COURSES · MATHEMATICS · GRADES 6–8

Pre-Algebra

First edition | 13 units | 73 lessons | About 12 hours of video

FIRST EDITION

Course guide

Pre-Algebra is the bridge between arithmetic and Algebra I. In one school year you will review and deepen what you know about whole numbers, factors, fractions and decimals, and then step into new ground: negative numbers, exponents and square roots, ratios and percents, variables, equations, graphing, geometry, and data and probability. By the end you will be ready to begin Algebra I with confidence.

The course has a simple approach. First, understand why: each lesson explains the reason behind a rule with number lines, models described in words, patterns and careful reasoning, not just a list of steps. Second, master the skills with worked examples and a daily practice set, and correct every mistake. Third, build habits: each lesson gives a short mental-math trick or a set of facts to recite until they are automatic. Along the way, short history notes introduce the people who shaped mathematics, from Euclid and Eratosthenes to Brahmagupta, al-Khwarizmi and Descartes.

> Most videos are from Math Antics. Eddie Woo and Math with Mr. J fill the gaps. Eddie Woo teaches in Australia, so he sometimes says "HCF" for GCF, "pronumeral" for variable, "gradient" for slope and "trapezium" for trapezoid. Each lesson notes where a video's words differ from the course.

Start with Lesson 1.1 →

Who it is for

Students in grades 6–8 (ages 11–14) who can already add, subtract, multiply and divide whole numbers fluently, including long division, and who know basic fractions.

How to use this course

1. The course takes one school year: 32 weeks, 5 days a week, about 45 minutes a day. There are 13 units and 73 lessons, plus a midterm in week 15 and a final exam

in week 32.

2. Daily rhythm: start with the lesson's habit (a few minutes of recitation or mental math), read the core ideas, study the two worked examples step by step, do the practice set in your notebook, check your answers, and correct every mistake until you understand it.
3. Some lessons take two days. On day 1, read the teaching, study the examples and do the first half of the practice set. On day 2, review the core ideas and your mistakes from day 1, then finish the second half of the practice set.
4. At the end of each unit, go through the mastery checklist. Review any "I can" statement you are unsure of, then take the unit test without help.
5. The midterm covers Units 1–6 and the final covers the whole course. Spend a few days reviewing old practice sets and unit tests before each exam.
6. Calculators: do Units 1–5 by hand, since the goal is fluent, careful arithmetic. From Unit 6 on, a basic or scientific calculator may be used for messy arithmetic and for checking, unless a problem says "by hand."
7. Younger students or anyone who needs more time may slow down. It is better to master each lesson than to finish on schedule; spread a lesson over an extra day whenever needed.

Scope and Sequence

UNIT 1 · WEEKS 1–2

Whole Numbers and Number Sense

Place value, the properties of addition and multiplication, the distributive property, the order of operations and estimation, with plenty of mental math.

1.1 Place Value and Our Number System

1.2 Properties of Addition and Multiplication

1.3 The Distributive Property and Mental Math

1.4 Order of Operations

1.5 Estimation and Rounding

UNIT 2 · WEEKS 2–5

Factors, Multiples and Primes

Divisibility rules, primes and prime factorization, the greatest common factor, Euclid's algorithm and the least common multiple.

2.1 Divisibility Rules

2.2 Prime and Composite Numbers

2.3 Prime Factorization

2.4 Greatest Common Factor

2.5 The Euclidean Algorithm

2.6 Least Common Multiple

UNIT 3 · WEEKS 5–8

Fractions

Equivalent fractions, simplifying and comparing, and all four operations with fractions and mixed numbers, including why each rule works.

3.1 Fractions and Equivalent Fractions
3.2 Simplifying and Comparing Fractions
3.3 Adding and Subtracting Fractions
3.4 Mixed Numbers
3.5 Multiplying Fractions
3.6 Dividing Fractions
3.7 Fraction Word Problems

UNIT 4 · WEEKS 8–10

Decimals

Decimal place value, the four operations with decimals, money and unit prices, and converting between fractions and decimals.

4.1 Decimals and Place Value
4.2 Adding and Subtracting Decimals
4.3 Multiplying Decimals
4.4 Dividing Decimals
4.5 Fractions and Decimals

UNIT 5 · WEEKS 10–13

Integers and Rational Numbers

Negative numbers, absolute value, and adding, subtracting, multiplying and dividing integers and other rational numbers, with the reasons behind the sign rules.

5.1 Negative Numbers and Absolute Value
5.2 Adding Integers
5.3 Subtracting Integers
5.4 Multiplying Integers
5.5 Dividing Integers
5.6 Operations with Rational Numbers

UNIT 6 · WEEKS 13–15

Exponents and Square Roots

Powers and the laws of exponents, scientific notation, square roots, estimating roots and the real number system.

6.1 Exponents and Powers
6.2 Laws of Exponents
6.3 Scientific Notation
6.4 Square Roots and Perfect Squares
6.5 Estimating Roots and Irrational Numbers

UNIT 7 · WEEKS 15–17

Ratios, Rates and Proportions

Ratios, unit rates and better buys, solving proportions, scale drawings and maps, and unit conversions.

7.1 Ratios

7.2 Rates and Unit Rates

7.3 Proportions

7.4 Scale Drawings and Maps

7.5 Unit Conversions

UNIT 8 · WEEKS 18–20

Percent

The meaning of percent, finding parts, percents and wholes, percent change, discounts, tax, tips and simple interest.

8.1 Understanding Percent

8.2 Finding a Percent of a Number

8.3 Percent Equations and Proportions

8.4 Percent Increase and Decrease

8.5 Discounts, Sales Tax, Tips and Markup

8.6 Simple Interest and Saving

UNIT 9 · WEEKS 20–22

Variables and Expressions

Using letters for numbers, evaluating and writing expressions, combining like terms and the distributive property with variables.

9.1 Variables and Expressions

9.2 Evaluating Expressions

9.3 Writing Expressions

9.4 Combining Like Terms

9.5 The Distributive Property with Variables

UNIT 10 · WEEKS 22–25

Equations and Inequalities

Solving one-step, two-step and multi-step equations, writing equations for word problems, and solving and graphing inequalities.

10.1 Equations and Solutions

10.2 One-Step Equations with Addition and Subtraction

10.3 One-Step Equations with Multiplication and Division

10.4 Two-Step Equations

10.5 Multi-Step Equations

10.6 Writing Equations for Word Problems

10.7 Inequalities

UNIT 11 · WEEKS 25–27

The Coordinate Plane and Graphing

Plotting points, tables and rules, graphing lines, slope, slope-intercept form and direct variation.

11.1 The Coordinate Plane

11.2 Tables, Rules and Graphs

11.3 Graphing Linear Equations

11.4 Slope

11.5 Slope-Intercept Form and Direct Variation

UNIT 12 · WEEKS 27–30

Geometry

Angles, triangles and polygons, perimeter and area, circles, volume and surface area, and the Pythagorean theorem.

12.1 Angles and Angle Relationships

12.2 Triangles and Polygons

12.3 Perimeter and Area

12.4 Circles

12.5 Volume and Surface Area

12.6 The Pythagorean Theorem

12.7 Using the Pythagorean Theorem

UNIT 13 · WEEKS 30–32

Data and Probability

Mean, median, mode and range, choosing the best measure, and the probability of simple and compound events.

13.1 Mean, Median, Mode and Range

13.2 Choosing a Measure and Outliers

13.3 Probability of Simple Events

13.4 Counting and Compound Events

Weekly pacing calendar

WEEK	PLAN
1	• Day 1: Lesson 1.1 Place Value and Our Number System • Day 2: Lesson 1.2 Properties of Addition and Multiplication • Day 3: Lesson 1.3 The Distributive Property and Mental Math (day 1 of 2) • Day 4: Lesson 1.3 The Distributive Property and Mental Math (day 2 of 2) • Day 5: Lesson 1.4 Order of Operations (day 1 of 2) *Families on a 4-day week can combine the two days of a two-day lesson when the student is ready, or stretch the course to about 40 weeks.*
2	• Day 1: Lesson 1.4 Order of Operations (day 2 of 2)

WEEK	PLAN
	• Day 2: Lesson 1.5 Estimation and Rounding • Day 3: Unit 1 review (mastery checklist and missed problems) • Day 4: Unit 1 test • Day 5: Lesson 2.1 Divisibility Rules
3	• Day 1: Lesson 2.2 Prime and Composite Numbers • Day 2: Lesson 2.3 Prime Factorization (day 1 of 2) • Day 3: Lesson 2.3 Prime Factorization (day 2 of 2) • Day 4: Lesson 2.4 Greatest Common Factor (day 1 of 2) • Day 5: Lesson 2.4 Greatest Common Factor (day 2 of 2)
4	• Day 1: Lesson 2.5 The Euclidean Algorithm (day 1 of 2) • Day 2: Lesson 2.5 The Euclidean Algorithm (day 2 of 2) • Day 3: Lesson 2.6 Least Common Multiple (day 1 of 2) • Day 4: Lesson 2.6 Least Common Multiple (day 2 of 2) • Day 5: Unit 2 review (mastery checklist and missed problems)
5	• Day 1: Unit 2 test • Day 2: Lesson 3.1 Fractions and Equivalent Fractions (day 1 of 2) • Day 3: Lesson 3.1 Fractions and Equivalent Fractions (day 2 of 2) • Day 4: Lesson 3.2 Simplifying and Comparing Fractions (day 1 of 2) • Day 5: Lesson 3.2 Simplifying and Comparing Fractions (day 2 of 2)
6	• Day 1: Lesson 3.3 Adding and Subtracting Fractions (day 1 of 2) • Day 2: Lesson 3.3 Adding and Subtracting Fractions (day 2 of 2) • Day 3: Lesson 3.4 Mixed Numbers (day 1 of 2) • Day 4: Lesson 3.4 Mixed Numbers (day 2 of 2) • Day 5: Lesson 3.5 Multiplying Fractions (day 1 of 2)
7	• Day 1: Lesson 3.5 Multiplying Fractions (day 2 of 2) • Day 2: Lesson 3.6 Dividing Fractions (day 1 of 2) • Day 3: Lesson 3.6 Dividing Fractions (day 2 of 2) • Day 4: Lesson 3.7 Fraction Word Problems (day 1 of 2) • Day 5: Lesson 3.7 Fraction Word Problems (day 2 of 2)
8	• Day 1: Unit 3 review (mastery checklist and missed problems) • Day 2: Unit 3 test • Day 3: Lesson 4.1 Decimals and Place Value

WEEK	PLAN
	• Day 4: Lesson 4.2 Adding and Subtracting Decimals (day 1 of 2) • Day 5: Lesson 4.2 Adding and Subtracting Decimals (day 2 of 2)
9	• Day 1: Lesson 4.3 Multiplying Decimals (day 1 of 2) • Day 2: Lesson 4.3 Multiplying Decimals (day 2 of 2) • Day 3: Lesson 4.4 Dividing Decimals (day 1 of 2) • Day 4: Lesson 4.4 Dividing Decimals (day 2 of 2) • Day 5: Lesson 4.5 Fractions and Decimals (day 1 of 2)
10	• Day 1: Lesson 4.5 Fractions and Decimals (day 2 of 2) • Day 2: Unit 4 review (mastery checklist and missed problems) • Day 3: Unit 4 test • Day 4: Lesson 5.1 Negative Numbers and Absolute Value • Day 5: Lesson 5.2 Adding Integers (day 1 of 2)
11	• Day 1: Lesson 5.2 Adding Integers (day 2 of 2) • Day 2: Lesson 5.3 Subtracting Integers (day 1 of 2) • Day 3: Lesson 5.3 Subtracting Integers (day 2 of 2) • Day 4: Lesson 5.4 Multiplying Integers (day 1 of 2) • Day 5: Lesson 5.4 Multiplying Integers (day 2 of 2)
12	• Day 1: Lesson 5.5 Dividing Integers (day 1 of 2) • Day 2: Lesson 5.5 Dividing Integers (day 2 of 2) • Day 3: Lesson 5.6 Operations with Rational Numbers (day 1 of 2) • Day 4: Lesson 5.6 Operations with Rational Numbers (day 2 of 2) • Day 5: Unit 5 review (mastery checklist and missed problems)
13	• Day 1: Unit 5 test • Day 2: Lesson 6.1 Exponents and Powers • Day 3: Lesson 6.2 Laws of Exponents (day 1 of 2) • Day 4: Lesson 6.2 Laws of Exponents (day 2 of 2) • Day 5: Lesson 6.3 Scientific Notation (day 1 of 2)
14	• Day 1: Lesson 6.3 Scientific Notation (day 2 of 2) • Day 2: Lesson 6.4 Square Roots and Perfect Squares (day 1 of 2) • Day 3: Lesson 6.4 Square Roots and Perfect Squares (day 2 of 2) • Day 4: Lesson 6.5 Estimating Roots and Irrational Numbers (day 1 of 2) • Day 5: Lesson 6.5 Estimating Roots and Irrational Numbers (day 2 of 2)

WEEK	PLAN
15	• Day 1: Unit 6 review (mastery checklist and missed problems) • Day 2: Unit 6 test • Day 3: Midterm review (Units 1–6) • Day 4: Midterm exam (Units 1–6) • Day 5: Lesson 7.1 Ratios *Midterm week. Use any spare time to correct missed problems from Units 1–6.*
16	• Day 1: Lesson 7.2 Rates and Unit Rates (day 1 of 2) • Day 2: Lesson 7.2 Rates and Unit Rates (day 2 of 2) • Day 3: Lesson 7.3 Proportions (day 1 of 2) • Day 4: Lesson 7.3 Proportions (day 2 of 2) • Day 5: Lesson 7.4 Scale Drawings and Maps (day 1 of 2)
17	• Day 1: Lesson 7.4 Scale Drawings and Maps (day 2 of 2) • Day 2: Lesson 7.5 Unit Conversions (day 1 of 2) • Day 3: Lesson 7.5 Unit Conversions (day 2 of 2) • Day 4: Unit 7 review (mastery checklist and missed problems) • Day 5: Unit 7 test
18	• Day 1: Lesson 8.1 Understanding Percent • Day 2: Lesson 8.2 Finding a Percent of a Number (day 1 of 2) • Day 3: Lesson 8.2 Finding a Percent of a Number (day 2 of 2) • Day 4: Lesson 8.3 Percent Equations and Proportions (day 1 of 2) • Day 5: Lesson 8.3 Percent Equations and Proportions (day 2 of 2)
19	• Day 1: Lesson 8.4 Percent Increase and Decrease (day 1 of 2) • Day 2: Lesson 8.4 Percent Increase and Decrease (day 2 of 2) • Day 3: Lesson 8.5 Discounts, Sales Tax, Tips and Markup (day 1 of 2) • Day 4: Lesson 8.5 Discounts, Sales Tax, Tips and Markup (day 2 of 2) • Day 5: Lesson 8.6 Simple Interest and Saving (day 1 of 2)
20	• Day 1: Lesson 8.6 Simple Interest and Saving (day 2 of 2) • Day 2: Unit 8 review (mastery checklist and missed problems) • Day 3: Unit 8 test • Day 4: Lesson 9.1 Variables and Expressions • Day 5: Lesson 9.2 Evaluating Expressions (day 1 of 2)

WEEK	PLAN
21	• Day 1: Lesson 9.2 Evaluating Expressions (day 2 of 2) • Day 2: Lesson 9.3 Writing Expressions (day 1 of 2) • Day 3: Lesson 9.3 Writing Expressions (day 2 of 2) • Day 4: Lesson 9.4 Combining Like Terms (day 1 of 2) • Day 5: Lesson 9.4 Combining Like Terms (day 2 of 2)
22	• Day 1: Lesson 9.5 The Distributive Property with Variables (day 1 of 2) • Day 2: Lesson 9.5 The Distributive Property with Variables (day 2 of 2) • Day 3: Unit 9 review (mastery checklist and missed problems) • Day 4: Unit 9 test • Day 5: Lesson 10.1 Equations and Solutions
23	• Day 1: Lesson 10.2 One-Step Equations with Addition and Subtraction • Day 2: Lesson 10.3 One-Step Equations with Multiplication and Division (day 1 of 2) • Day 3: Lesson 10.3 One-Step Equations with Multiplication and Division (day 2 of 2) • Day 4: Lesson 10.4 Two-Step Equations (day 1 of 2) • Day 5: Lesson 10.4 Two-Step Equations (day 2 of 2)
24	• Day 1: Lesson 10.5 Multi-Step Equations (day 1 of 2) • Day 2: Lesson 10.5 Multi-Step Equations (day 2 of 2) • Day 3: Lesson 10.6 Writing Equations for Word Problems (day 1 of 2) • Day 4: Lesson 10.6 Writing Equations for Word Problems (day 2 of 2) • Day 5: Lesson 10.7 Inequalities (day 1 of 2)
25	• Day 1: Lesson 10.7 Inequalities (day 2 of 2) • Day 2: Unit 10 review (mastery checklist and missed problems) • Day 3: Unit 10 test • Day 4: Lesson 11.1 The Coordinate Plane • Day 5: Lesson 11.2 Tables, Rules and Graphs (day 1 of 2)
26	• Day 1: Lesson 11.2 Tables, Rules and Graphs (day 2 of 2) • Day 2: Lesson 11.3 Graphing Linear Equations (day 1 of 2) • Day 3: Lesson 11.3 Graphing Linear Equations (day 2 of 2) • Day 4: Lesson 11.4 Slope (day 1 of 2) • Day 5: Lesson 11.4 Slope (day 2 of 2)
27	• Day 1: Lesson 11.5 Slope-Intercept Form and Direct Variation (day 1 of 2) • Day 2: Lesson 11.5 Slope-Intercept Form and Direct Variation (day 2 of 2)

WEEK	PLAN
	• Day 3: Unit 11 review (mastery checklist and missed problems) • Day 4: Unit 11 test • Day 5: Lesson 12.1 Angles and Angle Relationships
28	• Day 1: Lesson 12.2 Triangles and Polygons (day 1 of 2) • Day 2: Lesson 12.2 Triangles and Polygons (day 2 of 2) • Day 3: Lesson 12.3 Perimeter and Area (day 1 of 2) • Day 4: Lesson 12.3 Perimeter and Area (day 2 of 2) • Day 5: Lesson 12.4 Circles (day 1 of 2)
29	• Day 1: Lesson 12.4 Circles (day 2 of 2) • Day 2: Lesson 12.5 Volume and Surface Area (day 1 of 2) • Day 3: Lesson 12.5 Volume and Surface Area (day 2 of 2) • Day 4: Lesson 12.6 The Pythagorean Theorem (day 1 of 2) • Day 5: Lesson 12.6 The Pythagorean Theorem (day 2 of 2)
30	• Day 1: Lesson 12.7 Using the Pythagorean Theorem (day 1 of 2) • Day 2: Lesson 12.7 Using the Pythagorean Theorem (day 2 of 2) • Day 3: Unit 12 review (mastery checklist and missed problems) • Day 4: Unit 12 test • Day 5: Lesson 13.1 Mean, Median, Mode and Range
31	• Day 1: Lesson 13.2 Choosing a Measure and Outliers (day 1 of 2) • Day 2: Lesson 13.2 Choosing a Measure and Outliers (day 2 of 2) • Day 3: Lesson 13.3 Probability of Simple Events • Day 4: Lesson 13.4 Counting and Compound Events (day 1 of 2) • Day 5: Lesson 13.4 Counting and Compound Events (day 2 of 2)
32	• Day 1: Unit 13 review (mastery checklist and missed problems) • Day 2: Unit 13 test • Day 3: Final review (Units 1–6) • Day 4: Final review (Units 7–13) • Day 5: Final exam (Units 1–13) *Final exam week.*

Materials

Pencil and eraser

Do all math in pencil so mistakes can be corrected neatly.

Spiral notebook or composition book

Keep all practice, corrections and tests in one book so you can review your work before exams.

Graph paper

For lining up columns of digits, drawing area models and graphing in Unit 11.

Ruler (inches and centimeters)

For number lines, scale drawings, graphs and measuring in Unit 12.

Protractor

For measuring and drawing angles in Unit 12.

Drawing compass

For drawing circles in Unit 12.

Basic or scientific calculator

Used from Unit 6 on for messy arithmetic and for checking; Units 1–5 are done by hand.

Index cards

For recitation facts: perfect squares, primes, divisibility rules, fraction–decimal–percent equivalents, sign rules and formulas.

Fraction strips (optional)

Helpful for seeing equivalent fractions and comparing fractions in Unit 3.

Two dice and a coin (optional)

For probability experiments in Unit 13.

Tape measure and measuring cups (optional)

For hands-on measurement, recipe and carpentry problems.

Grading

Grades are based on daily practice, unit tests and two exams. Unit tests carry the most weight because they show what the student has truly mastered.

PART	%
Daily practice sets	20
Unit tests	50

PART	%
Midterm exam	15
Final exam	15

Scale: A 90–100, B 80–89, C 70–79, D 60–69, below 60 redo

- Grade daily practice for completion and corrections: full credit when every problem is attempted with work shown and every mistake has been corrected.
- If a unit test score is below 70, review the lessons behind the missed problems and retake a similar test; record the higher score, up to 80.
- Mastery matters more than speed. It is fine to take an extra day on a lesson or to repeat a unit before moving on.

A NOTE TO PARENTS

You do not need to be a math expert to teach this course. Every lesson explains its ideas step by step, and every worked example, practice problem, unit test and exam comes with its answer. Your job is to keep a steady routine, check work, and encourage careful thinking.

When your student is stuck, try not to do the problem for them. Ask questions instead: "What is the question asking?" "Can you estimate first?" "Which example is most like this problem?" "How can you check your answer, for example by substituting it back in?" Have them correct each mistake in the notebook and say what went wrong. A few minutes of daily recitation and mental math (the habit in each lesson) will build speed and confidence over the year.

Consistency matters more than speed. If a lesson takes longer, slow down; that is normal and good. Pre-algebra prepares students for Algebra I and the sciences, for the trades, and for managing money wisely, and it trains habits of careful, honest reasoning: saying exactly what you mean, showing your work, and checking that your answer is true.

UNIT 1

Whole Numbers and Number Sense

This first unit builds a strong foundation of number sense with whole numbers. You will see how our place-value system works and why it is such a good invention, learn the properties that let you rearrange sums and products, use the distributive property to do multiplication in your head, follow the order of operations, and estimate answers quickly by rounding.

These ideas are not just warm-ups. The properties of numbers and the order of operations are the rules of the road for all of algebra: when letters stand for numbers in Units 9 and 10, these same rules tell you what you may and may not do. Estimation will help you check your work in every unit that follows.

Parents: this unit is done by hand, without a calculator. Encourage your student to say mental-math steps out loud and to estimate before calculating; both habits pay off for years.

PREREQUISITES

- Add, subtract, multiply and divide whole numbers by hand, including long division.
- Know the addition and multiplication facts through 10×10 without hesitation.
- Read and write whole numbers into the thousands.
- Find the area of a rectangle as length times width.

MASTERY CHECKLIST

- ☐ I can read, write and compare whole numbers into the billions, and write them in expanded form.
- ☐ I can explain why place value with a zero makes arithmetic easier than Roman numerals.
- ☐ I can name and use the commutative, associative, identity and zero properties.
- ☐ I can show that subtraction and division are not commutative or associative.
- ☐ I can explain the distributive property with an area model.
- ☐ I can multiply mentally using the distributive property, such as

$7 \times$ and
98

15×12.

- ☐ I can evaluate squares and cubes.
- ☐ I can evaluate expressions using the order of operations, including nested grouping symbols and fraction bars.
- ☐ I can round a whole number to any place.
- ☐ I can estimate sums, differences, products and quotients and use estimates to check whether an answer is reasonable.

LESSONS IN THIS UNIT

1.1 Place Value and Our Number System

45 minutes

1.2	**Properties of Addition and Multiplication**	45 minutes
1.3	**The Distributive Property and Mental Math**	2 sessions of 45 minutes
1.4	**Order of Operations**	2 sessions of 45 minutes
1.5	**Estimation and Rounding**	45 minutes

Unit 1 test

You have 50 minutes. No calculator. Show your work for every problem, including mental-math steps when a problem asks for them. Give exact answers unless a problem asks you to round or estimate, and include units (such as dollars or miles) where they apply.

1. **5 pts**
   What is the value of the digit 6 in 2,468,105?

2. **5 pts**
   Write in standard form: four million, thirty thousand, eight hundred two.

3. **6 pts**
   List from least to greatest: 305,670; 350,067; 305,760.

4. **6 pts**
   Name the property shown: $8 \times (3 \times 25) = (8 \times 3) \times 25$.

5. **6 pts**
   Round 8,476,519 to the nearest ten thousand.

6. **7 pts**
   Find $4 \times 63 \times 25$ mentally. Show which properties you used.

7. **7 pts**
   Use the distributive property to find 8×97 mentally. Show your steps.

8. **7 pts**

   Find $9 \times 34 + 9 \times 66$ mentally.

9. **7 pts**

   Evaluate $5 + 3 \times 2^3$.

10. **8 pts**

    Evaluate $2 \times [18 - (1 + 2)^2] + 4^2 \div 8$.

11. **8 pts**

    The choir sells 48 tickets to its spring concert at \$15 each. Use the distributive property to find how much money the tickets bring in.

12. **8 pts**

    The Miller family drove 2,890 miles in 6 days. Use compatible numbers to estimate how many miles they drove each day.

13. **8 pts**

    Grace buys 3 bags of flour at \$4 each and 2 dozen eggs at \$3 a dozen. She pays with \$30. Write one expression for her change and evaluate it.

14. **12 pts**

    Henry says $3 + 4 \times 5 = 35$. Find the correct value and explain why everyone must follow the same order of operations.

ANSWER KEY

1. 60,000
2. 4,030,802
3. 305,670; 305,760; 350,067
4. Associative property of multiplication
5. 8,480,000
6. 6,300 (commutative and associative properties: $(4 \times 25) \times 63 = 100 \times 63$)
7. $8 \times 100 - 8 \times 3 = 800 - 24 = 776$
8. $9 \times (34 + 66) = 9 \times 100 = 900$
9. 29
10. 20
11. \$720
12. $3{,}000 \div 6 = 500$, about 500 miles a day
13. $30 - (3 \times 4 + 2 \times 3) = 12$, so her change is \$12
14. The correct value is 23: multiply first, $4 \times 5 = 20$, then $3 + 20 = 23$. Multiplication is a bundle of repeated addition (four fives), so it is done before adding. If people used different orders, the same written expression would have different answers, so we agree on one order so that every expression means exactly one number.

1.1 Place Value and Our Number System

OBJECTIVES

- Read and write large whole numbers in standard form, word form and expanded form.
- Compare and order large whole numbers using place value.
- Explain why place value makes arithmetic easier than a system like Roman numerals.

WATCH

Math Antics - Place Value

mathantics · 9:26

https://www.youtube.com/watch?v=T5Qf0qSSJFI

> **WHEN TO WATCH**
> After the Core ideas, before Example 1.

BACKUP VIDEO

Math Antics - Roman Numerals

mathantics · 11:45

https://www.youtube.com/watch?v=P_qD1GJ1dR8

Optional, after Example 2: Roman numerals show why place value makes arithmetic easier (objective 3).

KEY TERMS

digit

One of the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that we use to write every number.

place value

The value a digit has because of where it sits in a number. In 572, the 5 is worth 500 because it is in the hundreds place.

standard form

The usual way of writing a number with digits, such as 4,056.

expanded form

A number written as a sum showing the value of each digit, such as $4 \times 1{,}000 + 5 \times 10 + 6 \times 1$.

period

A group of three places separated by commas: the ones period, the thousands period, the millions period, the billions period.

CORE IDEAS

Our number system uses only ten digits, 0 through 9, yet it can write any whole number, no matter how large. The secret is **place value**: what a digit is worth depends on where it sits. In 555, the first 5 means 5 hundreds, the second means 5 tens, and the last means 5 ones.

Each place is worth **ten times** the place to its right: ones, tens, hundreds, thousands, ten thousands, hundred thousands, millions, and so on. Ten ones make a ten, ten tens make a hundred, ten hundreds make a thousand. That is why we call it a **base-ten** system.

To make big numbers easy to read, we split the digits into **periods** of three, separated by commas. Read each period like a number under 1,000, then say the period's name. So 52,306,014 is read "fifty-two million, three hundred six thousand, fourteen." We do not say "and" when reading whole numbers.

Zero is a placeholder. In 4,056 the 0 says "no hundreds," and it keeps the 4 in the thousands place. Without it we would have 456, a very different number.

To **compare** two whole numbers, first count digits: the number with more digits is larger. If they have the same number of digits, compare from the left, place by place, until two digits differ. The larger digit belongs to the larger number.

Why does place value make arithmetic easy? Roman numerals have no zero and no place value, so 1888 is written MDCCCLXXXVIII, using 13 symbols, and adding two Roman numerals means regrouping piles of letters. With place value, we line up the columns and use the same small set of addition and multiplication facts in every column. That is why the whole world now uses these numerals for arithmetic.

KEY FORMULAS

$$4{,}056 = 4 \times 1{,}000 + 0 \times 100 + 5 \times 10 + 6 \times 1$$

$$1 \text{ million} = 1{,}000 \text{ thousands} \qquad 1 \text{ billion} = 1{,}000 \text{ millions}$$

From history. The digits we use, including zero, were developed in India and passed on through scholars writing in Arabic, which is why we call them Hindu-Arabic numerals. In 1202 the Italian mathematician Leonardo of Pisa, known as Fibonacci, wrote the *Liber Abaci* (Book of Calculation), which taught Europeans to calculate with these numerals. Over the following centuries they replaced Roman numerals for everyday arithmetic.

Practice habit. Recite the place names out loud in groups of three until they are automatic: ones, tens, hundreds; thousands, ten thousands, hundred thousands; millions, ten millions, hundred millions; billions. Each step to the left is ten times bigger.

WORKED EXAMPLES

Example 1. Write "two million, sixty thousand, four hundred nine" in standard form and in expanded form.

Show solution

1. Millions period: "two million" gives 2.
2. Thousands period: "sixty thousand" gives 060. The 0 holds the hundred-thousands place.
3. Ones period: "four hundred nine" gives 409.
4. Standard form: 2,060,409.
5. Expanded form: $2 \times 1{,}000{,}000 + 6 \times 10{,}000 + 4 \times 100 + 9 \times 1$.
6. Check: $2{,}000{,}000 + 60{,}000 + 400 + 9 = 2{,}060{,}409$.

Answer: 2,060,409; expanded form $2 \times 1{,}000{,}000 + 6 \times 10{,}000 + 4 \times 100 + 9 \times 1$

Example 2. A state fair recorded its attendance for three years: 1,204,350 people, 1,240,305 people and 1,204,530 people. List the three attendance numbers from least to greatest.

Show solution

1. All three numbers have 7 digits, so compare from the left.
2. Millions: all have 1. Hundred thousands: all have 2.
3. Ten thousands: 0, 4 and 0. The 4 is largest, so 1,240,305 is the greatest.
4. Compare 1,204,350 and 1,204,530. Thousands: both 4. Hundreds: 3 and 5.
5. Since 3 is less than 5, 1,204,350 is less than 1,204,530.
6. Order: 1,204,350; 1,204,530; 1,240,305.

Answer: 1,204,350; 1,204,530; 1,240,305

PRACTICE

1. Easy What is the value of the digit 7 in 4,872,015?

   Show answer

   70,000 (seven ten thousands)

2. Easy Write in standard form: $60{,}000 + 3{,}000 + 40 + 8$.

   Show answer

   63,048

3. Easy Write 507,320 in words.

   Show answer

   five hundred seven thousand, three hundred twenty

4. Easy Which number is greater, 89,999 or 90,001?

   Show answer

   90,001

5. Medium Write in standard form: twelve million, four hundred thousand, seventy.

   Show answer

   12,400,070

6. **Medium** Write 8,050,603 in expanded form.

Show answer

$8 \times 1{,}000{,}000 + 5 \times 10{,}000 + 6 \times 100 + 3 \times 1$

7. **Medium** A church library has 4,312 books. The town library has 43,120 books. The town library has how many times as many books as the church library?

Show answer

10 times as many

8. **Medium** Grace read that three mountain peaks are 14,505 feet, 14,440 feet and 14,494 feet tall. List the heights from least to greatest.

Show answer

14,440 ft; 14,494 ft; 14,505 ft

9. **Hard** In Roman numerals, M = 1,000, C = 100 and I = 1. Write the year MCCII in our numerals. (It is the year Fibonacci wrote the *Liber Abaci*.)

Show answer

1202

10. **Hard** Use each of the digits 0, 3, 5, 7, 8 and 9 exactly once to write the largest possible six-digit number and the smallest possible six-digit number. (A six-digit number cannot begin with 0.)

Show answer

Largest 987,530; smallest 305,789

11. **Hard** Henry is counting the money from a church fundraiser. He has 4 bundles of $1,000, 7 bundles of $100, 12 ten-dollar bills and 9 one-dollar bills. How much money is there in all?

Show answer

$4,829

12. **Challenge** Ruth says the 0 in 4,056 doesn't matter, so she can leave it out. Explain why she is wrong and tell what number she would get.

Show answer

Without the 0 she would get 456. The 0 holds the hundreds place (it means "no hundreds"), which keeps the 4 in the thousands place. Leaving it out moves the 4 to the hundreds place and makes the number much smaller.

COMMON MISTAKES

- Forgetting zeros as placeholders when writing a number from words, such as writing "two million, sixty thousand, four hundred nine" as 2,60,409. Fill every period with exactly three digits: 2,060,409.
- Comparing only the first digits of numbers with different lengths, such as thinking 9,850 is greater than 12,004 because 9 is greater than 1. Count the digits first; more digits means a larger number.
- Saying "and" in the middle of a whole number ("four hundred and six"). Save "and" for the decimal point, which you will meet in Unit 4.

> **Make it a game · Million Dollar Digits.** Parent calls out a seven-digit number as the student writes it, digit by digit, with commas in place. Then the student writes it in word form and expanded form from memory, no peeking. Score one point per correct form, three per number. Swap roles: the student reads a number aloud and the parent writes. Try to reach 15 points in under ten minutes.

1.2 Properties of Addition and Multiplication

OBJECTIVES

- Name and give examples of the commutative, associative, identity and zero properties.
- Use the properties to reorder and regroup numbers for quick mental math.
- Show with an example that subtraction and division do not have the commutative or associative property.

WATCH

Multiplication Properties | Commutative, Associative, Identity, & Zero

Math with Mr. J · 3:37

https://www.youtube.com/watch?v=Doqt7bb8Gno

> **WHEN TO WATCH**
> Before Example 1: the commutative, associative, identity and zero properties (shown for multiplication).

Note: Mr. J shows the four properties with multiplication. The same ideas for addition are in the Core ideas.

BACKUP VIDEO

Math Antics - What Is Arithmetic?

mathantics · 11:23

https://www.youtube.com/watch?v=IwW0GJWKH98

Before the practice set: the commutative property for addition, and why subtraction and division are not commutative (objective 3).

KEY TERMS

commutative property

You can add or multiply two numbers in either order and get the same answer: $3 + 8 = 8 + 3$ and $4 \times 7 = 7 \times 4$.

associative property

When you add or multiply three numbers, you can group them either way: $(2 + 9) + 1 = 2 + (9 + 1)$.

identity property

Adding 0 or multiplying by 1 leaves a number unchanged. 0 is the identity for addition, and 1 is the identity for multiplication.

zero property of multiplication

Any number times 0 equals 0.

compatible numbers

Numbers that are easy to work with together, such as 25 and 4 (they multiply to 100) or 37 and 63 (they add to 100).

CORE IDEAS

A **property** is a rule that is true for every number. In the formulas below, the letters a, b and c simply stand for any numbers you like. Knowing these properties lets you rearrange a problem into an easier one.

The **commutative property** says order does not matter for adding or multiplying. Why is $4 \times 7 = 7 \times 4$? Picture 4 rows of 7 chairs. Walk around to the side and you see 7 rows of 4 chairs. Same chairs, same total, 28.

The **associative property** says grouping does not matter for adding or multiplying. Parentheses show which pair to do first: $(25 \times 4) \times 37$ and $25 \times (4 \times 37)$ both equal 3,700, but the first is much easier, because $25 \times 4 = 100$.

The **identity properties**: adding 0 changes nothing ($58 + 0 = 58$), and multiplying by 1 changes nothing ($58 \times 1 = 58$). The **zero property**: $58 \times 0 = 0$, because 58 groups of nothing is nothing.

Put the properties together for mental math. Look for **compatible numbers**, pairs that make 10, 100 or 1,000, and use the commutative and associative properties to bring them side by side. For $25 \times 37 \times 4$, swap to get $25 \times 4 \times 37 = 100 \times 37 = 3{,}700$.

Be careful: subtraction and division do not have these properties. $12 \div 4 = 3$, but $4 \div 12$ is only $\frac{1}{3}$. And $(12 - 5) - 2 = 5$, while $12 - (5 - 2) = 9$. Only rearrange sums and products.

KEY FORMULAS

$$a + b = b + a \qquad a \cdot b = b \cdot a$$

$$(a + b) + c = a + (b + c) \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$a + 0 = a \qquad a \cdot 1 = a$$

$$a \cdot 0 = 0$$

Practice habit. Memorize the friendly pairs that make round numbers: $25 \times 4 = 100$, $50 \times 2 = 100$, $20 \times 5 = 100$, $125 \times 8 = 1{,}000$, $250 \times 4 = 1{,}000$. Before you multiply or add a long list, scan it for these pairs.

WORKED EXAMPLES

Example 1. Find $25 \times 37 \times 4$ mentally.

Show solution

1. Notice that 25 and 4 are compatible: $25 \times 4 = 100$.
2. Commutative property: $25 \times 37 \times 4 = 25 \times 4 \times 37$.
3. Associative property: $(25 \times 4) \times 37 = 100 \times 37$.
4. $100 \times 37 = 3{,}700$.

Answer: 3,700

Example 2. On a trip, the Miller family drove 147 miles on Monday, 229 miles on Tuesday, 53 miles on Wednesday and 71 miles on Thursday. Use the properties of addition to find the total distance mentally.

Show solution

1. Look for pairs that make round numbers: $147 + 53 = 200$ and $229 + 71 = 300$.
2. Reorder and regroup: $(147 + 53) + (229 + 71)$.
3. $200 + 300 = 500$.
4. Check by adding in order: $147 + 229 = 376$, $376 + 53 = 429$, $429 + 71 = 500$.

Answer: 500 miles

PRACTICE

1. **Easy** Name the property shown: $9 + 14 = 14 + 9$.

   Show answer

   Commutative property of addition

2. **Easy** Name the property shown: $(6 \times 5) \times 2 = 6 \times (5 \times 2)$.

   Show answer

   Associative property of multiplication

3. **Easy** Find $0 \times 4{,}387$.

   Show answer

   0 (zero property of multiplication)

4. **Easy** What number goes in the box? $58 \times \square = 58$

   Show answer

   1 (identity property of multiplication)

5. **Medium** Find $50 \times 19 \times 2$ mentally.

   Show answer

   1,900

6. **Medium** Find $36 + 89 + 64 + 11$ mentally.

   Show answer

   200

7. **Medium** A carpenter buys 5 boxes of nails. Each box holds 7 bags, and each bag holds 20 nails. How many nails does the carpenter buy? Use the properties to find the answer mentally.

   Show answer

   700 nails

8. **Medium** Find $125 \times 13 \times 8$ mentally.

   Show answer

   13,000

9. Hard For a service project, 25 volunteers each pack 6 boxes of food, and each box holds 4 cans. How many cans do they pack in all?

Show answer

600 cans

10. Hard Is division associative? Find $(36 \div 6) \div 2$ and $36 \div (6 \div 2)$ and compare.

Show answer

No. $(36 \div 6) \div 2 = 6 \div 2 = 3$, but $36 \div (6 \div 2) = 36 \div 3 = 12$.

11. Hard Samuel saved \$18 in January, \$27 in February, \$32 in March and \$23 in April. Use compatible numbers to find how much he saved in all.

Show answer

\$100

12. Challenge Find $2 \times 3 \times 5 \times 7 \times 50 \times 20$ mentally.

Show answer

210,000

13. Challenge Explain why $7 \times 0 = 0$ by thinking about groups. Then explain why this is different from $7 + 0$.

Show answer

7×0 means 7 groups with nothing in each group, so there is nothing in all: 0. But $7 + 0$ means you start with 7 and add nothing, so you still have 7.

COMMON MISTAKES

- Rearranging subtraction or division as if they were addition or multiplication, for example treating $20 - 8$ as $8 - 20$. Only reorder and regroup sums and products.
- Mixing up the two identities: thinking $a \times 0 = a$ or $a + 1 = a$. Remember: 0 is the identity for adding, 1 is the identity for multiplying, and anything times 0 is 0.
- Moving numbers around but losing one along the way. After regrouping, count the numbers to be sure every one from the original problem is still there.

> **Make it a game · Property Detectives.** Write 12 quick mental-math problems on slips of paper, such as 4 + 9 + 6, 5 x 17 x 2, 83 x 0 and 1 x 56. Draw one at a time and race the clock: solve it aloud by regrouping, then name the property you used. Naming the property correctly earns a bonus point. Beat your own total on a second round.

1.3 The Distributive Property and Mental Math

OBJECTIVES

- Explain the distributive property using an area model of a rectangle.
- Multiply mentally by splitting a number into a sum or a difference, as in 7×98 and 15×12.
- Use the distributive property in reverse to simplify sums such as $6 \times 47 + 6 \times 53$.

WATCH

The Distributive Property In Arithmetic

mathantics · 10:06

https://www.youtube.com/watch?v=VZ0jG3W53nE

> **WHEN TO WATCH**
> After the area-model paragraph in the Core ideas, before Example 1.

Note: Eddie Woo teaches in Australia: he calls it the 'distributive law'.

BACKUP VIDEO

Mental Calculation Strategies (2 of 2: The Distributive Law)

Eddie Woo · 4:21

https://www.youtube.com/watch?v=Hcjb0_qu_0Q

Before the practice set: splitting numbers to multiply in your head.

KEY TERMS

distributive property

Multiplying a number by a sum gives the same result as multiplying the number by each part of the sum and then adding: $3 \times (10 + 2) = 3 \times 10 + 3 \times 2$.

area model

A picture of multiplication as the area of a rectangle, where the length and width are the two numbers being multiplied.

factor

A number being multiplied. In $6 \times 9 = 54$, the factors are 6 and 9.

product

The answer to a multiplication problem. In $6 \times 9 = 54$, the product is 54.

partial products

The smaller products you add together when you split a factor into parts, such as 70 and 21 in $7 \times 13 = 7 \times 10 + 7 \times 3 = 70 + 21$.

CORE IDEAS

Picture a rectangle 7 units tall and 13 units wide. Its area is 7×13. Now draw a line that cuts the width into 10 and 3. You get two smaller rectangles with areas $7 \times 10 = 70$ and $7 \times 3 = 21$. Cutting doesn't change the total area, so $7 \times 13 = 70 + 21 = 91$.

That picture is the **distributive property**: $7 \times (10 + 3) = 7 \times 10 + 7 \times 3$. The 7 is "distributed" to each part of the sum. You can also see it with groups: 7 groups of 13 is 7 groups of 10 plus 7 groups of 3.

It works with subtraction too. 98 is $100 - 2$, so 7 groups of 98 is 7 groups of 100 minus 7 groups of 2: $7 \times 98 = 700 - 14 = 686$. In the area model, you take a 7 by 100 rectangle and cut away a 7 by 2 strip.

Mental math strategy: split one factor into friendly pieces, usually a round number plus or minus a small number. For 15×12, think $15 \times 10 + 15 \times 2 = 150 + 30 = 180$. For 24×19, think $24 \times 20 - 24 \times 1 = 480 - 24 = 456$.

The property also works **in reverse**. If the same number multiplies two parts, you can pull it out: $6 \times 47 + 6 \times 53 = 6 \times (47 + 53) = 6 \times 100 = 600$. Watch for this shortcut whenever two products share a factor.

Remember that the outside number must multiply **every** part inside the parentheses. $6 \times (20 + 5)$ is $120 + 30$, not $120 + 5$.

KEY FORMULAS

$$a(b+c) = ab + ac$$

$$a(b-c) = ab - ac$$

$$7 \times 98 = 7 \times (100 - 2) = 700 - 14 = 686$$

Euclid connection. Euclid's *Elements*, Book II, Proposition 1, states the distributive property as a fact about rectangles: if one side is cut into pieces, the whole rectangle equals the sum of the smaller rectangles, exactly our area model.

Practice habit. Practice the "times 9, 99, 11, 101" tricks until they are quick: to multiply by 99, multiply by 100 and subtract one copy ($99 \times 6 = 600 - 6 = 594$); to multiply by 11, multiply by 10 and add one copy ($11 \times 23 = 230 + 23 = 253$).

WORKED EXAMPLES

Example 1. Find 7×98 mentally.

Show solution

1. Write 98 as a friendly difference: $98 = 100 - 2$.
2. Distribute: $7 \times (100 - 2) = 7 \times 100 - 7 \times 2$.
3. $7 \times 100 = 700$ and $7 \times 2 = 14$.
4. $700 - 14 = 686$.
5. Check: the answer should be a little less than $7 \times 100 = 700$, and 686 is.

Answer: 686

Example 2. A church orders 24 new hymnals that cost \$19 each. Use the distributive property to find the total cost mentally.

Show solution

1. The total cost is 24×19.
2. Write 19 as $20 - 1$: $24 \times (20 - 1) = 24 \times 20 - 24 \times 1$.
3. $24 \times 20 = 480$ and $24 \times 1 = 24$.
4. $480 - 24 = 456$.
5. Check another way: $19 \times 24 = 19 \times 20 + 19 \times 4 = 380 + 76 = 456$.

Answer: \$456

PRACTICE

1. Easy What number goes in the box? $6 \times (10 + 4) = 6 \times 10 + 6 \times \square$

   Show answer

   4

2. Easy Find 4×23 by thinking of it as $4 \times 20 + 4 \times 3$.

   Show answer

   92

3. Easy Find 8×52 mentally.

   Show answer

   416

4. Easy Find 5×99 mentally.

   Show answer

   495

5. Medium Find 15×12 mentally.

   Show answer

   180

6. Medium Find 9×63 mentally by thinking of 9 as $10 - 1$.

   Show answer

   567

7. **Medium** Find $7 \times 46 + 7 \times 54$ mentally.

   Show answer

   700

8. **Medium** Grace's vegetable garden is a rectangle 8 feet wide and 37 feet long. Find its area by splitting 37 into $30 + 7$.

   Show answer

   296 square feet

9. **Hard** Tickets to a choir concert cost $12 each. The Miller family's friends buy 35 tickets. Use the distributive property to find the total cost.

   Show answer

   $420

10. **Hard** Find 6×198 mentally.

    Show answer

    1,188

11. **Hard** A bakery packs cookies in boxes of 12. One day it fills 49 boxes. How many cookies is that? Use the distributive property.

    Show answer

    588 cookies

12. **Challenge** Find 101×47 mentally.

    Show answer

    4,747

13. **Challenge** Ruth worked out $6 \times (20 + 5)$ as $6 \times 20 + 5 = 125$. Explain her mistake and give the correct answer.

    Show answer

    Ruth multiplied only the 20 by 6. The distributive property says the 6 multiplies every part inside the parentheses: $6 \times 20 + 6 \times 5 = 120 + 30 = 150$. In groups: 6 groups of 25 is 6 groups of 20 plus 6 groups of 5.

COMMON MISTAKES

- Multiplying only the first part inside the parentheses, as in $6 \times (20 + 5) = 120 + 5$. Draw arrows from the outside number to every part inside.

- Subtracting the wrong amount when using a round number: 7×98 is $700 - 14$, not $700 - 2$. You took away 7 groups of 2, so subtract 7×2.
- Splitting into pieces that are not easier, such as $8 \times 52 = 8 \times 26 + 8 \times 26$. Split into a round number and a small number: $8 \times 50 + 8 \times 2$.

> **Make it a game · Split and Conquer.** Parent names a multiplication such as 7 x 98 or 15 x 12. You have 30 seconds to split one factor and solve it aloud, then sketch the area model on scrap paper to prove it. Score a point for each right answer and a bonus for a correct sketch. Then you give the parent three problems to solve the same way.

1.4 Order of Operations

OBJECTIVES

- Evaluate squares and cubes of whole numbers.
- Evaluate expressions with grouping symbols, exponents and all four operations in the correct order.
- Explain why everyone needs to follow the same order of operations.

WATCH

Math Antics - Order Of Operations

mathantics · 9:40

https://www.youtube.com/watch?v=dAgfnK528RA

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he may say 'BODMAS' (Brackets, Orders, Division/Multiplication, Addition/Subtraction). It is the same order as PEMDAS.

BACKUP VIDEO

Introduction to the Order of Operations (What comes first?)

Eddie Woo · 10:49

https://www.youtube.com/watch?v=rfQU5Uur_yY

If students keep doing operations left to right, rewatch before the practice set.

KEY TERMS

exponent

A small raised number that tells how many times to use the base as a factor. In 5^2, the exponent is 2.

base

The number being multiplied by itself. In 5^2, the base is 5.

square

A number times itself, written with exponent 2: $6^2 = 6 \times 6 = 36$. It is the area of a square with side 6.

cube

A number used as a factor three times, written with exponent 3: $2^3 = 2 \times 2 \times 2 = 8$. It is the volume of a cube with edge 2.

grouping symbols

Parentheses ( ), brackets [ ] and the fraction bar. They tell you to work out what is inside them first.

order of operations

The agreed order for doing the steps in an expression, so that everyone gets the same answer.

CORE IDEAS

An **exponent** is shorthand for repeated multiplication. 4^2 ("four squared") means $4 \times 4 = 16$, not 4×2. 4^3 ("four cubed") means $4 \times 4 \times 4 = 64$. The names come from geometry: a square with side 4 has area 4^2, and a cube with edge 4 has volume 4^3.

What is $3 + 4 \times 5$? If you go left to right you get $7 \times 5 = 35$. If you multiply first you get $3 + 20 = 23$. A math sentence must mean only one thing, just as a written sentence should, so mathematicians agreed on one order. The answer is **23**.

Why multiply before adding? Multiplication is a shortcut for repeated addition, so 4×5 is really one bundle, "four fives." The expression $3 + 4 \times 5$ means 3 plus four fives. In the same way, an exponent is a bundle of multiplications, so exponents come before multiplying.

The order: (1) work inside grouping symbols, innermost first; (2) evaluate exponents; (3) multiply and divide **from left to right**, whichever comes first; (4) add and subtract **from left to right**, whichever comes first.

Multiplication does not always come before division, and addition does not always come before subtraction. They are partners of equal rank. So $24 \div 4 \times 2 = 6 \times 2 = 12$, and $12 - 4 + 2 = 8 + 2 = 10$.

A **fraction bar** is a grouping symbol: work out the whole top and the whole bottom first, then divide. When grouping symbols are nested, like $2 \times [10 - (1 + 3)]$, start with the innermost parentheses and work outward.

KEY FORMULAS

$$5^2 = 5 \times 5 = 25 \qquad 2^3 = 2 \times 2 \times 2 = 8$$

$$\text{grouping} \;\to\; \text{exponents} \;\to\; \times \text{ and } \div \text{ (left to right)} \;\to\; + \text{ and } - \text{ (left to right)}$$

Practice habit. Recite the perfect squares from $1^2 = 1$ to $15^2 = 225$ and the cubes from $1^3 = 1$ to $5^3 = 125$ until they are automatic. Also recite the order aloud: "grouping, exponents, multiply and divide left to right, add and subtract left to right."

WORKED EXAMPLES

Example 1. Grace buys 3 notebooks at \$4 each and 2 pens at \$1 each. She pays with a \$20 bill. Write one expression for her change and evaluate it.

Show solution

1. Cost of the items: $3 \times 4 + 2 \times 1$. Put it in parentheses because it is subtracted as one amount.
2. Expression: $20 - (3 \times 4 + 2 \times 1)$.
3. Inside the parentheses, multiply first: $3 \times 4 = 12$ and $2 \times 1 = 2$.
4. Still inside: $12 + 2 = 14$.
5. Subtract: $20 - 14 = 6$.
6. Check: 3 notebooks cost \$12, 2 pens cost \$2, total \$14, and \$20 minus \$14 is \$6.

