the perimeter of the base times the height? Unfold the lateral faces and lay them side by side, as in the first figure. They form one long rectangle. Its height is h, and its width is the sum of the sides of the base, which is the perimeter P. So the lateral area is $L = Ph$, whatever the shape of the base. Adding the two bases, each of area B, gives the total surface area $S = Ph + 2B$.

A cylinder is like a prism whose base is a circle. Peel the label off a soup can and flatten it: it is a rectangle. Its height is the height h of the can, and its width is the distance once around the can, the circumference $2\pi r$ (Lesson 11.4). So the lateral area of a right cylinder is $L = 2\pi rh$, and adding the two circular bases of area πr^2 (Lesson 11.5) gives $S = 2\pi rh + 2\pi r^2$. Notice that this is the prism formula $S = Ph + 2B$ again, with $P = 2\pi r$ and $B = \pi r^2$.

Real problems rarely want the whole surface. A barn's walls are painted but not its floor; an open box has no lid; a can's label covers only the curved side. Before you compute, make a list of exactly which surfaces the problem needs, find the area of each, and add. Keep the units the same throughout: change inches to feet (or feet to inches) *before* you multiply.



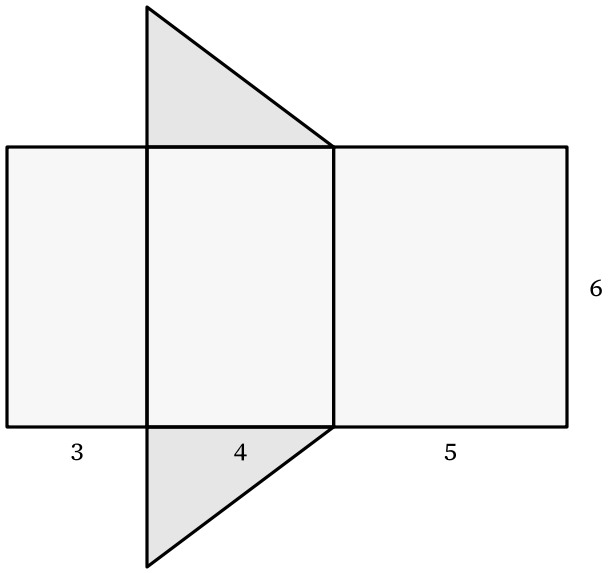
Net of a right triangular prism: the three lateral faces form one rectangle 3 + 4 + 5 = 12 wide and 6 high.



Net of a cylinder

THEOREM Surface Area of a Right Prism

A right prism with base perimeter P, base area B and height h has lateral area $L = Ph$ and total surface area $S = Ph + 2B$.



Net of a right triangular prism: the three lateral faces form one rectangle 3 + 4 + 5 = 12 wide and 6 high.

PARAGRAPH PROOF

Given: A right prism whose base is a polygon with sides $s_1, s_2, \ldots, s_n$, perimeter P and area B; the height of the prism is h

Prove: $L = Ph$ and $S = Ph + 2B$

Because the prism is right, its lateral edges are perpendicular to the bases, so each lateral face is a rectangle (definition of right prism). Each lateral face has one side that is a side of the base and an adjacent side that is a lateral edge of length h, so the faces have areas $s_1h, s_2h, \ldots, s_nh$ (area of a rectangle).

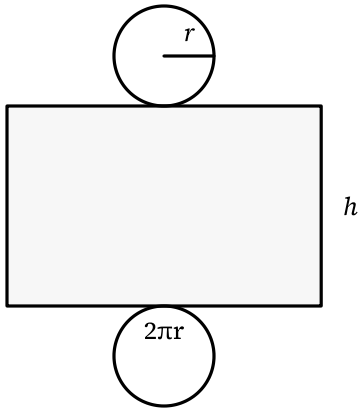
By the Area Addition Postulate, $L = s_1h + s_2h + \cdots + s_nh$. By the Distributive Property, $L = (s_1 + s_2 + \cdots + s_n)h$, and since the sum of the sides is the perimeter (definition of perimeter), $L = Ph$ by substitution.

The whole surface is made of the lateral faces and the two bases, which are congruent and so each have area B. By the Area Addition Postulate, $S = Ph + B + B = Ph + 2B$.

∎

THEOREM **Surface Area of a Right Cylinder**

A right cylinder with radius r and height h has lateral area $L = 2\pi rh$ and total surface area $S = 2\pi rh + 2\pi r^2$.



Net of a cylinder

PARAGRAPH PROOF

Given: A right cylinder with radius r and height h

Prove: $L = 2\pi rh$ and $S = 2\pi rh + 2\pi r^2$

The lateral area of a cylinder is the area of the rectangle its curved surface unrolls into (this is how lateral area of a cylinder is defined, just as a label peels off a can). One side of that rectangle is a segment of the curved surface perpendicular to the bases, so its length is the height h. The other side wraps exactly once around the base circle, so its length is the circumference $2\pi r$ (circumference formula, Lesson 11.4). By the area of a rectangle, $L = 2\pi r \cdot h = 2\pi rh$.

Each base is a circle of radius r with area πr^2 (area of a circle, Lesson 11.5). By the Area Addition Postulate, $S = 2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2$.

∎

KEY FORMULAS

$$L = Ph \qquad S = Ph + 2B \quad \text{(right prism)}$$

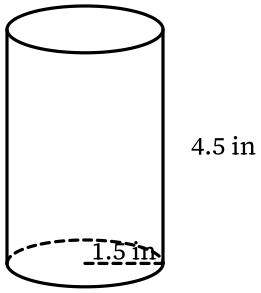
$$L = 2\pi rh \qquad S = 2\pi rh + 2\pi r^2 \quad \text{(right cylinder)}$$

From history. Euclid's Elements measures flat areas and compares volumes, but it never finds the area of a curved surface. That step was taken by Archimedes of Syracuse (about 250 BC). In his book On the Sphere and Cylinder he proved that the curved surface of a cylinder equals a circle whose radius is the mean proportional between the cylinder's height and the diameter of its base. In modern symbols that circle has area $\pi \cdot h \cdot 2r = 2\pi rh$, the same lateral area you find by peeling a label off a can.

Euclid connection. Euclid defines a prism in Book XI (Definition 13) as a solid bounded by planes, two of which are equal, similar and parallel, while the rest are parallelograms, and a cylinder (Definition 21) as the solid swept out by a rectangle turning about one side. Our surface area formulas rest on the area facts of Book I and Lesson 11.1, added face by face.

Practice habit. Sketch the net and write the dimensions on every face before computing anything, and list which surfaces the problem really needs. Label every answer with square units, and convert all measurements to the same unit before multiplying.

Example 1. A soup can has a diameter of 3 inches and a height of 4.5 inches. The paper label covers the curved side of the can exactly, with no overlap. Find the area of the label in terms of π and to the nearest tenth of a square inch.

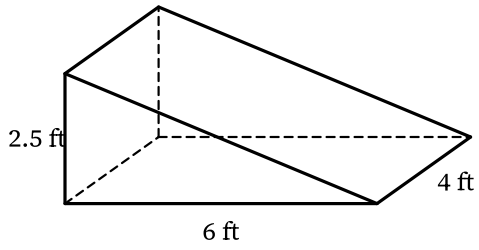


Show solution

1. The label is the lateral surface of a cylinder, so use $L = 2\pi rh$.
2. The radius is half the diameter: $r = 3 \div 2 = 1.5$ in. The height is $h = 4.5$ in.
3. $L = 2\pi(1.5)(4.5) = 13.5\pi \approx 42.4$ square inches.

Answer: $13.5\pi \approx 42.4$ square inches

Example 2. Henry builds a wooden loading ramp for the barn shaped like a triangular prism. Each triangular end is a right triangle with a horizontal leg of 6 feet and a vertical leg of 2.5 feet, and the ramp is 4 feet wide. He will paint every surface except the bottom, which rests on the ground. How many square feet will he paint?



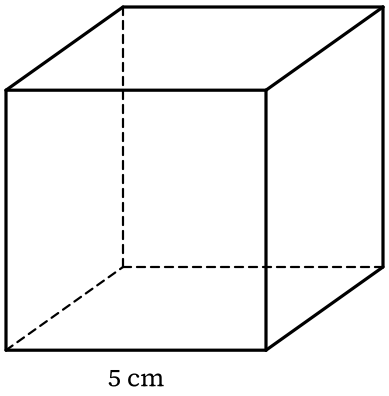
Show solution

1. Find the missing side of the triangle (the sloping top) with the Pythagorean theorem: $\sqrt{6^2 + 2.5^2} = \sqrt{36 + 6.25} = \sqrt{42.25} = 6.5$ ft.
2. The two triangular ends: $2 \times \frac{1}{2}(6)(2.5) = 15$ square feet.
3. The rectangles are 4 ft wide: the vertical back is $2.5 \times 4 = 10$, the sloping top is $6.5 \times 4 = 26$, and the bottom is $6 \times 4 = 24$ square feet. The bottom is not painted.
4. Painted area $= 15 + 10 + 26 = 51$ square feet. (Check: the whole surface is $15 + (6 + 2.5 + 6.5)(4) = 15 + 60 = 75$, and $75 - 24 = 51$.)

Answer: 51 square feet

PRACTICE

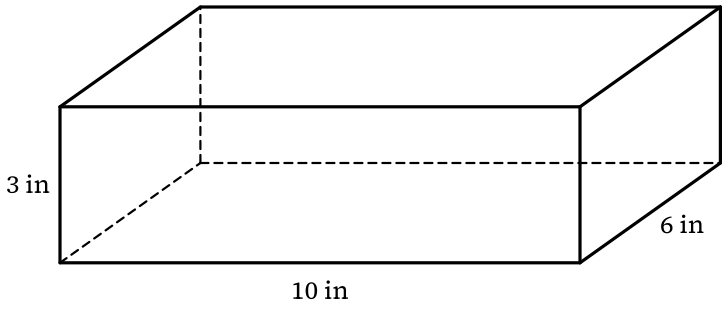
1. **Easy** Find the surface area of a cube with edges 5 cm long.



Show answer

$6 \times 5^2 = 150$ square centimeters

2. **Easy** Find the total surface area of a rectangular box 10 inches long, 6 inches wide and 3 inches high.



Show answer

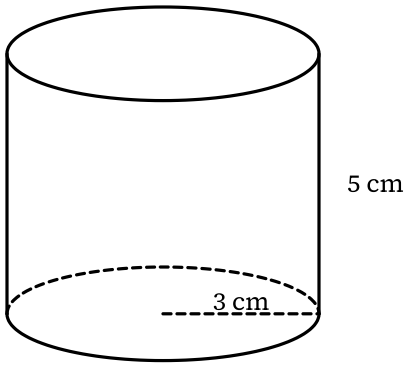
$2(10 \cdot 6 + 10 \cdot 3 + 6 \cdot 3) = 2(60 + 30 + 18) = 216$ square inches

3. **Easy** Find the lateral area of a right cylinder with radius 4 m and height 9 m. Give your answer in terms of π.

Show answer

$L = 2\pi(4)(9) = 72\pi$ square meters

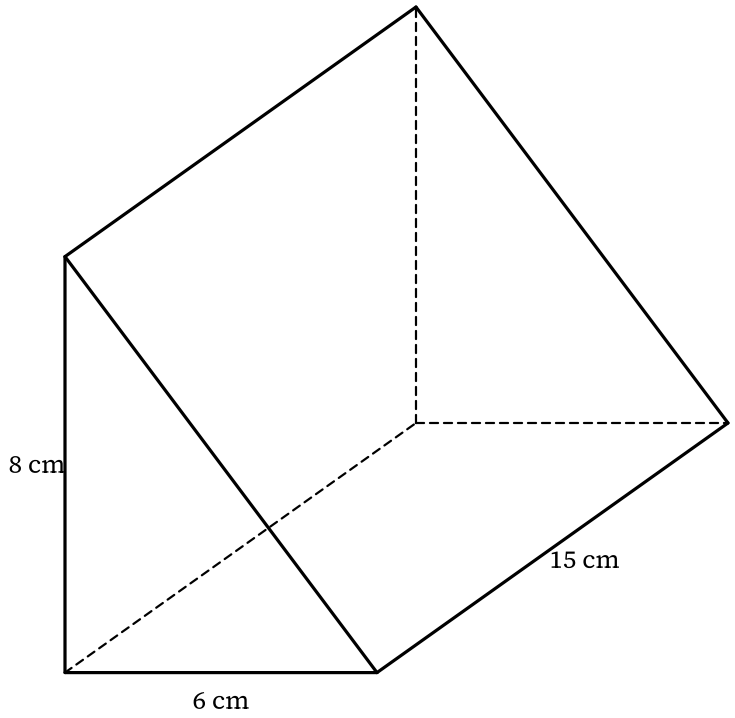
4. **Easy** Find the total surface area of the right cylinder in the figure, with radius 3 cm and height 5 cm. Give your answer in terms of π.



Show answer

$S = 2\pi(3)(5) + 2\pi(3)^2 = 30\pi + 18\pi = 48\pi$ square centimeters

5. **Medium** The bases of the right prism in the figure are right triangles with legs 6 cm and 8 cm, and the prism is 15 cm long. Find its total surface area.



Show answer

The hypotenuse is $\sqrt{36+64} = 10$ cm. $L = (6+8+10)(15) = 360$; $B = \frac{1}{2}(6)(8) = 24$. $S = 360 + 2(24) = 408$ square centimeters.

6. **Medium** Ruth is wrapping a gift box that is 14 inches long, 10 inches wide and 4 inches high. Ignoring overlap, what is the least amount of wrapping paper that will cover it?

Show answer

$2(14 \cdot 10 + 14 \cdot 4 + 10 \cdot 4) = 2(140 + 56 + 40) = 472$ square inches

7. **Medium** A can of peaches has a diameter of 8 cm and a height of 11 cm. The label covers the curved side exactly. Find the area of the label to the nearest tenth of a square centimeter.

Show answer

$L = 2\pi(4)(11) = 88\pi \approx 276.5$ square centimeters

8. **Medium** A right cylinder has radius 5 m and total surface area 120π square meters. Find its height.

Show answer

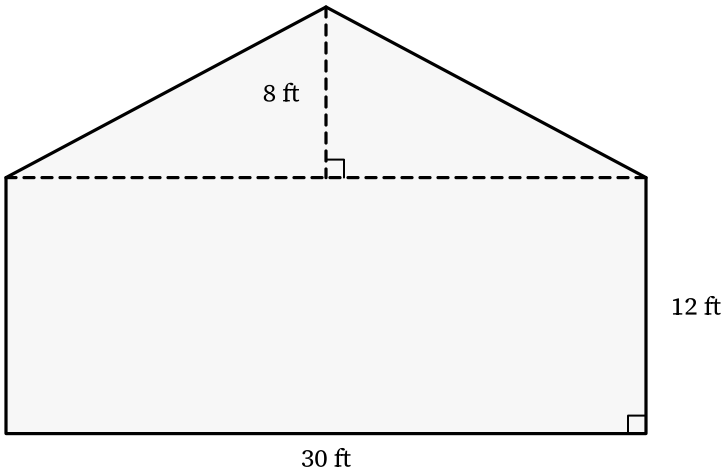
$2\pi(5)h + 2\pi(5)^2 = 120\pi$, so $10\pi h + 50\pi = 120\pi$, $10h = 70$, $h = 7$ m.

9. **Medium** Two right prisms are each 10 inches tall. One has a 5-inch by 5-inch square base; the other has an 8-inch by 2-inch rectangular base. Find the lateral area of each, and explain why they are equal even though the bases have different shapes.

Show answer

Both lateral areas are 200 square inches: $(5+5+5+5)(10) = 200$ and $(8+2+8+2)(10) = 200$. Unfolded, the lateral faces of each prism form one rectangle 10 in high whose width is the base perimeter. Both perimeters are 20 in, so both rectangles are 20 in by 10 in. Lateral area depends only on the perimeter and the height ($L = Ph$), not on the shape of the base.

10. Hard The Miller family's barn is 40 ft long and 30 ft wide, with side walls 12 ft high. At each end, a triangular gable rises 8 ft above the top of the walls, as in the end-wall figure. They will paint all four outside walls, including both gables, but not the roof. Ignore doors and windows. Find the area to be painted. If one gallon of paint covers 350 square feet, how many whole gallons must they buy?



End wall of the barn

Show answer

Side walls: $2(40 \times 12) = 960$. End walls: $2(30 \times 12 + \frac{1}{2} \times 30 \times 8) = 2(360 + 120) = 960$. Total 1,920 square feet. $1920 \div 350 \approx 5.49$, so they must buy 6 gallons.

11. Hard A right prism has a square base with edge x inches and a height of $2x$ inches. Its total surface area is 160 square inches. Find x.

Show answer

$S = 2x^2 + 4(x)(2x) = 2x^2 + 8x^2 = 10x^2 = 160$, so $x^2 = 16$ and $x = 4$ inches (the negative root is not a length).

12. Hard A wooden cube 4 inches on each edge has a hole 2 inches in diameter drilled straight through it, from the center of the top face to the center of the bottom face. Find the total surface area of the block, including the inside of the hole, to the nearest tenth of a square inch.

Show answer

Cube: $6 \times 16 = 96$. Remove two circles: $2\pi(1)^2 = 2\pi$. Add the inside wall of the hole, a cylinder's lateral area: $2\pi(1)(4) = 8\pi$. $S = 96 - 2\pi + 8\pi = 96 + 6\pi \approx 114.8$ square inches.

13. Challenge Prove that if both the radius and the height of a right cylinder are doubled, its total surface area is multiplied by 4. Let the first cylinder have radius r, height h and surface area S_1, and the second have radius $2r$, height $2h$ and surface area S_2.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Cylinder 1 has radius r, height h and surface area S_1; Cylinder 2 has radius $2r$, height $2h$ and surface area S_2

Prove: $S_2 = 4S_1$

	STATEMENTS	REASONS
1	$S_1 = 2\pi rh + 2\pi r^2$	Surface Area of a Right Cylinder
2	$S_2 = 2\pi(2r)(2h) + 2\pi(2r)^2$	Surface Area of a Right Cylinder
3	$S_2 = 8\pi rh + 8\pi r^2$	Properties of multiplication and exponents
4	$S_2 = 4(2\pi rh + 2\pi r^2)$	Distributive Property
5	$S_2 = 4S_1$	Substitution Property of Equality (row 1 into row 4)

∎

COMMON MISTAKES

- Using the diameter where the formula needs the radius. If a problem gives the diameter, divide by 2 first.

- Counting surfaces the problem does not want, or forgetting ones it does. Before computing, list the surfaces: an open-top box has one base, a label has none, a painted barn has no floor.
- Mixing units, such as a slab measured in feet and a thickness in inches. Convert everything to one unit before you multiply.
- Writing surface area in cubic units. Surface area is an area, so its units are square units.

> **Make it a game · Wrapping Paper Duel.** Choose a box and a can. Guess how much paper would wrap the box and how much covers the can's side, then measure and compute. For the can, unroll the label: its length is the circumference and its width is the height. Closest guess to the computed area scores. Explain L = 2 pi r h.

12.3 Surface Area of Pyramids and Cones

OBJECTIVES

- Find the slant height of a regular pyramid or a right cone with the Pythagorean theorem.
- Find the lateral area and total surface area of a regular pyramid, and explain why $L = \frac{1}{2}P\ell$.
- Find the lateral area and total surface area of a right cone, and explain why $L = \pi r\ell$ using its net.
- Solve practical problems about steeples, roofs and tents.

WATCH

Surface Area of a Cone (2 of 2: Deriving the formula)

Eddie Woo · 9:04

https://www.youtube.com/watch?v=yMph_c5bvNY

> **WHEN TO WATCH**
> After the cone formula: deriving L = πrℓ from the net.

BACKUP VIDEO

Surface Area of a Pyramid (1 of 2: Visualising the surfaces)

Eddie Woo · 4:33

https://www.youtube.com/watch?v=UF6j_D45Cj0

Before the pyramid examples: seeing every face of a pyramid.

KEY TERMS

regular pyramid

A pyramid whose base is a regular polygon and whose apex lies directly above the center of the base. Its lateral faces are congruent isosceles triangles.

height of a pyramid or cone

The perpendicular distance from the apex to the plane of the base.

slant height of a regular pyramid

The height of one lateral face: the length of the segment from the apex perpendicular to a side of the base. It is written ℓ.

lateral edge

An edge of a pyramid that joins the apex to a vertex of the base. It is longer than the slant height.

right cone

A cone whose apex lies directly above the center of its circular base.

slant height of a right cone

The distance ℓ from the apex to any point of the circle at the edge of the base.

CORE IDEAS

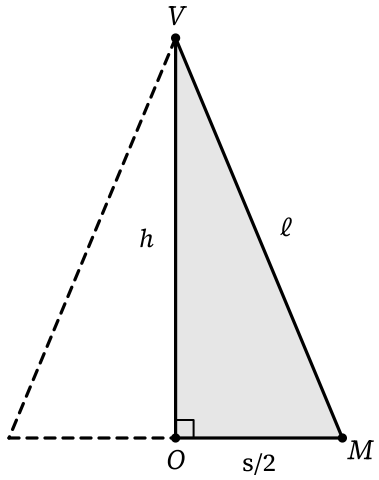
A pyramid's lateral faces are triangles, so their areas need a base and a height. The base of each face is a side of the pyramid's base. The height of each face is called the **slant height** ℓ, because it slants up the face from the middle of a base edge to the apex. Do not confuse it with the height h of the pyramid, which goes straight down from the apex to the center of the base.

The height, the slant height and the base are tied together by a right triangle hidden inside the pyramid (see the first figure). For a square pyramid with base edge s, the right triangle has the height h as one leg, half the base edge $\frac{1}{2}s$ as the other leg, and the slant height ℓ as the hypotenuse. So $\ell^2 = h^2 + \left(\frac{1}{2}s\right)^2$ by the Pythagorean theorem. Since the hypotenuse is the longest side, the slant height is always longer than the height.

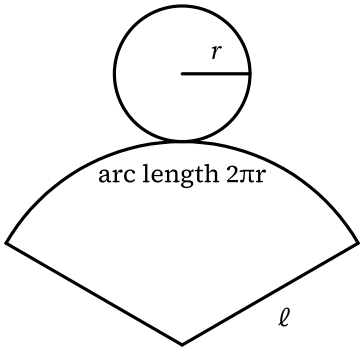
Why is the lateral area of a regular pyramid half the perimeter times the slant height? A regular pyramid with an n-gon base has n congruent triangular faces, each with base s and height ℓ, so each has area $\frac{1}{2}s\ell$. Together they have area $n \cdot \frac{1}{2}s\ell = \frac{1}{2}(ns)\ell = \frac{1}{2}P\ell$, because ns is the perimeter. Add the base for the total: $S = \frac{1}{2}P\ell + B$.

A cone is like a pyramid with a circular base. Cut a paper party hat straight up one side and flatten it, and you get a **sector** of a circle (the second figure). The sector's radius is the cone's slant height ℓ, and its arc is the edge of the hat, which wrapped once around the base circle, so the arc length is $2\pi r$. A full circle of radius ℓ has circumference $2\pi\ell$ and area $\pi\ell^2$; the sector is the fraction $\dfrac{2\pi r}{2\pi\ell} = \dfrac{r}{\ell}$ of that circle. So the lateral area is $\dfrac{r}{\ell} \cdot \pi\ell^2 = \pi r\ell$.

Compare the cone with the pyramid: $\frac{1}{2}P\ell$ with $P = 2\pi r$ gives $\frac{1}{2}(2\pi r)\ell = \pi r\ell$, the same formula. The total surface area of a cone adds the circular base: $S = \pi r\ell + \pi r^2$. For both solids, the step students most often skip is finding ℓ first. If a problem gives the height, draw the right triangle and find the slant height before using any area formula.



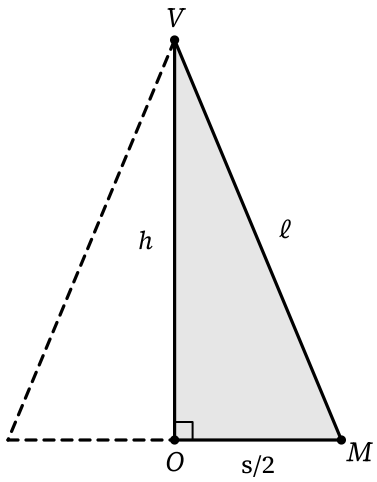
Inside a square pyramid: height h, half the base edge, slant height ℓ



Net of a right cone: a sector of radius ℓ and a circle of radius r

THEOREM Surface Area of a Regular Pyramid

A regular pyramid with base perimeter P, base area B and slant height ℓ has lateral area $L = \frac{1}{2}P\ell$ and total surface area $S = \frac{1}{2}P\ell + B$.



Inside a square pyramid: height h, half the base edge, slant height ℓ

PARAGRAPH PROOF

Given: A regular pyramid with apex V whose base is a regular n-gon with center O, side s, perimeter P and area B; slant height ℓ

Prove: $L = \frac{1}{2}P\ell$ and $S = \frac{1}{2}P\ell + B$

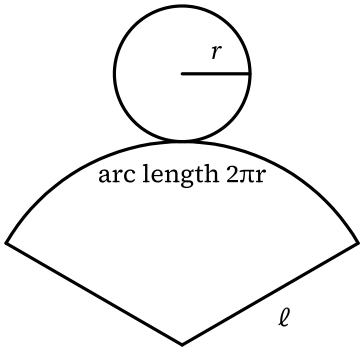
First, the lateral faces are congruent. Since V lies directly above O, $\overline{VO}$ is perpendicular to the base, so for every base vertex A, $\triangle VOA$ is a right triangle and $VA^2 = VO^2 + OA^2$ (Pythagorean Theorem). All the radii OA of a regular polygon are equal, so all the lateral edges VA are equal. Every lateral face therefore has two sides equal to the lateral edge and a third side equal to s, so the faces are congruent isosceles triangles (SSS), and they all have the same height, the slant height ℓ.

Each face has area $\frac{1}{2}s\ell$ (area of a triangle). By the Area Addition Postulate, the n faces have total area $L = n \cdot \frac{1}{2}s\ell = \frac{1}{2}(ns)\ell$. Since $ns = P$ (definition of perimeter), $L = \frac{1}{2}P\ell$ by substitution. Adding the base, $S = \frac{1}{2}P\ell + B$ (Area Addition Postulate).

∎

THEOREM **Surface Area of a Right Cone**

A right cone with base radius r and slant height ℓ has lateral area $L = \pi r \ell$ and total surface area $S = \pi r \ell + \pi r^2$.



Net of a right cone: a sector of radius ℓ and a circle of radius r

PARAGRAPH PROOF

Given: A right cone with base radius r and slant height ℓ

Prove: $L = \pi r \ell$ and $S = \pi r \ell + \pi r^2$

Cut the curved surface along one slant segment and lay it flat. Every point of the base circle is at distance ℓ from the apex (definition of slant height of a right cone), so the flattened surface is a sector of a circle with radius ℓ, and its arc is the base circle unrolled, with length $2\pi r$ (circumference formula).

The area of a sector is the same fraction of the full circle's area as its arc length is of the full circumference (Lesson 11.5). The full circle of radius ℓ has circumference $2\pi\ell$ and area $\pi\ell^2$, so $L = \dfrac{2\pi r}{2\pi\ell} \cdot \pi\ell^2 = \dfrac{r}{\ell} \cdot \pi\ell^2 = \pi r \ell$.

The base is a circle of area πr^2, so by the Area Addition Postulate $S = \pi r \ell + \pi r^2$.

∎

KEY FORMULAS

$$L = \tfrac{1}{2} P \ell \qquad S = \tfrac{1}{2} P \ell + B \quad \text{(regular pyramid)}$$

$$L = \pi r \ell \qquad S = \pi r \ell + \pi r^2 \quad \text{(right cone)}$$

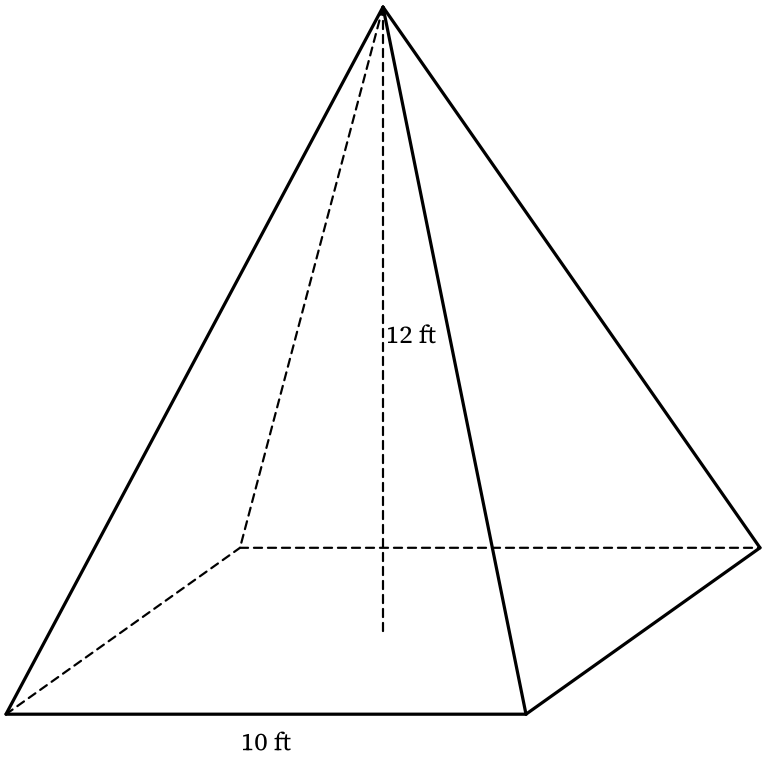
$$\ell^2 = h^2 + \left(\tfrac{1}{2} s\right)^2 \quad \text{(square pyramid)} \qquad \ell^2 = h^2 + r^2 \quad \text{(cone)}$$

From history. Archimedes, in On the Sphere and Cylinder, proved that the curved surface of a right cone equals a circle whose radius is the mean proportional between the cone's side (its slant height) and the radius of its base; in modern symbols that circle's area is $\pi \cdot r\ell$. Many medieval church steeples are tall square or octagonal pyramids, built of stone or of timber covered in slate, lead or wooden shingles. A builder ordering roofing for such a spire faced exactly the problem in this lesson: perimeter of the base, slant height, and half their product.

Euclid connection. Euclid defines a pyramid in Book XI (Definition 12) as a solid figure contained by planes, constructed from one plane to one point, and a cone (Definition 18) as the solid swept out by a right triangle turning about one of its legs. Every slant-height calculation in this lesson is an application of I.47, the Pythagorean theorem.

Practice habit. Before using an area formula, draw the right triangle hidden inside the solid (height, half the base edge or the radius, slant height), label it, and find ℓ first. Recite $L = \frac{1}{2}P\ell$ and $L = \pi r\ell$ until automatic, and label every answer in square units.

Example 1. A church steeple is a regular square pyramid with base edges 10 ft long and a height of 12 ft. How many square feet of shingles are needed to cover its four triangular faces?

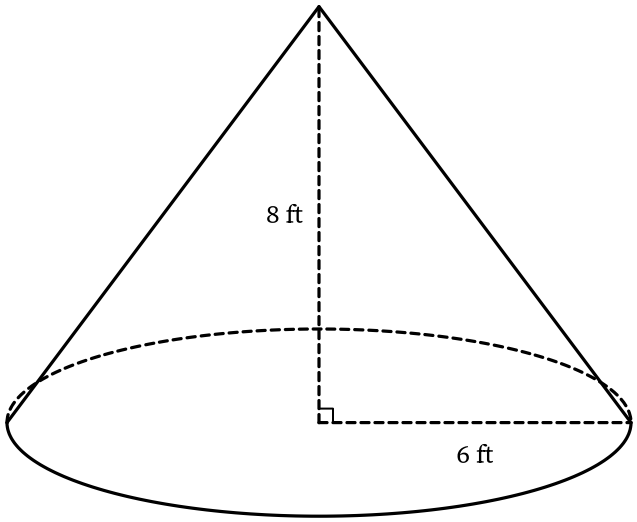


Show solution

1. The shingles cover the lateral area, $L = \frac{1}{2}P\ell$, so we need the slant height first.
2. The right triangle inside the pyramid has legs $h = 12$ and $\frac{1}{2}s = 5$, so $\ell = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$ ft.
3. The perimeter of the base is $P = 4 \times 10 = 40$ ft.
4. $L = \frac{1}{2}(40)(13) = 260$ square feet.

Answer: 260 square feet

Example 2. A round stone tower has a conical roof with base radius 6 ft and height 8 ft. Find the area of the roof to the nearest tenth of a square foot. Roofing shingles come in bundles that each cover $33\frac{1}{3}$ square feet. How many whole bundles must be bought?



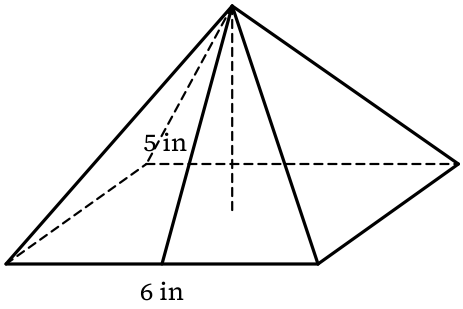
Show solution

1. The roof is the lateral surface of a cone, $L = \pi r \ell$. Find ℓ first: $\ell = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$ ft.
2. $L = \pi(6)(10) = 60\pi \approx 188.5$ square feet.
3. Bundles: $188.5 \div 33\frac{1}{3} \approx 5.65$. Five bundles are not enough, so buy 6 bundles.

Answer: $60\pi \approx 188.5$ square feet; 6 bundles

PRACTICE

1. **Easy** A regular square pyramid has base edges 6 in long and slant height 5 in. Find its total surface area.



Show answer

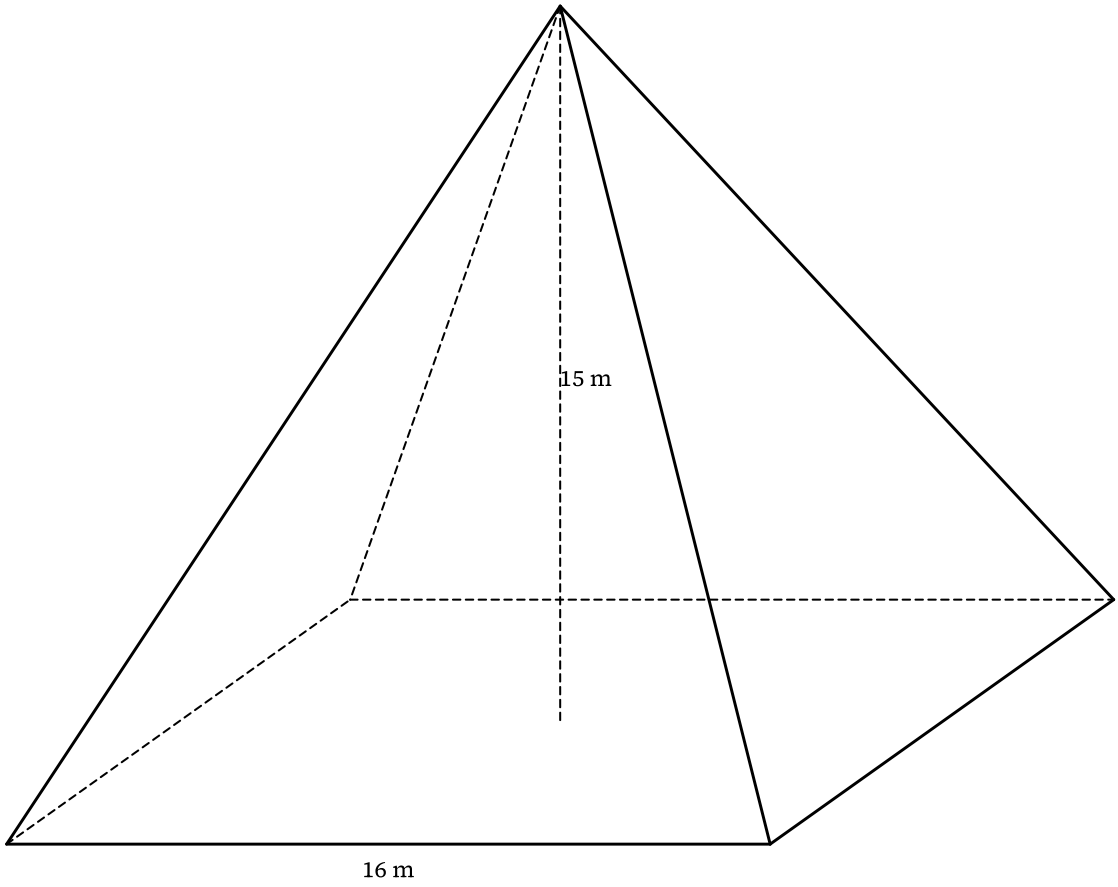
$L = \frac{1}{2}(24)(5) = 60$; $B = 36$; $S = 96$ square inches.

2. **Easy** A right cone has radius 3 cm and slant height 7 cm. Find its lateral area in terms of π.

Show answer

$L = \pi(3)(7) = 21\pi$ square centimeters

3. **Easy** A regular square pyramid has base edges 16 m long and height 15 m. Find its slant height.



Show answer

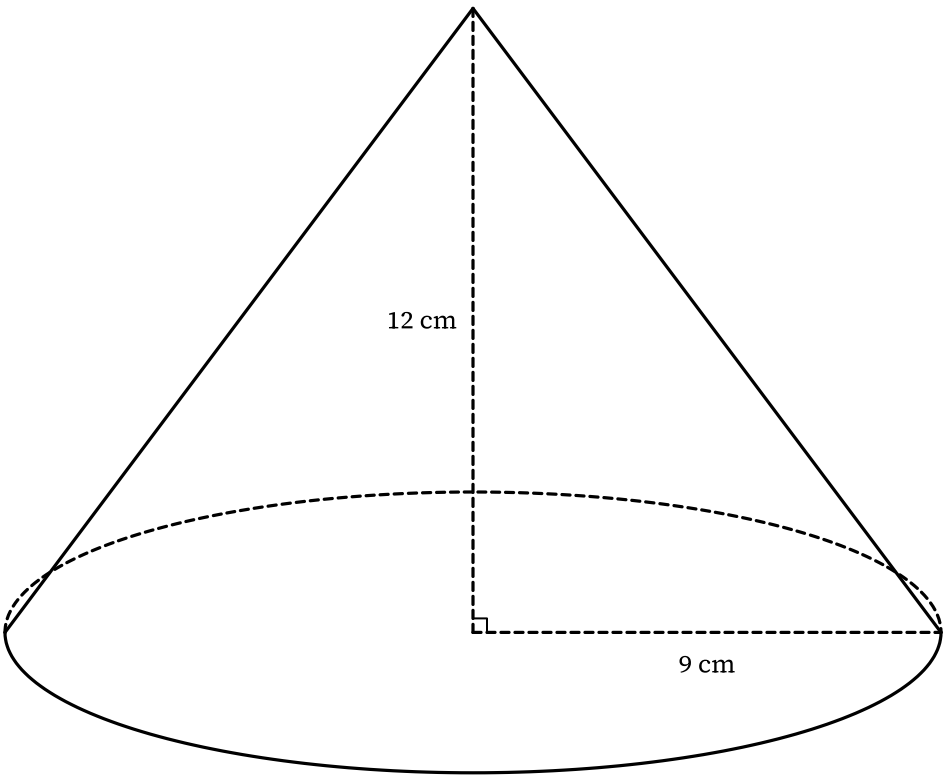
$\ell = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ m

4. **Easy** A right cone has radius 5 in and slant height 12 in. Find its total surface area in terms of π.

Show answer

$S = \pi(5)(12) + \pi(5)^2 = 60\pi + 25\pi = 85\pi$ square inches

5. **Medium** Find the total surface area of the right cone in the figure, with radius 9 cm and height 12 cm. Give your answer in terms of π.



Show answer

$\ell = \sqrt{144 + 81} = 15$. $S = \pi(9)(15) + \pi(9)^2 = 135\pi + 81\pi = 216\pi$ square centimeters.

6. **Medium** A chapel steeple is a regular hexagonal pyramid. Each edge of the hexagonal base is 4 ft long, and the slant height of each face is 15 ft. How many square feet of copper sheeting cover the six faces?

Show answer

$P = 6 \times 4 = 24$ ft, so $L = \frac{1}{2}(24)(15) = 180$ square feet.

7. **Medium** Grace's family pitches a conical tent with a floor diameter of 10 ft and a center pole 12 ft tall. How much canvas covers the sloping sides (not the floor), to the nearest tenth of a square foot?

   Show answer

   $r = 5$, $\ell = \sqrt{144 + 25} = 13$. $L = \pi(5)(13) = 65\pi \approx 204.2$ square feet.

8. **Medium** A right cone has radius 4 cm and lateral area 36π square centimeters. Find its slant height, and then its height in simplest radical form.

   Show answer

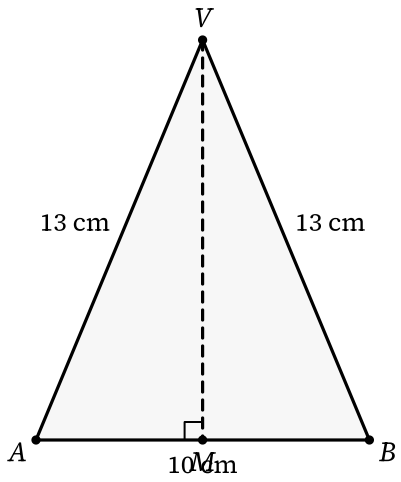
   $\pi(4)\ell = 36\pi$, so $\ell = 9$ cm. $h = \sqrt{9^2 - 4^2} = \sqrt{65}$ cm.

9. **Medium** Explain why the slant height of a right cone is always longer than its height.

   Show answer

   The height, the radius and the slant height form a right triangle with the right angle at the center of the base: the height and radius are the legs, and the slant height is the hypotenuse. By the Pythagorean theorem $\ell^2 = h^2 + r^2 > h^2$ (since $r > 0$), so $\ell > h$. The hypotenuse is always the longest side of a right triangle.

10. Hard A regular square pyramid has base edges 10 cm long, and each lateral edge is 13 cm long. The figure shows one lateral face. Find the total surface area of the pyramid.

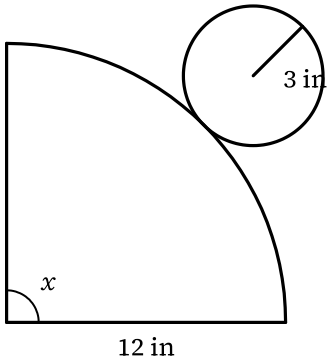


One lateral face

Show answer

The slant height bisects the base of the isosceles face, so $\ell = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$ cm. $L = \frac{1}{2}(40)(12) = 240$; $B = 100$; $S = 340$ square centimeters.

11. Hard A right cone has base radius 3 in and slant height 12 in. Its curved surface is cut open and laid flat into the sector in the figure. Find the central angle x of the sector.



Show answer

The arc has length $2\pi(3) = 6\pi$; the full circle of radius 12 has circumference 24π. The sector is $\dfrac{6\pi}{24\pi} = \dfrac{1}{4}$ of the circle, so $x = \frac{1}{4} \times 360^\circ = 90^\circ$.

12. Hard A round farm tower has a cylindrical wall 10 ft in diameter and 20 ft tall, topped by a conical roof 12 ft high. Find the total outside area of the wall and roof (no floor), to the nearest square foot.

Show answer

Wall: $2\pi(5)(20) = 200\pi$. Roof: $\ell = \sqrt{144 + 25} = 13$, so $\pi(5)(13) = 65\pi$. Total $265\pi \approx 833$ square feet.

13. **Challenge** The curved surface of a right cone, cut open and laid flat, is exactly a semicircle. The cone has base radius r, slant height ℓ and lateral area L. Prove that $\ell = 2r$ and $L = 2\pi r^2$ (so the curved surface has twice the area of the base).

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: The net of the curved surface of a right cone is a semicircle of radius ℓ; The cone has base radius r and lateral area L

Prove: $\ell = 2r$ and $L = 2\pi r^2$

	STATEMENTS	REASONS
1	The arc of the semicircle has length $\frac{1}{2}(2\pi\ell) = \pi\ell$	Arc length is the same fraction of the circumference as the arc is of the circle (Lesson 11.4)
2	The arc of the net has length $2\pi r$	The arc of the net wraps exactly once around the base circle (circumference formula)
3	$\pi\ell = 2\pi r$	Transitive Property of Equality
4	$\ell = 2r$	Division Property of Equality
5	$L = \pi r\ell$	Surface Area of a Right Cone
6	$L = \pi r(2r) = 2\pi r^2$	Substitution Property of Equality
7	$\ell = 2r$ and $L = 2\pi r^2$	Rows 4 and 6

∎

14. **Challenge** A lampshade is the curved side of a cone with its top cut off by a plane parallel to the base (a frustum). The top circle has radius 4 in, the bottom circle has radius 6 in, and the slant height along the shade is 5 in. Find the area of fabric on the shade to the nearest tenth of a square inch. (Hint: extend the shade to a full cone, and let x be the slant height of the small cone that was cut off.)

Show answer

A cut through the axis shows two similar right triangles (AA), so $\dfrac{x}{x+5} = \dfrac{4}{6}$, giving $6x = 4x + 20$ and $x = 10$. The full cone has $\ell = 15$. Fabric $= \pi(6)(15) - \pi(4)(10) = 90\pi - 40\pi = 50\pi \approx 157.1$ square inches.

COMMON MISTAKES

- Using the height h instead of the slant height ℓ in $L = \frac{1}{2}P\ell$ or $L = \pi r\ell$. The lateral area formulas need the slant height; find it from the right triangle first.
- Using the whole base edge s instead of half of it when finding the slant height of a square pyramid. The leg of the hidden right triangle runs from the center of the base to the middle of an edge, a distance of $\frac{1}{2}s$.
- Confusing the lateral edge with the slant height. The lateral edge goes to a corner of the base; the slant height goes to the midpoint of a base edge, and it is shorter.
- Forgetting the base in a total surface area, or including it when the problem (a roof, a tent, a lampshade) has no base.

> **Make it a game · Predict and Measure.** Roll a paper cone from a circle sector and measure its slant height with a ruler. Then predict the height using the Pythagorean theorem and the base radius, and measure to check. Next, parent gives a square pyramid with base 6 and height 4; find its slant height and lateral area. Closest prediction wins a point.

12.4 Volume of Prisms and Cylinders

OBJECTIVES

- Explain volume as the number of unit cubes that fill a solid, and use the Volume Postulates.
- State Cavalieri's Principle and use it to explain why $V = Bh$ holds for right and oblique prisms and cylinders.
- Find volumes of prisms, cylinders and composite solids, converting between cubic feet and cubic yards.
- Solve practical problems about tanks, grain bins and concrete.

WATCH

How do we measure the volume of a prism?

Eddie Woo · 10:59

https://www.youtube.com/watch?v=Ci2IBKgf4BA

> **WHEN TO WATCH**
> Before the Core ideas: volume as layers of unit cubes.

Note: Eddie Woo teaches in Australia: he says 'cross-sectional area' for base area. The backup uses Cavalieri's idea without naming it; the principle itself is taught from the Core ideas.

BACKUP VIDEO

Volume of an Oblique Prism - How to Find (Formula)

Mario's Math Tutoring · 3:25

https://www.youtube.com/watch?v=sHCiT2soUmI

Before the oblique solids: why a slanted stack of coins has the same volume as a straight one, then V = Bh for an oblique prism.

KEY TERMS

volume

The amount of space a solid fills, measured in cubic units: the number of unit cubes (cubes with edges 1 unit long) it would take to fill it.

Volume Postulates

Every solid has a unique positive volume; congruent solids have equal volumes; the volume of a solid is the sum of the volumes of nonoverlapping parts; a rectangular box with length ℓ, width w and height h has volume $\ell w h$.

Cavalieri's Principle

If two solids have the same height and the cross sections at every level have equal areas, then the two solids have equal volumes.

oblique prism or cylinder

A prism or cylinder that leans: its lateral edges (or its axis) are not perpendicular to the bases. Its height is still the perpendicular distance between the bases.

composite solid

A solid made of two or more simpler solids joined together or with one removed from another.

cubic yard

The volume of a cube 1 yard (3 feet) on each edge: $1\text{ yd}^3 = 3 \times 3 \times 3 = 27\text{ ft}^3$. Concrete, gravel and soil are sold by the cubic yard.

CORE IDEAS

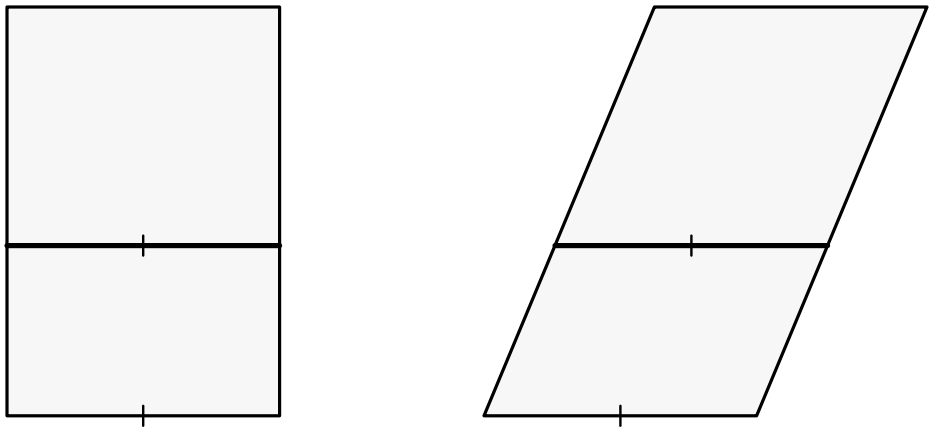
Area counts unit squares; **volume** counts unit cubes. A box 5 cm long, 4 cm wide and 3 cm high holds one layer of $5 \times 4 = 20$ centimeter cubes on its floor, and 3 such layers stack to the top, so it holds $20 \times 3 = 60$ cubes: its volume is 60 cubic centimeters (cm^3). The idea of the **Volume Postulates** is the same as the area postulates of Lesson 11.1: congruent solids have equal volumes, volumes of nonoverlapping pieces add, and a box has volume $\ell w h$.

Notice what the counting really did: (number of cubes in one layer) $\times$ (number of layers). The first factor is the area of the base, B, and the second is the height, h. So for a box, $V = Bh$. The same layer idea works for any right prism or cylinder, because every horizontal slice is a copy of the base: stack h unit-thick layers, each with volume B, and you get $V = Bh$. For a cylinder the base is a circle, so $V = \pi r^2 h$.

What about a solid that leans? Picture a neat stack of 20 coins, then push it so it leans like a tower of Pisa. Every coin is still the same coin, so the volume has not changed. This is **Cavalieri's Principle**: if two solids have the same height and every cross section at the same level has the same area, they have the same volume. An oblique prism or cylinder has every cross section congruent to its base, just like the right one with the same base and height, so its volume is also $V = Bh$. The height must be the perpendicular height, not the length of the slanted side.

Large volumes are measured in larger cubes. A cubic yard is not 3 cubic feet: a cube 1 yard on each edge is 3 ft by 3 ft by 3 ft, so it holds $3 \times 3 \times 3 = 27$ cubic feet. To change cubic feet to cubic yards, divide by 27. Similarly, $1 \text{ ft}^3 = 12^3 = 1728 \text{ in}^3$. The safest method is to convert every length to the final unit *before* multiplying.

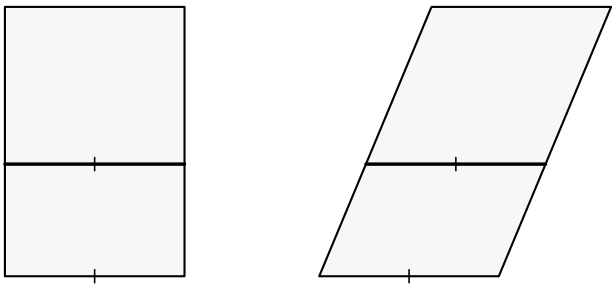
A **composite solid** is built from simpler ones. Find each part's volume and add, or, for a solid with a hole or a missing piece, subtract. A concrete step, an L-shaped barn wing, or a block with a pipe hole drilled through it are all handled this way. Many of these are themselves prisms with an unusual base: find the area of the base polygon (by cutting it into rectangles), then multiply by the length.



Same height, same cross section at every level: same volume

POSTULATE Cavalieri's Principle

If two solids have the same height, and the cross sections made by every plane parallel to the bases have equal areas at equal heights, then the two solids have equal volumes.

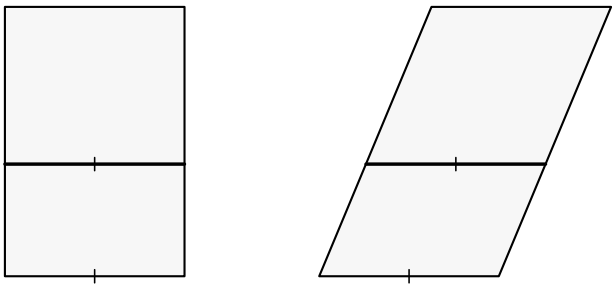


Same height, same cross section at every level: same volume

Accepted as a postulate in this course; a proof requires calculus. Bonaventura Cavalieri published it in 1635.

THEOREM Volume of a Prism or Cylinder

The volume of any prism or cylinder (right or oblique) with base area B and height h is $V = Bh$. For a cylinder with radius r, $V = \pi r^2 h$.



Same height, same cross section at every level: same volume

PARAGRAPH PROOF

Given: A prism or cylinder, right or oblique, with base area B and height h

Prove: $V = Bh$

Every cross section of a prism or cylinder parallel to its bases is congruent to the base (the solid has the same shape at every level), so every such cross section has area B.

Now consider a rectangular box whose base is a rectangle B units long and 1 unit wide, standing on the same plane with height h. Each of its horizontal cross sections is a B-by-1 rectangle, of area B. The two solids have the same height and equal cross-sectional areas at every level, so by Cavalieri's Principle they have equal volumes.

By the Volume Postulate for boxes, the box has volume $B \cdot 1 \cdot h = Bh$. Therefore the prism or cylinder has volume $V = Bh$. For a cylinder the base is a circle of area πr^2, so by substitution $V = \pi r^2 h$.

∎

KEY FORMULAS

$$V = Bh \quad \text{(any prism or cylinder)}$$

$$V = \ell w h \quad \text{(rectangular box)} \qquad V = \pi r^2 h \quad \text{(cylinder)}$$

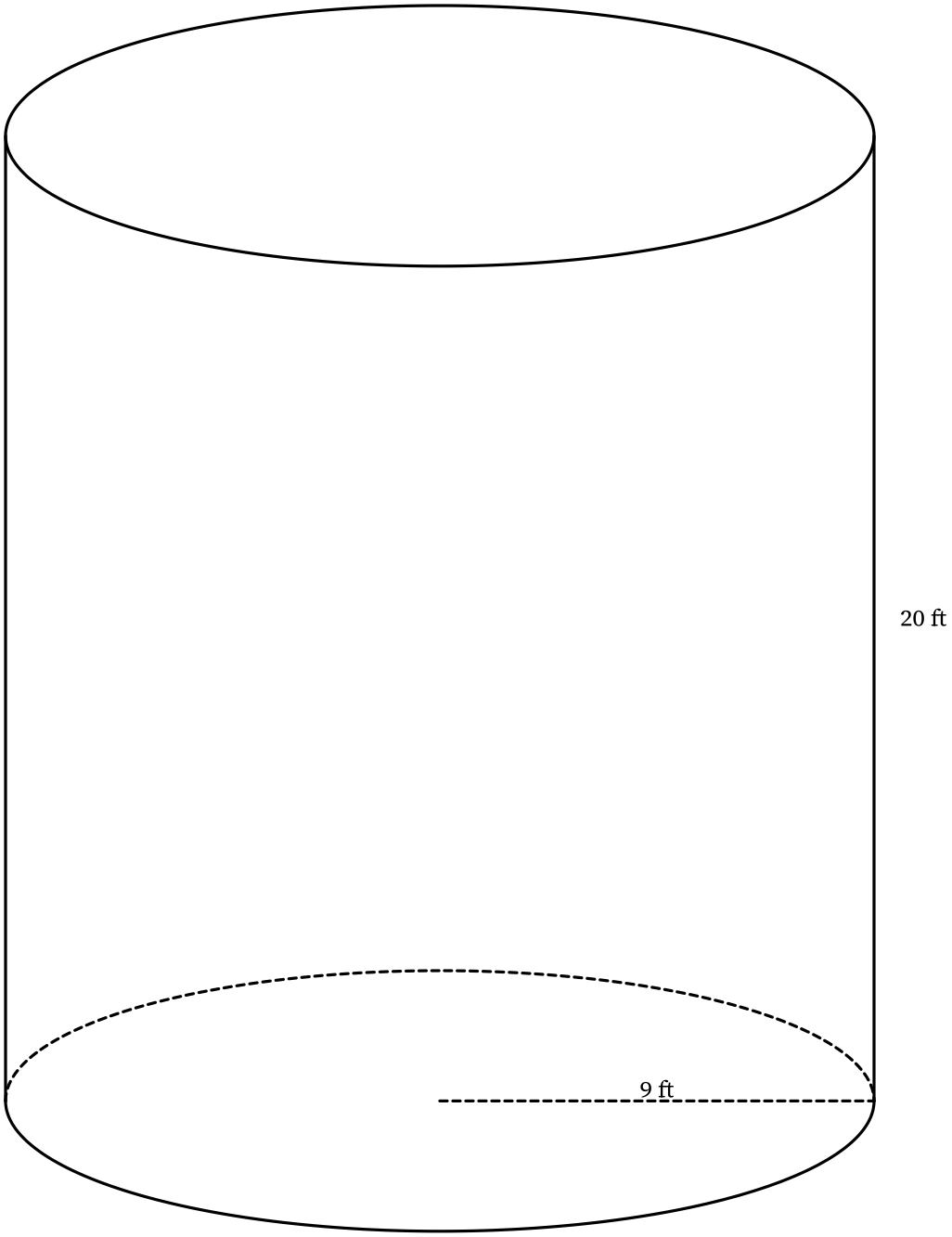
$$1\ \text{yd}^3 = 27\ \text{ft}^3 \qquad 1\ \text{ft}^3 = 1728\ \text{in}^3$$

From history. Bonaventura Cavalieri, an Italian mathematician and a student of Galileo's ideas, published his principle in 1635. He imagined a solid as a stack of infinitely many thin slices, like the pages of a book, and reasoned that two stacks with matching slices must hold the same amount of space. His method of slices was one of the steps that led, later in the same century, to the calculus of Newton and Leibniz.

Euclid connection. Euclid compares volumes rather than computing them with formulas. In Book XI he proves that parallelepipeds (slanted boxes) on equal bases and of the same height are equal to one another (XI.31), the same fact that Cavalieri's Principle gives us, and in Book XII (XII.11) that cones and cylinders of the same height are to one another as their bases.

Practice habit. Write $V = Bh$ first, then find B as a separate step with the right area formula for the base. Convert every length to the unit you want in the answer before you multiply, and label the answer in cubic units.

Example 1. A cylindrical grain bin has a diameter of 18 ft (radius 9 ft) and is filled with corn to a depth of 20 ft. Find the volume of the corn to the nearest cubic foot.



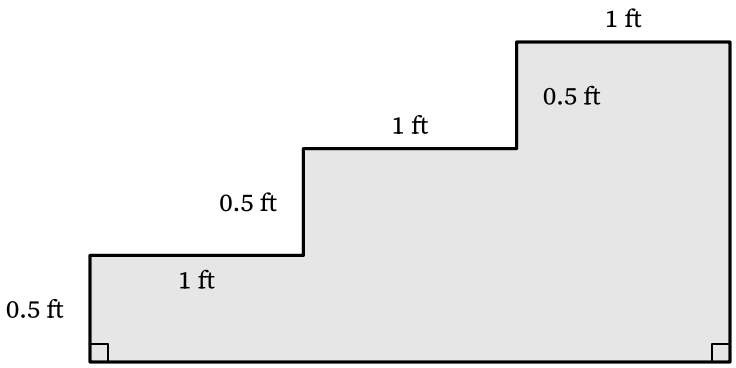
Show solution

1. The corn fills a cylinder, so $V = \pi r^2 h$.

2. $r = 9$ ft and $h = 20$ ft, so $V = \pi(9)^2(20) = \pi(81)(20) = 1620\pi$.
3. $1620\pi \approx 5089.38$, so the bin holds about 5,089 cubic feet of corn.

Answer: $1620\pi \approx 5{,}089$ cubic feet

Example 2. Samuel is pouring a set of three concrete steps that are 5 ft wide. Each step rises 0.5 ft and has a tread 1 ft deep, so the side of the steps has the shape shown in the figure. How many cubic feet of concrete are needed? How many cubic yards is that, to the nearest hundredth?



Side view of the steps (the steps are 5 ft wide)

Show solution

1. The steps are a prism whose base is the stair-shaped side, so $V = Bh$ with $h = 5$ ft (the width of the steps).
2. Cut the side into three vertical columns 1 ft wide: heights 0.5, 1 and 1.5 ft. $B = 1(0.5) + 1(1) + 1(1.5) = 3$ square feet.
3. $V = 3 \times 5 = 15$ cubic feet.
4. Convert: $15 \div 27 \approx 0.56$ cubic yard.

Answer: 15 cubic feet, about 0.56 cubic yard

PRACTICE

1. Easy Find the volume of a rectangular box 8 cm long, 5 cm wide and 3 cm high.

Show answer

$V = 8 \times 5 \times 3 = 120$ cubic centimeters

2. **Easy** Find the volume of a cylinder with radius 2 in and height 10 in. Give your answer in terms of π.

Show answer

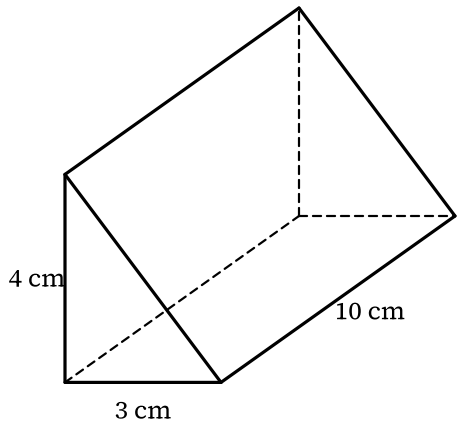
$V = \pi(2)^2(10) = 40\pi$ cubic inches

3. **Easy** How many cubic feet are in one cubic yard?

Show answer

27 cubic feet: a cube 1 yd on each edge is 3 ft by 3 ft by 3 ft, and $3 \times 3 \times 3 = 27$.

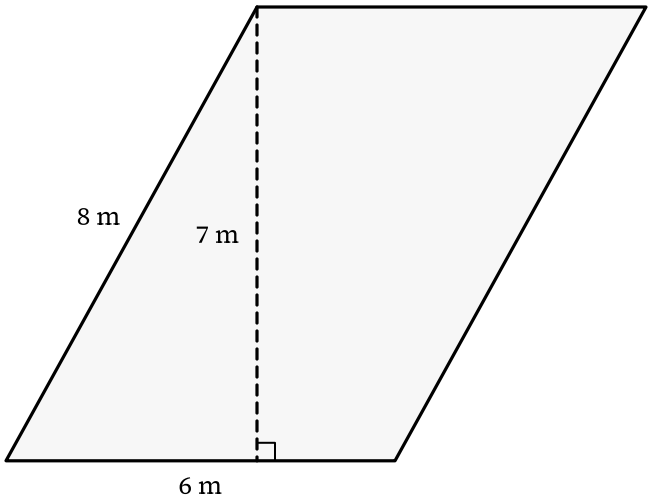
4. **Easy** The bases of the prism in the figure are right triangles with legs 3 cm and 4 cm, and the prism is 10 cm long. Find its volume.



Show answer

$B = \frac{1}{2}(3)(4) = 6$, so $V = 6 \times 10 = 60$ cubic centimeters.

5. **Medium** The figure shows the side view of an oblique cylinder. Its base has diameter 6 m, its slanted side is 8 m long, and its perpendicular height is 7 m. Find its volume in terms of π.



Side view of an oblique cylinder

Show answer

By Cavalieri's Principle use $V = \pi r^2 h$ with the perpendicular height: $V = \pi(3)^2(7) = 63\pi$ cubic meters. (The slanted length 8 m is not the height.)

6. **Medium** An aquarium is a rectangular box 30 in long, 12 in wide and 16 in high. One gallon is 231 cubic inches. How many gallons does the aquarium hold when full, to the nearest tenth?

Show answer

$V = 30 \times 12 \times 16 = 5760$ cubic inches; $5760 \div 231 \approx 24.9$ gallons.

7. **Medium** A cylinder has volume 200π cubic centimeters and radius 5 cm. Find its height.

Show answer

$\pi(5)^2 h = 200\pi$, so $25h = 200$ and $h = 8$ cm.

8. **Medium** Ruth stacks 20 identical quarters in a straight column. Henry stacks 20 more quarters in a column that leans to one side. Explain why the two stacks have the same volume.

Show answer

The stacks have the same height (20 coins thick). At every level, the cross section of each stack is one coin's face, a circle of the same area. By Cavalieri's Principle, solids with equal heights and equal cross-sectional areas at every level have equal volumes. (More simply, both stacks are made of the same 20 congruent coins, so by the Volume Postulates their volumes are equal.)

9. **Medium** A raised garden bed is a box 8 ft long and 4 ft wide, filled with soil to a depth of 10 inches. Find the volume of soil in cubic feet (to the nearest tenth) and in cubic yards (to the nearest hundredth).

Show answer

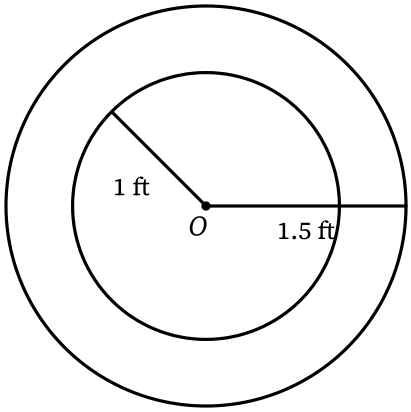
10 in $= \frac{10}{12}$ ft. $V = 8 \times 4 \times \frac{10}{12} \approx 26.7$ cubic feet; $26.\overline{6} \div 27 \approx 0.99$ cubic yard.

10. **Hard** A wooden block is a box 6 in long, 4 in wide and 3 in high. A hole 2 in in diameter is drilled straight down through it, from the top face to the bottom face. Find the volume of wood that remains, to the nearest tenth of a cubic inch.

Show answer

Box: $6 \times 4 \times 3 = 72$. Hole: a cylinder with $r = 1$ and $h = 3$, volume 3π. Remaining: $72 - 3\pi \approx 62.6$ cubic inches.

11. Hard A concrete drainage pipe is 8 ft long, with an outer diameter of 3 ft and an inner diameter of 2 ft. How many cubic feet of concrete does it contain, to the nearest tenth?



End view of the pipe (8 ft long)

Show answer

Outer cylinder minus inner: $\pi(1.5)^2(8) - \pi(1)^2(8) = 18\pi - 8\pi = 10\pi \approx 31.4$ cubic feet.

12. Hard A cylindrical rain barrel is replaced by one whose radius is twice as large but whose height is half as large. By what factor does the volume change?

Show answer

Old: $V = \pi r^2 h$. New: $\pi(2r)^2\left(\frac{1}{2}h\right) = \pi \cdot 4r^2 \cdot \frac{1}{2}h = 2\pi r^2 h$. The volume is multiplied by 2.

13. **Challenge** Two right cylinders have equal volumes. The first has radius $2r$ and height h_1; the second has radius r and height h_2. Prove that $h_2 = 4h_1$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: Cylinder 1 has radius $2r$ and height h_1; Cylinder 2 has radius r and height h_2; The cylinders have equal volumes

Prove: $h_2 = 4h_1$

	STATEMENTS	REASONS
1	$V_1 = \pi(2r)^2 h_1$ and $V_2 = \pi r^2 h_2$	Volume of a Prism or Cylinder
2	$V_1 = V_2$	Given
3	$\pi(2r)^2 h_1 = \pi r^2 h_2$	Substitution Property of Equality
4	$4\pi r^2 h_1 = \pi r^2 h_2$	Properties of exponents and multiplication
5	$4h_1 = h_2$	Division Property of Equality (divide by πr^2, which is not 0)
6	$h_2 = 4h_1$	Symmetric Property of Equality

∎

COMMON MISTAKES

- Using the slanted length of an oblique prism or cylinder as its height. By Cavalieri's Principle the formula $V = Bh$ needs the perpendicular height.
- Dividing cubic feet by 3 (or by 9) to get cubic yards. One cubic yard is $3 \times 3 \times 3 = 27$ cubic feet, so divide by 27.
- Multiplying feet by inches. A slab 12 ft by 15 ft by 4 in must use $\frac{4}{12} = \frac{1}{3}$ ft for the thickness.
- Using the diameter instead of the radius in $\pi r^2 h$, which makes the answer four times too large.

Make it a game · Slant the Stack. Stack twenty index cards into a neat prism, then push the stack sideways into an oblique prism without changing the cards. Measure both heights. Is the volume the same? Explain Cavalieri's Principle in your own words. Then parent names a box size; you find its volume in cubes and state V = Bh.

12.5 Volume of Pyramids and Cones

OBJECTIVES

- Find the volume of a pyramid or cone with $V = \frac{1}{3}Bh$, finding the height from the slant height when needed.
- Explain why a pyramid is one-third of the prism with the same base and height, and use Cavalieri's Principle to prove the cone formula.
- Find volumes of composite solids such as silos, rockets and flowerpots.
- Solve practical problems about gravel piles, funnels and grain storage.

WATCH

Volume of a Pyramid | Math with Mr. J

Math with Mr. J · 7:44

https://www.youtube.com/watch?v=JQD__LQTrYA

> **WHEN TO WATCH**
> Before Example 1: the volume of a pyramid.

Note: The Cavalieri argument for cones is taught from the Core ideas.

BACKUP VIDEO

Verifying Formulae for Cylinder, Cone & Sphere

Eddie Woo · 9:44

https://www.youtube.com/watch?v=XFtI8rwEegE

After the Core ideas: checking the one-third relationship for cones by experiment.

KEY TERMS

height of a pyramid or cone

The perpendicular distance from the apex to the plane of the base (not the slant height).

frustum

The part of a pyramid or cone that remains when the top is cut off by a plane parallel to the base, like a lampshade or a flowerpot.

method of exhaustion

The ancient Greek method, credited to Eudoxus, of proving an area or volume by trapping it between figures whose sizes come as close to it as desired; an early form of the limits used in calculus.

CORE IDEAS

Try this at home: make a hollow square pyramid and a hollow square prism with the same base and the same height, fill the pyramid with dry rice or sand, and pour it into the prism. It takes exactly three pyramids to fill the prism. A cone and a cylinder with the same base and height behave the same way. So the volume of a pyramid or cone is one-third of the matching prism or cylinder: $V = \frac{1}{3}Bh$.

Why one-third? Here is a case you can see completely. Pick one corner of a cube with edge s. Three faces of the cube do not touch that corner. Join each of those three faces to the chosen corner, and you get three square pyramids. Each has a face of the cube as its base and the chosen corner as its apex, so each has base area s^2 and height s. The three pyramids are congruent (each is the same pyramid turned to a different position), and together they fill the cube exactly. So each has volume $\frac{1}{3}s^3 = \frac{1}{3}(s^2)(s) = \frac{1}{3}Bh$.

For a general pyramid, Euclid gives a careful proof (XII.7) using the method of exhaustion. It needs ideas close to calculus, so we accept the pyramid formula in this course. The cube dissection and the pouring experiment are good evidence for it, and Cavalieri's Principle explains why the shape of the base does not matter: if two pyramids have equal base areas and equal heights, then at every level their cross sections are shrunken copies of their bases, shrunk by the same scale factor, so the cross sections have equal areas and the pyramids have equal volumes.

The same reasoning turns the pyramid formula into the cone formula. A cross section of a cone or pyramid at distance d from the apex is a copy of the base shrunk by the scale factor $\frac{d}{h}$, so its area is $\left(\frac{d}{h}\right)^2$ times the base area (Lesson 11.6). A cone with base area πr^2 and a pyramid with the same base area and height therefore

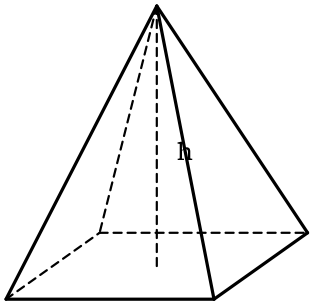
match at every level, and by Cavalieri's Principle the cone's volume is $\frac{1}{3}\pi r^2 h$. The proof is written out below.

As with surface area, read the problem carefully for which height you are given. The volume formula needs the perpendicular height h; if you are given the slant height, use the hidden right triangle and the Pythagorean theorem to find h first. For composite solids, such as a silo with a cone-shaped roof, find each part's volume separately and add.

POSTULATES AND THEOREMS

THEOREM **Volume of a Pyramid** Euclid XII.7

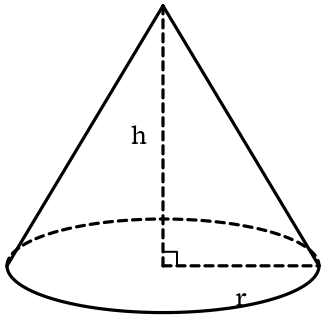
The volume of a pyramid with base area B and height h is $V = \frac{1}{3}Bh$.



Accepted without a complete proof in this course. Euclid proves that any pyramid is one third of the prism with the same base and equal height (the corollary of XII.7) using Eudoxus's method of exhaustion, which needs ideas close to calculus. The cube dissection in the core ideas proves the one-third in a special case.

THEOREM **Volume of a Cone** Euclid XII.10

The volume of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.



PARAGRAPH PROOF

Given: A cone with base radius r and height h

Prove: $V = \frac{1}{3}\pi r^2 h$

Place beside the cone a pyramid with height h whose base is a square with area πr^2 (side $r\sqrt{\pi}$), with both bases on the same plane and both apexes at the same height.

Take any plane parallel to the bases at distance d below the two apexes. In each solid, the cross section is the image of the base under a dilation centered at the apex with scale factor $\dfrac{d}{h}$, so it is similar to the base with that scale factor. By the area ratio of similar figures (Lesson 11.6), each cross section has area $\left(\dfrac{d}{h}\right)^2 \cdot \pi r^2$. So the two solids have the same height and equal cross-sectional areas at every level.

By Cavalieri's Principle the cone and the pyramid have equal volumes. By the Volume of a Pyramid theorem, the pyramid's volume is $\frac{1}{3}(\pi r^2)h$. Therefore the cone's volume is $V = \frac{1}{3}\pi r^2 h$.

∎

KEY FORMULAS

$$V = \tfrac{1}{3}Bh \quad \text{(any pyramid or cone)}$$

$$V = \tfrac{1}{3}\pi r^2 h \quad \text{(cone)}$$

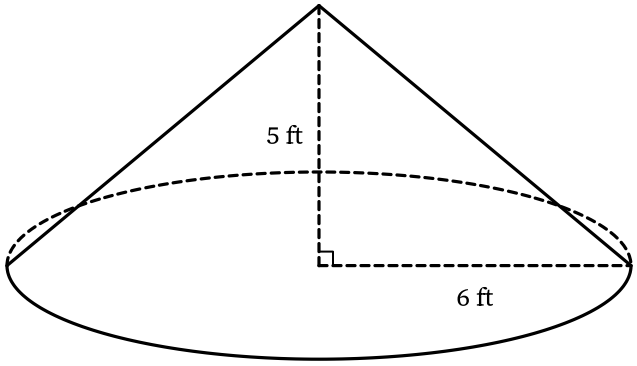
From history. Archimedes reports, in the preface to his work *The Method*, that Democritus first stated that a pyramid is one third of the prism and a cone one third of the cylinder with the same base and height, though without proof, and that Eudoxus of Cnidus was the first to prove these facts (Archimedes credits Eudoxus with these proofs in *On the Sphere and Cylinder* as well). Euclid's proofs in Book XII follow Eudoxus's method of exhaustion. The same idea of trapping a volume between simpler solids returned two thousand years later as the integral calculus.

Euclid connection. The corollary of XII.7 reads: 'any pyramid is a third part of the prism which has the same base with it and equal height.' XII.10 says the same of cones: 'any cone is a third part of the cylinder which has the same base with it and equal height.' Our cone proof by Cavalieri's Principle reaches Euclid's conclusion by comparing slices instead of by exhaustion.

Practice habit. Recite 'one-third base times height' for every pointed solid and 'base times height' for every solid with two bases, and check which kind you have before you compute. When a problem gives a slant height, draw and label the right triangle inside the solid and find the perpendicular height first.

WORKED EXAMPLES

Example 1. A pile of gravel is shaped like a cone with a base diameter of 12 ft and a height of 5 ft. Find its volume to the nearest tenth of a cubic foot, and convert it to cubic yards to the nearest tenth.

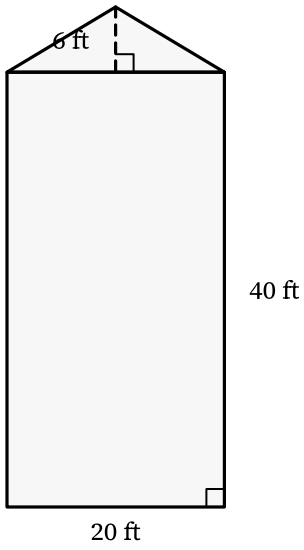


Show solution

1. The radius is $12 \div 2 = 6$ ft.
2. $V = \frac{1}{3}\pi(6)^2(5) = \frac{1}{3}\pi(36)(5) = 60\pi \approx 188.5$ cubic feet.
3. One cubic yard is 27 cubic feet: $60\pi \div 27 \approx 6.98 \approx 7.0$ cubic yards.

Answer: $60\pi \approx 188.5$ cubic feet, about 7.0 cubic yards

Example 2. A farm silo is a cylinder 20 ft in diameter and 40 ft tall, topped by a cone-shaped roof 6 ft high with the same diameter. Find the total volume of the silo, including the space under the roof, to the nearest cubic foot.



Side view of the silo

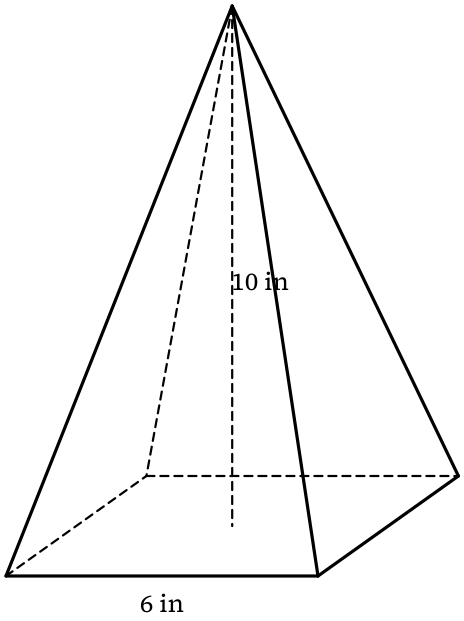
Show solution

1. Both parts have radius $r = 10$ ft.
2. Cylinder: $V = \pi(10)^2(40) = 4000\pi$ cubic feet.
3. Cone: $V = \frac{1}{3}\pi(10)^2(6) = 200\pi$ cubic feet.
4. Total: $4000\pi + 200\pi = 4200\pi \approx 13{,}195$ cubic feet.

Answer: $4200\pi \approx 13{,}195$ cubic feet

PRACTICE

1. **Easy** Find the volume of the square pyramid in the figure, with base edges 6 in and height 10 in.



Show answer

$V = \frac{1}{3}(36)(10) = 120$ cubic inches

2. **Easy** Find the volume of a cone with radius 3 cm and height 7 cm, in terms of π.

Show answer

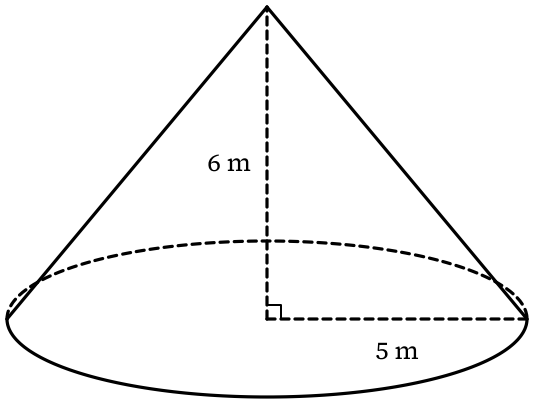
$V = \frac{1}{3}\pi(9)(7) = 21\pi$ cubic centimeters

3. **Easy** A prism has volume 90 cubic centimeters. What is the volume of a pyramid with the same base and the same height?

Show answer

$\frac{1}{3} \times 90 = 30$ cubic centimeters

4. **Easy** Find the volume of the cone in the figure, with radius 5 m and height 6 m, to the nearest tenth of a cubic meter.



Show answer

$V = \frac{1}{3}\pi(25)(6) = 50\pi \approx 157.1$ cubic meters

5. **Medium** The Great Pyramid of Giza originally had a square base about 230 m on each side and a height of about 146 m. Using these measurements, estimate its volume to the nearest thousand cubic meters.

Show answer

$V = \frac{1}{3}(230^2)(146) = \frac{1}{3}(52{,}900)(146) \approx 2{,}574{,}467$, about 2,574,000 cubic meters.

6. **Medium** A kitchen funnel is a cone with radius 4 cm and slant height 5 cm. If the spout is plugged, how many milliliters of water does the cone hold, to the nearest tenth? (1 cubic centimeter = 1 milliliter.)

Show answer

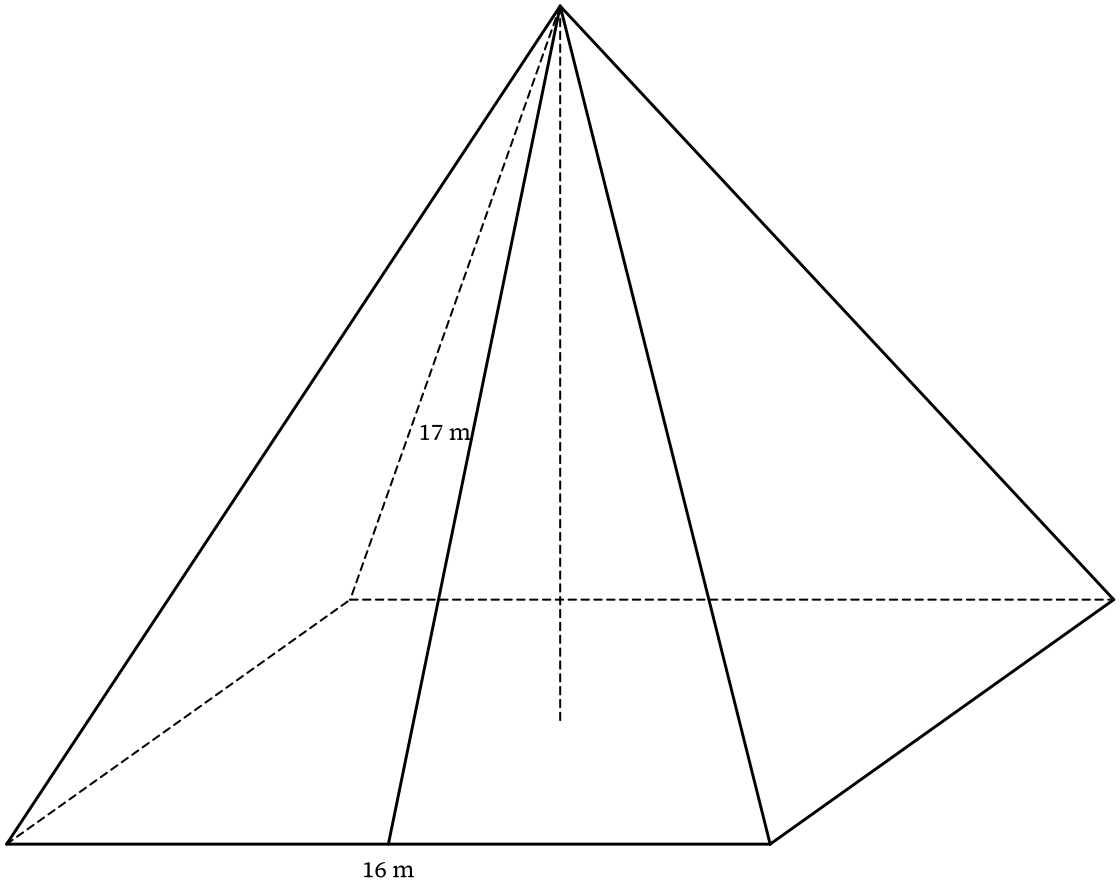
$h = \sqrt{5^2 - 4^2} = 3$ cm. $V = \frac{1}{3}\pi(16)(3) = 16\pi \approx 50.3$ mL.

7. **Medium** A cone has volume 100π cubic inches and radius 5 in. Find its height.

Show answer

$\frac{1}{3}\pi(25)h = 100\pi$, so $25h = 300$ and $h = 12$ in.

8. **Medium** A regular square pyramid has base edges 16 m and slant height 17 m. Find its volume.



Show answer

$h = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$ m. $V = \frac{1}{3}(256)(15) = 1280$ cubic meters.

9. **Medium** A cone-shaped paper cup and a cylindrical glass have the same radius and the same height. How many full cups of water does it take to fill the glass? Explain why.

Show answer

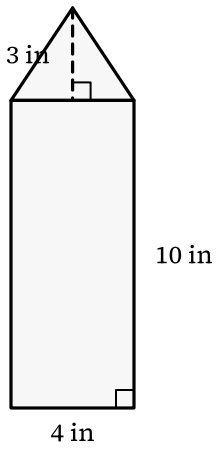
3 cups. The cone's volume is $\frac{1}{3}\pi r^2 h$ and the cylinder's is $\pi r^2 h$, so the glass holds exactly three times as much as the cup.

10. Hard A conical pile of gravel has a base diameter of 18 ft and a height of 6 ft. Find its volume in cubic yards to the nearest tenth. If gravel costs \$40 per cubic yard, what is the pile worth, to the nearest dollar?

Show answer

$V = \frac{1}{3}\pi(81)(6) = 162\pi$ cubic feet $= 162\pi \div 27 = 6\pi \approx 18.8$ cubic yards. Value: $40 \times 6\pi = 240\pi \approx \$754$.

11. Hard Henry's model rocket is a cylinder 4 in in diameter and 10 in long with a cone-shaped nose 3 in high, as in the figure. Find its total volume to the nearest tenth of a cubic inch.