Answer: \$6

Example 2. Evaluate $2 \times [3^3 - (4 + 2) \times 3] \div 6 + 1$.

Show solution

1. Innermost parentheses: $4 + 2 = 6$. Now $2 \times [3^3 - 6 \times 3] \div 6 + 1$.
2. Inside the brackets, exponent: $3^3 = 27$. Now $2 \times [27 - 6 \times 3] \div 6 + 1$.
3. Inside the brackets, multiply: $6 \times 3 = 18$, then subtract: $27 - 18 = 9$. Now $2 \times 9 \div 6 + 1$.
4. Multiply and divide left to right: $2 \times 9 = 18$, then $18 \div 6 = 3$.
5. Add: $3 + 1 = 4$.

Answer: 4

PRACTICE

1. **Easy** Evaluate 7^2.

   Show answer

   49

2. **Easy** Evaluate 4^3.

   Show answer

   64

3. **Easy** Evaluate $5 + 6 \times 3$.

   Show answer

   23

4. **Easy** Evaluate $(5 + 6) \times 3$.

   Show answer

   33

5. **Medium** Evaluate $36 \div 4 \times 3$.

   Show answer

   27

6. **Medium** Evaluate $60 - 2 \times 5^2 + 8$.

   Show answer

   18

7. **Medium** Evaluate $\dfrac{4 + 2^3}{15 - 3^2}$.

   Show answer

   2

8. **Medium** Ruth bakes 5 batches of 24 cookies. She gives 30 cookies to neighbors and divides the rest equally into 9 bags. Write one expression for the number of cookies in each bag and evaluate it.

   Show answer

   $(5 \times 24 - 30) \div 9 = 10$ cookies

9. Hard Henry buys 4 bags of mulch at \$5 each and 3 flats of flowers at \$12 each. He uses a coupon for \$6 off the total. Write one expression for what he pays and evaluate it.

   Show answer

   $4 \times 5 + 3 \times 12 - 6 = 50$, so he pays \$50

10. Hard Evaluate $3 \times [20 - (2 + 4)^2 \div 4] - 1^3$.

    Show answer

    32

11. Hard A farmer has 3 square pens that are each 6 feet on a side and 1 square pen that is 10 feet on a side. Write one expression for the total area of the pens and evaluate it.

    Show answer

    $3 \times 6^2 + 10^2 = 208$ square feet

12. Challenge Insert parentheses to make this statement true: $2 + 3 \times 4 + 1 = 25$.

    Show answer

    $(2 + 3) \times (4 + 1) = 25$

13. Challenge Samuel says $12 - 4 + 2 = 6$ because addition comes before subtraction. Explain his mistake and give the correct value.

    Show answer

    Addition and subtraction have equal rank, so you do them from left to right: $12 - 4 = 8$, then $8 + 2 = 10$. Samuel added $4 + 2$ first, which is like writing $12 - (4 + 2)$, a different expression.

COMMON MISTAKES

- Thinking $3^2 = 6$. The exponent tells how many times to use the base as a factor: $3^2 = 3 \times 3 = 9$.
- Always multiplying before dividing, or always adding before subtracting. Treat each pair as equals and go left to right: $24 \div 4 \times 2 = 12$, not 3.
- Ignoring the fraction bar as a grouping symbol. In $\frac{4 + 8}{6}$, add the whole top first: $\frac{12}{6} = 2$.

- Rewriting the whole expression carelessly and dropping a number. Copy every part you have not used yet on each new line.

> **Make it a game · Operations Obstacle Course.** Write ten expressions, each with grouping symbols, an exponent and at least two operations, on separate cards. Place them in a row across the floor. Stand at the first, solve it aloud from memory of the order, then step to the next. Any wrong answer sends you back one card. Time your trip and try to beat it.

1.5 Estimation and Rounding

OBJECTIVES

- Round whole numbers to any place.
- Estimate sums, differences, products and quotients by rounding, front-end estimation and compatible numbers.
- Use an estimate to decide whether an answer is reasonable.

WATCH

Math Antics - Rounding

mathantics · 10:41

https://www.youtube.com/watch?v=fd-E18EqSVk

> **WHEN TO WATCH**
> Before Example 1 (rounding).

BACKUP VIDEO

Rounding & Estimating (1 of 4: Why is it useful? - reupload)

Eddie Woo · 10:15

https://www.youtube.com/watch?v=WBAyTMyfvR0

Before Example 2: estimating answers by rounding first.

KEY TERMS

rounding

Replacing a number with a nearby number that is easier to use, such as replacing 6,482 with 6,500.

estimate

An answer that is close to the exact answer and is found quickly, often in your head.

front-end estimation

Estimating by keeping only the first (leftmost) digit of each number and changing the rest to zeros: 4,327 becomes 4,000.

compatible numbers

Nearby numbers chosen because they work together easily, such as replacing $2{,}738 \div 9$ with $2{,}700 \div 9$.

reasonable

Making sense. A reasonable answer is close to what an estimate says it should be.

CORE IDEAS

To **round** a number, ask: which of the two nearest round numbers is it closer to? Think of 6,482 on a number line between 6,400 and 6,500. The halfway mark is 6,450, and 6,482 is past it, so 6,482 rounds to 6,500 (to the nearest hundred).

The quick rule: underline the digit in the place you are rounding to and look at the digit just to its right. If that digit is **5 or more**, add 1 to the underlined digit; if it is **4 or less**, leave the underlined digit alone. Then change every digit to the right of it to 0. A 5 rounds up because by agreement the exact halfway point goes up.

Sometimes rounding up causes a chain: 39,962 to the nearest hundred. The hundreds digit is 9 and the next digit is 6, so the 9 must go up. 399 hundreds become 400 hundreds, which is 40,000.

To **estimate** a sum, difference, product or quotient, round the numbers first, then compute. Rounding to the greatest place is usually enough: $612 + 289 \approx 600 + 300 = 900$. The symbol $\approx$ means "is approximately equal to."

Two other tools: **front-end estimation** keeps only the first digit of each number ($4{,}327 + 2{,}815 \approx 4{,}000 + 2{,}000$), which is fast but always a bit low for sums. For division, use **compatible numbers**, numbers that divide evenly: $4{,}190 \div 7 \approx 4{,}200 \div 7 = 600$.

Why estimate when you can calculate exactly? Because an estimate catches mistakes. If you add 389, 412 and 205 and get 1,906, a quick estimate of $400 + 400 + 200 = 1{,}000$ tells you something went wrong. Estimate first, calculate second, and compare.

> **Practice habit.** Estimate first, every time. Before you work any arithmetic problem, say a rough answer out loud; after you finish, ask, "Is my answer close to my estimate?"

WORKED EXAMPLES

Example 1. Round 4,738,526 to the nearest thousand and to the nearest hundred thousand.

Show solution

1. Nearest thousand: the thousands digit is 8. The digit to its right is 5.
2. Since 5 is 5 or more, round the 8 up to 9 and change the digits after it to zeros: 4,739,000.
3. Nearest hundred thousand: the hundred-thousands digit is 7. The digit to its right is 3.
4. Since 3 is 4 or less, keep the 7 and change the digits after it to zeros: 4,700,000.

Answer: 4,739,000 and 4,700,000

Example 2. A church wants to buy 38 folding chairs that cost $47 each. It has $2,000 to spend. Use an estimate to decide whether $2,000 is enough.

Show solution

1. Round each number up to the nearest ten: 38 becomes 40 and 47 becomes 50.
2. Estimate: $40 \times 50 = 2{,}000$.
3. Both numbers were rounded up, so the real cost is less than $2,000.
4. So $2,000 is enough.
5. Check with the exact cost: $38 \times 47 = 40 \times 47 - 2 \times 47 = 1{,}880 - 94 = 1{,}786$, which is $1,786, less than $2,000.

Answer: Yes, $2,000 is enough (the exact cost is $1,786).

PRACTICE

1. Easy Round 6,482 to the nearest hundred.

   Show answer

   6,500

2. Easy Round 73,950 to the nearest thousand.

   Show answer

   74,000

3. **Easy** Round 2,649,300 to the nearest million.

Show answer

3,000,000

4. **Easy** Estimate $612 + 289$ by rounding each number to the nearest hundred.

Show answer

$600 + 300 = 900$

5. **Medium** Round 39,962 to the nearest hundred.

Show answer

40,000

6. **Medium** Estimate $7{,}815 - 3{,}290$ by rounding each number to the nearest thousand.

Show answer

$8{,}000 - 3{,}000 = 5{,}000$

7. **Medium** Use front-end estimation to estimate $4{,}327 + 2{,}815 + 1{,}604$.

Show answer

$4{,}000 + 2{,}000 + 1{,}000 = 7{,}000$

8. **Medium** Use compatible numbers to estimate $2{,}738 \div 9$.

Show answer

$2{,}700 \div 9 = 300$

9. **Hard** A farmer plants 23 rows of tomatoes with 58 plants in each row. Estimate the total number of plants by rounding each number to its greatest place.

Show answer

$20 \times 60 = 1{,}200$, about 1,200 plants

10. **Hard** A bakery used 4,190 eggs in 7 days. Use compatible numbers to estimate how many eggs it used each day.

Show answer

$4{,}200 \div 7 = 600$, about 600 eggs a day

11. Hard For a bake sale, Ruth counted 389 cookies, 412 muffins and 205 brownies. She added them and got 1,906 items. Use an estimate to decide whether her answer is reasonable.

Show answer

No. The estimate is $400 + 400 + 200 = 1{,}000$, so 1,906 is far too large. (The exact total is 1,006.)

12. Challenge What is the smallest whole number that rounds to 5,000 when rounded to the nearest hundred? What is the largest?

Show answer

Smallest 4,950; largest 5,049

13. Challenge Grace rounded 2,451 to the nearest thousand in two steps. First she rounded to the nearest hundred and got 2,500. Then she rounded 2,500 to the nearest thousand and got 3,000. Explain why her answer is wrong and give the correct answer.

Show answer

Always round the original number in one step. 2,451 is between 2,000 and 3,000, and the halfway mark is 2,500. Since 2,451 is less than 2,500, it is closer to 2,000. Rounding twice moved it past the halfway mark. The correct answer is 2,000.

COMMON MISTAKES

- Looking at the wrong digit. To round to the nearest hundred, look at the tens digit (the one just to the right of the hundreds), not the last digit of the number.
- Forgetting to change the digits after the rounding place to zeros, as in rounding 6,482 to 6,582 or to 65. Keep the number about the same size: 6,500.
- Rounding in several steps (2,451 to 2,500 to 3,000). Always round the original number directly to the place you want.
- Treating an estimate as the final answer when an exact answer is asked for. Use the estimate to check your exact work, not to replace it.

Make it a game · Close Enough Duel. Collect ten price tags, receipts or cereal-box numbers from the kitchen. Parent shows a pair, and each player estimates the sum or product in five seconds without writing, using rounding or compatible numbers. Then check on paper. The closer estimate wins a button. Whoever holds more buttons after ten rounds earns the title of Chief Estimator.

UNIT 1

Whole Numbers and Number Sense

This first unit builds a strong foundation of number sense with whole numbers. You will see how our place-value system works and why it is such a good invention, learn the properties that let you rearrange sums and products, use the distributive property to do multiplication in your head, follow the order of operations, and estimate answers quickly by rounding.

These ideas are not just warm-ups. The properties of numbers and the order of operations are the rules of the road for all of algebra: when letters stand for numbers in Units 9 and 10, these same rules tell you what you may and may not do. Estimation will help you check your work in every unit that follows.

Parents: this unit is done by hand, without a calculator. Encourage your student to say mental-math steps out loud and to estimate before calculating; both habits pay off for years.

PREREQUISITES

- Add, subtract, multiply and divide whole numbers by hand, including long division.
- Know the addition and multiplication facts through 10×10 without hesitation.
- Read and write whole numbers into the thousands.
- Find the area of a rectangle as length times width.

MASTERY CHECKLIST

- ☐ I can read, write and compare whole numbers into the billions, and write them in expanded form.
- ☐ I can explain why place value with a zero makes arithmetic easier than Roman numerals.
- ☐ I can name and use the commutative, associative, identity and zero properties.
- ☐ I can show that subtraction and division are not commutative or associative.
- ☐ I can explain the distributive property with an area model.
- ☐ I can multiply mentally using the distributive property, such as

$7 \times$ and
98

15×12.

- ☐ I can evaluate squares and cubes.
- ☐ I can evaluate expressions using the order of operations, including nested grouping symbols and fraction bars.
- ☐ I can round a whole number to any place.
- ☐ I can estimate sums, differences, products and quotients and use estimates to check whether an answer is reasonable.

LESSONS IN THIS UNIT

1.1 Place Value and Our Number System

45 minutes

1.2	**Properties of Addition and Multiplication**	45 minutes
1.3	**The Distributive Property and Mental Math**	2 sessions of 45 minutes
1.4	**Order of Operations**	2 sessions of 45 minutes
1.5	**Estimation and Rounding**	45 minutes

Unit 1 test

You have 50 minutes. No calculator. Show your work for every problem, including mental-math steps when a problem asks for them. Give exact answers unless a problem asks you to round or estimate, and include units (such as dollars or miles) where they apply.

1. **5 pts**
   What is the value of the digit 6 in 2,468,105?

2. **5 pts**
   Write in standard form: four million, thirty thousand, eight hundred two.

3. **6 pts**
   List from least to greatest: 305,670; 350,067; 305,760.

4. **6 pts**
   Name the property shown: $8 \times (3 \times 25) = (8 \times 3) \times 25$.

5. **6 pts**
   Round 8,476,519 to the nearest ten thousand.

6. **7 pts**
   Find $4 \times 63 \times 25$ mentally. Show which properties you used.

7. **7 pts**
   Use the distributive property to find 8×97 mentally. Show your steps.

8. **7 pts**

   Find $9 \times 34 + 9 \times 66$ mentally.

9. **7 pts**

   Evaluate $5 + 3 \times 2^3$.

10. **8 pts**

    Evaluate $2 \times [18 - (1 + 2)^2] + 4^2 \div 8$.

11. **8 pts**

    The choir sells 48 tickets to its spring concert at \$15 each. Use the distributive property to find how much money the tickets bring in.

12. **8 pts**

    The Miller family drove 2,890 miles in 6 days. Use compatible numbers to estimate how many miles they drove each day.

13. **8 pts**

    Grace buys 3 bags of flour at \$4 each and 2 dozen eggs at \$3 a dozen. She pays with \$30. Write one expression for her change and evaluate it.

14. **12 pts**

    Henry says $3 + 4 \times 5 = 35$. Find the correct value and explain why everyone must follow the same order of operations.

ANSWER KEY

1. 60,000
2. 4,030,802
3. 305,670; 305,760; 350,067
4. Associative property of multiplication
5. 8,480,000
6. 6,300 (commutative and associative properties: $(4 \times 25) \times 63 = 100 \times 63$)
7. $8 \times 100 - 8 \times 3 = 800 - 24 = 776$
8. $9 \times (34 + 66) = 9 \times 100 = 900$
9. 29
10. 20
11. \$720
12. $3{,}000 \div 6 = 500$, about 500 miles a day
13. $30 - (3 \times 4 + 2 \times 3) = 12$, so her change is \$12
14. The correct value is 23: multiply first, $4 \times 5 = 20$, then $3 + 20 = 23$. Multiplication is a bundle of repeated addition (four fives), so it is done before adding. If people used different orders, the same written expression would have different answers, so we agree on one order so that every expression means exactly one number.

UNIT 2

Factors, Multiples and Primes

In this unit you look inside whole numbers to see how they are built. You will learn quick rules for telling what divides a number. You will meet the prime numbers, the building blocks that every whole number is made from by multiplying, and you will break numbers into their prime factors.

With those tools you will find the greatest common factor (GCF) and the least common multiple (LCM) of two numbers. These answer practical questions, like how to cut boards into equal pieces or when two schedules will line up again. You will also learn the Euclidean algorithm, a method more than two thousand years old from Euclid's *Elements*, and see why it works.

This unit prepares directly for Unit 3. The GCF is what you use to put fractions in lowest terms, and the LCM is the least common denominator for adding and subtracting fractions. Work by hand throughout; the divisibility rules and the facts you memorize here will make fraction work much faster.

PREREQUISITES

- Recall multiplication facts through 12×12 quickly.
- Do long division of whole numbers by hand, including finding the remainder.
- Understand factors and multiples from arithmetic (for example, $3 \times 4 = 12$ means 3 and 4 are factors of 12).
- Use squares and cubes written with exponents, such as 5^2 and 2^3 (Lesson 1.4).
- Use the order of operations (Lesson 1.4).

MASTERY CHECKLIST

- ☐ I can use the divisibility rules for 2, 3, 4, 5, 6, 8, 9 and 10 without dividing.
- ☐ I can explain why the digit-sum rules for 3 and 9 work.
- ☐ I can tell whether a number is prime, composite or neither, and explain why 1 is neither.
- ☐ I can use the Sieve of Eratosthenes and recite the primes less than 50.
- ☐ I can find the prime factorization of a number with a factor tree or a division ladder and write it in exponent form.
- ☐ I can find the GCF of two or three numbers by listing factors or by using prime factorizations.
- ☐ I can use the Euclidean algorithm to find the GCF of two large numbers, and explain why it works.
- ☐ I can find the LCM of two or three numbers by listing multiples or by using prime factorizations.
- ☐ I can use GCF × LCM = product of the two numbers to find a missing GCF, LCM or number.
- ☐ I can decide whether a word problem calls for the GCF or the LCM, and solve it.

LESSONS IN THIS UNIT

2.1 Divisibility Rules 45 minutes

2.2	**Prime and Composite Numbers**	45 minutes
2.3	**Prime Factorization**	2 sessions of 45 minutes
2.4	**Greatest Common Factor**	2 sessions of 45 minutes
2.5	**The Euclidean Algorithm**	2 sessions of 45 minutes
2.6	**Least Common Multiple**	2 sessions of 45 minutes

Unit 2 test

You have 50 minutes. No calculator: work every problem by hand. Show your work, including factor trees, division ladders and each step of the Euclidean algorithm. All answers are whole numbers, so no rounding is needed. Write a complete sentence for word problems and the explanation problem.

1. **5 pts**
   Which of the numbers 2, 3, 4, 5, 6, 8, 9 and 10 divide 1,368 evenly? Use divisibility rules.

2. **5 pts**
   The four-digit number $5\square34$ has a missing digit. List every digit that could go in the box to make the number divisible by 3.

3. **5 pts**
   Is 97 prime?

4. **5 pts**
   Is 133 prime? If not, write it as a product of two smaller numbers.

5. **6 pts**
   Find the prime factorization of 540. Write it in exponent form.

6. **6 pts**
   What number has the prime factorization $2^4 \times 3^2 \times 5$?

7. **6 pts**

   Find the GCF of 36 and 60.

8. **8 pts**

   Mrs. Miller has 45 apples and 75 oranges. She wants to make identical fruit baskets using all of the fruit. What is the greatest number of baskets she can make, and how many apples and how many oranges go in each basket?

9. **8 pts**

   Use the Euclidean algorithm to find the GCF of 377 and 261. Show each step.

10. **6 pts**

    Find the LCM of 12 and 15.

11. **8 pts**

    Use prime factorization to find the LCM of 20 and 24.

12. **8 pts**

    Two buses leave the town square together at 7:00 a.m. One bus returns to the square every 15 minutes and the other every 20 minutes. How many minutes after 7:00 a.m. will both buses next be at the square together?

13. **8 pts**

    The GCF of 36 and 84 is 12. Use GCF × LCM = product of the two numbers to find the LCM of 36 and 84.

14. **8 pts**

    Since $91 = 2 \times 35 + 21$, explain why any number that divides both 91 and 35 must also divide 21. Why does this fact make the Euclidean algorithm work?

15. **8 pts**

    A rectangular room is 168 inches by 216 inches. Ruth's father wants to cover the floor with square tiles, all the same size and as large as possible, without cutting any tiles. What is the side length of each tile, and how many tiles are needed?

ANSWER KEY

1. 2, 3, 4, 6, 8 and 9
2. 0, 3, 6 or 9
3. Yes; 2, 3, 5 and 7 do not divide 97, and $11 \times 11 = 121 > 97$.
4. No; $133 = 7 \times 19$.
5. $2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$
6. 720
7. 12
8. 15 baskets, each with 3 apples and 5 oranges.
9. $377 = 1 \times 261 + 116$, $261 = 2 \times 116 + 29$, $116 = 4 \times 29 + 0$; GCF 29.
10. 60
11. $20 = 2^2 \times 5$ and $24 = 2^3 \times 3$, so the LCM is $2^3 \times 3 \times 5 = 120$.
12. 60 minutes (at 8:00 a.m.)
13. $36 \times 84 = 3024$ and $3024 \div 12 = 252$, so the LCM is 252.
14. A number that divides 35 also divides $2 \times 35 = 70$. If it divides both 91 and 70, it divides their difference, $91 - 70 = 21$. In the same way, anything dividing 35 and 21 divides $2 \times 35 + 21 = 91$. So the pair 91 and 35 has the same common divisors, and the same GCF, as the smaller pair 35 and 21. Each step of the algorithm keeps the GCF the same while making the numbers smaller.
15. Tiles 24 inches on a side; $7 \times 9 = 63$ tiles.

2.1 Divisibility Rules

OBJECTIVES

- Use the divisibility rules for 2, 3, 4, 5, 6, 8, 9 and 10 to test a whole number quickly, without dividing.
- Explain why the rules for 3 and 9 work by splitting a number into place values.
- Apply divisibility rules to decide whether objects can be shared or packed into equal groups with none left over.

WATCH

Divisibility Rules (2, 3, 4, & 5)

Math with Mr. J · 4:02

https://www.youtube.com/watch?v=cqnYjR46T58

> **WHEN TO WATCH**
> Before Example 1: the rules for 2, 3, 4 and 5.

Note: Neither video explains why the rules for 3 and 9 work (objective 2). Teach that from the place-value argument in the Core ideas.

BACKUP VIDEO

Divisibility Rules (6, 7, 8, 9, & 10)

Math with Mr. J · 5:42

https://www.youtube.com/watch?v=OPz99FFbo7M

Right after the primary: the rules for 6, 8, 9 and 10. He also gives a rule for 7, which this lesson skips.

KEY TERMS

divisible

A whole number is divisible by another if dividing gives a whole number with no remainder. For example, 18 is divisible by 3 because $18 \div 3 = 6$.

factor

A whole number that divides another whole number with no remainder. 3 is a factor of 18.

multiple

The result of multiplying a number by a whole number. 18 is a multiple of 3 because $3 \times 6 = 18$.

remainder

The amount left over when one whole number does not divide another evenly. $17 \div 5 = 3$ with remainder 2.

digit sum

The number you get by adding up all the digits of a number. The digit sum of 4,725 is $4 + 7 + 2 + 5 = 18$.

CORE IDEAS

A divisibility rule is a shortcut that tells you whether one number divides another evenly, without doing the long division. These rules will save you a great deal of time in this unit, when you look for factors, primes and common factors.

The rules for 2, 5 and 10 look only at the last digit. Every number is some tens plus its ones digit, and every ten is already divisible by 2, 5 and 10. So only the ones digit matters. A number is divisible by 2 if its last digit is 0, 2, 4, 6 or 8 (an even number); by 5 if its last digit is 0 or 5; and by 10 if its last digit is 0.

The rules for 4 and 8 look at the last two or last three digits. Since 100 is divisible by 4, all the hundreds in a number are already divisible by 4, so only the last two digits matter: 5,316 is divisible by 4 because 16 is. Since 1,000 is divisible by 8, only the last three digits matter for 8: 7,216 is divisible by 8 because $216 = 8 \times 27$.

The rules for 3 and 9 use the digit sum: a number is divisible by 3 if its digit sum is divisible by 3, and divisible by 9 if its digit sum is divisible by 9. Here is why. Each place value is one more than a number made of nines: $10 = 9 + 1$, $100 = 99 + 1$, $1000 = 999 + 1$. So $3456 = 3 \times 999 + 4 \times 99 + 5 \times 9 + (3 + 4 + 5 + 6)$. The parts with 999, 99 and 9 are all multiples of 9 (and of 3). What is left over is just the digit sum, $3 + 4 + 5 + 6 = 18$. Since 18 is divisible by 9, so is 3,456.

A number is divisible by 6 when it is divisible by both 2 and 3, because $6 = 2 \times 3$. So check that it is even AND that its digit sum is divisible by 3. Be careful: this trick works for 6 because 2 and 3 have no common factor other than 1. It does not work for 8: a number divisible by 2 and 4 is not always divisible by 8 (12 is an example).

KEY FORMULAS

$$10 = 9 + 1, \quad 100 = 99 + 1, \quad 1000 = 999 + 1$$

$$\text{divisible by } 6 \iff \text{divisible by 2 and by 3}$$

Practice habit. Recite the eight rules until they are automatic: 2 — last digit even; 3 — digit sum divisible by 3; 4 — last two digits; 5 — ends in 0 or 5; 6 — test for 2 and 3; 8 — last three digits; 9 — digit sum divisible by 9; 10 — ends in 0. Then practice by testing house numbers and license plates in your head.

WORKED EXAMPLES

Example 1. Which of the numbers 2, 3, 4, 5, 6, 8, 9 and 10 divide 2,340 evenly?

Show solution

1. The last digit is 0, so 2,340 is divisible by 2, by 5 and by 10.
2. Digit sum: $2 + 3 + 4 + 0 = 9$. Since 9 is divisible by both 3 and 9, so is 2,340.
3. Last two digits: 40, and $40 = 4 \times 10$, so 2,340 is divisible by 4.
4. It is divisible by 2 and by 3, so it is divisible by 6.
5. Last three digits: 340, and $340 \div 8 = 42$ remainder 4, so 2,340 is not divisible by 8.
6. Check one result: $2340 \div 9 = 260$, and $260 \times 9 = 2340$. ✓

Answer: 2, 3, 4, 5, 6, 9 and 10 (but not 8)

Example 2. The Miller family picked 1,512 apples. They want to pack every apple into bags holding the same number, with none left over. Use divisibility rules to decide whether bags of 8 work and whether bags of 9 work. For each size that works, find the number of bags.

Show solution

1. Test 8: look at the last three digits, 512. Since $512 = 8 \times 64$, 1,512 is divisible by 8.
2. Test 9: the digit sum is $1 + 5 + 1 + 2 = 9$, which is divisible by 9, so 1,512 is divisible by 9.
3. Bags of 8: $1512 \div 8 = 189$ bags.
4. Bags of 9: $1512 \div 9 = 168$ bags.
5. Check: $189 \times 8 = 1512$ ✓ and $168 \times 9 = 1512$ ✓

Answer: Both sizes work: 189 bags of 8, or 168 bags of 9.

PRACTICE

1. Easy Is 4,725 divisible by 5?

   Show answer

   Yes; it ends in 5.

2. Easy Is 3,618 divisible by 9?

   Show answer

   Yes; the digit sum is $3 + 6 + 1 + 8 = 18$, which is divisible by 9.

3. Easy Is 1,354 divisible by 4?

   Show answer

   No; the last two digits, 54, are not divisible by 4 ($54 \div 4 = 13$ remainder 2).

4. Easy Is 5,012 divisible by 3?

   Show answer

   No; the digit sum is $5 + 0 + 1 + 2 = 8$, which is not divisible by 3.

5. **Medium** List every number from the set 2, 3, 4, 5, 6, 8, 9, 10 that divides 780 evenly.

Show answer

2, 3, 4, 5, 6 and 10

6. **Medium** Use the rule for 6 to decide whether 2,316 is divisible by 6. If it is, find $2316 \div 6$ by hand.

Show answer

Yes (it is even and its digit sum is 12); $2316 \div 6 = 386$.

7. **Medium** Samuel is setting up 342 chairs for a church supper. He wants rows of 6 chairs with no chairs left over. Use divisibility rules to check that this works, then find how many rows he will make.

Show answer

It works (342 is even and $3 + 4 + 2 = 9$); 57 rows.

8. **Medium** Grace baked 156 muffins for a bake sale. She has boxes that hold 4, boxes that hold 6 and boxes that hold 9. Which box sizes could she use to pack all the muffins with none left over, using only one size of box?

Show answer

Boxes of 4 (39 boxes) or boxes of 6 (26 boxes); boxes of 9 do not work because the digit sum is 12.

9. **Hard** The number $47\square2$ is a four-digit number with one missing digit. What digit goes in the box to make the number divisible by 9?

Show answer

5 (the number is 4,752, with digit sum 18)

10. **Hard** Henry's family farm collected 2,016 eggs in one month. Egg cartons hold 12 eggs. Since $12 = 3 \times 4$, a number is divisible by 12 when it is divisible by both 3 and 4. Use this to check that the eggs fill cartons exactly, then find how many cartons are needed.

Show answer

It works ($2 + 0 + 1 + 6 = 9$ and 16 is divisible by 4); 168 cartons.

11. Hard What is the smallest four-digit number that is divisible by both 9 and 10?

Show answer

1,080

12. Challenge Explain why the rule for 3 works, using the number 471 as your example. (Hint: write $471 = 4 \times 100 + 7 \times 10 + 1$ and use $100 = 99 + 1$ and $10 = 9 + 1$.)

Show answer

$471 = 4 \times 99 + 7 \times 9 + (4 + 7 + 1)$. The parts 4×99 and 7×9 are multiples of 3, because 99 and 9 are. So 471 is divisible by 3 exactly when the leftover part, the digit sum $4 + 7 + 1 = 12$, is. Since 12 is divisible by 3, so is 471 (indeed $471 = 3 \times 157$).

COMMON MISTAKES

- Using the digit sum for 4 or 8. The digit sum works only for 3 and 9. For 4, look at the last two digits; for 8, the last three digits.
- Looking only at the last digit for 4. The number 314 ends in 4 but is not divisible by 4, because 14 is not. Always test the last two digits.
- Thinking that a number divisible by 2 and 4 must be divisible by 8. The number 12 is divisible by 2 and 4 but not 8. Use the last-three-digits rule for 8.
- Checking only one part of the rule for 6. A number like 15 has digit sum 6 but is odd, so it is not divisible by 6. It must pass both the test for 2 and the test for 3.

> **Make it a game · Divisible Detective.** Parent writes a big number, such as 3,672, on paper and covers it. Uncover for three seconds, then you declare which of 2, 3, 4, 5, 6, 8, 9 and 10 divide it, naming the rule for each. Check with a division afterward. Score one point per correct rule. Take a turn writing numbers to stump the parent.

2.2 Prime and Composite Numbers

OBJECTIVES

- Classify a whole number as prime, composite or neither, and explain why 1 is neither.
- Use the Sieve of Eratosthenes to find all the primes up to 100.
- Decide whether a number up to about 250 is prime by testing small prime divisors.

WATCH

Primes and Composites (2 of 3: Introducing Primes of Composites)

Eddie Woo · 11:35

https://www.youtube.com/watch?v=gYB9UeoyO3A

> **WHEN TO WATCH**
> After the Core ideas, before Example 1.

BACKUP VIDEO

The Sieve of Eratosthenes

Eddie Woo · 4:52

https://www.youtube.com/watch?v=Lj_SzTGr-G4

Right before students build their own Sieve of Eratosthenes up to 100 (objective 2).

KEY TERMS

prime number

A whole number greater than 1 whose only factors are 1 and itself. Examples: 2, 3, 5, 7, 11.

composite number

A whole number greater than 1 that has at least one factor other than 1 and itself. Examples: 4, 9, 15.

factor pair

Two whole numbers that multiply to give a number. The factor pairs of 12 are 1 and 12, 2 and 6, 3 and 4.

Sieve of Eratosthenes

A method for finding primes by listing numbers and crossing out the multiples of each prime in turn. The numbers left are the primes.

CORE IDEAS

Every whole number greater than 1 is either prime or composite. A prime number has exactly two factors: 1 and itself. The number 7 is prime, since only 1×7 makes 7. A composite number has more than two factors. The number 15 is composite, since $15 = 3 \times 5$.

The number 1 is neither prime nor composite. It has only one factor, itself, so it does not have the two factors a prime needs. It is not composite either, because it cannot be broken into smaller factors. Leaving 1 out also makes the next lesson work: each number then has only one way to be written as a product of primes.

The number 2 is the only even prime. Every other even number has 2 as a factor, so it has at least three factors (1, 2 and itself) and must be composite.

The Sieve of Eratosthenes finds every prime up to 100. Write the numbers 2 to 100. Circle 2 and cross out every other multiple of 2 (4, 6, 8, ...). Circle the next number not crossed out, 3, and cross out its other multiples. Do the same for 5 and then 7. You can stop there, because $11 \times 11 = 121$ is more than 100. Any composite number up to 100 has a factor smaller than 11, so it has already been crossed out. All the numbers left are prime.

To test whether one number is prime, divide it by the primes 2, 3, 5, 7, 11, ... in order, using the divisibility rules where you can. You only need to try primes whose square is not more than the number. Why? If a number is composite, write it as small factor × big factor. The small factor times itself is no more than small × big, which is the number. So a composite number always has a factor (and a prime factor) whose square is not more than the number, and you will find it in time. For example, to test 113, try 2, 3, 5 and 7. You can stop there because $11 \times 11 = 121$ is more than 113. None of them divide 113, so 113 is prime.

KEY FORMULAS

Primes less than 50: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$

From history. Eratosthenes of Cyrene was a Greek scholar of the third century BC who worked at the great library of Alexandria in Egypt. He is remembered for his sieve for finding primes and for measuring the size of the earth remarkably well. A little earlier, also in Alexandria, Euclid had proved in his *Elements* (Book IX, Proposition 20) that the primes never run out.

Euclid connection. Euclid's argument (Elements IX.20): take any list of primes, multiply them and add 1. For example, $2 \times 3 \times 5 + 1 = 31$. That new number leaves remainder 1 when divided by every prime on the list, so its prime factors are new primes, not on the list. No list of primes can ever be complete.

Practice habit. Recite the primes less than 50 until you can say them quickly: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. Remember that there are 15 of them and that 2 is the only even one.

WORKED EXAMPLES

Example 1. Is 91 prime? If not, write it as a product of two smaller numbers.

Show solution

1. Since $10 \times 10 = 100$ is more than 91, we only need to try the primes 2, 3, 5 and 7.
2. 91 is odd, so 2 is not a factor.
3. Digit sum $9 + 1 = 10$, not divisible by 3, so 3 is not a factor.
4. It does not end in 0 or 5, so 5 is not a factor.
5. Try 7: $91 \div 7 = 13$. So $91 = 7 \times 13$.
6. Check: $7 \times 13 = 91$ ✓

Answer: No; 91 is composite, because $91 = 7 \times 13$.

Example 2. Henry has 113 bricks. He wants to lay all of them in a rectangle with at least 2 rows and the same number of bricks in every row. Can he do it?

Show solution

1. A rectangle with equal rows means 113 = (number of rows) × (bricks per row), with both numbers at least 2. That is possible only if 113 is composite.
2. Since $11 \times 11 = 121$ is more than 113, test only 2, 3, 5 and 7.
3. 113 is odd: not divisible by 2.
4. Digit sum $1 + 1 + 3 = 5$: not divisible by 3.
5. It does not end in 0 or 5: not divisible by 5.
6. $113 \div 7 = 16$ remainder 1 (since $7 \times 16 = 112$): not divisible by 7.
7. So 113 is prime. Its only factor pair is 1 and 113.

Answer: No. 113 is prime, so the only rectangle is 1 row of 113 bricks.

PRACTICE

1. Easy Is 29 prime?

   Show answer

   Yes; its only factors are 1 and 29.

2. Easy Is 51 prime?

   Show answer

   No; $51 = 3 \times 17$.

3. Easy List all the factors of 18.

   Show answer

   1, 2, 3, 6, 9, 18

4. Easy List all the prime numbers between 20 and 40.

   Show answer

   23, 29, 31, 37

5. Medium Is 1 prime, composite or neither? Explain why.

Show answer

Neither. A prime has exactly two factors (1 and itself), but 1 has only one factor. A composite number has more than two factors, and 1 does not.

6. Medium Is 87 prime? If not, write it as a product of two smaller numbers.

Show answer

No; it is composite, since $87 = 3 \times 29$.

7. Medium Grace has 24 cookies to arrange on a tray in a rectangle with equal rows. How many different rectangles can she make? Count a 3-by-8 and an 8-by-3 rectangle as the same, and include the 1-by-24 rectangle.

Show answer

4 rectangles: 1 by 24, 2 by 12, 3 by 8, 4 by 6.

8. Medium A class has 31 students. Can the teacher split the class into equal teams, with at least 2 teams and at least 2 students on each team, and nobody left out?

Show answer

No; 31 is prime, so its only factor pair is 1 and 31.

9. Hard Is 143 prime? Show which primes you tested.

Show answer

No. Test 2, 3, 5, 7, 11 (since $13 \times 13 = 169 > 143$): $143 = 11 \times 13$.

10. Hard A choir director wants to arrange 36 singers in equal rows, with at least 2 rows and at least 2 singers in each row. How many different numbers of rows are possible?

Show answer

7 (2, 3, 4, 6, 9, 12 or 18 rows)

11. Hard Twin primes are two primes that differ by 2, like 11 and 13. How many pairs of twin primes are there with both numbers less than 50?

Show answer

6 pairs: 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43.

12. **Challenge** Is 221 prime? Explain which primes you need to test and why you can stop.

Show answer

No. Since $15 \times 15 = 225 > 221$, test primes up to 13: 2, 3, 5, 7 and 11 do not divide 221, but $221 = 13 \times 17$. So 221 is composite.

COMMON MISTAKES

- Calling 1 a prime number. A prime must have exactly two different factors; 1 has only one, so it is neither prime nor composite.
- Thinking every odd number is prime. Numbers like 9, 15, 21, 27, 51, 57, 87 and 91 are odd but composite. Always test small primes before deciding.
- Forgetting that 2 is prime. It is the only even prime, because its only factors are 1 and 2.
- Stopping too early when testing a number. Keep trying primes until the prime times itself is more than the number; for 143 you must reach 11.

> **Make it a game · Prime or Imposter.** Write numbers 2 through 60 on small paper squares and shuffle them. Draw a card, hold it up, and say prime or composite within five seconds, giving one factor pair as proof when it is composite. Wrong calls go back in the deck. Race to clear the pile, then play a second round trying to beat your time.

2.3 Prime Factorization

OBJECTIVES

- Find the prime factorization of a whole number using a factor tree or a division ladder.
- Write a prime factorization in exponent form and find the number it stands for.
- Use a prime factorization to decide whether one number divides another.

WATCH

Math Antics - Prime Factorization

mathantics · 6:40

https://www.youtube.com/watch?v=XGbOiYhHY2c

> **WHEN TO WATCH**
> Before Example 1 (factor trees).

Note: Eddie Woo teaches in Australia: he spells it 'factorisation'.

BACKUP VIDEO

Prime Factorisation (simple example with factor tree)

Eddie Woo · 2:33

https://www.youtube.com/watch?v=xjMd9eH5wX0

Before the practice set: a 2½-minute recap.

KEY TERMS

prime factorization

A number written as a product of prime numbers only. The prime factorization of 12 is $2 \times 2 \times 3$.

factor tree

A diagram that splits a number into two factors, then splits each composite factor again, until every branch ends in a prime.

division ladder

A way to factor by dividing a number by primes over and over, writing each quotient underneath, until the quotient is 1 or a prime.

exponent form

A way of writing repeated factors with exponents. The exponent tells how many times the base is used as a factor: $2 \times 2 \times 2 = 2^3$.

CORE IDEAS

Every composite number can be broken into smaller factors, and those factors can be broken again, until only primes are left. The result is the number's prime factorization. Primes are like the building blocks of the whole numbers: every whole number greater than 1 is built by multiplying primes.

A factor tree is one way to do this. To factor 360, split it into any two factors, say 36×10. Split 36 into 4×9 and 10 into 2×5. Then split 4 into 2×2 and 9 into 3×3. The primes at the ends of the branches are 2, 2, 3, 3, 2, 5. Put them in order: $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$.

A division ladder does the same job in a neat column. Divide by the smallest prime that works, write the quotient below, and repeat. Use the divisibility rules from Lesson 2.1 to see which prime works. When the quotient is a prime, you are done. The primes you divided by, together with that last prime, are the prime factorization.

Exponent form is shorter. In Unit 1 you saw that $5^2 = 5 \times 5$ and $5^3 = 5 \times 5 \times 5$. The same idea works for any number of factors: the exponent counts how many times the base is used as a factor. So $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$. A factor that appears only once gets no exponent (5 means 5^1). Always write the primes from smallest to largest.

Here is an important fact, sometimes called the Fundamental Theorem of Arithmetic: every whole number greater than 1 has exactly one prime factorization, apart from the order of the factors. It does not matter how you start the factor tree for 360. You could begin with 2×180 or 12×30, and you will still end with three 2s, two 3s and one 5. (This is one more reason 1 is not called prime. If it were, we could tack on as many 1s as we like, and factorizations would no longer be one of a kind.)

A prime factorization shows you everything that divides a number. One number divides another exactly when all of its prime factors appear in the other number's

factorization, at least as many times. For example, $360 = 2^3 \times 3^2 \times 5$ contains $12 = 2^2 \times 3$, so 12 divides 360. But it does not contain $27 = 3^3$, because 360 has only two 3s, so 27 does not divide 360.

KEY FORMULAS

$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$$

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

Practice habit. Learn the first powers of 2 and 3 by heart: 2, 4, 8, 16, 32, 64 and 3, 9, 27, 81. When you see one of these numbers inside a factorization, you will recognize it at once (for example, 64 is 2^6 and 81 is 3^4).

WORKED EXAMPLES

Example 1. Use a factor tree to find the prime factorization of 360. Write it in exponent form.

Show solution

1. Split 360 into two factors: $360 = 36 \times 10$.
2. Split 36: $36 = 4 \times 9$. Split 10: $10 = 2 \times 5$.
3. Split 4: $4 = 2 \times 2$. Split 9: $9 = 3 \times 3$.
4. Every branch now ends in a prime: 2, 2, 3, 3, 2, 5.
5. Write them in order: $2 \times 2 \times 2 \times 3 \times 3 \times 5$.
6. Exponent form: $2^3 \times 3^2 \times 5$.
7. Check: $8 \times 9 \times 5 = 72 \times 5 = 360$ ✓

Answer: $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$

Example 2. A farmer has 1,260 eggs. Use a division ladder to find the prime factorization of 1,260. Then use it to decide whether the eggs can be packed into flats of 18 with none left over. If so, how many flats?

Show solution

1. 1,260 is even: $1260 \div 2 = 630$.
2. 630 is even: $630 \div 2 = 315$.
3. 315 has digit sum 9, so divide by 3: $315 \div 3 = 105$.
4. 105 has digit sum 6, so divide by 3: $105 \div 3 = 35$.
5. 35 ends in 5: $35 \div 5 = 7$, and 7 is prime. Stop.
6. So $1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2^2 \times 3^2 \times 5 \times 7$.
7. Now $18 = 2 \times 3^2$. One 2 and two 3s are all inside the factorization of 1,260, so 18 divides 1,260.
8. The factors left over are $2 \times 5 \times 7 = 70$, so there are 70 flats.
9. Check: $18 \times 70 = 1260$ ✓

Answer: $1260 = 2^2 \times 3^2 \times 5 \times 7$; yes, the eggs fill 70 flats of 18.

PRACTICE

1. Easy Find the prime factorization of 12.

   Show answer

   $2 \times 2 \times 3 = 2^2 \times 3$

2. Easy Find the prime factorization of 45.

   Show answer

   $3 \times 3 \times 5 = 3^2 \times 5$

3. Easy The Miller family's orchard has $2^2 \times 3 \times 5$ apple trees. How many apple trees is that?

   Show answer

   60 trees

4. Easy Find the prime factorization of 98.

   Show answer

   $2 \times 7 \times 7 = 2 \times 7^2$

5. **Medium** Use a division ladder to find the prime factorization of 150. Write it in exponent form.

   Show answer

   $2 \times 3 \times 5 \times 5 = 2 \times 3 \times 5^2$

6. **Medium** Ruth says that both $2^2 \times 3 \times 7$ and $2 \times 6 \times 7$ are prime factorizations of 84. Is she right? Explain.

   Show answer

   No. Both products equal 84, but $2 \times 6 \times 7$ is not a prime factorization because 6 is not prime. Splitting 6 into 2×3 gives $2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$, the one and only prime factorization of 84.

7. **Medium** Find the prime factorization of 196. Write it in exponent form.

   Show answer

   $2 \times 2 \times 7 \times 7 = 2^2 \times 7^2$

8. **Medium** A hardware store has 840 nails. The prime factorization of 840 is $2^3 \times 3 \times 5 \times 7$. Packets hold 24 nails, and $24 = 2^3 \times 3$. Can all the nails be put into full packets? If so, how many packets?

   Show answer

   Yes; the factors left over are $5 \times 7 = 35$, so 35 packets.

9. **Hard** Find the prime factorization of 1,800. Write it in exponent form.

   Show answer

   $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 = 2^3 \times 3^2 \times 5^2$

10. **Hard** Samuel's rectangular garden has an area of 91 square feet. Its length and width are whole numbers of feet, and neither side is 1 foot. What are the length and width?

    Show answer

    7 feet by 13 feet (since $91 = 7 \times 13$)

11. **Hard** Find the prime factorization of 1,001.

    Show answer

    $7 \times 11 \times 13$

12. **Challenge** The prime factorization of 72 is $2^3 \times 3^2$. Every factor of 72 is made from zero to three 2s and zero to two 3s. Use this to find how many factors 72 has, then list them to check.

Show answer

There are 4 choices for the 2s and 3 choices for the 3s, so $4 \times 3 = 12$ factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

COMMON MISTAKES

- Stopping before every factor is prime. In $84 = 2 \times 6 \times 7$, the 6 must still be split into 2×3. Circle each prime as you reach it and keep going until every branch ends in a circle.
- Losing a factor when rewriting in exponent form. Count each prime carefully: three 2s is 2^3, not 2^2. Multiply your answer back out to check that you get the original number.
- Including 1 as a factor in the prime factorization. 1 is not prime, so it never appears in a prime factorization.
- Thinking 2^3 means 2×3. The exponent counts factors: $2^3 = 2 \times 2 \times 2 = 8$, not 6.

> **Make it a game · Tree Climbing Contest.** Parent picks a number between 20 and 200. You draw a factor tree on paper in under a minute, then write the factorization in exponent form from memory. Parent checks by multiplying out. Score three points for a correct tree and two for correct exponents. Switch: you pick a number and the parent tries to beat the clock.

2.4 Greatest Common Factor

OBJECTIVES

- Find the greatest common factor of two or three numbers by listing factors.
- Find the greatest common factor by using prime factorizations.
- Solve word problems about making equal groups or cutting lengths into equal pieces.

WATCH

GCF: Greatest Common Factor - Math Antics Extras

mathantics · 4:26

https://www.youtube.com/watch?v=CUEOL3_Wm3Y

> **WHEN TO WATCH**
> Before Example 1 (listing factors).

Note: Eddie Woo teaches in Australia: he says 'highest common factor (HCF)' and 'greatest common divisor'. Both mean the GCF.

BACKUP VIDEO

Greatest Common Divisor (Using Factor Trees to find the Highest Common Factor)

Eddie Woo · 11:00

https://www.youtube.com/watch?v=ui5yjN1riQA

Before Example 2: GCF from prime factorizations with factor trees.

KEY TERMS

common factor

A number that is a factor of two or more numbers. 3 is a common factor of 12 and 18.

greatest common factor (GCF)

The largest number that is a factor of each of the given numbers. The GCF of 12 and 18 is 6.

relatively prime

Two numbers are relatively prime when their only common factor is 1, so their GCF is 1. Example: 8 and 15. Neither number has to be prime itself.

CORE IDEAS

A common factor of two numbers divides both of them evenly. The greatest common factor, or GCF, is the largest such number. You will use the GCF in Unit 3 to put fractions in lowest terms. It also answers everyday questions about splitting things into equal groups.

Method 1, listing: write all the factors of each number, in factor pairs so you do not miss any, and pick the largest number on both lists. Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24. Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36. The common factors are 1, 2, 3, 4, 6 and 12, so the GCF is 12. Listing works well for small numbers.

Method 2, prime factorization: write each number as a product of primes and take the primes they share, each as many times as it appears in BOTH. For example, $48 = 2^4 \times 3$ and $72 = 2^3 \times 3^2$. Both have at least three 2s and at least one 3. So the GCF is $2^3 \times 3 = 24$. In exponent form: for each shared prime, use the smaller exponent.

Why does Method 2 work? From Lesson 2.3, a number divides 48 only if its prime factors are all found in 48. To divide both 48 and 72, its prime factors must be found in both. The biggest number you can build that way uses every shared prime as many times as both numbers allow.

In word problems, look for the clues *greatest*, *largest* or *most*, together with *equal groups*, *identical* or *no leftovers*. These usually mean GCF. If 24 roses and 40 daisies are to be made into identical bouquets with nothing left over, the number of bouquets must divide both 24 and 40. The most bouquets possible is the GCF, 8.

If two numbers share no prime factors, their GCF is 1 and they are called relatively prime. For example, $8 = 2^3$ and $15 = 3 \times 5$ share no primes, so their GCF is 1, even though neither number is prime.

KEY FORMULAS

$$48 = 2^4 \times 3, \quad 72 = 2^3 \times 3^2, \quad \text{GCF} = 2^3 \times 3 = 24$$

Practice habit. Before you compute, estimate: the GCF can never be bigger than the smaller number, and it must divide their difference too. For 48 and 72, the difference is 24, so the GCF is 24 or a factor of 24. Make it a habit to check this first.

WORKED EXAMPLES

Example 1. Find the GCF of 24 and 36 by listing factors.

Show solution

1. Factors of 24 (in pairs): 1 and 24, 2 and 12, 3 and 8, 4 and 6. So 1, 2, 3, 4, 6, 8, 12, 24.
2. Factors of 36 (in pairs): 1 and 36, 2 and 18, 3 and 12, 4 and 9, 6 and 6. So 1, 2, 3, 4, 6, 9, 12, 18, 36.
3. Common factors: 1, 2, 3, 4, 6, 12.
4. The greatest is 12.
5. Check: $24 \div 12 = 2$ and $36 \div 12 = 3$ ✓

Answer: 12

Example 2. Henry has two boards, one 84 inches long and one 126 inches long. He wants to cut both boards into pieces that are all the same length, as long as possible, with no wood left over. How long should each piece be, and how many pieces will he get in all?

Show solution

1. The piece length must divide both 84 and 126, and be as large as possible, so find the GCF.
2. $84 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$.
3. $126 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$.
4. Shared primes: one 2, one 3, one 7. GCF $= 2 \times 3 \times 7 = 42$.
5. The 84-inch board gives $84 \div 42 = 2$ pieces; the 126-inch board gives $126 \div 42 = 3$ pieces.
6. Total: $2 + 3 = 5$ pieces.
7. Check: $2 \times 42 = 84$ ✓ and $3 \times 42 = 126$ ✓

Answer: Each piece is 42 inches long; he gets 5 pieces in all.

PRACTICE

1. **Easy** Find the GCF of 12 and 18.

   Show answer

   6

2. **Easy** Find the GCF of 15 and 25.

   Show answer

   5

3. **Easy** Find the GCF of 8 and 21. Are 8 and 21 relatively prime?

   Show answer

   GCF 1; yes, they are relatively prime.

4. **Easy** Find the GCF of 16 and 40.

   Show answer

   8

5. **Medium** Use prime factorization to find the GCF of 48 and 80.

   Show answer

   $48 = 2^4 \times 3$ and $80 = 2^4 \times 5$, so the GCF is $2^4 = 16$.

6. **Medium** Grace has 24 roses and 40 daisies. She wants to make identical bouquets using all of the flowers. What is the greatest number of bouquets she can make, and how many roses and how many daisies will be in each?

   Show answer

   8 bouquets, each with 3 roses and 5 daisies.

7. **Medium** Find the GCF of 18, 30 and 42.

   Show answer

   6

8. Medium Explain why the GCF of two numbers can never be greater than the smaller of the two numbers.

Show answer

The GCF must be a factor of the smaller number, and a factor of a whole number can never be larger than the number itself. So the GCF is at most the smaller number.

9. Hard A rectangular floor is 60 inches by 84 inches. Samuel wants to cover it with square tiles, all the same size, as large as possible, without cutting any tiles. What is the side length of each tile, and how many tiles will he need?