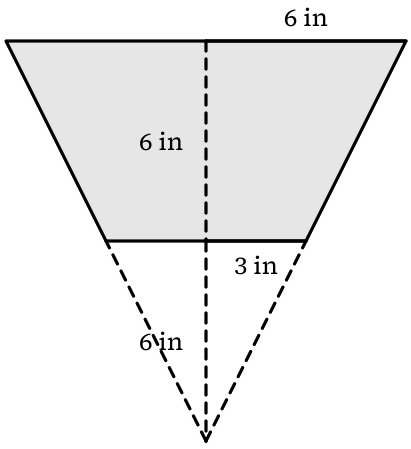


Side view of the rocket

Show answer

Cylinder: $\pi(2)^2(10) = 40\pi$. Cone: $\frac{1}{3}\pi(2)^2(3) = 4\pi$. Total $44\pi \approx 138.2$ cubic inches.

12. Hard A flowerpot is shaped like a frustum of a cone: its bottom has radius 3 in, its top has radius 6 in, and it is 6 in deep. As the figure shows, it is what remains when a cone of radius 6 in and height 12 in has its tip, a cone of radius 3 in and height 6 in, cut off. How much soil fills the pot, to the nearest tenth of a cubic inch?



Cut through the axis: the pot (shaded) and the removed tip (dashed)

Show answer

Large cone: $\frac{1}{3}\pi(36)(12) = 144\pi$. Small cone: $\frac{1}{3}\pi(9)(6) = 18\pi$. Pot: $144\pi - 18\pi = 126\pi \approx 395.8$ cubic inches.

13. **Challenge** A cone and a cylinder have the same radius r and equal volumes. The cone has height h_1 and the cylinder has height h_2. Prove that $h_1 = 3h_2$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: A cone with radius r and height h_1; A cylinder with radius r and height h_2; The cone and cylinder have equal volumes

Prove: $h_1 = 3h_2$

	STATEMENTS	REASONS
1	$V_{\text{cone}} = \frac{1}{3}\pi r^2 h_1$	Volume of a Cone
2	$V_{\text{cyl}} = \pi r^2 h_2$	Volume of a Prism or Cylinder
3	$V_{\text{cone}} = V_{\text{cyl}}$	Given
4	$\frac{1}{3}\pi r^2 h_1 = \pi r^2 h_2$	Substitution Property of Equality
5	$\pi r^2 h_1 = 3\pi r^2 h_2$	Multiplication Property of Equality
6	$h_1 = 3h_2$	Division Property of Equality (divide by πr^2, which is not 0)

∎

14. **Challenge** A cube has edges 6 cm long. Choose one corner of the cube, and join it to each of the three faces that do not touch that corner, forming three square pyramids that fill the cube. Find the volume of one pyramid with the pyramid formula, and check that the three volumes add up to the volume of the cube.

Show answer

Each pyramid has a 6-by-6 base and height 6 (the edge from the chosen corner perpendicular to that face): $V = \frac{1}{3}(36)(6) = 72$ cubic centimeters. Check: $3 \times 72 = 216 = 6^3$.

COMMON MISTAKES

- Forgetting the $\frac{1}{3}$. Ask: does the solid come to a point (pyramid or cone, use $\frac{1}{3}Bh$) or does it have two bases (prism or cylinder, use Bh)?

- Using the slant height in the volume formula. Volume needs the perpendicular height; find it from the slant height with the Pythagorean theorem.
- Squaring the diameter instead of the radius in $\frac{1}{3}\pi r^2 h$. Halve the diameter first.
- Dividing by 3 instead of 27 when changing cubic feet to cubic yards.

> **Make it a game · Pour Prediction.** If you have a cone-shaped paper cup and a cylindrical cup of the same opening and height, predict how many cone pours fill the cylinder (three), then try it with rice. No cups? Race to find volumes of three pyramids. Parent gives a cone with radius 3 and height 4; compute its volume.

12.6 Spheres and Similar Solids

OBJECTIVES

- Find the surface area and volume of spheres and hemispheres with $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.
- Prove the sphere volume formula with Cavalieri's Principle, and explain Archimedes' discovery that a sphere is two-thirds of its circumscribed cylinder.
- Use the ratios $a^2 : b^2$ and $a^3 : b^3$ to compare the surface areas and volumes of similar solids.
- Solve practical problems about domes, tanks, ball bearings and the earth.

WATCH

Cavalieri's Principle in 3D | Volume of a sphere |

Think Twice · 3:40

https://www.youtube.com/watch?v=0xCe6LGWWjU

> **WHEN TO WATCH**
> After the Core ideas: an animation of the Cavalieri argument for the volume of a sphere, using the cone formula. It has music but no narration; pause to compare the cross sections.

Note: Archimedes' 2/3 sphere-and-cylinder result is taught from the lesson.

BACKUP VIDEO

Measurements of Similar Solids

Eddie Woo · 8:12

https://www.youtube.com/watch?v=RRmVdo7Xlhc

Before the similar-solids examples: how area and volume scale.

KEY TERMS

sphere

The set of all points in space at a given distance (the **radius**) from a given point (the **center**). A **diameter** is a segment through the center with endpoints on the sphere.

great circle

The intersection of a sphere with a plane through its center. It is the largest circle on the sphere and has the same center and radius as the sphere; the equator is a great circle of the earth.

hemisphere

Half of a sphere, cut off by a plane through the center. Its flat face is a great circle.

similar solids

Solids of the same shape whose corresponding lengths (edges, radii, heights) all have the same ratio, called the **scale factor**.

CORE IDEAS

A **sphere** is the three-dimensional cousin of a circle: every point on it is the same distance r from the center. A plane through the center cuts it in a **great circle**, the biggest circle the sphere contains, and cuts it into two **hemispheres**. Any other plane that cuts through the sphere (not just touching it at a single point) cuts it in a smaller circle.

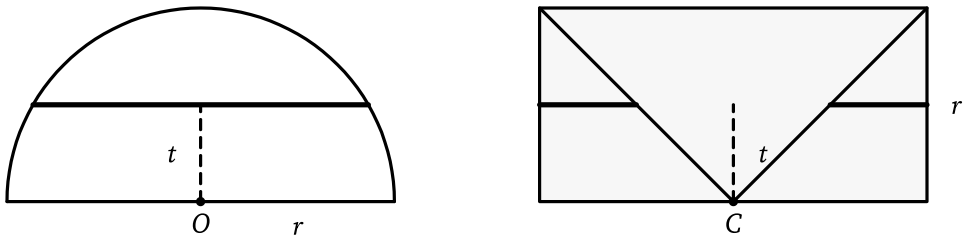
To find the volume of a sphere we compare a hemisphere with a cleverly chosen solid, using Cavalieri's Principle (see the first figure). Beside a hemisphere of radius r, stand a cylinder of radius r and height r, and hollow out of it a cone whose apex is at the center of the cylinder's bottom and whose base is the cylinder's top. Slice both solids at the same height t. The hemisphere's slice is a circle of radius $\sqrt{r^2 - t^2}$ (Pythagorean theorem), with area $\pi(r^2 - t^2)$. The other slice is a ring between a circle of radius r and a circle of radius t, with area $\pi r^2 - \pi t^2$. The areas are equal at every height! So the hemisphere has the same volume as the cylinder minus the cone: $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$, and the whole sphere has volume $\frac{4}{3}\pi r^3$.

The surface area of a sphere is $S = 4\pi r^2$, exactly four great circles. Archimedes put it this way: 'the surface of any sphere is four times its greatest circle.' Here is a reason that fits with the volume: imagine the sphere's surface divided into many tiny patches, each the base of a thin pyramid whose apex is the center. Each pyramid has

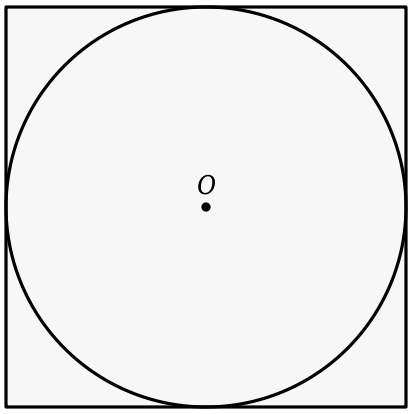
height r, so all together their volume is $\frac{1}{3}(\text{total base area})(r) = \frac{1}{3}Sr$. Setting $\frac{1}{3}Sr = \frac{4}{3}\pi r^3$ gives $S = 4\pi r^2$.

Put a sphere snugly inside a cylinder, touching the top, the bottom and the curved side (the second figure). The cylinder has radius r and height $2r$, so its volume is $\pi r^2(2r) = 2\pi r^3$, and $\frac{4}{3}\pi r^3$ is exactly $\frac{2}{3}$ of that. Its total surface area is $2\pi r(2r) + 2\pi r^2 = 6\pi r^2$, and $4\pi r^2$ is again exactly $\frac{2}{3}$. Archimedes discovered both facts and was so pleased that he asked for a sphere inside a cylinder to be carved on his tomb.

Two solids are **similar** when one is an enlargement of the other: every length is multiplied by the same scale factor k. Areas involve two lengths multiplied together, so they are multiplied by $k \cdot k = k^2$; volumes involve three, so they are multiplied by k^3. If two similar solids have scale factor $a : b$, their surface areas are in the ratio $a^2 : b^2$ and their volumes in the ratio $a^3 : b^3$. Double the size of a ball and it needs 4 times as much leather but holds 8 times as much air. This is why a large animal or building needs much thicker supports than a small one: its weight (volume) grows faster than the cross sections that carry it (area).



At every height t, both slices have area π(r² − t²).

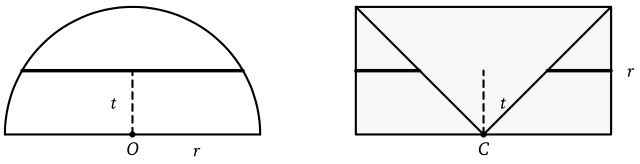


A sphere inside its circumscribed cylinder (side view)

POSTULATES AND THEOREMS

THEOREM **Volume of a Sphere**

The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.



At every height t, both slices have area π(r² – t²).

PARAGRAPH PROOF

Given: A sphere with radius r

Prove: $V = \frac{4}{3}\pi r^3$

Cut the sphere through its center into two congruent hemispheres, and set one with its flat face on a plane. Beside it on the same plane stand a cylinder with radius r and height r, and remove from the cylinder a cone whose base is the top of the cylinder (radius r) and whose apex is the center of the cylinder's bottom (height r). Call what remains the hollowed cylinder. Both solids have height r.

Slice both solids with a plane parallel to the base plane at height t, where $0 \le t \le r$. The slice of the hemisphere is a circle whose center lies directly above the center of the sphere. A point on the edge of that circle is at distance r from the sphere's center (definition of sphere), so the radius x of the slice satisfies $x^2 + t^2 = r^2$ (Pythagorean Theorem), and the slice has area $\pi x^2 = \pi(r^2 - t^2)$.

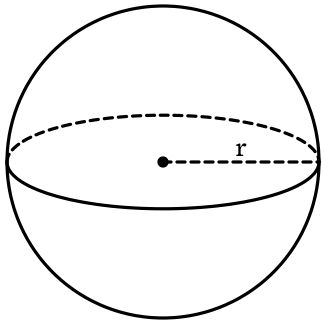
The slice of the hollowed cylinder is a ring. Its outer circle has radius r. Its inner circle is the slice of the cone at distance t from the apex; that slice is the cone's base dilated from the apex by the scale factor $\frac{t}{r}$, so its radius is $\frac{t}{r} \cdot r = t$. The ring has area $\pi r^2 - \pi t^2 = \pi(r^2 - t^2)$.

The solids have the same height and equal cross-sectional areas at every level, so by Cavalieri's Principle they have equal volumes. The hollowed cylinder has volume $\pi r^2 \cdot r - \frac{1}{3}\pi r^2 \cdot r = \frac{2}{3}\pi r^3$ (volumes of a cylinder and a cone, and the Volume Addition Postulate). So each hemisphere has volume $\frac{2}{3}\pi r^3$, and the sphere has volume $2 \cdot \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$.

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THEOREM **Surface Area of a Sphere**

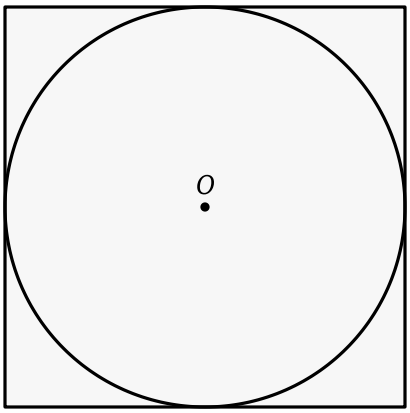
The surface area of a sphere with radius r is $S = 4\pi r^2$, four times the area of a great circle.



Proved by Archimedes in On the Sphere and Cylinder; we accept it without a complete proof in this course, since a sphere cannot be unrolled flat. The core ideas show why it agrees with the volume formula.

COROLLARY **Archimedes' Sphere and Cylinder**

A sphere has two-thirds the volume and two-thirds the total surface area of the cylinder circumscribed about it (radius r, height $2r$).



A sphere inside its circumscribed cylinder (side view)

PARAGRAPH PROOF

Given: A sphere of radius r inside a cylinder of radius r and height $2r$

Prove: $V_{\text{sphere}} = \frac{2}{3}V_{\text{cyl}}$

By the Volume of a Prism or Cylinder theorem, $V_{\text{cyl}} = \pi r^2(2r) = 2\pi r^3$, so $\frac{2}{3}V_{\text{cyl}} = \frac{2}{3}(2\pi r^3) = \frac{4}{3}\pi r^3$. By the Volume of a Sphere theorem, $V_{\text{sphere}} = \frac{4}{3}\pi r^3$. By the Transitive Property of Equality, $V_{\text{sphere}} = \frac{2}{3}V_{\text{cyl}}$. (The surface area part is proved in Practice Problem 14.)

∎

THEOREM **Similar Solids Theorem** Euclid XII.8, XII.12 and XII.18

If two similar solids have scale factor $a : b$, then their surface areas are in the ratio $a^2 : b^2$ and their volumes are in the ratio $a^3 : b^3$.

PARAGRAPH PROOF

Given: Two similar prisms, cylinders, pyramids, cones or spheres with scale factor $a : b$

Prove: Their surface areas are in the ratio $a^2 : b^2$ and their volumes in the ratio $a^3 : b^3$.

Let $k = \dfrac{b}{a}$. Every length of the second solid is k times the corresponding length of the first (definition of similar solids). Each surface area formula in this unit ($2\pi rh + 2\pi r^2$, $Ph + 2B$, $\frac{1}{2}P\ell + B$, $\pi r\ell + \pi r^2$, $4\pi r^2$) is a sum of terms, each a number times a product of two lengths (a base area B is itself such a sum, since corresponding faces are similar figures). Replacing each length x by kx multiplies each product of two lengths by k^2, so by the Distributive Property the whole surface area is multiplied by k^2, giving the ratio $1 : k^2 = a^2 : b^2$.

Each volume formula (Bh, $\frac{1}{3}Bh$, $\pi r^2 h$, $\frac{1}{3}\pi r^2 h$, $\frac{4}{3}\pi r^3$) is a number times a product of three lengths, since B is a product of two. Replacing each length x by kx multiplies the volume by k^3, giving the ratio $1 : k^3 = a^3 : b^3$.

∎

This proof covers the solids of this unit; for similar solids of any shape we accept the same result. Euclid proves the volume part for pyramids (XII.8), cones and cylinders (XII.12) and spheres (XII.18, 'spheres are to one another in the triplicate ratio of their respective diameters').

KEY FORMULAS

$$S = 4\pi r^2 \qquad V = \tfrac{4}{3}\pi r^3 \quad \text{(sphere)}$$

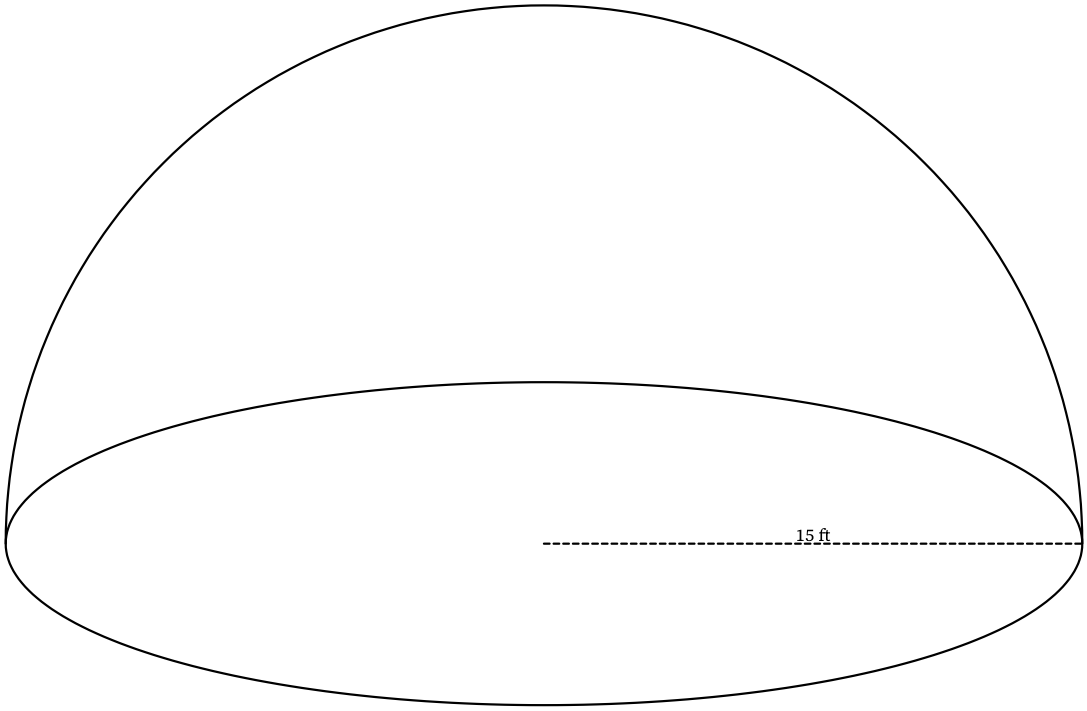
$$\text{scale factor } a : b \;\Rightarrow\; \text{surface areas } a^2 : b^2, \quad \text{volumes } a^3 : b^3$$

From history. Archimedes of Syracuse (about 250 BC) proved the sphere formulas in his book On the Sphere and Cylinder and regarded the two-thirds relationship between the sphere and its cylinder as one of his finest discoveries. He asked that a sphere inside a cylinder be carved on his tomb. The Roman orator Cicero reported that, while serving in Sicily, he found the forgotten, overgrown tomb by recognizing that very figure on a small column.

Euclid connection. Euclid does not find the volume of a sphere, but he proves how spheres compare: XII.18 states that 'spheres are to one another in the triplicate ratio of their respective diameters,' that is, volumes in the ratio $a^3 : b^3$. He proves the same triplicate ratio for similar pyramids (XII.8) and similar cones and cylinders (XII.12). Archimedes built on Book XII to find the actual formulas.

Practice habit. Recite $S = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$ and 'square for areas, cube for volumes' until automatic. Before using a ratio for similar solids, write the scale factor first, then square it or cube it according to whether the question asks about area or volume.

Example 1. The dome over a chapel is a hemisphere with radius 15 ft. How many square feet of copper are needed to cover its curved outer surface? Answer to the nearest tenth.



Show solution

1. The curved surface of a hemisphere is half of a sphere's surface: $\frac{1}{2}(4\pi r^2) = 2\pi r^2$. The flat circular floor is not covered.
2. $2\pi(15)^2 = 2\pi(225) = 450\pi$.
3. $450\pi \approx 1413.7$ square feet.

Answer: $450\pi \approx 1413.7$ square feet

Example 2. Two water tanks on a farm are similar, and their heights are 12 ft and 18 ft. The smaller tank holds 8,000 gallons and takes 12 gallons of paint to cover its outside. How much does the larger tank hold, and how much paint does it take?

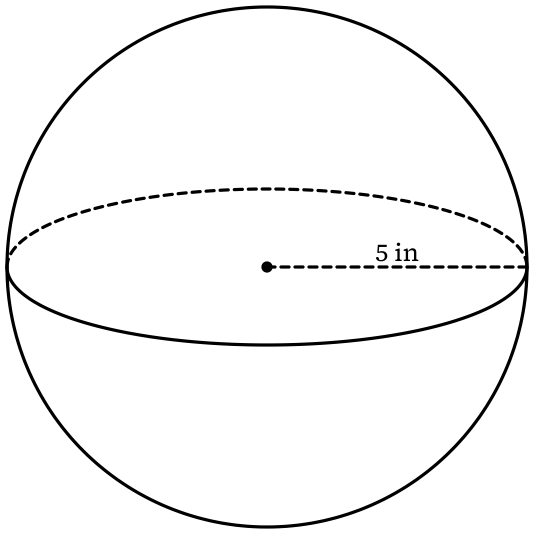
Show solution

1. Scale factor: $12 : 18 = 2 : 3$.
2. Volumes are in the ratio $2^3 : 3^3 = 8 : 27$, so the larger tank holds $8000 \times \frac{27}{8} = 27{,}000$ gallons.
3. Surface areas, and so amounts of paint, are in the ratio $2^2 : 3^2 = 4 : 9$, so the larger tank takes $12 \times \frac{9}{4} = 27$ gallons of paint.
4. Notice that the tank is only 1.5 times as tall, but holds 3.375 times as much water.

Answer: 27,000 gallons of water; 27 gallons of paint

PRACTICE

1. Easy Find the surface area of the sphere in the figure, with radius 5 in. Give your answer in terms of π.



Show answer

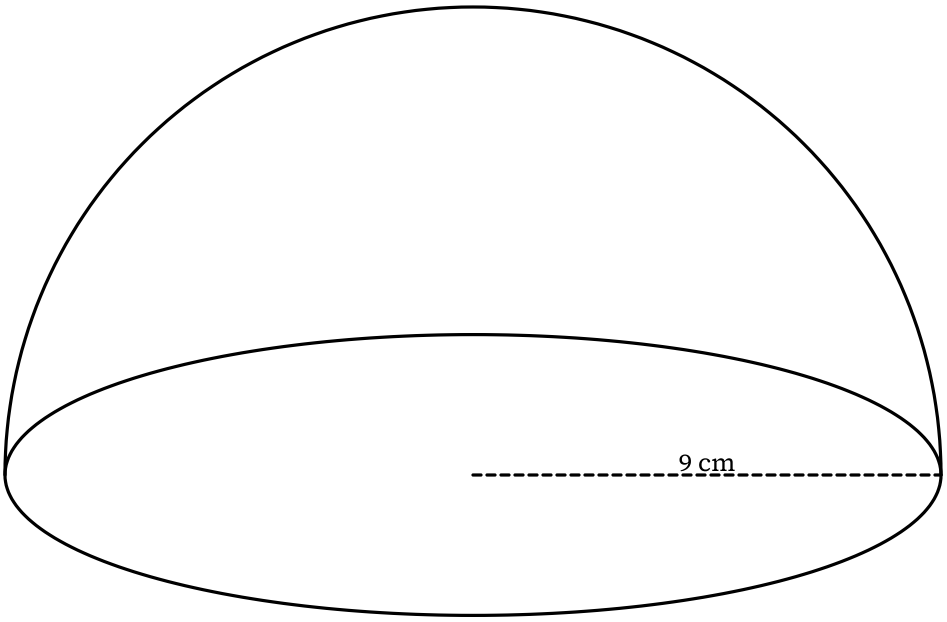
$S = 4\pi(5)^2 = 100\pi$ square inches

2. **Easy** Find the volume of a sphere with radius 6 cm, in terms of π.

Show answer

$V = \frac{4}{3}\pi(216) = 288\pi$ cubic centimeters

3. **Easy** Find the volume of the hemisphere in the figure, with radius 9 cm, in terms of π.



Show answer

$V = \frac{1}{2} \cdot \frac{4}{3}\pi(729) = 486\pi$ cubic centimeters

4. **Easy** Two similar solids have scale factor $2:5$. Write the ratio of their surface areas and the ratio of their volumes.

Show answer

Surface areas $4:25$; volumes $8:125$.

5. **Medium** The earth is nearly a sphere with a radius of about 3,960 miles. Estimate its surface area to the nearest million square miles.

Show answer

$S = 4\pi(3960)^2 \approx 197{,}060{,}797$, about 197,000,000 square miles.

6. **Medium** A steel ball bearing has a diameter of 2 cm. Find its volume to the nearest tenth of a cubic centimeter. Steel has a mass of about 7.9 grams per cubic centimeter; find the mass of the bearing to the nearest gram.

   Show answer

   $r = 1$, so $V = \frac{4}{3}\pi \approx 4.2$ cubic centimeters. Mass $\approx 4.189 \times 7.9 \approx 33$ grams.

7. **Medium** A sphere has volume 972π cubic inches. Find its radius.

   Show answer

   $\frac{4}{3}\pi r^3 = 972\pi$, so $r^3 = 729$ and $r = 9$ in.

8. **Medium** Two similar cones have heights 6 cm and 9 cm. The smaller cone has volume 48 cubic centimeters. Find the volume of the larger cone.

   Show answer

   Scale factor $6 : 9 = 2 : 3$, so volumes are $8 : 27$. $48 \times \dfrac{27}{8} = 162$ cubic centimeters.

9. **Medium** A new basketball has twice the diameter of a small toy ball of the same shape. How many times as much leather covers it, and how many times as much air does it hold? Explain why.

   Show answer

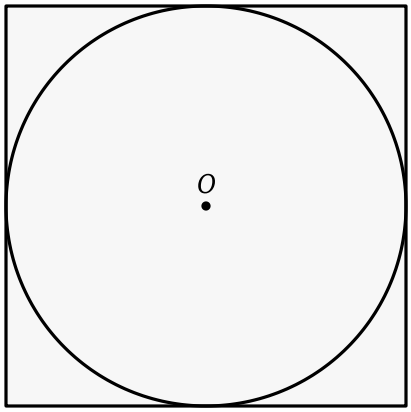
   4 times the leather and 8 times the air. The balls are similar with scale factor $1 : 2$. Surface area involves two lengths, so it is multiplied by $2^2 = 4$; volume involves three lengths, so it is multiplied by $2^3 = 8$. (From the formulas: $4\pi(2r)^2 = 4 \cdot 4\pi r^2$ and $\frac{4}{3}\pi(2r)^3 = 8 \cdot \frac{4}{3}\pi r^3$.)

10. **Hard** Two similar statues have surface areas in the ratio $49 : 81$. The smaller statue has a volume of 686 cubic inches. Find the volume of the larger statue.

    Show answer

    Scale factor $\sqrt{49} : \sqrt{81} = 7 : 9$, so volumes are $343 : 729$. $686 \times \dfrac{729}{343} = 2 \times 729 = 1458$ cubic inches.

11. Hard A sphere fits exactly inside a cylinder, touching the top, the bottom and the curved side, as in the figure. The cylinder has volume 54π cubic centimeters. Find the volume of the sphere in terms of π.

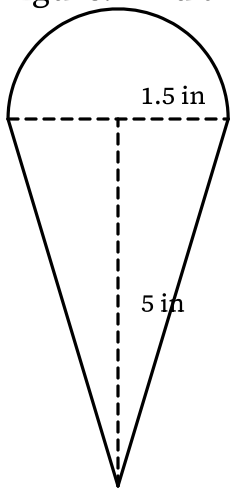


A sphere inside its circumscribed cylinder (side view)

Show answer

By Archimedes' result the sphere is $\frac{2}{3}$ of the cylinder: $\frac{2}{3}(54\pi) = 36\pi$ cubic centimeters. (Check: $2\pi r^3 = 54\pi$ gives $r = 3$, and $\frac{4}{3}\pi(27) = 36\pi$.)

12. Hard An ice cream cone is a cone with radius 1.5 in and height 5 in, filled with ice cream and topped with a hemisphere of ice cream of the same radius, as in the figure. Find the total volume of ice cream to the nearest tenth of a cubic inch.



Side view of the ice cream cone

Show answer

Cone: $\frac{1}{3}\pi(1.5)^2(5) = 3.75\pi$. Hemisphere: $\frac{2}{3}\pi(1.5)^3 = 2.25\pi$. Total $6\pi \approx 18.8$ cubic inches.

13. Hard A town's water tower holds water in a spherical tank with an inside diameter of 20 ft. One cubic foot holds about 7.48 gallons. About how many gallons does the tank hold, to the nearest thousand gallons?

Show answer

$V = \frac{4}{3}\pi(10)^3 \approx 4188.8$ cubic feet; $4188.8 \times 7.48 \approx 31{,}332$, about 31,000 gallons.

14. **Challenge** Prove the surface area half of Archimedes' discovery: a sphere of radius r has two-thirds the total surface area of the cylinder circumscribed about it, which has radius r and height $2r$.

Show answer

See the model proof.

TWO-COLUMN PROOF

Given: A sphere of radius r; The circumscribed cylinder has radius r and height $2r$
Prove: $S_{\text{sphere}} = \frac{2}{3} S_{\text{cyl}}$

	STATEMENTS	REASONS
1	$S_{\text{cyl}} = 2\pi r(2r) + 2\pi r^2$	Surface Area of a Right Cylinder
2	$S_{\text{cyl}} = 4\pi r^2 + 2\pi r^2 = 6\pi r^2$	Properties of multiplication; Distributive Property (combining like terms)
3	$\frac{2}{3} S_{\text{cyl}} = \frac{2}{3}(6\pi r^2) = 4\pi r^2$	Multiplication Property of Equality
4	$S_{\text{sphere}} = 4\pi r^2$	Surface Area of a Sphere
5	$S_{\text{sphere}} = \frac{2}{3} S_{\text{cyl}}$	Transitive Property of Equality (rows 3 and 4)

∎

COMMON MISTAKES

- Mixing up the sphere formulas. Surface area is measured in square units and has r^2: $4\pi r^2$. Volume is in cubic units and has r^3: $\frac{4}{3}\pi r^3$.
- Using the diameter in place of the radius. A ball 2 cm across has $r = 1$ cm, not 2 cm.
- Using the scale factor itself for areas or volumes. If the lengths double, areas are multiplied by 4 and volumes by 8, not by 2.
- Forgetting the flat face of a hemisphere, or including it when it should not be. Total surface area of a solid hemisphere is $2\pi r^2 + \pi r^2 = 3\pi r^2$; a dome's curved surface alone is $2\pi r^2$.

Make it a game · Archimedes' Challenge. Place a tennis ball in a tight-fitting can. Parent asks what fraction of the can the ball fills. Predict two thirds, then fill the remaining space with rice and compare. State Archimedes' discovery for both volume and surface area. Finally, parent gives a sphere of radius 3 and you compute its volume and surface area.

UNIT 12

Surface Area and Volume

In this unit geometry steps off the flat page and into space. You will name and sketch prisms, pyramids, cylinders, cones and spheres, unfold them into nets, slice them into cross sections, and meet Euler's formula and the five Platonic solids with which Euclid closes the Elements. Then you will learn to measure solids in two ways: surface area, the amount of material that covers a solid, and volume, the amount of space it fills.

As always in this course, the formulas are explained, not just memorized. Lateral areas come from unfolding nets; volumes come from stacking layers and from Cavalieri's Principle, which compares solids slice by slice; the one-third in the pyramid and cone formulas is Euclid's (XII.7 and XII.10); and the sphere formulas come from Archimedes, who asked for a sphere inside a cylinder to be carved on his tomb. The unit ends with similar solids, where areas grow by the square of the scale factor and volumes by the cube.

Expect many practical problems: painting a barn, shingling a steeple, ordering concrete by the cubic yard, filling a grain bin. A scientific calculator is allowed for π and decimal answers. Parents: the most common errors in this unit are using a diameter for a radius, a height for a slant height, and dividing by 3 instead of 27 to get cubic yards. A quick check of these three habits catches most mistakes.

PREREQUISITES

- Areas of rectangles, triangles, parallelograms, trapezoids and regular polygons (Lessons 11.1 to 11.3)
- Circumference, arc length and the areas of circles and sectors (Lessons 11.4 and 11.5)
- The ratio of areas of similar figures (Lesson 11.6)
- The Pythagorean theorem and special right triangles (Lessons 9.1 and 9.4)
- Similar triangles and dilations (Lessons 7.5, 8.2 and 8.3)
- Solving equations, including ones with squares and cubes, and rounding answers (Algebra I)

MASTERY CHECKLIST

- ☐ I can name prisms, pyramids, cylinders, cones and spheres and count the faces, edges and vertices of a polyhedron.
- ☐ I can use Euler's formula $F + V = E + 2$ and explain why there are exactly five Platonic solids.
- ☐ I can match a solid with its net, describe cross sections and solids of revolution, and construct the net of a regular tetrahedron with compass and straightedge.
- ☐ I can find the lateral area and surface area of right prisms and cylinders and explain why $L = Ph$ and $L = 2\pi rh$.
- ☐ I can find the slant height of a regular pyramid or a right cone with the Pythagorean theorem.
- ☐ I can find the lateral area and surface area of regular pyramids and right cones and explain where $L = \frac{1}{2}P\ell$ and $L = \pi r\ell$ come from.
- ☐ I can state Cavalieri's Principle and use it to explain why $V = Bh$ for right and oblique prisms and cylinders.

- ☐ I can find volumes of pyramids and cones with $V = \frac{1}{3}Bh$ and explain why the factor is one-third.
- ☐ I can find the surface area and volume of spheres and hemispheres and explain Archimedes' sphere-and-cylinder discovery.
- ☐ I can use the ratios $a^2 : b^2$ and $a^3 : b^3$ for the surface areas and volumes of similar solids.
- ☐ I can solve composite-solid and practical problems, converting units such as cubic feet to cubic yards correctly.

LESSONS IN THIS UNIT

12.1	**Solids, Nets and Cross Sections**	45 minutes
12.2	**Surface Area of Prisms and Cylinders**	2 sessions of 45 minutes
12.3	**Surface Area of Pyramids and Cones**	2 sessions of 45 minutes
12.4	**Volume of Prisms and Cylinders**	2 sessions of 45 minutes
12.5	**Volume of Pyramids and Cones**	2 sessions of 45 minutes
12.6	**Spheres and Similar Solids**	2 sessions of 45 minutes

Unit 12 Test

You have 60 minutes. A scientific calculator is allowed. Show your work for every problem: write the formula you use, substitute, and then compute. Give exact

answers in terms of π or in simplest radical form when a problem asks for them; otherwise round as the problem directs and include units (square units for area, cubic units for volume). In the proof, give a reason for every step. For Problem 15 you need a compass, a straightedge and a centimeter ruler.

1. **5 pts**

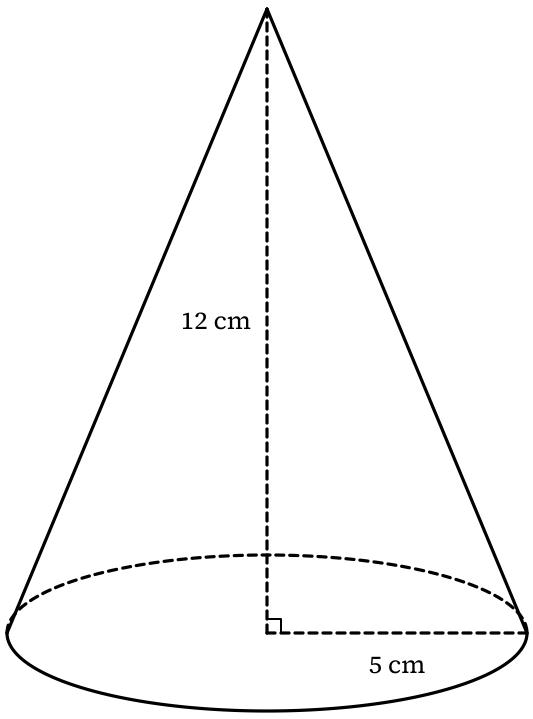
   A pyramid has a pentagon for its base. Find its number of faces, vertices and edges, in that order.

2. **5 pts**

   A convex polyhedron has 9 faces and 16 edges. How many vertices does it have?

3. **6 pts**

   The right cone in the figure has radius 5 cm and height 12 cm. It is sliced by a vertical plane through its apex and its axis. Name the shape of the cross section and find its area.

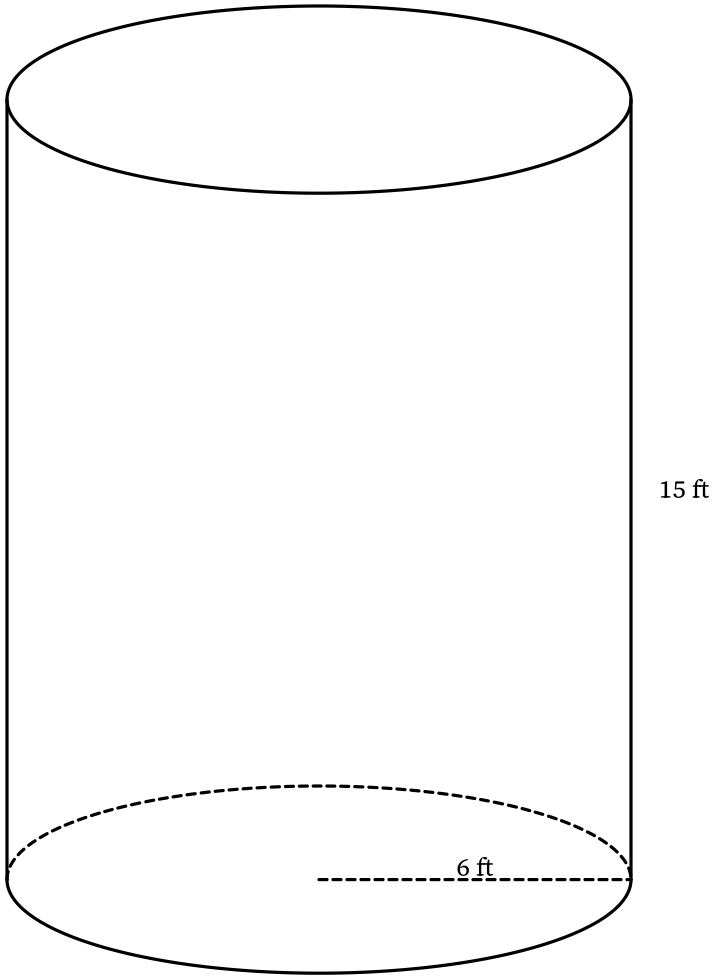


4. **6 pts**

   Find the total surface area of a rectangular box 9 cm long, 7 cm wide and 2 cm high.

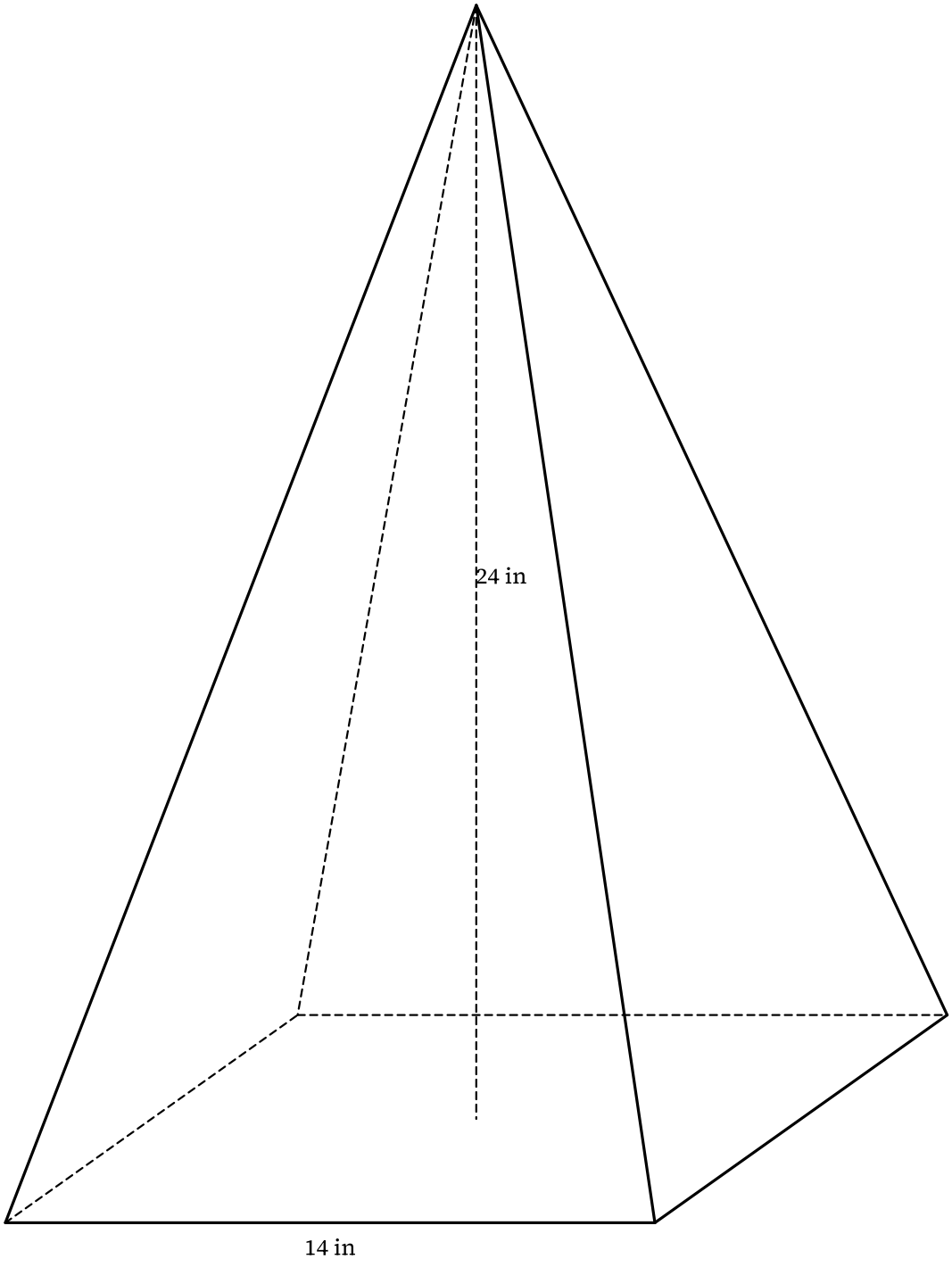
5. **7 pts**

   A cylindrical water tank with radius 6 ft and height 15 ft stands on the ground, as in the figure. Its curved side and its flat top will be painted, but not its bottom. Find the area to be painted to the nearest tenth of a square foot. If one gallon of paint covers 300 square feet, how many whole gallons are needed?



6. **7 pts**

A regular square pyramid has base edges 14 in and height 24 in. Find its slant height and its total surface area.



7. **7 pts**

   A right cone has radius 8 cm and height 15 cm. Find its total surface area in terms of π.

8. **7 pts**

   The Miller family is pouring a concrete driveway 40 ft long, 12 ft wide and 4 inches thick. How many cubic yards of concrete does it take, to the nearest tenth? If they order 6 cubic yards at \$150 per cubic yard, what is the cost?

9. **7 pts**

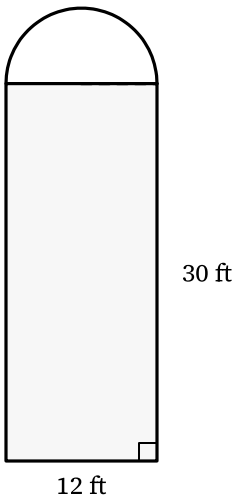
   Grace stacks 30 identical coins, each 2.4 cm in diameter and 0.2 cm thick, into a column, then nudges the column so it leans. Find the volume of the leaning stack to the nearest tenth of a cubic centimeter, and explain why leaning does not change the volume.

10. **7 pts**

    The attic of a gazebo is the space under a square pyramid roof with base edges 12 ft and height 5 ft. Find the volume of the attic.