Show answer

Tiles 12 inches on a side; $5 \times 7 = 35$ tiles.

10. Hard Use prime factorization to find the GCF of 126 and 180.

Show answer

$126 = 2 \times 3^2 \times 7$ and $180 = 2^2 \times 3^2 \times 5$, so the GCF is $2 \times 3^2 = 18$.

11. Hard A church youth group is packing care kits. They have 54 toothbrushes, 72 bars of soap and 90 washcloths. Every kit must be identical, and every item must be used. What is the greatest number of kits they can pack, and how many of each item goes in each kit?

Show answer

18 kits, each with 3 toothbrushes, 4 bars of soap and 5 washcloths.

12. Challenge Find the GCF of $2^3 \times 3^2 \times 5$ and $2^2 \times 3^4 \times 7$. Give your answer in exponent form and as a whole number.

Show answer

$2^2 \times 3^2 = 36$

COMMON MISTAKES

- Giving a common factor that is not the greatest. For 24 and 36, both 6 and 12 are common factors, but the GCF is 12. List all the factors in pairs so you do not miss the largest one.
- Taking the larger exponent instead of the smaller one. For $2^4 \times 3$ and $2^3 \times 3^2$, the GCF uses 2^3 and 3^1, because each number must have enough of every prime. Use the smaller exponent of each shared prime.

- Including primes that only one number has. In $84 = 2^2 \times 3 \times 7$ and $90 = 2 \times 3^2 \times 5$, the 5 and the 7 are not shared, so they are not in the GCF (which is 6).
- Answering the wrong question in a word problem. In the bouquet problem, the GCF (8) is the number of bouquets, but the number of roses in each bouquet is $24 \div 8 = 3$. Reread the question before writing your answer.

> **Make it a game · Greatest Common Quest.** Pick two numbers under 100 from a deck of homemade cards. Both players race to find the greatest common factor, one using factor lists and the other prime factorizations. Compare answers and methods, and the first correct finisher wins the round. Trade methods each round. Play to five wins, then try three numbers at once for a final challenge.

2.5 The Euclidean Algorithm

OBJECTIVES

- Find the GCF of two numbers by repeated division with remainders (the Euclidean algorithm).
- Explain why the algorithm works: a common divisor of two numbers also divides the remainder.
- Use the Euclidean algorithm to solve GCF word problems with large numbers.

WATCH

Euclid's Algorithm - Numberphile

Numberphile · 12:45

https://www.youtube.com/watch?v=6Y3jHHE_hbA

> **WHEN TO WATCH**
> After the Core ideas, before Example 1. It shows the repeated-division steps.

Note: Alternate-tier primary (Numberphile). No approved channel has a Euclidean-algorithm video, and the backup is a repeat of Lesson 2.4.

BACKUP VIDEO

GCF: Greatest Common Factor - Math Antics Extras

mathantics · 4:26

https://www.youtube.com/watch?v=CUEOL3_Wm3Y

Review only: the listing method from Lesson 2.4, to compare with the faster Euclidean algorithm.

KEY TERMS

algorithm

A step-by-step method that always gives the answer if you follow the steps. Long division is an algorithm.

quotient

The whole-number result of a division. In $50 \div 15$, the quotient is 3.

remainder

What is left over after dividing. In $50 \div 15$, the remainder is 5, because $50 = 3 \times 15 + 5$. The remainder is always smaller than the number you divide by.

Euclidean algorithm

A method for finding the GCF of two numbers: divide the larger by the smaller, then divide the smaller by the remainder, and keep going until the remainder is 0. The last remainder that is not 0 is the GCF.

CORE IDEAS

Listing factors or finding prime factorizations gets hard when the numbers are large, like 1,183 and 1,456. The Euclidean algorithm finds the GCF using nothing but division, and it is often much faster.

Here are the steps. Divide the larger number by the smaller one and note the remainder. Then replace the pair with (smaller number, remainder) and divide again. Repeat until the remainder is 0. The last remainder that was not 0 is the GCF. (If the very first division has remainder 0, the GCF is the smaller number.)

Example: the GCF of 252 and 105. First, $252 = 2 \times 105 + 42$. Next, $105 = 2 \times 42 + 21$. Then $42 = 2 \times 21 + 0$. The remainder is now 0, so the last nonzero remainder, 21, is the GCF. Check: $252 = 21 \times 12$ and $105 = 21 \times 5$.

Why does this work? Suppose a number d divides both 252 and 105. Then d divides $2 \times 105 = 210$, and so it divides the difference $252 - 210 = 42$, the remainder. (If two numbers are both made of groups of d, their difference is too.) Going the other way, a number that divides both 105 and 42 also divides $2 \times 105 + 42 = 252$. So the pair 252 and 105 has exactly the same common divisors as the pair 105 and 42, and therefore the same GCF.

Each step keeps the GCF the same while making the numbers smaller. The numbers cannot shrink forever, so eventually a remainder is 0. At that point one number divides the other, like 21 dividing 42, and the GCF of such a pair is simply the smaller number. That is why the last nonzero remainder is the GCF of the original pair.

KEY FORMULAS

$$\text{larger} = \text{quotient} \times \text{smaller} + \text{remainder}$$

$$\text{GCF}(\text{larger}, \text{smaller}) = \text{GCF}(\text{smaller}, \text{remainder})$$

From history. This is one of the oldest algorithms still in everyday use. Euclid of Alexandria described it around 300 BC in his *Elements*, Book VII, Propositions 1 and 2. Euclid wrote it as repeatedly subtracting the smaller number from the larger; dividing with a remainder is a quicker way of doing the same subtractions all at once.

Euclid connection. In Elements VII.2, Euclid shows how to find the greatest common measure of two numbers that are not relatively prime, by taking the smaller away from the larger again and again. Our repeated division is the same method, and his reasoning — whatever measures both numbers also measures what is left over — is the reason it works.

Practice habit. Write every step in the same form, larger = quotient × smaller + remainder, and line the steps up in a column. Each new line uses the last two numbers of the line above. Then check the final GCF by dividing it into both original numbers.

WORKED EXAMPLES

Example 1. Use the Euclidean algorithm to find the GCF of 252 and 105.

Show solution

1. Divide 252 by 105: $252 = 2 \times 105 + 42$.
2. Divide 105 by 42: $105 = 2 \times 42 + 21$.
3. Divide 42 by 21: $42 = 2 \times 21 + 0$.
4. The remainder is 0, so the GCF is the last nonzero remainder, 21.
5. Check: $252 \div 21 = 12$ and $105 \div 21 = 5$ ✓

Answer: 21

Example 2. The Miller family is planting trees along two straight sides of their land. One side is 1,071 feet long and the other is 462 feet long. They want the same spacing between trees on both sides, with a tree at each end of each side, and they want the spacing (a whole number of feet) to be as large as possible. How far apart should the trees be?

Show solution

1. The spacing must divide both 1,071 and 462 exactly, and be as large as possible, so find the GCF.
2. Divide 1,071 by 462: $1071 = 2 \times 462 + 147$.
3. Divide 462 by 147: $462 = 3 \times 147 + 21$.
4. Divide 147 by 21: $147 = 7 \times 21 + 0$.
5. The last nonzero remainder is 21, so the GCF is 21.
6. Check: $1071 \div 21 = 51$ and $462 \div 21 = 22$ ✓

Answer: 21 feet apart

PRACTICE

1. Easy Write 50 in the form $50 = q \times 15 + r$, where r is less than 15. What are the quotient q and the remainder r?

   Show answer

   $50 = 3 \times 15 + 5$; quotient 3, remainder 5.

2. Easy Use the Euclidean algorithm to find the GCF of 48 and 18.

   Show answer

   $48 = 2 \times 18 + 12$, $18 = 1 \times 12 + 6$, $12 = 2 \times 6 + 0$; GCF 6.

3. Easy Use the Euclidean algorithm to find the GCF of 35 and 21.

   Show answer

   $35 = 1 \times 21 + 14$, $21 = 1 \times 14 + 7$, $14 = 2 \times 7 + 0$; GCF 7.

4. Easy A teacher has 100 pencils and 75 erasers. She wants to make identical supply bags with nothing left over. Use the Euclidean algorithm to find the greatest number of bags she can make.

Show answer

$100 = 1 \times 75 + 25$, $75 = 3 \times 25 + 0$; 25 bags.

5. Medium Use the Euclidean algorithm to find the GCF of 168 and 120.

Show answer

$168 = 1 \times 120 + 48$, $120 = 2 \times 48 + 24$, $48 = 2 \times 24 + 0$; GCF 24.

6. Medium Use the Euclidean algorithm to find the GCF of 391 and 299.

Show answer

$391 = 1 \times 299 + 92$, $299 = 3 \times 92 + 23$, $92 = 4 \times 23 + 0$; GCF 23.

7. Medium Grace has two ribbons, 144 inches and 210 inches long. She wants to cut both into pieces of equal length, as long as possible, with no ribbon wasted. Use the Euclidean algorithm to find the length of each piece.

Show answer

$210 = 1 \times 144 + 66$, $144 = 2 \times 66 + 12$, $66 = 5 \times 12 + 6$, $12 = 2 \times 6 + 0$; each piece is 6 inches.

8. Medium Use the Euclidean algorithm to find the GCF of 221 and 143.

Show answer

$221 = 1 \times 143 + 78$, $143 = 1 \times 78 + 65$, $78 = 1 \times 65 + 13$, $65 = 5 \times 13 + 0$; GCF 13.

9. Hard Use the Euclidean algorithm to find the GCF of 1,071 and 1,029.

Show answer

$1071 = 1 \times 1029 + 42$, $1029 = 24 \times 42 + 21$, $42 = 2 \times 21 + 0$; GCF 21.

10. Hard A farm has two water tanks holding 1,183 gallons and 1,456 gallons. Samuel wants to find the largest bucket size, in whole gallons, that could empty each tank exactly with full buckets. Use the Euclidean algorithm.

Show answer

$1456 = 1 \times 1183 + 273$, $1183 = 4 \times 273 + 91$, $273 = 3 \times 91 + 0$; a 91-gallon bucket.

11. Hard Since $252 = 2 \times 105 + 42$, the Euclidean algorithm says that 252 and 105 have the same GCF as 105 and 42. Explain why any number that divides both 252 and 105 must also divide 42.

Show answer

If a number divides 105, it divides $2 \times 105 = 210$. If it divides both 252 and 210, it divides their difference, $252 - 210 = 42$, because the difference of two multiples of a number is also a multiple of it. So every common divisor of 252 and 105 also divides 42.

12. Challenge Use the Euclidean algorithm to find the GCF of 3,289 and 2,233. Check your answer by dividing.

Show answer

$3289 = 1 \times 2233 + 1056$, $2233 = 2 \times 1056 + 121$, $1056 = 8 \times 121 + 88$, $121 = 1 \times 88 + 33$, $88 = 2 \times 33 + 22$, $33 = 1 \times 22 + 11$, $22 = 2 \times 11 + 0$; GCF 11. Check: $3289 = 11 \times 299$ and $2233 = 11 \times 203$.

COMMON MISTAKES

- Giving the last remainder (0) or the last quotient as the answer. The GCF is the last remainder that is NOT zero — the number you divided by in the final step.
- Dividing the wrong pair in the next step. After $252 = 2 \times 105 + 42$, the next division is 105 by 42 (the old divisor by the remainder), not 252 by 42.
- Making an arithmetic slip in one division, which ruins every step after it. Check each line by multiplying and adding: $2 \times 105 + 42$ should equal 252.
- Forgetting to check the answer. Always divide the GCF into both original numbers; if either division leaves a remainder, go back and find the error.

> **Make it a game · Remainder Relay.** Write a pair of numbers, such as 252 and 105, at the top of a sheet. Take turns being the divider: one divides, writes the remainder, and passes the paper. Keep going until the remainder is zero. Then explain aloud why the last divisor is the GCF. Make up a harder pair for the other player to try.

2.6 Least Common Multiple

OBJECTIVES

- Find the least common multiple of two or three numbers by listing multiples and by prime factorization.
- Use the fact that GCF × LCM equals the product of two numbers, and explain why it is true.
- Solve word problems about schedules or packages that line up.

WATCH

LCM: Basic Method - Math Antics Extras

mathantics · 3:58

https://www.youtube.com/watch?v=Nu6AC6X3_E4

> **WHEN TO WATCH**
> Before Example 1 (listing multiples).

BACKUP VIDEO

LCM: Prime Factorization Method - Math Antics Extras

mathantics · 4:42

https://www.youtube.com/watch?v=SXPsfr-Fnu4

Before the practice set: the LCM from prime factorizations.

KEY TERMS

multiple

The result of multiplying a number by 1, 2, 3, and so on. The multiples of 6 are 6, 12, 18, 24, ...

common multiple

A number that is a multiple of each of two or more numbers. 24 and 48 are common multiples of 6 and 8.

least common multiple (LCM)

The smallest number (other than 0) that is a multiple of each of the given numbers. The LCM of 6 and 8 is 24.

CORE IDEAS

The least common multiple, or LCM, is the smallest number that each of the given numbers divides into evenly. You will use it in Unit 3 as the least common denominator when you add fractions. It also answers everyday questions like "When will these two schedules line up again?"

Method 1, listing: write the multiples of each number until you find one on both lists. Multiples of 6: 6, 12, 18, 24, 30, ... Multiples of 8: 8, 16, 24, ... The first number on both lists is 24, so the LCM of 6 and 8 is 24. A shortcut: list multiples of the larger number only, and stop at the first one the smaller number divides.

Method 2, prime factorization: the LCM must contain each number's prime factorization inside it. So take every prime that appears in either number, using the LARGER exponent. For $18 = 2 \times 3^2$ and $24 = 2^3 \times 3$, the LCM needs three 2s (for 24) and two 3s (for 18): $2^3 \times 3^2 = 72$. Compare the GCF, which uses the SMALLER exponents: $2 \times 3 = 6$.

Here is a useful fact for two numbers: GCF × LCM = the product of the two numbers. For 18 and 24: $6 \times 72 = 432$ and $18 \times 24 = 432$. Why? For each prime, the GCF takes the smaller exponent and the LCM takes the larger one. Together they use each prime's two exponents exactly once — the same factors as multiplying the two numbers. So when you know the GCF, you can find the LCM: LCM = product ÷ GCF.

In word problems, look for things that repeat on different cycles and the question *when will they happen together again?* or *what is the smallest number that works for both?* These usually mean LCM. Compare with the GCF problems of Lesson 2.4, which split things into the largest equal groups. A quick test: GCF answers are no bigger than the given numbers; LCM answers are no smaller.

KEY FORMULAS

$$\mathrm{GCF}(a, b) \times \mathrm{LCM}(a, b) = a \times b$$

$$18 = 2 \times 3^2, \quad 24 = 2^3 \times 3, \quad \mathrm{LCM} = 2^3 \times 3^2 = 72$$

Practice habit. Say it until it sticks: "GCF — smaller exponents, shared primes only. LCM — larger exponents, every prime." Then check every LCM by the product rule: GCF × LCM should equal the product of the two numbers.

WORKED EXAMPLES

Example 1. Find the LCM of 6 and 8 by listing multiples.

Show solution

1. List multiples of the larger number, 8: 8, 16, 24, 32, ...
2. Does 6 divide 8? No. 16? No. 24? Yes, $24 = 6 \times 4$.
3. So the LCM is 24.
4. Check: $24 \div 6 = 4$ and $24 \div 8 = 3$ ✓

Answer: 24

Example 2. At a church, one bell rings every 12 minutes and another rings every 18 minutes. Both ring together at 9:00 a.m. How many minutes later will they next ring together, and at what time?

Show solution

1. The bells ring together at times that are multiples of both 12 and 18, so find the LCM.
2. $12 = 2^2 \times 3$ and $18 = 2 \times 3^2$.
3. Use every prime with its larger exponent: $2^2 \times 3^2 = 4 \times 9 = 36$.
4. Check with the product rule: the GCF is $2 \times 3 = 6$, and $6 \times 36 = 216 = 12 \times 18$ ✓
5. So they next ring together 36 minutes after 9:00 a.m., at 9:36 a.m.

Answer: 36 minutes later, at 9:36 a.m.

PRACTICE

1. Easy Find the LCM of 4 and 6.

   Show answer

   12

2. Easy Find the LCM of 5 and 7.

   Show answer

   35

3. Easy Find the LCM of 3 and 9.

Show answer

9

4. Easy Find the LCM of 8 and 12.

Show answer

24

5. Medium Use prime factorization to find the LCM of 18 and 30.

Show answer

$18 = 2 \times 3^2$ and $30 = 2 \times 3 \times 5$, so the LCM is $2 \times 3^2 \times 5 = 90$.

6. Medium For a church picnic, Ruth buys hot dogs in packs of 10 and buns in packs of 8. She wants exactly the same number of hot dogs and buns. What is the smallest number of each she can buy, and how many packs of hot dogs and how many packs of buns is that?

Show answer

40 of each: 4 packs of hot dogs and 5 packs of buns.

7. Medium Find the LCM of 6, 10 and 15.

Show answer

30

8. Medium The GCF of 24 and 36 is 12. Use the fact that GCF × LCM equals the product of the two numbers to find the LCM of 24 and 36.

Show answer

$24 \times 36 = 864$ and $864 \div 12 = 72$, so the LCM is 72.

9. Hard Grace waters her ferns every 4 days, her violets every 6 days and her cactus every 10 days. She watered all three today. In how many days will she next water all three on the same day?

Show answer

60 days

10. Hard Two numbers have a GCF of 6 and an LCM of 90. One of the numbers is 18. What is the other number?

Show answer

$6 \times 90 = 540$ and $540 \div 18 = 30$, so the other number is 30.

11. Hard Explain, using prime factorizations, why GCF × LCM equals 12×18.

Show answer

$12 = 2^2 \times 3$ and $18 = 2 \times 3^2$. The GCF takes the smaller exponent of each prime, $2^1 \times 3^1 = 6$, and the LCM takes the larger, $2^2 \times 3^2 = 36$. Together they use 2^1 and 2^2 and 3^1 and 3^2 — exactly the factors in 12×18. So $6 \times 36 = 216 = 12 \times 18$.

12. Challenge In Henry's workshop, a gear with 36 teeth turns a larger gear with 48 teeth. A chalk mark is placed where the two gears touch. Each tooth that passes moves both gears by one tooth. How many full turns must the smaller gear make before the two chalk marks line up again at the same spot for the first time?

Show answer

The marks line up after LCM(36, 48) = 144 teeth have passed, which is $144 \div 36 = 4$ turns of the smaller gear (and $144 \div 48 = 3$ turns of the larger gear).

COMMON MISTAKES

- Just multiplying the two numbers. $6 \times 8 = 48$ is a common multiple of 6 and 8, but not the least one; the LCM is 24. Multiplying gives the LCM only when the numbers are relatively prime, like 5 and 7.
- Mixing up GCF and LCM. The GCF is no bigger than the smaller number; the LCM is no smaller than the larger number. If your LCM of 18 and 24 comes out as 6, you have found the GCF instead.
- Using the smaller exponent or leaving out a prime when finding the LCM by factorization. For $18 = 2 \times 3^2$ and $24 = 2^3 \times 3$, the LCM needs 2^3 and 3^2. Check that each original number divides your LCM.
- Using GCF × LCM = product for three numbers. The rule is for two numbers only. For three numbers, find the LCM by prime factorization or by listing.

Make it a game · Multiples Marathon. Pick two numbers like 6 and 8 and say their multiples in alternating turns, counting up as fast as you can. The first time a number appears that both players can claim, stop and shout it: that is the LCM. Then verify with prime factors. Play five rounds with new pairs, scoring a point for each first shout.

UNIT 2

Factors, Multiples and Primes

In this unit you look inside whole numbers to see how they are built. You will learn quick rules for telling what divides a number. You will meet the prime numbers, the building blocks that every whole number is made from by multiplying, and you will break numbers into their prime factors.

With those tools you will find the greatest common factor (GCF) and the least common multiple (LCM) of two numbers. These answer practical questions, like how to cut boards into equal pieces or when two schedules will line up again. You will also learn the Euclidean algorithm, a method more than two thousand years old from Euclid's *Elements*, and see why it works.

This unit prepares directly for Unit 3. The GCF is what you use to put fractions in lowest terms, and the LCM is the least common denominator for adding and subtracting fractions. Work by hand throughout; the divisibility rules and the facts you memorize here will make fraction work much faster.

PREREQUISITES

- Recall multiplication facts through 12×12 quickly.
- Do long division of whole numbers by hand, including finding the remainder.
- Understand factors and multiples from arithmetic (for example, $3 \times 4 = 12$ means 3 and 4 are factors of 12).
- Use squares and cubes written with exponents, such as 5^2 and 2^3 (Lesson 1.4).
- Use the order of operations (Lesson 1.4).

MASTERY CHECKLIST

- ☐ I can use the divisibility rules for 2, 3, 4, 5, 6, 8, 9 and 10 without dividing.
- ☐ I can explain why the digit-sum rules for 3 and 9 work.
- ☐ I can tell whether a number is prime, composite or neither, and explain why 1 is neither.
- ☐ I can use the Sieve of Eratosthenes and recite the primes less than 50.
- ☐ I can find the prime factorization of a number with a factor tree or a division ladder and write it in exponent form.
- ☐ I can find the GCF of two or three numbers by listing factors or by using prime factorizations.
- ☐ I can use the Euclidean algorithm to find the GCF of two large numbers, and explain why it works.
- ☐ I can find the LCM of two or three numbers by listing multiples or by using prime factorizations.
- ☐ I can use GCF × LCM = product of the two numbers to find a missing GCF, LCM or number.
- ☐ I can decide whether a word problem calls for the GCF or the LCM, and solve it.

LESSONS IN THIS UNIT

2.1 Divisibility Rules 45 minutes

2.2	**Prime and Composite Numbers**	45 minutes
2.3	**Prime Factorization**	2 sessions of 45 minutes
2.4	**Greatest Common Factor**	2 sessions of 45 minutes
2.5	**The Euclidean Algorithm**	2 sessions of 45 minutes
2.6	**Least Common Multiple**	2 sessions of 45 minutes

Unit 2 test

You have 50 minutes. No calculator: work every problem by hand. Show your work, including factor trees, division ladders and each step of the Euclidean algorithm. All answers are whole numbers, so no rounding is needed. Write a complete sentence for word problems and the explanation problem.

1. **5 pts**
   Which of the numbers 2, 3, 4, 5, 6, 8, 9 and 10 divide 1,368 evenly? Use divisibility rules.

2. **5 pts**
   The four-digit number $5\square34$ has a missing digit. List every digit that could go in the box to make the number divisible by 3.

3. **5 pts**
   Is 97 prime?

4. **5 pts**
   Is 133 prime? If not, write it as a product of two smaller numbers.

5. **6 pts**
   Find the prime factorization of 540. Write it in exponent form.

6. **6 pts**
   What number has the prime factorization $2^4 \times 3^2 \times 5$?

7. **6 pts**

   Find the GCF of 36 and 60.

8. **8 pts**

   Mrs. Miller has 45 apples and 75 oranges. She wants to make identical fruit baskets using all of the fruit. What is the greatest number of baskets she can make, and how many apples and how many oranges go in each basket?

9. **8 pts**

   Use the Euclidean algorithm to find the GCF of 377 and 261. Show each step.

10. **6 pts**

    Find the LCM of 12 and 15.

11. **8 pts**

    Use prime factorization to find the LCM of 20 and 24.

12. **8 pts**

    Two buses leave the town square together at 7:00 a.m. One bus returns to the square every 15 minutes and the other every 20 minutes. How many minutes after 7:00 a.m. will both buses next be at the square together?

13. **8 pts**

    The GCF of 36 and 84 is 12. Use GCF × LCM = product of the two numbers to find the LCM of 36 and 84.

14. **8 pts**

    Since $91 = 2 \times 35 + 21$, explain why any number that divides both 91 and 35 must also divide 21. Why does this fact make the Euclidean algorithm work?

15. **8 pts**

    A rectangular room is 168 inches by 216 inches. Ruth's father wants to cover the floor with square tiles, all the same size and as large as possible, without cutting any tiles. What is the side length of each tile, and how many tiles are needed?

ANSWER KEY

1. 2, 3, 4, 6, 8 and 9
2. 0, 3, 6 or 9
3. Yes; 2, 3, 5 and 7 do not divide 97, and $11 \times 11 = 121 > 97$.
4. No; $133 = 7 \times 19$.
5. $2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$
6. 720
7. 12
8. 15 baskets, each with 3 apples and 5 oranges.
9. $377 = 1 \times 261 + 116$, $261 = 2 \times 116 + 29$, $116 = 4 \times 29 + 0$; GCF 29.
10. 60
11. $20 = 2^2 \times 5$ and $24 = 2^3 \times 3$, so the LCM is $2^3 \times 3 \times 5 = 120$.
12. 60 minutes (at 8:00 a.m.)
13. $36 \times 84 = 3024$ and $3024 \div 12 = 252$, so the LCM is 252.
14. A number that divides 35 also divides $2 \times 35 = 70$. If it divides both 91 and 70, it divides their difference, $91 - 70 = 21$. In the same way, anything dividing 35 and 21 divides $2 \times 35 + 21 = 91$. So the pair 91 and 35 has the same common divisors, and the same GCF, as the smaller pair 35 and 21. Each step of the algorithm keeps the GCF the same while making the numbers smaller.
15. Tiles 24 inches on a side; $7 \times 9 = 63$ tiles.

UNIT 3

Fractions

Fractions are everywhere: in recipes, on rulers and tape measures, in sharing a pan of brownies, and in dividing up a garden. In this unit you will learn what a fraction really means: part of a whole, a point on the number line, and a division. Then you will learn to simplify, compare, add, subtract, multiply and divide fractions and mixed numbers.

The unit explains *why* each rule works, not only how. You will see why fractions need a common denominator before they are added, why we multiply tops and bottoms (using an area model), and why dividing by a fraction is the same as multiplying by its reciprocal. The work in Unit 2 on factors, GCF and LCM is used all through this unit: the GCF simplifies fractions and the LCM gives the least common denominator.

Parents: this unit is done by hand, without a calculator. Fractions are the foundation for decimals (Unit 4), ratios and proportions (Unit 7), percents (Unit 8) and solving equations in algebra, so steady practice here pays off for years.

PREREQUISITES

- Multiplication and division facts through 12×12
- Divisibility rules (Lesson 2.1)
- Finding the greatest common factor (Lessons 2.4 and 2.5)
- Finding the least common multiple (Lesson 2.6)
- Long division with remainders

MASTERY CHECKLIST

- ☐ I can explain a fraction as part of a whole, as a point on the number line, and as a division.
- ☐ I can build equivalent fractions and fill in a missing numerator or denominator.
- ☐ I can write a fraction in lowest terms using the GCF.
- ☐ I can compare and order fractions using common denominators or cross products, and explain why cross products work.
- ☐ I can add and subtract fractions using the least common denominator and explain why a common denominator is needed.
- ☐ I can convert between mixed numbers and improper fractions.
- ☐ I can add and subtract mixed numbers, including regrouping.
- ☐ I can multiply fractions and mixed numbers, canceling first, and explain the area model.
- ☐ I can divide fractions and mixed numbers and explain why we multiply by the reciprocal.
- ☐ I can choose the right operation for a fraction word problem, scale a recipe, and check that my answer makes sense.

LESSONS IN THIS UNIT

3.1 Fractions and Equivalent Fractions

2 sessions of 45 minutes

3.2	**Simplifying and Comparing Fractions**	2 sessions of 45 minutes
3.3	**Adding and Subtracting Fractions**	2 sessions of 45 minutes
3.4	**Mixed Numbers**	2 sessions of 45 minutes
3.5	**Multiplying Fractions**	2 sessions of 45 minutes
3.6	**Dividing Fractions**	2 sessions of 45 minutes
3.7	**Fraction Word Problems**	2 sessions of 45 minutes

Unit 3 test

You have 60 minutes. Do not use a calculator. Show your work for every problem. Write all fraction answers in lowest terms, and write answers greater than 1 as mixed numbers unless the problem says otherwise. Include units in answers to word problems.

1. **5 pts**
   Fill in the box: $\frac{4}{9} = \frac{\square}{36}$.

2. **5 pts**
   Write $5 \div 8$ as a fraction.

3. **6 pts**
   Write $\frac{54}{72}$ in lowest terms.

4. **6 pts**
   Order from least to greatest: $\frac{3}{5}, \frac{4}{7}, \frac{5}{9}$.

5. **6 pts**
   Add: $\frac{5}{8} + \frac{1}{6}$.

6. **6 pts**

   Subtract: $\frac{9}{10} - \frac{2}{5}$.

7. **7 pts**

   Subtract: $6\frac{1}{5} - 2\frac{3}{4}$.

8. **7 pts**

   Add: $3\frac{5}{6} + 4\frac{3}{8}$.

9. **7 pts**

   Multiply: $\frac{4}{9} \times \frac{15}{16}$.

10. **7 pts**

    Multiply: $1\frac{3}{5} \times 2\frac{1}{2}$.

11. **7 pts**

    Divide: $\frac{5}{6} \div \frac{10}{9}$.

12. **8 pts**

    A rope is $13\frac{1}{2}$ feet long. Henry cuts it into pieces that are each $2\frac{1}{4}$ feet long. How many pieces does he get?

13. **8 pts**

    A rice recipe serves 4 people and uses $\frac{2}{3}$ cup of rice. How much rice is needed to serve 10 people?

14. **8 pts**

    Samuel earned \$84 doing farm chores. He saves $\frac{1}{4}$ of it. Then he spends $\frac{2}{3}$ of the rest on tools. How much money does he have left?

15. **7 pts**

    Grace says $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$. Explain why she is wrong and give the correct sum.

ANSWER KEY

1. 16
2. $\frac{5}{8}$
3. The GCF is 18, so $\frac{54}{72} = \frac{3}{4}$.
4. $\frac{5}{9}, \frac{4}{7}, \frac{3}{5}$
5. $\frac{15}{24} + \frac{4}{24} = \frac{19}{24}$
6. $\frac{9}{10} - \frac{4}{10} = \frac{5}{10} = \frac{1}{2}$
7. $5\frac{24}{20} - 2\frac{15}{20} = 3\frac{9}{20}$
8. $3\frac{20}{24} + 4\frac{9}{24} = 7\frac{29}{24} = 8\frac{5}{24}$
9. $\frac{1}{3} \times \frac{5}{4} = \frac{5}{12}$
10. $\frac{8}{5} \times \frac{5}{2} = 4$
11. $\frac{5}{6} \times \frac{9}{10} = \frac{3}{4}$
12. $\frac{27}{2} \times \frac{4}{9} = 6$ pieces
13. Scale factor $\frac{10}{4} = \frac{5}{2}$; $\frac{5}{2} \times \frac{2}{3} = \frac{5}{3} = 1\frac{2}{3}$ cups
14. Saves \$21; rest \$63; tools \$42; left \$21
15. Model answer: Halves and thirds are different-size pieces, so they cannot be counted together until they are rewritten as the same size. Adding the denominators is wrong because the denominator only names the size of the pieces. Also, $\frac{2}{5}$ is less than $\frac{1}{2}$ alone, so it cannot be the sum. Correct: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

3.1 Fractions and Equivalent Fractions

OBJECTIVES

- Describe a fraction as part of a whole, as a point on the number line, and as a division.
- Build equivalent fractions by multiplying or dividing the numerator and denominator by the same number.
- Find the missing number that makes two fractions equivalent.

WATCH

What Are Fractions? - Math Antics

mathantics · 12:05

https://www.youtube.com/watch?v=I1u3pM9g9o8

> **WHEN TO WATCH**
> Before the Core ideas: numerator, denominator, parts of a whole, and fractions as division.

BACKUP VIDEO

Equivalent & Simplified Fractions

Eddie Woo · 3:36

https://www.youtube.com/watch?v=Ue1_s85G0JU

Before Example 1: building equivalent fractions.

KEY TERMS

fraction

A number that names equal parts of a whole, written with one number over another, like $\frac{3}{4}$.

numerator

The top number of a fraction. It tells how many of the equal parts you have.

denominator

The bottom number of a fraction. It tells how many equal parts make one whole. It can never be 0.

unit fraction

A fraction with numerator 1, like $\frac{1}{5}$. It names just one of the equal parts.

equivalent fractions

Fractions that look different but name the same amount, like $\frac{1}{2}$ and $\frac{3}{6}$.

CORE IDEAS

A fraction names equal parts of a whole. In $\frac{3}{4}$, the denominator 4 says the whole is cut into 4 equal parts, and the numerator 3 says you have 3 of those parts. If a pie is cut into 4 equal slices and you eat 3, you ate $\frac{3}{4}$ of the pie. The parts must be equal, or the fraction does not mean anything.

A fraction is also a point on the number line. To find $\frac{3}{4}$, cut the space from 0 to 1 into 4 equal steps. Each step is $\frac{1}{4}$. Count 3 steps from 0 and you land on $\frac{3}{4}$. You can keep counting past 1: the 5th step is $\frac{5}{4}$, so fractions can be greater than 1.

A fraction is also a division: $\frac{3}{4} = 3 \div 4$. Here is why. If 3 loaves of bread are shared equally by 4 people, cut each loaf into 4 equal pieces. Each person gets one piece from each loaf, which is 3 pieces of size $\frac{1}{4}$, or $\frac{3}{4}$ of a loaf. So the fraction bar means "divided by."

Equivalent fractions name the same amount. Picture a candy bar cut into 2 equal pieces; you take 1, which is $\frac{1}{2}$. Now cut every piece into 3 smaller pieces. The bar has $2 \times 3 = 6$ pieces, and your share has $1 \times 3 = 3$ pieces. You still have the same amount of candy, so $\frac{1}{2} = \frac{3}{6}$. Cutting every piece into the same number of smaller pieces multiplies both the numerator and the denominator by that number.

So to build an equivalent fraction, multiply the top and the bottom by the same whole number (not 0). You may also divide the top and bottom by the same number, which glues small pieces back into bigger ones. To fill in a missing number, ask: "What did the denominator get multiplied by?" Then do the same thing to the numerator.

Never add the same number to the top and bottom. $\frac{1}{2}$ and $\frac{2}{3}$ are not equal: on the number line, $\frac{2}{3}$ is farther from 0. Only multiplying or dividing both parts by the same number keeps the amount the same.

KEY FORMULAS

$$\frac{a}{b} = a \div b$$

$$\frac{a}{b} = \frac{a \times k}{b \times k} \quad (k \text{ a whole number, not } 0)$$

From history. The way we write fractions, one number over another with a bar between, was used by Arabic-speaking mathematicians in the Middle Ages. Fibonacci's book *Liber Abaci* (1202), which taught Europeans to use Hindu-Arabic numerals, also used the fraction bar and helped it spread across Europe.

Practice habit. Recite fraction families until they are automatic: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$, then do the same for $\frac{1}{3}$ and $\frac{1}{4}$. Each time, say what number both parts were multiplied by.

WORKED EXAMPLES

Example 1. Fill in the box so the fractions are equivalent: $\frac{3}{5} = \frac{\square}{20}$.

Show solution

1. Ask what the denominator was multiplied by: $5 \times 4 = 20$.
2. Multiply the numerator by the same number: $3 \times 4 = 12$.
3. So $\frac{3}{5} = \frac{12}{20}$.
4. Check: $\frac{3}{5}$ means 3 of 5 equal parts. Cutting each part into 4 pieces gives 20 pieces in all and 12 pieces in your share, so the amount has not changed.

Answer: 12

Example 2. The Miller family cuts a pie into 6 equal slices and eats 4 of them. Mr. Miller says they ate $\frac{2}{3}$ of the pie. Is he right?

Show solution

1. The family ate 4 of 6 equal slices, which is $\frac{4}{6}$ of the pie.
2. Build an equivalent fraction from $\frac{2}{3}$ with denominator 6: $3 \times 2 = 6$.
3. Multiply the numerator by the same number: $2 \times 2 = 4$, so $\frac{2}{3} = \frac{4}{6}$.
4. Picture it: if the pie were cut into 3 big slices, eating 2 of them is the same as eating 4 of 6 half-size slices.
5. Since $\frac{4}{6} = \frac{2}{3}$, Mr. Miller is right.

Answer: Yes. $\frac{4}{6} = \frac{2}{3}$, so they ate $\frac{2}{3}$ of the pie.

PRACTICE

1. Easy Ruth cuts a pizza into 8 equal slices and eats 3 of them. What fraction of the pizza did she eat?

   Show answer

   $\frac{3}{8}$

2. Easy Fill in the box: $\frac{2}{3} = \frac{\square}{12}$.

   Show answer

   8

3. Easy Fill in the box: $\frac{5}{6} = \frac{20}{\square}$.

   Show answer

   24

4. Easy Write $7 \div 4$ as a fraction.

   Show answer

   $\frac{7}{4}$

5. **Medium** The space from 0 to 1 on a number line is cut into 5 equal steps by tick marks. What fraction is at the 3rd tick mark to the right of 0?

   Show answer

   $\frac{3}{5}$

6. **Medium** Are $\frac{4}{6}$ and $\frac{10}{15}$ equivalent fractions? (Hint: try writing each one with denominator 3.)

   Show answer

   Yes. Both equal $\frac{2}{3}$.

7. **Medium** Henry shares 3 loaves of bread equally among 4 families. What fraction of a loaf does each family get?

   Show answer

   $\frac{3}{4}$ of a loaf

8. **Medium** Fill in the box: $\frac{3}{4} = \frac{\square}{28}$.

   Show answer

   21

9. **Hard** Grace's garden has 12 equal beds. She wants tomatoes in $\frac{3}{4}$ of the beds. Use an equivalent fraction with denominator 12 to find how many beds get tomatoes.

   Show answer

   $\frac{3}{4} = \frac{9}{12}$, so 9 beds

10. **Hard** Find the fraction equivalent to $\frac{2}{7}$ whose numerator and denominator add up to 36.

    Show answer

    $\frac{8}{28}$ (numerator 8, denominator 28)

11. Hard Explain why multiplying the numerator and denominator of a fraction by the same number gives an equivalent fraction. Use $\frac{1}{3} = \frac{2}{6}$ as your example.

Show answer

Model answer: $\frac{1}{3}$ is 1 of 3 equal parts. If I cut every part into 2 smaller pieces, the whole now has 6 pieces and my share has 2 pieces. I still have the same amount, so $\frac{1}{3} = \frac{2}{6}$. Multiplying top and bottom by 2 just counts the same amount using smaller pieces.

12. Challenge A number line from 0 to 2 has each whole cut into 6 equal steps. What fraction names the 9th tick mark to the right of 0? Then write the same number as an equivalent fraction with denominator 2.

Show answer

$\frac{9}{6}$, which equals $\frac{3}{2}$

COMMON MISTAKES

- Adding the same number to the top and bottom, such as writing $\frac{2}{3} = \frac{3}{4}$. Adding changes the amount. Always multiply (or divide) both parts by the same number.
- Multiplying only the denominator, such as writing $\frac{3}{5} = \frac{3}{20}$. Whatever you do to the bottom, do to the top.
- Writing a sharing problem upside down. "3 loaves shared by 4 people" is $3 \div 4 = \frac{3}{4}$, not $\frac{4}{3}$. The thing being shared goes on top.
- Using unequal parts. A fraction only makes sense when the whole is cut into equal parts, so draw your pictures with equal pieces.

> **Make it a game · Fraction Charades.** Write 12 fractions on cards. Draw one and act it out or draw it without words: fold paper into parts and shade, or step along a floor number line. Your partner guesses the fraction, then names two equivalent fractions for it. Score a point for each correct guess and a bonus for equivalents. Switch roles after six cards.

3.2 Simplifying and Comparing Fractions

OBJECTIVES

- Simplify a fraction to lowest terms by dividing the numerator and denominator by their GCF.
- Compare and order fractions using common denominators or cross products.
- Explain why the cross-product test for comparing fractions works.

WATCH

Math Antics - Simplifying Fractions

mathantics · 8:31

https://www.youtube.com/watch?v=AtBUQH8Tkqc

> **WHEN TO WATCH**
> Before Example 1 (simplifying).

BACKUP VIDEO

Math Antics - Comparing Fractions

mathantics · 6:31

https://www.youtube.com/watch?v=KNdUJQ_qd4U

Before Example 2 (comparing fractions).

KEY TERMS

lowest terms

A fraction is in lowest terms (simplest form) when the numerator and denominator have no common factor except 1, like $\frac{3}{4}$.

simplify

To rewrite a fraction as an equivalent fraction with smaller numbers by dividing the top and bottom by a common factor.

greatest common factor (GCF)

The largest whole number that divides two numbers evenly. The GCF of 36 and 48 is 12.

common denominator

A denominator that two or more fractions share, so their pieces are the same size.

cross products
For $\frac{a}{b}$ and $\frac{c}{d}$, the products $a \times d$ and $b \times c$, found by multiplying across diagonally.

CORE IDEAS

Simplifying is building equivalent fractions in reverse. Instead of cutting pieces smaller, you glue pieces together into bigger ones. Dividing the numerator and denominator by the same number does not change the amount: $\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$.

The quickest way to reach lowest terms is to divide by the GCF in one step. For $\frac{36}{48}$, the GCF of 36 and 48 is 12, so $\frac{36}{48} = \frac{3}{4}$. You may also divide by smaller common factors several times ($\frac{36}{48} = \frac{18}{24} = \frac{9}{12} = \frac{3}{4}$); just keep going until the only common factor left is 1.

To compare fractions with the same denominator, compare the numerators: the pieces are the same size, so more pieces means more. $\frac{7}{9} > \frac{5}{9}$. When the numerators are the same, the fraction with the smaller denominator is larger, because its pieces are bigger: $\frac{3}{5} > \frac{3}{8}$.

When the denominators are different, rewrite the fractions with a common denominator so the pieces are the same size. To compare $\frac{2}{3}$ and $\frac{5}{8}$, use 24: $\frac{2}{3} = \frac{16}{24}$ and $\frac{5}{8} = \frac{15}{24}$. Since 16 > 15, $\frac{2}{3} > \frac{5}{8}$.

Cross products are a shortcut for exactly that. To compare $\frac{a}{b}$ and $\frac{c}{d}$, use the common denominator $b \times d$. The first fraction becomes $\frac{a \times d}{b \times d}$ and the second becomes $\frac{c \times b}{b \times d}$. The denominators now match, so you only need to compare the new numerators, $a \times d$ and $b \times c$. Write each cross product on the same side as the fraction whose numerator it uses, and the larger cross product marks the larger fraction.

KEY FORMULAS

$$\frac{a}{b} = \frac{a \div g}{b \div g} \quad (g = \text{GCF of } a \text{ and } b)$$

$$\frac{a}{b} > \frac{c}{d} \text{ exactly when } a \times d > b \times c$$

Euclid connection. For big numbers, Euclid's method from *Elements* VII.1–2 finds the GCF quickly. For $\frac{91}{119}$: $119 = 91 \times 1 + 28$, $91 = 28 \times 3 + 7$, $28 = 7 \times 4 + 0$, so the GCF is 7 and $\frac{91}{119} = \frac{13}{17}$.

Practice habit. Before simplifying, run the divisibility rules for 2, 3, 5 and 10 on the top and bottom. For a quick comparison, use $\frac{1}{2}$ as a benchmark: double the numerator and see whether it is more or less than the denominator.

WORKED EXAMPLES

Example 1. Write $\frac{36}{48}$ in lowest terms.

Show solution

1. Find the GCF. Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36. Factors of 48: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. The GCF is 12.
2. Divide the numerator by 12: $36 \div 12 = 3$.
3. Divide the denominator by 12: $48 \div 12 = 4$.
4. So $\frac{36}{48} = \frac{3}{4}$. The only common factor of 3 and 4 is 1, so this is lowest terms.
5. Check by building back up: $3 \times 12 = 36$ and $4 \times 12 = 48$.

Answer: $\frac{3}{4}$

Example 2. At basketball practice, Samuel made 7 of his 9 free throws, and Henry made 10 of his 13. Who made the larger fraction of his free throws?

Show solution

1. Samuel made $\frac{7}{9}$ and Henry made $\frac{10}{13}$.
2. Cross product for Samuel's fraction: $7 \times 13 = 91$.
3. Cross product for Henry's fraction: $9 \times 10 = 90$.
4. Since $91 > 90$, $\frac{7}{9} > \frac{10}{13}$.
5. Check with the common denominator $9 \times 13 = 117$: $\frac{7}{9} = \frac{91}{117}$ and $\frac{10}{13} = \frac{90}{117}$. Samuel's is larger by just $\frac{1}{117}$.

Answer: Samuel, because $\frac{7}{9} > \frac{10}{13}$

PRACTICE

1. Easy Write $\frac{6}{8}$ in lowest terms.

   Show answer

   $\frac{3}{4}$

2. Easy Write $\frac{15}{25}$ in lowest terms.

   Show answer

   $\frac{3}{5}$

3. Easy Which is greater, $\frac{5}{9}$ or $\frac{7}{9}$?

   Show answer

   $\frac{7}{9}$

4. Easy Write $\frac{20}{28}$ in lowest terms.

   Show answer

   $\frac{5}{7}$

5. **Medium** Write $\frac{48}{72}$ in lowest terms using the GCF.

Show answer

The GCF is 24, so $\frac{48}{72} = \frac{2}{3}$.

6. **Medium** Use cross products to decide which is greater, $\frac{3}{4}$ or $\frac{5}{7}$.

Show answer

$3 \times 7 = 21$ and $4 \times 5 = 20$, so $\frac{3}{4}$ is greater.

7. **Medium** A bushel basket holds 40 apples, and 16 of them are red. What fraction of the apples are red? Write it in lowest terms.

Show answer

$$\frac{16}{40} = \frac{2}{5}$$

8. **Medium** Is $\frac{14}{21}$ in lowest terms? If not, simplify it.

Show answer

No. The GCF is 7, so $\frac{14}{21} = \frac{2}{3}$.

9. **Hard** On Saturday morning Samuel ran $\frac{2}{3}$ mile, Ruth ran $\frac{5}{8}$ mile, and Henry ran $\frac{7}{12}$ mile. List the three distances from shortest to longest.

Show answer

With denominator 24: $\frac{16}{24}$, $\frac{15}{24}$, $\frac{14}{24}$. Shortest to longest: $\frac{7}{12}$ (Henry), $\frac{5}{8}$ (Ruth), $\frac{2}{3}$ (Samuel).

10. **Hard** In the workshop, one board is $\frac{7}{8}$ inch thick and another is $\frac{5}{6}$ inch thick. Which board is thicker?

Show answer

$7 \times 6 = 42$ and $8 \times 5 = 40$, so the $\frac{7}{8}$-inch board is thicker.

11. Hard Use the Euclidean algorithm to find the GCF of 85 and 102, then write $\frac{85}{102}$ in lowest terms.

Show answer

$102 = 85 \times 1 + 17$ and $85 = 17 \times 5 + 0$, so the GCF is 17 and $\frac{85}{102} = \frac{5}{6}$.

12. Challenge Explain why the cross-product test works. Use $\frac{2}{5}$ and $\frac{3}{7}$ as your example.

Show answer

Model answer: Rewrite both fractions with the common denominator $5 \times 7 = 35$. Then $\frac{2}{5} = \frac{2 \times 7}{35} = \frac{14}{35}$ and $\frac{3}{7} = \frac{3 \times 5}{35} = \frac{15}{35}$. With the same denominator, the larger numerator is the larger fraction. The new numerators, 14 and 15, are exactly the cross products, so comparing cross products is the same as comparing with a common denominator. Here $\frac{3}{7}$ is larger.

13. Challenge Find a fraction with denominator 30 that is between $\frac{3}{5}$ and $\frac{2}{3}$.

Show answer

$\frac{3}{5} = \frac{18}{30}$ and $\frac{2}{3} = \frac{20}{30}$, so $\frac{19}{30}$ is between them.

COMMON MISTAKES

- Stopping too soon, such as simplifying $\frac{36}{48}$ to $\frac{18}{24}$ and quitting. Check that the top and bottom have no common factor but 1, or divide by the GCF in one step.
- Thinking a bigger denominator means a bigger fraction. $\frac{1}{8}$ is smaller than $\frac{1}{3}$ because eighths are smaller pieces. Picture the pieces before you decide.
- Mixing up which cross product goes with which fraction. Write each product next to the fraction whose numerator you used; that fraction is larger when its product is larger.
- Subtracting instead of dividing when simplifying, such as turning $\frac{6}{8}$ into $\frac{4}{6}$ by taking 2 from each. Only dividing both by the same number keeps the amount the same.

> **Make it a game · Simplify Showdown.** Deal yourselves each five cards with fractions like 18/24 or 15/45. Flip one at a time, and each player simplifies it aloud, naming the GCF used. First to the right answer takes the card. Then lay your cards in order from least to greatest. Whoever has the longer correctly ordered row wins.

3.3 Adding and Subtracting Fractions

OBJECTIVES

- Explain why fractions must have a common denominator before they can be added or subtracted.
- Find the least common denominator of two or three fractions.
- Add and subtract fractions and write the answer in lowest terms.

WATCH

Math Antics - Adding and Subtracting Fractions

mathantics · 4:22

https://www.youtube.com/watch?v=5juto2ze8Lg

> **WHEN TO WATCH**
> After the Core ideas, before Example 1.

BACKUP VIDEO

Math Antics - Common Denominator LCD

mathantics · 5:23

https://www.youtube.com/watch?v=pZEmFSP3Z0I

Before Example 2: finding the least common denominator.

KEY TERMS

like fractions

Fractions with the same denominator, like $\frac{2}{7}$ and $\frac{3}{7}$.

unlike fractions

Fractions with different denominators, like $\frac{1}{4}$ and $\frac{2}{3}$.

least common denominator (LCD)

The smallest number that all the denominators divide into evenly. It is the LCM of the denominators.

least common multiple (LCM)

The smallest number that is a multiple of each of the given numbers. The LCM of 4 and 6 is 12.

CORE IDEAS

Adding like fractions is counting pieces of the same size. Two sevenths plus three sevenths is five sevenths, just as 2 apples plus 3 apples is 5 apples: $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$. Add

the numerators and keep the denominator. The denominator is the *name* of the piece (sevenths), so it does not get added.

Unlike fractions are pieces of different sizes, and you cannot count them together directly. What is one fourth plus two thirds? "Three somethings" makes no sense, because fourths and thirds are different sizes. First cut both into the same size of piece.

To do that, find a common denominator: a number both denominators divide into. For fourths and thirds, use twelfths, since 12 is the LCM of 4 and 3. Then $\frac{1}{4} = \frac{3}{12}$ and $\frac{2}{3} = \frac{8}{12}$, and $\frac{3}{12} + \frac{8}{12} = \frac{11}{12}$. Any common denominator works (24 would also work), but the least common denominator keeps the numbers small.

Subtraction works the same way: rewrite with a common denominator, then subtract the numerators. $\frac{3}{4} - \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$.

Always finish by simplifying, and check that your answer is reasonable. $\frac{1}{4} + \frac{2}{3}$ should be a little less than 1, because $\frac{2}{3}$ needs $\frac{1}{3}$ more to make 1 and $\frac{1}{4}$ is less than $\frac{1}{3}$. The answer $\frac{11}{12}$ fits.

KEY FORMULAS

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

$$\text{LCD} = \text{LCM of the denominators}$$

Practice habit. Memorize the common LCDs so you can find them instantly: 2 and 3 → 6, 3 and 4 → 12, 4 and 6 → 12, 6 and 8 → 24, 5 and 3 → 15, 8 and 12 → 24. Before you compute, estimate the answer using the benchmarks 0, $\frac{1}{2}$ and 1.

WORKED EXAMPLES

Example 1. Add: $\frac{1}{4} + \frac{2}{3}$.

Show solution

1. Estimate: $\frac{1}{4}$ plus a bit more than $\frac{1}{2}$ is a little less than 1.
2. The LCD of 4 and 3 is 12.
3. Rewrite: $\frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12}$.
4. Rewrite: $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.
5. Add the numerators: $\frac{3}{12} + \frac{8}{12} = \frac{11}{12}$.
6. Check: $\frac{11}{12}$ is in lowest terms and is a little less than 1, matching the estimate.

Answer: $\frac{11}{12}$

Example 2. Ruth has $\frac{5}{6}$ cup of milk. She uses $\frac{3}{8}$ cup in a batch of pancakes. How much milk is left?