11. **8 pts**

    A grain silo is a cylinder 12 ft in diameter and 30 ft tall, topped by a hemispherical dome with the same diameter, as in the figure. Find its total volume to the nearest cubic foot.



Side view of the silo

12. **7 pts**

A sphere has volume 288π cubic meters. Find its radius and then its surface area in terms of π.

13. **8 pts**

Ruth makes two similar clay pitchers whose heights are in the ratio $3:4$. The larger pitcher holds 64 ounces. The smaller pitcher needs 18 square inches of glaze on its outside. How many ounces does the smaller pitcher hold, and how much glaze does the larger one need?

14. **6 pts**

Prove that a sphere of radius r and a cone with base radius r and height $4r$ have equal volumes.

15. **7 pts**

With compass and straightedge (and a ruler only to measure the first segment), construct a net for a regular tetrahedron whose edges are 3 cm long. List your steps, and explain why the net folds into a solid whose four faces are congruent equilateral triangles.

ANSWER KEY

1. $F = 6$, $V = 6$, $E = 10$ (and $6 + 6 = 10 + 2$).
2. $9 + V = 16 + 2$, so $V = 9$ vertices.
3. An isosceles triangle with base $2 \times 5 = 10$ cm and height 12 cm; area $\frac{1}{2}(10)(12) = 60$ square centimeters.
4. $2(9 \cdot 7 + 9 \cdot 2 + 7 \cdot 2) = 2(63 + 18 + 14) = 190$ square centimeters
5. Side: $2\pi(6)(15) = 180\pi$; top: $\pi(6)^2 = 36\pi$. Total $216\pi \approx 678.6$ square feet. $678.6 \div 300 \approx 2.26$, so 3 gallons.
6. $\ell = \sqrt{24^2 + 7^2} = \sqrt{625} = 25$ in. $S = \frac{1}{2}(56)(25) + 14^2 = 700 + 196 = 896$ square inches.
7. $\ell = \sqrt{64 + 225} = 17$. $S = \pi(8)(17) + \pi(8)^2 = 136\pi + 64\pi = 200\pi$ square centimeters.
8. $V = 40 \times 12 \times \frac{4}{12} = 160$ cubic feet $= 160 \div 27 \approx 5.9$ cubic yards. Cost: $6 \times 150 =$ \$900.
9. The stack is $30 \times 0.2 = 6$ cm tall with radius 1.2 cm: $V = \pi(1.2)^2(6) = 8.64\pi \approx 27.1$ cubic centimeters. The leaning stack has the same height as the straight one, and at every level its cross section is the same circle, of the same area; by Cavalieri's Principle the volumes are equal.
10. $V = \frac{1}{3}(144)(5) = 240$ cubic feet
11. $r = 6$. Cylinder: $\pi(36)(30) = 1080\pi$. Hemisphere: $\frac{2}{3}\pi(216) = 144\pi$. Total $1224\pi \approx 3845$ cubic feet.
12. $\frac{4}{3}\pi r^3 = 288\pi$, so $r^3 = 216$ and $r = 6$ m. $S = 4\pi(36) = 144\pi$ square meters.
13. Volumes are in the ratio $27 : 64$, so the smaller holds $64 \times \frac{27}{64} = 27$ ounces. Areas are in the ratio $9 : 16$, so the larger needs $18 \times \frac{16}{9} = 32$ square inches of glaze.
14. See the model proof.

TWO-COLUMN PROOF

Given: A sphere with radius r; A cone with base radius r and height $4r$

Prove: $V_{\text{sphere}} = V_{\text{cone}}$

	STATEMENTS	REASONS
1	$V_{\text{sphere}} = \frac{4}{3}\pi r^3$	Volume of a Sphere
2	$V_{\text{cone}} = \frac{1}{3}\pi r^2(4r)$	Volume of a Cone

	STATEMENTS	REASONS
3	$V_{\text{cone}} = \frac{4}{3}\pi r^3$	Properties of multiplication and exponents
4	$V_{\text{sphere}} = V_{\text{cone}}$	Transitive and Symmetric Properties of Equality (rows 1 and 3)

∎

15. Draw $\overline{AB}$ 6 cm long. Draw circles centered at A and at B with radius AB; they meet at C; draw $\overline{AC}$ and $\overline{BC}$ (Euclid I.1). Bisect each side with the perpendicular bisector construction (I.10) to find midpoints D, E, F of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, and draw $\overline{DE}$, $\overline{EF}$, $\overline{FD}$. Why: $AC = AB = BC$ as radii of the two circles, so $\triangle ABC$ is equilateral with sides 6 cm. The half-sides are 3 cm, and by the Triangle Midsegment Theorem each midsegment is $\frac{1}{2}(6) = 3$ cm. So the four small triangles each have three 3-cm sides: they are congruent (SSS) equilateral triangles, and folding along $\overline{DE}$, $\overline{EF}$, $\overline{FD}$ brings A, B and C together to form a regular tetrahedron.

UNIT 13

Coordinate Geometry and Proof

This last unit brings together two great traditions: Euclid's geometry of figures and reasons, and Descartes' idea of naming every point by a pair of numbers. You will gather the coordinate tools (distance, midpoint, slope and partitions), use them to prove what kind of triangle or quadrilateral a set of points forms, and then place general figures on axes to prove theorems for every triangle or parallelogram at once.

Parents: the algebra here is all from Algebra I. What is new is how it is used, as evidence in a proof. Students should state what they will prove, show every computation, and name the definition or theorem the numbers satisfy. The unit ends with a look back over the whole course, from Euclid's postulates to coordinate proofs, in preparation for the final exam.

PREREQUISITES

- Use the Pythagorean Theorem and its converse (Lessons 9.1 and 9.2).
- Find slopes and use them to recognize parallel and perpendicular lines (Lesson 3.5).
- Know the properties and tests for parallelograms, rectangles, rhombuses, squares, trapezoids and kites (Unit 6).
- Simplify square roots and expand expressions such as $(a+b)^2$.
- Write two-column, paragraph and flow proofs with a reason for every step.

MASTERY CHECKLIST

- ☐ I can find the distance, midpoint and slope for two points and explain why the Distance Formula comes from the Pythagorean Theorem.
- ☐ I can find the point that partitions a directed segment in a given ratio, and divide a segment in a given ratio with compass and straightedge.
- ☐ I can find the perimeter and area of a polygon on the coordinate plane, using the box method when no side is horizontal or vertical.
- ☐ I can use slopes to decide whether lines are parallel or perpendicular.
- ☐ I can prove that a triangle with given vertices is isosceles or right.
- ☐ I can prove that a quadrilateral with given vertices is a parallelogram, rectangle, rhombus, square, trapezoid or kite, or show that it is not.
- ☐ I can find a missing vertex of a parallelogram.
- ☐ I can place a general figure on coordinate axes without assuming anything extra, using coordinates like

$(2a, 0)$ to avoid fractions.

- ☐ I can write coordinate proofs of general theorems such as the Midsegment Theorem and the

diagonal properties of parallelograms and rectangles.

- ☐ I can compare a coordinate proof with Euclid's synthetic proof of the same theorem.

LESSONS IN THIS UNIT

13.1	**Coordinate Tools: Distance, Midpoint, Slope and Partitions**	45 minutes
13.2	**Classifying Figures with Coordinates**	2 sessions of 45 minutes
13.3	**Coordinate Proofs**	2 sessions of 45 minutes

Unit 13 test

You have 60 minutes. A scientific calculator is allowed for square roots and decimal approximations. You will need a compass and straightedge for Problem 12. Show all your work. Give exact answers (simplest radical form) unless a problem asks you to round. For every proof, state what is given and what you will prove, and give a reason for every step; in coordinate proofs, show each computation. The test is worth 100 points.

1. **5 pts**
   For $P(-4, 3)$ and $Q(8, -2)$, find (a) the distance PQ and (b) the midpoint of $\overline{PQ}$.

2. **6 pts**
   Line ℓ passes through $A(1, 4)$ and $B(5, -2)$. Line m passes through $C(0, 1)$ and $D(6, 5)$. Are the lines parallel, perpendicular, or neither? Show the slopes.

3. **6 pts**
   Find the point P that partitions the directed segment from $A(-3, -4)$ to $B(12, 6)$ in the ratio $2 : 3$.

4. **6 pts**
   Parallelogram $EFGH$ has vertices $E(-1, -2)$, $F(4, -1)$ and $G(6, 3)$. Find the coordinates of H.

5. **6 pts**

The line through $(-1, 2)$ and $(3, k)$ is perpendicular to the line through $(0, 0)$ and $(2, -4)$. Find k.

6. **7 pts**

On a ship's chart measured in nautical miles, a fishing boat at $S(2, 1)$ sails in a straight line to the harbor at $H(26, 11)$, as in the figure. (a) How far is the trip? (b) Where is the boat when it has gone $\frac{3}{4}$ of the way?

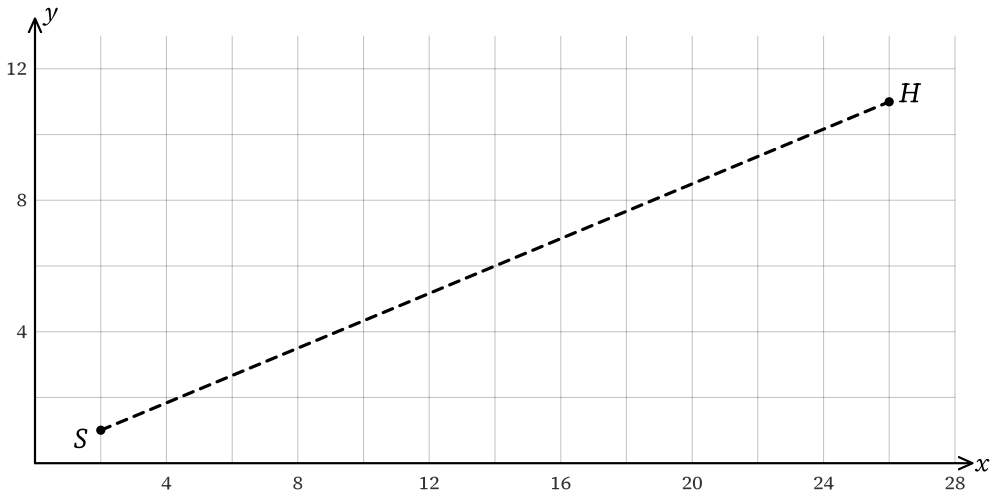
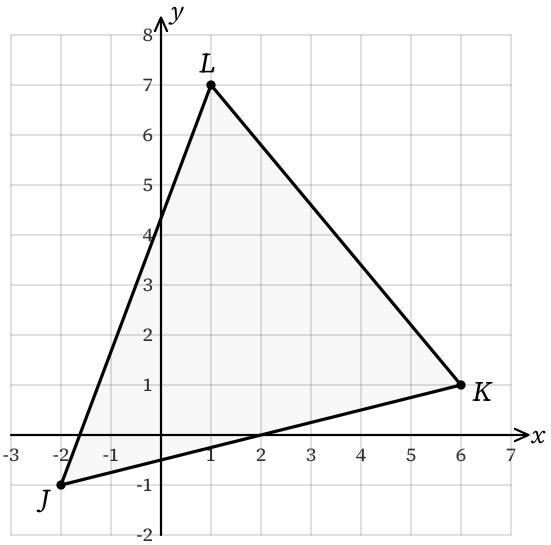


Chart in nautical miles

7. **8 pts**

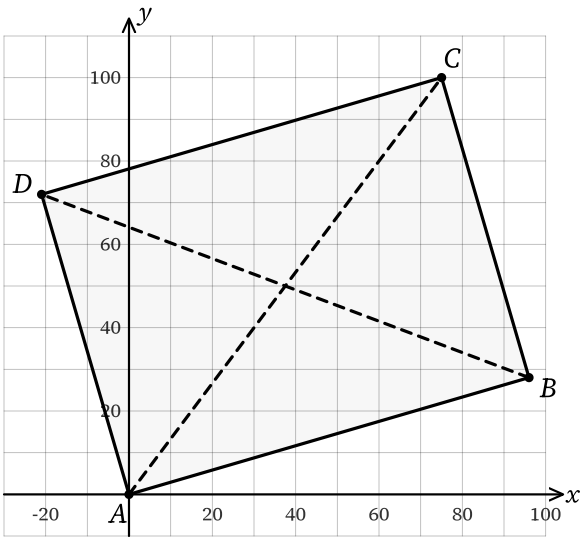
On the plan of a church courtyard, measured in meters, a triangular flower bed has corners $J(-2, -1)$, $K(6, 1)$ and $L(1, 7)$, as in the figure. Find (a) the area of the bed and (b) its perimeter to the nearest tenth of a meter.



Coordinates in meters

8. **7 pts**

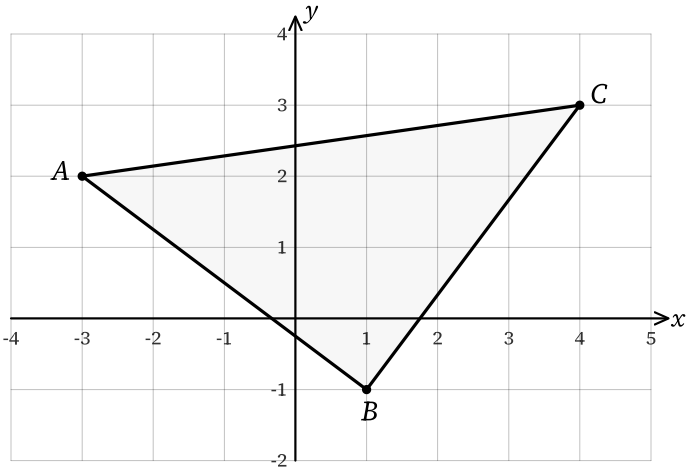
A coach marks the corners of a practice field at $A(0, 0)$, $B(96, 28)$, $C(75, 100)$ and $D(-21, 72)$, measured in yards, as in the figure. She has already checked that both pairs of opposite sides are congruent. (a) Find the length of diagonal $\overline{AC}$. (b) Find the length of diagonal $\overline{BD}$. (c) Is the field a rectangle?



Coordinates in yards

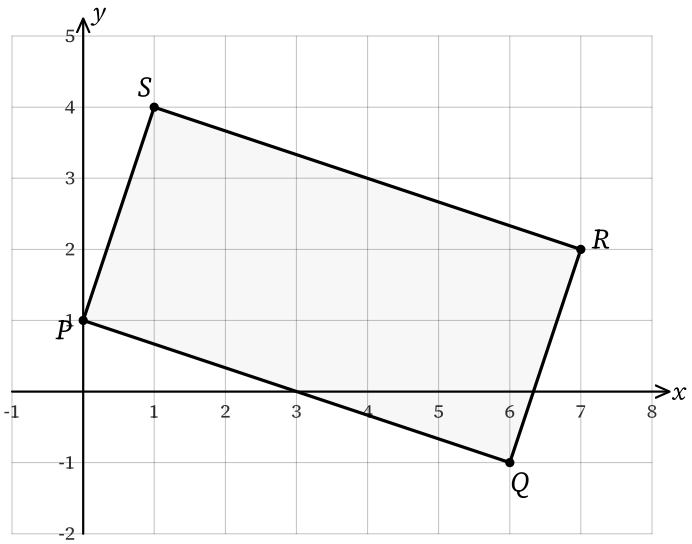
9. **8 pts**

Prove that $\triangle ABC$ with $A(-3, 2)$, $B(1, -1)$ and $C(4, 3)$ is an isosceles right triangle. Use a two-column proof.



10. **8 pts**

Prove that quadrilateral $PQRS$ with $P(0, 1)$, $Q(6, -1)$, $R(7, 2)$ and $S(1, 4)$ is a rectangle but not a square. Use a flow proof.



11. **6 pts**

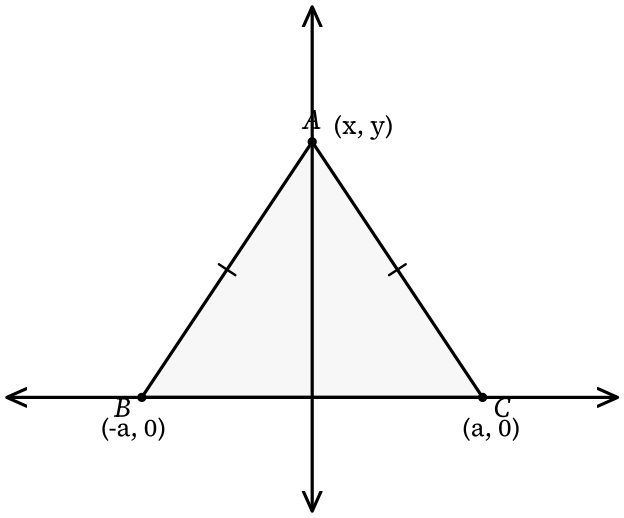
Henry shows that one pair of opposite sides of a quadrilateral have equal slopes and concludes that it is a parallelogram. Explain why this is not enough, and give the coordinates of a quadrilateral that shows he could be wrong.

12. **7 pts**

Draw a segment $\overline{AB}$ about 9 cm long. With compass and straightedge only, construct the point P on $\overline{AB}$ with $AP : PB = 1 : 2$. List your steps and explain why the construction works.

13. **10 pts**

Prove with coordinates: in an isosceles triangle, the median to the base is perpendicular to the base. Place $\triangle ABC$ with base $B(-a, 0)$, $C(a, 0)$, where $a > 0$, and apex $A(x, y)$ with $y > 0$ and $AB = AC$. Do not assume that A lies on the y-axis. Write a paragraph proof.



14. **10 pts**

Prove with coordinates: if the diagonals of a parallelogram are congruent, the parallelogram is a rectangle. Place parallelogram $ABCD$ with $A(0, 0)$, $B(a, 0)$ and $D(b, c)$, where $a > 0$ and $c > 0$. Use a two-column proof.

ANSWER KEY

1. (a) $PQ = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$. (b) $\left(\dfrac{-4+8}{2}, \dfrac{3+(-2)}{2}\right) = (2, 0.5)$.
2. Slope of $\ell = \dfrac{-6}{4} = -\dfrac{3}{2}$; slope of $m = \dfrac{4}{6} = \dfrac{2}{3}$. The product is -1, so the lines are perpendicular.
3. $k = \dfrac{2}{5}$; run 15, rise 10; $P = (-3+6,\ -4+4) = (3, 0)$
4. $H = E + G - F = (-1+6-4,\ -2+3-(-1)) = (1, 2)$
5. The second line has slope -2, so the first must have slope $\dfrac{1}{2}$: $\dfrac{k-2}{4} = \dfrac{1}{2}$, so $k = 4$.
6. (a) $SH = \sqrt{24^2 + 10^2} = \sqrt{676} = 26$ nautical miles. (b) $\dfrac{3}{4}$ of the run 24 is 18 and of the rise 10 is 7.5, so the boat is at $(20, 8.5)$.
7. (a) Box method: the rectangle from $x = -2$ to 6 and $y = -1$ to 7 has area $8 \times 8 = 64$. Subtract the outside right triangles: 8 (legs 8, 2), 15 (legs 5, 6) and 12 (legs 3, 8). Area $= 64 - 35 = 29$ square meters. (b) $JK = \sqrt{68} \approx 8.246$, $KL = \sqrt{61} \approx 7.810$, $LJ = \sqrt{73} \approx 8.544$; perimeter ≈ 24.6 meters.
8. (a) $AC = \sqrt{75^2 + 100^2} = 125$ yd. (b) $BD = \sqrt{(-117)^2 + 44^2} = \sqrt{15625} = 125$ yd. (c) Yes: both pairs of opposite sides are congruent, so it is a parallelogram, and a parallelogram with congruent diagonals is a rectangle.
9. See the model proof: $AB = BC = 5$ and $\overline{AB} \perp \overline{BC}$.

TWO-COLUMN PROOF

Given: $A(-3, 2)$, $B(1, -1)$, $C(4, 3)$

Prove: $\triangle ABC$ is an isosceles right triangle

	STATEMENTS	REASONS
1	$A(-3, 2)$, $B(1, -1)$, $C(4, 3)$	Given
2	$AB = \sqrt{4^2 + (-3)^2} = 5$ and $BC = \sqrt{3^2 + 4^2} = 5$	Distance Formula
3	$\overline{AB} \cong \overline{BC}$	Definition of congruent segments
4	$\triangle ABC$ is isosceles	Definition of isosceles triangle
5	Slope of $\overline{AB} = -\dfrac{3}{4}$; slope of $\overline{BC} = \dfrac{4}{3}$	Slope formula

	STATEMENTS	REASONS
6	$\overline{AB} \perp \overline{BC}$	Slopes with product $-\frac{3}{4} \cdot \frac{4}{3} = -1$
7	$\angle ABC$ is a right angle	Definition of perpendicular lines
8	$\triangle ABC$ is an isosceles right triangle	Definitions of isosceles triangle and right triangle

∎

10. See the model proof: opposite slopes are equal, consecutive slopes multiply to -1, and $PQ = \sqrt{40} \neq QR = \sqrt{10}$.

FLOW PROOF

Given: $P(0,1)$, $Q(6,-1)$, $R(7,2)$, $S(1,4)$

Prove: $PQRS$ is a rectangle but not a square

(1) $P(0,1)$, $Q(6,-1)$, $R(7,2)$, $S(1,4)$
Given

(2) Slopes: $\overline{PQ}$: $-\frac{1}{3}$; $\overline{QR}$: 3; $\overline{RS}$: $-\frac{1}{3}$; $\overline{SP}$: 3
Slope formula
from (1)

(3) $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$
Lines with equal slopes are parallel
from (2)

(4) $PQRS$ is a parallelogram
Definition of parallelogram
from (3)

(5) Each pair of consecutive sides is perpendicular
Slopes with product $-\frac{1}{3} \cdot 3 = -1$
from (2)

(6) $\angle P$, $\angle Q$, $\angle R$, $\angle S$ are right angles
Definition of perpendicular lines
from (5)

(7) $PQRS$ is a rectangle
Definition of rectangle

from (4), (6)

(8) $PQ = \sqrt{6^2 + (-2)^2} = \sqrt{40}$ and $QR = \sqrt{1^2 + 3^2} = \sqrt{10}$
Distance Formula
from (1)

(9) $PQRS$ is not a square
A square has four congruent sides, but $PQ \neq QR$
from (8)

(10) $PQRS$ is a rectangle but not a square
Steps 7 and 9 together
from (7), (9)

∎

11. Equal slopes for one pair of sides prove only that those two sides are parallel. A parallelogram needs both pairs parallel (or one pair both parallel and congruent, Euclid I.33). Example: $A(0,0)$, $B(6,0)$, $C(4,3)$, $D(1,3)$: $\overline{AB}$ and $\overline{DC}$ both have slope 0, but $\overline{BC}$ has slope $-\dfrac{3}{2}$ and $\overline{AD}$ has slope 3, so this is a trapezoid, not a parallelogram. (Any correct example earns credit.)
12. Steps: (1) Draw a ray from A not along $\overline{AB}$. (2) With a fixed compass width, mark three equal segments along the ray from A: X_1, X_2, X_3. (3) Draw $\overline{X_3B}$. (4) At X_1, copy $\angle AX_3B$ to draw a line through X_1 parallel to $\overline{X_3B}$; it meets $\overline{AB}$ at P. Why: in $\triangle AX_3B$, the line through X_1 is parallel to side $\overline{X_3B}$, so by the Triangle Proportionality Theorem (Euclid VI.2) $AP : PB = AX_1 : X_1X_3 = 1 : 2$.
13. See the model proof: $AB = AC$ forces $x = 0$, so the median from A lies along the y-axis.

    PARAGRAPH PROOF

    Given: $\triangle ABC$ with $B(-a, 0)$, $C(a, 0)$, $A(x, y)$, $a > 0$, $y > 0$; $\overline{AB} \cong \overline{AC}$
    Prove: The median from A to $\overline{BC}$ is perpendicular to $\overline{BC}$
    Since $AB = AC$, the Distance Formula gives $\sqrt{(x+a)^2 + y^2} = \sqrt{(x-a)^2 + y^2}$. Squaring both sides and subtracting y^2 gives $(x+a)^2 = (x-a)^2$, that is, $x^2 + 2ax + a^2 = x^2 - 2ax + a^2$. So $4ax = 0$, and since $a \neq 0$, $x = 0$. Thus $A = (0, y)$. By the Midpoint Formula, the midpoint of $\overline{BC}$ is $\left(\dfrac{-a+a}{2}, \dfrac{0+0}{2}\right) = (0, 0)$, so the median from A is the segment from $(0, y)$ to $(0, 0)$. Both endpoints have x-coordinate 0, so the median is vertical. The base $\overline{BC}$ lies on the x-axis, so it is horizontal.

A vertical line and a horizontal line are perpendicular, so the median to the base is perpendicular to the base.

∎

14. See the model proof: $AC = BD$ forces $b = 0$, so the sides are horizontal and vertical.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram with $A(0,0)$, $B(a,0)$, $D(b,c)$; $a > 0$, $c > 0$; $\overline{AC} \cong \overline{BD}$

Prove: $ABCD$ is a rectangle

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram with $A(0,0)$, $B(a,0)$, $D(b,c)$, and $\overline{AC} \cong \overline{BD}$	Given
2	$C = (a+b, c)$	Fourth Vertex of a Parallelogram (Lesson 13.2): $C = B + D - A$
3	$AC^2 = (a+b)^2 + c^2$ and $BD^2 = (b-a)^2 + c^2$	Distance Formula, squared
4	$(a+b)^2 + c^2 = (b-a)^2 + c^2$	Definition of congruent segments ($AC = BD$) and Substitution
5	$4ab = 0$	Expand both sides; Subtraction Property of Equality
6	$b = 0$	Division Property of Equality ($a \neq 0$)
7	$D = (0, c)$ and $C = (a, c)$	Substitution
8	$\overline{AB}$ and $\overline{DC}$ are horizontal; $\overline{AD}$ and $\overline{BC}$ are vertical	$\overline{AB}$ lies on $y = 0$ and $\overline{DC}$ on $y = c$; $\overline{AD}$ lies on $x = 0$ and $\overline{BC}$ on $x = a$
9	$\angle A$, $\angle B$, $\angle C$, $\angle D$ are right angles	A horizontal line and a vertical line are perpendicular; definition of perpendicular lines

	STATEMENTS	REASONS
10	$ABCD$ is a rectangle	Definition of rectangle (a parallelogram with four right angles)

■

13.1 Coordinate Tools: Distance, Midpoint, Slope and Partitions

OBJECTIVES

- Use the distance, midpoint and slope formulas together to describe segments in the coordinate plane.
- Find the point that partitions a directed segment in a given ratio.
- Find the perimeter and area of polygons whose vertices are given by coordinates.
- Decide with slopes whether sides of a figure are parallel or perpendicular.

WATCH

Deriving the Distance Formula

Eddie Woo · 10:26

https://www.youtube.com/watch?v=99MRLjoj5Tk

> **WHEN TO WATCH**
> Before Example 1: deriving the distance formula from Pythagoras.

Note: Eddie Woo teaches in Australia: he says 'interval' for segment and 'gradient' for slope.

BACKUP VIDEO

Point Along a Directed Segment

Mario's Math Tutoring · 2:02

https://www.youtube.com/watch?v=z4elAAM8Q1s

Before the partition examples: finding the point 3/5 of the way along a directed segment.

KEY TERMS

Distance Formula

The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Midpoint Formula

The midpoint of the segment joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

slope

For a nonvertical line through (x_1, y_1) and (x_2, y_2), the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$ (rise over run). Vertical lines have no slope.

directed segment

A segment with a starting point and an ending point, such as the segment from A to B. The direction matters when we partition it.

partition

A point P on $\overline{AB}$ partitions the directed segment from A to B in the ratio $a : b$ when $AP : PB = a : b$.

CORE IDEAS

René Descartes' great idea was to name every point by two numbers. Once points have coordinates, questions about lengths, midpoints and parallel lines become arithmetic. In this lesson you gather the four coordinate tools you have met before (distance, midpoint, slope and partitions) and use them together. In the next two lessons you will use them to prove things.

Distance is the Pythagorean Theorem in disguise. To find the distance from $A(x_1, y_1)$ to $B(x_2, y_2)$, draw a horizontal leg and a vertical leg to make a right triangle with hypotenuse $\overline{AB}$. The horizontal leg is $|x_2 - x_1|$ long and the vertical leg is $|y_2 - y_1|$ long, so $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Squaring removes any negative signs, which is why it does not matter which point you call the first.

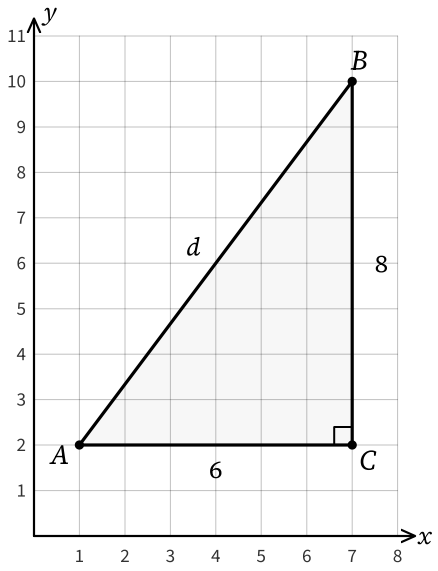
The midpoint is the average. Halfway from A to B you have gone half the run and half the rise, so each coordinate of the midpoint is halfway between the two coordinates: their average. The proof below checks this with the Distance Formula.

Slope tells direction. Slope is rise over run, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. From Lesson 3.5: two nonvertical lines are parallel exactly when their slopes are equal, and perpendicular exactly when their slopes multiply to -1. A horizontal line (slope 0) is perpendicular to a vertical line (no slope). Always subtract in the same order on top and bottom.

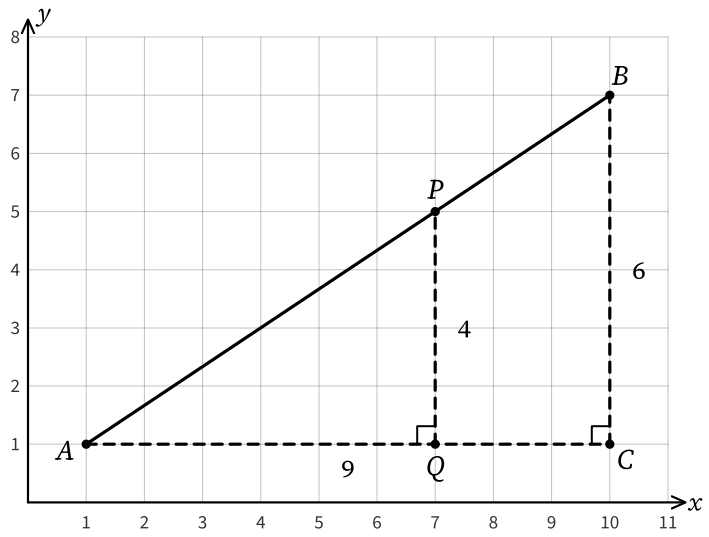
Partitioning a segment. To find the point P with $AP : PB = a : b$, notice that P is $\dfrac{a}{a + b}$ of the way from A to B (the whole segment has $a + b$ equal parts and AP is a of them). So start at A and add that fraction of the run and that fraction of the rise. Why does this land on the segment in the right place? The small right triangle from

A to P and the large one from A to B have the same shape, and parallel lines cut segments proportionally (Lesson 8.5), so going a fraction k of the run and of the rise means going the same fraction k of the distance. Notice that you start at A: the ratio $1:3$ from A is a different point from the ratio $1:3$ from B.

Perimeter and area on the grid. Perimeter is the sum of the side lengths found with the Distance Formula. For area, if a side is horizontal or vertical, use it as a base and read the height off the grid. If no side is, use the *box method*: enclose the figure in the smallest rectangle with horizontal and vertical sides, then subtract the right triangles (and any rectangles) that lie outside the figure. This works because of the Area Addition Postulate (Lesson 11.1): the rectangle's area is the sum of the areas of the non-overlapping pieces.



The horizontal and vertical legs make a right triangle with hypotenuse AB.

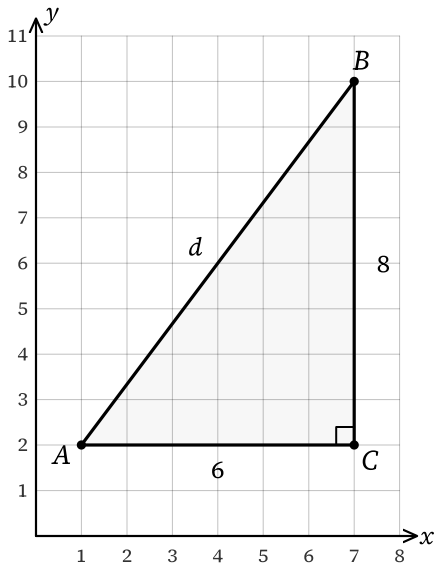


P is two-thirds of the way from A to B: two-thirds of the run 9 and two-thirds of the rise 6.

POSTULATES AND THEOREMS

THEOREM **Distance Formula**

The distance between $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is* $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



The horizontal and vertical legs make a right triangle with hypotenuse AB.

PARAGRAPH PROOF

Given: $A(x_1, y_1)$ and $B(x_2, y_2)$

Prove: $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Suppose first that $x_1 \neq x_2$ and $y_1 \neq y_2$. Let C be the point (x_2, y_1). Then $\overline{AC}$ is horizontal and $\overline{BC}$ is vertical. Horizontal and vertical lines are perpendicular, so $\angle ACB$ is a right angle and $\triangle ACB$ is a right triangle with hypotenuse $\overline{AB}$.

On a horizontal line, the distance between two points is the absolute value of the difference of their x-coordinates (Ruler Postulate), so $AC = |x_2 - x_1|$. In the same way, $BC = |y_2 - y_1|$. By the Pythagorean Theorem, $AB^2 = AC^2 + BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, because squaring a number and squaring its absolute value give the same result. Since AB is positive, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

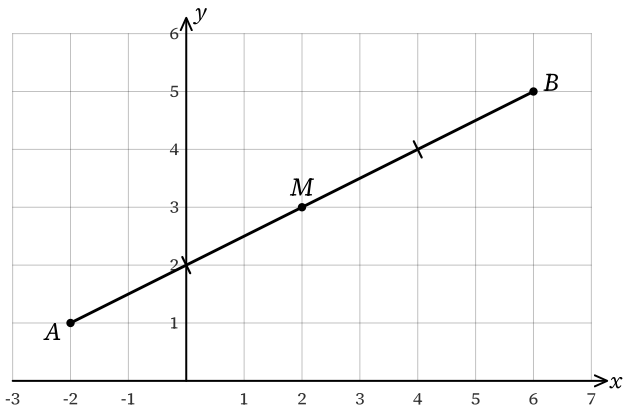
If $y_1 = y_2$, the two points lie on one horizontal line, and $AB = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2 + 0^2}$, which is the formula again. The case $x_1 = x_2$ is the same with the roles of x and y exchanged. So the formula is true in every case.

∎

This is Euclid's I.47 (the Pythagorean Theorem) applied to a right triangle whose legs are horizontal and vertical.

THEOREM **Midpoint Formula**

The midpoint of the segment joining $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is* $M\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.



PARAGRAPH PROOF

Given: $A(x_1, y_1)$ and $B(x_2, y_2)$; $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Prove: M is the midpoint of $\overline{AB}$

The x-coordinate of M minus the x-coordinate of A is $\dfrac{x_1 + x_2}{2} - x_1 = \dfrac{x_2 - x_1}{2}$, and in the same way the difference of the y-coordinates is $\dfrac{y_2 - y_1}{2}$. By the Distance Formula, $AM = \sqrt{\dfrac{(x_2 - x_1)^2}{4} + \dfrac{(y_2 - y_1)^2}{4}} = \dfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \dfrac{1}{2}AB$.

Likewise $x_2 - \dfrac{x_1 + x_2}{2} = \dfrac{x_2 - x_1}{2}$ and $y_2 - \dfrac{y_1 + y_2}{2} = \dfrac{y_2 - y_1}{2}$, so $MB = \dfrac{1}{2}AB$ as well. Therefore $AM + MB = AB$.

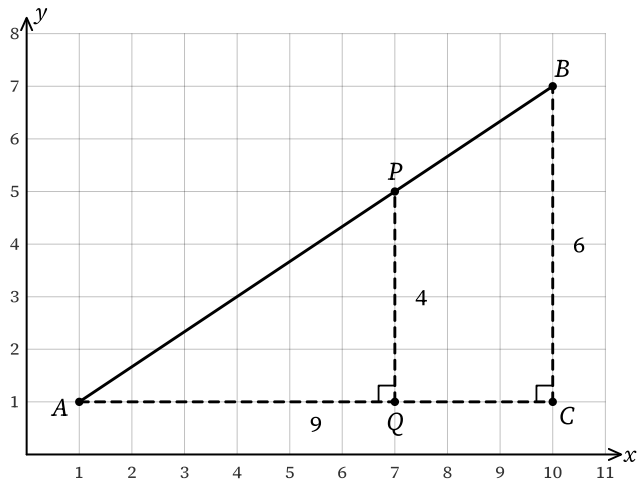
This forces M to lie on $\overline{AB}$. If M were not on line AB, the Triangle Inequality Theorem would give $AM + MB > AB$. If M were on line AB but outside the

segment, say beyond B, then $AM = AB + BM > AB$ (Segment Addition Postulate), which is impossible since $AM = \frac{1}{2}AB$; beyond A is impossible for the same reason. So M is on $\overline{AB}$ with $AM = MB$, and by the definition of midpoint, M is the midpoint of $\overline{AB}$.

∎

THEOREM **Partition Formula**

Let $0 < k < 1$. The point $P = (x_1 + k(x_2 - x_1),\ y_1 + k(y_2 - y_1))$ lies on $\overline{AB}$ with $AP = k \cdot AB$. Taking $k = \frac{a}{a+b}$ gives the point that partitions the directed segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ in the ratio $a : b$.



P is two-thirds of the way from A to B: two-thirds of the run 9 and two-thirds of the rise 6.

PARAGRAPH PROOF

Given: $A(x_1, y_1)$, $B(x_2, y_2)$ and $0 < k < 1$; $P = (x_1 + k(x_2 - x_1),\ y_1 + k(y_2 - y_1))$
Prove: P is on $\overline{AB}$ with $AP = k \cdot AB$ and $PB = (1 - k) \cdot AB$
From A to P the coordinates change by $k(x_2 - x_1)$ and $k(y_2 - y_1)$. By the Distance Formula, $AP = \sqrt{k^2(x_2 - x_1)^2 + k^2(y_2 - y_1)^2} = k\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = k \cdot AB$, since $k > 0$.

From P to B the coordinates change by $x_2 - x_1 - k(x_2 - x_1) = (1 - k)(x_2 - x_1)$ and, in the same way, $(1 - k)(y_2 - y_1)$. Since $1 - k > 0$, the same computation gives $PB = (1 - k) \cdot AB$.

So $AP + PB = k \cdot AB + (1 - k) \cdot AB = AB$, and exactly as in the proof of the Midpoint Formula this means P lies on $\overline{AB}$. Finally, $AP : PB = k : (1 - k)$. When $k = \frac{a}{a+b}$, we get $1 - k = \frac{b}{a+b}$, so $AP : PB = a : b$.

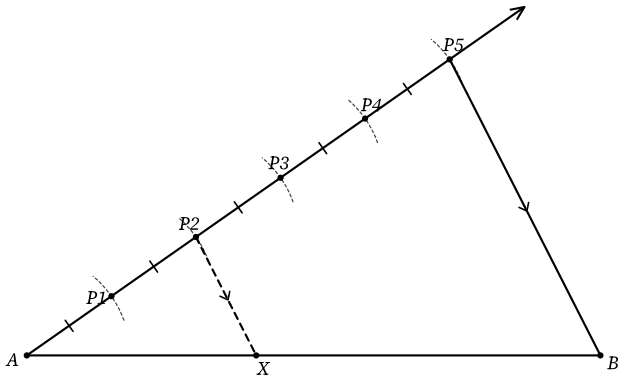
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The midpoint is the special case $k = \frac{1}{2}$, the ratio $1 : 1$.

CONSTRUCTIONS

CONSTRUCTION **Divide a segment in a given ratio (here 2 : 3)**

Euclid VI.9, VI.10



1. Draw the given segment $\overline{AB}$. From A, draw a ray that is not along $\overline{AB}$.
2. Set your compass to any convenient width. Starting at A, mark off 5 equal segments along the ray (since $2 + 3 = 5$), naming the marks P_1, P_2, P_3, P_4, P_5. Keep the compass width fixed.
3. Draw $\overline{P_5B}$.
4. Through P_2 (the end of the first 2 parts), construct a line parallel to $\overline{P_5B}$ by copying $\angle AP_5B$ at P_2 (Euclid I.23 and I.31). Call the point where it meets $\overline{AB}$ X.
5. Then X divides $\overline{AB}$ so that $AX : XB = 2 : 3$.

Why it works. In $\triangle AP_5B$, the line $\overleftrightarrow{P_2X}$ is parallel to the side $\overline{P_5B}$, so by the Triangle Proportionality Theorem (Lesson 8.5, Euclid VI.2) it divides the other two sides proportionally: $AX : XB = AP_2 : P_2P_5 = 2 : 3$. This is the compass-

and-straightedge twin of the Partition Formula.

KEY FORMULAS

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

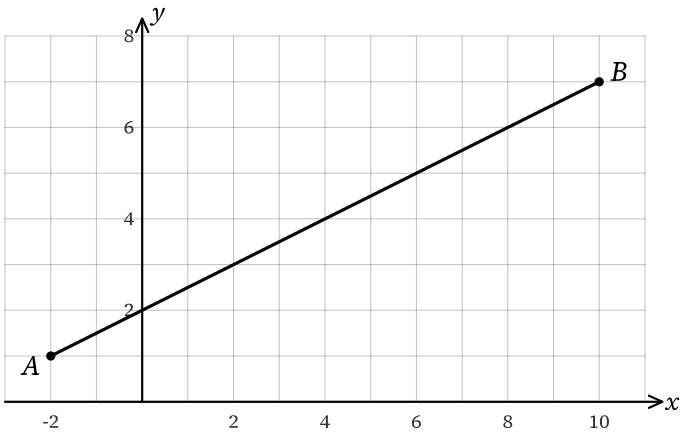
$$P = (x_1 + k(x_2 - x_1),\ y_1 + k(y_2 - y_1)), \quad k = \frac{a}{a + b}$$

From history. René Descartes published *La Géométrie* in 1637, showing how to describe curves by equations; Pierre de Fermat, a French lawyer, worked out similar ideas independently at about the same time. Locating places by two numbers is much older: in the second century AD, Ptolemy's *Geography* listed thousands of places by latitude and longitude. Surveyors still lay out fields and roads by measuring east-west and north-south distances from a fixed corner, just as you do on a coordinate grid.

Euclid connection. The Distance Formula is Euclid's I.47 ('the square on the side subtending the right angle is equal to the squares on the sides containing the right angle') applied to a horizontal and a vertical leg. Partitioning a segment is the coordinate form of Euclid VI.9 and VI.10, where Euclid cuts off a prescribed part of a straight line by drawing a parallel.

Practice habit. Recite the four formulas until they are automatic, and say the Distance Formula as 'the Pythagorean Theorem in disguise.' Before computing, plot the points on a quick sketch, label which point is (x_1, y_1), and estimate the answer so you can check it.

Example 1. Find the point P that partitions the directed segment from $A(-2, 1)$ to $B(10, 7)$ in the ratio $2 : 1$. Then check your answer with the Distance Formula.