Show solution

1. "How much is left" means subtract: $\frac{5}{6} - \frac{3}{8}$.
2. Multiples of 8: 8, 16, 24. The first one that 6 divides into is 24, so the LCD is 24.
3. Rewrite: $\frac{5}{6} = \frac{20}{24}$ and $\frac{3}{8} = \frac{9}{24}$.
4. Subtract the numerators: $\frac{20}{24} - \frac{9}{24} = \frac{11}{24}$.
5. Check by adding back: $\frac{11}{24} + \frac{9}{24} = \frac{20}{24} = \frac{5}{6}$.

Answer: $\frac{11}{24}$ cup

PRACTICE

1. Easy Add: $\frac{2}{9} + \frac{5}{9}$.

   Show answer

   $\frac{7}{9}$

2. Easy Subtract and simplify: $\frac{7}{10} - \frac{3}{10}$.

Show answer

$$\frac{4}{10} = \frac{2}{5}$$

3. Easy Add: $\frac{1}{2} + \frac{1}{3}$.

Show answer

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

4. Easy Subtract: $\frac{3}{4} - \frac{1}{8}$.

Show answer

$$\frac{6}{8} - \frac{1}{8} = \frac{5}{8}$$

5. Medium Add: $\frac{3}{8} + \frac{1}{6}$.

Show answer

$$\frac{9}{24} + \frac{4}{24} = \frac{13}{24}$$

6. Medium Subtract and simplify: $\frac{7}{12} - \frac{1}{3}$.

Show answer

$$\frac{7}{12} - \frac{4}{12} = \frac{3}{12} = \frac{1}{4}$$

7. Medium Henry walks $\frac{3}{10}$ mile to the store and then $\frac{2}{5}$ mile from the store to church. How far does he walk in all?

Show answer

$$\frac{3}{10} + \frac{4}{10} = \frac{7}{10} \text{ mile}$$

8. Medium Subtract: $\frac{4}{5} - \frac{2}{3}$.

Show answer

$$\frac{12}{15} - \frac{10}{15} = \frac{2}{15}$$

9. Hard In the Millers' garden, $\frac{1}{2}$ is planted in corn, $\frac{1}{3}$ in beans and $\frac{1}{12}$ in squash. The rest is for flowers. What fraction of the garden is for flowers?

Show answer

Planted: $\frac{6}{12} + \frac{4}{12} + \frac{1}{12} = \frac{11}{12}$. Flowers: $\frac{12}{12} - \frac{11}{12} = \frac{1}{12}$ of the garden.

10. Hard Samuel has a board $\frac{7}{8}$ yard long. He cuts off a piece $\frac{1}{6}$ yard long. How long is the board that is left? (Ignore the width of the saw cut.)

Show answer

$\frac{21}{24} - \frac{4}{24} = \frac{17}{24}$ yard

11. Hard Grace says $\frac{1}{3} + \frac{1}{4} = \frac{2}{7}$. Explain why she is wrong and give the correct sum.

Show answer

Model answer: She added the denominators, but the denominator only names the size of the pieces. Also, $\frac{2}{7}$ is less than $\frac{1}{3}$ by itself, so it cannot be the sum. Using twelfths: $\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.

12. Challenge Work from left to right and simplify: $\frac{5}{6} - \frac{3}{10} + \frac{2}{15}$.

Show answer

LCD 30: $\frac{25}{30} - \frac{9}{30} + \frac{4}{30} = \frac{20}{30} = \frac{2}{3}$

COMMON MISTAKES

- Adding the denominators, as in $\frac{1}{3} + \frac{1}{4} = \frac{2}{7}$. The denominator is the name of the piece; add only the numerators, after the pieces are the same size.
- Changing the denominator but not the numerator, as in writing $\frac{2}{3}$ as $\frac{2}{12}$. Multiply the top by the same number you used on the bottom.
- Forgetting to simplify the answer. After adding or subtracting, check whether the top and bottom share a factor, as in $\frac{3}{12} = \frac{1}{4}$.
- Skipping the estimate. A quick check against 0, $\frac{1}{2}$ and 1 catches answers that are far too big or too small.

Make it a game · Common Denominator Race. Write 10 pairs of unlike fractions on strips. Draw a strip, and both of you race to find the least common denominator and the sum or difference, writing it down. The first to show a correct, fully simplified answer wins the strip. Bonus point for explaining aloud why the denominators had to match. Play to six strips.

3.4 Mixed Numbers

OBJECTIVES

- Convert between mixed numbers and improper fractions and explain why the method works.
- Add mixed numbers, regrouping when the fraction parts add up to 1 or more.
- Subtract mixed numbers, regrouping a whole when the fraction being subtracted is larger.

WATCH

Math Antics - Mixed Numbers

mathantics · 12:47

https://www.youtube.com/watch?v=qk2oP6FZ6HA

> **WHEN TO WATCH**
> Before Example 1 (converting mixed numbers and improper fractions).

BACKUP VIDEO

Math Antics - Subtracting Mixed Numbers

mathantics · 9:13

https://www.youtube.com/watch?v=RqUPSZLwHH0

Before the subtraction example: regrouping a whole when subtracting.

KEY TERMS

proper fraction

A fraction whose numerator is smaller than its denominator, like $\frac{3}{5}$. It is less than 1.

improper fraction

A fraction whose numerator is equal to or larger than its denominator, like $\frac{7}{4}$. It is 1 or more.

mixed number

A whole number and a proper fraction written together, like $2\frac{1}{3}$, which means $2 + \frac{1}{3}$.

regrouping

Trading 1 whole for an equal number of fraction pieces (such as $1 = \frac{4}{4}$), or trading a full set of pieces for 1 whole.

CORE IDEAS

A mixed number is a whole number plus a fraction: $2\frac{1}{3}$ means $2 + \frac{1}{3}$. You see mixed numbers in recipes ($1\frac{1}{2}$ cups) and on rulers ($5\frac{3}{8}$ inches).

To change $2\frac{1}{3}$ to an improper fraction, count thirds. Each whole is $\frac{3}{3}$, so 2 wholes are $2 \times 3 = 6$ thirds. Add the 1 extra third to get 7 thirds: $2\frac{1}{3} = \frac{7}{3}$. The rule is: whole times denominator, plus numerator, over the same denominator.

To change an improper fraction to a mixed number, divide. For $\frac{17}{5}$: $17 \div 5 = 3$ remainder 2. That means 17 fifths make 3 full wholes (15 fifths) with 2 fifths left over, so $\frac{17}{5} = 3\frac{2}{5}$.

To add mixed numbers, add the whole numbers and add the fractions (using a common denominator). If the fraction part comes out to 1 or more, regroup. For $4\frac{5}{6} + 2\frac{2}{6}$: the wholes give 6 and the fractions give $\frac{7}{6} = 1\frac{1}{6}$, so the sum is $7\frac{1}{6}$.

To subtract, first get a common denominator. If the top fraction is too small to subtract from, borrow 1 whole and turn it into fraction pieces. In $5\frac{1}{4} - 2\frac{3}{4}$, you cannot take $\frac{3}{4}$ from $\frac{1}{4}$, so rewrite $5\frac{1}{4}$ as $4 + \frac{4}{4} + \frac{1}{4} = 4\frac{5}{4}$. Now $4\frac{5}{4} - 2\frac{3}{4} = 2\frac{2}{4} = 2\frac{1}{2}$. This is just like borrowing a ten in whole-number subtraction, except the whole becomes 4 fourths instead of 10 ones.

Another method that always works is to change both mixed numbers to improper fractions, add or subtract, and change back. It uses bigger numbers, but you never have to regroup.

KEY FORMULAS

$$a\tfrac{b}{c} = \frac{a \times c + b}{c}$$

$$1 = \frac{c}{c} \quad \text{(used for regrouping)}$$

> **Practice habit.** Estimate first by rounding each mixed number to the nearest whole: $5\frac{1}{4} - 2\frac{2}{3}$ is about $5 - 3 = 2$. Also practice saying conversions out loud: "two and one third is seven thirds."

WORKED EXAMPLES

Example 1. Write $\frac{23}{4}$ as a mixed number.

Show solution

1. Divide the numerator by the denominator: $23 \div 4 = 5$ remainder 3.
2. The 5 is the number of wholes, since $5 \times 4 = 20$ fourths make 5 wholes.
3. The remainder 3 is the number of fourths left over.
4. So $\frac{23}{4} = 5\frac{3}{4}$.
5. Check: $5 \times 4 + 3 = 23$, so $5\frac{3}{4} = \frac{23}{4}$.

Answer: $5\frac{3}{4}$

Example 2. Grace has $5\frac{1}{4}$ yards of ribbon. She uses $2\frac{2}{3}$ yards to decorate for a church dinner. How much ribbon is left?

Show solution

1. Estimate: about $5 - 3 = 2$ yards.
2. The LCD of 4 and 3 is 12: $5\frac{1}{4} = 5\frac{3}{12}$ and $2\frac{2}{3} = 2\frac{8}{12}$.
3. You cannot take $\frac{8}{12}$ from $\frac{3}{12}$, so regroup: $5\frac{3}{12} = 4 + \frac{12}{12} + \frac{3}{12} = 4\frac{15}{12}$.
4. Subtract the fractions: $\frac{15}{12} - \frac{8}{12} = \frac{7}{12}$.
5. Subtract the wholes: $4 - 2 = 2$. The difference is $2\frac{7}{12}$ yards.
6. Check by adding: $2\frac{7}{12} + 2\frac{8}{12} = 4\frac{15}{12} = 5\frac{3}{12} = 5\frac{1}{4}$. It also matches the estimate of about 2.

Answer: $2\frac{7}{12}$ yards

PRACTICE

1. Easy Write $2\frac{1}{3}$ as an improper fraction.

   Show answer

   $\frac{7}{3}$

2. **Easy** Write $\dfrac{17}{5}$ as a mixed number.

Show answer

$3\frac{2}{5}$

3. **Easy** Add: $1\frac{2}{7} + 3\frac{4}{7}$.

Show answer

$4\frac{6}{7}$

4. **Easy** Subtract and simplify: $5\frac{5}{8} - 2\frac{3}{8}$.

Show answer

$3\frac{2}{8} = 3\frac{1}{4}$

5. **Medium** Add: $4\frac{5}{6} + 2\frac{1}{3}$.

Show answer

$4\frac{5}{6} + 2\frac{2}{6} = 6\frac{7}{6} = 7\frac{1}{6}$

6. **Medium** Subtract: $6 - 2\frac{3}{8}$.

Show answer

$5\frac{8}{8} - 2\frac{3}{8} = 3\frac{5}{8}$

7. **Medium** Henry worked in the garden $1\frac{3}{4}$ hours on Saturday and $2\frac{1}{2}$ hours on Monday. How many hours did he work in all?

Show answer

$1\frac{3}{4} + 2\frac{2}{4} = 3\frac{5}{4} = 4\frac{1}{4}$ hours

8. **Medium** Subtract and simplify: $7\frac{1}{6} - 3\frac{1}{2}$.

Show answer

$6\frac{7}{6} - 3\frac{3}{6} = 3\frac{4}{6} = 3\frac{2}{3}$

9. **Hard** A board is $8\frac{1}{4}$ feet long. Samuel cuts off $3\frac{5}{8}$ feet. How long is the piece that is left? (Ignore the width of the saw cut.)

Show answer

$7\frac{10}{8} - 3\frac{5}{8} = 4\frac{5}{8}$ feet

10. Hard Add: $3\frac{2}{3} + 2\frac{3}{4} + 1\frac{5}{6}$.

Show answer

LCD 12: $3\frac{8}{12} + 2\frac{9}{12} + 1\frac{10}{12} = 6\frac{27}{12} = 8\frac{3}{12} = 8\frac{1}{4}$

11. Hard Ruth has $4\frac{1}{4}$ cups of flour. Her bread recipe uses $2\frac{1}{3}$ cups. How much flour will she have left?

Show answer

$3\frac{15}{12} - 2\frac{4}{12} = 1\frac{11}{12}$ cups

12. Hard To find $5\frac{1}{4} - 2\frac{3}{4}$, Grace rewrites $5\frac{1}{4}$ as $4\frac{5}{4}$. Explain why these two numbers are equal and why she needs to do this.

Show answer

Model answer: She took 1 of the 5 wholes and cut it into 4 fourths. That leaves 4 wholes, and the fourths become $\frac{4}{4} + \frac{1}{4} = \frac{5}{4}$, so the amount is the same. She needs to do it because $\frac{1}{4}$ is too small to take $\frac{3}{4}$ away from, but $\frac{5}{4}$ is not. Then $4\frac{5}{4} - 2\frac{3}{4} = 2\frac{2}{4} = 2\frac{1}{2}$.

13. Challenge The Miller family is painting a fence 20 feet long. They paint $6\frac{2}{3}$ feet on Monday and $7\frac{3}{4}$ feet on Tuesday. How many feet are still unpainted?

Show answer

Painted: $6\frac{8}{12} + 7\frac{9}{12} = 13\frac{17}{12} = 14\frac{5}{12}$ feet. Unpainted: $19\frac{12}{12} - 14\frac{5}{12} = 5\frac{7}{12}$ feet.

COMMON MISTAKES

- Regrouping 1 whole as 10 pieces, as in whole-number subtraction. In $5\frac{1}{4}$, one whole is $\frac{4}{4}$, not $\frac{10}{4}$; the whole becomes as many pieces as the denominator says.
- Subtracting the smaller fraction from the larger one no matter where it is, as in $5\frac{1}{4} - 2\frac{3}{4} = 3\frac{2}{4}$. When the top fraction is smaller, regroup first.
- Leaving an improper fraction inside a mixed number, as in $6\frac{7}{6}$. Turn the extra pieces into a whole: $6\frac{7}{6} = 7\frac{1}{6}$.
- Converting $2\frac{1}{3}$ as $\frac{2+1}{3}$ or $\frac{2\times 1}{3}$. Multiply the whole by the denominator, then add the numerator: $\frac{2\times 3+1}{3} = \frac{7}{3}$.

Make it a game · Improper Switcheroo. Make 12 cards, six with mixed numbers and six with improper fractions. Shuffle them face down. Flip one and convert it aloud in 20 seconds, explaining each step. Parent sneaks in one wrong step in their own demonstration for you to catch. Score a point for every correct conversion and a point for every caught mistake.

3.5 Multiplying Fractions

OBJECTIVES

- Explain with an area model why we multiply numerators and multiply denominators.
- Multiply fractions and whole numbers, canceling common factors before multiplying.
- Multiply mixed numbers by first writing them as improper fractions.

WATCH

Math Antics - Multiplying Fractions

mathantics · 5:42

https://www.youtube.com/watch?v=qmfXyR7Z6Lk

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Multiplying fractions: why does the rule work?

Eddie Woo · 11:40

https://www.youtube.com/watch?v=lO5KTovme18

After the area-model paragraph: why the multiply-across rule works.

KEY TERMS

product

The answer to a multiplication problem.

area model

A picture of a multiplication as a rectangle: the two numbers are the side lengths and the product is the area.

"of"

In fraction problems, "of" means multiply: $\frac{1}{2}$ of 10 means $\frac{1}{2} \times 10 = 5$.

canceling

Dividing a numerator and a denominator by a common factor before you multiply, so the numbers stay small.

Multiplying a whole number by a fraction is repeated addition: $3 \times \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5}$. Only the numerator is multiplied, because you are counting more fifths, not changing their size.

A fraction of a fraction uses the word "of", and "of" means times. Half of a third of a cake is $\frac{1}{2} \times \frac{1}{3}$. Cut a third into 2 equal pieces, and each piece is $\frac{1}{6}$ of the cake, so $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

The area model shows why the rule works. Draw a square with sides 1 unit long. Cut it into 3 equal columns and shade 2 of them to show $\frac{2}{3}$. Cut it into 4 equal rows and shade 3 of them to show $\frac{3}{4}$. The square now has $3 \times 4 = 12$ small boxes, so each box is $\frac{1}{12}$; that is why the denominators multiply. The part shaded both ways is 2 columns by 3 rows, or $2 \times 3 = 6$ boxes; that is why the numerators multiply. So $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$.

Canceling saves work. In $\frac{2}{3} \times \frac{3}{4}$, the 3 on top and the 3 on the bottom divide to 1, and the 2 on top and the 4 on the bottom divide to 1 and 2. You get $\frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$ right away. You may cancel any numerator with any denominator, because the result is the same as simplifying at the end.

To multiply mixed numbers, change them to improper fractions first. Do not multiply the wholes and the fractions separately: $2\frac{1}{2} \times 2\frac{1}{2}$ is not $4\frac{1}{4}$. It is $\frac{5}{2} \times \frac{5}{2} = \frac{25}{4} = 6\frac{1}{4}$. A square $2\frac{1}{2}$ feet on a side has a 2 by 2 part, two 2 by $\frac{1}{2}$ strips, and a small $\frac{1}{2}$ by $\frac{1}{2}$ corner, and the separate method forgets the two strips.

Multiplying by a fraction less than 1 makes a number smaller, because you are taking only part of it. $\frac{3}{4}$ of 20 is 15, which is less than 20. Use this to check whether your answer is reasonable.

KEY FORMULAS

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$n \times \frac{a}{b} = \frac{n \times a}{b}$$

Practice habit. Find a fraction of a whole number mentally by dividing by the denominator, then multiplying by the numerator: $\frac{3}{4}$ of 24 is $24 \div 4 = 6$, then $6 \times 3 = 18$. Say "of means times" each time you see the word.

WORKED EXAMPLES

Example 1. Multiply: $\frac{2}{3} \times \frac{3}{4}$. Use an area model, then check by canceling.

Show solution

1. Draw a unit square. Cut it into 3 equal columns and shade 2 of them for $\frac{2}{3}$.
2. Cut the same square into 4 equal rows and shade 3 of them for $\frac{3}{4}$.
3. Count all the small boxes: $3 \times 4 = 12$.
4. Count the boxes shaded both ways: $2 \times 3 = 6$. The product is $\frac{6}{12} = \frac{1}{2}$.
5. Check by canceling: the 3s cancel to 1, and 2 with 4 becomes 1 with 2, giving $\frac{1 \times 1}{1 \times 2} = \frac{1}{2}$.

Answer: $\frac{1}{2}$

Example 2. Grace's muffin recipe uses $2\frac{1}{4}$ cups of flour for one batch. She wants to make $1\frac{2}{3}$ batches for a bake sale. How much flour does she need?

Show solution

1. Estimate: about $2 \times 2 = 4$ cups.
2. Write as improper fractions: $2\frac{1}{4} = \frac{9}{4}$ and $1\frac{2}{3} = \frac{5}{3}$.
3. Set up the product: $\frac{9}{4} \times \frac{5}{3}$.
4. Cancel 9 on top with 3 on the bottom: $9 \div 3 = 3$ and $3 \div 3 = 1$.
5. Multiply: $\frac{3 \times 5}{4 \times 1} = \frac{15}{4}$.
6. Change to a mixed number: $15 \div 4 = 3$ remainder 3, so $3\frac{3}{4}$ cups.
7. Check: $3\frac{3}{4}$ is close to the estimate of 4.

Answer: $3\frac{3}{4}$ cups

PRACTICE

1. Easy Multiply: $\frac{1}{2} \times \frac{3}{5}$.

   Show answer

   $\frac{3}{10}$

2. Easy Multiply: $\frac{2}{3} \times \frac{4}{7}$.

   Show answer

   $\frac{8}{21}$

3. Easy Multiply: $4 \times \frac{2}{9}$.

   Show answer

   $\frac{8}{9}$

4. Easy There are 20 students in a homeschool co-op, and $\frac{3}{4}$ of them sing in the choir. How many students sing in the choir?

   Show answer

   15 students

5. **Medium** Cancel first, then multiply: $\frac{5}{6} \times \frac{3}{10}$.

Show answer

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

6. **Medium** Cancel first, then multiply: $\frac{7}{8} \times \frac{4}{21}$.

Show answer

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

7. **Medium** Henry's garden plot is $\frac{2}{3}$ acre. He plants potatoes on $\frac{3}{5}$ of it. How many acres are planted with potatoes?

Show answer

$$\frac{2}{3} \times \frac{3}{5} = \frac{6}{15} = \frac{2}{5} \text{ acre}$$

8. **Medium** Multiply: $2\frac{1}{2} \times 1\frac{1}{5}$.

Show answer

$$\frac{5}{2} \times \frac{6}{5} = \frac{30}{10} = 3$$

9. **Hard** Multiply: $3\frac{1}{3} \times 2\frac{1}{4}$.

Show answer

$$\frac{10}{3} \times \frac{9}{4} = \frac{5 \times 3}{1 \times 2} = \frac{15}{2} = 7\frac{1}{2}$$

10. **Hard** Samuel has a board $5\frac{1}{4}$ feet long. He uses $\frac{2}{3}$ of it to make a shelf. How long is the shelf?

Show answer

$$\frac{2}{3} \times \frac{21}{4} = \frac{7}{2} = 3\frac{1}{2} \text{ feet}$$

11. Hard Ruth says $2\frac{1}{2} \times 2\frac{1}{2} = 4\frac{1}{4}$ because $2 \times 2 = 4$ and $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Explain what is wrong and give the correct product.

Show answer

Model answer: Picture a square $2\frac{1}{2}$ feet on each side. It has a 2 by 2 part (area 4), two strips that are 2 by $\frac{1}{2}$ (area 1 each), and a $\frac{1}{2}$ by $\frac{1}{2}$ corner (area $\frac{1}{4}$). Ruth forgot the two strips. The correct product is $\frac{5}{2} \times \frac{5}{2} = \frac{25}{4} = 6\frac{1}{4}$.

12. Challenge Multiply: $\frac{3}{4} \times \frac{8}{9} \times 1\frac{1}{2}$.

Show answer

$$\frac{3}{4} \times \frac{8}{9} \times \frac{3}{2} = \frac{72}{72} = 1$$

COMMON MISTAKES

- Finding a common denominator before multiplying. That step is for adding and subtracting only. To multiply, just multiply top times top and bottom times bottom.
- Multiplying the whole number by both the numerator and the denominator, as in $4 \times \frac{2}{9} = \frac{8}{36}$. Write the whole number as $\frac{4}{1}$ first so you see that only the numerator is multiplied.
- Multiplying the parts of mixed numbers separately, as in $2\frac{1}{2} \times 2\frac{1}{2} = 4\frac{1}{4}$. Always change mixed numbers to improper fractions first.
- Canceling two numerators with each other. You may cancel only a numerator with a denominator.

> **Make it a game · Area Model Draw.** Draw it to prove it. Parent calls a product such as 2/3 x 3/4. You quickly sketch a unit square, shade the area model, and read the product from the picture. Then multiply on paper, canceling first, and check that both agree. Score two points for matching answers. Race for six correct rounds, then swap with the parent.

3.6 Dividing Fractions

OBJECTIVES

- Explain why dividing by a fraction is the same as multiplying by its reciprocal.
- Divide fractions, whole numbers and mixed numbers, and check by multiplying.
- Solve "how many groups?" and sharing word problems with fraction division.

WATCH

Math Antics - Dividing Fractions

mathantics · 5:13

https://www.youtube.com/watch?v=4lkq3DgvmJo

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Dividing fractions: why do we "keep change flip"?

Eddie Woo · 6:29

https://www.youtube.com/watch?v=_jFWu_whDjI

After Example 1: why 'keep, change, flip' (multiplying by the reciprocal) works.

KEY TERMS

reciprocal

The number you get by flipping a fraction upside down. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. A number times its reciprocal is 1.

dividend

The number being divided. In $6 \div 2 = 3$, the dividend is 6.

divisor

The number you divide by. In $6 \div 2 = 3$, the divisor is 2.

quotient

The answer to a division problem. In $6 \div 2 = 3$, the quotient is 3.

CORE IDEAS

Division asks "how many groups?" $12 \div 3$ asks how many 3s fit in 12. In the same way, $3 \div \frac{1}{4}$ asks how many fourths fit in 3. Each whole holds 4 fourths, so 3 wholes

hold $3 \times 4 = 12$ fourths. Dividing by $\frac{1}{4}$ gave the same answer as multiplying by 4.

Try one more: $\frac{3}{4} \div \frac{1}{8}$ asks how many eighths fit in $\frac{3}{4}$. Since $\frac{3}{4} = \frac{6}{8}$, the answer is 6. And $\frac{3}{4} \times 8 = \frac{24}{4} = 6$, so once again dividing by a fraction matched multiplying by its flip.

Here is why this always works. If you multiply both numbers in a division by the same number, the quotient does not change: $12 \div 4 = 3$ and $24 \div 8 = 3$. So in $\frac{2}{5} \div \frac{3}{4}$, multiply both numbers by $\frac{4}{3}$, the reciprocal of the divisor. The divisor becomes $\frac{3}{4} \times \frac{4}{3} = 1$, and the problem becomes $\left(\frac{2}{5} \times \frac{4}{3}\right) \div 1$. Dividing by 1 changes nothing, so the answer is just $\frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$.

That gives the rule: to divide by a fraction, multiply by its reciprocal. Keep the first fraction, change $\div$ to $\times$, and flip only the second fraction. After flipping, you may cancel as usual. A whole number like 7 is $\frac{7}{1}$, so its reciprocal is $\frac{1}{7}$.

For mixed numbers, change them to improper fractions first, then multiply by the reciprocal of the divisor. Always check by multiplying: the quotient times the divisor should give back the dividend.

Dividing by a number less than 1 gives an answer bigger than the number you started with, because many small pieces fit inside it. $5 \div \frac{1}{2} = 10$, since 10 halves make 5. If your quotient seems too small, check that you flipped the right fraction.

KEY FORMULAS

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

$$\frac{c}{d} \times \frac{d}{c} = 1$$

Practice habit. Before you compute, ask "Is the divisor more or less than 1?" If it is less than 1, the answer will be bigger than the dividend; if more than 1, smaller. Then check every quotient by multiplying it by the divisor.

WORKED EXAMPLES

Example 1. Divide: $\frac{3}{4} \div \frac{1}{8}$.

Show solution

1. Think: how many eighths fit in $\frac{3}{4}$? Since $\frac{3}{4} = \frac{6}{8}$, expect 6.
2. Multiply by the reciprocal of $\frac{1}{8}$, which is $\frac{8}{1}$: $\frac{3}{4} \times \frac{8}{1}$.
3. Cancel 8 with 4: $8 \div 4 = 2$ and $4 \div 4 = 1$.
4. Multiply: $\frac{3 \times 2}{1 \times 1} = 6$.
5. Check: $6 \times \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$.

Answer: 6

Example 2. Ruth has $8\frac{3}{4}$ cups of flour. Each loaf of bread needs $1\frac{1}{4}$ cups. How many loaves can she make?

Show solution

1. "How many loaves" asks how many groups of $1\frac{1}{4}$ fit in $8\frac{3}{4}$, so divide.
2. Estimate: about $9 \div 1 = 9$, but a little less because each loaf needs more than 1 cup.
3. Write as improper fractions: $8\frac{3}{4} = \frac{35}{4}$ and $1\frac{1}{4} = \frac{5}{4}$.
4. Multiply by the reciprocal: $\frac{35}{4} \times \frac{4}{5}$.
5. Cancel the 4s, and 35 with 5 gives 7: $\frac{7}{1} \times \frac{1}{1} = 7$.
6. Check: $7 \times 1\frac{1}{4} = 7 + \frac{7}{4} = 7 + 1\frac{3}{4} = 8\frac{3}{4}$.

Answer: 7 loaves

PRACTICE

1. Easy Write the reciprocal of $\frac{5}{8}$.

   Show answer

   $\frac{8}{5}$

2. Easy Divide: $4 \div \frac{1}{3}$.

Show answer

$4 \times 3 = 12$

3. Easy Divide: $\frac{2}{5} \div \frac{3}{4}$.

Show answer

$\frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$

4. Easy Divide: $\frac{5}{6} \div \frac{1}{6}$.

Show answer

5

5. Medium Divide: $\frac{3}{8} \div \frac{9}{10}$.

Show answer

$\frac{3}{8} \times \frac{10}{9} = \frac{5}{12}$

6. Medium Divide: $\frac{7}{9} \div 7$.

Show answer

$\frac{7}{9} \times \frac{1}{7} = \frac{1}{9}$

7. Medium Grace has a ribbon 6 yards long. She cuts it into pieces that are each $\frac{3}{4}$ yard long. How many pieces does she get?

Show answer

$6 \times \frac{4}{3} = 8$ pieces

8. Medium Divide: $2\frac{2}{3} \div 1\frac{1}{3}$.

Show answer

$\frac{8}{3} \times \frac{3}{4} = 2$

9. Hard Divide: $3\frac{1}{3} \div 2\frac{1}{2}$.

Show answer

$\frac{10}{3} \times \frac{2}{5} = \frac{4}{3} = 1\frac{1}{3}$

10. Hard Samuel has a board $10\frac{1}{2}$ feet long. He wants to cut it into shelves that are each $1\frac{3}{4}$ feet long. How many shelves can he cut? (Ignore the width of the saw cuts.)

Show answer

$\frac{21}{2} \times \frac{4}{7} = 6$ shelves

11. Hard Ruth has $\frac{3}{4}$ of a pan of brownies left. She shares it equally among 6 people. What fraction of a whole pan does each person get?

Show answer

$\frac{3}{4} \div 6 = \frac{3}{4} \times \frac{1}{6} = \frac{1}{8}$ of a pan

12. Hard Henry is surprised that $5 \div \frac{1}{2} = 10$, because he thought dividing always makes a number smaller. Explain why the answer is bigger than 5.

Show answer

Model answer: $5 \div \frac{1}{2}$ asks how many halves fit in 5. Each whole holds 2 halves, so 5 wholes hold 10 halves. When you divide by a number less than 1, the pieces are small, so more of them fit, and the answer is bigger than the number you started with.

13. Challenge Find the missing number: $\square \div \frac{3}{4} = 2\frac{2}{3}$.

Show answer

Division is undone by multiplication: $2\frac{2}{3} \times \frac{3}{4} = \frac{8}{3} \times \frac{3}{4} = 2$. The missing number is 2. Check: $2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$.

COMMON MISTAKES

- Flipping the first fraction instead of the divisor. Keep the first number as it is and flip only the number you are dividing by.
- Canceling diagonally before flipping, as in canceling the 3 and the 6 in $\frac{3}{4} \div \frac{1}{6}$, which gives the wrong answer $\frac{1}{2}$. Rewrite as a multiplication first ($\frac{3}{4} \times \frac{6}{1}$), then cancel the 6 with the 4 to get $\frac{9}{2} = 4\frac{1}{2}$.

- Flipping a mixed number without converting it, as in turning $1\frac{1}{4}$ into $1\frac{4}{1}$. Change it to $\frac{5}{4}$ first, so the reciprocal is $\frac{4}{5}$.
- Choosing the wrong operation in a word problem. "How many pieces of size $\frac{3}{4}$ fit?" is division, not multiplication. Check your answer by multiplying back.

> **Make it a game · Flip and Multiply Duel.** Write 12 division problems with fractions and mixed numbers on cards. Draw one, and you have 40 seconds to flip the divisor, multiply, simplify and check by multiplying back. Say aloud why flipping works. A correct answer with a correct explanation earns two points. Try for ten points before the parent reaches eight.

3.7 Fraction Word Problems

OBJECTIVES

- Decide whether a fraction word problem calls for adding, subtracting, multiplying or dividing.
- Scale a recipe up or down by multiplying each ingredient by a scale factor.
- Solve multi-step fraction problems about cooking, carpentry, sharing and money, and check that the answer makes sense.

WATCH

Multiplying and Dividing Fractions Word Problems | Solving Fraction Word Problems | Math with Mr. J

Math with Mr. J · 12:31

https://www.youtube.com/watch?v=yqXKHkm8bZw

> **WHEN TO WATCH**
> Before Example 1: deciding when a word problem calls for multiplying or dividing (cupcake batches, paint, walking distance).

Note: Partial fit: neither video scales a recipe (objective 2), but the multiply-a-fraction examples use the same skill.

BACKUP VIDEO

Adding and Subtracting Fractions Word Problems | Solving Fraction Word Problems | Math with Mr. J

Math with Mr. J · 11:04

https://www.youtube.com/watch?v=JgQSMwNOWCI

Before the practice set: word problems that call for adding or subtracting.

KEY TERMS

scale factor

The number you multiply by to make a recipe or other amount bigger or smaller. To go from 4 servings to 6 servings, the scale factor is $\frac{6}{4} = \frac{3}{2}$.

multi-step problem

A word problem that needs two or more calculations, where the answer to one step is used in the next.

the rest (the remainder)

What is left after part has been used or given away. A fraction "of the rest" is a fraction of that leftover amount, not of the original.

CORE IDEAS

Every word problem asks you to choose an operation. Putting amounts together or finding a total means **add**. Finding how much is left or how much more means **subtract**. Taking a fraction *of* an amount, or repeating an amount several times, means **multiply**. Asking how many pieces of a certain size fit, or sharing equally, means **divide**.

Use the same plan every time: read the problem and name what it asks; estimate with whole numbers; choose the operation; compute; check the answer against your estimate; and answer in a sentence with units (cups, feet, dollars).

To scale a recipe, find the scale factor by dividing the number of servings you want by the number the recipe makes. Then multiply every ingredient by that same factor. A recipe for 12 cookies made into 30 cookies has scale factor $\frac{30}{12} = \frac{5}{2}$, so every ingredient is multiplied by $2\frac{1}{2}$.

Carpentry problems often use inches and fractions like eighths and sixteenths. Lengths placed end to end add; cutting a piece off subtracts; cutting a board into equal pieces divides. When a board does not divide evenly, the whole-number part of the quotient tells how many full pieces you get, and you can multiply back to find the leftover.

In multi-step problems, watch the word *rest*. If Henry gives away $\frac{1}{10}$ of his money and then spends $\frac{1}{3}$ *of the rest*, the $\frac{1}{3}$ is taken from what is left, not from the starting amount. Work one step at a time and label each result.

KEY FORMULAS

$$\text{scale factor} = \frac{\text{servings wanted}}{\text{servings the recipe makes}}$$

$$\text{new amount} = \text{scale factor} \times \text{original amount}$$

Practice habit. Estimate first with whole numbers, and write the unit next to every number in your work. Before computing, say the operation out loud: "of means multiply," "how many fit means divide."

WORKED EXAMPLES

Example 1. A cookie recipe makes 12 cookies and uses $\frac{3}{4}$ cup of sugar. Grace wants to make 30 cookies for a church bake sale. How much sugar does she need?

Show solution

1. Find the scale factor: $\frac{30}{12} = \frac{5}{2}$.
2. Estimate: $2\frac{1}{2}$ times a bit less than 1 cup is a bit less than $2\frac{1}{2}$ cups.
3. Multiply the sugar by the scale factor: $\frac{5}{2} \times \frac{3}{4} = \frac{15}{8}$.
4. Change to a mixed number: $15 \div 8 = 1$ remainder 7, so $1\frac{7}{8}$ cups.
5. Check: $1\frac{7}{8}$ is a bit less than $2\frac{1}{2}$, matching the estimate.

Answer: $1\frac{7}{8}$ cups of sugar

Example 2. Henry earned \$60 mowing lawns. He gives $\frac{1}{10}$ of it as an offering at church. Then he spends $\frac{1}{3}$ of the rest on a book. How much money does he have left?

Show solution

1. Offering: $\frac{1}{10} \times 60 = 6$, so he gives \$6.
2. The rest: $60 - 6 = 54$, so he has \$54.
3. Book: $\frac{1}{3} \times 54 = 18$, so the book costs \$18.
4. Left: $54 - 18 = 36$, so he has \$36.
5. Check: $6 + 18 + 36 = 60$.

Answer: \$36

PRACTICE

1. Easy A muffin recipe calls for $\frac{2}{3}$ cup of oil. Ruth doubles the recipe. How much oil does she need?

   Show answer

   $2 \times \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$ cups

2. Easy The church choir has 32 members, and $\frac{1}{4}$ of them sing alto. How many altos are in the choir?

   Show answer

   8 altos

3. Easy Samuel glues a board $\frac{7}{8}$ inch thick on top of a sheet of plywood $\frac{1}{2}$ inch thick. How thick are the two together? (Ignore the glue.)

   Show answer

   $\frac{7}{8} + \frac{4}{8} = \frac{11}{8} = 1\frac{3}{8}$ inches

4. Easy Grace has 3 cups of raisins. She puts $\frac{1}{4}$ cup in each snack bag. How many bags can she fill?

   Show answer

   $3 \div \frac{1}{4} = 12$ bags

5. Medium A pancake recipe serves 8 people and uses $1\frac{1}{2}$ cups of milk. How much milk is needed to serve 12 people?

   Show answer

   Scale factor $\frac{12}{8} = \frac{3}{2}$; $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4} = 2\frac{1}{4}$ cups

6. Medium A board is 36 inches long. Samuel cuts 4 pieces from it, each $8\frac{3}{4}$ inches long. How much of the board is left? (Ignore the width of the saw cuts.)

   Show answer

   $4 \times \frac{35}{4} = 35$ inches used; $36 - 35 = 1$ inch left

7. Medium A farmer has 40 acres. He plants $\frac{3}{8}$ of the land in corn and $\frac{1}{4}$ in wheat. The rest is hay. How many acres are hay?

Show answer

Corn: 15 acres; wheat: 10 acres; hay: $40 - 25 = 15$ acres

8. Medium A hiking trail is $5\frac{1}{4}$ miles long. The Miller family has hiked $3\frac{2}{3}$ miles. How much farther do they have to go?

Show answer

$4\frac{15}{12} - 3\frac{8}{12} = 1\frac{7}{12}$ miles

9. Hard Grace bakes $4\frac{1}{2}$ dozen cookies for the bake sale and sells $\frac{5}{6}$ of them. How many cookies does she sell? (1 dozen = 12.)

Show answer

$4\frac{1}{2} \times 12 = 54$ cookies; $\frac{5}{6} \times 54 = 45$ cookies sold

10. Hard Samuel cuts a 30-inch board into pieces $4\frac{3}{4}$ inches long. How many full pieces can he cut, and how long is the leftover piece? (Ignore the width of the saw cuts.)

Show answer

$30 \div \frac{19}{4} = \frac{120}{19} = 6\frac{6}{19}$, so 6 full pieces. They use $6 \times 4\frac{3}{4} = 28\frac{1}{2}$ inches, leaving $1\frac{1}{2}$ inches.

11. Hard A rain barrel holds 60 gallons and is $\frac{3}{4}$ full. Henry uses $\frac{1}{3}$ of the water in it to water the garden. How many gallons are left in the barrel?

Show answer

Water: $\frac{3}{4} \times 60 = 45$ gallons; used: $\frac{1}{3} \times 45 = 15$ gallons; left: 30 gallons

12. Hard Ruth wants to know how many $\frac{3}{4}$-cup servings are in 6 cups of soup. Her brother says to multiply $6 \times \frac{3}{4}$. Explain why that is the wrong operation, and find the correct answer.

Show answer

Model answer: The question asks how many $\frac{3}{4}$-cup servings fit in 6 cups, which is a "how many groups" question, so it calls for division. Multiplying gives $4\frac{1}{2}$, which is too small: each serving is less than a cup, so there must be more than 6 servings. The correct answer is $6 \div \frac{3}{4} = 6 \times \frac{4}{3} = 8$ servings.

13. Challenge A biscuit recipe serves 6 and uses $2\frac{1}{4}$ cups of flour and $\frac{3}{4}$ cup of buttermilk. Ruth has plenty of flour but only $1\frac{1}{2}$ cups of buttermilk. If she uses all her buttermilk, how many people can she serve, and how much flour will she need?

Show answer

Scale factor: $1\frac{1}{2} \div \frac{3}{4} = 2$. She can serve $2 \times 6 = 12$ people and needs $2 \times 2\frac{1}{4} = 4\frac{1}{2}$ cups of flour.

COMMON MISTAKES

- Taking a fraction of the original amount when the problem says "of the rest." Find what is left first, then take the fraction of that.
- Scaling only one ingredient. When a recipe changes size, multiply every ingredient by the same scale factor.
- Multiplying when the problem asks how many pieces fit. "How many $\frac{3}{4}$-cup servings?" is division. If your answer is smaller than you expect, check the operation.
- Giving a mixed-number answer when the question needs a whole number, such as $6\frac{6}{19}$ shelves. Decide what the answer means: here only 6 full pieces can be cut.

> **Make it a game · Recipe Rescale.** Pick a recipe card from the kitchen with fractional amounts. Parent names a scale factor, such as half, double or one and a half. You rescale every ingredient on paper within three minutes and then explain which operation you used for each. Check by measuring one ingredient with a real cup. Score a point per ingredient.

UNIT 3

Fractions

Fractions are everywhere: in recipes, on rulers and tape measures, in sharing a pan of brownies, and in dividing up a garden. In this unit you will learn what a fraction really means: part of a whole, a point on the number line, and a division. Then you will learn to simplify, compare, add, subtract, multiply and divide fractions and mixed numbers.

The unit explains *why* each rule works, not only how. You will see why fractions need a common denominator before they are added, why we multiply tops and bottoms (using an area model), and why dividing by a fraction is the same as multiplying by its reciprocal. The work in Unit 2 on factors, GCF and LCM is used all through this unit: the GCF simplifies fractions and the LCM gives the least common denominator.

Parents: this unit is done by hand, without a calculator. Fractions are the foundation for decimals (Unit 4), ratios and proportions (Unit 7), percents (Unit 8) and solving equations in algebra, so steady practice here pays off for years.

PREREQUISITES

- Multiplication and division facts through 12×12
- Divisibility rules (Lesson 2.1)
- Finding the greatest common factor (Lessons 2.4 and 2.5)
- Finding the least common multiple (Lesson 2.6)
- Long division with remainders

MASTERY CHECKLIST

- ☐ I can explain a fraction as part of a whole, as a point on the number line, and as a division.
- ☐ I can build equivalent fractions and fill in a missing numerator or denominator.
- ☐ I can write a fraction in lowest terms using the GCF.
- ☐ I can compare and order fractions using common denominators or cross products, and explain why cross products work.
- ☐ I can add and subtract fractions using the least common denominator and explain why a common denominator is needed.
- ☐ I can convert between mixed numbers and improper fractions.
- ☐ I can add and subtract mixed numbers, including regrouping.
- ☐ I can multiply fractions and mixed numbers, canceling first, and explain the area model.
- ☐ I can divide fractions and mixed numbers and explain why we multiply by the reciprocal.
- ☐ I can choose the right operation for a fraction word problem, scale a recipe, and check that my answer makes sense.

LESSONS IN THIS UNIT

3.1 Fractions and Equivalent Fractions

2 sessions of 45 minutes

3.2	**Simplifying and Comparing Fractions**	2 sessions of 45 minutes
3.3	**Adding and Subtracting Fractions**	2 sessions of 45 minutes
3.4	**Mixed Numbers**	2 sessions of 45 minutes
3.5	**Multiplying Fractions**	2 sessions of 45 minutes
3.6	**Dividing Fractions**	2 sessions of 45 minutes
3.7	**Fraction Word Problems**	2 sessions of 45 minutes

Unit 3 test

You have 60 minutes. Do not use a calculator. Show your work for every problem. Write all fraction answers in lowest terms, and write answers greater than 1 as mixed numbers unless the problem says otherwise. Include units in answers to word problems.

1. **5 pts**

   Fill in the box: $\frac{4}{9} = \frac{\square}{36}$.

2. **5 pts**

   Write $5 \div 8$ as a fraction.

3. **6 pts**

   Write $\frac{54}{72}$ in lowest terms.

4. **6 pts**

   Order from least to greatest: $\frac{3}{5}, \frac{4}{7}, \frac{5}{9}$.

5. **6 pts**

   Add: $\frac{5}{8} + \frac{1}{6}$.

6. **6 pts**

   Subtract: $\frac{9}{10} - \frac{2}{5}$.

7. **7 pts**

   Subtract: $6\frac{1}{5} - 2\frac{3}{4}$.

8. **7 pts**

   Add: $3\frac{5}{6} + 4\frac{3}{8}$.

9. **7 pts**

   Multiply: $\frac{4}{9} \times \frac{15}{16}$.

10. **7 pts**

    Multiply: $1\frac{3}{5} \times 2\frac{1}{2}$.

11. **7 pts**

    Divide: $\frac{5}{6} \div \frac{10}{9}$.

12. **8 pts**

    A rope is $13\frac{1}{2}$ feet long. Henry cuts it into pieces that are each $2\frac{1}{4}$ feet long. How many pieces does he get?

13. **8 pts**

    A rice recipe serves 4 people and uses $\frac{2}{3}$ cup of rice. How much rice is needed to serve 10 people?

14. **8 pts**

    Samuel earned \$84 doing farm chores. He saves $\frac{1}{4}$ of it. Then he spends $\frac{2}{3}$ of the rest on tools. How much money does he have left?

15. **7 pts**

    Grace says $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$. Explain why she is wrong and give the correct sum.

ANSWER KEY

1. 16
2. $\frac{5}{8}$
3. The GCF is 18, so $\frac{54}{72} = \frac{3}{4}$.
4. $\frac{5}{9}, \frac{4}{7}, \frac{3}{5}$
5. $\frac{15}{24} + \frac{4}{24} = \frac{19}{24}$
6. $\frac{9}{10} - \frac{4}{10} = \frac{5}{10} = \frac{1}{2}$
7. $5\frac{24}{20} - 2\frac{15}{20} = 3\frac{9}{20}$
8. $3\frac{20}{24} + 4\frac{9}{24} = 7\frac{29}{24} = 8\frac{5}{24}$
9. $\frac{1}{3} \times \frac{5}{4} = \frac{5}{12}$
10. $\frac{8}{5} \times \frac{5}{2} = 4$
11. $\frac{5}{6} \times \frac{9}{10} = \frac{3}{4}$
12. $\frac{27}{2} \times \frac{4}{9} = 6$ pieces
13. Scale factor $\frac{10}{4} = \frac{5}{2}$; $\frac{5}{2} \times \frac{2}{3} = \frac{5}{3} = 1\frac{2}{3}$ cups
14. Saves $21; rest $63; tools $42; left $21
15. Model answer: Halves and thirds are different-size pieces, so they cannot be counted together until they are rewritten as the same size. Adding the denominators is wrong because the denominator only names the size of the pieces. Also, $\frac{2}{5}$ is less than $\frac{1}{2}$ alone, so it cannot be the sum. Correct: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

UNIT 4

Decimals

Decimals are fractions written in the language of place value. In this unit you will read, compare and round decimals, then add, subtract, multiply and divide them by hand. You will see that decimals follow the same place-value pattern you already know from whole numbers, simply continued to the right of the decimal point into tenths, hundredths and thousandths.

We will not just learn the rules; we will see why they work. Lining up decimal points means adding pieces of the same size. The decimal places add when you multiply because tenths times hundredths gives thousandths. And you may multiply the divisor and dividend by the same power of 10 because that makes an equivalent fraction. Along the way you will use decimals the way families do every day: prices, making change, unit prices, measuring boards and weighing food.

The unit ends by connecting decimals back to fractions from Unit 3. You will convert both ways and discover which fractions make terminating decimals and which make repeating ones, using the prime factorization you learned in Unit 2. These skills prepare you for negative decimals in Unit 5 and for percents in Unit 8.

PREREQUISITES

- Place value and rounding of whole numbers (Lessons 1.1 and 1.5)
- Long multiplication and long division of whole numbers by hand
- Prime factorization (Lesson 2.3) and the greatest common factor (Lesson 2.4)
- Simplifying fractions and writing equivalent fractions (Lessons 3.1 and 3.2)
- Mixed numbers and improper fractions (Lesson 3.4)
- Multiplying fractions (Lesson 3.5)

MASTERY CHECKLIST

- ☐ I can read and write decimals to the thousandths place in words, standard form and expanded form.
- ☐ I can compare and order decimals by lining up place values.
- ☐ I can round a decimal to the nearest whole number, tenth or hundredth.
- ☐ I can add and subtract decimals by lining up the decimal points, and explain why.
- ☐ I can make change by subtracting and check it by counting up.
- ☐ I can multiply decimals and explain why the decimal places add.
- ☐ I can multiply and divide by 10, 100 and 1,000 mentally.
- ☐ I can divide decimals, including by a decimal divisor, and explain why multiplying both numbers by a power of 10 is allowed.
- ☐ I can find unit prices and choose the better buy.
- ☐ I can convert fractions to decimals and decimals to fractions in lowest terms.
- ☐ I can write repeating decimals with bar notation.
- ☐ I can tell whether a fraction gives a terminating decimal by looking at the prime factors of its denominator, and explain why.

LESSONS IN THIS UNIT

4.1	**Decimals and Place Value**	45 minutes
4.2	**Adding and Subtracting Decimals**	2 sessions of 45 minutes
4.3	**Multiplying Decimals**	2 sessions of 45 minutes
4.4	**Dividing Decimals**	2 sessions of 45 minutes
4.5	**Fractions and Decimals**	2 sessions of 45 minutes

Unit 4 test

You have 50 minutes. No calculator: do all work by hand. Show your work for every problem, including an estimate where it helps. Round money to the nearest cent, and round other answers only when a problem tells you to. Use bar notation for repeating decimals.

1. **5 pts**
   Order from least to greatest: 3.09, 3.9, 3.091, 3.019.

2. **5 pts**
   Round 27.6849 to the nearest hundredth.

3. **6 pts**
   Add: $14.3 + 2.68 + 0.5$.

4. **6 pts**
   Subtract: $12 - 5.47$.

5. **10 pts**
   Samuel buys a hammer for $12.89 and a box of nails for $4.36. He pays with a $20 bill. How much change should he get?

6. **6 pts**
   Multiply: 3.4×0.06.

7. **6 pts**

   Multiply mentally: 0.725×100.

8. **8 pts**

   Cheese costs \$4.80 per pound. Ruth buys 2.5 pounds for a church potluck. How much does she pay?

9. **6 pts**

   Divide: $9.12 \div 0.3$.

10. **10 pts**

    A 16-ounce jar of peanut butter costs \$3.52 and a 28-ounce jar costs \$5.88. Find the unit price (cost per ounce) of each jar. Which is the better buy?

11. **6 pts**

    Write $\frac{7}{8}$ as a decimal.

12. **6 pts**

    Write 0.64 as a fraction in lowest terms.

13. **10 pts**

    Write $\frac{7}{12}$ as a decimal using bar notation.

14. **10 pts**

    One of the fractions $\frac{9}{40}$ and $\frac{2}{9}$ is a terminating decimal and the other is a repeating decimal. Which is which? Explain using prime factors, and write both decimals.

ANSWER KEY

1. 3.019, 3.09, 3.091, 3.9
2. 27.68
3. 17.48
4. 6.53
5. $2.75
6. 0.204
7. 72.5
8. $12.00
9. 30.4
10. 16-ounce jar: $0.22 per ounce; 28-ounce jar: $0.21 per ounce. The 28-ounce jar is the better buy.
11. 0.875
12. $\frac{16}{25}$
13. $0.58\overline{3}$
14. Both fractions are in lowest terms. $40 = 2^3 \times 5$ has only 2s and 5s, so $\frac{9}{40}$ can be rewritten with a denominator of 1,000: $\frac{9}{40} = \frac{225}{1000} = 0.225$, which terminates. $9 = 3^2$ has a factor of 3, and no power of 10 is a multiple of 3, so $\frac{2}{9}$ repeats: $\frac{2}{9} = 0.\overline{2}$.

4.1 Decimals and Place Value

OBJECTIVES

- Read and write decimals to the thousandths place in words, in standard form and in expanded form.
- Compare and order decimals by lining up place values.
- Round decimals to the nearest whole number, tenth or hundredth.

WATCH

Math Antics - Decimal Place Value

mathantics · 11:51

https://www.youtube.com/watch?v=KG6ILNOiMgM

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Comparing Decimal Numbers - Math Antics Extras

mathantics · 5:04

https://www.youtube.com/watch?v=rALUd3wW29s

Before Example 2 (comparing and ordering decimals).

KEY TERMS

decimal

A number written with a decimal point, using place value to show parts of a whole, such as 3.25.

decimal point

The dot that separates the whole-number part (on the left) from the parts smaller than one (on the right).

tenths

The first place to the right of the decimal point. One tenth is one of 10 equal parts of a whole: $0.1 = \frac{1}{10}$.

hundredths

The second place to the right of the decimal point. One hundredth is one of 100 equal parts: $0.01 = \frac{1}{100}$.

thousandths

The third place to the right of the decimal point. One thousandth is one of 1,000 equal parts: $0.001 = \frac{1}{1000}$.

CORE IDEAS

You already know that each place in a whole number is worth 10 times the place to its right: ones, tens, hundreds. Decimals simply keep that pattern going to the right. Each place to the right of the ones place is worth one tenth of the place before it: tenths, then hundredths, then thousandths.

Think of money. One dollar is the whole. A dime is one tenth of a dollar, so it is \$0.10. A penny is one hundredth of a dollar, so it is \$0.01. That is why we write prices with two digits after the decimal point: those two places count dimes and pennies.

To read a decimal, read the whole-number part, say "and" for the decimal point, then read the digits after the point as a whole number and finish with the name of the last place. So 4.307 is read "four and three hundred seven thousandths." In expanded form, $4.307 = 4 + 0.3 + 0.007$ (there are no hundredths, so that place is simply left out).

To compare decimals, line up the decimal points and compare place by place from the left. Writing extra zeros at the end helps, because zeros at the end of a decimal do not change its value: $0.7 = 0.70 = 0.700$ (7 dimes is the same amount as 70 pennies). Now compare 0.7 and 0.69: as 0.70 and 0.69, you can see 70 hundredths is more than 69 hundredths. More digits does not mean bigger!

To round a decimal, find the place you are rounding to and look at the digit just to its right. If that digit is 5 or more, round up; if it is 4 or less, keep the digit the same. Then drop every digit after the rounding place. For example, 8.364 rounded to the nearest tenth is 8.4, because the digit after the tenths place is 6.

KEY FORMULAS

$$0.1 = \frac{1}{10} \qquad 0.01 = \frac{1}{100} \qquad 0.001 = \frac{1}{1000}$$

$$0.7 = 0.70 = 0.700$$

From history. Fractions with denominators of 10, 100 and 1,000 were used long before the decimal point became common. In 1585 the Flemish mathematician and engineer Simon Stevin published a short book called *The Tenth*, which urged merchants, surveyors and everyone else to use decimal fractions for everyday calculations. His notation was clumsier than ours, but within a few generations the decimal point became standard.

Practice habit. Recite the place names to the right of the decimal point until they are automatic: tenths, hundredths, thousandths. Whenever you see a decimal with two places, picture dimes and pennies.

WORKED EXAMPLES

Example 1. Order these decimals from least to greatest: 0.6, 0.06, 0.606, 0.66.

Show solution

1. Write each number with three decimal places so they line up: 0.600, 0.060, 0.606, 0.660.
2. Now compare them as thousandths: 600, 60, 606 and 660.
3. From least to greatest: 60, 600, 606, 660.
4. Change back to the original numbers: 0.06, 0.6, 0.606, 0.66.

Answer: 0.06, 0.6, 0.606, 0.66

Example 2. At the county fair, Ruth's pumpkin weighed 12.386 pounds. Round the weight to the nearest tenth of a pound and to the nearest hundredth of a pound.

Show solution

1. Nearest tenth: the tenths digit is 3. The digit to its right is 8.
2. Since 8 is 5 or more, round the 3 up to 4 and drop the rest: 12.4 pounds.
3. Nearest hundredth: the hundredths digit is 8. The digit to its right is 6.
4. Since 6 is 5 or more, round the 8 up to 9 and drop the rest: 12.39 pounds.
5. Check: both answers are close to 12.386, so they are reasonable.

Answer: 12.4 pounds to the nearest tenth; 12.39 pounds to the nearest hundredth

PRACTICE

1. Easy Write 4.307 in words.

   Show answer

   four and three hundred seven thousandths

2. Easy What digit is in the hundredths place of 6.285?

   Show answer

   8

3. Easy Write as a decimal: seventeen and nine hundredths.

   Show answer

   17.09

4. Easy Which is greater, 0.7 or 0.69?

   Show answer

   0.7 (because 0.70 is greater than 0.69)

5. Medium Write 3.452 in expanded form.

   Show answer

   $3 + 0.4 + 0.05 + 0.002$

6. Medium Round 8.364 to the nearest tenth.

   Show answer

   8.4

7. Medium Round 15.0962 to the nearest hundredth.

   Show answer

   15.10

8. Medium Order from least to greatest: 2.5, 2.05, 2.505, 2.055.

   Show answer

   2.05, 2.055, 2.5, 2.505

9. Hard Grace kept a rain gauge in her garden. She recorded 0.35 inch on Monday, 0.4 inch on Tuesday, 0.308 inch on Wednesday and 0.385 inch on Thursday. How much rain fell on the rainiest day, and which day was it?

   Show answer

   0.4 inch, on Tuesday

10. Hard Gas stations often show prices to the thousandth of a dollar. A sign shows gas at $3.479 per gallon. Round this price to the nearest cent.

    Show answer

    $3.48

11. Hard At a church field day, Samuel's long jump was 4.27 meters, Henry's was 4.3 meters and Peter's was 4.207 meters. List the jumps from longest to shortest.

    Show answer

    4.3 m (Henry), 4.27 m (Samuel), 4.207 m (Peter)

12. Challenge What decimal is exactly halfway between 0.34 and 0.35?

    Show answer

    0.345

13. Challenge Henry says that 0.125 is greater than 0.5 because 125 is greater than 5. Explain why Henry is wrong.

    Show answer

    Henry compared the digits as if they were whole numbers. Line up the places: 0.5 = 0.500, which is 500 thousandths, while 0.125 is only 125 thousandths. Also, 0.5 has 5 tenths and 0.125 has only 1 tenth. So 0.5 is greater.

COMMON MISTAKES

- Thinking a decimal with more digits is bigger (for example, saying 0.125 > 0.5). Avoid this by writing zeros at the end so both numbers have the same number of decimal places, then comparing.
- Reading 4.307 as "four point three hundred seven" and losing track of its size. Say "and" for the decimal point and name the last place: "four and three hundred seven thousandths."
- When rounding, changing the digits after the rounding place instead of dropping them (writing 8.364 as 8.400 and then forgetting which place you rounded to).

Round, then drop every digit to the right of the rounding place.

- When rounding a 9 up, forgetting to carry (for example, rounding 15.0962 to the nearest hundredth as 15.010). A 9 that rounds up becomes 10, so the 0 to its left becomes 1: 15.10.

> **Make it a game · Decimal Place Hunt.** Write 15 decimals to the thousandths on cards. Draw one, read it aloud in words, then write it in expanded form without looking at the card. Score a point for each fully correct card. Afterward, lay five cards on the table and arrange them from least to greatest in 20 seconds. Beat your own count next round.

4.2 Adding and Subtracting Decimals

OBJECTIVES

- Add and subtract decimals by hand by lining up the decimal points.
- Explain why the decimal points must be lined up.
- Solve money problems, including making change by counting up.

WATCH

Math Antics - Decimal Arithmetic

mathantics · 12:15

https://www.youtube.com/watch?v=kwh4SD1ToFc

> **WHEN TO WATCH**
> Watch the first 3½ minutes (adding and subtracting) before Example 1.

Note: Shared video: the same Math Antics 'Decimal Arithmetic' video is used in Lessons 4.2–4.4. Each lesson watches its own section.

BACKUP VIDEO

Math Antics - Dollars And Cents

mathantics · 13:55

https://www.youtube.com/watch?v=DbYbIB4m3RM

Before the money example: dollars and cents.

KEY TERMS

line up the decimal points
Write the numbers in a column so the decimal points are directly under each other; then every digit sits under a digit of the same place value.

placeholder zero
A zero written at the end of a decimal to fill an empty place, such as writing 3.5 as 3.50. It does not change the value.

change
The money you get back when you pay more than the price. Change = amount paid − total cost.

counting up
A way to make change by starting at the price and adding coins and bills until you reach the amount paid.

CORE IDEAS

When you add whole numbers, you add ones to ones and tens to tens. Decimals work exactly the same way: tenths are added to tenths and hundredths to hundredths. You cannot add 3 dimes and 5 pennies and get 8 of anything useful; they are different sizes.

That is WHY we line up the decimal points. When the points are lined up, each column holds one place value, so you are always adding or subtracting the same-sized pieces. Lining up the right-hand digits (as you might with whole numbers) puts tenths under hundredths, which gives a wrong answer.

Fill empty places with placeholder zeros. To add 12.6 + 3.45, write 12.60 + 3.45. To subtract 10 − 4.36, write 10 as 10.00. The zeros do not change the values, but they keep the columns neat and make regrouping (borrowing) clear.

Then add or subtract just as with whole numbers, carrying or regrouping as needed, and bring the decimal point straight down into the answer.

Money is the most common use of decimals. To make change, you can subtract (amount paid − total), or count up from the price the way a cashier does: add pennies to reach a multiple of 5 or 10 cents, then larger coins to reach a whole dollar, then bills. Both methods should give the same answer, so one checks the other.

Always estimate first. Round each number to a whole number (or nearest dollar) and add or subtract. If your exact answer is far from your estimate, look for a misplaced decimal point.

KEY FORMULAS

$$\text{change} = \text{amount paid} - \text{total cost}$$

> **Practice habit.** Estimate before you compute: round each decimal to the nearest whole number and find a quick answer. Then practice making change from $1, $5, $10 and $20 by counting up out loud.

WORKED EXAMPLES

Example 1. Add: $12.6 + 3.45 + 0.008$.

Show solution

1. Estimate: about $13 + 3 + 0 = 16$.
2. Write each number with three decimal places: 12.600, 3.450, 0.008.
3. Line up the decimal points and add each column from the right: thousandths $0 + 0 + 8 = 8$; hundredths $0 + 5 + 0 = 5$; tenths $6 + 4 + 0 = 10$, write 0 and carry 1; ones $2 + 3 + 0 + 1 = 6$; tens 1.
4. Bring the decimal point straight down: 16.058.
5. Check: 16.058 is close to the estimate of 16.

Answer: 16.058

Example 2. Ruth buys a loaf of bread for \$3.79, a gallon of milk for \$2.49 and a bag of apples for \$4.35. She pays with a \$20 bill. How much change should she get?

Show solution

1. Estimate: about $4 + 2 + 4 = 10$ dollars, so the change should be about \$10.
2. Add the first two prices: $3.79 + 2.49 = 6.28$.
3. Add the apples: $6.28 + 4.35 = 10.63$. The total is \$10.63.
4. Subtract from the amount paid: $20.00 - 10.63 = 9.37$.
5. Check by counting up from \$10.63: 2 pennies make \$10.65, a dime makes \$10.75, a quarter makes \$11.00, and 9 dollars make \$20.00. That is $0.02 + 0.10 + 0.25 + 9 = 9.37$.

Answer: \$9.37

PRACTICE

1. Easy Add: $4.5 + 2.3$.

   Show answer

   6.8

2. Easy Add: $7.82 + 1.4$.

Show answer

9.22

3. Easy Subtract: $9.6 - 3.2$.

Show answer

6.4

4. Easy Add: $15 + 0.75$.

Show answer

15.75

5. Medium Subtract: $8.03 - 2.7$.

Show answer

5.33

6. Medium Subtract: $10 - 4.36$.

Show answer

5.64

7. Medium Add: $6.007 + 12.08 + 0.9$.

Show answer

18.987

8. Medium Henry ran 2.75 miles on Monday and 3.4 miles on Tuesday. How far did he run in all?

Show answer

6.15 miles

9. Hard Subtract: $20.5 - 7.68$.

Show answer

12.82

10. Hard Grace buys school supplies that cost $6.85 in all. She pays with a $10 bill. How much change should she get?

    Show answer

    $3.15

11. Hard Mr. Miller has a board 8 feet long. He cuts off one piece 2.375 feet long and another piece 3.5 feet long. Ignoring the width of the saw cuts, how long is the board that is left?

    Show answer

    2.125 feet

12. Challenge The Miller family has $50 to spend on groceries for a church potluck. They buy chicken for $18.47, rolls for $4.29, salad for $7.85 and lemonade for $5.60. How much of the $50 is left?

    Show answer

    $13.79

13. Challenge Samuel added $3.5 + 1.25$ by lining up the last digits on the right and got 1.60. Explain his mistake and find the correct sum.

    Show answer

    Samuel lined up the 5 tenths of 3.5 under the 5 hundredths of 1.25, so he added different place values. He should line up the decimal points: $3.50 + 1.25 = 4.75$. His answer is also unreasonable, since 3.5 + 1.25 must be more than 3.5.

COMMON MISTAKES

- Lining up the last digits instead of the decimal points (writing 3.5 + 1.25 as 1.60). Always line up the decimal points and fill empty places with zeros.
- Subtracting the smaller digit from the larger in each column when the top digit is smaller (for example, getting $8.03 - 2.7 = 6.73$ instead of 5.33). Write 2.7 as 2.70, and when a top digit is smaller, regroup (borrow) from the place to its left.
- Forgetting to bring the decimal point down into the answer, or putting it in the wrong place. Estimate first so you can see whether your answer is about the right size.
- When making change, forgetting to add up all the items before subtracting from the amount paid. Find the total first, then subtract, then check by counting up.

Make it a game · Line Up the Points. Parent reads a decimal addition or subtraction problem aloud, such as 12.4 minus 3.875, and you write it vertically with placeholder zeros and solve it. Score one point for correct alignment and one for the answer. Parent slips in one problem with a wrong alignment to see if you catch it. Play for ten points.

4.3 Multiplying Decimals

OBJECTIVES

- Multiply decimals by hand and place the decimal point correctly.
- Explain, using fractions with denominators of 10, 100 and 1,000, why the numbers of decimal places add.
- Multiply decimals by 10, 100 and 1,000 mentally by moving the decimal point.

WATCH

Math Antics - Decimal Arithmetic

mathantics · 12:15

https://www.youtube.com/watch?v=kwh4SD1ToFc

> **WHEN TO WATCH**
> Watch from about 3:30 to 7:30 (multiplication) before Example 1.

Note: Shared video: this lesson uses the multiplication section of 'Decimal Arithmetic'.

BACKUP VIDEO

Multiply and Divide by Powers of 10 - Math Antics Extras

mathantics · 4:54

https://www.youtube.com/watch?v=qJB19dAZPpY

Before the practice set: multiplying and dividing by 10, 100 and 1,000 by moving the decimal point.

KEY TERMS

decimal places

The number of digits to the right of the decimal point. For example, 2.45 has 2 decimal places.

factor

A number being multiplied. In 2.4×0.13, the factors are 2.4 and 0.13.

product

The answer to a multiplication problem.

power of 10

A number such as 10, 100 or 1,000 made by multiplying 10 by itself; also written 10^1, 10^2, 10^3.

CORE IDEAS

Every decimal can be written as a fraction with a denominator of 10, 100, 1,000 and so on. For example, $0.3 = \frac{3}{10}$ and $0.13 = \frac{13}{100}$. That fact explains the whole rule for multiplying decimals.

Look at 0.3×0.2. As fractions, this is $\frac{3}{10} \times \frac{2}{10} = \frac{6}{100} = 0.06$. The numerators multiply to give 6, and the denominators multiply to give 100. One decimal place times one decimal place gives hundredths, which is two decimal places.

In the same way, $2.4 \times 0.13 = \frac{24}{10} \times \frac{13}{100} = \frac{312}{1000} = 0.312$. Tenths times hundredths gives thousandths, because $10 \times 100 = 1000$. That is WHY the decimal places add: $1 + 2 = 3$ places.

So here is the method: (1) ignore the decimal points and multiply as whole numbers; (2) count the total number of decimal places in both factors; (3) place the decimal point so the product has that many decimal places, writing extra zeros on the left if needed. You do not need to line up the decimal points when multiplying.

Multiplying by 10 makes every digit worth 10 times as much, so each digit moves one place to the left; it looks like the decimal point moves one place to the right. Multiplying by 100 moves it two places right, and by 1,000 three places right. For example, $5.27 \times 100 = 527$.

Estimate first to check where the decimal point goes. For 3.5×1.29, think $3.5 \times 1 = 3.5$ and 3.5×1.3 is a little more than 4, so an answer like 45.15 or 0.4515 must be wrong.

KEY FORMULAS

$$\frac{3}{10} \times \frac{2}{10} = \frac{6}{100}, \text{ so } 0.3 \times 0.2 = 0.06$$

$$\text{decimal places in the product} = \text{decimal places in the first factor} + \text{decimal places in the second factor}$$

$$\times 10: \text{ point moves 1 place right} \qquad \times 100: \text{ 2 places} \qquad \times 1000: \text{ 3 places}$$

> **Practice habit.** Before every decimal multiplication, say an estimate out loud using whole numbers. Then practice multiplying by 10, 100 and 1,000 mentally: say the new number after moving the point 1, 2 or 3 places to the right.

WORKED EXAMPLES

Example 1. Multiply: 2.4×0.13.

Show solution

1. Estimate: 0.13 is a little more than 0.1, and $2.4 \times 0.1 = 0.24$, so the product should be a little more than 0.24.
2. Multiply as whole numbers: $24 \times 13 = 312$.
3. Count decimal places: 2.4 has 1 and 0.13 has 2, for a total of $1 + 2 = 3$.
4. Place the point so there are 3 decimal places: 0.312.
5. Check with fractions: $\frac{24}{10} \times \frac{13}{100} = \frac{312}{1000} = 0.312$. It also matches the estimate.

Answer: 0.312

Example 2. Grace buys 3.5 pounds of apples at \$1.29 per pound. How much does she pay, to the nearest cent?

Show solution

1. Cost = pounds × price per pound = 3.5×1.29.
2. Estimate: 3.5×1.3 is about 4.5, so the cost is about \$4.50.
3. Multiply as whole numbers: $35 \times 129 = 4515$.
4. Count decimal places: 3.5 has 1 and 1.29 has 2, for a total of 3. So the product is 4.515.
5. Round to the nearest cent (hundredth): the digit after the hundredths place is 5, so round up to 4.52.
6. Check: \$4.52 is close to the estimate of \$4.50.

Answer: \$4.52

PRACTICE

1. Easy Multiply: 0.4×3.

   Show answer

   1.2

2. Easy Multiply: 0.6×0.7.

Show answer

0.42

3. Easy Multiply mentally: 5.27×10.

Show answer

52.7

4. Easy Multiply mentally: 3.8×100.

Show answer

380

5. Medium Multiply: 1.5×2.6.

Show answer

3.9

6. Medium Multiply: 0.05×0.3.

Show answer

0.015

7. Medium Multiply mentally: 4.125×1000.

Show answer

4,125

8. Medium Notebooks cost $2.35 each. How much do 6 notebooks cost?

Show answer

$14.10

9. Hard Multiply: 12.4×0.25.

Show answer

3.1

10. Hard Mrs. Miller fills her car with 11.5 gallons of gas at $3.20 per gallon. How much does she pay?

Show answer

$36.80

11. Hard Samuel's vegetable garden is a rectangle 6.5 meters long and 4.2 meters wide. What is its area?

Show answer

27.3 square meters

12. Challenge Multiply: 0.08×0.004.

Show answer

0.00032

13. Challenge Explain, using fractions, why $0.5 \times 0.5 = 0.25$ and not 2.5. Then explain why the product is smaller than 0.5.

Show answer

$0.5 \times 0.5 = \frac{5}{10} \times \frac{5}{10} = \frac{25}{100} = 0.25$; tenths times tenths gives hundredths, so there are 2 decimal places. The product is smaller than 0.5 because we are taking half of 0.5, and half of something is less than the whole thing.

COMMON MISTAKES

- Lining up the decimal points and bringing the point straight down, as in addition (getting $0.6 \times 0.7 = 4.2$). In multiplication, count the total decimal places in the factors instead: $0.6 \times 0.7 = 0.42$.
- Forgetting to write zeros on the left when the product has too few digits (writing $0.05 \times 0.3 = 0.15$). The whole-number product is 15 and you need 3 decimal places, so write 0.015.
- Moving the decimal point the wrong way when multiplying by 10, 100 or 1,000. Multiplying by a power of 10 makes a number bigger, so the point moves to the right.
- Expecting the product to always be bigger than both factors. When you multiply by a number less than 1, the product is smaller. Estimate first so you know what size to expect.

> **Make it a game · Decimal Places Detective.** Parent says a multiplication such as 0.4 x 0.06. Before you calculate, you predict the number of decimal places in the answer and say why it works using tenths and hundredths. Then multiply and check. Score a point for each correct prediction and a point for each correct product. Play to eight points, then trade roles.

4.4 Dividing Decimals

OBJECTIVES

- Divide a decimal by a whole number by long division, writing extra zeros when needed.
- Divide by a decimal by multiplying both numbers by the same power of 10, and explain why this is allowed.
- Find unit prices and use them to decide which item is the better buy.

WATCH

Math Antics - Decimal Arithmetic

mathantics · 12:15

https://www.youtube.com/watch?v=kwh4SD1ToFc

> **WHEN TO WATCH**
> Watch from about 7:30 to the end (division, including dividing by a decimal) before Example 1.

Note: Shared video: this lesson uses the division section of 'Decimal Arithmetic'.

BACKUP VIDEO

How to Find Unit Prices | Unit Price Problems

Math with Mr. J · 5:10

https://www.youtube.com/watch?v=kFNEqLWy1P4

Before the unit-price example (objective 3): finding unit prices to compare items.

KEY TERMS

dividend

The number being divided. In $7.56 \div 4$, the dividend is 7.56.

divisor

The number you divide by. In $7.56 \div 4$, the divisor is 4.

quotient

The answer to a division problem.

unit price

The price of one unit of something, such as the cost per ounce or per pound. Unit price = total price ÷ number of units.

CORE IDEAS

Dividing a decimal by a whole number works just like long division with whole numbers. The only new step: put the decimal point in the quotient directly above the decimal point in the dividend, then divide as usual. For $7.56 \div 4$, you get 1.89, because $4 \times 1.89 = 7.56$.

If the division does not come out even, write zeros at the end of the dividend and keep dividing. A zero at the end of a decimal does not change its value. For example, $5 \div 8$: write 5 as 5.000 and divide to get 0.625.

Dividing BY a decimal is harder, so we change the problem into one with a whole-number divisor. Multiply both the divisor and the dividend by the same power of 10. For $4.8 \div 0.6$, multiply both by 10 to get $48 \div 6 = 8$.

WHY is that allowed? A division can be written as a fraction: $4.8 \div 0.6 = \dfrac{4.8}{0.6}$. Multiplying the top and bottom of a fraction by the same number gives an equivalent fraction, so $\dfrac{4.8}{0.6} = \dfrac{48}{6}$. Here is another way to see it: asking how many 0.6-meter pieces fit in 4.8 meters is the same as asking how many 6-decimeter pieces fit in 48 decimeters. Changing the unit changes both numbers, but not the answer.

In practice: move the decimal point in the divisor to the right until the divisor is a whole number, then move the point in the dividend the same number of places (adding zeros if needed). Then divide.

A unit price tells you the cost of one unit, such as one ounce. To compare two sizes of the same product, find each unit price; the lower unit price is the better buy.

KEY FORMULAS

$$\frac{4.8}{0.6} = \frac{4.8 \times 10}{0.6 \times 10} = \frac{48}{6} = 8$$

$$\text{unit price} = \frac{\text{total price}}{\text{number of units}}$$

> **Practice habit.** Check every division by multiplying: quotient × divisor should equal the dividend. Also practice dividing by 10, 100 and 1,000 mentally by moving the decimal point 1, 2 or 3 places to the left.

WORKED EXAMPLES

Example 1. Divide: $7.56 \div 4$.

Show solution

1. Estimate: $8 \div 4 = 2$, so the answer should be a little less than 2.
2. Put the decimal point in the quotient right above the decimal point in 7.56.
3. 4 goes into 7 one time (4); $7 - 4 = 3$. Bring down 5 to make 35.
4. 4 goes into 35 eight times (32); $35 - 32 = 3$. Bring down 6 to make 36.
5. 4 goes into 36 nine times (36); the remainder is 0. The quotient is 1.89.
6. Check: $1.89 \times 4 = 7.56$.

Answer: 1.89

Example 2. Henry has a rope 8.4 meters long. He cuts it into pieces that are each 0.35 meter long for a camp knot-tying lesson. How many pieces does he get?

Show solution

1. Number of pieces = $8.4 \div 0.35$.
2. Make the divisor whole: move the point in 0.35 two places right to get 35.
3. Move the point in 8.4 the same two places right (add a zero) to get 840.
4. Divide: $840 \div 35 = 24$.
5. Check: $24 \times 0.35 = 8.40$. Correct.

Answer: 24 pieces

PRACTICE

1. Easy Divide: $9.6 \div 3$.

   Show answer

   3.2

2. Easy Divide: $14.5 \div 5$.

   Show answer

   2.9

3. Easy Divide mentally: $47.2 \div 10$.

Show answer

4.72

4. Easy Divide: $6.3 \div 0.7$.

Show answer

9

5. Medium Divide: $1.44 \div 1.2$.

Show answer

1.2

6. Medium Divide: $3.6 \div 0.04$.

Show answer

90

7. Medium Divide by hand, writing zeros as needed: $5 \div 8$.

Show answer

0.625

8. Medium Four friends share a lunch bill of $13.80 equally. How much does each friend pay?

Show answer

$3.45

9. Hard Divide: $0.552 \div 0.24$.

Show answer

2.3

10. Hard A 12-ounce box of oatmeal costs $3.48 and an 18-ounce box costs $4.86. Find the unit price (cost per ounce) of each box. Which is the better buy?

Show answer

12-ounce box: $0.29 per ounce; 18-ounce box: $0.27 per ounce. The 18-ounce box is the better buy.

11. Hard The Millers drove 247.5 miles on a trip and used 11 gallons of gas. How many miles did they drive per gallon?

Show answer

22.5 miles per gallon

12. Hard Explain why $4.8 \div 0.6$ has the same answer as $48 \div 6$.

Show answer

We can write the division as a fraction, $\frac{4.8}{0.6}$. Multiplying the top and bottom by 10 gives the equivalent fraction $\frac{48}{6}$, which has the same value. So both quotients equal 8. (Also: $8 \times 0.6 = 4.8$ and $8 \times 6 = 48$.)

13. Challenge Ruth pays \$2.85 for 1.5 pounds of grapes. At the same price per pound, how much would 4 pounds of grapes cost?

Show answer

\$7.60 (the price is \$1.90 per pound)

14. Challenge Divide by hand: $7 \div 0.16$.

Show answer

43.75

COMMON MISTAKES

- Moving the decimal point in the divisor but not in the dividend (turning $3.6 \div 0.04$ into $3.6 \div 4$). Whatever you do to the divisor, do the same to the dividend: $360 \div 4 = 90$.
- Putting the decimal point in the wrong place in the quotient. Place it directly above the decimal point in the dividend before you start dividing, and estimate to check the size.
- Stopping with a remainder instead of writing zeros and continuing. With decimals, write zeros after the last digit of the dividend and keep dividing until the remainder is 0 (or until you have enough places to round).
- Dividing the wrong way for a unit price (dividing ounces by dollars). Unit price is price ÷ number of units, so the dollars go on top.

Make it a game · Best Buy Shelf Quiz. Collect eight food packages from the pantry, each showing a price or an invented price tag and a weight. Pick two and divide to find the unit price of each, working long division with decimals. Name the better buy. Score a point per correct unit price. Play against the parent, using a stopwatch for the speed bonus.

4.5 Fractions and Decimals

OBJECTIVES

- Convert fractions and mixed numbers to decimals by division, and decimals to fractions in lowest terms.
- Recognize terminating and repeating decimals and write repeating decimals with a bar.
- Explain why a fraction in lowest terms gives a terminating decimal exactly when its denominator has no prime factors other than 2 and 5.

WATCH

Math Antics - Fractions and Decimals

mathantics · 9:15

https://www.youtube.com/watch?v=Mst8iZjIpFE

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Math Antics - Convert any Fraction to a Decimal

mathantics · 6:43

https://www.youtube.com/watch?v=do_IbHId2Os

Before Example 2: turning any fraction into a decimal by division, including repeating decimals.

KEY TERMS

terminating decimal

A decimal that ends, such as 0.375. The division eventually has a remainder of 0.

repeating decimal

A decimal in which a digit or group of digits repeats forever, such as $0.333\ldots$

repetend

The digit or group of digits that repeats in a repeating decimal.

bar notation

A bar drawn over the repetend to show that it repeats forever: $0.\overline{3} = 0.333\ldots$ and $0.8\overline{3} = 0.8333\ldots$

CORE IDEAS

A fraction is a division: $\frac{3}{8}$ means $3 \div 8$. So to change a fraction to a decimal, divide the numerator by the denominator, writing zeros after the decimal point as needed. $3 \div 8 = 0.375$.

To change a decimal to a fraction, read it by its place value and write that fraction, then simplify with the GCF. For example, 0.36 is 36 hundredths, so $0.36 = \frac{36}{100} = \frac{9}{25}$. A decimal greater than 1 becomes a mixed number: $2.125 = 2\frac{125}{1000} = 2\frac{1}{8}$.

Sometimes the division never ends. For $1 \div 3$, the remainder is always 1, so the digit 3 repeats forever: $\frac{1}{3} = 0.333\ldots = 0.\overline{3}$. Why must a decimal either stop or repeat? When you divide by a whole number, say 12, every remainder is a whole number less than 12. Sooner or later a remainder comes up a second time, and from then on the same digits come out again and again. (If the remainder is 0, the decimal terminates.)

Which fractions terminate? A terminating decimal is a fraction with a denominator of 10, 100, 1,000 and so on. These numbers are made only of 2s and 5s: $10 = 2 \times 5$, $100 = 2^2 \times 5^2$, $1000 = 2^3 \times 5^3$. So a fraction can be rewritten with a denominator of 10, 100 or 1,000 only if its denominator (in lowest terms) has no prime factors except 2 and 5. For example, $\frac{3}{8} = \frac{3}{2^3}$; multiply top and bottom by $5^3 = 125$ to get $\frac{375}{1000} = 0.375$.

If the denominator in lowest terms has any other prime factor, such as 3, 7 or 11, no power of 10 can ever be a multiple of it, so the decimal cannot terminate; it repeats. Be sure to simplify first: $\frac{6}{15}$ looks like it has a 3 in the denominator, but $\frac{6}{15} = \frac{2}{5} = 0.4$, which terminates.

KEY FORMULAS

$$\frac{a}{b} = a \div b$$

$$\frac{1}{2} = 0.5 \quad \frac{1}{4} = 0.25 \quad \frac{3}{4} = 0.75 \quad \frac{1}{5} = 0.2 \quad \frac{1}{8} = 0.125 \quad \frac{1}{3} = 0.\overline{3} \quad \frac{2}{3} = 0.\overline{6}$$

$$\text{Lowest terms: terminates} \iff \text{denominator} = 2^m \times 5^n$$

Practice habit. Recite the fraction–decimal equivalents until they are automatic: halves, fourths, fifths, eighths and thirds (for example, $\frac{1}{8} = 0.125$, $\frac{3}{8} = 0.375$, $\frac{5}{8} = 0.625$, $\frac{7}{8} = 0.875$, $\frac{2}{3} = 0.\overline{6}$).

WORKED EXAMPLES

Example 1. (a) Write $\frac{3}{8}$ as a decimal. (b) Write 0.36 as a fraction in lowest terms.

Show solution

1. (a) Divide $3 \div 8$. Write 3 as 3.000.
2. 8 goes into 30 three times (24), remainder 6. 8 goes into 60 seven times (56), remainder 4. 8 goes into 40 five times (40), remainder 0.
3. So $\frac{3}{8} = 0.375$. It terminates because $8 = 2^3$ has only 2s.
4. (b) 0.36 is 36 hundredths: $0.36 = \frac{36}{100}$.
5. The GCF of 36 and 100 is 4: $\frac{36 \div 4}{100 \div 4} = \frac{9}{25}$.
6. Check: $9 \div 25 = 0.36$.

Answer: (a) 0.375 (b) $\frac{9}{25}$

Example 2. Grace's kitchen scale shows weights as decimals. Her recipe calls for $\frac{5}{12}$ pound of butter. Write $\frac{5}{12}$ as a decimal using bar notation, then round it to the nearest hundredth so she can read it on the scale.

Show solution

1. First predict: $12 = 2^2 \times 3$ has a factor of 3, and $\frac{5}{12}$ is in lowest terms, so the decimal will repeat.
2. Divide $5 \div 12$. 12 goes into 50 four times (48), remainder 2.
3. 12 goes into 20 one time (12), remainder 8.
4. 12 goes into 80 six times (72), remainder 8. The remainder 8 repeats, so the 6 repeats forever.
5. So $\frac{5}{12} = 0.41666\ldots = 0.41\overline{6}$.
6. Round to the nearest hundredth: the digit after the hundredths place is 6, so round up to 0.42 pound.

Answer: $0.41\overline{6}$, which is about 0.42 pound

PRACTICE

1. Easy Write $\frac{3}{4}$ as a decimal.

   Show answer

   0.75

2. Easy Write 0.6 as a fraction in lowest terms.

   Show answer

   $\frac{3}{5}$

3. Easy Write 0.45 as a fraction in lowest terms.

   Show answer

   $\frac{9}{20}$

4. Easy Write $\frac{1}{3}$ as a decimal using bar notation.

   Show answer

   $0.\overline{3}$

5. **Medium** Write 2.125 as a mixed number in lowest terms.

Show answer

$2\frac{1}{8}$

6. **Medium** Write $\frac{5}{6}$ as a decimal using bar notation.

Show answer

$0.8\overline{3}$

7. **Medium** On a family road trip, the Millers have driven $\frac{5}{16}$ of the way to Grandma's house. Write this fraction as a decimal.

Show answer

0.3125

8. **Medium** A carpenter's tape measure is marked in eighths of an inch, but Mr. Miller's plan says a shelf is 2.375 inches thick. Write 2.375 as a mixed number in eighths.

Show answer

$2\frac{3}{8}$ inches

9. **Hard** Without dividing, decide whether $\frac{7}{20}$ is a terminating decimal. Then find the decimal.

Show answer

Yes; $20 = 2^2 \times 5$ has only 2s and 5s. $\frac{7}{20} = \frac{35}{100} = 0.35$.

10. **Hard** Without dividing, decide whether $\frac{4}{15}$ is a terminating decimal.

Show answer

No; $\frac{4}{15}$ is in lowest terms and $15 = 3 \times 5$ has a factor of 3, so it repeats ($0.2\overline{6}$).

11. **Hard** Is $\frac{6}{15}$ a terminating decimal? Explain.

Show answer

Yes. In lowest terms, $\frac{6}{15} = \frac{2}{5}$, and 5 is the only prime factor of the denominator, so it terminates: $\frac{2}{5} = 0.4$.

12. Hard Henry answered 17 out of 20 questions correctly on a quiz. Write the fraction of questions he answered correctly as a decimal.

Show answer

0.85

13. Challenge Write $\frac{3}{11}$ as a decimal using bar notation.

Show answer

$0.\overline{27}$

14. Challenge Explain why $\frac{1}{7}$ can never be written as a terminating decimal.

Show answer

A terminating decimal is a fraction with a denominator of 10, 100, 1,000 and so on, and those numbers have only 2s and 5s as prime factors. None of them is a multiple of 7, so $\frac{1}{7}$ cannot be rewritten with such a denominator. When you divide $1 \div 7$, the remainder is never 0, so the digits go on forever and repeat: $\frac{1}{7} = 0.\overline{142857}$.

COMMON MISTAKES

- Dividing the denominator by the numerator (computing $8 \div 3$ for $\frac{3}{8}$). The fraction bar means top ÷ bottom: $\frac{3}{8} = 3 \div 8$.
- Putting the bar over the wrong digits, such as writing $\frac{5}{6}$ as $0.\overline{83}$. Only the 3 repeats ($0.8333\ldots$), so write $0.8\overline{3}$.
- Deciding whether a fraction terminates before simplifying it. Always reduce to lowest terms first; $\frac{6}{15}$ simplifies to $\frac{2}{5}$, which terminates.
- Forgetting to simplify when changing a decimal to a fraction (leaving $0.36 = \frac{36}{100}$). Divide the numerator and denominator by their GCF.

> **Make it a game · Repeat or Stop.** Write 12 fractions on cards. Draw one, predict aloud whether it will terminate or repeat, then divide to find out. If it repeats, write it with bar notation. A correct prediction earns a point and a correct decimal earns another. Then convert three decimals back to fractions in lowest terms. Reach 12 points to win.

UNIT 4

Decimals

Decimals are fractions written in the language of place value. In this unit you will read, compare and round decimals, then add, subtract, multiply and divide them by hand. You will see that decimals follow the same place-value pattern you already know from whole numbers, simply continued to the right of the decimal point into tenths, hundredths and thousandths.

We will not just learn the rules; we will see why they work. Lining up decimal points means adding pieces of the same size. The decimal places add when you multiply because tenths times hundredths gives thousandths. And you may multiply the divisor and dividend by the same power of 10 because that makes an equivalent fraction. Along the way you will use decimals the way families do every day: prices, making change, unit prices, measuring boards and weighing food.

The unit ends by connecting decimals back to fractions from Unit 3. You will convert both ways and discover which fractions make terminating decimals and which make repeating ones, using the prime factorization you learned in Unit 2. These skills prepare you for negative decimals in Unit 5 and for percents in Unit 8.

PREREQUISITES

- Place value and rounding of whole numbers (Lessons 1.1 and 1.5)
- Long multiplication and long division of whole numbers by hand
- Prime factorization (Lesson 2.3) and the greatest common factor (Lesson 2.4)
- Simplifying fractions and writing equivalent fractions (Lessons 3.1 and 3.2)
- Mixed numbers and improper fractions (Lesson 3.4)
- Multiplying fractions (Lesson 3.5)

MASTERY CHECKLIST

- ☐ I can read and write decimals to the thousandths place in words, standard form and expanded form.
- ☐ I can compare and order decimals by lining up place values.
- ☐ I can round a decimal to the nearest whole number, tenth or hundredth.
- ☐ I can add and subtract decimals by lining up the decimal points, and explain why.
- ☐ I can make change by subtracting and check it by counting up.
- ☐ I can multiply decimals and explain why the decimal places add.
- ☐ I can multiply and divide by 10, 100 and 1,000 mentally.
- ☐ I can divide decimals, including by a decimal divisor, and explain why multiplying both numbers by a power of 10 is allowed.
- ☐ I can find unit prices and choose the better buy.
- ☐ I can convert fractions to decimals and decimals to fractions in lowest terms.
- ☐ I can write repeating decimals with bar notation.
- ☐ I can tell whether a fraction gives a terminating decimal by looking at the prime factors of its denominator, and explain why.

LESSONS IN THIS UNIT

4.1	**Decimals and Place Value**	45 minutes
4.2	**Adding and Subtracting Decimals**	2 sessions of 45 minutes
4.3	**Multiplying Decimals**	2 sessions of 45 minutes
4.4	**Dividing Decimals**	2 sessions of 45 minutes
4.5	**Fractions and Decimals**	2 sessions of 45 minutes

Unit 4 test

You have 50 minutes. No calculator: do all work by hand. Show your work for every problem, including an estimate where it helps. Round money to the nearest cent, and round other answers only when a problem tells you to. Use bar notation for repeating decimals.

1. **5 pts**
   Order from least to greatest: 3.09, 3.9, 3.091, 3.019.

2. **5 pts**
   Round 27.6849 to the nearest hundredth.

3. **6 pts**
   Add: $14.3 + 2.68 + 0.5$.

4. **6 pts**
   Subtract: $12 - 5.47$.

5. **10 pts**
   Samuel buys a hammer for $12.89 and a box of nails for $4.36. He pays with a $20 bill. How much change should he get?

6. **6 pts**
   Multiply: 3.4×0.06.

7. **6 pts**

   Multiply mentally: 0.725×100.

8. **8 pts**

   Cheese costs \$4.80 per pound. Ruth buys 2.5 pounds for a church potluck. How much does she pay?

9. **6 pts**

   Divide: $9.12 \div 0.3$.

10. **10 pts**

    A 16-ounce jar of peanut butter costs \$3.52 and a 28-ounce jar costs \$5.88. Find the unit price (cost per ounce) of each jar. Which is the better buy?

11. **6 pts**

    Write $\frac{7}{8}$ as a decimal.

12. **6 pts**

    Write 0.64 as a fraction in lowest terms.

13. **10 pts**

    Write $\frac{7}{12}$ as a decimal using bar notation.

14. **10 pts**

    One of the fractions $\frac{9}{40}$ and $\frac{2}{9}$ is a terminating decimal and the other is a repeating decimal. Which is which? Explain using prime factors, and write both decimals.

ANSWER KEY

1. 3.019, 3.09, 3.091, 3.9
2. 27.68
3. 17.48
4. 6.53
5. $2.75
6. 0.204
7. 72.5
8. $12.00
9. 30.4
10. 16-ounce jar: $0.22 per ounce; 28-ounce jar: $0.21 per ounce. The 28-ounce jar is the better buy.
11. 0.875
12. $\frac{16}{25}$
13. $0.58\overline{3}$
14. Both fractions are in lowest terms. $40 = 2^3 \times 5$ has only 2s and 5s, so $\frac{9}{40}$ can be rewritten with a denominator of 1,000: $\frac{9}{40} = \frac{225}{1000} = 0.225$, which terminates. $9 = 3^2$ has a factor of 3, and no power of 10 is a multiple of 3, so $\frac{2}{9}$ repeats: $\frac{2}{9} = 0.\overline{2}$.

UNIT 5

Integers and Rational Numbers

Until now every number in this course has been zero or greater. In this unit the number line stretches to the left of zero. Negative numbers let us describe temperatures below zero, places below sea level and money that is owed, and they make subtraction always possible: $3 - 8$ finally has an answer.

Students learn to compare integers and find absolute value, and then to add, subtract, multiply and divide them. Each sign rule is explained, not just memorized: with the number line, with Brahmagupta's "debts and fortunes," with number patterns and with the distributive property, which shows why a negative times a negative must be positive. The unit ends by applying the same rules to negative fractions and decimals and to the order of operations.

Parents: this unit is done by hand, without a calculator. The sign rules here are used in every later unit, especially in exponents, equations, graphing and Algebra I, so steady practice now pays off for years.

PREREQUISITES

- Place a whole number, fraction or decimal on a number line (Lessons 3.1 and 4.1).
- Use the commutative, associative and distributive properties (Lessons 1.2 and 1.3).
- Follow the order of operations (Lesson 1.4).
- Add, subtract, multiply and divide fractions and mixed numbers (Unit 3).
- Add, subtract, multiply and divide decimals (Unit 4).

MASTERY CHECKLIST

- ☐ I can use negative numbers to describe temperatures, elevations and money owed.
- ☐ I can find the opposite and the absolute value of a number.
- ☐ I can compare and order integers using a number line.
- ☐ I can add integers with the same sign or different signs.
- ☐ I can subtract integers by adding the opposite, and explain why that works.
- ☐ I can multiply integers and explain why a negative times a negative is positive.
- ☐ I can find the sign of a product by counting negative factors.
- ☐ I can divide integers and explain why dividing by zero is undefined.
- ☐ I can add, subtract, multiply and divide negative fractions, mixed numbers and decimals.
- ☐ I can use the order of operations with negative numbers.
- ☐ I can solve word problems involving gains and losses, changes and averages.

LESSONS IN THIS UNIT

5.1	**Negative Numbers and Absolute Value**	45 minutes
5.2	**Adding Integers**	2 sessions of 45 minutes
5.3	**Subtracting Integers**	2 sessions of 45 minutes

5.4	**Multiplying Integers**	2 sessions of 45 minutes
5.5	**Dividing Integers**	2 sessions of 45 minutes
5.6	**Operations with Rational Numbers**	2 sessions of 45 minutes

Unit 5 test

You have 50 minutes. Do not use a calculator. Show your work for every problem. Give fraction answers in lowest terms. Write negative answers with a minus sign, and include units where the problem has them.

1. **5 pts**
   Order from least to greatest: $-3,\ 5,\ -8,\ 0,\ -1$.

2. **5 pts**
   Find $|-21| - |9|$.

3. **5 pts**
   Find $-13 + 8$.

4. **5 pts**
   Find $-17 + (-26)$.

5. **5 pts**
   Find $6 - 15$.

6. **5 pts**
   Find $-12 - (-20)$.

7. **5 pts**
   Find $(-9)(-6)$.

8. **6 pts**
   Find $(-2)(5)(-3)(-1)$.

9. **6 pts**

   Find $-72 \div (-8)$.

10. **6 pts**

    Find $-45 \div 5 \times (-2)$. (Work from left to right.)

11. **7 pts**

    At 4 p.m. the temperature was $11°\text{F}$. By 4 a.m. it was $-16°\text{F}$. Find the change in temperature (new minus old).

12. **7 pts**

    Samuel's account had \$30 in it. He paid a bill of \$52, and then deposited \$15. Write the sum $30 + (-52) + 15$ and find his final balance.

13. **7 pts**

    Find $-\dfrac{3}{5} + \dfrac{1}{2}$.

14. **7 pts**

    Find $-4 + 6 \times (-3 + 1) \div (-4)$.

15. **9 pts**

    Ruth recorded the low temperature on five winter nights: $-6°\text{F}$, $-2°\text{F}$, $3°\text{F}$, $-7°\text{F}$ and $-8°\text{F}$. Find the average (mean) low temperature.

16. **10 pts**

    Explain why $(-4)(-3) = 12$. Use either a pattern or the distributive property.

ANSWER KEY

1. $-8,\ -3,\ -1,\ 0,\ 5$
2. 12
3. -5
4. -43
5. -9
6. 8
7. 54
8. -30
9. 9
10. 18
11. -27 degrees (it fell 27 degrees)
12. -7 dollars (the account is overdrawn by \$7)
13. $-\dfrac{1}{10}$
14. -1
15. $-4°\text{F}$
16. Pattern: $(-4)(2) = -8$, $(-4)(1) = -4$, $(-4)(0) = 0$; each step the product goes up by 4, so $(-4)(-1) = 4$, $(-4)(-2) = 8$, $(-4)(-3) = 12$. Or: $(-4)[3 + (-3)] = (-4)(0) = 0$, and by the distributive property it equals $-12 + (-4)(-3)$, so $(-4)(-3)$ must be 12.

5.1 Negative Numbers and Absolute Value

OBJECTIVES

- Place positive and negative numbers on a number line and name the opposite of any number.
- Compare and order integers, and find the absolute value of a number.
- Use negative numbers to describe real situations such as cold temperatures, places below sea level and money owed.

WATCH

Math Antics - Negative Numbers

mathantics · 8:27

https://www.youtube.com/watch?v=OAoLCXpao6s

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Math Antics - Absolute Value

mathantics · 13:15

https://www.youtube.com/watch?v=BrYy1bgh3Y0

Before Example 2 (absolute value).

KEY TERMS

negative number

A number less than zero. It is written with a minus sign in front, like -5, and sits to the left of 0 on the number line.

positive number

A number greater than zero, like 5. It sits to the right of 0 on the number line. Zero itself is neither positive nor negative.

integer

Any whole number, its opposite, or zero: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

opposites

Two numbers that are the same distance from 0 but on different sides, like 7 and -7.

absolute value

The distance a number is from 0 on the number line. Distance is never negative, so $|-7| = 7$ and $|7| = 7$.

CORE IDEAS

Until now, every number you have used was zero or more. But some questions need numbers *less* than zero. If the temperature is 5 degrees below zero, or a diver is 30 feet below the surface of the sea, or you owe your brother 4 dollars, you need a way to say "below" or "owed." We use **negative numbers**: -5 degrees, -30 feet, -4 dollars.

Picture the number line. Zero sits in the middle. The positive numbers 1, 2, 3, ... march off to the right, and the negative numbers $-1, -2, -3, \ldots$ march off to the left, the same spacing apart. Each positive number has a mirror-image partner on the left side: 3 and -3 are **opposites**. The opposite of 0 is 0.

On a number line, numbers get larger as you move right and smaller as you move left. So -2 is greater than -9, because -2 is to the right of -9. This makes sense with money: owing 2 dollars is better than owing 9 dollars. It also makes sense with weather: $-2°\text{F}$ is warmer than $-9°\text{F}$. Every negative number is less than 0, and every positive number is greater than every negative number.

The **absolute value** of a number is its distance from 0. We write it with two straight bars: $|-6| = 6$ because -6 is 6 steps from zero, and $|6| = 6$ too. Absolute value tells you *how far*, not *which way*. A debt of 50 dollars and a savings of 50 dollars both have size 50; the sign tells you which side of zero you are on.

Be careful: a bigger absolute value does not mean a bigger number when the numbers are negative. $|-10| = 10$ is larger than $|-2| = 2$, yet $-10 < -2$. The number -10 is farther from zero, but it is farther to the *left*, so it is smaller.

KEY FORMULAS

$$|a| = \text{distance from } a \text{ to } 0, \quad \text{so } |-a| = |a|$$

$$\text{On a number line: farther right means greater.}$$

From history. In the year 628 the Indian mathematician Brahmagupta wrote a book that gave clear rules for working with zero and with negative numbers. He called positive numbers "fortunes" and negative numbers "debts," and he explained how to add, subtract and multiply them. Many European mathematicians were still uneasy about negative numbers more than a thousand years later.

Practice habit. Picture the number line every time you compare two numbers: whichever is farther to the right is greater. Say it aloud as you work: "Right is more, left is less."

WORKED EXAMPLES

Example 1. Order these integers from least to greatest: $-5,\ 2,\ -1,\ 0,\ -8$.

Show solution

1. Find the negative numbers: -5, -1 and -8. They are all less than 0 and 2.
2. Among the negatives, the one farthest left on the number line is smallest. -8 is farthest left, then -5, then -1.
3. Next comes 0, and then the positive number 2.
4. Check: each number is to the right of the one before it on the number line.

Answer: $-8,\ -5,\ -1,\ 0,\ 2$

Example 2. Grace records the morning temperature on her family's farm: Monday -7°F, Tuesday 3°F, Wednesday -12°F, Thursday -1°F. List the temperatures from coldest to warmest. Then find the absolute value of the coldest temperature and explain what it tells you.

Show solution

1. Colder means lower, which means farther left on the number line.
2. The negatives are -7, -12 and -1. Farthest left is -12, then -7, then -1.
3. The only positive temperature, 3, is the warmest.
4. Coldest to warmest: $-12, -7, -1, 3$.
5. The coldest temperature is -12°F, and $|-12| = 12$. It tells you Wednesday was 12 degrees away from zero, on the cold side.

Answer: -12°F, -7°F, -1°F, 3°F; $|-12| = 12$, so Wednesday was 12 degrees below zero.

PRACTICE

1. Easy What is the opposite of 8?

   Show answer

   -8

2. Easy Find $|-13|$.

   Show answer

   13

3. Easy Henry owes his father $25. Write an integer to stand for this amount.

   Show answer

   -25

4. Easy Which is greater, -4 or -9?

   Show answer

   -4

5. Medium Order from least to greatest: $3,\ -7,\ 0,\ -2,\ 5$.

   Show answer

   $-7,\ -2,\ 0,\ 3,\ 5$

6. **Medium** Find $|-6| + |4|$.

Show answer

10

7. **Medium** At sunrise the temperature in a mountain town was $-8°$F. At noon it was $-3°$F. Which temperature was warmer?

Show answer

$-3°$F

8. **Medium** A diver swims 40 feet below the surface of the sea. Write an integer for the diver's position, using 0 for sea level.

Show answer

-40 feet

9. **Hard** Which two integers have an absolute value of 7?

Show answer

-7 and 7

10. **Hard** Which number is farther from 0 on the number line, -15 or 12?

Show answer

-15 (it is 15 units from 0; 12 is only 12 units away)

11. **Hard** How many integers are strictly between -4 and 3? (Do not count -4 or 3.)

Show answer

6 (they are $-3, -2, -1, 0, 1, 2$)

12. **Challenge** Samuel says, "Since 10 is greater than 2, -10 must be greater than -2." Explain why Samuel is wrong.

Show answer

On the number line, -10 is to the left of -2, so $-10 < -2$. Think of money: owing 10 dollars is worse than owing 2 dollars. A larger absolute value only means farther from zero, and for negative numbers that means farther to the left, so smaller.

COMMON MISTAKES

- Thinking -10 is greater than -2 because 10 is greater than 2. Always picture the number line: the number farther right is greater.

- Writing an absolute value as negative, such as $|-5| = -5$. Absolute value is a distance, so it is never negative: $|-5| = 5$.
- Thinking zero is positive. Zero is neither positive nor negative; it is the dividing point between them.