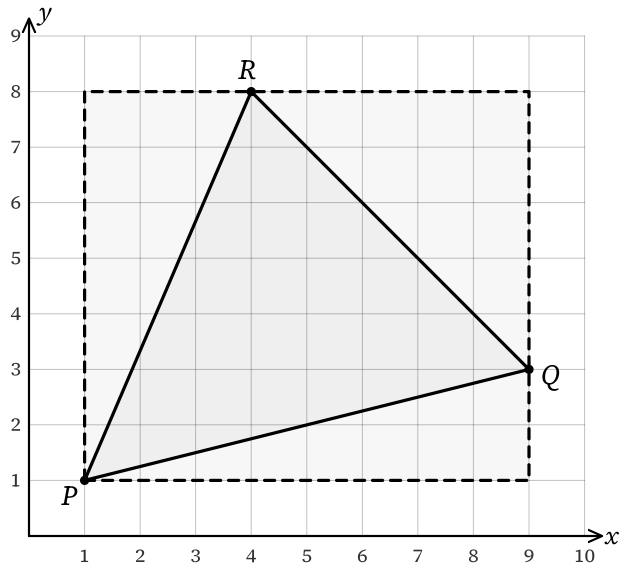


Show solution

1. The ratio $2 : 1$ splits the segment into $2 + 1 = 3$ equal parts, and AP is 2 of them, so P is $k = \dfrac{2}{3}$ of the way from A to B.
2. Run: $10 - (-2) = 12$. Rise: $7 - 1 = 6$. Two-thirds of the run is 8 and two-thirds of the rise is 4.
3. Start at A and add: $P = (-2 + 8,\ 1 + 4) = (6, 5)$.
4. Check: $AP = \sqrt{8^2 + 4^2} = \sqrt{80} = 4\sqrt{5}$ and $PB = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$. Indeed $AP = 2 \cdot PB$, so $AP : PB = 2 : 1$.

Answer: $P(6, 5)$

Example 2. Grace is planning a triangular flower bed. On her garden plan each grid unit is 1 foot, and the corners are $P(1, 1)$, $Q(9, 3)$ and $R(4, 8)$. (a) How many feet of edging does she need to go all the way around the bed, to the nearest tenth of a foot? (b) What is the area of the bed?



1 grid unit = 1 foot

Show solution

1. Side lengths by the Distance Formula: $PQ = \sqrt{8^2 + 2^2} = \sqrt{68} \approx 8.246$; $QR = \sqrt{(-5)^2 + 5^2} = \sqrt{50} \approx 7.071$; $RP = \sqrt{3^2 + 7^2} = \sqrt{58} \approx 7.616$.
2. Perimeter: $8.246 + 7.071 + 7.616 \approx 22.933$, so about 22.9 feet of edging.
3. No side is horizontal or vertical, so use the box method. The smallest rectangle around the triangle runs from $x = 1$ to $x = 9$ and $y = 1$ to $y = 8$: its area is $8 \times 7 = 56$ square feet.
4. Subtract the three right triangles outside the bed: under $\overline{PQ}$ (legs 8 and 2) has area 8; right of $\overline{QR}$ (legs 5 and 5) has area 12.5; left of $\overline{RP}$ (legs 3 and 7) has area 10.5. Total $8 + 12.5 + 10.5 = 31$.
5. Area of the bed: $56 - 31 = 25$ square feet.

Answer: (a) about 22.9 ft of edging; (b) 25 square feet

PRACTICE

1. Easy Find the distance between $(-3, 2)$ and $(5, -4)$.

   Show answer

   $$\sqrt{8^2 + (-6)^2} = \sqrt{100} = 10$$

2. Easy Find the midpoint of the segment joining $(-7, 4)$ and $(3, -10)$.

   Show answer

   $$\left(\frac{-7 + 3}{2}, \frac{4 + (-10)}{2}\right) = (-2, -3)$$

3. Easy Find the slope of the line through $(2, -5)$ and $(6, 7)$.

   Show answer

   $$m = \frac{7 - (-5)}{6 - 2} = \frac{12}{4} = 3$$

4. Easy $M(4, -1)$ is the midpoint of $\overline{AB}$, and A is $(-2, 5)$. Find the coordinates of B.

   Show answer

   From A to M the coordinates change by $+6$ and -6; do it again from M: $B(10, -7)$.

5. Medium Find the point P that partitions the directed segment from $A(1, -3)$ to $B(13, 5)$ in the ratio $3 : 1$.

   Show answer

   $k = \frac{3}{4}$; run 12, rise 8; $P = (1 + 9,\ -3 + 6) = (10, 3)$

6. Medium Find the two points that divide the segment from $A(-4, 7)$ to $B(8, -2)$ into three congruent parts (the trisection points). Give the point nearer A first.

   Show answer

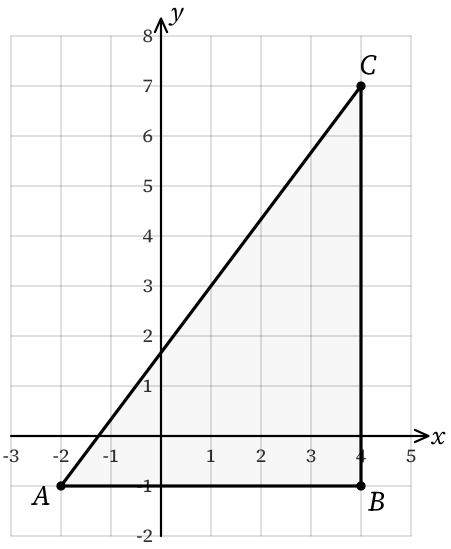
   Run 12, rise -9; one third is $(4, -3)$. The points are $(0, 4)$ and $(4, 1)$.

7. **Medium** On Henry's trail map, each unit is 1 kilometer. He walks in a straight line from camp $C(-4, -2)$ to a lookout $L(11, 18)$, and plans to rest $\frac{2}{5}$ of the way there. (a) What are the coordinates of the rest stop? (b) How far is the rest stop from camp?

Show answer

(a) Run 15, rise 20; $\frac{2}{5}$ of each is 6 and 8, so the rest stop is $(2, 6)$. (b) $CL = \sqrt{15^2 + 20^2} = 25$ km, and $\frac{2}{5}$ of 25 is 10 km.

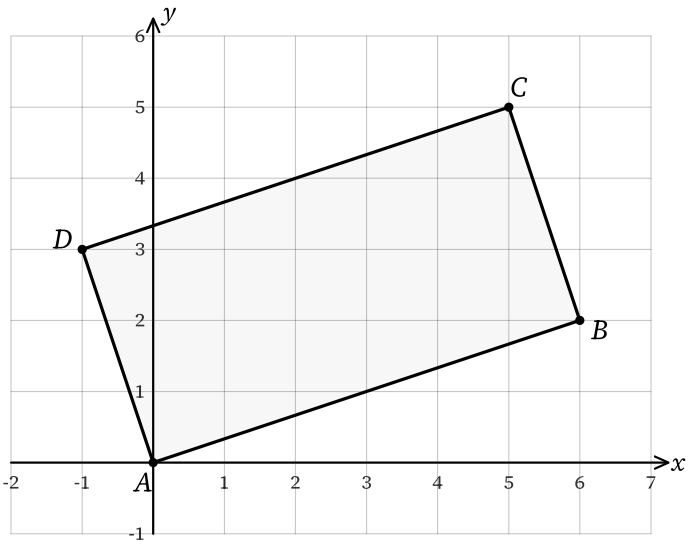
8. **Medium** In the figure, $\triangle ABC$ has vertices $A(-2, -1)$, $B(4, -1)$ and $C(4, 7)$. Find (a) its perimeter and (b) its area.



Show answer

$AB = 6$, $BC = 8$, $AC = \sqrt{6^2 + 8^2} = 10$. (a) Perimeter $= 24$ units. (b) $\overline{AB}$ is horizontal and $\overline{BC}$ is vertical, so they are perpendicular legs: area $= \frac{1}{2}(6)(8) = 24$ square units.

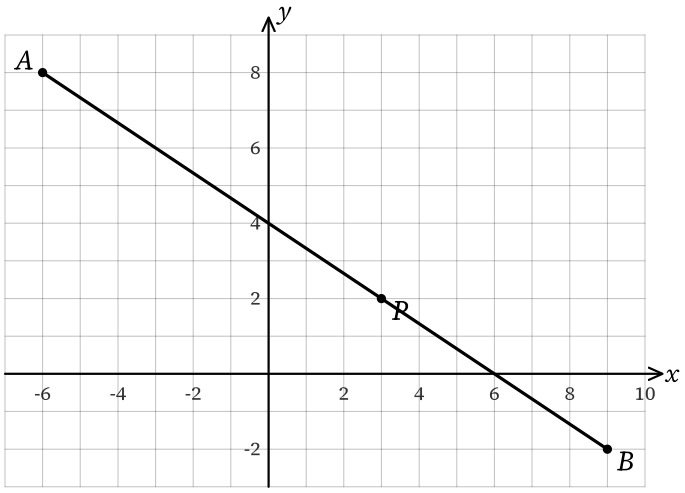
9. **Medium** Quadrilateral $ABCD$ in the figure has vertices $A(0,0)$, $B(6,2)$, $C(5,5)$ and $D(-1,3)$. Find the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, in that order. Which sides are parallel, and which are perpendicular?



Show answer

Slopes: AB: $\frac{1}{3}$; BC: -3; CD: $\frac{1}{3}$; DA: -3. So $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$, and every pair of sides meeting at a vertex is perpendicular, because $\frac{1}{3} \cdot (-3) = -1$.

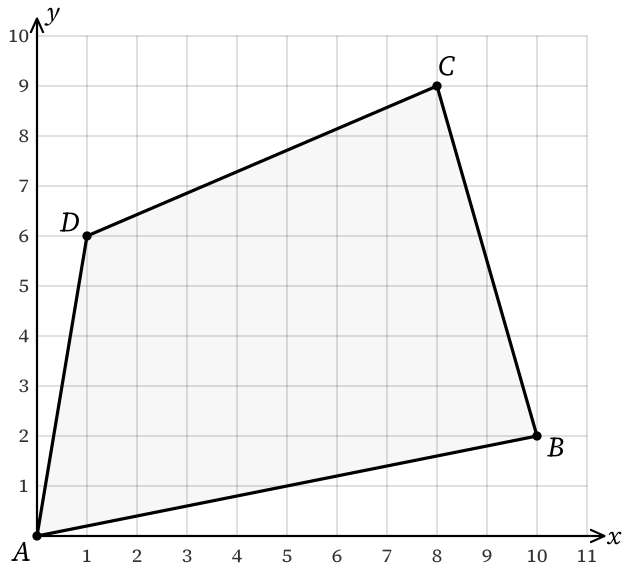
10. **Medium** In the figure, $P(3, 2)$ lies on the segment from $A(-6, 8)$ to $B(9, -2)$. What fraction of the way from A to B is P? Then give the ratio $AP : PB$.



Show answer

The full run is 15 and the run from A to P is 9, so P is $\frac{9}{15} = \frac{3}{5}$ of the way (the rise agrees: -6 out of -10). So $AP : PB = 3 : 2$.

11. Hard Samuel surveys a four-sided pasture. On his map each grid unit is 10 meters, and the corners are $A(0,0)$, $B(10,2)$, $C(8,9)$ and $D(1,6)$, as in the figure. Find the area of the pasture in square meters.



1 grid unit = 10 meters

Show answer

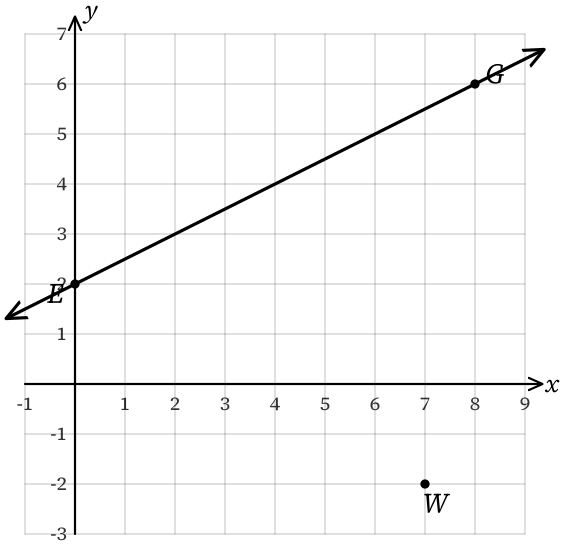
Box method: the rectangle from $x = 0$ to 10 and $y = 0$ to 9 has area 90 square units. Outside the pasture are right triangles of area 10 (legs 10, 2), 7 (legs 2, 7), 10.5 (legs 7, 3) and 3 (legs 1, 6), plus a 1-by-3 rectangle in the upper left of area 3. Total 33.5, so the pasture is $90 - 33.5 = 56.5$ square units. Each square unit is $10 \times 10 = 100$ square meters, so the area is 5650 square meters.

12. Hard Ruth says that to find the point one third of the way from $A(x_1, y_1)$ to $B(x_2, y_2)$ you can use $\left(\dfrac{x_1 + x_2}{3}, \dfrac{y_1 + y_2}{3}\right)$. Test her rule with $A(3,0)$ and $B(9,6)$, find the correct point, and explain why her rule does not work.

Show answer

Ruth's rule gives $(4, 2)$. The correct point starts at A and adds one third of the run (6) and of the rise (6): $(3 + 2,\ 0 + 2) = (5, 2)$. Her rule fails because it does not start from A; dividing a sum by 3 is not 'one third of the way.' (It works for the midpoint only because the average $\dfrac{x_1 + x_2}{2}$ happens to equal $x_1 + \dfrac{1}{2}(x_2 - x_1)$.) In general the point is $x_1 + \dfrac{1}{3}(x_2 - x_1) = \dfrac{2x_1 + x_2}{3}$.

13. Hard On a farm map with units in yards, a straight water line runs through $E(0, 2)$ and $G(8, 6)$. A well is at $W(7, -2)$, as in the figure. A pipe will run from the well to the water line along the shortest possible path. (a) At what point does the pipe meet the water line? (b) How long is the pipe, to the nearest tenth of a yard?



Water line through E and G; well at W (units in yards)

Show answer

The shortest path is perpendicular to the line (Lesson 3.4). The water line has slope $\frac{1}{2}$ and equation $y = \frac{1}{2}x + 2$. The perpendicular through W has slope -2: $y = -2x + 12$. Solving $\frac{1}{2}x + 2 = -2x + 12$ gives $x = 4$, $y = 4$. (a) $(4, 4)$. (b) Length $= \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5} \approx 6.7$ yards.

14. Challenge The point $P(2, 5)$ partitions the directed segment from $A(-1, 7)$ to B in the ratio $1 : 3$. Find the coordinates of B.

Show answer

P is $\frac{1}{4}$ of the way from A to B. From A to P the coordinates change by $(3, -2)$, which is one quarter of the full change, so the full change is $(12, -8)$ and $B = (-1 + 12,\ 7 - 8) = (11, -1)$.

COMMON MISTAKES

- Mixing the order of subtraction in the slope formula, such as $\frac{y_2 - y_1}{x_1 - x_2}$, which flips the sign. Always subtract in the same order on top and bottom.

- Using $\frac{a}{b}$ instead of $\frac{a}{a+b}$ for a partition. The ratio $2:1$ means two-thirds of the way, not 'twice the way.' Count the total number of parts first.
- Starting a partition from the wrong end, or averaging as in the midpoint rule. Always start at the first point of the directed segment and add the fraction of the run and rise.
- Taking square roots too early: $\sqrt{8^2+6^2}$ is 10, not $8+6=14$. Add the squares first, then take one square root.

> **Make it a game · Three Tools Sprint.** Parent gives two points, such as (1, 2) and (7, 10). In 30 seconds, find the distance, the midpoint and the slope. Then find the point two fifths of the way from the first point to the second. Five pairs per round. Record your time and beat it the next round. Check partition answers by plotting on graph paper.

13.2 Classifying Figures with Coordinates

OBJECTIVES

- Prove with distances and slopes that a triangle with given vertices is isosceles or right.
- Prove that a quadrilateral with given vertices is a parallelogram, rectangle, rhombus, square, trapezoid or kite, or show that it is not.
- Write a clear coordinate argument that names the definition or theorem each computation proves.
- Find a missing vertex of a parallelogram.

WATCH

Understanding a triangle through Coordinate Geometry

Eddie Woo · 10:36

https://www.youtube.com/watch?v=RRGkuCNhZWA

> **WHEN TO WATCH**
> Before Example 1: using lengths and gradients to classify a triangle.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Coordinate Geometry - example test question

Eddie Woo · 7:21

https://www.youtube.com/watch?v=KxDSx_MTIy8

Before the practice set: a full coordinate-geometry test question.

KEY TERMS

coordinate argument

A proof about a figure with given coordinates: compute the needed lengths, slopes or midpoints, then state which definition or theorem the numbers satisfy.

rhombus

A quadrilateral with four congruent sides.

rectangle

A parallelogram with four right angles.

square

A quadrilateral with four congruent sides and four right angles (a figure that is both a rhombus and a rectangle).

trapezoid

A quadrilateral with exactly one pair of parallel sides, called the bases; the other two sides are the legs. An isosceles trapezoid has congruent legs.

kite

A quadrilateral with two pairs of consecutive congruent sides, where opposite sides are not congruent.

CORE IDEAS

A figure on a grid can *look* like a square and still be slightly off, just as a drawing can mislead in a synthetic proof. Coordinates let you settle the question with exact numbers. In this lesson you prove what kind of figure you have by computing lengths, slopes and midpoints, and then naming the definition or theorem those numbers satisfy.

Triangles. To show a triangle is *isosceles*, find two congruent sides with the Distance Formula (*equilateral*: all three). To show it is *right*, either find two sides whose slopes multiply to -1 (or one horizontal and one vertical side), or show that $a^2 + b^2 = c^2$ for the squared side lengths and use the Converse of the Pythagorean Theorem (Euclid I.48). Working with squared lengths avoids square roots.

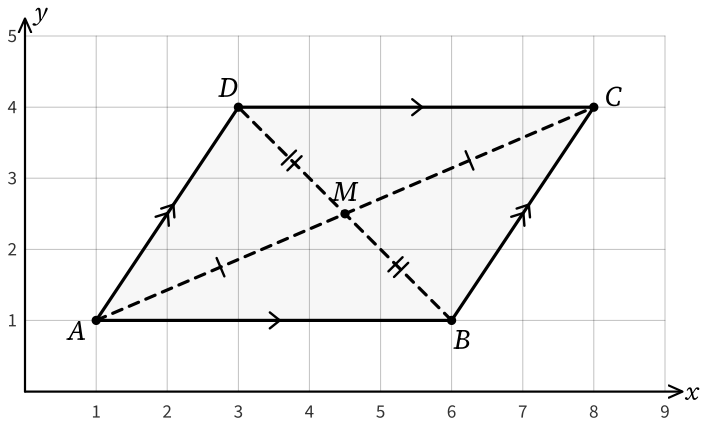
Quadrilaterals. Match each claim to a test you already know. *Parallelogram:* both pairs of opposite sides have equal slopes (the definition), or the diagonals have the same midpoint (they bisect each other, Lesson 6.3), or one pair of opposite sides is both parallel and congruent (Euclid I.33). *Rectangle:* a parallelogram whose consecutive sides are perpendicular, or a parallelogram with congruent diagonals (Lesson 6.4). *Rhombus:* four congruent sides. *Square:* four congruent sides and four right angles. *Trapezoid:* exactly one pair of parallel sides, so you must also show the other pair is NOT parallel. *Kite:* two pairs of consecutive congruent sides, with opposite sides not congruent.

Writing a clear coordinate argument. (1) Say what you will prove and which definition or theorem you will use. (2) Show each computation, labeled: '$AB =$

$\sqrt{4^2 + 2^2} = \sqrt{20}$.' (3) Say what the numbers mean: equal slopes mean parallel lines; slopes with product -1 mean perpendicular lines; equal lengths mean congruent segments. (4) Conclude by naming the definition or theorem. A reader should be able to check every number.

Showing that a figure is NOT a certain type needs only one failed condition. For example, if two consecutive sides have slopes 0 and $\frac{4}{3}$, that angle is not a right angle, so the figure is not a rectangle or a square. One honest counterexample settles it.

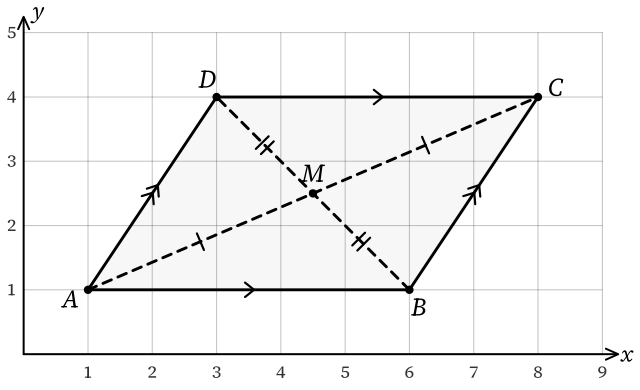
Finding a missing vertex. The diagonals of parallelogram $ABCD$ bisect each other, so diagonals $\overline{AC}$ and $\overline{BD}$ have the same midpoint. Setting the two midpoints equal gives $D = A + C - B$, coordinate by coordinate (proved below). Why: going from B to A is the same move as going from C to D. If you are told only three vertices and not their order, there are three possible fourth vertices, one opposite each given vertex.



POSTULATES AND THEOREMS

THEOREM **Fourth Vertex of a Parallelogram**

If $ABCD$ is a parallelogram with $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ and $D(x_D, y_D)$, then $x_D = x_A + x_C - x_B$ and $y_D = y_A + y_C - y_B$.



TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram with vertices $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$, $D(x_D, y_D)$

Prove: $x_D = x_A + x_C - x_B$ and $y_D = y_A + y_C - y_B$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram	Given
2	$\overline{AC}$ and $\overline{BD}$ bisect each other	Diagonals of a parallelogram bisect each other (Lesson 6.2)
3	$\overline{AC}$ and $\overline{BD}$ have the same midpoint	Definition of bisect (the point where they meet is the midpoint of each)
4	$\left(\frac{x_A + x_C}{2}, \frac{y_A + y_C}{2}\right) = \left(\frac{x_B + x_D}{2}, \frac{y_B + y_D}{2}\right)$	Midpoint Formula
5	$x_A + x_C = x_B + x_D$ and $y_A + y_C = y_B + y_D$	Multiplication Property of Equality
6	$x_D = x_A + x_C - x_B$ and $y_D = y_A + y_C - y_B$	Subtraction Property of Equality

∎

THEOREM **Slopes of Parallel and Perpendicular Lines**

Two nonvertical lines are parallel if and only if they have equal slopes, and perpendicular if and only if the product of their slopes is -1. A horizontal line and a vertical line are perpendicular.

Proved in Lesson 3.5.

THEOREM **Converse of the Pythagorean Theorem** Euclid I.48

If the square of one side of a triangle equals the sum of the squares of the other two sides, the angle opposite that side is a right angle.

Proved in Lesson 9.2.

THEOREM **One Pair of Sides Parallel and Congruent** Euclid I.33

If one pair of opposite sides of a quadrilateral are both parallel and congruent, the quadrilateral is a parallelogram.

Proved in Lesson 6.3. With coordinates: equal slopes and equal lengths for one pair of opposite sides.

KEY FORMULAS

$$\text{parallel: } m_1 = m_2 \qquad \text{perpendicular: } m_1 \cdot m_2 = -1$$

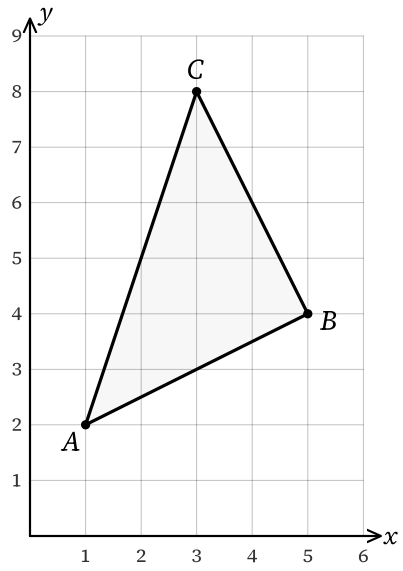
$$\text{Fourth vertex of parallelogram } ABCD\text{: } D = (x_A + x_C - x_B,\ y_A + y_C - y_B)$$

Euclid connection. Euclid sorts quadrilaterals in Book I, Definition 22: 'a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled.' Coordinates check exactly his two conditions, equal sides (Distance Formula) and right angles (slopes). Euclid's classes leave each other out (his rhombus is never a square), while ours include each other (every square is a rhombus).

Practice habit. Before computing anything, write the definition or theorem you will use and list exactly which numbers it needs ('rhombus: four equal sides'). Then label every computation so a reader can check it, and end with a sentence naming the definition or theorem.

WORKED EXAMPLES

Example 1. Prove that $\triangle ABC$ with vertices $A(1, 2)$, $B(5, 4)$ and $C(3, 8)$ is an isosceles right triangle.



Show solution

1. Plan: 'isosceles' needs two equal sides (Distance Formula); 'right' needs two perpendicular sides (slopes).
2. The sketch suggests $AB = BC$ and a right angle at B, but we may not trust the sketch, so compute AB, BC and the slopes of $\overline{AB}$ and $\overline{BC}$.
3. Conclude with the definitions of isosceles and right triangles.

TWO-COLUMN PROOF

Given: $A(1, 2)$, $B(5, 4)$, $C(3, 8)$

Prove: $\triangle ABC$ is an isosceles right triangle

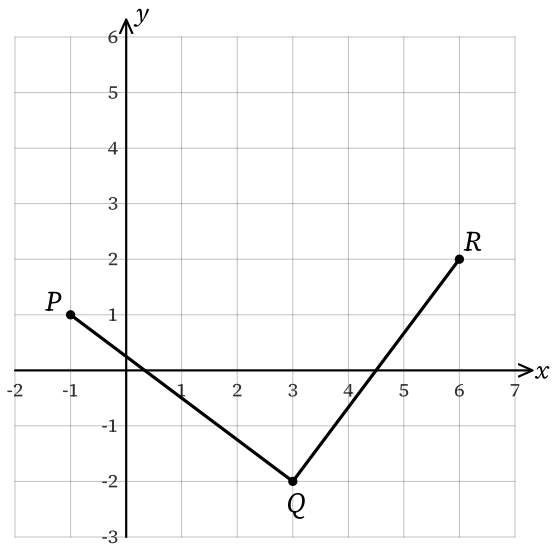
	STATEMENTS	REASONS
1	$A(1, 2)$, $B(5, 4)$, $C(3, 8)$	Given
2	$AB = \sqrt{(5-1)^2 + (4-2)^2} = \sqrt{20}$	Distance Formula
3	$BC = \sqrt{(3-5)^2 + (8-4)^2} = \sqrt{20}$	Distance Formula
4	$\overline{AB} \cong \overline{BC}$	Definition of congruent segments (equal lengths)
5	$\triangle ABC$ is isosceles	Definition of isosceles triangle

	STATEMENTS	REASONS
6	Slope of $\overline{AB} = \frac{2}{4} = \frac{1}{2}$; slope of $\overline{BC} = \frac{4}{-2} = -2$	Slope formula
7	$\overline{AB} \perp \overline{BC}$	Slopes with product $\frac{1}{2} \cdot (-2) = -1$ (Lesson 3.5)
8	$\angle ABC$ is a right angle	Definition of perpendicular lines
9	$\triangle ABC$ is an isosceles right triangle	Definitions of isosceles triangle and right triangle

∎

Answer: See the model proof: $AB = BC = \sqrt{20}$ and $\overline{AB} \perp \overline{BC}$.

Example 2. Three vertices of square $PQRS$ are $P(-1, 1)$, $Q(3, -2)$ and $R(6, 2)$. Find the fourth vertex S, and then prove that $PQRS$ is a square.



Show solution

1. A square is a parallelogram, so by the Fourth Vertex theorem $S = P + R - Q = (-1 + 6 - 3,\ 1 + 2 - (-2)) = (2, 5)$.
2. Plan the proof: four equal sides make a rhombus; a rhombus is a parallelogram; a parallelogram with congruent diagonals is a rectangle; a rhombus that is a rectangle is a square.
3. Compute the four sides and the two diagonals.

PARAGRAPH PROOF

Given: $P(-1, 1)$, $Q(3, -2)$, $R(6, 2)$, $S(2, 5)$

Prove: $PQRS$ is a square

By the Distance Formula, $PQ = \sqrt{4^2 + (-3)^2} = 5$, $QR = \sqrt{3^2 + 4^2} = 5$, $RS = \sqrt{(-4)^2 + 3^2} = 5$ and $SP = \sqrt{(-3)^2 + (-4)^2} = 5$. All four sides are congruent, so $PQRS$ is a rhombus by definition. Its opposite sides are congruent, so it is a parallelogram (Lesson 6.3).

The diagonals are $PR = \sqrt{7^2 + 1^2} = \sqrt{50}$ and $QS = \sqrt{(-1)^2 + 7^2} = \sqrt{50}$, so they are congruent. A parallelogram with congruent diagonals is a rectangle (Lesson 6.4), so $PQRS$ has four right angles.

Therefore $PQRS$ has four congruent sides and four right angles, so it is a square by the definition of a square.

∎

Answer: $S(2,5)$, and $PQRS$ is a square (see the model proof).

PRACTICE

1. Easy Is the triangle with vertices $A(0,0)$, $B(6,0)$ and $C(3,5)$ isosceles? Justify with distances.

   Show answer

   Yes. $CA = \sqrt{3^2+5^2} = \sqrt{34}$ and $CB = \sqrt{(-3)^2+5^2} = \sqrt{34}$, so $\overline{CA} \cong \overline{CB}$.

2. Easy Does $\triangle JKL$ with $J(-1,1)$, $K(2,3)$ and $L(4,0)$ have a right angle? Use slopes.

   Show answer

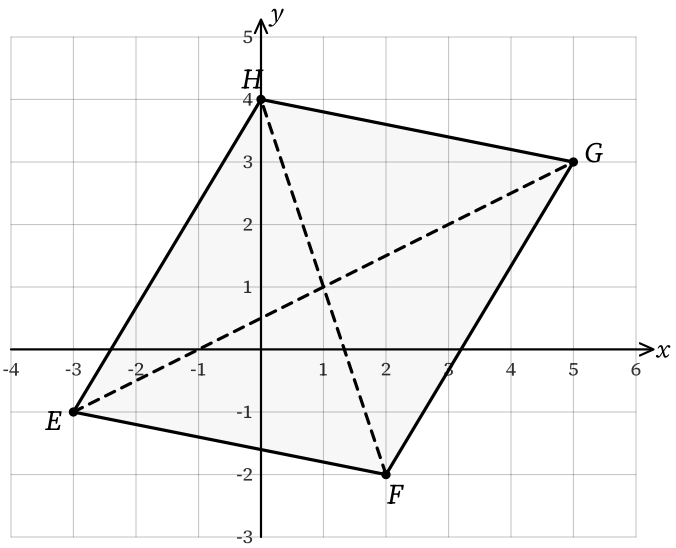
   Yes. Slope of $\overline{JK} = \dfrac{2}{3}$ and slope of $\overline{KL} = \dfrac{-3}{2}$; their product is -1, so $\angle K$ is a right angle.

3. Easy Parallelogram $ABCD$ has $A(1,1)$, $B(6,1)$ and $C(8,5)$. Find D.

   Show answer

   $D = (1+8-6,\ 1+5-1) = (3,5)$

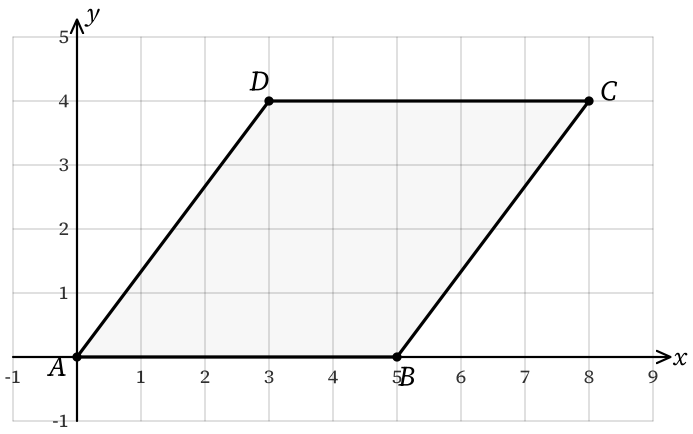
4. **Easy** Quadrilateral $EFGH$ in the figure has $E(-3, -1)$, $F(2, -2)$, $G(5, 3)$ and $H(0, 4)$. Find the midpoints of diagonals $\overline{EG}$ and $\overline{FH}$. Is $EFGH$ a parallelogram?



Show answer

Both midpoints are $(1, 1)$, so the diagonals bisect each other and $EFGH$ is a parallelogram (Lesson 6.3). Yes.

5. **Medium** Prove that quadrilateral $ABCD$ with $A(0,0)$, $B(5,0)$, $C(8,4)$ and $D(3,4)$ is a rhombus but not a square.



Show answer

See the model proof: all sides are 5, but $\angle B$ is not a right angle.

PARAGRAPH PROOF

Given: $A(0,0)$, $B(5,0)$, $C(8,4)$, $D(3,4)$

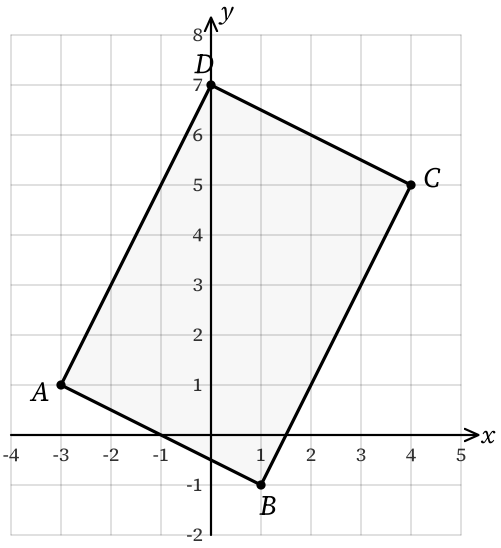
Prove: $ABCD$ is a rhombus but not a square

By the Distance Formula, $AB = 5$ (a horizontal segment from $x = 0$ to $x = 5$), $BC = \sqrt{3^2 + 4^2} = 5$, $CD = 5$ (horizontal from $x = 8$ to $x = 3$) and $DA = \sqrt{3^2 + 4^2} = 5$. All four sides are congruent, so $ABCD$ is a rhombus by definition.

A square must have four right angles. $\overline{AB}$ is horizontal, so the only line perpendicular to it through B is vertical. But $\overline{BC}$ has slope $\frac{4}{3}$, so it is not vertical, and $\angle ABC$ is not a right angle. Therefore $ABCD$ is not a square: it is a rhombus but not a square.

∎

6. **Medium** Prove that quadrilateral $ABCD$ with $A(-3, 1)$, $B(1, -1)$, $C(4, 5)$ and $D(0, 7)$ is a rectangle.



Show answer

See the model proof: opposite sides have equal slopes and consecutive slopes multiply to -1.

TWO-COLUMN PROOF

Given: $A(-3, 1)$, $B(1, -1)$, $C(4, 5)$, $D(0, 7)$

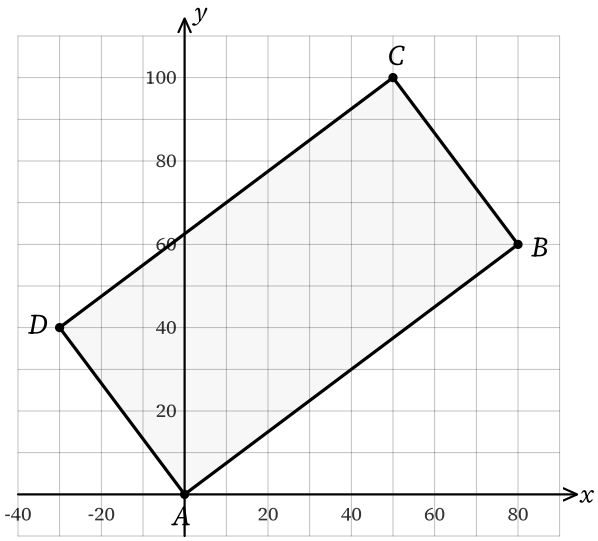
Prove: $ABCD$ is a rectangle

	STATEMENTS	REASONS
1	$A(-3, 1)$, $B(1, -1)$, $C(4, 5)$, $D(0, 7)$	Given
2	Slope of $\overline{AB} = \dfrac{-2}{4} = -\dfrac{1}{2}$; slope of $\overline{BC} = \dfrac{6}{3} = 2$	Slope formula
3	Slope of $\overline{CD} = \dfrac{2}{-4} = -\dfrac{1}{2}$; slope of $\overline{DA} = \dfrac{-6}{-3} = 2$	Slope formula
4	$\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$	Lines with equal slopes are parallel
5	$ABCD$ is a parallelogram	Definition of parallelogram
6	$\overline{AB} \perp \overline{BC}, \overline{BC} \perp \overline{CD}, \overline{CD} \perp \overline{DA}, \overline{DA} \perp \overline{AB}$	Each pair of slopes has product $\left(-\dfrac{1}{2}\right)(2) = -1$

	STATEMENTS	REASONS
7	$\angle A$, $\angle B$, $\angle C$ and $\angle D$ are right angles	Definition of perpendicular lines
8	$ABCD$ is a rectangle	Definition of rectangle

∎

7. **Medium** Henry stakes out a building lot. Measured in feet from a corner post, the corners are $A(0, 0)$, $B(80, 60)$, $C(50, 100)$ and $D(-30, 40)$, as in the figure. (a) Is the lot a rectangle? (b) Find its area in square feet.

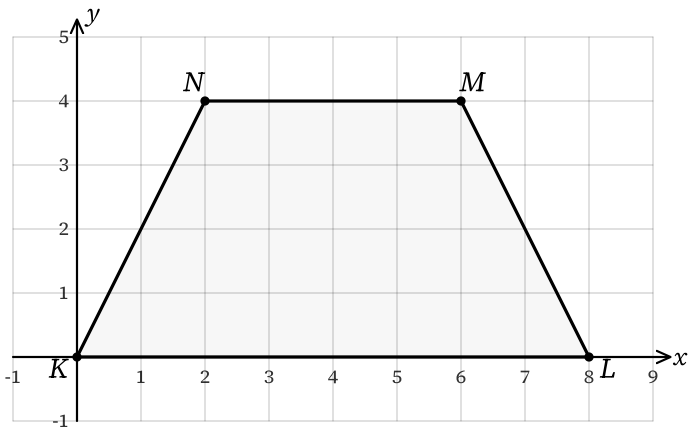


Coordinates in feet

Show answer

(a) Yes. Slopes: AB: $\frac{3}{4}$; BC: $-\frac{4}{3}$; CD: $\frac{3}{4}$; DA: $-\frac{4}{3}$. Opposite sides are parallel and consecutive sides are perpendicular, so it is a rectangle. (b) $AB = \sqrt{80^2 + 60^2} = 100$ ft and $BC = \sqrt{30^2 + 40^2} = 50$ ft, so the area is $100 \times 50 = 5000$ square feet.

8. **Medium** Prove that quadrilateral $KLMN$ with $K(0,0)$, $L(8,0)$, $M(6,4)$ and $N(2,4)$ is an isosceles trapezoid. Use a flow proof.



Show answer

See the model proof: exactly one pair of sides is parallel and the legs are both $\sqrt{20}$.

FLOW PROOF

Given: $K(0,0)$, $L(8,0)$, $M(6,4)$, $N(2,4)$

Prove: $KLMN$ is an isosceles trapezoid

(1) $K(0,0)$, $L(8,0)$, $M(6,4)$, $N(2,4)$
Given

(2) Slope of $\overline{KL} = 0$ and slope of $\overline{NM} = 0$
Slope formula
from (1)

(3) $\overline{KL} \parallel \overline{NM}$
Lines with equal slopes are parallel
from (2)

(4) Slope of $\overline{LM} = \dfrac{4}{-2} = -2$ and slope of $\overline{KN} = \dfrac{4}{2} = 2$
Slope formula
from (1)

(5) $\overline{LM}$ is not parallel to $\overline{KN}$
Parallel lines have equal slopes; these slopes are unequal
from (4)

(6) $KLMN$ is a trapezoid

Definition of trapezoid (exactly one pair of parallel sides)
from (3), (5)

(7) $LM = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$ and $KN = \sqrt{2^2 + 4^2} = \sqrt{20}$
Distance Formula
from (1)

(8) $\overline{LM} \cong \overline{KN}$
Definition of congruent segments
from (7)

(9) $KLMN$ is an isosceles trapezoid
Definition of isosceles trapezoid (congruent legs)
from (6), (8)

∎

9. Hard Three vertices of a parallelogram are $A(1, 2)$, $B(5, 3)$ and $C(3, 6)$, but the order of the vertices is not given. Find all three possible locations of the fourth vertex. List them in this order: the one opposite B, the one opposite C, the one opposite A.

Show answer

Opposite B: $A + C - B = (-1, 5)$. Opposite C: $A + B - C = (3, -1)$. Opposite A: $B + C - A = (7, 7)$.

10. Hard Use the Converse of the Pythagorean Theorem to prove that $\triangle ABC$ with $A(-2, 1)$, $B(4, -1)$ and $C(5, 2)$ is a right triangle with right angle $\angle B$.

Show answer

See the model proof: $AB^2 + BC^2 = 40 + 10 = 50 = AC^2$.

TWO-COLUMN PROOF

Given: $A(-2, 1)$, $B(4, -1)$, $C(5, 2)$

Prove: $\triangle ABC$ is a right triangle with right angle $\angle B$

	STATEMENTS	REASONS
1	$A(-2, 1)$, $B(4, -1)$, $C(5, 2)$	Given
2	$AB^2 = 6^2 + (-2)^2 = 40$	Distance Formula (squared)
3	$BC^2 = 1^2 + 3^2 = 10$	Distance Formula (squared)
4	$AC^2 = 7^2 + 1^2 = 50$	Distance Formula (squared)
5	$AB^2 + BC^2 = AC^2$	Substitution Property of Equality ($40 + 10 = 50$)
6	$\triangle ABC$ is a right triangle with right angle $\angle B$	Converse of the Pythagorean Theorem (Euclid I.48); $\angle B$ is opposite $\overline{AC}$

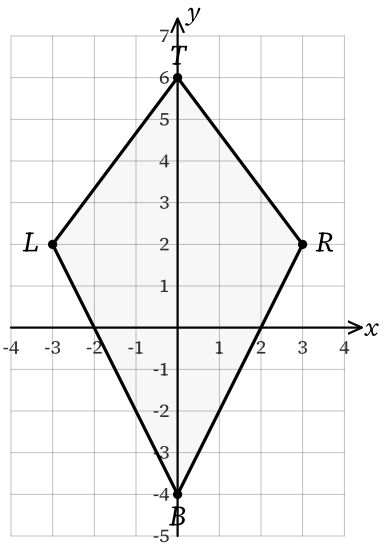
∎

11. Hard Grace says: 'If the diagonals of a quadrilateral are congruent, it must be a rectangle.' Use the trapezoid $KLMN$ from Problem 8 to explain why she is wrong.

Show answer

Its diagonals are $KM = \sqrt{6^2 + 4^2} = \sqrt{52}$ and $LN = \sqrt{(-6)^2 + 4^2} = \sqrt{52}$, so they are congruent, yet $KLMN$ is not a rectangle (it is not even a parallelogram, since $\overline{LM}$ and $\overline{KN}$ are not parallel). The rectangle test in Lesson 6.4 applies only to figures already known to be parallelograms.

12. Hard Ruth is cutting a piece of stained glass. On her pattern, measured in inches, the corners are $T(0, 6)$, $R(3, 2)$, $B(0, -4)$ and $L(-3, 2)$, as in the figure. (a) Is $TRBL$ a kite? (b) Find the area of the glass piece in square inches.



Coordinates in inches

Show answer

(a) Yes. $TR = \sqrt{3^2 + (-4)^2} = 5$ and $LT = \sqrt{3^2 + 4^2} = 5$; $RB = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$ and $BL = \sqrt{(-3)^2 + 6^2} = \sqrt{45}$. Two pairs of consecutive sides are congruent, and opposite sides (5 and $\sqrt{45}$) are not. (b) The diagonals are $TB = 10$ (vertical) and $RL = 6$ (horizontal), so the area is $\frac{1}{2}(10)(6) = 30$ square inches.

13. **Challenge** Quadrilateral $ABCD$ has $A(0,0)$, $B(6,2)$, $C(4,k)$ and $D(-2,6)$. (a) Find the value of k that makes $ABCD$ a parallelogram. (b) For that value of k, prove that $ABCD$ is a square.

Show answer

(a) A quadrilateral is a parallelogram exactly when its diagonals bisect each other (Lessons 6.2 and 6.3). Midpoint of $\overline{AC}$: $\left(2, \frac{k}{2}\right)$; midpoint of $\overline{BD}$: $(2,4)$. So $\frac{k}{2} = 4$ and $k = 8$. (b) See the model proof.

TWO-COLUMN PROOF

Given: $A(0,0)$, $B(6,2)$, $C(4,8)$, $D(-2,6)$ (using $k = 8$ from part (a))

Prove: $ABCD$ is a square

	STATEMENTS	REASONS
1	$A(0,0)$, $B(6,2)$, $C(4,8)$, $D(-2,6)$	Given (part (a))
2	$AB = \sqrt{6^2 + 2^2} = \sqrt{40}$ and $BC = \sqrt{(-2)^2 + 6^2} = \sqrt{40}$	Distance Formula
3	$CD = \sqrt{(-6)^2 + (-2)^2} = \sqrt{40}$ and $DA = \sqrt{2^2 + (-6)^2} = \sqrt{40}$	Distance Formula
4	$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$	Definition of congruent segments
5	Slopes: $\overline{AB}$: $\frac{1}{3}$; $\overline{BC}$: -3; $\overline{CD}$: $\frac{1}{3}$; $\overline{DA}$: -3	Slope formula
6	$\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$, $\overline{CD} \perp \overline{DA}$, $\overline{DA} \perp \overline{AB}$	Each pair of slopes has product $\frac{1}{3} \cdot (-3) = -1$
7	$\angle A$, $\angle B$, $\angle C$ and $\angle D$ are right angles	Definition of perpendicular lines
8	$ABCD$ is a square	Definition of square (four congruent sides and four right angles)

∎

COMMON MISTAKES

- Trusting the look of the grid. A figure that looks like a square may be a rhombus or a rectangle; compute the sides and the slopes before you name it.

- Calling a figure with four congruent sides a square. That proves only a rhombus; you also need a right angle (or congruent diagonals in a parallelogram).
- Showing one pair of opposite sides parallel and concluding 'parallelogram.' You need both pairs parallel, or one pair both parallel AND congruent (Euclid I.33). For a trapezoid you must also show the other pair is NOT parallel.
- Using the wrong vertex in the fourth-vertex rule. For parallelogram $ABCD$, D is opposite B, so $D = A + C - B$; subtract the vertex that is opposite the one you want.

> **Make it a game · Pay Per Clue.** Parent secretly draws four points on graph paper, naming them A, B, C and D. You buy clues with paper chips: a distance or a slope costs one chip each. Decide whether the quadrilateral is a parallelogram, rectangle, rhombus, square, trapezoid or kite, and justify. Fewest chips used wins. Swap roles.

13.3 Coordinate Proofs

OBJECTIVES

- Place a general figure on coordinate axes so that the algebra is simple without assuming anything extra.
- Prove general theorems with coordinates, including the Midsegment Theorem, the diagonals of a parallelogram and a rectangle, the midpoint of the hypotenuse, and the midpoints of any quadrilateral.
- Compare a coordinate proof with Euclid's synthetic proof of the same theorem.
- Review the course's path from Euclid's postulates to Descartes' coordinates.

WATCH

Coordinate Methods (2 of 3: Setting up the proof)

Eddie Woo · 8:40

https://www.youtube.com/watch?v=Zr4kp5iVnJk

> **WHEN TO WATCH**
> Before Example 1: placing a figure on the axes to set up a coordinate proof.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Coordinate Methods (1 of 3: Random quadrilaterals & magic parallelograms)

Eddie Woo · 7:53

https://www.youtube.com/watch?v=pjTwBL7nMnk

After the Varignon example: why the midpoints of any quadrilateral make a parallelogram.

KEY TERMS

coordinate proof

A proof of a general theorem in which the figure is placed on coordinate axes with variable coordinates, and the conclusion is shown by algebra with the distance, midpoint and slope formulas.

synthetic proof

A proof that reasons directly from definitions, postulates and earlier theorems about figures, without coordinates, as Euclid did.

general position

A placement whose variable coordinates can describe every figure of the given kind, not just a special one.

CORE IDEAS

In Lesson 13.2 you proved facts about *one* figure with numbered vertices. Now you will prove facts about *every* figure of a kind: every triangle, every parallelogram. To do that, the coordinates must be variables, such as $(2a, 0)$ and $(2b, 2c)$, so that the algebra covers all cases at once. A proof with numbers checks one example; a proof with variables proves the theorem.

Placing the figure. Slides, turns and flips (translations, rotations and reflections: the rigid motions of Unit 7) do not change lengths, angle measures, midpoints or parallel lines. So we may move any figure so that one vertex is at the origin and one side lies along the positive x-axis, and, by flipping over the x-axis if necessary, so that the rest of the figure lies above the x-axis. We lose nothing: whatever we prove about the moved figure is true of the original. Then use the figure's own features: put a right angle at the origin with its legs on the axes; put an isosceles triangle's base on the x-axis with its midpoint at the origin.

Do not assume too much. The coordinates must describe a *general* figure of the given kind. For an arbitrary triangle, $(0, 0)$, $(2a, 0)$, $(2b, 2c)$ works, but $(0, 0)$, $(2a, 0)$, $(0, 2b)$ secretly makes the triangle right, so it would prove the theorem only for right triangles. Any vertex that is not free (like the fourth vertex of a parallelogram) must be *derived* from the definition, not guessed from the picture.

Avoid fractions. Midpoints divide by 2, so when a proof uses midpoints, give the coordinates as $2a$, $2b$, $2c$. Then the midpoint of $(0, 0)$ and $(2b, 2c)$ is simply (b, c). When thirds appear (the centroid), use $6a$, $6b$, $6c$ for the same reason.

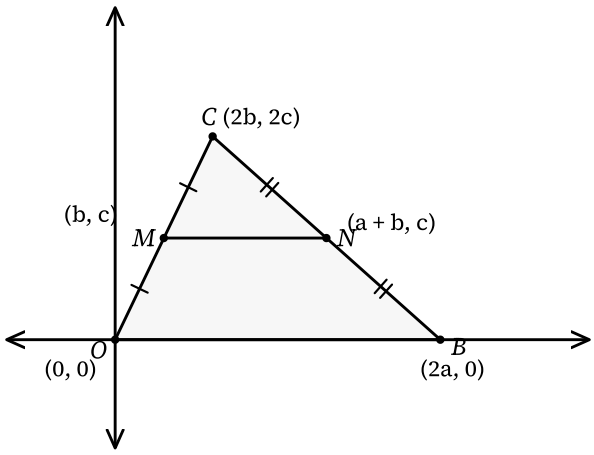
Coordinate proof and Euclid's proof. Euclid proves the Midsegment Theorem's parallel part through proportions (VI.2), and he proves facts about parallelograms with congruent triangles (I.34). A coordinate proof computes instead. Each has its strength: a synthetic proof shows *why* a figure must be so, while a coordinate proof is mechanical and very hard to get wrong. But coordinates do not replace Euclid; they stand on him. The Distance Formula is I.47, and the slope tests rest on the similar triangles of Book VI.

From Euclid to Descartes: looking back over the course. You began where Euclid began, with undefined terms, definitions, five postulates and five common notions, and learned to demonstrate rather than merely observe. From them you built angle facts and parallels (I.29 and the fifth postulate), congruent triangles (I.4, I.8, I.26), similar triangles (Book VI), circles (Book III), area and volume (Books I, XII). Descartes' idea of naming points by numbers then let algebra serve geometry. In this last lesson the two traditions meet: every coordinate tool you use was proved from Euclid's foundations.

POSTULATES AND THEOREMS

THEOREM **Triangle Midsegment Theorem (coordinate proof)**

The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.



PARAGRAPH PROOF

Given: $\triangle OBC$ with $O(0,0)$, $B(2a,0)$, $C(2b,2c)$, where $a>0$ and $c>0$; M is the midpoint of $\overline{OC}$ and N is the midpoint of $\overline{BC}$

Prove: $\overline{MN} \parallel \overline{OB}$ and $MN = \frac{1}{2}OB$

Placement: any triangle can be moved so that one vertex O is at the origin, side $\overline{OB}$ lies on the positive x-axis, and C lies above the x-axis. So the coordinates $O(0,0)$, $B(2a,0)$, $C(2b,2c)$ with $a>0$, $c>0$ describe every triangle.

By the Midpoint Formula, $M = \left(\frac{0+2b}{2}, \frac{0+2c}{2}\right) = (b,c)$ and $N = \left(\frac{2a+2b}{2}, \frac{0+2c}{2}\right) = (a+b,c)$.

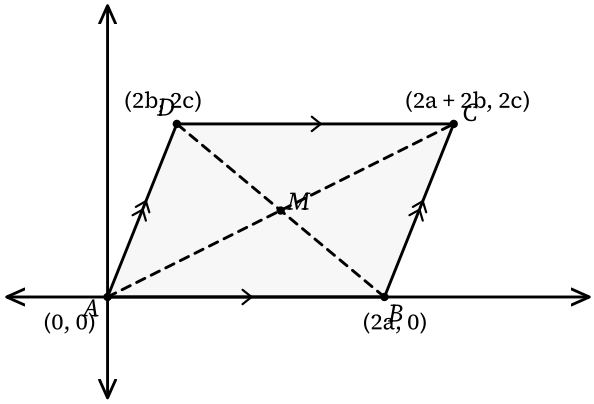
M and N both have y-coordinate c, so $\overline{MN}$ has slope 0, and $\overline{OB}$, on the x-axis, also has slope 0. The lines are different because $c \neq 0$, so lines with equal slopes are parallel: $\overline{MN} \parallel \overline{OB}$.

Since $\overline{MN}$ is horizontal, $MN = |(a+b) - b| = a$, while $OB = 2a$. Therefore $MN = \dfrac{1}{2}OB$.

∎

THEOREM **Diagonals of a Parallelogram (coordinate proof)**

The diagonals of a parallelogram bisect each other.



TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram

Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram	Given
2	Place $A(0,0)$, $B(2a,0)$ and $D(2b,2c)$ with $a>0$, $c>0$; call $C(x,y)$	Rigid motions preserve lengths, midpoints and parallels, so we may choose the axes this way
3	$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	Definition of parallelogram
4	$y = 2c$	$\overline{AB}$ is horizontal, so the parallel $\overline{DC}$ is horizontal (equal slopes) and C is level with D
5	$x = 2a + 2b$	$\overline{BC} \parallel \overline{AD}$: if $b \neq 0$, equal slopes give $\dfrac{2c}{x-2a} = \dfrac{2c}{2b}$, so $x - 2a = 2b$; if $b = 0$, both are vertical and $x = 2a$
6	Midpoint of $\overline{AC}$ is $\left(\dfrac{2a+2b}{2}, \dfrac{2c}{2}\right) = (a+b, c)$	Midpoint Formula

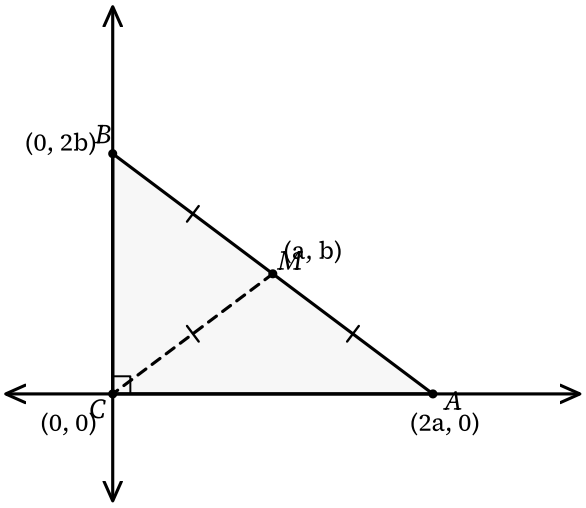
	STATEMENTS	REASONS
7	Midpoint of $\overline{BD}$ is $\left(\frac{2a+2b}{2}, \frac{0+2c}{2}\right) = (a+b, c)$	Midpoint Formula
8	$\overline{AC}$ and $\overline{BD}$ have the same midpoint	Transitive Property of Equality
9	$\overline{AC}$ and $\overline{BD}$ bisect each other	Definition of bisect (each diagonal passes through the midpoint of the other)

∎

This theorem was proved with congruent triangles in Lesson 6.2. Notice that the coordinates of C were derived from the definition of a parallelogram, not read from the picture.

THEOREM **Midpoint of the Hypotenuse**

The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.



TWO-COLUMN PROOF

Given: $\triangle ACB$ with right angle $\angle C$; M is the midpoint of hypotenuse $\overline{AB}$

Prove: $MA = MB = MC$

	STATEMENTS	REASONS
1	$\angle C$ is a right angle and M is the midpoint of $\overline{AB}$	Given
2	Place $C(0,0)$, $A(2a,0)$, $B(0,2b)$ with $a>0$, $b>0$	The legs are perpendicular, so they can be laid along the perpendicular axes

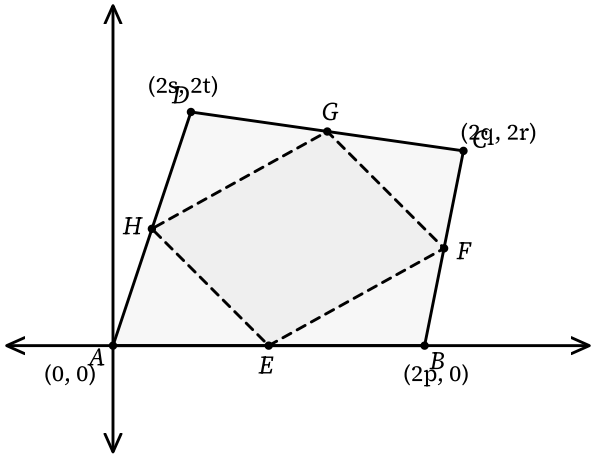
	STATEMENTS	REASONS
3	$M = (a, b)$	Midpoint Formula
4	$MA = \sqrt{(2a - a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$	Distance Formula
5	$MB = \sqrt{(0 - a)^2 + (2b - b)^2} = \sqrt{a^2 + b^2}$	Distance Formula
6	$MC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$	Distance Formula
7	$MA = MB = MC$	Substitution Property of Equality

∎

So the circle with the hypotenuse as diameter passes through the right-angle vertex, This is the converse of Euclid III.31, which says that an angle inscribed in a semicircle is right.

THEOREM **Varignon's Theorem (Midpoint Quadrilateral)**

The midpoints of the sides of any quadrilateral, joined in order, form a parallelogram.



E, F, G and H are the midpoints of the sides.

PARAGRAPH PROOF

Given: Quadrilateral $ABCD$ with $A(0, 0)$, $B(2p, 0)$, $C(2q, 2r)$, $D(2s, 2t)$; E, F, G, H are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$

Prove: $EFGH$ is a parallelogram

Placement: move the quadrilateral so that A is at the origin and B is on the positive x-axis. The other two vertices can be anywhere, so they get four separate variables; doubling them avoids fractions.

By the Midpoint Formula, $E = (p, 0)$, $F = (p + q, r)$, $G = (q + s, r + t)$ and $H = (s, t)$.

The midpoint of diagonal $\overline{EG}$ is $\left(\frac{p+q+s}{2}, \frac{r+t}{2}\right)$, and the midpoint of diagonal $\overline{FH}$ is $\left(\frac{p+q+s}{2}, \frac{r+t}{2}\right)$, the same point. So the diagonals of $EFGH$ bisect each other, and a quadrilateral whose diagonals bisect each other is a parallelogram (Lesson 6.3). Therefore $EFGH$ is a parallelogram.

∎

Named for the French mathematician Pierre Varignon (1654–1722). A synthetic proof uses the Midsegment Theorem twice: $\overline{EF}$ and $\overline{HG}$ are both parallel to diagonal $\overline{AC}$ and half as long, so $EFGH$ has a pair of opposite sides that are parallel and congruent (Euclid I.33).

KEY FORMULAS

General triangle: $(0, 0),\ (2a, 0),\ (2b, 2c)$

Parallelogram: $(0, 0),\ (2a, 0),\ (2a + 2b, 2c),\ (2b, 2c)$

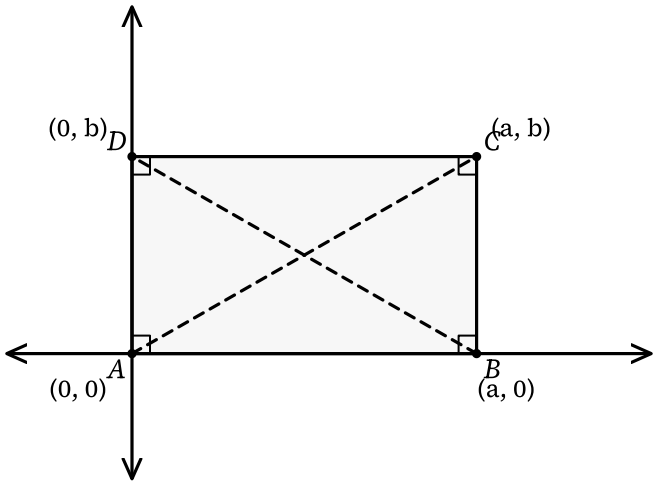
Right triangle: $(0, 0),\ (2a, 0),\ (0, 2b)$ Rectangle: $(0, 0),\ (a, 0),\ (a, b),\ (0, b)$

From history. Descartes published *La Géométrie* in 1637 as one of three essays attached to his *Discourse on Method*; in it he showed how geometric problems could be turned into equations. His method, and Fermat's at about the same time, prepared the way for the calculus of Newton and Leibniz later in the century. The two approaches support each other: when David Hilbert set out careful foundations for Euclid's geometry in 1899, he showed that his axioms could not contradict one another by building a model of them out of pairs of numbers, that is, out of coordinates.

Euclid connection. Euclid's proofs are synthetic: in I.34 he shows the opposite sides of a parallelogram are equal using congruent triangles, and the second half of VI.2 contains the parallel part of the Midsegment Theorem as the case where the sides are cut in half. A coordinate proof reaches the same conclusions by algebra, but its formulas (the Distance Formula from I.47, the slope tests from similar triangles) were first proved in Euclid's way.

> **Practice habit.** Before placing a figure, write down exactly what the hypothesis gives you and check that your coordinates add nothing extra, such as a hidden right angle or equal sides. Draw the figure large and write the coordinates at every vertex, then plan backward from the prove statement: which distances, slopes or midpoints will show it?

Example 1. Prove with coordinates: the diagonals of a rectangle are congruent.



Show solution

1. Place vertex A at the origin and B on the positive x-axis. The right angle at A puts D on the y-axis.
2. Derive C from the right angles at B and D, rather than reading it from the picture.
3. No midpoints are needed, so use a and b (not $2a$ and $2b$), and compare AC and BD with the Distance Formula.

TWO-COLUMN PROOF

Given: $ABCD$ is a rectangle

Prove: $\overline{AC} \cong \overline{BD}$

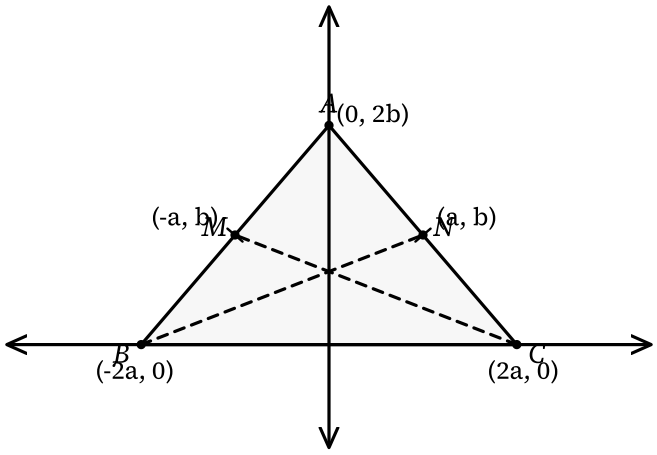
	STATEMENTS	REASONS
1	$ABCD$ is a rectangle	Given
2	Place $A(0,0)$ and $B(a,0)$ with $a > 0$, and D above the x-axis	Rigid motions (slides, turns and, if needed, a flip over the x-axis) preserve lengths, so we may choose the axes this way
3	$\angle A$, $\angle B$, $\angle D$ are right angles	Definition of rectangle
4	$D = (0, b)$ with $b > 0$	$\overline{AD} \perp \overline{AB}$ at A, and the y-axis is the only line perpendicular to

	STATEMENTS	REASONS
		the x-axis at the origin; D is above the x-axis (step 2)
5	$C = (a, b)$	$\overline{BC} \perp \overline{AB}$, so $\overline{BC}$ is vertical ($x = a$); $\overline{DC} \perp \overline{AD}$, so $\overline{DC}$ is horizontal ($y = b$)
6	$AC = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$	Distance Formula
7	$BD = \sqrt{(0-a)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$	Distance Formula
8	$AC = BD$	Substitution Property of Equality
9	$\overline{AC} \cong \overline{BD}$	Definition of congruent segments

∎

Answer: See the model proof: both diagonals have length $\sqrt{a^2 + b^2}$.

Example 2. Prove with coordinates: in an isosceles triangle, the medians drawn to the two legs are congruent. (In $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$, let M be the midpoint of $\overline{AB}$ and N the midpoint of $\overline{AC}$; prove $\overline{CM} \cong \overline{BN}$.)



Show solution

1. Put the base $\overline{BC}$ on the x-axis with its midpoint at the origin: $B(-2a, 0)$, $C(2a, 0)$. Call the apex $A(x, y)$ with $y > 0$; do not assume it is on the y-axis.
2. Use $AB = AC$ to prove that $x = 0$.
3. Find M and N, then compare CM and BN with the Distance Formula. Write the proof as a flow proof.

FLOW PROOF

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$; M is the midpoint of $\overline{AB}$; N is the midpoint of $\overline{AC}$
Prove: $\overline{CM} \cong \overline{BN}$

(1) $B(-2a, 0)$, $C(2a, 0)$, $A(x, y)$ with $a > 0$, $y > 0$; $AB = AC$
Given, with axes chosen so the base lies on the x-axis with its midpoint at the origin

(2) $(x + 2a)^2 + y^2 = (x - 2a)^2 + y^2$
Distance Formula; squaring equal lengths
from (1)

(3) $8ax = 0$, so $x = 0$
Expand and subtract; $a \neq 0$
from (2)

(4) $A = (0, 2b)$, writing $y = 2b$ with $b > 0$

Substitution
from (3)

(5) $M = (-a, b)$ and $N = (a, b)$
Midpoint Formula
from (1), (4)

(6) $CM = \sqrt{(-a - 2a)^2 + (b - 0)^2} = \sqrt{9a^2 + b^2}$
Distance Formula
from (1), (5)

(7) $BN = \sqrt{(a + 2a)^2 + (b - 0)^2} = \sqrt{9a^2 + b^2}$
Distance Formula
from (1), (5)

(8) $CM = BN$
Substitution Property of Equality
from (6), (7)

(9) $\overline{CM} \cong \overline{BN}$
Definition of congruent segments
from (8)

∎

Answer: See the model proof: both medians have length $\sqrt{9a^2 + b^2}$.

PRACTICE

1. Easy Right triangle ABC has vertices $C(0, 0)$, $A(8, 0)$ and $B(0, 6)$. Find the midpoint M of the hypotenuse and the common distance $MA = MB = MC$.

   Show answer

   $M = (4, 3)$; $MA = \sqrt{4^2 + 3^2} = 5$, $MB = \sqrt{4^2 + 3^2} = 5$, $MC = \sqrt{4^2 + 3^2} = 5$. The common distance is 5.

2. Easy A rectangle has width a and height b. Place it with one vertex at the origin and two sides along the positive axes, and give the coordinates of all four vertices. Why does this placement not lose generality?

   Show answer

   $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$. Any rectangle can be slid and turned (rigid motions) into this position without changing its lengths or angles, and a and b can be any positive numbers, so every rectangle is described.

3. **Easy** Check the Midsegment Theorem for the triangle $O(0,0)$, $B(10,0)$, $C(4,6)$: find the midpoint M of $\overline{OC}$, the midpoint N of $\overline{BC}$, and the length MN.

Show answer

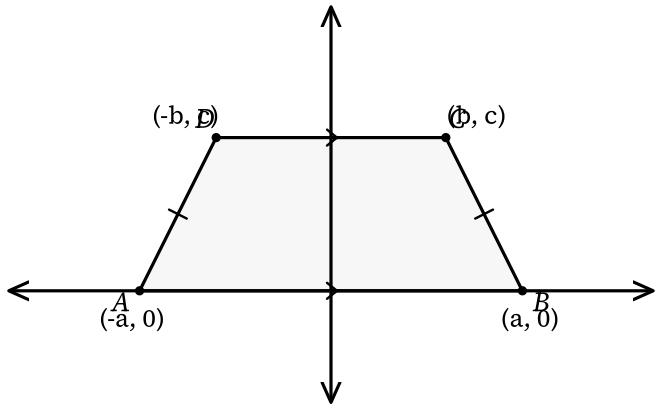
$M = (2,3)$, $N = (7,3)$, and $MN = 5$, which is half of $OB = 10$; $\overline{MN}$ is horizontal like $\overline{OB}$.

4. **Easy** In the proof of the Midsegment Theorem we used $O(0,0)$, $B(2a,0)$, $C(2b,2c)$. (a) Why use $2a$, $2b$, $2c$ instead of a, b, c? (b) Why would $O(0,0)$, $B(2a,0)$, $C(0,2b)$ be a poor choice for proving the theorem for every triangle?

Show answer

(a) The proof uses midpoints, which divide by 2; doubling the coordinates makes every midpoint free of fractions, such as (b,c). (b) Those coordinates put C on the y-axis, so $\angle O$ would be a right angle. The proof would then cover only right triangles, not every triangle.

5. **Medium** An isosceles trapezoid $ABCD$ is placed with the y-axis as its line of symmetry: $A(-a, 0)$, $B(a, 0)$, $C(b, c)$, $D(-b, c)$, where a, b, c are positive and $a \neq b$. Prove that its diagonals are congruent.



Show answer

See the model proof: both diagonals have length $\sqrt{(a+b)^2 + c^2}$.

TWO-COLUMN PROOF

Given: Isosceles trapezoid $ABCD$ with $A(-a, 0)$, $B(a, 0)$, $C(b, c)$, $D(-b, c)$; $a, b, c > 0$

Prove: $\overline{AC} \cong \overline{BD}$

	STATEMENTS	REASONS
1	$A(-a, 0)$, $B(a, 0)$, $C(b, c)$, $D(-b, c)$	Given
2	$AC = \sqrt{(b-(-a))^2 + (c-0)^2} = \sqrt{(a+b)^2 + c^2}$	Distance Formula
3	$BD = \sqrt{(-b-a)^2 + (c-0)^2} = \sqrt{(a+b)^2 + c^2}$	Distance Formula; $(-b-a)^2 = (a+b)^2$
4	$AC = BD$	Substitution Property of Equality
5	$\overline{AC} \cong \overline{BD}$	Definition of congruent segments

∎

6. **Medium** Square $ABCD$ is placed with $A(0,0)$, $B(a,0)$, $C(a,a)$, $D(0,a)$, where $a > 0$. Write a paragraph proof that its diagonals are congruent and perpendicular.

Show answer

See the model proof: both diagonals are $a\sqrt{2}$ long and their slopes are 1 and -1.

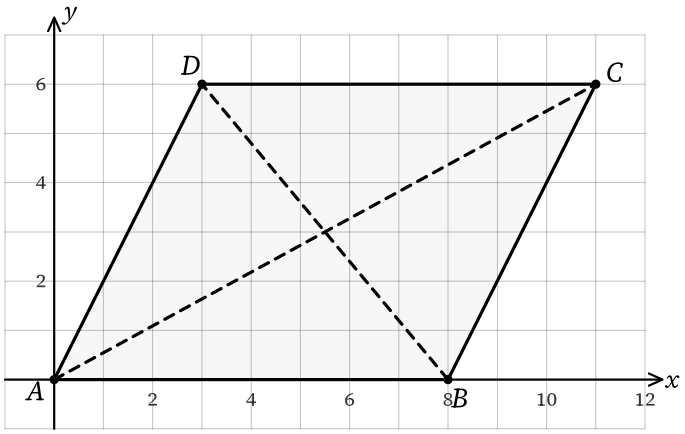
PARAGRAPH PROOF

Given: Square $ABCD$ with $A(0,0)$, $B(a,0)$, $C(a,a)$, $D(0,a)$, $a > 0$
Prove: $\overline{AC} \cong \overline{BD}$ and $\overline{AC} \perp \overline{BD}$
By the Distance Formula, $AC = \sqrt{a^2 + a^2} = a\sqrt{2}$ and $BD = \sqrt{(0-a)^2 + (a-0)^2} = a\sqrt{2}$. The lengths are equal, so $\overline{AC} \cong \overline{BD}$ by the definition of congruent segments.

The slope of $\overline{AC}$ is $\dfrac{a-0}{a-0} = 1$ and the slope of $\overline{BD}$ is $\dfrac{a-0}{0-a} = -1$. Their product is -1, so $\overline{AC} \perp \overline{BD}$ (Lesson 3.5). Therefore the diagonals of a square are congruent and perpendicular.

∎

7. **Medium** Henry builds a garden trellis in the shape of parallelogram $ABCD$. On his plan, in feet, $A(0,0)$, $B(8,0)$, $C(11,6)$ and $D(3,6)$, as in the figure. He nails the two diagonal braces $\overline{AC}$ and $\overline{BD}$ together where they cross. (a) At what point do they cross? (b) How long is the longer brace, to the nearest tenth of a foot?

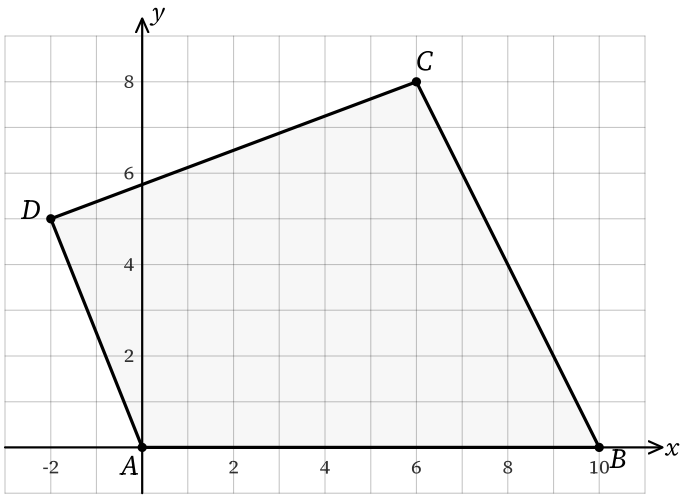


Coordinates in feet

Show answer

(a) The diagonals of a parallelogram bisect each other, so they cross at the midpoint of $\overline{AC}$: $(5.5, 3)$ (the midpoint of $\overline{BD}$ is the same point). (b) $AC = \sqrt{11^2 + 6^2} = \sqrt{157} \approx 12.5$ ft; $BD = \sqrt{5^2 + 6^2} = \sqrt{61} \approx 7.8$ ft. The longer brace is about 12.5 ft.

8. **Medium** Grace's garden bed is a quadrilateral with corners $A(0,0)$, $B(10,0)$, $C(6,8)$ and $D(-2,5)$, in feet, as in the figure. She lays a gravel path that joins the midpoints of the four sides in order. How many feet long is the whole path?

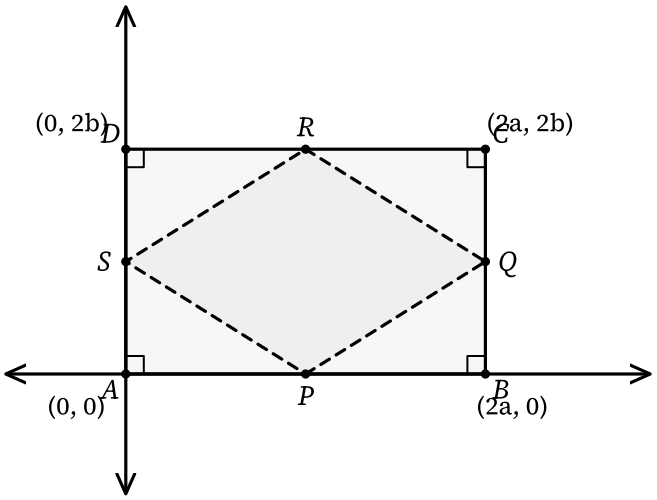


Coordinates in feet

Show answer

The midpoints are $(5,0)$, $(8,4)$, $(2,6.5)$ and $(-1,2.5)$. The path's sides are 5, 6.5, 5 and 6.5 feet, for a total of 23 feet. (Shortcut: each side is half a diagonal by the Midsegment Theorem, so the path equals $AC + BD = 10 + 13 = 23$ feet.)

9. Hard Prove with coordinates: the midpoints of the sides of a rectangle are the vertices of a rhombus. Use $A(0,0)$, $B(2a,0)$, $C(2a,2b)$, $D(0,2b)$ with $a, b > 0$, and let P, Q, R, S be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.



Show answer

See the model proof: all four sides of $PQRS$ have length $\sqrt{a^2+b^2}$.

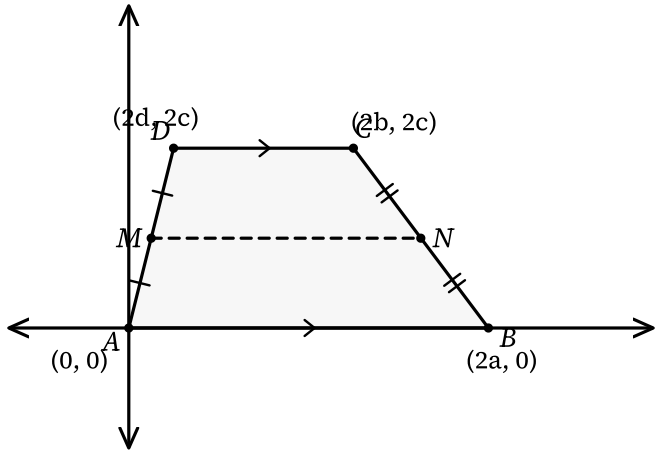
TWO-COLUMN PROOF

Given: Rectangle $ABCD$ with $A(0,0)$, $B(2a,0)$, $C(2a,2b)$, $D(0,2b)$; P, Q, R, S are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$

Prove: $PQRS$ is a rhombus

	STATEMENTS	REASONS
1	$A(0,0)$, $B(2a,0)$, $C(2a,2b)$, $D(0,2b)$; P, Q, R, S are midpoints	Given
2	$P(a,0)$, $Q(2a,b)$, $R(a,2b)$, $S(0,b)$	Midpoint Formula
3	$PQ = \sqrt{a^2+b^2}$ and $QR = \sqrt{(-a)^2+b^2} = \sqrt{a^2+b^2}$	Distance Formula
4	$RS = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$ and $SP = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$	Distance Formula
5	$\overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{SP}$	Definition of congruent segments
6	$PQRS$ is a rhombus	Definition of rhombus

10. Hard Prove the Trapezoid Midsegment Theorem with coordinates, using a flow proof. Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{DC}$, with $A(0,0)$, $B(2a,0)$, $C(2b,2c)$, $D(2d,2c)$, where $a>0$, $c>0$ and $b>d$. M is the midpoint of $\overline{AD}$ and N is the midpoint of $\overline{BC}$. Prove that $\overline{MN}$ is parallel to both bases and $MN=\frac{1}{2}(AB+DC)$.



Show answer

See the model proof: $MN=a+b-d=\frac{1}{2}(2a+2b-2d)$.

FLOW PROOF

Given: Trapezoid $ABCD$ with $A(0,0)$, $B(2a,0)$, $C(2b,2c)$, $D(2d,2c)$; $a>0$, $c>0$, $b>d$; M is the midpoint of $\overline{AD}$; N is the midpoint of $\overline{BC}$

Prove: $\overline{MN}\parallel\overline{AB}$, $\overline{MN}\parallel\overline{DC}$ and $MN=\frac{1}{2}(AB+DC)$

(1) $A(0,0)$, $B(2a,0)$, $C(2b,2c)$, $D(2d,2c)$; M, N midpoints of $\overline{AD}$, $\overline{BC}$
Given

(2) $M=(d,c)$ and $N=(a+b,c)$
Midpoint Formula
from (1)

(3) $\overline{MN}$, $\overline{AB}$ and $\overline{DC}$ all have slope 0
Slope formula (each pair of endpoints has equal y-coordinates)
from (1), (2)

(4) $\overline{MN}\parallel\overline{AB}$ and $\overline{MN}\parallel\overline{DC}$
Equal slopes; the lines $y=c, y=0, y=2c$ are different because $c\neq 0$

from (3)

(5) $MN = (a + b) - d$
Distance on a horizontal line; positive since $a > 0$ and $b > d$
from (2)

(6) $AB = 2a$ and $DC = 2b - 2d$
Distance on a horizontal line
from (1)

(7) $\frac{1}{2}(AB + DC) = \frac{1}{2}(2a + 2b - 2d) = a + b - d$
Substitution and Distributive Property
from (6)

(8) $MN = \frac{1}{2}(AB + DC)$
Transitive Property of Equality
from (5), (7)

(9) $\overline{MN} \parallel \overline{AB}$, $\overline{MN} \parallel \overline{DC}$ and $MN = \frac{1}{2}(AB + DC)$
Steps 4 and 8 together
from (4), (8)

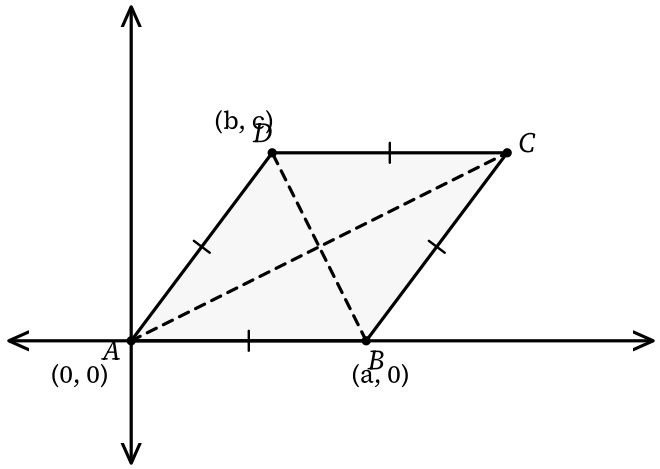
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11. **Hard** Samuel 'proves' that the diagonals of every rectangle are congruent by drawing the rectangle $(0, 0)$, $(4, 0)$, $(4, 3)$, $(0, 3)$ and showing both diagonals are 5. Explain why this is not a proof of the theorem, and say what he should do instead.

Show answer

He has checked only one rectangle, a 4-by-3 one. A single example can support a conjecture (inductive reasoning) but cannot prove a statement about every rectangle. He should use variable coordinates $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$, which describe every rectangle, and show both diagonals equal $\sqrt{a^2 + b^2}$, as in Worked Example 1.

12. Hard Prove with coordinates: the diagonals of a rhombus are perpendicular. Place rhombus $ABCD$ with $A(0,0)$, $B(a,0)$ and $D(b,c)$, where $a>0$ and $c>0$.



Show answer

See the model proof: the slopes of the diagonals multiply to $\dfrac{c^2}{b^2-a^2}=-1$ because $b^2+c^2=a^2$.

TWO-COLUMN PROOF

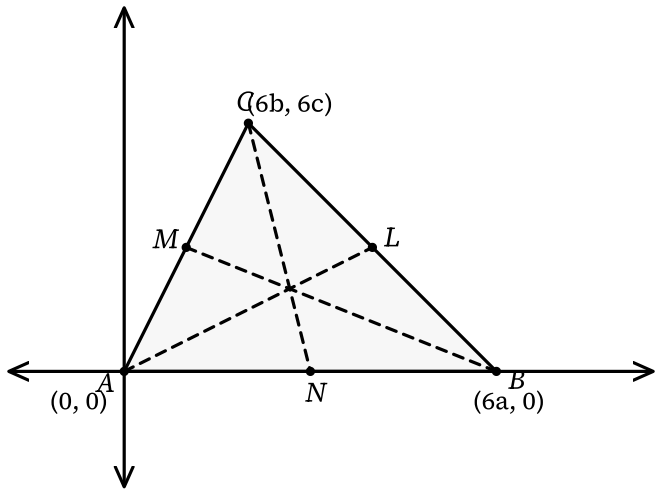
Given: $ABCD$ is a rhombus with $A(0,0)$, $B(a,0)$, $D(b,c)$; $a>0$, $c>0$
Prove: $\overline{AC}\perp\overline{BD}$

	STATEMENTS	REASONS
1	$ABCD$ is a rhombus with $A(0,0)$, $B(a,0)$, $D(b,c)$	Given
2	$ABCD$ is a parallelogram	A rhombus is a parallelogram (Lesson 6.4)
3	$C=(a+b,c)$	Fourth Vertex of a Parallelogram (Lesson 13.2): $C=B+D-A$
4	$AD=AB$, so $b^2+c^2=a^2$	Definition of rhombus; Distance Formula, squared
5	$b\neq a$ and $b\neq -a$	If $b=\pm a$, then $b^2=a^2$ and step 4 gives $c=0$, but $c>0$

	STATEMENTS	REASONS
6	Slope of $\overline{AC} = \frac{c}{a+b}$; slope of $\overline{BD} = \frac{c}{b-a}$	Slope formula (denominators are not 0 by step 5)
7	Product of the slopes $= \frac{c^2}{(a+b)(b-a)} = \frac{c^2}{b^2-a^2}$	Multiplication
8	$b^2 - a^2 = -c^2$	Subtraction Property of Equality (step 4)
9	Product of the slopes $= \frac{c^2}{-c^2} = -1$	Substitution Property of Equality
10	$\overline{AC} \perp \overline{BD}$	Lines whose slopes have product -1 are perpendicular (Lesson 3.5)

∎

13. **Challenge** Prove with coordinates that the three medians of any triangle meet at one point, which is two-thirds of the way from each vertex to the midpoint of the opposite side. Use $A(0,0)$, $B(6a,0)$, $C(6b,6c)$ with $a>0$, $c>0$, and write a paragraph proof.



Show answer

See the model proof: the point two-thirds of the way along each median is $(2a+2b, 2c)$.