> **Make it a game · Number Line Walk.** Tape a strip of paper on the floor and mark it from -10 to 10. Parent calls a number: you stand on it. Then answer the opposite, then the absolute value, then which of two numbers is greater. A wrong answer sends you back to zero. See how many rounds you can survive in a row.

5.2 Adding Integers

OBJECTIVES

- Add integers using a number line and the "debts and fortunes" model.
- Explain and use the rules for adding two integers with the same sign or with different signs.
- Solve word problems that combine gains and losses.

WATCH

Math Antics - Adding & Subtracting Integers

mathantics · 11:23

https://www.youtube.com/watch?v=_BgblvF90UE

> **WHEN TO WATCH**
> Watch the addition part before Example 1.

Note: Shared video: Lessons 5.2 and 5.3 use the same Math Antics video, the addition part here and the subtraction part in 5.3.

BACKUP VIDEO

Adding and Subtracting Integers (Applying operations to integers and visualising with Number line)

Eddie Woo · 11:58

https://www.youtube.com/watch?v=IgNWnYgNujg

Before the practice set: adding on a number line.

KEY TERMS

sum

The result of adding numbers together.

additive inverse

Another name for the opposite of a number. A number plus its additive inverse is always 0, as in $6 + (-6) = 0$.

fortune

An amount you have, shown by a positive number. (Brahmagupta's word.)

debt

An amount you owe, shown by a negative number. (Brahmagupta's word.)

CORE IDEAS

The number line model. To add, start at the first number. Adding a positive number means moving *right*; adding a negative number means moving *left*. For $-3 + 5$, start at -3 and move 5 steps right. You land on 2, so $-3 + 5 = 2$. For $4 + (-6)$, start at 4 and move 6 steps left. You land on -2.

The debts and fortunes model. Think of positive numbers as money you have (fortunes) and negative numbers as money you owe (debts). If you owe 3 dollars and then owe 4 more, you owe 7: $-3 + (-4) = -7$. If you have 8 dollars and owe 5, after paying the debt you have 3 left: $8 + (-5) = 3$. If you have 2 dollars and owe 9, after paying what you can, you still owe 7: $2 + (-9) = -7$.

Same signs. When both numbers are positive or both are negative, they push the same way. Add their absolute values and keep the common sign. Two debts make a bigger debt: $-12 + (-15) = -27$.

Different signs. When one number is positive and one is negative, they work against each other and partly cancel. Subtract the smaller absolute value from the larger, and give the answer the sign of the number with the larger absolute value. For $-14 + 9$: $14 - 9 = 5$, and the debt (14) is bigger than the fortune (9), so the answer is -5.

Opposites cancel. A number plus its opposite is zero: $7 + (-7) = 0$. Paying off a debt of 7 dollars with 7 dollars leaves you with nothing owed and nothing left. This is why we call opposites *additive inverses*.

When you add a long list of integers, the commutative and associative properties from Unit 1 still work. You may add all the positives, add all the negatives, and then combine the two totals.

KEY FORMULAS

Same signs: add the absolute values, keep the sign.

Different signs: subtract the absolute values, take the sign of the larger absolute value.

$$a + (-a) = 0$$

From history. Brahmagupta's rules from 628 include "a debt added to a debt is a debt" and "a fortune added to a fortune is a fortune." He also said that the sum of a fortune and an equal debt is zero. These are the same rules you are learning today.

Practice habit. Before adding, say the sign of the answer first: "Debt bigger, so negative" or "Fortune bigger, so positive." Then find the size. Recite the two rules until automatic: same signs add and keep; different signs subtract and take the larger's sign.

WORKED EXAMPLES

Example 1. Find $-7 + 3$.

Show solution

1. The signs are different, so the numbers partly cancel.
2. Absolute values: $|-7| = 7$ and $|3| = 3$. Subtract: $7 - 3 = 4$.
3. The larger absolute value belongs to -7, so the answer is negative.
4. $-7 + 3 = -4$.
5. Check on the number line: start at -7, move 3 right: $-6, -5, -4$.

Answer: -4

Example 2. On four plays, Samuel's football team gains 6 yards, loses 9 yards, loses 4 yards and then gains 12 yards. What is the team's total change in yards?

Show solution

1. Write gains as positive and losses as negative: $6 + (-9) + (-4) + 12$.
2. Add the positives: $6 + 12 = 18$.
3. Add the negatives: $-9 + (-4) = -13$.
4. Combine: $18 + (-13)$. Different signs: $18 - 13 = 5$, and 18 is larger, so the result is positive: 5.
5. Check left to right: $6 + (-9) = -3$; $-3 + (-4) = -7$; $-7 + 12 = 5$. It matches.

Answer: A gain of 5 yards $(+5)$

PRACTICE

1. Easy Find $-5 + (-3)$.

   Show answer

   -8

2. Easy Find $8 + (-3)$.

   Show answer

   5

3. Easy Find $-9 + 4$.

   Show answer

   -5

4. Easy Find $-6 + 6$.

   Show answer

   0

5. Medium Find $-14 + (-27)$.

   Show answer

   -41

6. **Medium** Find $23 + (-38)$.

Show answer

-15

7. **Medium** Early one winter morning the temperature was $-9°$F. By afternoon it had risen 15 degrees. What was the afternoon temperature?

Show answer

$6°$F

8. **Medium** Find $-12 + 5 + (-8) + 20$.

Show answer

5

9. **Hard** The Miller family's account had \$40 in it. They paid a bill of \$65, so the account was overdrawn, and then they deposited \$32. Write the sum $40 + (-65) + 32$ and find the final balance.

Show answer

\$7

10. **Hard** A submarine is at -250 feet (250 feet below sea level). It rises 85 feet and then dives 120 feet. What is its new position?

Show answer

-285 feet (285 feet below sea level)

11. **Hard** What number goes in the box? $-17 + \square = 4$

Show answer

21

12. **Challenge** Find $-1 + 2 + (-3) + 4 + (-5) + 6 + \cdots + (-19) + 20$. (The numbers 1 through 20 are added, with every odd number made negative.)

Show answer

10 (there are 10 pairs like $-1 + 2$, and each pair equals 1)

13. **Challenge** Explain, using debts and fortunes, why $-8 + 5$ is negative, and why its size is 3.

Show answer

Owing 8 dollars and having 5 dollars: you use the 5 dollars to pay part of the debt, and you still owe 3 dollars. Because the debt was bigger than the fortune, the result is a debt, -3; its size is $8 - 5 = 3$.

COMMON MISTAKES

- Adding the absolute values when the signs are different, as in $-9 + 4 = -13$. When signs differ, the numbers cancel partly, so subtract: $9 - 4 = 5$, then take the sign of -9: -5.
- Giving the answer the wrong sign. Before computing, ask which number has the larger absolute value; its sign is the sign of the answer.
- Moving the wrong way on the number line. Adding a negative number means moving left; adding a positive number means moving right.

> **Make it a game · Debts and Fortunes.** Use paper chips or beans: dark chips are debts and light chips are fortunes. Parent calls an addition such as -7 + 4. You lay out the chips, cancel pairs, and say what remains and what the sign is. Score a point per correct answer. Then write the rule for same signs and different signs from memory.

5.3 Subtracting Integers

OBJECTIVES

- Subtract integers by adding the opposite.
- Explain why subtracting a negative number gives the same result as adding a positive number.
- Use subtraction of integers to find changes and differences in temperature, elevation and money.

WATCH

Math Antics - Adding & Subtracting Integers

mathantics · 11:23

https://www.youtube.com/watch?v=_BgblvF90UE

> **WHEN TO WATCH**
> Watch the subtraction part (adding the opposite) before Example 1.

Note: Shared video: this lesson uses the subtraction part of the 5.2 video.

BACKUP VIDEO

Subtracting Integers with the Number Line

Eddie Woo · 12:04

https://www.youtube.com/watch?v=9vd9sl1qufI

Before Example 2: subtracting integers on a number line.

KEY TERMS

difference

The result of subtracting one number from another.

adding the opposite

Rewriting a subtraction as an addition by changing the number being subtracted to its opposite: $a - b = a + (-b)$.

change

How much a quantity went up or down, found by new value minus old value. A positive change is an increase; a negative change is a decrease.

CORE IDEAS

Look at this pattern. Each time the number being subtracted goes down by 1, the answer goes up by 1: $5 - 3 = 2$, $5 - 2 = 3$, $5 - 1 = 4$, $5 - 0 = 5$. If the pattern continues, the next lines must be $5 - (-1) = 6$ and $5 - (-2) = 7$. So subtracting a negative number makes the answer *bigger*, exactly as if we had added the positive number.

Debts and fortunes explain why. Suppose your worth is 5 dollars, but someone was counting a 2-dollar debt against you that you never really owed. If that debt is taken away, you are 2 dollars better off: $5 - (-2) = 5 + 2 = 7$. Taking away a debt is the same as giving a fortune. In the same way, taking away a fortune is the same as giving a debt: $5 - 2 = 5 + (-2) = 3$.

This gives one rule for every subtraction: **subtracting a number is the same as adding its opposite.** So $a - b = a + (-b)$. Change the subtraction sign to addition, change the second number to its opposite, and then use the addition rules from Lesson 5.2. Only the second number changes; the first number stays exactly as it is.

Subtraction also measures distance and change. The difference $a - b$ tells how far a is above b. If the high temperature is $7°\text{F}$ and the low is $-13°\text{F}$, the high is $7 - (-13) = 7 + 13 = 20$ degrees above the low. On a number line, from -13 up to 0 is 13 steps, and from 0 up to 7 is 7 more, for 20 in all.

To find a change, always compute *new value minus old value*. A negative answer means the quantity went down.

KEY FORMULAS

$$a - b = a + (-b)$$

$$a - (-b) = a + b$$

$$\text{change} = \text{new value} - \text{old value}$$

From history. Brahmagupta wrote in 628 that when a debt is taken away from zero, the result is a fortune, and when a fortune is taken away from zero, the result is a debt. In our symbols that is $0 - (-3) = 3$ and $0 - 3 = -3$.

Practice habit. Say "keep, change, opposite" for every subtraction: keep the first number, change minus to plus, and write the opposite of the second number. Then use the addition rules.

WORKED EXAMPLES

Example 1. Find $4 - 9$.

Show solution

1. Keep 4, change subtraction to addition, use the opposite of 9: $4 + (-9)$.
2. Different signs: $9 - 4 = 5$, and -9 has the larger absolute value, so the answer is negative.
3. $4 - 9 = -5$.
4. Check by adding back: $-5 + 9 = 4$. Correct.

Answer: -5

Example 2. On a January day in North Dakota, the high temperature was $7°$F and the low was $-13°$F. How many degrees warmer was the high than the low?

Show solution

1. "How much warmer" means high minus low: $7 - (-13)$.
2. Keep, change, opposite: $7 + 13$.
3. $7 + 13 = 20$.
4. Check on the number line: from -13 up to 0 is 13 degrees, and from 0 up to 7 is 7 degrees; $13 + 7 = 20$.

Answer: 20 degrees

PRACTICE

1. Easy Find $3 - 8$.

Show answer

-5

2. Easy Find $-6 - 4$.

Show answer

-10

3. Easy Find $5 - (-7)$.

Show answer

12

4. Easy Find $-9 - (-2)$.

Show answer

-7

5. Medium Find $-15 - (-15)$.

Show answer

0

6. Medium Find $0 - 11$.

Show answer

-11

7. Medium Find $-23 - 19$.

Show answer

-42

8. Medium A hilltop on Ruth's map is at an elevation of 340 feet. A nearby valley floor is at -45 feet (45 feet below sea level). How many feet higher is the hilltop than the valley floor?

Show answer

385 feet

9. Hard Find $-8 - (-3) + 6 - 10$.

Show answer

-9

10. Hard At noon the temperature was -4°F. By midnight it was -17°F. Find the change in temperature (new minus old).

Show answer

-13 degrees (it fell 13 degrees)

11. Hard What number goes in the box? $\square - (-6) = 2$

Show answer

-4 (check: $-4 + 6 = 2$)

12. Hard A pelican flies 18 feet above the surface of the sea, directly over a diver who is 32 feet below the surface. How far apart are they?

Show answer

50 feet

13. Challenge Find $7 - (-4)$ and $-4 - 7$. What do you notice about the two answers?

Show answer

11 and -11; switching the order of a subtraction gives the opposite answer.

14. Challenge Explain why $5 - (-3) = 8$. Use either a pattern or debts and fortunes.

Show answer

Pattern: $5 - 2 = 3$, $5 - 1 = 4$, $5 - 0 = 5$; each time we subtract one less, the answer grows by 1, so $5 - (-1) = 6$, $5 - (-2) = 7$, $5 - (-3) = 8$. Or: taking away a debt of 3 dollars makes you 3 dollars better off, so it is the same as adding 3.

COMMON MISTAKES

- Changing the sign of the first number as well as the second. In $-9 - (-2)$ only the second number changes: $-9 + 2 = -7$, not $9 + 2$.
- Forgetting to change the second number after changing minus to plus, as in $5 - (-7) = 5 + (-7)$. Keep, change, *opposite*: $5 + 7 = 12$.
- Subtracting in the wrong order when finding a change. Always compute new value minus old value, and check whether the answer should be an increase or a decrease.

Make it a game · Add the Opposite. Write 16 integer subtraction problems on cards. Draw a card, rewrite it as adding the opposite, and answer, all in 15 seconds. Parent holds a timer and calls the next card. Then explain aloud why subtracting a negative gives the same result as adding a positive. Score a point per card. Beat your total next round.

5.4 Multiplying Integers

OBJECTIVES

- Multiply integers using the sign rules.
- Explain why a negative times a negative is positive, using both a pattern and the distributive property.
- Find the sign of a product of several integers by counting the negative factors.

WATCH

Math Antics - Integer Multiplication & Division

mathantics · 9:51

https://www.youtube.com/watch?v=K_tPbVPfHgk

> **WHEN TO WATCH**
> Watch the multiplication part before Example 1.

Note: Shared video: Lessons 5.4 and 5.5 use the same Math Antics video.

BACKUP VIDEO

Ascending and Descending Integers (2 of 3: Noticing patterns in multiplications)

Eddie Woo · 8:50

https://www.youtube.com/watch?v=yqhoH7pn0KQ

After Example 1: the pattern argument for why a negative times a negative is positive (objective 2).

KEY TERMS

product

The result of multiplying numbers together.

factor

One of the numbers being multiplied. In $(-3)(4) = -12$, the factors are -3 and 4.

sign rules

The rules that tell whether a product is positive or negative: same signs give a positive product; different signs give a negative product.

CORE IDEAS

Positive times negative. Multiplication is repeated addition, so $3 \times (-4)$ means three debts of 4 dollars: $(-4) + (-4) + (-4) = -12$. Because order does not matter in multiplication (the commutative property), $(-4) \times 3$ is also -12. A positive times a negative is negative.

Negative times negative: the pattern. Look at this list. Each time the second factor goes down by 1, the product goes *up* by 3: $(-3)(2) = -6$, $(-3)(1) = -3$, $(-3)(0) = 0$. To keep the pattern going, the next lines must be $(-3)(-1) = 3$ and $(-3)(-2) = 6$. So a negative times a negative is positive.

Negative times negative: the distributive property. We know $2 + (-2) = 0$, so $(-3) \times [2 + (-2)] = (-3) \times 0 = 0$. But the distributive property says this also equals $(-3)(2) + (-3)(-2) = -6 + (-3)(-2)$. For $-6 + (-3)(-2)$ to equal 0, the product $(-3)(-2)$ must be the opposite of -6, which is 6. The rules of arithmetic you already trust *force* a negative times a negative to be positive.

The sign rules. Multiply the absolute values, then decide the sign. Same signs (both positive or both negative): the product is positive. Different signs: the product is negative. Anything times 0 is 0.

Many factors. Each pair of negative factors makes a positive. So count the negative factors: an *even* number of them gives a positive product, and an *odd* number gives a negative product. For $(-2)(-3)(-4)$ there are three negatives (odd), so the product is -24.

In real life, a negative times a negative often means "going back in time" while something is decreasing. If a tank loses 5 gallons each hour, then 2 hours *ago* it had $(-5)(-2) = 10$ more gallons than it has now.

KEY FORMULAS

$$(+)(+) = + \qquad (-)(-) = + \qquad (+)(-) = - \qquad (-)(+) = -$$

$$a \times 0 = 0$$

Even number of negative factors: positive. Odd number: negative.

From history. Brahmagupta's rules of 628 state that the product of two debts is a fortune, the product of a debt and a fortune is a debt, and the product of two fortunes is a fortune. Mathematicians argued for centuries about why "debt times debt" should be a fortune; the distributive-property argument in this lesson is the reason accepted today.

Practice habit. Recite the sign rules until automatic: "Same signs, positive; different signs, negative." When multiplying several numbers, count the negatives first and write down the sign before you multiply.

WORKED EXAMPLES

Example 1. Find $(-8)(-7)$.

Show solution

1. Multiply the absolute values: $8 \times 7 = 56$.
2. The signs are the same (both negative), so the product is positive.
3. $(-8)(-7) = 56$.

Answer: 56

Example 2. On a cold evening the temperature on Henry's porch drops steadily by 3 degrees every hour. Right now it reads 2°F. What did it read 5 hours ago?

Show solution

1. The rate of change is -3 degrees per hour (a drop).
2. Five hours ago is a change in time of -5 hours (going back).
3. Change in temperature: $(-3)(-5) = 15$. Going back in time, the temperature was 15 degrees *higher*.
4. Temperature 5 hours ago: $2 + 15 = 17$°F.
5. Check by going forward: start at 17 and drop 3 degrees five times: $17 - 15 = 2$. Correct.

Answer: 17°F

PRACTICE

1. Easy Find $(-4)(6)$.

   Show answer

   -24

2. Easy Find $(-7)(-3)$.

   Show answer

   21

3. Easy Find $9 \times (-8)$.

   Show answer

   -72

4. Easy Find $(-1)(-15)$.

   Show answer

   15

5. Medium Find $(-12)(5)$.

   Show answer

   -60

6. Medium Find $(-2)(-3)(-4)$.

   Show answer

   -24

7. Medium A hiker walks down a steep trail, losing 45 feet of elevation each minute for 8 minutes. Write the change in elevation as a product and find it.

   Show answer

   $(-45)(8) = -360$ feet (down 360 feet)

8. Medium Ruth takes \$6 out of her savings jar every week for 7 weeks to buy art supplies. What integer describes the total change in her savings?

   Show answer

   -42 dollars (\$42 less)

9. Hard Find $(-3)(4)(-5)(-1)$.

Show answer

-60

10. Hard What number goes in the box? $(-9) \times \square = 54$

Show answer

-6

11. Hard The temperature inside a freezer is falling 4 degrees every hour. Right now it is 0°F. Use multiplication of integers to find what the temperature was 3 hours ago.

Show answer

12°F, since $(-4)(-3) = 12$

12. Hard Find $(-5)(-2)(-10) + (-6)(-6)$.

Show answer

-64 (because $-100 + 36 = -64$)

13. Challenge Use the distributive property to explain why $(-2)(-5) = 10$. (Hint: start with $(-2) \times [5 + (-5)]$.)

Show answer

Since $5 + (-5) = 0$, we have $(-2) \times [5 + (-5)] = 0$. By the distributive property it also equals $(-2)(5) + (-2)(-5) = -10 + (-2)(-5)$. For $-10 + (-2)(-5)$ to be 0, $(-2)(-5)$ must be 10.

COMMON MISTAKES

- Mixing up the addition and multiplication rules, such as writing $(-3)(-4) = -12$ because "two negatives make a bigger negative." That is true for adding, $-3 + (-4) = -7$, but for multiplying, two negatives give a positive: $(-3)(-4) = 12$.
- Getting the sign wrong with three or more factors. Count the negative factors: even means positive, odd means negative.
- Forgetting that any number times 0 is 0, even a negative number: $(-25)(0) = 0$.

Make it a game · Sign Rule Charades. Assign gestures: thumbs up means positive, thumbs down means negative. Parent calls an integer multiplication such as -6 x -3. You show the sign of the answer with your gesture before saying the product. Wrong gesture, no point. Then give the pattern reason: -3 x 2, -3 x 1, -3 x 0, and so on. Play for ten points.

5.5 Dividing Integers

OBJECTIVES

- Divide integers using the same sign rules as multiplication, and explain why those rules hold.
- Explain why division by zero is undefined, while zero divided by a nonzero number is 0.
- Solve word problems that share a loss equally or find an average change.

WATCH

Math Antics - Integer Multiplication & Division

mathantics · 9:51

https://www.youtube.com/watch?v=K_tPbVPfHgk

> **WHEN TO WATCH**
> Watch the division part before Example 1.

Note: Shared video: this lesson uses the division part of the 5.4 video.

BACKUP VIDEO

Dividing by zero?

Eddie Woo · 9:09

https://www.youtube.com/watch?v=J2z5uzqxJNU

Before Example 2: why you cannot divide by zero (objective 2).

KEY TERMS

quotient

The result of dividing one number by another.

dividend

The number being divided. In $-20 \div 4$, the dividend is -20.

divisor

The number you divide by. In $-20 \div 4$, the divisor is 4.

undefined

Having no meaning or value. Dividing by zero is undefined because no number works as the answer.

CORE IDEAS

Division undoes multiplication. Saying $-20 \div 4 = \square$ is the same as asking $\square \times 4 = -20$. The only number that works is -5, so $-20 \div 4 = -5$. Every division fact is really a multiplication fact read backward.

This is why division follows the same sign rules as multiplication. $-20 \div (-4) = 5$, because $5 \times (-4) = -20$. And $20 \div (-4) = -5$, because $(-5) \times (-4) = 20$. Same signs give a positive quotient; different signs give a negative quotient. Divide the absolute values and then attach the sign.

A fraction bar also means division, so $\dfrac{-36}{9} = -4$ and $\dfrac{-36}{-9} = 4$. Notice $\dfrac{-36}{9}$, $\dfrac{36}{-9}$ and $-\dfrac{36}{9}$ are all equal to -4.

Zero divided by a number. $0 \div 7 = 0$, because $0 \times 7 = 0$. Sharing nothing among 7 people gives each person nothing.

A number divided by zero. Try $6 \div 0 = \square$. That would mean $\square \times 0 = 6$. But every number times 0 is 0, never 6, so no number works. And $0 \div 0$ fails in a different way: $\square \times 0 = 0$ is true for *every* number, so there is no single answer. Either way, division by zero has no answer. We say it is **undefined**, and zero can never be a divisor.

Division of integers is useful for sharing a loss equally (a debt split among several people) and for finding an average: add the values and divide by how many there are. A negative average means the values were mostly below zero.

KEY FORMULAS

$$\frac{+}{+} = + \qquad \frac{-}{-} = + \qquad \frac{+}{-} = - \qquad \frac{-}{+} = -$$

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

$$0 \div a = 0 \ (a \neq 0); \qquad a \div 0 \text{ is undefined}$$

From history. Brahmagupta in 628 gave the sign rules for division of fortunes and debts, matching the rules for multiplication. He also wrote about dividing by zero and said that zero divided by zero is zero. Later mathematicians saw that this does not fit the other rules, and today we say that division by zero is undefined.

> **Practice habit.** Check every division by multiplying back: quotient times divisor must equal the dividend. For $-42 \div 6 = -7$, say "-7 times 6 is -42, check."

WORKED EXAMPLES

Example 1. Find $-48 \div (-6)$.

Show solution

1. Divide the absolute values: $48 \div 6 = 8$.
2. The signs are the same (both negative), so the quotient is positive.
3. $-48 \div (-6) = 8$.
4. Check: $8 \times (-6) = -48$. Correct.

Answer: 8

Example 2. Samuel recorded the morning temperature for five days: $-4°\text{F}$, $3°\text{F}$, $-9°\text{F}$, $-6°\text{F}$ and $1°\text{F}$. What was the average (mean) morning temperature?

Show solution

1. Add the temperatures: $-4 + 3 + (-9) + (-6) + 1$.
2. Positives: $3 + 1 = 4$. Negatives: $-4 + (-9) + (-6) = -19$.
3. Sum: $4 + (-19) = -15$.
4. Divide by the number of days: $-15 \div 5 = -3$.
5. Check: $-3 \times 5 = -15$. Correct.

Answer: $-3°\text{F}$

PRACTICE

1. Easy Find $-36 \div 9$.

   Show answer

   -4

2. Easy Find $-42 \div (-7)$.

   Show answer

   6

3. **Easy** Find $56 \div (-8)$.

Show answer

-7

4. **Easy** Find $0 \div (-12)$.

Show answer

0

5. **Medium** Find $\dfrac{-81}{-9}$.

Show answer

9

6. **Medium** Find $-100 \div 25$.

Show answer

-4

7. **Medium** A diver goes from the surface down to -84 feet in 7 equal stages. What integer describes the change in depth during each stage?

Show answer

-12 feet

8. **Medium** Find $(-6)(4) \div (-3)$.

Show answer

8

9. **Hard** Four brothers broke a neighbor's window playing ball and together owe \$96 to replace it. Write the debt as -96 and divide it equally among the four brothers. What integer stands for each brother's share?

Show answer

-24 (each owes \$24)

10. **Hard** What number goes in the box? $\square \div (-5) = 11$

Show answer

-55

11. Hard Find $-144 \div 12 \div (-3)$. (Work from left to right.)

Show answer

4

12. Hard Grace recorded these low temperatures for six winter nights: $-8°\text{F}$, $-3°\text{F}$, $2°\text{F}$, $-11°\text{F}$, $-5°\text{F}$, $-5°\text{F}$. Find the average (mean) low temperature.

Show answer

$-5°\text{F}$

13. Challenge Explain why $-7 \div 0$ is undefined but $0 \div (-7) = 0$.

Show answer

$0 \div (-7) = 0$ because $0 \times (-7) = 0$. But $-7 \div 0 = \square$ would mean $\square \times 0 = -7$, and every number times 0 is 0, so no number works. Since there is no answer, dividing by zero is undefined.

COMMON MISTAKES

- Using the addition rules instead of the multiplication rules, such as $-42 \div (-7) = -6$. Two negatives divided give a positive: 6.
- Saying $5 \div 0 = 0$ or $5 \div 0 = 5$. Division by zero is undefined; check by multiplying back and you will see no number works.
- Confusing $0 \div 5$ with $5 \div 0$. Zero divided by a nonzero number is 0; a number divided by zero is undefined.

> **Make it a game · Undefined Duel.** Prepare 14 cards of integer division problems, including a few with zero. Draw a card and answer in five seconds. Zero in the divisor means shout undefined and explain why no number works. Zero divided by a nonzero number is 0. Score a point per right answer. Then each of you writes one tricky problem to stump the other.

5.6 Operations with Rational Numbers

OBJECTIVES

- Add, subtract, multiply and divide negative fractions, mixed numbers and decimals using the integer sign rules.
- Evaluate expressions with integers and rational numbers using the order of operations.
- Compare and order positive and negative fractions and decimals.

WATCH

Adding and Subtracting Negative Fractions | Math with Mr. J

Math with Mr. J · 14:41

https://www.youtube.com/watch?v=KcVJqDp2YUI

> **WHEN TO WATCH**
> Before Example 1: adding and subtracting negative fractions and mixed numbers.

Note: The order of operations and comparing rational numbers (objectives 2 and 3) review earlier lessons. See Lesson 1.4 and Unit 5.

BACKUP VIDEO

How to Multiply Negative Fractions | Math with Mr. J

Math with Mr. J · 5:20

https://www.youtube.com/watch?v=fR-wfrvgP2A

Before the multiplication and division examples: multiplying negative fractions. His companion video 'How to Divide Negative Fractions' (youtube.com/watch?v=2A4OkcsGtFU) covers division.

KEY TERMS

rational number

Any number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and b is not 0. Integers, fractions, mixed numbers, terminating decimals and repeating decimals, positive or negative, are all rational numbers.

order of operations

The agreed order for doing arithmetic: grouping symbols first, then exponents, then multiplication and division from left to right, then addition and subtraction from left to right.

negative fraction

A fraction less than zero, such as $-\frac{3}{4}$. It can be written three ways: $-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}$.

CORE IDEAS

Negative numbers are not only integers. A temperature can be $-2.5°\text{F}$, and a water level can drop $\frac{3}{4}$ of an inch. A **rational number** is any number you can write as a fraction of two integers (with a nonzero bottom). Every integer is rational, because $-6 = \frac{-6}{1}$, and so is every decimal that ends or repeats, such as $-0.25 = -\frac{1}{4}$.

The good news: nothing new has to be learned about signs. **The sign rules for integers work for all rational numbers.** First decide the sign using the rules from Lessons 5.2 to 5.5; then do the fraction or decimal arithmetic you learned in Units 3 and 4 with the absolute values. For example, -0.6×0.5: different signs, so negative; $0.6 \times 0.5 = 0.3$; answer -0.3.

Why is $-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}$? A fraction bar means division. $(-3) \div 4$ and $3 \div (-4)$ both have different signs, so both equal the negative number -0.75, which is $-\frac{3}{4}$. Usually we write the minus sign in front of the fraction or in the numerator, never in the denominator.

To add or subtract negative fractions, get a common denominator just as before, then add or subtract the numerators using the integer rules: $-\frac{2}{3} + \frac{1}{4} = -\frac{8}{12} + \frac{3}{12} = \frac{-8+3}{12} = -\frac{5}{12}$. With a negative mixed number, it is safest to change it to an improper fraction first, because $-2\frac{1}{2}$ means $-(2 + \frac{1}{2})$, the whole thing is negative.

The order of operations does not change when negatives appear: grouping symbols, then exponents, then multiplication and division left to right, then addition and subtraction left to right. Put parentheses around negative numbers that follow an operation sign, as in $8 - 3 \times (-4)$, so the two signs do not run together. A fraction bar acts like grouping symbols: work out the whole top and the whole bottom before dividing.

To compare negative fractions and decimals, change them to decimals (or to a common denominator) and picture the number line. $-\frac{3}{4} = -0.75$ is less than -0.7, because -0.75 is farther left.

KEY FORMULAS

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

Same signs: positive product or quotient. Different signs: negative.

Order: grouping, exponents, $\times$ and $\div$ left to right, $+$ and $-$ left to right

Practice habit. Sign first, size second. Before any computation with fractions or decimals, decide and write down the sign of the answer, then work with the positive numbers. Recite the order of operations aloud before each multi-step problem.

WORKED EXAMPLES

Example 1. Find $-\frac{2}{3} + \frac{1}{4}$.

Show solution

1. The least common denominator of 3 and 4 is 12.
2. Rewrite: $-\frac{2}{3} = -\frac{8}{12}$ and $\frac{1}{4} = \frac{3}{12}$.
3. Add the numerators with the integer rules: $-8 + 3 = -5$.
4. The sum is $-\frac{5}{12}$, already in lowest terms.
5. Check the sign: $\frac{2}{3}$ is bigger than $\frac{1}{4}$, and it is the negative one, so the answer should be negative. It is.

Answer: $-\frac{5}{12}$

Example 2. At 6 a.m. the thermometer at Ruth's house read -2.5°F. The temperature rose 1.5 degrees each hour for 4 hours, and then a cold wind made it fall 3.25 degrees. Write one expression for the final temperature and evaluate it.

Show solution

1. Expression: $-2.5 + 4 \times 1.5 - 3.25$.
2. Multiply first: $4 \times 1.5 = 6$. Now $-2.5 + 6 - 3.25$.
3. Add and subtract left to right: $-2.5 + 6 = 3.5$.
4. $3.5 - 3.25 = 0.25$.
5. Check: after 4 hours it was $-2.5 + 6 = 3.5$°F, which is reasonable; falling 3.25 degrees leaves it just above zero.

Answer: 0.25°F

PRACTICE

1. Easy Find $-3.7 + 1.2$.

   Show answer

   -2.5

2. **Easy** Find -0.6×0.5.

Show answer

-0.3

3. **Easy** Find $-\frac{1}{2} - \frac{1}{3}$.

Show answer

$-\frac{5}{6}$

4. **Easy** Find $\left(-\frac{3}{4}\right) \times \left(-\frac{8}{9}\right)$.

Show answer

$\frac{2}{3}$

5. **Medium** Find $-4.8 \div (-0.6)$.

Show answer

8

6. **Medium** Find $-\frac{5}{6} \div \frac{10}{3}$.

Show answer

$-\frac{1}{4}$

7. **Medium** Find $8 - 3 \times (-4)$.

Show answer

20

8. **Medium** Find $-2\frac{1}{2} + 4\frac{1}{3}$.

Show answer

$1\frac{5}{6}$

9. **Medium** Grace has \$12.40. She wants to buy 3 notebooks that cost \$4.85 each. Evaluate $12.40 - 3 \times 4.85$. What does the answer tell Grace?

Show answer

-2.15; she is \$2.15 short.

10. Hard Over four weeks, the water level of the Miller family's farm pond changed by -0.35 ft, $+0.12$ ft, -0.48 ft and $+0.20$ ft. What was the total change in the water level?

Show answer

-0.51 ft (it went down 0.51 ft)

11. Hard Find $(-6 + 2) \times 5 - (-9)$.

Show answer

-11

12. Hard Find $\dfrac{-18 \div (-3) + 4}{-2}$.

Show answer

-5

13. Hard Henry measured the temperature at noon on four winter days: $-3.5°\text{F}$, $2.0°\text{F}$, $-1.5°\text{F}$ and $-4.0°\text{F}$. Find the average (mean) noon temperature.

Show answer

$-1.75°\text{F}$

14. Challenge Order from least to greatest: $-0.7,\ -\dfrac{3}{4},\ -\dfrac{2}{3},\ 0.5,\ -0.07$.

Show answer

$-\dfrac{3}{4},\ -0.7,\ -\dfrac{2}{3},\ -0.07,\ 0.5$

15. Challenge Explain why $\dfrac{-3}{4}$ and $\dfrac{3}{-4}$ are the same number.

Show answer

A fraction bar means division. $(-3) \div 4$ and $3 \div (-4)$ each divide numbers with different signs, so each is negative, and each has size $3 \div 4 = 0.75$. Both equal -0.75, which is $-\dfrac{3}{4}$.

COMMON MISTAKES

- Treating a negative mixed number as part negative and part positive, as in $-2\frac{1}{2} = -2 + \frac{1}{2}$. The minus sign applies to the whole number: $-2\frac{1}{2} = -\dfrac{5}{2}$. Change to an improper fraction first.

- Working left to right without following the order of operations, as in $8 - 3 \times (-4) = 5 \times (-4) = -20$. Multiply first: $8 - (-12) = 20$.
- Ordering negative decimals by their digits, thinking $-0.75 > -0.7$ because 75 > 7. Picture the number line: -0.75 is farther left, so it is less.
- Losing a sign halfway through a long problem. Decide and write the sign at each step before working out the size.

> **Make it a game · Rational Relay.** Write 12 problems with negative fractions, mixed numbers and decimals on cards. Place them in a row. Solve one at a time, stepping forward only if the answer is correct. Parent checks with a second method. Add one order-of-operations problem as the finish line. Time your relay and then try to cut the time.

UNIT 5

Integers and Rational Numbers

Until now every number in this course has been zero or greater. In this unit the number line stretches to the left of zero. Negative numbers let us describe temperatures below zero, places below sea level and money that is owed, and they make subtraction always possible: $3 - 8$ finally has an answer.

Students learn to compare integers and find absolute value, and then to add, subtract, multiply and divide them. Each sign rule is explained, not just memorized: with the number line, with Brahmagupta's "debts and fortunes," with number patterns and with the distributive property, which shows why a negative times a negative must be positive. The unit ends by applying the same rules to negative fractions and decimals and to the order of operations.

Parents: this unit is done by hand, without a calculator. The sign rules here are used in every later unit, especially in exponents, equations, graphing and Algebra I, so steady practice now pays off for years.

PREREQUISITES

- Place a whole number, fraction or decimal on a number line (Lessons 3.1 and 4.1).
- Use the commutative, associative and distributive properties (Lessons 1.2 and 1.3).
- Follow the order of operations (Lesson 1.4).
- Add, subtract, multiply and divide fractions and mixed numbers (Unit 3).
- Add, subtract, multiply and divide decimals (Unit 4).

MASTERY CHECKLIST

- ☐ I can use negative numbers to describe temperatures, elevations and money owed.
- ☐ I can find the opposite and the absolute value of a number.
- ☐ I can compare and order integers using a number line.
- ☐ I can add integers with the same sign or different signs.
- ☐ I can subtract integers by adding the opposite, and explain why that works.
- ☐ I can multiply integers and explain why a negative times a negative is positive.
- ☐ I can find the sign of a product by counting negative factors.
- ☐ I can divide integers and explain why dividing by zero is undefined.
- ☐ I can add, subtract, multiply and divide negative fractions, mixed numbers and decimals.
- ☐ I can use the order of operations with negative numbers.
- ☐ I can solve word problems involving gains and losses, changes and averages.

LESSONS IN THIS UNIT

5.1	**Negative Numbers and Absolute Value**	45 minutes
5.2	**Adding Integers**	2 sessions of 45 minutes
5.3	**Subtracting Integers**	2 sessions of 45 minutes

5.4	**Multiplying Integers**	2 sessions of 45 minutes
5.5	**Dividing Integers**	2 sessions of 45 minutes
5.6	**Operations with Rational Numbers**	2 sessions of 45 minutes

Unit 5 test

You have 50 minutes. Do not use a calculator. Show your work for every problem. Give fraction answers in lowest terms. Write negative answers with a minus sign, and include units where the problem has them.

1. **5 pts**
   Order from least to greatest: $-3,\ 5,\ -8,\ 0,\ -1$.

2. **5 pts**
   Find $|-21| - |9|$.

3. **5 pts**
   Find $-13 + 8$.

4. **5 pts**
   Find $-17 + (-26)$.

5. **5 pts**
   Find $6 - 15$.

6. **5 pts**
   Find $-12 - (-20)$.

7. **5 pts**
   Find $(-9)(-6)$.

8. **6 pts**
   Find $(-2)(5)(-3)(-1)$.

9. **6 pts**

   Find $-72 \div (-8)$.

10. **6 pts**

    Find $-45 \div 5 \times (-2)$. (Work from left to right.)

11. **7 pts**

    At 4 p.m. the temperature was $11°\text{F}$. By 4 a.m. it was $-16°\text{F}$. Find the change in temperature (new minus old).

12. **7 pts**

    Samuel's account had \$30 in it. He paid a bill of \$52, and then deposited \$15. Write the sum $30 + (-52) + 15$ and find his final balance.

13. **7 pts**

    Find $-\dfrac{3}{5} + \dfrac{1}{2}$.

14. **7 pts**

    Find $-4 + 6 \times (-3 + 1) \div (-4)$.

15. **9 pts**

    Ruth recorded the low temperature on five winter nights: $-6°\text{F}$, $-2°\text{F}$, $3°\text{F}$, $-7°\text{F}$ and $-8°\text{F}$. Find the average (mean) low temperature.

16. **10 pts**

    Explain why $(-4)(-3) = 12$. Use either a pattern or the distributive property.

ANSWER KEY

1. $-8,\ -3,\ -1,\ 0,\ 5$
2. 12
3. -5
4. -43
5. -9
6. 8
7. 54
8. -30
9. 9
10. 18
11. -27 degrees (it fell 27 degrees)
12. -7 dollars (the account is overdrawn by \$7)
13. $-\dfrac{1}{10}$
14. -1
15. $-4°\text{F}$
16. Pattern: $(-4)(2) = -8$, $(-4)(1) = -4$, $(-4)(0) = 0$; each step the product goes up by 4, so $(-4)(-1) = 4$, $(-4)(-2) = 8$, $(-4)(-3) = 12$. Or: $(-4)[3 + (-3)] = (-4)(0) = 0$, and by the distributive property it equals $-12 + (-4)(-3)$, so $(-4)(-3)$ must be 12.

UNIT 6

Exponents and Square Roots

In this unit you learn a powerful shorthand. An exponent tells how many times to multiply a number by itself, and a few simple laws, each one explained by writing out the factors, let you multiply and divide powers quickly. You will use powers of 10 to write very large and very small numbers in scientific notation, the way scientists and engineers do.

Then you will turn squaring around. A square root answers the question "what number times itself gives this?" You will learn the perfect squares to 400 by heart, estimate square roots that are not whole numbers, and meet irrational numbers such as $\sqrt{2}$ and π, whose decimals never end or repeat. The unit closes by putting every kind of number you have studied into one family, the real numbers.

These skills are the doorway to algebra and geometry. Exponents appear in every algebra course, scientific notation in every science course, and square roots in the Pythagorean theorem (Unit 12). Parents: the key habits are reciting the powers of 2, the powers of 10 and the perfect squares to 400, and always asking "what is the base?" before evaluating a power with a minus sign.

PREREQUISITES

- Squares, cubes and the order of operations (Lesson 1.4)
- Multiplying and dividing fractions (Lessons 3.5 and 3.6)
- Multiplying and dividing decimals, and multiplying by powers of 10 (Lessons 4.3 and 4.4)
- Terminating and repeating decimals (Lesson 4.5)
- Multiplying integers and the sign rules (Lessons 5.4 and 5.5)

MASTERY CHECKLIST

- ☐ I can write repeated multiplication with an exponent and evaluate powers, including powers of 10.
- ☐ I can explain the difference between $(-3)^2$ and -3^2 and evaluate each.
- ☐ I can use the product, quotient and power-of-a-power rules and explain each one by writing out factors.
- ☐ I can explain why a nonzero number to the zero power equals 1.
- ☐ I can explain why $10^{-n} = \frac{1}{10^n}$.
- ☐ I can convert numbers between standard form and scientific notation, and compare them.
- ☐ I can multiply numbers in scientific notation and write the answer in correct scientific notation.
- ☐ I can find square roots of perfect squares up to 400 and of fractions made of perfect squares.
- ☐ I can estimate a square root between two whole numbers and to the nearest tenth.
- ☐ I can tell rational numbers from irrational numbers and place a number in the real number system.

LESSONS IN THIS UNIT

6.1	**Exponents and Powers**	45 minutes
6.2	**Laws of Exponents**	2 sessions of 45 minutes
6.3	**Scientific Notation**	2 sessions of 45 minutes
6.4	**Square Roots and Perfect Squares**	2 sessions of 45 minutes
6.5	**Estimating Roots and Irrational Numbers**	2 sessions of 45 minutes

Unit 6 test

You have 60 minutes. No calculator. Show all your work. Give answers in scientific notation when a problem asks for it, and round square-root estimates to the nearest tenth when told to. Write units in your answers to word problems.

1. **5 pts**
   Evaluate 3^5.

2. **6 pts**
   Evaluate $(-4)^2$ and -4^2.

3. **6 pts**
   Write $2^5 \times 2^3$ as a single power, then evaluate it.

4. **6 pts**
   Write $\dfrac{(7^3)^2}{7^4}$ as a single power, then evaluate it.

5. **5 pts**
   Evaluate $12^0 + 10^{-1}$. Write your answer as a decimal.

6. **7 pts**
   Write 0.000056 in scientific notation.

7. **7 pts**

   Multiply $(4 \times 10^{6})(3 \times 10^{-2})$. Give your answer in scientific notation.

8. **8 pts**

   A grain elevator holds about 4×10^{5} bushels of wheat. One bushel holds about 1×10^{6} kernels. About how many kernels are in the full elevator? Give your answer in scientific notation.

9. **6 pts**

   Evaluate $\sqrt{196}$.

10. **6 pts**

    Evaluate $\sqrt{\dfrac{25}{64}}$.

11. **7 pts**

    A square vegetable garden has an area of 289 square feet. How many feet of fencing are needed to go around all four sides?

12. **8 pts**

    Estimate $\sqrt{85}$ to the nearest tenth. Show how you decided.

13. **7 pts**

    Grace makes a square picture frame with an area of 45 square inches. How long is each side, to the nearest tenth of an inch?

14. **5 pts**

    Is $\sqrt{48}$ a rational number? Answer yes or no, and give a reason.

15. **11 pts**

    Explain why $4^{0} = 1$. Use the quotient rule or a pattern of powers of 4.

ANSWER KEY

1. 243
2. $(-4)^2 = 16$ and $-4^2 = -16$
3. $2^8 = 256$
4. $7^2 = 49$
5. $1 + 0.1 = 1.1$
6. 5.6×10^{-5}
7. $12 \times 10^4 = 1.2 \times 10^5$
8. About 4×10^{11} kernels
9. 14
10. $\frac{5}{8}$
11. 68 feet (each side is 17 feet; $4 \times 17 = 68$)
12. 9.2 ($9.2^2 = 84.64$, $9.3^2 = 86.49$, and $9.25^2 = 85.5625$ is more than 85)
13. About 6.7 inches ($6.7^2 = 44.89$, $6.8^2 = 46.24$, and $6.75^2 = 45.5625$ is more than 45)
14. No; 48 is not a perfect square, so $\sqrt{48}$ is irrational
15. By the quotient rule, $\frac{4^3}{4^3} = 4^{3-3} = 4^0$. But $\frac{4^3}{4^3} = \frac{64}{64} = 1$, because any nonzero number divided by itself is 1. So $4^0 = 1$. The pattern agrees: $4^2 = 16$, $4^1 = 4$, and dividing by 4 again gives $4^0 = 1$.

6.1 Exponents and Powers

OBJECTIVES

- Write repeated multiplication in exponent form and evaluate powers, including powers of 10.
- Evaluate powers with negative bases and explain the difference between $(-3)^2$ and -3^2.

WATCH

Math Antics - Intro To Exponents (aka Indices)

mathantics · 10:05

https://www.youtube.com/watch?v=-zUmvpkhvW8

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Exponents with Negative Bases | Math with Mr. J

Math with Mr. J · 9:02

https://www.youtube.com/watch?v=Yg_bC19aNjY

Before the negative-base example (objective 2): why $(-5)^2 = 25$ but $-5^2 = -25$.

KEY TERMS

power

A number written with a base and an exponent, such as 5^3; it also means the value, so 125 is a power of 5.

base

The number being multiplied over and over. In 5^3 the base is 5.

exponent

The small raised number that tells how many times the base is used as a factor. In 5^3 the exponent is 3.

power of 10

A number such as $10^1 = 10$, $10^2 = 100$ or $10^3 = 1{,}000$ made by multiplying 10 by itself.

squared and cubed

Special names for exponents 2 and 3: 6^2 is read "6 squared" and 6^3 is read "6 cubed."

CORE IDEAS

In Lesson 1.4 you met squares and cubes. Now we use any whole-number exponent. An exponent is a shorthand for repeated multiplication, just as multiplication is a shorthand for repeated addition. Instead of writing $2 \times 2 \times 2 \times 2 \times 2$, we write 2^5 and read it "2 to the fifth power." The base 2 is the factor, and the exponent 5 counts how many times it appears.

Be careful: 2^5 is NOT 2×5. It is $2 \times 2 \times 2 \times 2 \times 2 = 32$, while $2 \times 5 = 10$. Powers grow fast. An exponent of 1 means the base appears once, so $7^1 = 7$.

Powers of 10 are the easiest powers of all, and they are the backbone of place value. $10^2 = 100$, $10^3 = 1{,}000$, $10^6 = 1{,}000{,}000$. The exponent tells you how many zeros follow the 1. That is why $4 \times 10^3 = 4{,}000$: multiplying by 10^3 moves every digit three places to the left.

A negative number can be a base too, but it must be inside parentheses. $(-3)^2$ means $(-3) \times (-3) = 9$, because a negative times a negative is positive (Lesson 5.4). With more factors, the signs pair up: an even number of negative factors gives a positive answer, and an odd number gives a negative answer. So $(-2)^4 = 16$ but $(-2)^3 = -8$.

Without parentheses, the meaning changes. In -3^2, the exponent belongs only to the 3. The order of operations says exponents come first, and the minus sign in front is like multiplying by -1 afterward. So $-3^2 = -(3 \times 3) = -9$. The parentheses decide what the base is: in $(-3)^2$ the base is -3; in -3^2 the base is 3.

KEY FORMULAS

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}$$

$$10^n = 1\underbrace{00\ldots0}_{n \text{ zeros}}$$

$$(-3)^2 = 9 \qquad -3^2 = -9$$

Even number of negative factors: positive. Odd number of negative factors: negative.

Practice habit. Recite the powers of 2 up to 2^{10} (2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024) until they are automatic. Before evaluating any power with a minus sign, ask out loud: "What is the base?"

WORKED EXAMPLES

Example 1. Evaluate $(-2)^4$ and -2^4.

Show solution

1. In $(-2)^4$ the parentheses make the base -2: $(-2) \times (-2) \times (-2) \times (-2)$.
2. Pair the factors: $(-2) \times (-2) = 4$ and $(-2) \times (-2) = 4$.
3. $4 \times 4 = 16$, so $(-2)^4 = 16$. Four negative factors (an even number) gives a positive answer.
4. In -2^4 the base is only 2. Do the exponent first: $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
5. Then apply the minus sign: $-2^4 = -16$.

Answer: $(-2)^4 = 16$ and $-2^4 = -16$

Example 2. Grace starts a savings jar. On day 1 she puts in 1 cent, and each day after that she puts in twice as much as the day before. How much does she put in on day 11? Give your answer in dollars.

Show solution

1. Day 1: 1 cent. Day 2: 2 cents. Day 3: 4 cents. Day 4: 8 cents.
2. Look for the pattern: day 2 is 2^1, day 3 is 2^2, day 4 is 2^3. The exponent is one less than the day number.
3. So day 11 is 2^{10} cents.
4. $2^{10} = 1{,}024$ (from the powers of 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024).
5. 1,024 cents is \$10.24.
6. Check: day 10 would be $2^9 = 512$ cents, and doubling 512 gives 1,024. ✓

Answer: \$10.24

PRACTICE

1. Easy Write $5 \times 5 \times 5 \times 5$ in exponent form, then evaluate it.

   Show answer

   $5^4 = 625$

2. Easy Evaluate 2^6.

   Show answer

   64

3. Easy Write 10^5 as a whole number.

   Show answer

   100,000

4. Easy In 7^3, name the base and the exponent, then evaluate it.

   Show answer

   Base 7, exponent 3; $7^3 = 343$

5. Medium Evaluate $(-3)^2$.

   Show answer

   9

6. Medium Evaluate -3^2.

   Show answer

   -9

7. Medium Evaluate $(-2)^5$.

   Show answer

   -32

8. Medium Write 6×10^4 as a whole number.

   Show answer

   60,000

9. Hard The church prayer chain works like this: in round 1 the pastor calls 3 people. In round 2 each of those people calls 3 new people, and so on. How many people are called in round 4 alone?

Show answer

$3^4 = 81$ people

10. Hard A wooden storage box is a cube 6 inches long on each edge. Its volume is the edge length cubed. Find the volume.

Show answer

$6^3 = 216$ cubic inches

11. Hard Evaluate $2 \times 3^2 - (-2)^3$. Follow the order of operations.

Show answer

26

12. Hard Henry folds a sheet of paper in half, then in half again, 7 times in all. Each fold doubles the number of layers. How many layers thick is the folded paper?

Show answer

$2^7 = 128$ layers

13. Challenge Without multiplying it all out, is $(-1)^{25}$ equal to 1 or -1? Then explain why $(-5)^2$ and -5^2 are not equal.

Show answer

$(-1)^{25} = -1$ because 25 negative factors is an odd number. $(-5)^2$ has base -5, so it is $(-5) \times (-5) = 25$. In -5^2 the base is only 5, so it is $-(5 \times 5) = -25$. The parentheses change the base.

COMMON MISTAKES

- Multiplying the base by the exponent, as in $2^5 = 10$. Write out the factors ($2 \times 2 \times 2 \times 2 \times 2$) until the meaning is automatic.
- Treating -3^2 as $(-3)^2$. Without parentheses the base is only 3, so square first and then apply the minus sign: $-3^2 = -9$.
- Getting the sign wrong on a negative base. Count the negative factors: even gives positive, odd gives negative.

- Writing the wrong number of zeros for a power of 10. The exponent equals the number of zeros: $10^5 = 100{,}000$ has five zeros.

> **Make it a game · Power Charades.** Parent calls a power such as 2 to the 5th. You show the repeated multiplication by stepping five times and doubling a pile of paper chips, then say the value. Then the tricky pair: write (-3) squared and the negative of 3 squared and explain the difference. Score a point per correct answer. Race to ten.