PARAGRAPH PROOF

Given: $\triangle ABC$ with $A(0,0)$, $B(6a,0)$, $C(6b,6c)$; $a>0$, $c>0$; L, M, N are the midpoints of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$

Prove: Medians $\overline{AL}$, $\overline{BM}$, $\overline{CN}$ share the point two-thirds of the way from each vertex

Placement: any triangle can be moved so that A is at the origin, B is on the positive x-axis and C is above it. We use multiples of 6 because the proof divides by 2 (midpoints) and by 3 (thirds). By the Midpoint Formula, $L=(3a+3b,3c)$, $M=(3b,3c)$ and $N=(3a,0)$.

By the Partition Formula (Lesson 13.1) with $k=\dfrac{2}{3}$, the point two-thirds of the way from A to L is $\left(0+\dfrac{2}{3}(3a+3b),\ 0+\dfrac{2}{3}(3c)\right)=(2a+2b,2c)$.

Two-thirds of the way from B to M: $\left(6a+\dfrac{2}{3}(3b-6a),\ 0+\dfrac{2}{3}(3c)\right)=(6a+2b-4a,\ 2c)=(2a+2b,2c)$.

Two-thirds of the way from C to N: $\left(6b + \frac{2}{3}(3a - 6b),\ 6c + \frac{2}{3}(0 - 6c)\right) = (6b + 2a - 4b,\ 6c - 4c) = (2a + 2b, 2c)$.

All three computations give the same point $G(2a + 2b, 2c)$, and by the Partition Formula G lies on each median. So the three medians meet at G, and G is two-thirds of the way from each vertex to the midpoint of the opposite side.

∎

COMMON MISTAKES

- Choosing coordinates that make the figure special. Proving a fact about 'any triangle' with $(0, 0)$, $(2a, 0)$, $(0, 2b)$ proves it only for right triangles. Ask: can my coordinates describe every figure of this kind?
- Proving with numbers. Checking one triangle with numbered vertices is an example, not a proof; use variables.
- Assuming a vertex from the picture. The fourth vertex of a parallelogram, or the apex of an isosceles triangle, must be derived from the definition (equal slopes, equal lengths) before you use it.
- Forgetting to double. If a proof uses midpoints and the coordinates are a, b, c, fractions pile up; start with $2a$, $2b$, $2c$ (or $6a$, $6b$, $6c$ when thirds appear).

> **Make it a game · Spot the Assumption.** Parent proposes placements on axes: a rectangle at (0,0), (a,0), (a,b), (0,b); a general triangle with a right angle at the origin; a parallelogram with a vertex at (a, b) and (a + c, b). You decide which assume too much and give a better placement. Then prove one theorem aloud with coordinates.

UNIT 13

Coordinate Geometry and Proof

This last unit brings together two great traditions: Euclid's geometry of figures and reasons, and Descartes' idea of naming every point by a pair of numbers. You will gather the coordinate tools (distance, midpoint, slope and partitions), use them to prove what kind of triangle or quadrilateral a set of points forms, and then place general figures on axes to prove theorems for every triangle or parallelogram at once.

Parents: the algebra here is all from Algebra I. What is new is how it is used, as evidence in a proof. Students should state what they will prove, show every computation, and name the definition or theorem the numbers satisfy. The unit ends with a look back over the whole course, from Euclid's postulates to coordinate proofs, in preparation for the final exam.

PREREQUISITES

- Use the Pythagorean Theorem and its converse (Lessons 9.1 and 9.2).
- Find slopes and use them to recognize parallel and perpendicular lines (Lesson 3.5).
- Know the properties and tests for parallelograms, rectangles, rhombuses, squares, trapezoids and kites (Unit 6).
- Simplify square roots and expand expressions such as $(a+b)^2$.
- Write two-column, paragraph and flow proofs with a reason for every step.

MASTERY CHECKLIST

- ☐ I can find the distance, midpoint and slope for two points and explain why the Distance Formula comes from the Pythagorean Theorem.
- ☐ I can find the point that partitions a directed segment in a given ratio, and divide a segment in a given ratio with compass and straightedge.
- ☐ I can find the perimeter and area of a polygon on the coordinate plane, using the box method when no side is horizontal or vertical.
- ☐ I can use slopes to decide whether lines are parallel or perpendicular.
- ☐ I can prove that a triangle with given vertices is isosceles or right.
- ☐ I can prove that a quadrilateral with given vertices is a parallelogram, rectangle, rhombus, square, trapezoid or kite, or show that it is not.
- ☐ I can find a missing vertex of a parallelogram.
- ☐ I can place a general figure on coordinate axes without assuming anything extra, using coordinates like

$(2a, 0)$ to avoid fractions.

- ☐ I can write coordinate proofs of general theorems such as the Midsegment Theorem and the

diagonal properties of parallelograms and rectangles.

- ☐ I can compare a coordinate proof with Euclid's synthetic proof of the same theorem.

LESSONS IN THIS UNIT

13.1	**Coordinate Tools: Distance, Midpoint, Slope and Partitions**	45 minutes
13.2	**Classifying Figures with Coordinates**	2 sessions of 45 minutes
13.3	**Coordinate Proofs**	2 sessions of 45 minutes

Unit 13 test

You have 60 minutes. A scientific calculator is allowed for square roots and decimal approximations. You will need a compass and straightedge for Problem 12. Show all your work. Give exact answers (simplest radical form) unless a problem asks you to round. For every proof, state what is given and what you will prove, and give a reason for every step; in coordinate proofs, show each computation. The test is worth 100 points.

1. **5 pts**
   For $P(-4, 3)$ and $Q(8, -2)$, find (a) the distance PQ and (b) the midpoint of $\overline{PQ}$.

2. **6 pts**
   Line ℓ passes through $A(1, 4)$ and $B(5, -2)$. Line m passes through $C(0, 1)$ and $D(6, 5)$. Are the lines parallel, perpendicular, or neither? Show the slopes.

3. **6 pts**
   Find the point P that partitions the directed segment from $A(-3, -4)$ to $B(12, 6)$ in the ratio $2 : 3$.

4. **6 pts**
   Parallelogram $EFGH$ has vertices $E(-1, -2)$, $F(4, -1)$ and $G(6, 3)$. Find the coordinates of H.

5. **6 pts**

The line through $(-1, 2)$ and $(3, k)$ is perpendicular to the line through $(0, 0)$ and $(2, -4)$. Find k.

6. **7 pts**

On a ship's chart measured in nautical miles, a fishing boat at $S(2, 1)$ sails in a straight line to the harbor at $H(26, 11)$, as in the figure. (a) How far is the trip? (b) Where is the boat when it has gone $\frac{3}{4}$ of the way?

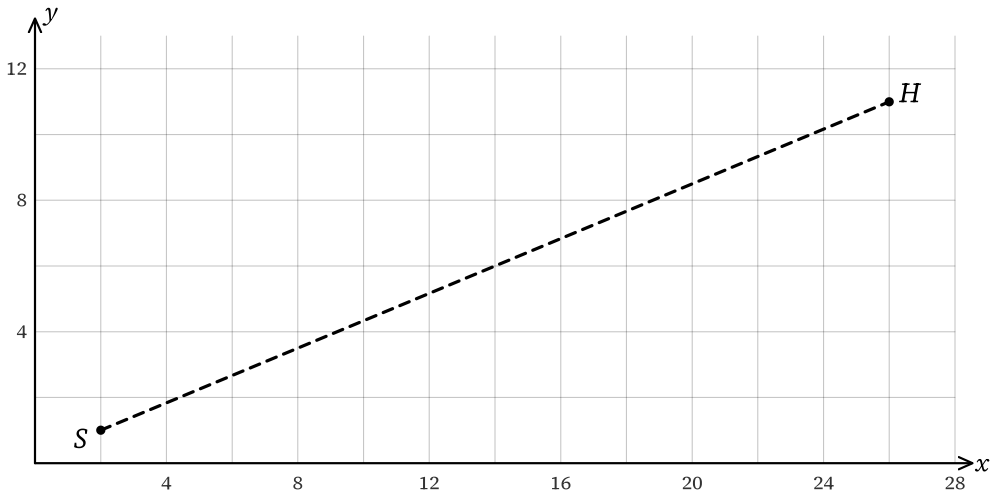
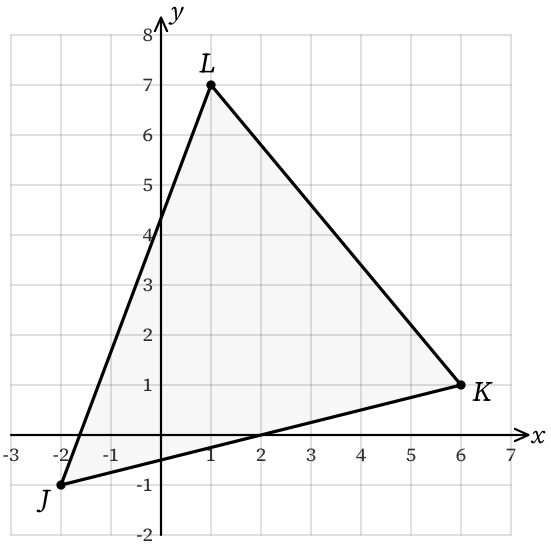


Chart in nautical miles

7. **8 pts**

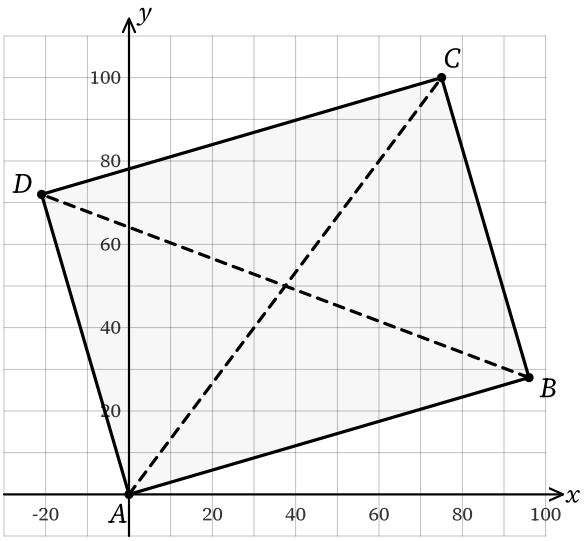
On the plan of a church courtyard, measured in meters, a triangular flower bed has corners $J(-2, -1)$, $K(6, 1)$ and $L(1, 7)$, as in the figure. Find (a) the area of the bed and (b) its perimeter to the nearest tenth of a meter.



Coordinates in meters

8. **7 pts**

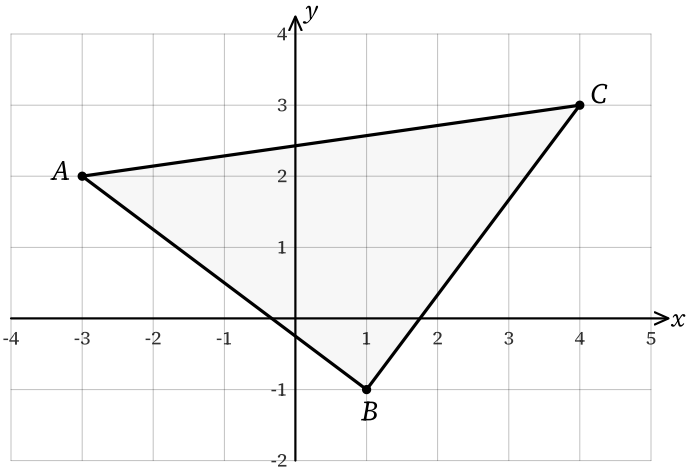
A coach marks the corners of a practice field at $A(0,0)$, $B(96,28)$, $C(75,100)$ and $D(-21,72)$, measured in yards, as in the figure. She has already checked that both pairs of opposite sides are congruent. (a) Find the length of diagonal $\overline{AC}$. (b) Find the length of diagonal $\overline{BD}$. (c) Is the field a rectangle?



Coordinates in yards

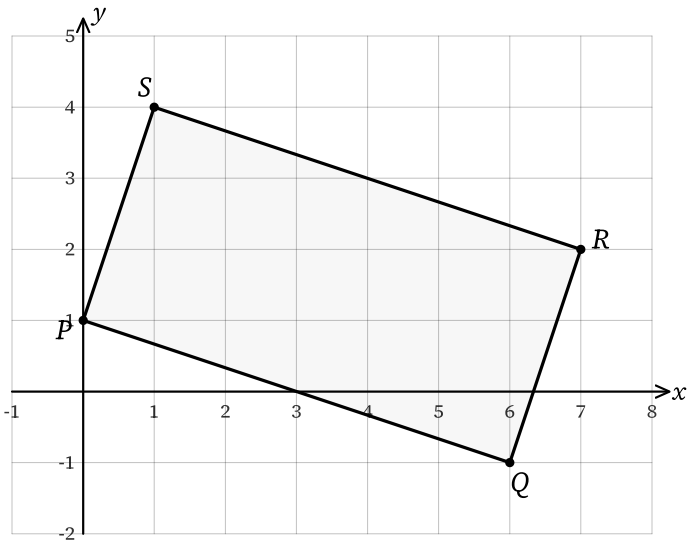
9. **8 pts**

Prove that $\triangle ABC$ with $A(-3,2)$, $B(1,-1)$ and $C(4,3)$ is an isosceles right triangle. Use a two-column proof.



10. **8 pts**

Prove that quadrilateral $PQRS$ with $P(0, 1)$, $Q(6, -1)$, $R(7, 2)$ and $S(1, 4)$ is a rectangle but not a square. Use a flow proof.



11. **6 pts**

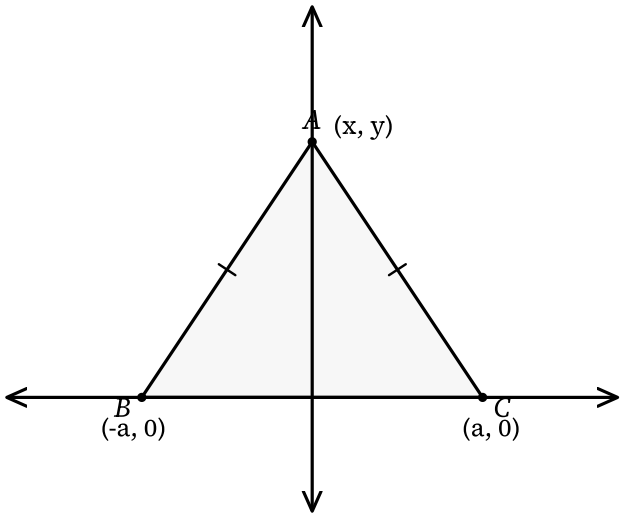
Henry shows that one pair of opposite sides of a quadrilateral have equal slopes and concludes that it is a parallelogram. Explain why this is not enough, and give the coordinates of a quadrilateral that shows he could be wrong.

12. **7 pts**

Draw a segment $\overline{AB}$ about 9 cm long. With compass and straightedge only, construct the point P on $\overline{AB}$ with $AP : PB = 1 : 2$. List your steps and explain why the construction works.

13. **10 pts**

Prove with coordinates: in an isosceles triangle, the median to the base is perpendicular to the base. Place $\triangle ABC$ with base $B(-a, 0)$, $C(a, 0)$, where $a > 0$, and apex $A(x, y)$ with $y > 0$ and $AB = AC$. Do not assume that A lies on the y-axis. Write a paragraph proof.



14. **10 pts**

Prove with coordinates: if the diagonals of a parallelogram are congruent, the parallelogram is a rectangle. Place parallelogram $ABCD$ with $A(0, 0)$, $B(a, 0)$ and $D(b, c)$, where $a > 0$ and $c > 0$. Use a two-column proof.

ANSWER KEY

1. (a) $PQ = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$. (b) $\left(\dfrac{-4+8}{2}, \dfrac{3+(-2)}{2}\right) = (2, 0.5)$.
2. Slope of $\ell = \dfrac{-6}{4} = -\dfrac{3}{2}$; slope of $m = \dfrac{4}{6} = \dfrac{2}{3}$. The product is -1, so the lines are perpendicular.
3. $k = \dfrac{2}{5}$; run 15, rise 10; $P = (-3+6,\ -4+4) = (3, 0)$
4. $H = E + G - F = (-1+6-4,\ -2+3-(-1)) = (1, 2)$
5. The second line has slope -2, so the first must have slope $\dfrac{1}{2}$: $\dfrac{k-2}{4} = \dfrac{1}{2}$, so $k = 4$.
6. (a) $SH = \sqrt{24^2 + 10^2} = \sqrt{676} = 26$ nautical miles. (b) $\dfrac{3}{4}$ of the run 24 is 18 and of the rise 10 is 7.5, so the boat is at $(20, 8.5)$.
7. (a) Box method: the rectangle from $x = -2$ to 6 and $y = -1$ to 7 has area $8 \times 8 = 64$. Subtract the outside right triangles: 8 (legs 8, 2), 15 (legs 5, 6) and 12 (legs 3, 8). Area $= 64 - 35 = 29$ square meters. (b) $JK = \sqrt{68} \approx 8.246$, $KL = \sqrt{61} \approx 7.810$, $LJ = \sqrt{73} \approx 8.544$; perimeter ≈ 24.6 meters.
8. (a) $AC = \sqrt{75^2 + 100^2} = 125$ yd. (b) $BD = \sqrt{(-117)^2 + 44^2} = \sqrt{15625} = 125$ yd. (c) Yes: both pairs of opposite sides are congruent, so it is a parallelogram, and a parallelogram with congruent diagonals is a rectangle.
9. See the model proof: $AB = BC = 5$ and $\overline{AB} \perp \overline{BC}$.

TWO-COLUMN PROOF

Given: $A(-3, 2)$, $B(1, -1)$, $C(4, 3)$

Prove: $\triangle ABC$ is an isosceles right triangle

	STATEMENTS	REASONS
1	$A(-3, 2)$, $B(1, -1)$, $C(4, 3)$	Given
2	$AB = \sqrt{4^2 + (-3)^2} = 5$ and $BC = \sqrt{3^2 + 4^2} = 5$	Distance Formula
3	$\overline{AB} \cong \overline{BC}$	Definition of congruent segments
4	$\triangle ABC$ is isosceles	Definition of isosceles triangle
5	Slope of $\overline{AB} = -\dfrac{3}{4}$; slope of $\overline{BC} = \dfrac{4}{3}$	Slope formula

	STATEMENTS	REASONS
6	$\overline{AB} \perp \overline{BC}$	Slopes with product $-\frac{3}{4} \cdot \frac{4}{3} = -1$
7	$\angle ABC$ is a right angle	Definition of perpendicular lines
8	$\triangle ABC$ is an isosceles right triangle	Definitions of isosceles triangle and right triangle

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10. See the model proof: opposite slopes are equal, consecutive slopes multiply to -1, and $PQ = \sqrt{40} \neq QR = \sqrt{10}$.

FLOW PROOF

Given: $P(0, 1)$, $Q(6, -1)$, $R(7, 2)$, $S(1, 4)$

Prove: $PQRS$ is a rectangle but not a square

(1) $P(0, 1)$, $Q(6, -1)$, $R(7, 2)$, $S(1, 4)$
Given

(2) Slopes: $\overline{PQ}$: $-\frac{1}{3}$; $\overline{QR}$: 3; $\overline{RS}$: $-\frac{1}{3}$; $\overline{SP}$: 3
Slope formula
from (1)

(3) $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$
Lines with equal slopes are parallel
from (2)

(4) $PQRS$ is a parallelogram
Definition of parallelogram
from (3)

(5) Each pair of consecutive sides is perpendicular
Slopes with product $-\frac{1}{3} \cdot 3 = -1$
from (2)

(6) $\angle P$, $\angle Q$, $\angle R$, $\angle S$ are right angles
Definition of perpendicular lines
from (5)

(7) $PQRS$ is a rectangle
Definition of rectangle

from (4), (6)

(8) $PQ = \sqrt{6^2 + (-2)^2} = \sqrt{40}$ and $QR = \sqrt{1^2 + 3^2} = \sqrt{10}$
Distance Formula
from (1)

(9) $PQRS$ is not a square
A square has four congruent sides, but $PQ \neq QR$
from (8)

(10) $PQRS$ is a rectangle but not a square
Steps 7 and 9 together
from (7), (9)

∎

11. Equal slopes for one pair of sides prove only that those two sides are parallel. A parallelogram needs both pairs parallel (or one pair both parallel and congruent, Euclid I.33). Example: $A(0,0)$, $B(6,0)$, $C(4,3)$, $D(1,3)$: $\overline{AB}$ and $\overline{DC}$ both have slope 0, but $\overline{BC}$ has slope $-\frac{3}{2}$ and $\overline{AD}$ has slope 3, so this is a trapezoid, not a parallelogram. (Any correct example earns credit.)
12. Steps: (1) Draw a ray from A not along $\overline{AB}$. (2) With a fixed compass width, mark three equal segments along the ray from A: X_1, X_2, X_3. (3) Draw $\overline{X_3B}$. (4) At X_1, copy $\angle AX_3B$ to draw a line through X_1 parallel to $\overline{X_3B}$; it meets $\overline{AB}$ at P. Why: in $\triangle AX_3B$, the line through X_1 is parallel to side $\overline{X_3B}$, so by the Triangle Proportionality Theorem (Euclid VI.2) $AP : PB = AX_1 : X_1X_3 = 1 : 2$.
13. See the model proof: $AB = AC$ forces $x = 0$, so the median from A lies along the y-axis.

    PARAGRAPH PROOF

    Given: $\triangle ABC$ with $B(-a, 0)$, $C(a, 0)$, $A(x, y)$, $a > 0$, $y > 0$; $\overline{AB} \cong \overline{AC}$
    Prove: The median from A to $\overline{BC}$ is perpendicular to $\overline{BC}$
    Since $AB = AC$, the Distance Formula gives $\sqrt{(x+a)^2 + y^2} = \sqrt{(x-a)^2 + y^2}$. Squaring both sides and subtracting y^2 gives $(x+a)^2 = (x-a)^2$, that is, $x^2 + 2ax + a^2 = x^2 - 2ax + a^2$. So $4ax = 0$, and since $a \neq 0$, $x = 0$. Thus $A = (0, y)$. By the Midpoint Formula, the midpoint of $\overline{BC}$ is $\left(\frac{-a+a}{2}, \frac{0+0}{2}\right) = (0, 0)$, so the median from A is the segment from $(0, y)$ to $(0, 0)$. Both endpoints have x-coordinate 0, so the median is vertical. The base $\overline{BC}$ lies on the x-axis, so it is horizontal.

A vertical line and a horizontal line are perpendicular, so the median to the base is perpendicular to the base.

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14. See the model proof: $AC = BD$ forces $b = 0$, so the sides are horizontal and vertical.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram with $A(0,0)$, $B(a,0)$, $D(b,c)$; $a > 0$, $c > 0$; $\overline{AC} \cong \overline{BD}$

Prove: $ABCD$ is a rectangle

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram with $A(0,0)$, $B(a,0)$, $D(b,c)$, and $\overline{AC} \cong \overline{BD}$	Given
2	$C = (a+b, c)$	Fourth Vertex of a Parallelogram (Lesson 13.2): $C = B + D - A$
3	$AC^2 = (a+b)^2 + c^2$ and $BD^2 = (b-a)^2 + c^2$	Distance Formula, squared
4	$(a+b)^2 + c^2 = (b-a)^2 + c^2$	Definition of congruent segments ($AC = BD$) and Substitution
5	$4ab = 0$	Expand both sides; Subtraction Property of Equality
6	$b = 0$	Division Property of Equality ($a \neq 0$)
7	$D = (0, c)$ and $C = (a, c)$	Substitution
8	$\overline{AB}$ and $\overline{DC}$ are horizontal; $\overline{AD}$ and $\overline{BC}$ are vertical	$\overline{AB}$ lies on $y = 0$ and $\overline{DC}$ on $y = c$; $\overline{AD}$ lies on $x = 0$ and $\overline{BC}$ on $x = a$
9	$\angle A$, $\angle B$, $\angle C$, $\angle D$ are right angles	A horizontal line and a vertical line are perpendicular; definition of perpendicular lines

	STATEMENTS	REASONS
10	$ABCD$ is a rectangle	Definition of rectangle (a parallelogram with four right angles)

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COURSE EXAM

Midterm Exam

Covers Units 1–6. Time: 90 minutes.

You have 90 minutes. No calculator is allowed. Show all your work. In every proof, give a reason for every statement. Give exact answers (in simplest radical form where needed) unless a problem asks you to round. Use a compass and straightedge for the construction, and leave your construction arcs showing. Figures are drawn to scale unless marked otherwise, but a figure never proves anything: use only the given information and the marks on it. The exam is worth 100 points.

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1. **3 pts** **Lesson 1.3**

   Consider the true statement: "If two angles form a linear pair, then they are supplementary." (a) Write its converse. (b) Is the converse true? If it is false, give a counterexample.

2. **3 pts** **Lesson 1.4**

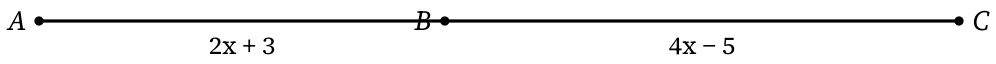
   Is this argument valid? Name the law it uses or the error it makes. "If a quadrilateral is a square, then it has four right angles. The Miller family's garden plot is a quadrilateral with four right angles. Therefore the garden plot is a square."

3. **4 pts** **Lesson 1.5**

   Write a two-column proof. Given: $3(x - 4) = 5x + 8$. Prove: $x = -10$.

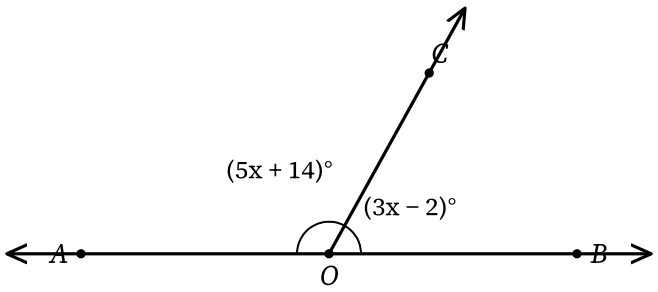
4. **4 pts** **Lesson 2.2**

   In the figure, B is between A and C on $\overline{AC}$, with $AB = 2x + 3$, $BC = 4x - 5$ and $AC = 34$. Find BC.



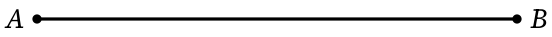
5. **4 pts** **Lesson 2.4**

In the figure, A, O and B are collinear with O between A and B, $m\angle AOC = (5x + 14)^\circ$ and $m\angle COB = (3x - 2)^\circ$. Find $m\angle AOC$.



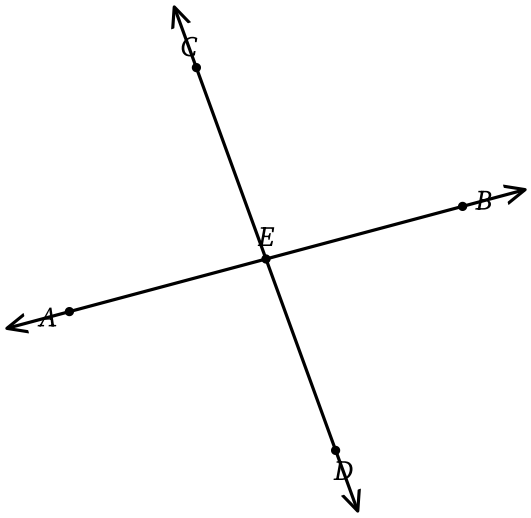
6. **5 pts** **Lesson 2.5**

Copy $\overline{AB}$ from the figure onto your paper. Using only a compass and straightedge, construct the perpendicular bisector of $\overline{AB}$. List your steps, and explain why the line you drew is the perpendicular bisector.



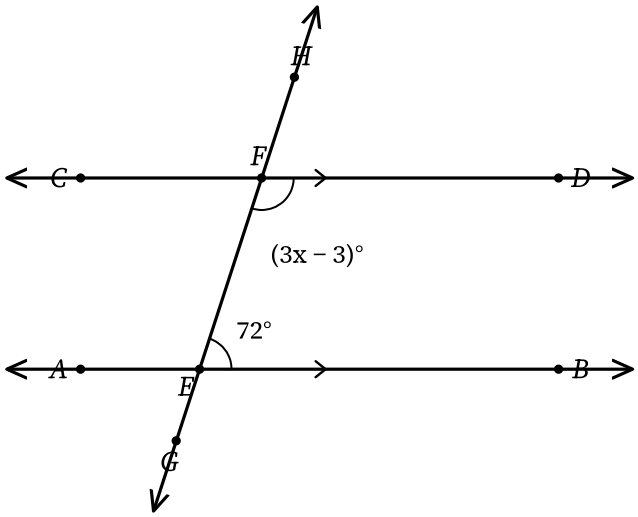
7. **5 pts** **Lesson 2.6**

Write a paragraph proof. Given: lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, as in the figure. Prove: $\angle AEC \cong \angle BED$. (Do not simply quote the Vertical Angles Theorem; prove it.)



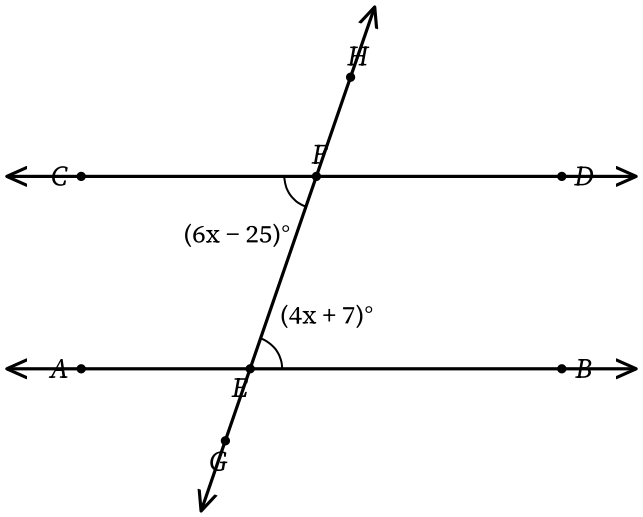
8. **3 pts** **Lesson 3.2**

In the figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, and the transversal $\overleftrightarrow{GH}$ crosses them at E and F, with B and D on the same side of the transversal. If $m\angle BEF = 72^\circ$ and $m\angle DFE = (3x - 3)^\circ$, find x.



9. **4 pts** **Lesson 3.3**

In the figure, the transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F, with $m\angle BEF = (4x + 7)^\circ$ and $m\angle CFE = (6x - 25)^\circ$. For what value of x is $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$? Name the theorem that guarantees it.



10. **4 pts** **Lesson 3.4**

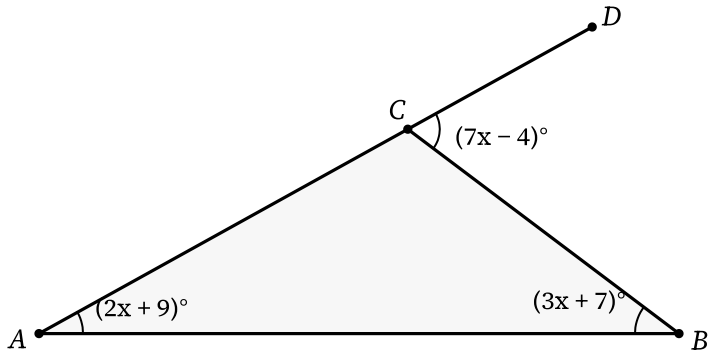
Samuel wants to lay the shortest possible straight pipe from a spring S to a straight irrigation ditch. Where on the ditch should the pipe end? Explain why that pipe is shorter than any other straight pipe from S to the ditch.

11. **4 pts** **Lesson 3.5**

Write an equation in slope-intercept form of the line through $(4, -1)$ that is perpendicular to the line $y = 2x + 5$.

12. **4 pts** **Lesson 4.1**

In the figure, side $\overline{AC}$ of $\triangle ABC$ is extended through C to D. $m\angle BAC = (2x + 9)^\circ$, $m\angle ABC = (3x + 7)^\circ$ and $m\angle BCD = (7x - 4)^\circ$. Find $m\angle ACB$.

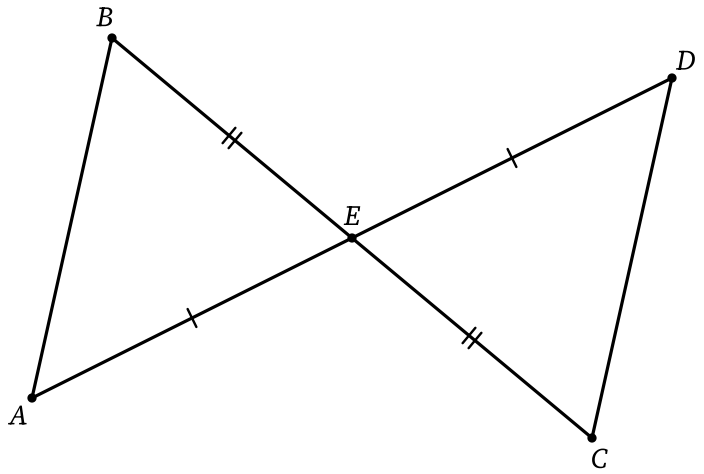


13. **3 pts** **Lesson 4.2**

$\triangle DEF \cong \triangle KLM$, with $DE = 2y - 3$ and $KL = y + 9$. (a) Find y. (b) Find DE.

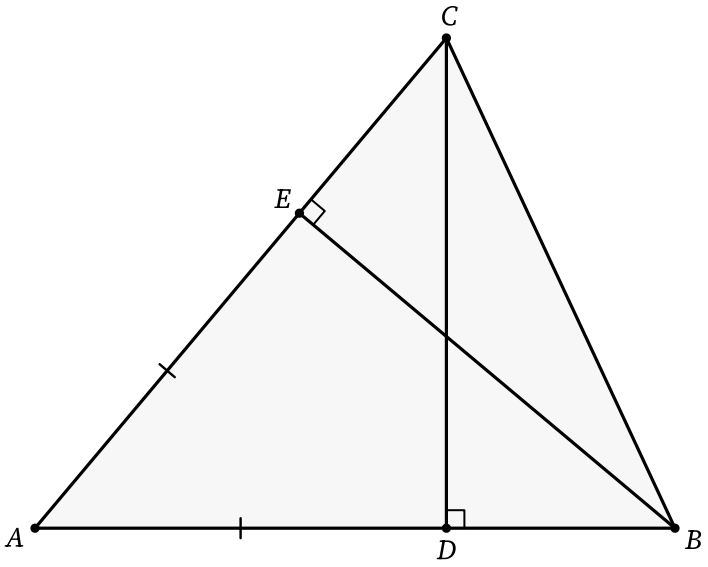
14. **6 pts** **Lesson 4.5**

Write a two-column proof. Given: E is the midpoint of $\overline{AD}$ and of $\overline{BC}$, as in the figure. Prove: $\overline{AB} \parallel \overline{CD}$.



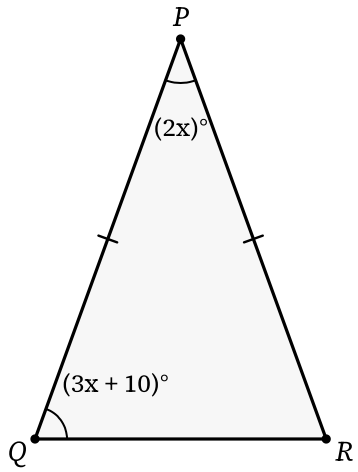
15. **5 pts** **Lesson 4.4**

Write a flow proof. Given: in $\triangle ABC$, $\overline{BE} \perp \overline{AC}$ at E, $\overline{CD} \perp \overline{AB}$ at D, and $\overline{AD} \cong \overline{AE}$, as in the figure. Prove: $\triangle ABE \cong \triangle ACD$.



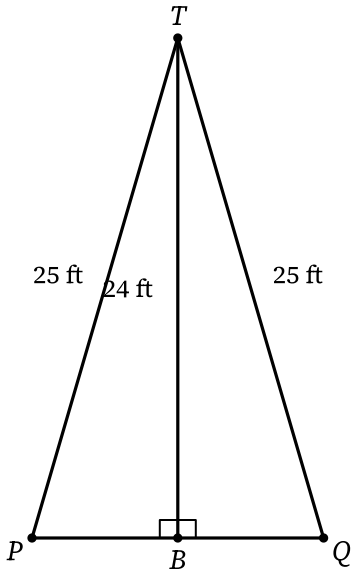
16. **4 pts** **Lesson 4.6**

The triangular front face of a church steeple is $\triangle PQR$ with $PQ = PR$, as in the figure. If $m\angle Q = (3x + 10)^\circ$ and $m\angle P = (2x)^\circ$, find the angle at the peak, $m\angle P$.



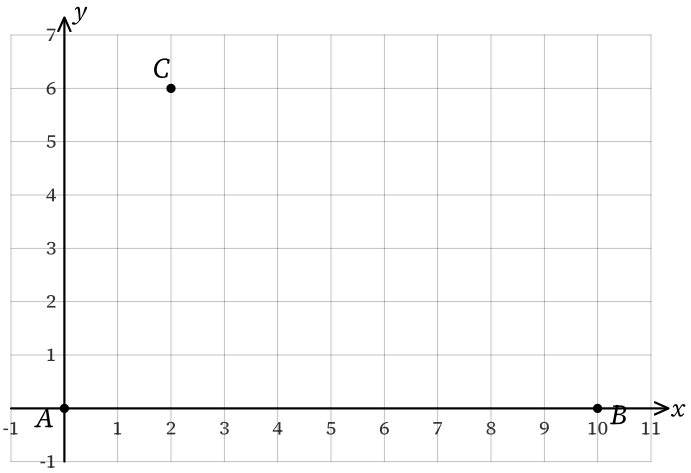
17. **4 pts** **Lesson 4.7**

A flagpole $\overline{TB}$, 24 ft tall, stands perpendicular to level ground. Two guy wires, each 25 ft long, run from the top T to stakes P and Q on opposite sides of the pole, with P, B and Q on one line, as in the figure. (a) Name the congruence theorem that proves $\triangle TBP \cong \triangle TBQ$. (b) How far apart are the two stakes?



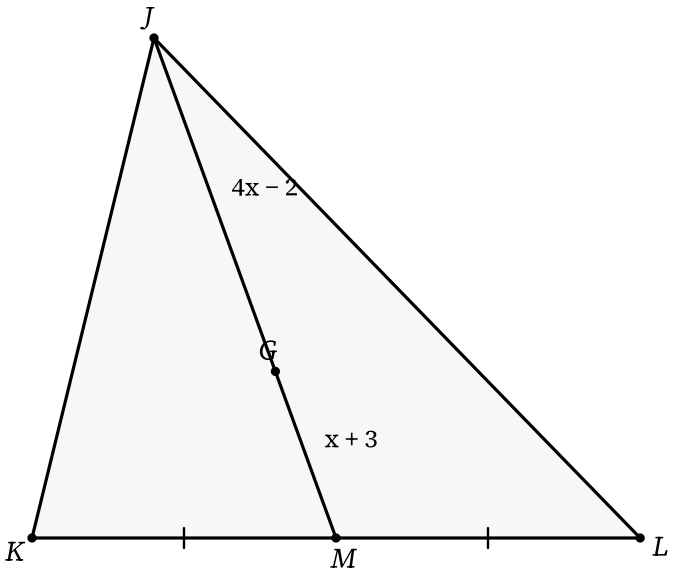
18. **4 pts** **Lesson 5.2**

On a map, three farmhouses stand at $A(0, 0)$, $B(10, 0)$ and $C(2, 6)$, where each unit is 100 yards, as in the figure. The families want one well that is the same distance from all three houses. (a) Find the coordinates of the well. (b) About how many yards will the well be from each house?



19. **4 pts** **Lesson 5.3**

In the figure, G is the centroid of $\triangle JKL$ and M is the midpoint of $\overline{KL}$. If $JG = 4x - 2$ and $GM = x + 3$, find JM.



20. **3 pts** **Lesson 5.5**

In $\triangle ABC$, $m\angle A = 48^\circ$ and $m\angle B = 71^\circ$. List the sides of the triangle from shortest to longest.

21. **3 pts** **Lesson 5.6**

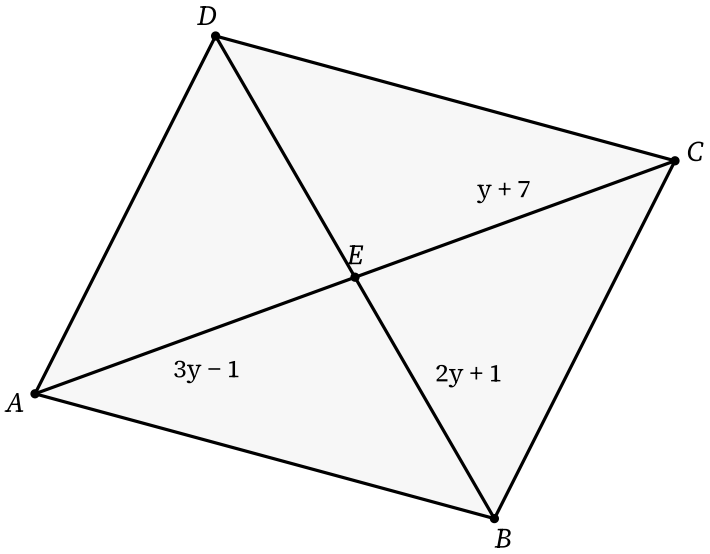
Two sides of a triangular garden bed measure 11 ft and 17 ft. Write an inequality for the possible lengths x (in feet) of the third side.

22. **3 pts** **Lesson 6.1**

The stone frame of a rose window is a regular polygon, and each of its interior angles measures 160°. How many sides does the frame have?

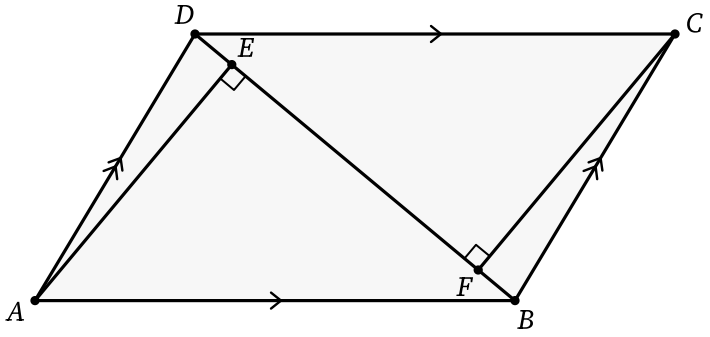
23. **3 pts** **Lesson 6.2**

In parallelogram $ABCD$ in the figure, the diagonals meet at E, with $AE = 3y - 1$, $EC = y + 7$ and $BE = 2y + 1$. Find BD.



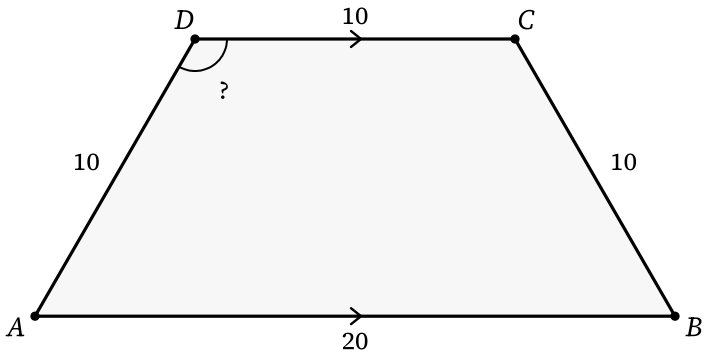
24. **6 pts** **Lesson 6.2**

Write a two-column proof. Given: $ABCD$ is a parallelogram, $\overline{AE} \perp \overline{BD}$ at E, and $\overline{CF} \perp \overline{BD}$ at F, as in the figure. Prove: $\overline{AE} \cong \overline{CF}$.



25. **5 pts** **Lesson 6.5**

Challenge. In trapezoid $ABCD$ in the figure, $\overline{AB} \parallel \overline{DC}$, $AB = 20$, $DC = 10$ and $AD = BC = 10$. Find $m\angle ADC$. (Hint: draw a segment through D parallel to $\overline{BC}$.)



ANSWER KEY

1. (a) If two angles are supplementary, then they form a linear pair. (b) False. Counterexample: a 110° angle at one corner of a field and a 70° angle at another corner are supplementary, but they are not adjacent, so they do not form a linear pair.

2. No, the argument is not valid. It affirms the conclusion (it reasons from the converse). A rectangular plot 20 ft by 30 ft has four right angles but is not a square.

3. See the model proof.

   TWO-COLUMN PROOF

   Given: $3(x - 4) = 5x + 8$

   Prove: $x = -10$

	STATEMENTS	REASONS
1	$3(x - 4) = 5x + 8$	Given
2	$3x - 12 = 5x + 8$	Distributive Property
3	$-12 = 2x + 8$	Subtraction Property of Equality (subtract $3x$)
4	$-20 = 2x$	Subtraction Property of Equality (subtract 8)
5	$-10 = x$	Division Property of Equality (divide by 2)
6	$x = -10$	Symmetric Property of Equality

   ∎

4. By the Segment Addition Postulate, $(2x + 3) + (4x - 5) = 34$, so $6x - 2 = 34$ and $x = 6$. Then $BC = 4(6) - 5 = 19$.

5. $\angle AOC$ and $\angle COB$ form a linear pair, so they are supplementary: $(5x + 14) + (3x - 2) = 180$, $8x + 12 = 180$, $x = 21$. So $m\angle AOC = 5(21) + 14 = 119^\circ$.

6. Steps: (1) Open the compass to a width greater than half of AB. (2) With the point on A, draw arcs above and below $\overline{AB}$. (3) Keeping the same width, put the point on B and draw arcs that cross the first two arcs at P (above) and Q (below). (4) Draw $\overleftrightarrow{PQ}$. Why it works: the same compass width gives $PA = PB$ and $QA = QB$, so P and Q are each equidistant from A and B. By the Converse of

the Perpendicular Bisector Theorem, both lie on the perpendicular bisector of $\overline{AB}$, and since two points determine a line, $\overleftrightarrow{PQ}$ is the perpendicular bisector (Euclid I.10).

7. See the model proof.

   PARAGRAPH PROOF

   Given: Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E

   Prove: $\angle AEC \cong \angle BED$

   Because A, E and B lie on one line with E between A and B, the angles $\angle AEC$ and $\angle CEB$ form a linear pair, so by the Linear Pair Theorem they are supplementary: $m\angle AEC + m\angle CEB = 180°$. In the same way, C, E and D lie on one line, so $\angle CEB$ and $\angle BED$ form a linear pair and $m\angle CEB + m\angle BED = 180°$.

   By the Transitive Property (or Substitution) of Equality, $m\angle AEC + m\angle CEB = m\angle CEB + m\angle BED$. Subtracting $m\angle CEB$ from both sides (Subtraction Property of Equality) gives $m\angle AEC = m\angle BED$. Angles with equal measures are congruent, so $\angle AEC \cong \angle BED$.

   ∎

8. $\angle BEF$ and $\angle DFE$ are same-side interior angles, so they are supplementary: $72 + (3x - 3) = 180$, $3x = 111$, $x = 37$.

9. $\angle BEF$ and $\angle CFE$ are alternate interior angles. Set them equal: $4x + 7 = 6x - 25$, so $x = 16$ and each angle measures $71°$. By the Converse of the Alternate Interior Angles Theorem (Euclid I.27), the lines are parallel.

10. The pipe should end at F, the foot of the perpendicular from S to the ditch. If X is any other point on the ditch, then $\triangle SFX$ has a right angle at F. The acute angles of a right triangle are complementary, so $\angle SXF$ is smaller than the right angle $\angle SFX$. In a triangle the larger side lies opposite the larger angle (Euclid I.19), so $SX > SF$. Therefore the perpendicular pipe $\overline{SF}$ is the shortest.

11. The slope of $y = 2x + 5$ is 2, so a perpendicular line has slope $-\dfrac{1}{2}$. Point-slope form: $y + 1 = -\dfrac{1}{2}(x - 4)$, so $y = -\dfrac{1}{2}x + 1$.

12. By the Exterior Angle Theorem, $7x - 4 = (2x + 9) + (3x + 7)$, so $7x - 4 = 5x + 16$ and $x = 10$. Then $m\angle BCD = 66°$, and since $\angle ACB$ and $\angle BCD$ form a linear pair, $m\angle ACB = 180° - 66° = 114°$.

13. (a) $\overline{DE}$ corresponds to $\overline{KL}$, so $2y - 3 = y + 9$ and $y = 12$. (b) $DE = 2(12) - 3 = 21$.

14. See the model proof.

TWO-COLUMN PROOF

Given: E is the midpoint of $\overline{AD}$; E is the midpoint of $\overline{BC}$

Prove: $\overline{AB} \parallel \overline{CD}$

	STATEMENTS	REASONS
1	E is the midpoint of $\overline{AD}$	Given
2	$\overline{AE} \cong \overline{DE}$	Definition of midpoint
3	E is the midpoint of $\overline{BC}$	Given
4	$\overline{BE} \cong \overline{CE}$	Definition of midpoint
5	$\angle AEB \cong \angle DEC$	Vertical Angles Theorem
6	$\triangle AEB \cong \triangle DEC$	SAS
7	$\angle BAE \cong \angle CDE$	CPCTC
8	$\overline{AB} \parallel \overline{CD}$	Converse of the Alternate Interior Angles Theorem (transversal $\overline{AD}$)

∎

15. See the model proof.

FLOW PROOF

Given: $\overline{BE} \perp \overline{AC}$ at E; $\overline{CD} \perp \overline{AB}$ at D; $\overline{AD} \cong \overline{AE}$

Prove: $\triangle ABE \cong \triangle ACD$

(1) $\overline{BE} \perp \overline{AC}$ at E
Given

(2) $\overline{CD} \perp \overline{AB}$ at D
Given

(3) $\overline{AE} \cong \overline{AD}$
Given

(4) $\angle AEB$ is a right angle
Definition of perpendicular lines
from (1)

(5) $\angle ADC$ is a right angle
Definition of perpendicular lines

from (2)

(6) $\angle AEB \cong \angle ADC$
All right angles are congruent
from (4), (5)

(7) $\angle BAE \cong \angle CAD$
Reflexive Property of Congruence (both are $\angle BAC$)

(8) $\triangle ABE \cong \triangle ACD$
ASA
from (7), (3), (6)

∎

16. By the Isosceles Triangle Theorem, $m\angle R = m\angle Q = (3x + 10)^\circ$. By the Triangle Angle Sum Theorem, $(3x + 10) + (3x + 10) + 2x = 180$, so $8x + 20 = 180$ and $x = 20$. Then $m\angle P = 40^\circ$.
17. (a) HL: both triangles have a right angle at B, the hypotenuses $\overline{TP}$ and $\overline{TQ}$ are congruent (25 ft), and they share the leg $\overline{TB}$. (b) $BP = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$ ft, and $BQ = BP = 7$ ft by CPCTC, so $PQ = 14$ ft.
18. (a) The well belongs at the circumcenter, where the perpendicular bisectors of the sides meet. The perpendicular bisector of $\overline{AB}$ is $x = 5$. The midpoint of $\overline{AC}$ is $(1, 3)$ and the slope of $\overline{AC}$ is 3, so its perpendicular bisector is $y - 3 = -\frac{1}{3}(x - 1)$. At $x = 5$, $y = 3 - \frac{4}{3} = \frac{5}{3}$. The well is at $\left(5, \frac{5}{3}\right)$. (b) Its distance from A is $\sqrt{5^2 + \left(\frac{5}{3}\right)^2} = \dfrac{5\sqrt{10}}{3} \approx 5.27$ units, or about 527 yards, and it is the same distance from B and C.
19. The centroid is two-thirds of the way from a vertex to the opposite midpoint, so $JG = 2 \cdot GM$: $4x - 2 = 2(x + 3)$, $2x = 8$, $x = 4$. Then $JG = 14$, $GM = 7$ and $JM = 21$.
20. $m\angle C = 180^\circ - 48^\circ - 71^\circ = 61^\circ$. The smaller side lies opposite the smaller angle, so from shortest to longest: $\overline{BC}$, $\overline{AB}$, $\overline{AC}$.
21. By the Triangle Inequality Theorem, $17 - 11 < x < 17 + 11$, so $6 < x < 28$.
22. Each exterior angle measures $180^\circ - 160^\circ = 20^\circ$. The exterior angles add to 360°, so the polygon has $360 \div 20 = 18$ sides.
23. The diagonals of a parallelogram bisect each other, so $3y - 1 = y + 7$ and $y = 4$. Then $BE = 9$, $ED = BE = 9$, and $BD = 18$.
24. See the model proof.

TWO-COLUMN PROOF

Given: $ABCD$ is a parallelogram; $\overline{AE} \perp \overline{BD}$ at E; $\overline{CF} \perp \overline{BD}$ at F

Prove: $\overline{AE} \cong \overline{CF}$

	STATEMENTS	REASONS
1	$ABCD$ is a parallelogram	Given
2	$\overline{AB} \cong \overline{CD}$	Opposite sides of a parallelogram are congruent
3	$\overline{AB} \parallel \overline{DC}$	Definition of parallelogram
4	$\angle ABE \cong \angle CDF$	Alternate Interior Angles Theorem (transversal $\overline{BD}$)
5	$\overline{AE} \perp \overline{BD}, \overline{CF} \perp \overline{BD}$	Given
6	$\angle AEB$ and $\angle CFD$ are right angles	Definition of perpendicular lines
7	$\angle AEB \cong \angle CFD$	All right angles are congruent
8	$\triangle AEB \cong \triangle CFD$	AAS
9	$\overline{AE} \cong \overline{CF}$	CPCTC

∎

25. Draw $\overline{DE} \parallel \overline{BC}$ with E on $\overline{AB}$. Then $DEBC$ has both pairs of opposite sides parallel, so it is a parallelogram, and $DE = CB = 10$ and $EB = DC = 10$. So $AE = 20 - 10 = 10$, and $\triangle ADE$ is equilateral ($AD = DE = AE = 10$), which makes $m\angle A = 60°$. Since $\overline{AB} \parallel \overline{DC}$, $\angle A$ and $\angle ADC$ are same-side interior angles, so $m\angle ADC = 180° - 60° = 120°$.

COURSE EXAM

Final Exam

Covers Units 1–13. Time: 2 hours.

You have 2 hours. A scientific calculator is allowed; set it to degree mode. Show all your work. In every proof, give a reason for every statement. Give exact answers (in simplest radical form or in terms of π) unless a problem asks you to round, and then round as directed. Label units in your answers. Use a compass and straightedge for any construction. Figures are drawn to scale unless marked otherwise, but a figure never proves anything: use only the given information and the marks on it. The exam is worth 100 points.

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1. **3 pts** **Lesson 1.3**

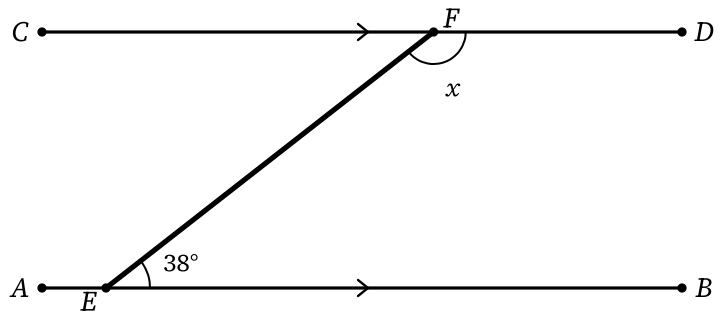
   Write the contrapositive of this true statement, and tell whether the contrapositive is true: "If a quadrilateral is a rhombus, then its diagonals are perpendicular."

2. **3 pts** **Lesson 2.4**

   The supplement of an angle measures 18° more than three times its complement. Find the measure of the angle.

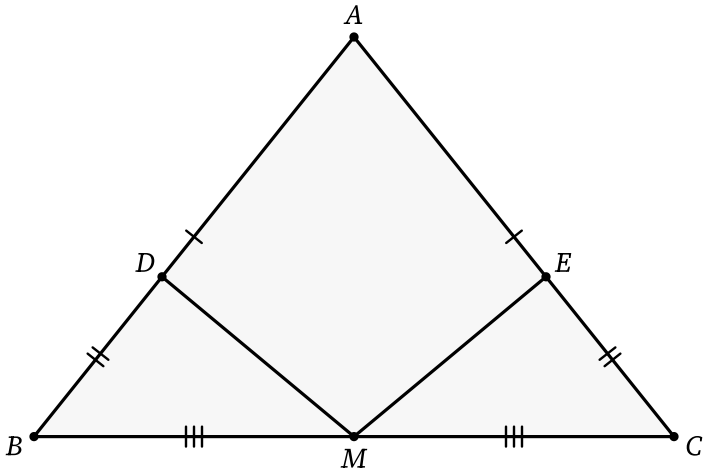
3. **3 pts** **Lesson 3.2**

   Henry nails a diagonal brace $\overline{EF}$ across the two parallel horizontal boards $\overline{AB}$ and $\overline{CD}$ of a gate, as in the figure, where B and D are at the same end of the gate. The brace makes $m\angle BEF = 38^\circ$ with the bottom board. Find $x = m\angle DFE$, the angle the brace makes with the top board.



4. **5 pts** **Lesson 4.6**

Write a two-column proof. Given: in $\triangle ABC$, $\overline{AB} \cong \overline{AC}$, M is the midpoint of $\overline{BC}$, and points D on $\overline{AB}$ and E on $\overline{AC}$ satisfy $\overline{BD} \cong \overline{CE}$, as in the figure. Prove: $\overline{MD} \cong \overline{ME}$.



5. **3 pts** **Lesson 5.6**

Grace has three straight sticks that are 4.2 cm, 5.9 cm and 10.3 cm long. Can she lay them end to end to form a triangle? Explain why or why not.

6. **3 pts** **Lesson 6.1**

The frame of a stained-glass window is a regular polygon, and each of its interior angles measures 144°. How many sides does the frame have?

7. **3 pts** **Lesson 6.4**

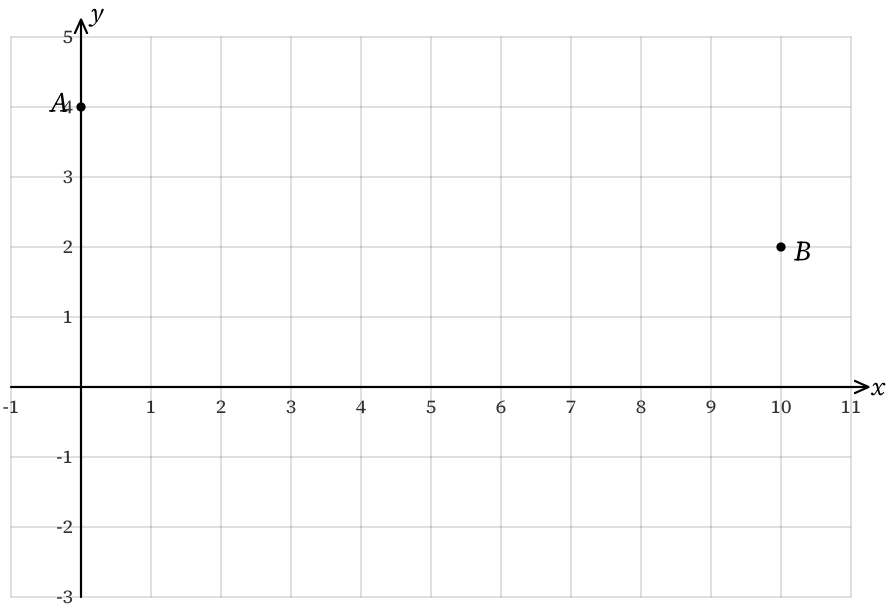
In rectangle $ABCD$, $AC = 5x - 7$ and $BD = 3x + 11$. Find AC.

8. **3 pts** **Lesson 7.3**

Find the image of $P(3, -5)$ under a rotation of 90° counterclockwise about the origin.

9. **5 pts** **Lesson 7.2**

Challenge. On a map in which each unit is 10 meters, a farmhouse is at $A(0, 4)$ and a barn is at $B(10, 2)$, both on the same side of a straight creek that runs along the x-axis, as in the figure. Ruth will walk from the house straight to a point P on the creek to fill a bucket, then straight to the barn. (a) Find the x-coordinate of the point P that makes her walk as short as possible. (b) Find the length of that shortest walk in meters, to the nearest tenth.



10. **3 pts** **Lesson 7.5**

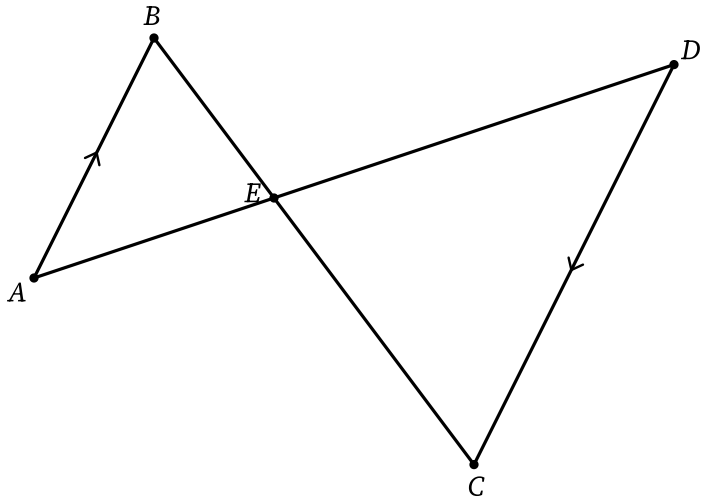
A dilation with center at the origin maps $A(4, -6)$ to $A'(10, -15)$. (a) Find the scale factor k. (b) Find the image of $B(-2, 8)$ under the same dilation.

11. **3 pts** **Lesson 8.3**

At the same time of day, Samuel, who is 6 ft tall, casts a shadow 4.5 ft long, and the church steeple casts a shadow 63 ft long. How tall is the steeple?

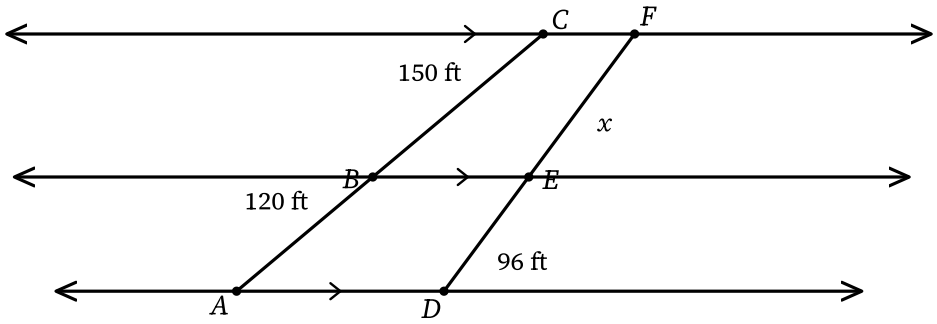
12. **5 pts** **Lesson 8.3**

Write a two-column proof. Given: $\overline{AB} \parallel \overline{CD}$, and $\overline{AD}$ and $\overline{BC}$ intersect at E, as in the figure. Prove: $AE \cdot CE = BE \cdot DE$.



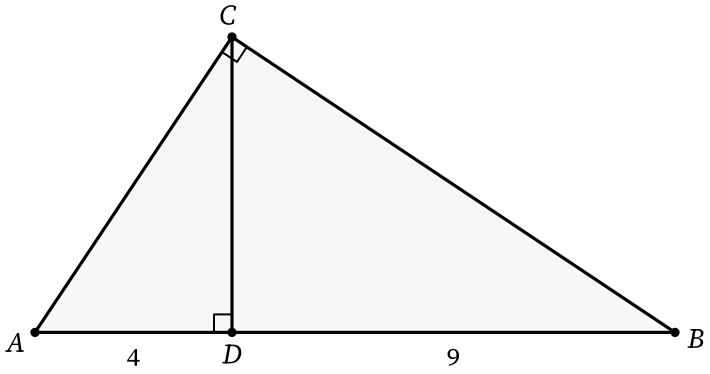
13. **3 pts** **Lesson 8.5**

In the figure, three parallel streets cross two straight roads. Along the first road the lot frontages are $AB = 120$ ft and $BC = 150$ ft. Along the second road the first frontage is $DE = 96$ ft. Find the second frontage EF.



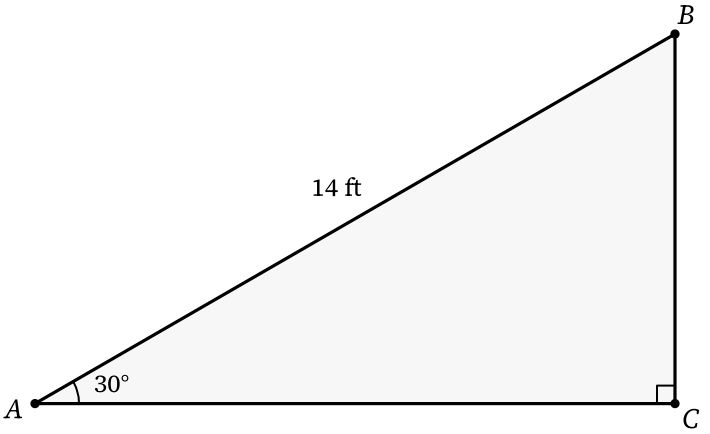
14. **3 pts** **Lesson 9.3**

In right $\triangle ABC$ in the figure, $\angle ACB$ is a right angle and $\overline{CD}$ is the altitude to the hypotenuse, with $AD = 4$ and $DB = 9$. Find (a) CD and (b) AC, in simplest radical form.



15. **3 pts** **Lesson 9.4**

A roof rafter $\overline{AB}$ is 14 ft long and makes a 30° angle with the horizontal ceiling joist $\overline{AC}$; the post $\overline{BC}$ is vertical, as in the figure. Find exactly (a) the rise BC and (b) the run AC.



16. **3 pts** **Lesson 9.5**

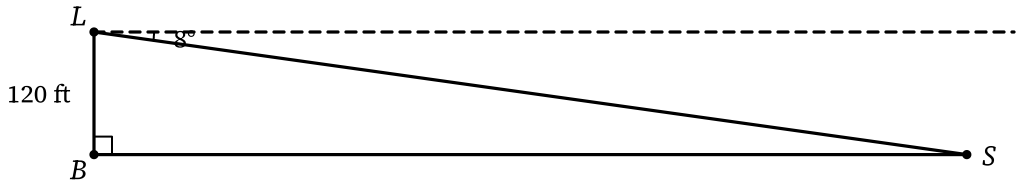
Explain why, in every right triangle that has a 34° acute angle, the ratio $\dfrac{\text{opposite leg}}{\text{adjacent leg}}$ for that angle is the same number. (This is why $\tan 34^\circ$ has one value.)

17. **3 pts** **Lesson 9.6**

A shed roof rises 9 ft over a horizontal run of 14 ft. To the nearest tenth of a degree, what angle does the roof make with the horizontal?

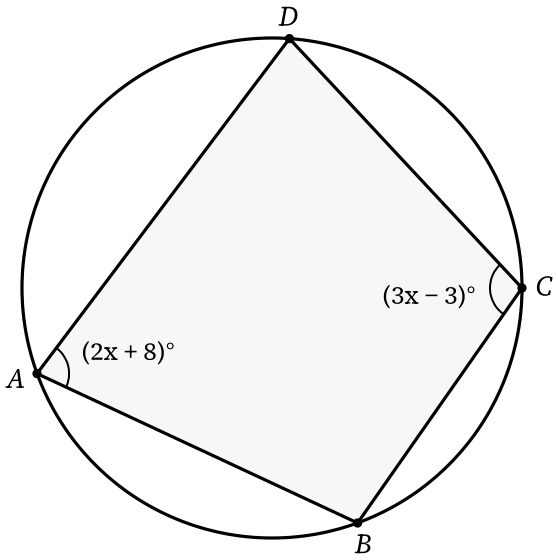
18. **3 pts** **Lesson 9.7**

From the top L of a lighthouse 120 ft tall, the keeper sees a boat S at an angle of depression of 8°, as in the figure. How far is the boat from the base B of the lighthouse, to the nearest foot?



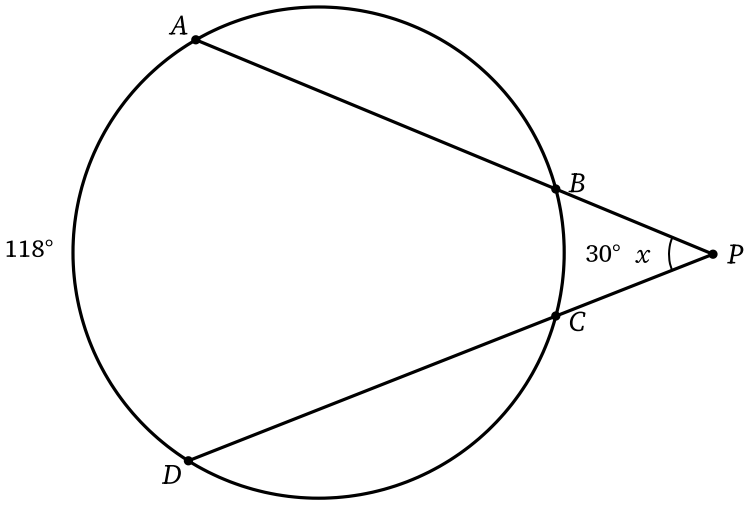
19. **3 pts** **Lesson 10.4**

Quadrilateral $ABCD$ is inscribed in a circle, as in the figure, with $m\angle DAB = (2x + 8)^\circ$ and $m\angle BCD = (3x - 3)^\circ$. Find $m\angle BCD$.



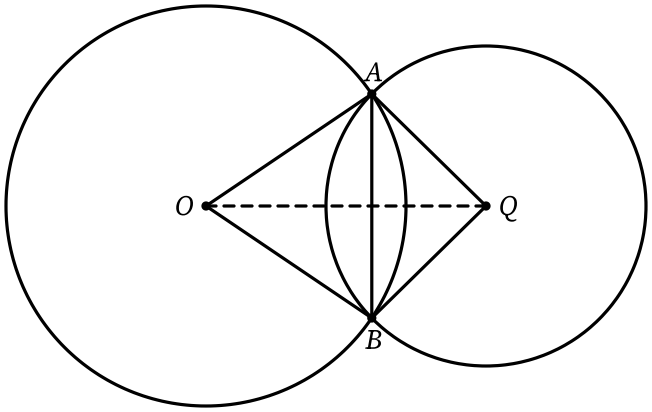
20. **3 pts** **Lesson 10.5**

In the figure, two secants are drawn from a point P outside the circle; one meets the circle at B and A, the other at C and D. The intercepted arcs measure $m\overset{\frown}{AD} = 118°$ and $m\overset{\frown}{BC} = 30°$. Find $x = m\angle BPC$.



21. **5 pts** **Lesson 10.3**

Write a paragraph proof. Given: Two different circles, circle O and circle Q, meet at the two points A and B, as in the figure. Prove: $\overleftrightarrow{OQ}$ is the perpendicular bisector of the common chord $\overline{AB}$.



22. **3 pts** **Lesson 10.6**

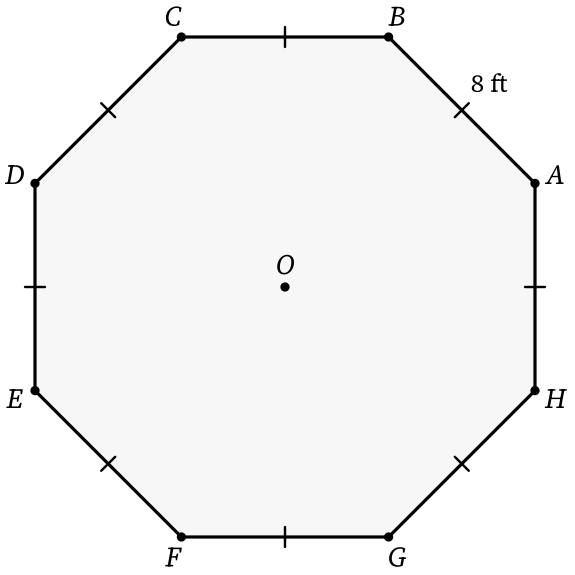
A sailor climbs to a lookout 60 ft above the sea. Taking the earth's radius as 3960 miles, how far away is the horizon, to the nearest tenth of a mile? (The line of sight to the horizon is tangent to the earth; 1 mile = 5280 ft.)

23. **3 pts** **Lesson 10.7**

Find the center and radius of the circle $x^2 + y^2 - 6x + 10y + 9 = 0$.

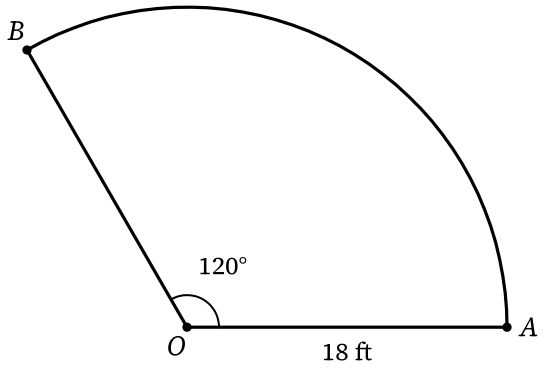
24. **3 pts** **Lesson 11.3**

The floor of an octagonal gazebo is a regular octagon with sides 8 ft long, as in the figure. Find the area of the floor to the nearest square foot.



25. **3 pts** **Lesson 11.5**

A lawn sprinkler waters a sector with radius 18 ft and central angle $120°$, as in the figure. Find the area of the lawn it waters, in terms of π.



26. **3 pts** **Lesson 11.6**

Two stained-glass panes are similar. Their areas are 48 square inches and 108 square inches. The smaller pane is 10 inches tall. How tall is the larger pane?

27. **3 pts** **Lesson 12.5**

A grain silo is a cylinder with radius 6 ft and height 20 ft, topped by a cone with the same radius and a height of 5 ft. Find the volume of the silo in terms of π.

28. **3 pts** **Lesson 12.6**

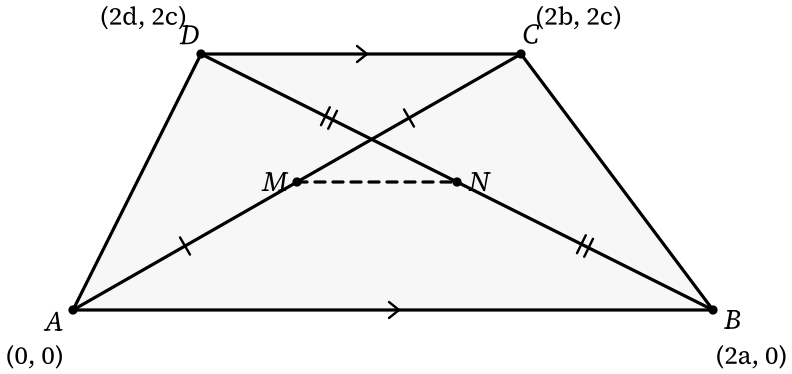
Two water tanks are similar solids. The smaller tank is 6 ft tall and holds 320 gallons. The larger tank is 9 ft tall. How many gallons does the larger tank hold?

29. **3 pts** **Lesson 13.1**

Find the coordinates of the point P on $\overline{AB}$ with $AP : PB = 1 : 2$, where $A(-2, 3)$ and $B(10, -6)$.

30. **5 pts** **Lesson 13.3**

Challenge. Write a coordinate proof. Given: $ABCD$ is a trapezoid with bases $\overline{AB}$ and $\overline{DC}$, where $AB > DC$, and M and N are the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$, as in the figure. Prove: $\overline{MN} \parallel \overline{AB}$ and $MN = \frac{1}{2}(AB - DC)$. (Use $A(0,0)$, $B(2a,0)$, $C(2b,2c)$ and $D(2d,2c)$.)



ANSWER KEY

1. If the diagonals of a quadrilateral are not perpendicular, then the quadrilateral is not a rhombus. It is true, because a conditional and its contrapositive are logically equivalent and the original statement is true.
2. Let the angle measure $x°$. Then $180 - x = 3(90 - x) + 18$, so $180 - x = 288 - 3x$, $2x = 108$ and $x = 54$. The angle measures $54°$. (Check: supplement $126°$, complement $36°$, and $3(36) + 18 = 126$.)
3. $\angle BEF$ and $\angle DFE$ are same-side interior angles of parallel lines, so they are supplementary: $x = 180 - 38 = 142$. So $m\angle DFE = 142°$.
4. See the model proof.

   TWO-COLUMN PROOF

   Given: $\overline{AB} \cong \overline{AC}$; M is the midpoint of $\overline{BC}$; D is on $\overline{AB}$, E is on $\overline{AC}$, and $\overline{BD} \cong \overline{CE}$

   Prove: $\overline{MD} \cong \overline{ME}$

	STATEMENTS	REASONS
1	$\overline{AB} \cong \overline{AC}$	Given
2	$\angle B \cong \angle C$	Isosceles Triangle Theorem
3	M is the midpoint of $\overline{BC}$	Given
4	$\overline{BM} \cong \overline{CM}$	Definition of midpoint
5	$\overline{BD} \cong \overline{CE}$	Given
6	$\triangle DBM \cong \triangle ECM$	SAS (rows 5, 2, 4; $\angle B$ is included between $\overline{BD}$ and $\overline{BM}$, and $\angle C$ between $\overline{CE}$ and $\overline{CM}$)
7	$\overline{MD} \cong \overline{ME}$	CPCTC

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5. No. By the Triangle Inequality Theorem, the two shorter sides must add to more than the longest side, but $4.2 + 5.9 = 10.1 < 10.3$.
6. Each exterior angle measures $180° - 144° = 36°$, and the exterior angles add to $360°$, so the frame has $360 \div 36 = 10$ sides.
7. The diagonals of a rectangle are congruent, so $5x - 7 = 3x + 11$ and $x = 9$. Then $AC = 5(9) - 7 = 38$.

8. The rule is $(x, y) \to (-y, x)$, so $P'(5, 3)$.

9. (a) Reflect B in the x-axis to $B'(10, -2)$. The x-axis is the perpendicular bisector of $\overline{BB'}$, so $PB = PB'$ for every point P on the creek, and the walk $AP + PB = AP + PB'$ is shortest when P lies on $\overline{AB'}$ (Triangle Inequality). The line AB' is $y = 4 - 0.6x$, which meets the x-axis at $x = \dfrac{20}{3} \approx 6.67$. (b) The shortest walk is $AB' = \sqrt{10^2 + 6^2} = \sqrt{136} = 2\sqrt{34} \approx 11.66$ units, or about 116.6 m.

10. (a) $k = \dfrac{10}{4} = \dfrac{-15}{-6} = 2.5$. (b) $B' = (2.5 \cdot (-2),\ 2.5 \cdot 8) = (-5, 20)$.

11. The sun's rays make equal angles with the level ground and both Samuel and the steeple stand vertically, so the triangles are similar by AA. $\dfrac{h}{63} = \dfrac{6}{4.5}$, so $h = 84$ ft.

12. See the model proof.

    TWO-COLUMN PROOF

    Given: $\overline{AB} \parallel \overline{CD}$; $\overline{AD}$ and $\overline{BC}$ intersect at E

    Prove: $AE \cdot CE = BE \cdot DE$

	STATEMENTS	REASONS
1	$\overline{AB} \parallel \overline{CD}$	Given
2	$\angle BAE \cong \angle CDE$	Alternate Interior Angles Theorem (transversal $\overline{AD}$)
3	$\angle AEB \cong \angle DEC$	Vertical Angles Theorem
4	$\triangle AEB \sim \triangle DEC$	AA Similarity Postulate
5	$\dfrac{AE}{DE} = \dfrac{BE}{CE}$	Corresponding sides of similar triangles are proportional
6	$AE \cdot CE = BE \cdot DE$	Cross Products Property

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13. Parallel lines cut transversals proportionally, so $\dfrac{AB}{BC} = \dfrac{DE}{EF}$: $\dfrac{120}{150} = \dfrac{96}{EF}$, so $EF = \dfrac{150 \times 96}{120} = 120$ ft.

14. (a) The altitude is the geometric mean of the two pieces of the hypotenuse: $CD = \sqrt{4 \cdot 9} = 6$. (b) Each leg is the geometric mean of the hypotenuse and the adjacent piece: $AC = \sqrt{4 \cdot 13} = \sqrt{52} = 2\sqrt{13}$.

15. $\triangle ABC$ is a 30°-60°-90° triangle with hypotenuse 14 ft. (a) The shorter leg, opposite 30°, is half the hypotenuse: $BC = 7$ ft. (b) The longer leg is $\sqrt{3}$ times

the shorter leg: $AC = 7\sqrt{3}$ ft.

16. Any two such right triangles have a pair of congruent right angles and a pair of congruent 34° angles, so they are similar by AA. Corresponding sides of similar triangles are proportional, so $\frac{a}{a'} = \frac{b}{b'}$ for the opposite legs a, a' and adjacent legs b, b', which rearranges to $\frac{a}{b} = \frac{a'}{b'}$. The ratio depends only on the angle, so $\tan 34^\circ \approx 0.6745$ in every such triangle.

17. $\tan\theta = \frac{9}{14}$, so $\theta = \tan^{-1}\left(\frac{9}{14}\right) \approx 32.7^\circ$.

18. The angle of depression equals the angle of elevation from the boat (alternate interior angles), so $m\angle LSB = 8^\circ$ and $\tan 8^\circ = \frac{120}{BS}$. Then $BS = \frac{120}{\tan 8^\circ} \approx 853.8$, about 854 ft.

19. Opposite angles of an inscribed quadrilateral are supplementary: $(2x + 8) + (3x - 3) = 180$, so $5x + 5 = 180$ and $x = 35$. Then $m\angle BCD = 3(35) - 3 = 102^\circ$.

20. An angle formed outside a circle by two secants is half the difference of the intercepted arcs: $x = \frac{1}{2}(118 - 30) = 44$. So $m\angle BPC = 44^\circ$.

21. See the model proof.

    PARAGRAPH PROOF

    Given: Circle O and circle Q are different circles; The circles meet at A and B
    Prove: $\overleftrightarrow{OQ}$ is the perpendicular bisector of $\overline{AB}$
    Draw the radii $\overline{OA}$, $\overline{OB}$, $\overline{QA}$ and $\overline{QB}$. All radii of a circle are congruent, so $OA = OB$ and $QA = QB$. Thus O and Q are each equidistant from A and B.
    By the Converse of the Perpendicular Bisector Theorem (Lesson 5.1), every point equidistant from A and B lies on the perpendicular bisector of $\overline{AB}$, so both O and Q lie on it.
    O and Q are different points: if the circles had the same center, then both radii would equal OA, and the circles would be the same circle, which they are not.
    Through two points there is exactly one line, so the perpendicular bisector of $\overline{AB}$ is $\overleftrightarrow{OQ}$.

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22. The height is $\frac{60}{5280} = \frac{1}{88}$ mile. The line from the lookout through the earth's center is a secant with outside part $\frac{1}{88}$ and whole length $\frac{1}{88} + 7920$. By the Tangent-Secant Theorem, $d^2 = \frac{1}{88}\left(\frac{1}{88} + 7920\right) \approx 90.0001$, so $d \approx 9.49$, about 9.5 miles.

23. Complete the square: $(x^2 - 6x + 9) + (y^2 + 10y + 25) = -9 + 9 + 25$, so $(x - 3)^2 + (y + 5)^2 = 25$. The center is $(3, -5)$ and the radius is 5.

24. The central angle is $360^\circ \div 8 = 45^\circ$, and the apothem bisects it and the side, so $\tan 22.5^\circ = \dfrac{4}{a}$ and $a = \dfrac{4}{\tan 22.5^\circ} \approx 9.657$ ft. The perimeter is $8 \times 8 = 64$ ft, so $A = \dfrac{1}{2}aP \approx \dfrac{1}{2}(9.657)(64) \approx 309$ square feet.

25. $A = \dfrac{120}{360} \cdot \pi \cdot 18^2 = \dfrac{1}{3} \cdot 324\pi = 108\pi$ square feet (about 339.3 square feet).

26. The ratio of areas is $\dfrac{108}{48} = \dfrac{9}{4}$, so the scale factor is $\sqrt{\dfrac{9}{4}} = \dfrac{3}{2}$. The larger pane is $10 \times \dfrac{3}{2} = 15$ inches tall.

27. Cylinder: $\pi(6^2)(20) = 720\pi$. Cone: $\dfrac{1}{3}\pi(6^2)(5) = 60\pi$. Total: 780π cubic feet (about 2450.4 cubic feet).

28. The scale factor is $\dfrac{9}{6} = \dfrac{3}{2}$, so the volumes are in the ratio $\left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$. The larger tank holds $320 \times \dfrac{27}{8} = 1080$ gallons.

29. P is one-third of the way from A to B: $P = \left(-2 + \dfrac{1}{3}(12),\ 3 + \dfrac{1}{3}(-9)\right) = (2, 0)$.

30. $\overline{MN} \parallel \overline{AB}$ and $MN = \frac{1}{2}(AB - DC)$; see the model proof.

    PARAGRAPH PROOF

    Given: $ABCD$ is a trapezoid with bases $\overline{AB}$ and $\overline{DC}$, $AB > DC$; M is the midpoint of $\overline{AC}$ and N is the midpoint of $\overline{BD}$

    Prove: $\overline{MN} \parallel \overline{AB}$ and $MN = \dfrac{1}{2}(AB - DC)$

    Place the trapezoid with $A(0, 0)$, $B(2a, 0)$, $C(2b, 2c)$ and $D(2d, 2c)$, where $a > 0$, $c > 0$ and $b > d$. Any such trapezoid can be placed this way: put the longer base $\overline{AB}$ on the positive x-axis with A at the origin, flipping over the x-axis if necessary so that C and D lie above it. The other base is parallel to $\overline{AB}$, so C and D have the same y-coordinate. Because the vertices of $ABCD$ go around in order, the legs $\overline{AD}$ and $\overline{BC}$ do not cross, so D is to the left of C just as A is to the left of B; that is, $b > d$. Then $AB = 2a$ and $DC = 2b - 2d$, and $AB > DC$ means $a > b - d$, that is, $a + d > b$. Doubling the coordinates keeps the midpoints free of fractions.

    By the Midpoint Formula, $M = \left(\dfrac{0 + 2b}{2}, \dfrac{0 + 2c}{2}\right) = (b, c)$ and $N = \left(\dfrac{2a + 2d}{2}, \dfrac{0 + 2c}{2}\right) = (a + d, c)$.

M and N have the same y-coordinate, so $\overline{MN}$ lies on the horizontal line $y = c$ and has slope 0. $\overline{AB}$ lies on the x-axis, which also has slope 0, and $y = c$ is a different line because $c > 0$. Different lines with equal slopes are parallel, so $\overline{MN} \parallel \overline{AB}$.

Since $a + d > b$, the horizontal distance from M to N is $MN = (a + d) - b = a - b + d$. Also $\frac{1}{2}(AB - DC) = \frac{1}{2}(2a - 2b + 2d) = a - b + d$. By the Transitive Property of Equality, $MN = \frac{1}{2}(AB - DC)$.

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