6.2 Laws of Exponents

OBJECTIVES

- Use the product, quotient and power-of-a-power rules to simplify expressions with whole-number exponents.
- Explain each rule by writing out the factors.
- Explain why any nonzero number to the zero power equals 1.

WATCH

Algebra Basics: Laws Of Exponents - Math Antics

mathantics · 13:46

https://www.youtube.com/watch?v=LkhPRz7Hocg

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he calls these the 'index laws'.

BACKUP VIDEO

Index Laws / "Shortcuts" (2 of 3: All About Zero)

Eddie Woo · 7:25

https://www.youtube.com/watch?v=AHEQA55w0lk

Before the zero-power paragraph: why any nonzero number to the zero power is 1 (objective 3).

KEY TERMS

product rule

When you multiply powers with the same base, keep the base and add the exponents: $2^3 \times 2^4 = 2^7$.

quotient rule

When you divide powers with the same base, keep the base and subtract the exponents: $5^6 \div 5^2 = 5^4$.

power-of-a-power rule

When you raise a power to another power, keep the base and multiply the exponents: $(3^2)^4 = 3^8$.

zero exponent

Any number except 0 raised to the zero power equals 1: $9^0 = 1$.

same base

Powers have the same base when the number being multiplied over and over is the same, like 7^2 and 7^5. The laws in this lesson work only for the same base.

CORE IDEAS

You do not need to memorize the laws of exponents as magic. Each one comes from writing out the factors and counting. In this lesson the letter a stands for any base and m and n for whole-number exponents, just as letters stood for any numbers in the properties of Unit 1.

Product rule. $2^3 \times 2^4$ means $(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$. Count the 2s: there are $3 + 4 = 7$ of them, so the product is 2^7. Multiplying powers of the same base puts all the factors into one long row, so the exponents add.

Quotient rule. $\dfrac{5^6}{5^2}$ means six 5s on top and two 5s on the bottom. Each 5 on the bottom cancels one 5 on top, because $\dfrac{5}{5} = 1$. That leaves $6 - 2 = 4$ fives, so the answer is 5^4. Dividing powers of the same base removes factors, so the exponents subtract. (The base cannot be 0, since we cannot divide by 0.)

Power-of-a-power rule. $(3^2)^4$ means $3^2 \times 3^2 \times 3^2 \times 3^2$: four groups, each with two 3s. That is $4 \times 2 = 8$ threes, so the answer is 3^8. Repeated groups of factors means the exponents multiply.

Zero exponent. What should 7^0 mean? Use the quotient rule on $\dfrac{7^3}{7^3}$: subtracting exponents gives 7^0. But any nonzero number divided by itself is 1. So $7^0 = 1$. A pattern agrees: $2^3 = 8$, $2^2 = 4$, $2^1 = 2$; each step down divides by 2, so the next step is $2^0 = 2 \div 2 = 1$. This works for every base except 0. (We leave 0^0 undefined in this course.)

Watch out: the rules work only for the **same base**. $2^3 \times 3^2$ cannot be combined into one power; just evaluate it: $8 \times 9 = 72$. And a power-of-a-power multiplies exponents, but a product of powers adds them, so $(2^3)^2 = 2^6$ while $2^3 \times 2^2 = 2^5$.

KEY FORMULAS

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0,\ m \geq n)$$

$$(a^m)^n = a^{m \cdot n}$$

$$a^0 = 1 \quad (a \neq 0)$$

Practice habit. Recite the three rules as a chant: "Multiply powers, add exponents. Divide powers, subtract exponents. Power of a power, multiply exponents." Whenever you are unsure, write out the factors for a small example and count.

WORKED EXAMPLES

Example 1. Write $3^4 \times 3^2$ as a single power, then evaluate it.

Show solution

1. Write out the factors: $(3 \times 3 \times 3 \times 3) \times (3 \times 3)$.
2. Count the 3s: $4 + 2 = 6$. So $3^4 \times 3^2 = 3^6$.
3. Evaluate: $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$.
4. Check: $3^4 \times 3^2 = 81 \times 9 = 729$. ✓

Answer: $3^6 = 729$

Example 2. A large water tank is shaped like a cube. Each edge is 10^2 centimeters long. The volume of a cube is the edge length cubed. Write the volume as a single power of 10, then as a whole number.

Show solution

1. Volume $= (\text{edge})^3 = (10^2)^3$.
2. Power of a power: multiply the exponents, $2 \times 3 = 6$.
3. So the volume is 10^6 cubic centimeters.
4. 10^6 is a 1 followed by six zeros: 1,000,000.
5. Check: the edge is 100 cm, and $100 \times 100 \times 100 = 1{,}000{,}000$. ✓

Answer: $10^6 = 1{,}000{,}000$ cubic centimeters

PRACTICE

1. **Easy** Write $2^3 \times 2^4$ as a single power, then evaluate it.

   Show answer

   $2^7 = 128$

2. **Easy** Write $10^5 \times 10^3$ as a single power of 10, then as a whole number.

   Show answer

   $10^8 = 100{,}000{,}000$

3. **Easy** Write $\dfrac{5^6}{5^4}$ as a single power, then evaluate it.

   Show answer

   $5^2 = 25$

4. **Easy** Evaluate 9^0.

   Show answer

   1

5. **Medium** Write $(3^2)^3$ as a single power, then evaluate it.

   Show answer

   $3^6 = 729$

6. **Medium** The letter x stands for any nonzero number. Simplify $x^4 \cdot x^5$ as a single power of x.

   Show answer

   x^9

7. **Medium** Write $(-2)^3 \times (-2)^2$ as a single power, then evaluate it.

   Show answer

   $(-2)^5 = -32$

8. **Medium** Evaluate $4^0 + 4^1 + 4^2$.

   Show answer

   21

9. Hard A hardware store receives a crate holding 10^2 boxes of nails. Each box holds 10^3 nails. Write the total number of nails as a power of 10, then as a whole number.

Show answer

$10^5 = 100{,}000$ nails

10. Hard Ruth plants a square herb garden that is 2^3 feet on each side. The area of a square is the side length squared. Write the area as a single power of 2, then find it in square feet.

Show answer

$(2^3)^2 = 2^6 = 64$ square feet

11. Hard A seed company has 3^8 seeds and puts 3^5 seeds in each packet. How many packets can it fill? Use the quotient rule, then evaluate.

Show answer

$3^3 = 27$ packets

12. Hard Write $\dfrac{(5^2)^3}{5^4}$ as a single power, then evaluate it.

Show answer

$5^2 = 25$

13. Challenge Samuel says $2^3 \times 3^2 = 6^5$. Is he right? Evaluate both sides to check.

Show answer

No. $2^3 \times 3^2 = 8 \times 9 = 72$, but $6^5 = 7{,}776$. The product rule works only when the bases are the same.

14. Challenge Explain why $6^0 = 1$ and not 0.

Show answer

By the quotient rule, $\dfrac{6^2}{6^2} = 6^{2-2} = 6^0$. But $\dfrac{6^2}{6^2} = \dfrac{36}{36} = 1$, since any nonzero number divided by itself is 1. So 6^0 must equal 1. The pattern $6^2 = 36$, $6^1 = 6$, $6^0 = 6 \div 6 = 1$ agrees.

COMMON MISTAKES

- Multiplying the bases in the product rule, as in $2^3 \times 2^4 = 4^7$. The base stays the same; only the exponents change: 2^7.

- Mixing up the rules: adding exponents for a power of a power. $(3^2)^3$ is three groups of two 3s, so multiply: 3^6, not 3^5.
- Saying $a^0 = 0$. Any nonzero number to the zero power is 1. Remember $\frac{7^3}{7^3} = 1$.
- Using the rules with different bases. $2^3 \times 3^2$ cannot be combined; evaluate each power and multiply.

> **Make it a game · Exponent Laws Lightning.** Write a dozen expressions like x to the 3 times x to the 4, or (a squared) cubed, on cards. Draw a card, simplify aloud in ten seconds, and name the law you used. Then expand one result by writing out all the factors to prove it. Score a point per correct simplification and a bonus for the proof.

6.3 Scientific Notation

OBJECTIVES

- Explain why $10^{-n} = \frac{1}{10^n}$ and write negative powers of 10 as fractions and decimals.
- Convert very large and very small numbers between standard form and scientific notation, and compare them.
- Multiply numbers written in scientific notation and give the answer in scientific notation.

WATCH

Math Antics - Scientific Notation

mathantics · 14:28

https://www.youtube.com/watch?v=bXkewQ7WEdI

> **WHEN TO WATCH**
> After the paragraph on negative powers of 10, before Example 1.

Note: Eddie Woo teaches in Australia: in Australia and the UK, 'standard form' means scientific notation. In this course (US usage), 'standard form' means an ordinary number like 4,500,000.

BACKUP VIDEO

Scientific (Standard) Notation (1 of 2: Simplifying large numbers with scientific notation)

Eddie Woo · 8:53

https://www.youtube.com/watch?v=bz3dIzFS2XY

Before the practice set: more conversions.

KEY TERMS

scientific notation

A way to write a number as a number from 1 up to (but not including) 10, times a power of 10. Example: $4{,}500{,}000 = 4.5 \times 10^6$.

standard form

The usual way of writing a number with all its digits, such as 4,500,000 or 0.00072.

coefficient

In scientific notation, the first number, such as the 4.5 in 4.5×10^6. It must be at least 1 and less than 10.

negative exponent

An exponent less than zero. For powers of 10, 10^{-n} means $\frac{1}{10^n}$, so $10^{-3} = \frac{1}{1{,}000} = 0.001$.

CORE IDEAS

Scientists, engineers and map makers work with numbers that are huge (the distance to the sun) or tiny (the thickness of a sheet of paper). Writing all those zeros is slow and easy to get wrong. **Scientific notation** packs the zeros into a power of 10: $150{,}000{,}000 = 1.5 \times 10^8$.

To handle small numbers we need negative exponents. Look at the pattern: $10^3 = 1{,}000$, $10^2 = 100$, $10^1 = 10$, $10^0 = 1$. Each time the exponent goes down by 1, we divide by 10. Keep going: $10^{-1} = \frac{1}{10}$, $10^{-2} = \frac{1}{100}$, $10^{-3} = \frac{1}{1{,}000}$. So $10^{-n} = \frac{1}{10^n}$. A negative exponent does not make the number negative; it makes it a small positive fraction.

The quotient rule agrees: $\frac{10^2}{10^5} = 10^{2-5} = 10^{-3}$, and $\frac{100}{100{,}000} = \frac{1}{1{,}000}$. The product rule still works too: $10^5 \times 10^{-3} = 10^{5+(-3)} = 10^2$.

Converting to scientific notation. Move the decimal point so that exactly one nonzero digit is to its left. Count how many places you moved it; that count is the exponent. If the original number is 10 or more, the exponent is positive: $4{,}500{,}000 = 4.5 \times 10^6$ (moved 6 places). If the number is less than 1, the exponent is negative: $0.00072 = 7.2 \times 10^{-4}$ (moved 4 places). To go back to standard form, move the decimal point the other way: right for a positive exponent, left for a negative one.

Comparing. When both numbers are in scientific notation, compare the exponents first; the bigger exponent means the bigger number, because each step is a factor of 10. Only if the exponents are equal do you compare the coefficients. So 4.1×10^5 is greater than 9.8×10^4, even though 9.8 is more than 4.1.

Multiplying. Use the commutative and associative properties to regroup: multiply the coefficients, and multiply the powers of 10 by adding exponents. If the new coefficient is 10 or more, adjust it: $(3 \times 10^4)(5 \times 10^6) = 15 \times 10^{10} = 1.5 \times 10^{11}$, because $15 = 1.5 \times 10^1$.

KEY FORMULAS

$$10^{-n} = \frac{1}{10^n}$$

$$\text{scientific notation: } a \times 10^n, \quad 1 \le a < 10$$

$$(a \times 10^m)(b \times 10^n) = (a \times b) \times 10^{m+n}$$

Practice habit. Recite the powers of 10 from 10^3 down to 10^{-3}: "one thousand, one hundred, ten, one, one tenth, one hundredth, one thousandth." Before converting, say whether the number is big or small: big means a positive exponent, small (less than 1) means a negative exponent.

WORKED EXAMPLES

Example 1. Write 4,500,000 and 0.00072 in scientific notation.

Show solution

1. 4,500,000: put the decimal point after the first digit to get 4.5.
2. Count the places from 4.5 back to 4,500,000: the point moves 6 places. The number is big, so the exponent is positive: 4.5×10^6.
3. 0.00072: move the decimal point right until one nonzero digit is in front of it: 7.2.
4. The point moved 4 places. The number is less than 1, so the exponent is negative: 7.2×10^{-4}.
5. Check: $4.5 \times 1{,}000{,}000 = 4{,}500{,}000$ ✓ and $7.2 \times \frac{1}{10{,}000} = 0.00072$ ✓.

Answer: 4.5×10^6 and 7.2×10^{-4}

Example 2. Light travels about 3×10^5 kilometers per second. Sunlight takes about 5×10^2 seconds to reach Earth. About how far is the sun from Earth? Give your answer in scientific notation.

Show solution

1. Distance = speed × time $= (3 \times 10^5)(5 \times 10^2)$.
2. Multiply the coefficients: $3 \times 5 = 15$.
3. Multiply the powers of 10 by adding exponents: $10^5 \times 10^2 = 10^7$.
4. So the product is 15×10^7. But 15 is not less than 10.
5. Rewrite $15 = 1.5 \times 10^1$, so $15 \times 10^7 = 1.5 \times 10^8$.
6. Check in standard form: $300{,}000 \times 500 = 150{,}000{,}000$, and $1.5 \times 10^8 = 150{,}000{,}000$. ✓

Answer: About 1.5×10^8 kilometers

PRACTICE

1. Easy Write 10^{-3} as a decimal.

   Show answer

   $$\frac{1}{1{,}000} = 0.001$$

2. Easy Write 3,200,000 in scientific notation.

   Show answer

   3.2×10^6

3. Easy Write 6.05×10^4 in standard form.

   Show answer

   60,500

4. Easy Write 0.0009 in scientific notation.

   Show answer

   9×10^{-4}

5. Medium Write 2.7×10^{-3} in standard form.

   Show answer

   0.0027

6. **Medium** Which is greater, 4.1×10^5 or 9.8×10^4?

   Show answer

   4.1×10^5 (410,000 is greater than 98,000)

7. **Medium** The number 23×10^4 is not in scientific notation because 23 is not less than 10. Rewrite it in correct scientific notation.

   Show answer

   2.3×10^5

8. **Medium** Multiply $(2 \times 10^3)(4 \times 10^5)$. Give your answer in scientific notation.

   Show answer

   8×10^8

9. **Hard** The average distance from Earth to the moon is about 384,000 kilometers. Write this distance in scientific notation.

   Show answer

   3.84×10^5 kilometers

10. **Hard** A sheet of printer paper is about 0.0001 meter thick. A ream has 500 sheets. How thick is a ream, in meters? Give your answer in scientific notation.

    Show answer

    5×10^{-2} meter (0.05 meter)

11. **Hard** Order from least to greatest: 5.2×10^{-3}, 4.9×10^{-2}, 8×10^{-4}.

    Show answer

    8×10^{-4}, 5.2×10^{-3}, 4.9×10^{-2}

12. **Hard** A person's heart beats about 1×10^5 times a day. About how many times does it beat in a year of 365 days? Write 365 in scientific notation first, then multiply. Give your answer in scientific notation.

    Show answer

    $(1 \times 10^5)(3.65 \times 10^2) = 3.65 \times 10^7$ beats

13. **Challenge** Multiply $(6 \times 10^5)(2 \times 10^{-3})$. Give your answer in scientific notation and in standard form.

Show answer

$12 \times 10^2 = 1.2 \times 10^3 = 1{,}200$

14. **Challenge** Explain why 10^{-2} equals $\frac{1}{100}$ and not -100.

Show answer

In the pattern $10^2 = 100$, $10^1 = 10$, $10^0 = 1$, each time the exponent goes down by 1 we divide by 10. So $10^{-1} = \frac{1}{10}$ and $10^{-2} = \frac{1}{100}$. The negative exponent means dividing by 10 again and again, which gives a small positive number, not a negative one.

COMMON MISTAKES

- Thinking a negative exponent makes a negative number. $10^{-2} = 0.01$, a small positive number.
- Leaving the coefficient 10 or larger, as in 15×10^7. Rewrite it so the coefficient is from 1 up to 10: 1.5×10^8.
- Moving the decimal point the wrong way. Ask first: is the number big or small? Big numbers get positive exponents; numbers less than 1 get negative exponents.
- Comparing coefficients before exponents. Look at the exponents first; 4.1×10^5 is greater than 9.8×10^4.

> **Make it a game · Big and Tiny Numbers.** Gather ten facts, such as the distance to the moon, the width of a human hair or the population of a city, from a book or atlas. Draw a fact card and convert its number between standard form and scientific notation in 30 seconds. Then put three cards in order from smallest to largest. Score a point per correct conversion.

6.4 Square Roots and Perfect Squares

OBJECTIVES

- Explain a square root as the inverse of squaring and find square roots of perfect squares up to 400.
- Find square roots of fractions whose numerator and denominator are perfect squares.
- Use the fact that $\sqrt{a^2} = a$ for $a \geq 0$ and solve word problems about square areas.

WATCH

Math Antics - Exponents and Square Roots

mathantics · 11:09

https://www.youtube.com/watch?v=B4zejSI8zho

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Square Roots & Cube Roots (1 of 3: Introducing Index Notation)

Eddie Woo · 4:31

https://www.youtube.com/watch?v=nofSLaxQVgU

Before the practice set: square roots and perfect squares. It also covers cube roots, which are not in this lesson.

KEY TERMS

square root

A number that, when multiplied by itself, gives the original number. A square root of 49 is 7, because $7 \times 7 = 49$.

radical sign

The symbol $\sqrt{\ }$. It means the principal (nonnegative) square root, so $\sqrt{49} = 7$.

perfect square

A number that is a whole number times itself, such as 1, 4, 9, 16 and 25.

inverse operation

An operation that undoes another. Taking the square root undoes squaring.

principal square root

The nonnegative square root of a number. Both 7 and -7 square to 49, but $\sqrt{49}$ names only 7.

CORE IDEAS

Picture a square tile floor with an area of 49 square feet. How long is each side? You need a number that times itself makes 49. That number is 7, since $7 \times 7 = 49$. We say 7 is the **square root** of 49 and write $\sqrt{49} = 7$. Squaring turns a side into an area; the square root turns an area back into a side. Each undoes the other, so they are **inverse operations**.

A **perfect square** is a whole number squared. The perfect squares up to 400 are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400 (the squares of 1 through 20). Knowing these by heart makes square roots instant: $\sqrt{225} = 15$, $\sqrt{361} = 19$.

Every positive number actually has two square roots, one positive and one negative, because $(-7) \times (-7) = 49$ too. The radical sign $\sqrt{\ }$ always means the nonnegative one, called the **principal square root**. To name the negative root, put a minus sign outside: $-\sqrt{49} = -7$. What about $\sqrt{-49}$? No number we know times itself gives a negative number (positive × positive and negative × negative are both positive), so $\sqrt{-49}$ is not a real number. Also, $\sqrt{0} = 0$.

Square roots of fractions. Since $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$, we have $\sqrt{\frac{9}{25}} = \frac{3}{5}$. Take the square root of the top and of the bottom. Decimals work the same way if you think of them as fractions: $0.36 = \frac{36}{100}$, so $\sqrt{0.36} = \frac{6}{10} = 0.6$. Check: $0.6 \times 0.6 = 0.36$.

Undoing a square. If a is 0 or positive, then $\sqrt{a^2} = a$, because squaring a and then taking the principal root brings you back where you started: $\sqrt{8^2} = \sqrt{64} = 8$. Squaring a square root also undoes it: $(\sqrt{81})^2 = 9^2 = 81$.

Order of operations. A radical sign acts like grouping symbols: do everything under it first. $\sqrt{9 + 16} = \sqrt{25} = 5$, which is NOT the same as $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

KEY FORMULAS

$$\sqrt{a} = b \text{ means } b \times b = a \text{ and } b \geq 0$$

$$\sqrt{a^2} = a \quad (a \geq 0)$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Perfect squares to 400: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400

Practice habit. Recite the perfect squares from $1^2 = 1$ to $20^2 = 400$ until they are automatic. A handy check: the last digit of a perfect square is always 0, 1, 4, 5, 6 or 9, so a number ending in 2, 3, 7 or 8 is never a perfect square.

WORKED EXAMPLES

Example 1. Evaluate $\sqrt{\frac{49}{144}}$.

Show solution

1. Take the square root of the top: $\sqrt{49} = 7$, since $7 \times 7 = 49$.
2. Take the square root of the bottom: $\sqrt{144} = 12$, since $12 \times 12 = 144$.
3. So $\sqrt{\frac{49}{144}} = \frac{7}{12}$.
4. Check: $\frac{7}{12} \times \frac{7}{12} = \frac{49}{144}$. ✓

Answer: $\frac{7}{12}$

Example 2. The Miller family is building a square patio with an area of 256 square feet. They want to put a border of edging all the way around it. How many feet of edging do they need?

Show solution

1. The area of a square is side × side, so the side is $\sqrt{256}$.
2. $16 \times 16 = 256$, so each side is 16 feet.
3. The edging goes around all 4 sides: $4 \times 16 = 64$ feet.
4. Check: $16^2 = 256$ square feet ✓, and $16 + 16 + 16 + 16 = 64$ ✓.

Answer: 64 feet

PRACTICE

1. Easy Evaluate $\sqrt{81}$.

   Show answer

   9

2. Easy Evaluate $\sqrt{169}$.

   Show answer

   13

3. Easy Evaluate $-\sqrt{64}$.

   Show answer

   -8

4. Easy Is 200 a perfect square? Answer yes or no.

   Show answer

   No ($14^2 = 196$ and $15^2 = 225$, so no whole number squared gives 200)

5. Medium Evaluate $\sqrt{\frac{16}{81}}$.

   Show answer

   $\frac{4}{9}$

6. **Medium** Evaluate $\sqrt{0.49}$.

Show answer

0.7

7. **Medium** Evaluate $\sqrt{17^2}$.

Show answer

17

8. **Medium** Evaluate $\sqrt{36 + 64}$.

Show answer

$\sqrt{100} = 10$

9. **Hard** A square quilt has an area of 324 square inches. How long is each side?

Show answer

18 inches

10. **Hard** Henry has 225 square tiles, all the same size. Can he arrange all of them into one square with no tiles left over? If so, how many tiles are along each side?

Show answer

Yes; 15 tiles on each side ($15 \times 15 = 225$)

11. **Hard** A square vegetable garden has an area of 121 square yards. Ruth wants to fence all four sides. Fencing costs $6 per yard. How much will the fence cost?

Show answer

$264 (each side is 11 yards; $4 \times 11 = 44$ yards; $44 \times 6 = 264$)

12. **Hard** Evaluate $3\sqrt{25} - \sqrt{4} \times \sqrt{9}$.

Show answer

$3 \times 5 - 2 \times 3 = 15 - 6 = 9$

13. **Challenge** Evaluate $\sqrt{2\frac{1}{4}}$. (Hint: write the mixed number as an improper fraction first.)

Show answer

$\sqrt{\frac{9}{4}} = \frac{3}{2} = 1\frac{1}{2}$

14. **Challenge** Grace says $\sqrt{9+16} = \sqrt{9} + \sqrt{16}$. Is she right? Explain.

Show answer

No. $\sqrt{9+16} = \sqrt{25} = 5$, but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$. The radical sign is a grouping symbol, so you must add under it first; square roots do not split apart over addition.

COMMON MISTAKES

- Dividing by 2 instead of taking the square root, as in $\sqrt{64} = 32$. Ask: what number times itself gives 64? The answer is 8.
- Splitting a square root over addition, as in $\sqrt{9+16} = 3 + 4$. Work under the radical sign first: $\sqrt{25} = 5$.
- Writing $\sqrt{49} = -7$ or ± 7. The radical sign means only the nonnegative root, 7. Write $-\sqrt{49}$ when you want -7.
- Taking the square root of only the top of a fraction. $\sqrt{\frac{9}{25}}$ is $\frac{3}{5}$, not $\frac{3}{25}$; take the root of both the top and the bottom.

> **Make it a game · Square Root Sprint.** Write perfect squares up to 400 on index cards, with matching root cards on the back of the next set. Race to match each square with its root, out loud, from memory. Then flip it: parent says a root and you say the square. Beat your fastest time for all 20 squares. Add a few square-root fractions for the bonus round.

6.5 Estimating Roots and Irrational Numbers

OBJECTIVES

- Estimate a square root between two consecutive whole numbers and to the nearest tenth by reasoning with perfect squares.
- Tell rational numbers from irrational numbers and explain the difference.
- Describe the real number system and place a number in every set it belongs to.

WATCH

Irrational Numbers Explained | Math with Mr. J

Math with Mr. J · 8:19

https://www.youtube.com/watch?v=GupzujexBFA

> **WHEN TO WATCH**
> Before the rational/irrational section: why the square root of a non-perfect square (such as $\sqrt{50}$) is irrational.

Note: Weak fit: neither video shows estimating a square root between two whole numbers (objective 1). Teach that from Example 1.

BACKUP VIDEO

Irrational Numbers - Math Antics Extras

mathantics · 5:25

https://www.youtube.com/watch?v=wg4YXMe5ccs

Before Example 2 (number sets), as a second short explanation.

KEY TERMS

consecutive whole numbers

Whole numbers that come one right after the other, like 7 and 8.

rational number

A number that can be written as a fraction $\frac{a}{b}$ of two integers, with b not 0. Its decimal either ends or repeats: $\frac{3}{4} = 0.75$, $\frac{1}{3} = 0.\overline{3}$, $-5 = \frac{-5}{1}$.

irrational number

A number that cannot be written as a fraction of two integers. Its decimal goes on forever with no repeating pattern. Examples: $\sqrt{2}$, $\sqrt{50}$, π.

real numbers

All the rational and irrational numbers together; every point on the number line is a real number.

integers

The whole numbers and their opposites: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

CORE IDEAS

Most whole numbers are not perfect squares, so their square roots are not whole numbers. You can still pin them down. Since $49 < 50 < 64$, the square root of 50 is between $\sqrt{49} = 7$ and $\sqrt{64} = 8$. Because 50 is much closer to 49 than to 64, $\sqrt{50}$ is just a little more than 7. Always reason this way first, before reaching for a calculator.

To the nearest tenth. Square some tenths. $7.0^2 = 49$ and $7.1^2 = 50.41$, so $\sqrt{50}$ is between 7.0 and 7.1. To decide which one it rounds to, square the halfway number: $7.05^2 = 49.7025$. Since 50 is more than 49.7025, $\sqrt{50}$ is past the halfway point, so $\sqrt{50} \approx 7.1$. (A calculator shows 7.0710678..., which agrees.)

Why does the calculator's decimal for $\sqrt{50}$ never stop? Numbers come in two kinds. A **rational number** can be written as a fraction of two integers, like $\frac{3}{4}$, $-2\frac{1}{2} = \frac{-5}{2}$, or $7 = \frac{7}{1}$. In Lesson 4.5 you saw that the decimal of a fraction always either ends or repeats. An **irrational number** cannot be written as such a fraction, so its decimal goes on forever with no repeating block.

The square root of a whole number is either a whole number (when the number is a perfect square) or irrational. So $\sqrt{36} = 6$ is rational, but $\sqrt{2}$, $\sqrt{13}$ and $\sqrt{50}$ are irrational. The number π is irrational too; 3.14 and $\frac{22}{7}$ are only rational approximations of it. Careful: a long decimal that repeats, like $0.\overline{142857}$, is rational.

The real number system. The counting numbers $1, 2, 3, \ldots$ sit inside the whole numbers $0, 1, 2, \ldots$, which sit inside the integers, which sit inside the rational numbers. The rational and the irrational numbers together make up the **real numbers**, which fill the whole number line. Every real number is either rational or

irrational, never both. For example, -4 is an integer, a rational number and a real number; $\sqrt{7}$ is irrational and real.

KEY FORMULAS

$$\text{If } a^2 < n < b^2, \text{ then } a < \sqrt{n} < b$$

$$\text{rational: } \frac{a}{b} \text{ with integers } a, b \text{ and } b \neq 0$$

Counting numbers ⊂ whole numbers ⊂ integers ⊂ rational numbers ⊂ real numbers; irrational numbers ⊂ real numbers

From history. The followers of Pythagoras, in ancient Greece, believed that everything could be described by ratios of whole numbers. Then they discovered that the diagonal of a square cannot be measured as a ratio of whole numbers with its side; in our language, $\sqrt{2}$ is not a ratio of whole numbers. This was the first known irrational number, and it changed how the Greeks thought about number and length.

Practice habit. Before estimating any square root, recite the two perfect squares on either side of the number ("50 is between 49 and 64, so $\sqrt{50}$ is between 7 and 8"). Estimate first, then check by squaring your estimate.

WORKED EXAMPLES

Example 1. Estimate $\sqrt{50}$ to the nearest tenth without a calculator.

Show solution

1. Find the nearby perfect squares: $7^2 = 49$ and $8^2 = 64$. So $\sqrt{50}$ is between 7 and 8, close to 7.
2. Try tenths: $7.1^2 = 50.41$, which is more than 50. So $\sqrt{50}$ is between 7.0 and 7.1.
3. Square the halfway number: $7.05^2 = 49.7025$, which is less than 50.
4. So $\sqrt{50}$ is between 7.05 and 7.1, and it rounds to 7.1.
5. Check with a calculator: $\sqrt{50} \approx 7.071$, which rounds to 7.1. ✓

Answer: $\sqrt{50} \approx 7.1$

Example 2. The Miller family wants a square garden plot with an area of 90 square feet. How long should each side be, to the nearest tenth of a foot?

Show solution

1. The side is $\sqrt{90}$ feet.
2. Nearby perfect squares: $9^2 = 81$ and $10^2 = 100$. So the side is between 9 and 10 feet, a little closer to 9 than to 10.
3. Try tenths: $9.4^2 = 88.36$ (too small) and $9.5^2 = 90.25$ (too big). So $\sqrt{90}$ is between 9.4 and 9.5.
4. Square the halfway number: $9.45^2 = 89.3025$, which is less than 90.
5. So $\sqrt{90}$ is past the halfway point and rounds to 9.5.
6. Check: $9.5 \times 9.5 = 90.25$, very close to 90. ✓

Answer: About 9.5 feet

PRACTICE

1. Easy Between which two consecutive whole numbers is $\sqrt{30}$?

   Show answer

   5 and 6 (since $25 < 30 < 36$)

2. Easy Between which two consecutive whole numbers is $\sqrt{75}$?

   Show answer

   8 and 9 (since $64 < 75 < 81$)

3. Easy Is $\sqrt{36}$ a rational number? Answer yes or no.

   Show answer

   Yes; $\sqrt{36} = 6 = \frac{6}{1}$

4. Easy Is $\sqrt{13}$ a rational number? Answer yes or no.

   Show answer

   No; 13 is not a perfect square, so $\sqrt{13}$ is irrational

5. Medium Is $0.\overline{6}$ a rational number? Answer yes or no.

   Show answer

   Yes; it repeats, and $0.\overline{6} = \frac{2}{3}$

6. **Medium** To which whole number is $\sqrt{40}$ closest?

Show answer

6 (40 is 4 more than 36 but 9 less than 49)

7. **Medium** Estimate $\sqrt{20}$ to the nearest tenth without a calculator.

Show answer

4.5 ($4.4^2 = 19.36$, $4.5^2 = 20.25$, and $4.45^2 = 19.8025$ is less than 20)

8. **Medium** From the list whole numbers, integers, rational numbers, irrational numbers, real numbers, name every set that -4 belongs to.

Show answer

Integers, rational numbers and real numbers

9. **Hard** Ruth has a square rug with an area of 60 square feet. How long is each side, to the nearest tenth of a foot? Reason first, then check with a calculator.

Show answer

About 7.7 feet ($7.7^2 = 59.29$, $7.8^2 = 60.84$, and $7.75^2 = 60.0625$ is more than 60)

10. **Hard** A square tablecloth for the church potluck has an area of 2 square meters. How long is each side, to the nearest tenth of a meter?

Show answer

About 1.4 meters ($1.4^2 = 1.96$, $1.5^2 = 2.25$, and $1.45^2 = 2.1025$ is more than 2)

11. **Hard** A carpenter cuts a square board with an area of 150 square inches. How long is each side, to the nearest tenth of an inch?

Show answer

About 12.2 inches ($12.2^2 = 148.84$, $12.3^2 = 151.29$, and $12.25^2 = 150.0625$ is more than 150)

12. **Hard** Order from least to greatest: $\sqrt{10}$, 3.2, π, $\frac{10}{3}$. You may use a calculator to check.

Show answer

π, $\sqrt{10}$, 3.2, $\frac{10}{3}$ (about 3.1416, 3.1623, 3.2, 3.3333)

13. **Challenge** Which number in this list is irrational: $\sqrt{49}$, $0.\overline{6}$, $\sqrt{11}$, $\frac{22}{7}$? Give its value to the nearest tenth.

Show answer

$\sqrt{11}$, about 3.3

14. **Challenge** Explain the difference between a rational number and an irrational number, and give one example of each.

Show answer

A rational number can be written as a fraction of two integers (with a nonzero bottom), and its decimal ends or repeats, like $\frac{3}{4} = 0.75$. An irrational number cannot be written that way, and its decimal goes on forever without repeating, like $\sqrt{2} = 1.41421\ldots$

COMMON MISTAKES

- Thinking $\sqrt{50}$ is about 25 because 25 is half of 50. A square root is the number that times itself gives 50, so it is a little more than 7.
- Rounding to the nearest tenth by only comparing distances to the two squares. Square the halfway number (like 7.05^2) to decide which way to round.
- Calling every long decimal irrational. A decimal that repeats, like $0.\overline{142857}$, is rational; only a decimal that never ends and never repeats is irrational.
- Thinking $\pi = 3.14$ exactly. 3.14 and $\frac{22}{7}$ are rational approximations; π itself is irrational.

> **Make it a game · Between Two Squares.** Parent calls a number such as 70. You say between which two consecutive whole numbers its square root falls, then estimate to the nearest tenth and test by squaring. Score two points for the bracket and a point for the decimal estimate. Then sort ten cards of numbers into rational and irrational. Play to fifteen points.

UNIT 6

Exponents and Square Roots

In this unit you learn a powerful shorthand. An exponent tells how many times to multiply a number by itself, and a few simple laws, each one explained by writing out the factors, let you multiply and divide powers quickly. You will use powers of 10 to write very large and very small numbers in scientific notation, the way scientists and engineers do.

Then you will turn squaring around. A square root answers the question "what number times itself gives this?" You will learn the perfect squares to 400 by heart, estimate square roots that are not whole numbers, and meet irrational numbers such as $\sqrt{2}$ and π, whose decimals never end or repeat. The unit closes by putting every kind of number you have studied into one family, the real numbers.

These skills are the doorway to algebra and geometry. Exponents appear in every algebra course, scientific notation in every science course, and square roots in the Pythagorean theorem (Unit 12). Parents: the key habits are reciting the powers of 2, the powers of 10 and the perfect squares to 400, and always asking "what is the base?" before evaluating a power with a minus sign.

PREREQUISITES

- Squares, cubes and the order of operations (Lesson 1.4)
- Multiplying and dividing fractions (Lessons 3.5 and 3.6)
- Multiplying and dividing decimals, and multiplying by powers of 10 (Lessons 4.3 and 4.4)
- Terminating and repeating decimals (Lesson 4.5)
- Multiplying integers and the sign rules (Lessons 5.4 and 5.5)

MASTERY CHECKLIST

- ☐ I can write repeated multiplication with an exponent and evaluate powers, including powers of 10.
- ☐ I can explain the difference between $(-3)^2$ and -3^2 and evaluate each.
- ☐ I can use the product, quotient and power-of-a-power rules and explain each one by writing out factors.
- ☐ I can explain why a nonzero number to the zero power equals 1.
- ☐ I can explain why $10^{-n} = \frac{1}{10^n}$.
- ☐ I can convert numbers between standard form and scientific notation, and compare them.
- ☐ I can multiply numbers in scientific notation and write the answer in correct scientific notation.
- ☐ I can find square roots of perfect squares up to 400 and of fractions made of perfect squares.
- ☐ I can estimate a square root between two whole numbers and to the nearest tenth.
- ☐ I can tell rational numbers from irrational numbers and place a number in the real number system.

LESSONS IN THIS UNIT

6.1	**Exponents and Powers**	45 minutes
6.2	**Laws of Exponents**	2 sessions of 45 minutes
6.3	**Scientific Notation**	2 sessions of 45 minutes
6.4	**Square Roots and Perfect Squares**	2 sessions of 45 minutes
6.5	**Estimating Roots and Irrational Numbers**	2 sessions of 45 minutes

Unit 6 test

You have 60 minutes. No calculator. Show all your work. Give answers in scientific notation when a problem asks for it, and round square-root estimates to the nearest tenth when told to. Write units in your answers to word problems.

1. **5 pts**
   Evaluate 3^5.

2. **6 pts**
   Evaluate $(-4)^2$ and -4^2.

3. **6 pts**
   Write $2^5 \times 2^3$ as a single power, then evaluate it.

4. **6 pts**
   Write $\dfrac{(7^3)^2}{7^4}$ as a single power, then evaluate it.

5. **5 pts**
   Evaluate $12^0 + 10^{-1}$. Write your answer as a decimal.

6. **7 pts**
   Write 0.000056 in scientific notation.

7. **7 pts**

   Multiply $(4 \times 10^{6})(3 \times 10^{-2})$. Give your answer in scientific notation.

8. **8 pts**

   A grain elevator holds about 4×10^{5} bushels of wheat. One bushel holds about 1×10^{6} kernels. About how many kernels are in the full elevator? Give your answer in scientific notation.

9. **6 pts**

   Evaluate $\sqrt{196}$.

10. **6 pts**

    Evaluate $\sqrt{\dfrac{25}{64}}$.

11. **7 pts**

    A square vegetable garden has an area of 289 square feet. How many feet of fencing are needed to go around all four sides?

12. **8 pts**

    Estimate $\sqrt{85}$ to the nearest tenth. Show how you decided.

13. **7 pts**

    Grace makes a square picture frame with an area of 45 square inches. How long is each side, to the nearest tenth of an inch?

14. **5 pts**

    Is $\sqrt{48}$ a rational number? Answer yes or no, and give a reason.

15. **11 pts**

    Explain why $4^{0} = 1$. Use the quotient rule or a pattern of powers of 4.

ANSWER KEY

1. 243
2. $(-4)^2 = 16$ and $-4^2 = -16$
3. $2^8 = 256$
4. $7^2 = 49$
5. $1 + 0.1 = 1.1$
6. 5.6×10^{-5}
7. $12 \times 10^4 = 1.2 \times 10^5$
8. About 4×10^{11} kernels
9. 14
10. $\frac{5}{8}$
11. 68 feet (each side is 17 feet; $4 \times 17 = 68$)
12. 9.2 ($9.2^2 = 84.64$, $9.3^2 = 86.49$, and $9.25^2 = 85.5625$ is more than 85)
13. About 6.7 inches ($6.7^2 = 44.89$, $6.8^2 = 46.24$, and $6.75^2 = 45.5625$ is more than 45)
14. No; 48 is not a perfect square, so $\sqrt{48}$ is irrational
15. By the quotient rule, $\frac{4^3}{4^3} = 4^{3-3} = 4^0$. But $\frac{4^3}{4^3} = \frac{64}{64} = 1$, because any nonzero number divided by itself is 1. So $4^0 = 1$. The pattern agrees: $4^2 = 16$, $4^1 = 4$, and dividing by 4 again gives $4^0 = 1$.

UNIT 7

Ratios, Rates and Proportions

This unit is about comparing quantities. A ratio compares two amounts, such as 3 cups of lemon juice to 5 cups of water. A rate compares amounts in different units, such as miles per hour or dollars per pound. A proportion says two ratios are equal, and it lets you scale a recipe, plan a potluck, read a map or build a model.

You will learn to write and simplify ratios, find unit rates and better buys, solve proportions with cross products (and see why cross products work), use scales on maps and blueprints, and convert units by multiplying by a fraction equal to 1. These skills are used every day in the kitchen, the store, the workshop and on the road.

Ratios build on your work with fractions and equivalent fractions in Unit 3 and decimals in Unit 4. They lead straight into percent in Unit 8, since a percent is a ratio out of 100, and later into slope and rates of change in Unit 11. A basic calculator may be used for messy division in this unit.

PREREQUISITES

- Find the GCF of two numbers (Lesson 2.4).
- Build equivalent fractions and simplify fractions (Lessons 3.1 and 3.2).
- Multiply and divide fractions and mixed numbers (Lessons 3.5 and 3.6).
- Multiply and divide decimals, including money (Lessons 4.3 and 4.4).
- Round decimals to a given place (Lesson 4.1).

MASTERY CHECKLIST

- ☐ I can write a ratio three ways and tell whether it is part-to-part or part-to-whole.
- ☐ I can write a ratio in simplest form and find equivalent ratios.
- ☐ I can complete a ratio table and use it to solve a word problem.
- ☐ I can find a unit rate, including speed and unit price.
- ☐ I can compare unit prices to find the better buy.
- ☐ I can tell whether two ratios form a proportion by using cross products.
- ☐ I can explain why the cross products of a proportion are equal.
- ☐ I can solve a proportion with a missing number and check my answer.
- ☐ I can use a map scale or blueprint scale to find actual lengths and drawing lengths.
- ☐ I can find a scale factor by writing both lengths in the same unit.
- ☐ I can convert units by multiplying by a conversion factor and explain why the quantity does not change.
- ☐ I can convert a rate, such as miles per hour to feet per second.

LESSONS IN THIS UNIT

7.1 Ratios

45 minutes

7.2	**Rates and Unit Rates**	2 sessions of 45 minutes
7.3	**Proportions**	2 sessions of 45 minutes
7.4	**Scale Drawings and Maps**	2 sessions of 45 minutes
7.5	**Unit Conversions**	2 sessions of 45 minutes

Unit 7 test

You have 50 minutes. You may use a basic calculator. Show your work for every problem, including how you set up each ratio, proportion or conversion. Write units in your answers. Round money to the nearest cent and other answers as each problem directs. Use these facts as needed: 12 inches = 1 foot, 3 feet = 1 yard, 5,280 feet = 1 mile, 4 quarts = 1 gallon, 2 pints = 1 quart, 2 cups = 1 pint.

1. **5 pts**

   A youth group has 16 boys and 20 girls. Write the ratio of boys to girls in simplest form.

2. **5 pts**

   A granola recipe uses 3 cups of oats for every 2 cups of raisins. Use a ratio table to find how many cups of raisins go with 12 cups of oats.

3. **6 pts**

   The ratio of boys to girls in a choir is $4 : 5$. The choir has 36 members. How many boys are in the choir?

4. **6 pts**

   Henry and Samuel share $84 from mowing lawns in the ratio $3 : 4$, with Samuel getting the larger share. How much does Samuel get?

5. **6 pts**

   Seven pounds of apples cost $9.73. What is the price per pound?

6. **6 pts**

   A bus travels 156 miles in 2.4 hours. What is its average speed in miles per hour?

7. **7 pts**

   A 24-ounce jar of peanut butter costs \$4.32. A 40-ounce jar costs \$6.80. Find the price per ounce of each jar. Which is the better buy?

8. **5 pts**

   Do $\frac{4}{10}$ and $\frac{6}{15}$ form a proportion? Answer yes or no and show the cross products.

9. **6 pts**

   Find the number in the box: $\frac{9}{12} = \frac{\square}{28}$.

10. **7 pts**

    Three bags of garden soil cover 45 square feet. How many bags are needed to cover 165 square feet?

11. **8 pts**

    Use the proportion $\frac{3}{5} = \frac{6}{10}$ to explain why the cross products of a proportion are equal.

12. **6 pts**

    A map has the scale 1 inch : 30 miles. Two towns are 4.5 inches apart on the map. What is the actual distance between them?

13. **7 pts**

    A blueprint uses the scale $\frac{1}{4}$ inch : 1 foot. A kitchen is 16 feet long and 13 feet wide. How long and how wide is the kitchen on the blueprint?

14. **5 pts**

    Convert 5.5 feet to inches.

15. **7 pts**

    Grace is making 2.5 gallons of soup for a family gathering. How many cups is that?

16. **8 pts**

A country road has a speed limit of 30 miles per hour. How fast is 30 miles per hour in feet per second? Show your conversion factors.

ANSWER KEY

1. $4 : 5$
2. 8 cups of raisins
3. 16 boys
4. \$48
5. \$1.39 per pound
6. 65 miles per hour
7. \$0.18 per ounce and \$0.17 per ounce; the 40-ounce jar is the better buy.
8. Yes ($4 \times 15 = 60$ and $10 \times 6 = 60$)
9. 21
10. 11 bags
11. Rewrite both fractions over the common denominator $5 \times 10 = 50$: $\frac{3}{5} = \frac{3 \times 10}{50} = \frac{30}{50}$ and $\frac{6}{10} = \frac{6 \times 5}{50} = \frac{30}{50}$. Equal fractions with the same denominator have equal numerators, and those numerators, 3×10 and 6×5, are the cross products.
12. 135 miles
13. 4 inches by $3\frac{1}{4}$ inches
14. 66 inches
15. 40 cups
16. 44 feet per second

7.1 Ratios

OBJECTIVES

- Write a ratio in three ways and tell whether it compares part to part or part to whole.
- Find equivalent ratios and write ratios in simplest form.
- Complete ratio tables and use them to solve word problems.

WATCH

Math Antics - Ratios And Rates

mathantics · 8:49

https://www.youtube.com/watch?v=RQ2nYUBVvqI

> **WHEN TO WATCH**
> Watch the ratio part before Example 1.

Note: Shared video: Lessons 7.1 and 7.2 use the same Math Antics 'Ratios and Rates' video.

BACKUP VIDEO

Introduction to Ratios (2 of 2: Equivalent Ratios)

Eddie Woo · 4:43

https://www.youtube.com/watch?v=tKXaO8fVoRY

Before Example 2: equivalent ratios and simplest form.

KEY TERMS

ratio

A comparison of two quantities by division. The ratio of 3 to 5 can be written 3 to 5, 3:5 or $\frac{3}{5}$.

terms of a ratio

The two numbers being compared. In 3:5, the terms are 3 and 5.

part-to-part ratio

A ratio that compares one part of a group to another part, such as boys to girls.

part-to-whole ratio

A ratio that compares one part of a group to the whole group, such as boys to all students.

equivalent ratios

Ratios that make the same comparison, like 2:3 and 4:6. You get one from the other by multiplying or dividing both terms by the same nonzero number.

ratio table

A table with two rows (or columns) that lists equivalent ratios side by side.

CORE IDEAS

A **ratio** compares two amounts by division. If a basket holds 3 apples and 5 pears, the ratio of apples to pears is 3 to 5. You can write it three ways: 3 to 5, 3:5, or $\frac{3}{5}$. All three are read "3 to 5."

Order matters. The ratio of apples to pears is 3:5, but the ratio of pears to apples is 5:3. Always say the words in order ("apples to pears") before you write the numbers.

Some ratios compare **part to part** (apples to pears, 3:5). Others compare **part to whole** (apples to all the fruit, 3:8, since $3 + 5 = 8$). A part-to-whole ratio is the same thing as a fraction of the group: $\frac{3}{8}$ of the fruit is apples.

Equivalent ratios make the same comparison. If every bag has 3 apples and 5 pears, then 2 bags have 6 apples and 10 pears, and 3 bags have 9 apples and 15 pears. So 3:5, 6:10 and 9:15 are equivalent. Why does this work? Each bag keeps the same mix, so you are just counting more copies of the same group. That means you multiply *both* terms by the same number, exactly like building equivalent fractions.

To write a ratio in **simplest form**, divide both terms by their greatest common factor. For 12:18 the GCF is 6, so $12 : 18 = 2 : 3$. Adding the same number to both terms does *not* give an equivalent ratio: 2:3 and 3:4 are different mixes.

A **ratio table** organizes equivalent ratios. Put one quantity in the top row and the other in the bottom row, and multiply (or divide) both entries in a column by the same number to make a new column. Ratio tables are a quick, reliable way to scale a recipe or a mixture up or down.

KEY FORMULAS

$$a \text{ to } b \;=\; a : b \;=\; \frac{a}{b}$$

$$a : b = (a \times n) : (b \times n) \quad \text{for any nonzero number } n$$

From history. Greek mathematicians thought carefully about ratios as ways to compare sizes, and Book V of Euclid's Elements is devoted to ratio and proportion. Our English word *ratio* comes from a Latin word meaning "reckoning" or "reason."

Practice habit. Before writing any ratio, say its words aloud in order, such as "cows to horses." Keep your multiplication facts through 12×12 quick, because every ratio table is just multiplication facts in two rows.

WORKED EXAMPLES

Example 1. A bag holds 8 red marbles and 12 blue marbles. Write the ratio of red marbles to blue marbles in simplest form. Then write the ratio of red marbles to all the marbles in simplest form.

Show solution

1. Red to blue is a part-to-part ratio: $8 : 12$.
2. The GCF of 8 and 12 is 4. Divide both terms by 4: $8 : 12 = 2 : 3$.
3. Find the whole: $8 + 12 = 20$ marbles in all.
4. Red to all is a part-to-whole ratio: $8 : 20$.
5. The GCF of 8 and 20 is 4. Divide both terms by 4: $8 : 20 = 2 : 5$.
6. Check: for every 2 red there are 3 blue, so every group of 5 marbles has 2 red. That matches 2:5.

Answer: Red to blue is $2 : 3$ (or $\frac{2}{3}$); red to all is $2 : 5$ (or $\frac{2}{5}$).

Example 2. Ruth makes lemonade with 3 cups of lemon juice for every 5 cups of water. For the church picnic she will use 40 cups of water. Use a ratio table to find how many cups of lemon juice she needs. How many cups of lemonade will she make in all?

Show solution

1. Set up the table. Top row: lemon juice 3. Bottom row: water 5.
2. The water must go from 5 to 40. Since $40 \div 5 = 8$, multiply both rows by 8.
3. Lemon juice: $3 \times 8 = 24$ cups. Water: $5 \times 8 = 40$ cups.
4. Total lemonade: $24 + 40 = 64$ cups.
5. Check: $24 : 40$ simplifies (divide by 8) to $3 : 5$, the original recipe.

Answer: 24 cups of lemon juice; 64 cups of lemonade in all.

PRACTICE

1. Easy A farm has 7 cows and 4 horses. Write the ratio of cows to horses in three ways.

   Show answer

   7 to 4, $7 : 4$, $\frac{7}{4}$

2. Easy Write the ratio $12 : 18$ in simplest form.

   Show answer

   $2 : 3$

3. Easy A class has 25 students, and 15 of them are girls. Write the ratio of boys to girls in simplest form.

   Show answer

   $2 : 3$ (10 boys to 15 girls)

4. Easy Fill in the box to make equivalent ratios: $4 : 7 = 20 : \square$.

   Show answer

   35

5. **Medium** A pancake recipe uses 2 cups of flour for every 3 eggs. Complete the ratio table. Flour (cups): 2, 4, 6, 10. Eggs: 3, ?, ?, ?.

Show answer

6, 9, 15 eggs

6. **Medium** Are the ratios $6 : 8$ and $15 : 20$ equivalent? Answer yes or no.

Show answer

Yes. Both simplify to $3 : 4$.

7. **Medium** A garden has 18 tomato plants and 12 pepper plants. Write the ratio of pepper plants to all the plants in simplest form.

Show answer

$2 : 5$ (12 peppers out of 30 plants)

8. **Medium** Which ratio is NOT equivalent to $3 : 4$: $6 : 8$, $9 : 12$, $12 : 15$, or $15 : 20$?

Show answer

$12 : 15$ (it simplifies to $4 : 5$)

9. **Medium** Henry says $2 : 3$ and $3 : 4$ are equivalent because in both ratios the second number is one more than the first. Explain why he is wrong.

Show answer

Equivalent ratios come from multiplying or dividing both terms by the same number, not adding. $2 : 3 = 8 : 12$ but $3 : 4 = 9 : 12$, so with the same second term the first terms differ. They are different mixes.

10. **Hard** The ratio of boys to girls in a choir is $3 : 5$. There are 40 singers in all. How many girls are in the choir?

Show answer

25 girls

11. **Hard** For every \$5 Samuel spends, he saves \$3. Last month he saved \$27. How much did he spend last month?

Show answer

\$45

12. Hard A concrete mix uses cement, sand and gravel in the ratio $1 : 2 : 3$. A builder wants 18 buckets of mix in all. How many buckets of sand are needed?

Show answer

6 buckets of sand

13. Challenge A bin holds red and green apples in the ratio $4 : 3$. Grace adds 6 more green apples, and now the numbers of red and green apples are equal. How many red apples are in the bin?

Show answer

24 red apples (one "part" is 6 apples, so red is $4 \times 6 = 24$ and green went from 18 to 24)

COMMON MISTAKES

- Writing the terms in the wrong order. If the question says "horses to cows," the number of horses comes first. Say the words aloud, then write.
- Mixing up part-to-part and part-to-whole. For "boys to all students," add the parts first to find the whole.
- Adding the same number to both terms to get an equivalent ratio. Only multiplying or dividing both terms by the same number keeps the ratio the same.
- Leaving a ratio unsimplified when simplest form is asked for. Divide both terms by their GCF, not just by any common factor.

> **Make it a game · Ratio Recipe Race.** Make a ratio table together using a pretend lemonade recipe of lemon juice to water. Parent calls a ratio, and you fill in three equivalent ratios as quickly as you can, then simplify. Also say whether the ratio is part to part or part to whole. Score a point per correct row. Race to ten points.

7.2 Rates and Unit Rates

OBJECTIVES

- Write a rate as a fraction and find its unit rate.
- Find speeds and unit prices and use them to solve word problems.
- Compare unit prices to decide which item is the better buy.

WATCH

Math Antics - Ratios And Rates

mathantics · 8:49

https://www.youtube.com/watch?v=RQ2nYUBVvqI

> **WHEN TO WATCH**
> Watch the rates section, after the ratio examples, before Example 1.

Note: Shared video: this lesson uses the rates part of the 7.1 video. Eddie Woo teaches in Australia: prices in the backup are in Australian dollars.

BACKUP VIDEO

Calculating Best Buys (how to use two comparable methods)

Eddie Woo · 5:14

https://www.youtube.com/watch?v=bXPPX1PR_KU

Before the 'better buy' example: comparing unit prices.

KEY TERMS

rate

A ratio that compares two quantities measured in different units, such as 120 miles in 2 hours.

unit rate

A rate that tells how much for just one unit, such as 60 miles per 1 hour. The word *per* means "for each."

speed

A rate that compares distance to time, such as miles per hour.

unit price

The cost of one unit of an item, such as the price for one ounce or one pound.

better buy

The choice with the lower unit price, so you pay less for each unit.

CORE IDEAS

A **rate** is a ratio that compares two different kinds of quantities: miles and hours, dollars and pounds, pages and minutes. Like any ratio, a rate can be written as a fraction. Reading 120 pages in 4 hours is the rate $\frac{120 \text{ pages}}{4 \text{ hours}}$.

A **unit rate** tells how much for *one* unit of the second quantity. To find it, divide the top by the bottom: $120 \div 4 = 30$, so the rate is 30 pages per hour. Why divide? The fraction $\frac{120}{4}$ is equivalent to $\frac{30}{1}$, because we divided both terms by 4, just as with equivalent ratios.

Speed is a unit rate: distance divided by time. A car that goes 195 miles in 3 hours averages $195 \div 3 = 65$ miles per hour. Once you know a unit rate, you can multiply to find more: in 5 hours at 65 miles per hour, the car goes $65 \times 5 = 325$ miles. You can also divide: to go 260 miles takes $260 \div 65 = 4$ hours.

A **unit price** is the cost for one unit: price divided by quantity. Shoppers use unit prices to find the **better buy**. A bigger package usually costs more in total, so comparing total prices tells you nothing. Comparing the price of *one* ounce (or one pound, or one can) puts the two choices on equal footing.

Watch the units. Always divide in the order the question asks ("dollars per pound" means dollars on top, pounds on the bottom), and make sure both items are measured in the same unit before comparing. You may use a calculator for messy division in this unit; round money to the nearest cent unless the problem says otherwise.

KEY FORMULAS

$$\text{unit rate} = \frac{\text{first quantity}}{\text{second quantity}}$$

$$\text{speed} = \frac{\text{distance}}{\text{time}} \qquad \text{distance} = \text{speed} \times \text{time}$$

$$\text{unit price} = \frac{\text{total price}}{\text{number of units}}$$

Practice habit. Estimate a unit rate in your head before you compute it. For example, for \$4.86 for 18 ounces, think $18 \times \$0.25 = \$4.50$ and $18 \times \$0.30 = \$5.40$, so the unit price must be between 25 and 30 cents an ounce.

WORKED EXAMPLES

Example 1. The Miller family drives 195 miles in 3 hours. What is their average speed in miles per hour?

Show solution

1. Write the rate as a fraction: $\dfrac{195 \text{ miles}}{3 \text{ hours}}$.
2. Divide to get miles for 1 hour: $195 \div 3 = 65$.
3. Check: $65 \times 3 = 195$. Correct.

Answer: 65 miles per hour

Example 2. A 12-ounce box of cereal costs \$3.48. An 18-ounce box of the same cereal costs \$4.86. Find each unit price in dollars per ounce. Which box is the better buy?

Show solution

1. Small box: $\$3.48 \div 12 = \$0.29$ per ounce.
2. Check: $0.29 \times 12 = 3.48$. Correct.
3. Large box: $\$4.86 \div 18 = \$0.27$ per ounce.
4. Check: $0.27 \times 18 = 4.86$. Correct.
5. Compare: \$0.27 is less than \$0.29, so the large box costs less per ounce.

Answer: The 12-ounce box costs \$0.29 per ounce and the 18-ounce box costs \$0.27 per ounce, so the 18-ounce box is the better buy.

PRACTICE

1. Easy Samuel reads 120 pages in 4 hours. What is his unit rate in pages per hour?

   Show answer

   30 pages per hour

2. **Easy** Five pounds of apples cost $6.45. What is the price per pound?

   Show answer

   $1.29 per pound

3. **Easy** A faucet fills a 40-gallon tub in 8 minutes. How many gallons per minute does it pour?

   Show answer

   5 gallons per minute

4. **Easy** A car goes 240 miles on 8 gallons of gasoline. Write this rate as a fraction, then find the unit rate in miles per gallon.

   Show answer

   $\frac{240 \text{ miles}}{8 \text{ gallons}}$; 30 miles per gallon

5. **Medium** Grace earns $58.50 for 6.5 hours of babysitting. How much does she earn per hour?

   Show answer

   $9.00 per hour

6. **Medium** At one store, 3 cans of beans cost $2.67. At another, 5 cans cost $4.25. Find the price per can at each store. Which is the better buy?

   Show answer

   $0.89 per can and $0.85 per can; 5 cans for $4.25 is the better buy.

7. **Medium** A train travels at 72 miles per hour. How far does it travel in 3.5 hours?

   Show answer

   252 miles

8. Medium Henry says, "The bigger package always costs more, so the smaller package is always the better buy." Explain why comparing total prices does not tell you the better buy.

Show answer

The packages hold different amounts, so their total prices are not a fair comparison. You must find the price for one unit (the unit price) of each and compare those. The bigger package costs more in total but may cost less per ounce.

9. Hard Henry walks $2\frac{1}{2}$ miles in $\frac{3}{4}$ of an hour. What is his speed in miles per hour? Give your answer as a mixed number.

Show answer

$3\frac{1}{3}$ miles per hour

10. Hard A 2-pound bag of rice costs \$3.20. A 40-ounce bag of the same rice costs \$4.40. There are 16 ounces in 1 pound. Find the price per ounce of each bag. Which is the better buy?

Show answer

The 2-pound (32-ounce) bag costs \$0.10 per ounce; the 40-ounce bag costs \$0.11 per ounce. The 2-pound bag is the better buy.

11. Hard A pump moves water at 12 gallons per minute. How many hours will it take to fill a 4,320-gallon water tank on a farm?

Show answer

6 hours (360 minutes)

12. Challenge On a trip, the Miller family drove 135 miles in $2\frac{1}{4}$ hours, then 110 miles in 2 hours. What was their average speed for the whole trip? Round to the nearest tenth of a mile per hour.

Show answer

About 57.6 miles per hour (245 miles in 4.25 hours)

COMMON MISTAKES

- Dividing in the wrong order. "Miles per hour" means miles divided by hours. Write the rate as a fraction with labels before dividing.

- Comparing total prices instead of unit prices. Always find the price for one unit of each item, then compare.
- Comparing items measured in different units, such as pounds and ounces. Convert to the same unit first.
- Averaging the two speeds to find the speed for a whole trip. Instead, divide the total distance by the total time.

> **Make it a game · Unit Rate Showdown.** Parent invents ten rates, such as 150 miles in 3 hours or $4.80 for 6 apples. Race to find the unit rate in your head or on paper before the parent counts down from ten. Then compare two rates and name the better buy. Score a point per correct rate and an extra for the better buy. Play to twelve.

7.3 Proportions

OBJECTIVES

- Decide whether two ratios form a proportion by simplifying or by comparing cross products.
- Explain why the cross products of a proportion are equal.
- Solve proportions with a missing number and use them to solve word problems.

WATCH

Math Antics - Proportions

mathantics · 10:30

https://www.youtube.com/watch?v=USmit5zUGas

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Ratio Problems (Solving by Two Methods)

Eddie Woo · 7:38

https://www.youtube.com/watch?v=q9OWPSkY1p4

Before the word problems: two ways to set up and solve a ratio problem.

KEY TERMS

proportion

A statement that two ratios are equal, such as $\frac{2}{3} = \frac{8}{12}$.

cross products

In $\frac{a}{b} = \frac{c}{d}$, the two products $a \times d$ and $b \times c$, found by multiplying across the equal sign on the diagonals.

missing term

The one unknown number in a proportion, shown in this lesson by a box $\square$.

CORE IDEAS

A **proportion** says two ratios are equal: $\frac{2}{3} = \frac{8}{12}$. One way to test a proportion is to simplify both ratios. Since $\frac{8}{12}$ simplifies to $\frac{2}{3}$, these ratios are equal and form a proportion.

A faster test uses **cross products**. In $\frac{2}{3} = \frac{8}{12}$, multiply on the diagonals: $2 \times 12 = 24$ and $3 \times 8 = 24$. The cross products are equal, so the ratios form a proportion. In $\frac{3}{7}$ and $\frac{9}{20}$, the cross products are $3 \times 20 = 60$ and $7 \times 9 = 63$. They are not equal, so these ratios do not form a proportion.

Why do cross products work? Rewrite both fractions with the common denominator you get by multiplying the two denominators. For $\frac{2}{3}$ and $\frac{8}{12}$ that denominator is $3 \times 12 = 36$: $\frac{2}{3} = \frac{2 \times 12}{36} = \frac{24}{36}$ and $\frac{8}{12} = \frac{8 \times 3}{36} = \frac{24}{36}$. When two fractions have the same denominator, they are equal exactly when their numerators are equal. Those new numerators, 2×12 and 8×3, are the cross products. So equal fractions always have equal cross products, and equal cross products always mean equal fractions.

To **solve a proportion** with a missing term, set the cross products equal and then divide. For $\frac{3}{4} = \frac{\square}{20}$: the cross products give $4 \times \square = 3 \times 20 = 60$. Ask "4 times what is 60?" The answer is $60 \div 4 = 15$. Check: $\frac{15}{20}$ simplifies to $\frac{3}{4}$.

In word problems, set up the two ratios **the same way**. If the first ratio is $\frac{\text{cups of flour}}{\text{cookies}}$, the second must also be $\frac{\text{cups of flour}}{\text{cookies}}$. Writing labels on your fractions keeps the units lined up. Then estimate: your answer should be bigger if you are making more, smaller if you are making less.

KEY FORMULAS

$$\frac{a}{b} = \frac{c}{d} \quad \text{exactly when} \quad a \times d = b \times c \qquad (b \neq 0,\ d \neq 0)$$

$$\frac{3}{4} = \frac{\square}{20} \ \Rightarrow\ 4 \times \square = 3 \times 20 \ \Rightarrow\ \square = 60 \div 4 = 15$$

Euclid connection. In Book VII of the Elements, Euclid defines four numbers as proportional when the first is the same multiple, or the same part or parts, of the second as the third is of the fourth. That is the same idea as our equal ratios.

Practice habit. Check every proportion answer by substitution: put your number in the box and confirm that the two cross products are equal. Estimate first so you know whether the answer should be bigger or smaller than the number you started with.

WORKED EXAMPLES

Example 1. Do the ratios $\frac{6}{9}$ and $\frac{10}{15}$ form a proportion?

Show solution

1. Find one cross product: $6 \times 15 = 90$.
2. Find the other cross product: $9 \times 10 = 90$.
3. The cross products are equal, so the ratios form a proportion.
4. Check by simplifying: $\frac{6}{9} = \frac{2}{3}$ and $\frac{10}{15} = \frac{2}{3}$. They match.

Answer: Yes, $\frac{6}{9} = \frac{10}{15}$ is a proportion.

Example 2. Four tickets to the county fair cost \$30. At the same rate, how much do 10 tickets cost?

Show solution

1. Set up the ratios the same way: $\frac{\text{tickets}}{\text{dollars}}$. So $\frac{4}{30} = \frac{10}{\square}$.
2. Cross products: $4 \times \square = 30 \times 10 = 300$.
3. Divide: $\square = 300 \div 4 = 75$.
4. Estimate check: 10 tickets is a bit more than twice 4 tickets, and \$75 is a bit more than twice \$30. Reasonable.
5. Substitution check: $4 \times 75 = 300$ and $30 \times 10 = 300$. Equal.

Answer: \$75

PRACTICE

1. **Easy** Do $\frac{2}{5}$ and $\frac{8}{20}$ form a proportion? Answer yes or no.

   Show answer

   Yes ($2 \times 20 = 40$ and $5 \times 8 = 40$)

2. **Easy** Do $\frac{3}{7}$ and $\frac{9}{20}$ form a proportion? Answer yes or no.

   Show answer

   No ($3 \times 20 = 60$ but $7 \times 9 = 63$)

3. **Easy** Find the number in the box: $\frac{\square}{8} = \frac{15}{24}$.

   Show answer

   5

4. **Easy** Find the number in the box: $\frac{5}{6} = \frac{20}{\square}$.

   Show answer

   24

5. **Medium** Find the number in the box: $\frac{7}{\square} = \frac{21}{36}$.

   Show answer

   12

6. **Medium** Find the number in the box: $\frac{2.5}{4} = \frac{\square}{10}$.

   Show answer

   6.25

7. **Medium** A recipe uses 3 cups of flour to make 24 cookies. How many cups of flour are needed to make 60 cookies?

   Show answer

   $7\frac{1}{2}$ cups

8. **Medium** A farm truck uses 6 gallons of gasoline to travel 108 miles. How many gallons will it use to travel 270 miles?

   Show answer

   15 gallons

9. Hard Use the proportion $\frac{2}{3} = \frac{4}{6}$ to explain why the cross products of a proportion are always equal.

Show answer

Rewrite both fractions over the common denominator $3 \times 6 = 18$: $\frac{2}{3} = \frac{2 \times 6}{18} = \frac{12}{18}$ and $\frac{4}{6} = \frac{4 \times 3}{18} = \frac{12}{18}$. Equal fractions with the same denominator must have equal numerators, and those numerators, 2×6 and 4×3, are exactly the cross products.

10. Hard For the church potluck, 5 pans of lasagna feed 40 people. If 112 people are expected, how many pans of lasagna are needed?

Show answer

14 pans

11. Hard At the same time of day, a 6-foot fence post casts a 4-foot shadow and a tree casts a 22-foot shadow. How tall is the tree?

Show answer

33 feet

12. Hard Two-thirds of a gallon of paint covers 150 square feet. How many gallons are needed to cover 375 square feet? Give your answer as a mixed number.

Show answer

$1\frac{2}{3}$ gallons

13. Challenge In a class, the ratio of students who walk to school to students who ride to school is $3 : 5$. Every student either walks or rides. If 12 students walk, how many students are in the class?

Show answer

32 students (12 walk and 20 ride)

COMMON MISTAKES

- Setting up the two ratios in different orders, such as $\frac{\text{miles}}{\text{gallons}} = \frac{\text{gallons}}{\text{miles}}$. Label both fractions and make the labels match.
- Multiplying straight across (top times top) instead of on the diagonals. Cross products pair each numerator with the *other* denominator.

- Stopping after finding the cross product. When $4 \times \square = 60$, you still must divide: $\square = 60 \div 4 = 15$.
- Forgetting to check. Put your answer in the box and confirm the cross products are equal, and ask whether the size of the answer makes sense.

> **Make it a game · Proportion or Fraud?.** Write 12 pairs of ratios on cards, half true proportions and half impostors. Draw one and decide in 20 seconds whether it is a proportion, proving it with cross products or simplifying. Then solve a missing-term card. Wrong guesses give the parent a point. Play to nine points and then swap sides.

7.4 Scale Drawings and Maps

OBJECTIVES

- Use a map scale or drawing scale to find actual lengths and drawing lengths.
- Find the scale factor of a drawing or model by writing both lengths in the same unit.
- Solve word problems with maps, blueprints and scale models.

WATCH

Scale Diagrams (Using Ratios to find a scale and hence draw a scale diagram)

Eddie Woo · 6:33

https://www.youtube.com/watch?v=EDS5lfRFARg

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: his scales are metric (for example 1:100 in millimetres and metres), so convert examples as needed.

BACKUP VIDEO

Building Plans (1 of 3: Introduction to Building Plans and relation to scaled diagrams)

Eddie Woo · 7:37

https://www.youtube.com/watch?v=_ZuFzgiyPWU

Before the blueprint problems: reading building plans.

KEY TERMS

scale drawing

A drawing of a real object in which every length is shrunk (or enlarged) by the same ratio. Maps and blueprints are scale drawings.

scale

The ratio that tells how a length on the drawing compares to the real length, such as 1 inch : 25 miles.

scale factor

The scale written with the same unit on both sides, so it has no units, such as $1 : 48$ or $\frac{1}{48}$.

scale model

A three-dimensional copy of an object, such as a model ship or model barn, built to a scale.

blueprint

A scale drawing that shows the plan of a building, used by carpenters and builders.

CORE IDEAS

A **scale drawing** shows a real object with every length multiplied by the same number. A map might use the **scale** 1 inch : 25 miles. That means 1 inch on the map stands for 25 miles on the ground, 2 inches stand for 50 miles, and so on. Because every length uses the same ratio, the shapes on the map have the same shape as the real places, just smaller.

A scale is a ratio, so you can solve scale problems with a ratio table or a proportion. Keep the labels lined up: $\frac{\text{map inches}}{\text{real miles}} = \frac{\text{map inches}}{\text{real miles}}$. Often it is quickest to multiply: with 1 inch : 25 miles, a distance of 3.5 inches on the map is $3.5 \times 25 = 87.5$ miles.

To go the other way, from real length to drawing length, divide. On a blueprint with scale $\frac{1}{4}$ inch : 1 foot, each foot is drawn as $\frac{1}{4}$ inch, so a 14-foot wall is drawn $14 \times \frac{1}{4} = 3\frac{1}{2}$ inches long. Another way to think of it: 4 feet are drawn as 1 inch, so $14 \div 4 = 3\frac{1}{2}$ inches.

The **scale factor** compares lengths in the **same unit**, so it has no units. For $\frac{1}{4}$ inch : 1 foot, change 1 foot to 12 inches: $\frac{1}{4} : 12$. Multiply both terms by 4 to get $1 : 48$. So the drawing is $\frac{1}{48}$ the real size, and the real building is 48 times the drawing.

Be careful with **area**. If every length on a drawing is $\frac{1}{10}$ the real length, a square on the drawing is $\frac{1}{10}$ as long *and* $\frac{1}{10}$ as wide as the real square. So its area is only $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ of the real area. To find a real area, first find the real lengths, then multiply.

KEY FORMULAS

$$\frac{\text{drawing length}}{\text{actual length}} = \text{scale}$$

$$\text{scale factor} = \frac{\text{drawing length}}{\text{actual length}} \quad \text{(both in the same unit)}$$

Practice habit. Memorize 12 inches = 1 foot and practice the multiples of 12 (24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144), since they turn inch-to-foot scales into scale factors. Before computing, say whether the answer should be bigger (going to real life) or smaller (going to the drawing).

WORKED EXAMPLES

Example 1. A map has the scale 1 inch : 25 miles. Two towns are 3.5 inches apart on the map. How far apart are the towns in real life?

Show solution

1. Each inch on the map stands for 25 miles.
2. Multiply: $3.5 \times 25 = 87.5$ miles.
3. Check with a proportion: $\frac{1}{25} = \frac{3.5}{87.5}$, and the cross products are $1 \times 87.5 = 87.5$ and $25 \times 3.5 = 87.5$. Equal.

Answer: 87.5 miles

Example 2. A carpenter's blueprint uses the scale $\frac{1}{4}$ inch : 1 foot. A bedroom is 14 feet long and 12 feet wide. How long and how wide is the bedroom on the blueprint? What is the scale factor?

Show solution

1. Each foot is drawn as $\frac{1}{4}$ inch.
2. Length: $14 \times \frac{1}{4} = \frac{14}{4} = 3\frac{1}{2}$ inches.
3. Width: $12 \times \frac{1}{4} = 3$ inches.
4. For the scale factor, use the same unit: 1 foot = 12 inches, so the scale is $\frac{1}{4} : 12$.
5. Multiply both terms by 4: $1 : 48$.
6. Check: $3\frac{1}{2} \times 48 = 168$ inches, and $168 \div 12 = 14$ feet. Correct.

Answer: $3\frac{1}{2}$ inches by 3 inches; scale factor $1 : 48$ (or $\frac{1}{48}$)

PRACTICE

1. Easy A map has the scale 1 cm : 5 km. Two lakes are 7 cm apart on the map. What is the actual distance?

   Show answer

   35 km

2. Easy A drawing of a barn uses the scale 1 inch : 8 feet. A wall is 3 inches long on the drawing. How long is the real wall?

   Show answer

   24 feet

3. Easy A model car is built with scale factor $1 : 24$. The real car is 168 inches long. How long is the model?

   Show answer

   7 inches

4. **Easy** A map has the scale 1 inch : 20 miles. Two cities are 90 miles apart. How far apart are they on the map?

   Show answer

   4.5 inches

5. **Medium** A blueprint uses the scale $\frac{1}{4}$ inch : 1 foot. A wall measures $5\frac{1}{2}$ inches on the blueprint. How long is the real wall?

   Show answer

   22 feet

6. **Medium** Grace drew a plan of the vegetable garden with the scale 2 inches : 5 feet. A garden bed is 9 inches long on her plan. How long is the real bed?

   Show answer

   22.5 feet

7. **Medium** A model barn is 18 inches tall. The real barn is 27 feet tall. Using 12 inches = 1 foot, find the scale factor of the model.

   Show answer

   $1 : 18$ (27 feet = 324 inches, and $18 : 324 = 1 : 18$)

8. **Medium** Henry is building a model of the church with the scale 1 inch : 6 feet. The real steeple is 54 feet tall. How tall should the model steeple be?

   Show answer

   9 inches

9. **Medium** Explain why a scale of 1 inch : 2 feet has a scale factor of $1 : 24$, not $1 : 2$.

   Show answer

   A scale factor compares lengths in the same unit. Since 2 feet = 24 inches, the scale is 1 inch : 24 inches, so every real length is 24 times the drawing length, not 2 times.

10. Hard A map has the scale 1 inch : 15 miles. The Miller family's route measures $2\frac{3}{4}$ inches on the map. If they drive at 55 miles per hour, how many minutes will the trip take?

Show answer

45 minutes (41.25 miles at 55 miles per hour is $\frac{3}{4}$ hour)

11. Hard A blueprint uses the scale $\frac{1}{2}$ inch : 1 foot. A room measures 6 inches by $4\frac{1}{2}$ inches on the blueprint. What is the area of the real room in square feet?

Show answer

108 square feet (the room is 12 feet by 9 feet)

12. Hard On a map with the scale 1 inch : 40 miles, a road is 5 inches long. Samuel redraws the map with the scale 1 inch : 25 miles. How long is the road on his new map?

Show answer

8 inches (the road is 200 miles long)

13. Challenge A square garden is drawn 2 inches by 2 inches on a plan with the scale 1 inch : 10 feet. Grace says that since the drawing has an area of 4 square inches and each inch stands for 10 feet, the real garden has an area of 40 square feet. Find the real area and explain her mistake.

Show answer

400 square feet. The real garden is 20 feet by 20 feet. Each square inch on the plan stands for a square 10 feet by 10 feet, which is 100 square feet, not 10.

COMMON MISTAKES

- Dividing when you should multiply (or the reverse). Going from the drawing to real life makes lengths bigger; going from real life to the drawing makes them smaller. Decide which before you compute.
- Finding a scale factor without changing to the same unit. Convert feet to inches (or miles to inches) first, so the units cancel.
- Using the length scale for area. Find the real lengths first, then multiply them to get the real area.

Make it a game · Map Quest Measure. Draw a quick treasure map of your house on paper with a scale such as one inch equals four feet. Parent hides an object and names a distance, such as 28 feet. You work out how long that distance is on the map and use a ruler to mark the spot. Trade roles and make a new map.

7.5 Unit Conversions

OBJECTIVES

- Explain why multiplying by a conversion factor does not change a quantity.
- Convert lengths, weights and capacities within the customary and metric systems and between them.
- Convert rates, such as miles per hour to feet per second, using more than one conversion factor.

WATCH

Math Antics - Units of Distance

mathantics · 13:56

https://www.youtube.com/watch?v=cKbmvLv-FRo

> **WHEN TO WATCH**
> Before Example 1 (converting lengths).

Note: Eddie Woo teaches in Australia: his examples are metric (km/h to m/s).

BACKUP VIDEO

Rates (3 of 3: Converting one type of rate to another)

Eddie Woo · 3:56

https://www.youtube.com/watch?v=htLPazUW3ZY

Before the rate-conversion example (objective 3).

KEY TERMS

conversion factor

A fraction whose top and bottom are equal amounts in different units, such as $\frac{12 \text{ in}}{1 \text{ ft}}$. It is equal to 1.

customary system

The system of units used in everyday life in the United States: inches, feet, yards, miles, ounces, pounds, cups, quarts, gallons.

metric system

The system of units based on powers of 10: millimeters, centimeters, meters, kilometers, grams, kilograms, milliliters, liters.

unit canceling

Crossing out a unit that appears on both the top and the bottom of a product of fractions, just as you cancel a common factor.

CORE IDEAS

Since 12 inches and 1 foot are the same length, the fraction $\frac{12\text{ in}}{1\text{ ft}}$ has the same amount on top and bottom. Any number divided by itself is 1, so this fraction **equals 1**. That is why it is called a **conversion factor**: multiplying by it changes the units but not the actual size, because multiplying by 1 never changes a quantity.

To convert, choose the conversion factor that puts the **old unit on the bottom**, so it cancels. To change 7.5 feet to inches: $7.5\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}} = 90\text{ in}$. The feet cancel, leaving inches. To change 64 ounces to pounds, use $\frac{1\text{ lb}}{16\text{ oz}}$: $64 \div 16 = 4$ pounds.

A quick check: converting to a **smaller** unit gives a **bigger** number (7.5 feet is 90 inches), and converting to a **larger** unit gives a **smaller** number (64 ounces is 4 pounds). If your answer goes the wrong way, you probably flipped the factor.

Metric units are built on powers of 10, so metric conversions just move the decimal point. 1 kilometer = 1,000 meters, 1 meter = 100 centimeters, 1 centimeter = 10 millimeters, 1 kilogram = 1,000 grams and 1 liter = 1,000 milliliters. For example, $3.2\text{ km} = 3.2 \times 1{,}000 = 3{,}200$ meters. Between systems, use 1 inch = 2.54 centimeters (exact), 1 mile ≈ 1.61 kilometers and 1 kilogram ≈ 2.2 pounds.

You can multiply by **several** conversion factors in a row. To change a rate like 60 miles per hour to feet per second, change miles to feet *and* hours to seconds: $\frac{60\text{ mi}}{1\text{ h}} \times \frac{5{,}280\text{ ft}}{1\text{ mi}} \times \frac{1\text{ h}}{3{,}600\text{ s}}$. Miles and hours cancel, leaving feet per second. Each factor equals 1, so the speed itself never changes. A calculator is fine for the arithmetic.

KEY FORMULAS

$$\frac{12\text{ in}}{1\text{ ft}} = 1 \qquad \frac{3\text{ ft}}{1\text{ yd}} = 1 \qquad \frac{5{,}280\text{ ft}}{1\text{ mi}} = 1$$

$$16\text{ oz} = 1\text{ lb} \qquad 2{,}000\text{ lb} = 1\text{ ton}$$

$$8\text{ fl oz} = 1\text{ c} \qquad 2\text{ c} = 1\text{ pt} \qquad 2\text{ pt} = 1\text{ qt} \qquad 4\text{ qt} = 1\text{ gal}$$

$$1\text{ km} = 1{,}000\text{ m} \qquad 1\text{ m} = 100\text{ cm} \qquad 1\text{ kg} = 1{,}000\text{ g} \qquad 1\text{ L} = 1{,}000\text{ mL}$$

$$1\text{ in} = 2.54\text{ cm} \qquad 1\text{ mi} \approx 1.61\text{ km} \qquad 1\text{ kg} \approx 2.2\text{ lb}$$

Practice habit. Recite the customary facts until they are automatic: 12 inches in a foot, 3 feet in a yard, 5,280 feet in a mile, 16 ounces in a pound, 2 cups in a pint, 2 pints in a quart, 4 quarts in a gallon. Before converting, say "smaller unit, bigger number" or "bigger unit, smaller number."

WORKED EXAMPLES

Example 1. Samuel's new board is 7.5 feet long. How many inches long is it?

Show solution

1. Feet are bigger than inches, so expect a bigger number.
2. Use the conversion factor with feet on the bottom: $\frac{12\text{ in}}{1\text{ ft}}$.
3. Multiply: $7.5\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}} = 90\text{ in}$.
4. Check: $90 \div 12 = 7.5$. Correct.

Answer: 90 inches

Example 2. A car travels at 60 miles per hour. How fast is that in feet per second?

Show solution

1. Write the rate as a fraction: $\dfrac{60\text{ mi}}{1\text{ h}}$.
2. Change miles to feet with $\dfrac{5{,}280\text{ ft}}{1\text{ mi}}$ (miles on the bottom so they cancel).
3. Change hours to seconds with $\dfrac{1\text{ h}}{3{,}600\text{ s}}$ (hours on top so they cancel; 1 hour = 60 minutes = $60 \times 60 = 3{,}600$ seconds).
4. Multiply: $\dfrac{60 \times 5{,}280\text{ ft}}{3{,}600\text{ s}} = \dfrac{316{,}800\text{ ft}}{3{,}600\text{ s}}$.
5. Divide: $316{,}800 \div 3{,}600 = 88$.
6. Check: $88 \times 3{,}600 = 316{,}800$ feet in an hour, and $316{,}800 \div 5{,}280 = 60$ miles. Correct.

Answer: 88 feet per second

PRACTICE

1. Easy Convert 4 yards to feet.

   Show answer

   12 feet

2. Easy Convert 64 ounces to pounds.

   Show answer

   4 pounds

3. Easy Convert 3.2 kilometers to meters.

   Show answer

   3,200 meters

4. Easy Convert 250 centimeters to meters.

   Show answer

   2.5 meters

5. **Medium** Ruth needs 3 gallons of punch for the church picnic. How many cups is that?

Show answer

48 cups

6. **Medium** A shelf bracket is 10 inches long. Using 1 inch = 2.54 centimeters, how long is it in centimeters?

Show answer

25.4 centimeters

7. **Medium** Samuel runs a 5-kilometer race. Using 1 mile ≈ 1.61 kilometers, about how many miles does he run? Round to the nearest tenth.

Show answer

About 3.1 miles

8. **Medium** Grace's kitten weighs 3.6 kilograms. Using 1 kilogram ≈ 2.2 pounds, about how many pounds does the kitten weigh?

Show answer

About 7.92 pounds

9. **Medium** Henry says multiplying 5 feet by $\frac{12 \text{ in}}{1 \text{ ft}}$ makes the length 12 times longer. Explain why the length does not change.

Show answer

12 inches and 1 foot are the same length, so the fraction $\frac{12 \text{ in}}{1 \text{ ft}}$ equals 1. Multiplying by 1 does not change a quantity; it only changes the units. The number gets bigger (60) because inches are smaller units, but 60 inches is the same length as 5 feet.

10. **Hard** A runner's speed is 66 feet per second. How fast is that in miles per hour?

Show answer

45 miles per hour

11. Hard The Miller family needs 150 feet of fencing for their garden. The fencing costs $4.50 per yard. How much will the fencing cost?

    Show answer

    $225.00 (150 feet is 50 yards)

12. Hard A leaky faucet drips 2 cups of water every hour. How many gallons of water does it waste in one day (24 hours)?

    Show answer

    3 gallons (48 cups, and 16 cups make a gallon)

13. Challenge A rectangular room is 15 feet long and 12 feet wide. Carpet costs $24 per square yard. A square yard is 3 feet by 3 feet. How much will it cost to carpet the room?

    Show answer

    $480 (180 square feet is 20 square yards, since 1 square yard = 9 square feet)

COMMON MISTAKES

- Flipping the conversion factor, so the old unit does not cancel. Write the units and make sure the unit you are getting rid of is on the bottom (or on top, if it started on the bottom).
- Not checking the size of the answer. Changing to a smaller unit must give a bigger number; changing to a bigger unit must give a smaller number.
- Using 3 square feet in a square yard. A square yard is 3 feet by 3 feet, which is 9 square feet.
- Converting only one unit in a rate. For miles per hour to feet per second, both miles and hours must be changed.

> **Make it a game · Conversion Chain Challenge.** Write conversion factors on cards, such as 12 inches per foot, 16 ounces per pound, 1000 millimeters per meter. Parent calls a conversion challenge, such as 3 yards to inches. You build a chain of factors and cancel units out loud. Score a point for each correct answer. Add a customary-to-metric card for the finale.

UNIT 7

Ratios, Rates and Proportions

This unit is about comparing quantities. A ratio compares two amounts, such as 3 cups of lemon juice to 5 cups of water. A rate compares amounts in different units, such as miles per hour or dollars per pound. A proportion says two ratios are equal, and it lets you scale a recipe, plan a potluck, read a map or build a model.

You will learn to write and simplify ratios, find unit rates and better buys, solve proportions with cross products (and see why cross products work), use scales on maps and blueprints, and convert units by multiplying by a fraction equal to 1. These skills are used every day in the kitchen, the store, the workshop and on the road.

Ratios build on your work with fractions and equivalent fractions in Unit 3 and decimals in Unit 4. They lead straight into percent in Unit 8, since a percent is a ratio out of 100, and later into slope and rates of change in Unit 11. A basic calculator may be used for messy division in this unit.

PREREQUISITES

- Find the GCF of two numbers (Lesson 2.4).
- Build equivalent fractions and simplify fractions (Lessons 3.1 and 3.2).
- Multiply and divide fractions and mixed numbers (Lessons 3.5 and 3.6).
- Multiply and divide decimals, including money (Lessons 4.3 and 4.4).
- Round decimals to a given place (Lesson 4.1).

MASTERY CHECKLIST

- ☐ I can write a ratio three ways and tell whether it is part-to-part or part-to-whole.
- ☐ I can write a ratio in simplest form and find equivalent ratios.
- ☐ I can complete a ratio table and use it to solve a word problem.
- ☐ I can find a unit rate, including speed and unit price.
- ☐ I can compare unit prices to find the better buy.
- ☐ I can tell whether two ratios form a proportion by using cross products.
- ☐ I can explain why the cross products of a proportion are equal.
- ☐ I can solve a proportion with a missing number and check my answer.
- ☐ I can use a map scale or blueprint scale to find actual lengths and drawing lengths.
- ☐ I can find a scale factor by writing both lengths in the same unit.
- ☐ I can convert units by multiplying by a conversion factor and explain why the quantity does not change.
- ☐ I can convert a rate, such as miles per hour to feet per second.

LESSONS IN THIS UNIT

7.1 Ratios

45 minutes

7.2	**Rates and Unit Rates**	2 sessions of 45 minutes
7.3	**Proportions**	2 sessions of 45 minutes
7.4	**Scale Drawings and Maps**	2 sessions of 45 minutes
7.5	**Unit Conversions**	2 sessions of 45 minutes

Unit 7 test

You have 50 minutes. You may use a basic calculator. Show your work for every problem, including how you set up each ratio, proportion or conversion. Write units in your answers. Round money to the nearest cent and other answers as each problem directs. Use these facts as needed: 12 inches = 1 foot, 3 feet = 1 yard, 5,280 feet = 1 mile, 4 quarts = 1 gallon, 2 pints = 1 quart, 2 cups = 1 pint.

1. **5 pts**
   A youth group has 16 boys and 20 girls. Write the ratio of boys to girls in simplest form.

2. **5 pts**
   A granola recipe uses 3 cups of oats for every 2 cups of raisins. Use a ratio table to find how many cups of raisins go with 12 cups of oats.

3. **6 pts**
   The ratio of boys to girls in a choir is $4 : 5$. The choir has 36 members. How many boys are in the choir?

4. **6 pts**
   Henry and Samuel share $84 from mowing lawns in the ratio $3 : 4$, with Samuel getting the larger share. How much does Samuel get?

5. **6 pts**
   Seven pounds of apples cost $9.73. What is the price per pound?

6. **6 pts**
   A bus travels 156 miles in 2.4 hours. What is its average speed in miles per hour?

7. **7 pts**

   A 24-ounce jar of peanut butter costs \$4.32. A 40-ounce jar costs \$6.80. Find the price per ounce of each jar. Which is the better buy?

8. **5 pts**

   Do $\frac{4}{10}$ and $\frac{6}{15}$ form a proportion? Answer yes or no and show the cross products.

9. **6 pts**

   Find the number in the box: $\frac{9}{12} = \frac{\square}{28}$.

10. **7 pts**

    Three bags of garden soil cover 45 square feet. How many bags are needed to cover 165 square feet?

11. **8 pts**

    Use the proportion $\frac{3}{5} = \frac{6}{10}$ to explain why the cross products of a proportion are equal.

12. **6 pts**

    A map has the scale 1 inch : 30 miles. Two towns are 4.5 inches apart on the map. What is the actual distance between them?

13. **7 pts**

    A blueprint uses the scale $\frac{1}{4}$ inch : 1 foot. A kitchen is 16 feet long and 13 feet wide. How long and how wide is the kitchen on the blueprint?

14. **5 pts**

    Convert 5.5 feet to inches.

15. **7 pts**

    Grace is making 2.5 gallons of soup for a family gathering. How many cups is that?

16. **8 pts**

A country road has a speed limit of 30 miles per hour. How fast is 30 miles per hour in feet per second? Show your conversion factors.

ANSWER KEY

1. $4 : 5$
2. 8 cups of raisins
3. 16 boys
4. \$48
5. \$1.39 per pound
6. 65 miles per hour
7. \$0.18 per ounce and \$0.17 per ounce; the 40-ounce jar is the better buy.
8. Yes ($4 \times 15 = 60$ and $10 \times 6 = 60$)
9. 21
10. 11 bags
11. Rewrite both fractions over the common denominator $5 \times 10 = 50$: $\frac{3}{5} = \frac{3 \times 10}{50} = \frac{30}{50}$ and $\frac{6}{10} = \frac{6 \times 5}{50} = \frac{30}{50}$. Equal fractions with the same denominator have equal numerators, and those numerators, 3×10 and 6×5, are the cross products.
12. 135 miles
13. 4 inches by $3\frac{1}{4}$ inches
14. 66 inches
15. 40 cups
16. 44 feet per second

UNIT 8

Percent

Percent means "per hundred." It is one of the most useful ideas in everyday math: sale prices, sales tax, tips, test scores, harvests that grow from one year to the next, and the interest on savings and loans are all described with percents. In this unit you will learn what a percent really is, how to move easily among percents, fractions and decimals, and how to find a percent of a number in your head or on paper.

Then you will put percents to work. You will solve any percent problem by finding the part, the percent or the whole, measure how much an amount has grown or shrunk, figure store and restaurant bills, and see how money grows in a savings account with simple and compound interest.

This unit builds directly on fractions and decimals (Units 3 and 4) and on ratios and proportions (Unit 7), since a percent is simply a ratio out of 100. The habit of setting up a problem carefully, estimating first and checking at the end will serve you well in Algebra, where the percent equation becomes a true equation with a variable.

PREREQUISITES

- Multiply and divide decimals, including by powers of 10 (Lessons 4.3–4.4).
- Convert between fractions and decimals (Lesson 4.5).
- Simplify fractions and multiply a fraction by a whole number (Lessons 3.2 and 3.5).
- Write ratios and find unit rates (Lessons 7.1–7.2).
- Solve a proportion with cross products (Lesson 7.3).

MASTERY CHECKLIST

- ☐ I can explain that a percent is a ratio that compares a number to 100.
- ☐ I can convert among percents, fractions and decimals, including percents greater than 100% and less than 1%.
- ☐ I can find a percent of a number using the decimal method or the fraction method.
- ☐ I can find 10%, 5%, 1%, 25% and 50% of a number in my head and combine them.
- ☐ I can find a missing part, percent or whole using the percent equation or a percent proportion.
- ☐ I can find the percent of increase or decrease and explain why the original amount is the base.
- ☐ I can find a new amount after a percent increase or decrease.
- ☐ I can find sale prices, sales tax, tips, markups and total costs.
- ☐ I can work with successive discounts and explain why they do not simply add.
- ☐ I can use $I = Prt$ to find simple interest and the balance of a savings account or loan.
- ☐ I can make a year-by-year table to find compound interest and compare it with simple interest.

LESSONS IN THIS UNIT

8.1	**Understanding Percent**	45 minutes
8.2	**Finding a Percent of a Number**	2 sessions of 45 minutes
8.3	**Percent Equations and Proportions**	2 sessions of 45 minutes
8.4	**Percent Increase and Decrease**	2 sessions of 45 minutes
8.5	**Discounts, Sales Tax, Tips and Markup**	2 sessions of 45 minutes
8.6	**Simple Interest and Saving**	2 sessions of 45 minutes

Unit 8 test

You have 50 minutes. You may use a basic calculator, but do problems 1–4 by hand or in your head. Show your work for every problem. Round money to the nearest cent and percents to the nearest tenth of a percent unless the problem says otherwise.

1. **5 pts**
   Write 0.6% as a decimal.

2. **5 pts**
   Write $\frac{7}{20}$ as a percent.

3. **6 pts**
   Find 15% of 260 in your head. Show how you used 10% and 5%.

4. **5 pts**
   Find 125% of 48.

5. **7 pts**
   21 is what percent of 60?

6. **7 pts**
   24 is 40% of what number?

7. 7 pts

   At the church bake sale, Grace's family baked 54 pies. That was 45% of all the pies sold at the sale. How many pies were sold at the bake sale in all?

8. 7 pts

   A number increases from 40 to 52. What is the percent of increase?

9. 8 pts

   The price of a bag of chicken feed dropped from $25 to $22. What was the percent of decrease?

10. 8 pts

    A sweater regularly costs $64. It is on sale for 25% off, and the sales tax is 6%. What is the total cost?

11. 8 pts

    Henry's family has a restaurant bill of $38 and leaves a 15% tip. How much do they pay in all?

12. 8 pts

    A camping tent regularly costs $150. It is marked 20% off, and then an extra 10% is taken off the sale price. What is the final price?

13. 7 pts

    Samuel deposits $900 in a savings account that pays 4% simple interest per year. How much interest will he earn in 3 years?

14. 7 pts

    Ruth saves $400 in an account that pays 5% interest compounded once a year. Make a year-by-year table and find her balance after 2 years.

15. 5 pts

    A $100 price is raised by 10%, and then the new price is lowered by 10%. Is the final price $100? Explain why or why not.

ANSWER KEY

1. 0.006
2. 35%
3. 39 (10% is 26 and 5% is 13)
4. 60
5. 35%
6. 60
7. 120 pies
8. 30%
9. 12%
10. $50.88
11. $43.70
12. $108
13. $108
14. $441 (balances $420, then $441)
15. No. The 10% increase is taken from $100, giving $110. The 10% decrease is taken from the larger amount $110, which takes off $11, so the final price is $99. The two percents are taken from different bases.

8.1 Understanding Percent

OBJECTIVES

- Explain that a percent is a ratio that compares a number to 100.
- Convert among percents, fractions and decimals.
- Write and interpret percents greater than 100% and less than 1%.

WATCH

Math Antics - What Are Percentages?

mathantics · 8:53

https://www.youtube.com/watch?v=JeVSmq1Nrpw

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Math Antics - Percents And Equivalent Fractions

mathantics · 8:37

https://www.youtube.com/watch?v=kmVfZ9o-2gg

Before Example 2: converting between percents and fractions.

KEY TERMS

percent

A ratio that compares a number to 100. The symbol % means "out of 100" or "per hundred," so 23% means 23 out of 100.

per hundred

For every hundred. The word *percent* comes from the Latin *per centum*, "by the hundred."

whole

The full amount you are comparing to. The whole is always 100%.

CORE IDEAS

A **percent** is a ratio with a second term of 100. If 23 of the 100 squares on a hundred grid are shaded, then $\frac{23}{100}$ of the grid is shaded, and we say 23% is shaded. Because

every percent is "out of 100," percents make it easy to compare amounts that started with different wholes.

Percent to decimal and back. Since 23% means $\frac{23}{100}$, and $\frac{23}{100} = 0.23$, you change a percent to a decimal by dividing by 100: move the decimal point two places to the left. So $7\% = 0.07$ and $125\% = 1.25$. To go the other way, multiply by 100: move the decimal point two places to the right. So $0.4 = 40\%$ and $0.035 = 3.5\%$.

Percent to fraction. Write the percent over 100 and simplify: $45\% = \frac{45}{100} = \frac{9}{20}$.
Fraction to percent. If you can, build an equivalent fraction with denominator 100: $\frac{3}{5} = \frac{60}{100} = 60\%$. If not, divide to get a decimal and then multiply by 100: $\frac{3}{8} = 0.375 = 37.5\%$.

More than 100%. 100% is one whole. So 150% is one whole and a half more: $150\% = 1.5 = 1\frac{1}{2}$. If a garden produced 150% of last year's harvest, it produced one and a half times as much.

Less than 1%. 1% is one hundredth, 0.01. A percent smaller than 1% is a part of one hundredth. For example, $0.5\% = 0.005$, which is half of one hundredth, or 5 out of every 1,000. Do not confuse 0.5% with 0.5, which is 50%.

KEY FORMULAS

$$n\% = \frac{n}{100}$$

$$\text{percent} \to \text{decimal: divide by } 100 \qquad \text{decimal} \to \text{percent: multiply by } 100$$

$$100\% = 1 \text{ whole}$$

From history. The word *percent* comes from the Latin *per centum*, meaning "by the hundred." Latin was the language of learning in Europe for many centuries, and merchants and scholars used it when they wrote about trade, taxes and interest. Our word *century* (100 years) and *cent* (one hundredth of a dollar) come from the same Latin root.

Practice habit. Recite these fraction–decimal–percent equivalents until they are automatic: $\frac{1}{2} = 0.5 = 50\%$, $\frac{1}{4} = 0.25 = 25\%$, $\frac{3}{4} = 0.75 = 75\%$, $\frac{1}{5} = 0.2 = 20\%$, $\frac{1}{10} = 0.1 = 10\%$, $\frac{1}{8} = 0.125 = 12.5\%$, $\frac{1}{3} = 0.\overline{3} = 33\frac{1}{3}\%$.

WORKED EXAMPLES

Example 1. Write $\frac{3}{4}$ as a decimal and as a percent.

Show solution

1. Build a fraction with denominator 100: multiply top and bottom by 25.
2. $\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100}$.
3. As a decimal, $\frac{75}{100} = 0.75$.
4. As a percent, 75 out of 100 is 75%.
5. Check: $0.75 \times 100 = 75$, so $0.75 = 75\%$.

Answer: 0.75 and 75%

Example 2. A bakery baked 400 loaves of bread last month, and 3 of them were burned. What percent of the loaves were burned?

Show solution

1. Write the part over the whole: $\frac{3}{400}$.
2. To get a denominator of 100, divide top and bottom by 4: $\frac{3 \div 4}{400 \div 4} = \frac{0.75}{100}$.
3. So the burned loaves are 0.75 out of every 100, which is 0.75%.
4. Check with a decimal: $3 \div 400 = 0.0075$, and $0.0075 \times 100 = 0.75$.
5. This is less than 1%, which makes sense: 1% of 400 loaves would be 4 loaves, and only 3 burned.

Answer: 0.75%

PRACTICE

1. Easy Write 7% as a decimal.

   Show answer

   0.07

2. Easy Write 0.08 as a percent.

   Show answer

   8%

3. Easy Write 45% as a fraction in lowest terms.

Show answer

$\frac{9}{20}$

4. Easy Write $\frac{3}{5}$ as a percent.

Show answer

60%

5. Medium Write $\frac{3}{8}$ as a percent.

Show answer

37.5%

6. Medium Write 150% as a mixed number in lowest terms.

Show answer

$1\frac{1}{2}$

7. Medium Write 0.4% as a decimal.

Show answer

0.004

8. Medium Grace planted 25 tomato plants, and 18 of them have ripe tomatoes. What percent of the plants have ripe tomatoes?

Show answer

72%

9. Hard Order from least to greatest: 0.6, $\frac{5}{8}$, 59%, 0.065.

Show answer

0.065, 59%, 0.6, $\frac{5}{8}$ (as decimals: $0.065 < 0.59 < 0.6 < 0.625$)

10. Hard Henry answered $\frac{2}{3}$ of the questions on his history quiz correctly. What percent is that? Round to the nearest tenth of a percent.

Show answer

66.7%

11. Hard The choir hoped to sell 80 tickets to its spring concert and actually sold 100 tickets. What percent of the goal did the choir sell?

Show answer

125% ($\frac{100}{80} = 1.25$)

12. Hard Samuel says that $\frac{1}{2}\%$ and 50% are the same thing. Explain why he is wrong.

Show answer

50% means 50 out of 100, which is 0.5, or half of the whole. But $\frac{1}{2}\%$ means one half of one hundredth, which is 0.005, or 5 out of 1,000. 50% is 100 times as large as $\frac{1}{2}\%$.

13. Challenge Write $\frac{1}{8}\%$ as a decimal.

Show answer

0.00125 (since $\frac{1}{8}\% = 0.125\%$ and $0.125 \div 100 = 0.00125$)

COMMON MISTAKES

- Moving the decimal point the wrong way. Remember that a percent is a bigger-looking number than its decimal: 35% becomes 0.35, not 35.00 or 3500. Ask: is the decimal smaller than 1 when the percent is smaller than 100%?
- Writing 5% as 0.5. Since $5\% = \frac{5}{100}$, it is 0.05. Always fill in two places when you move the decimal point.
- Thinking a percent can never be more than 100%. 100% is one whole, and you can have more than one whole: 200% is twice the whole.
- Treating 0.5% as half. 0.5% is 0.005; half is 50%. Watch for the % sign.

> **Make it a game · Percent Switch Cards.** Make a deck of cards with a fraction, a decimal and a percent on separate cards. Shuffle and deal five face down. Flip one card and race to say the other two forms from memory. Then match all the triples you can on the table. The first to match four complete triples wins the round. Play best of three.

8.2 Finding a Percent of a Number

OBJECTIVES

- Find a percent of a number by changing the percent to a decimal or to a fraction and multiplying.
- Find 10%, 5%, 1%, 25% and 50% of a number mentally and combine them to find other percents.
- Solve word problems that ask for a percent of an amount, including percents greater than 100%.

WATCH

Math Antics - Finding A Percent Of A Number

mathantics · 7:32

https://www.youtube.com/watch?v=rR95Cbcjzus

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Strategies for Mentally Calculating Percentages

Eddie Woo · 10:50

https://www.youtube.com/watch?v=f0hDmH901rw

Before the mental-math section: building up from 10%, 5% and 1%.

KEY TERMS

percent of a number

A part of a number found by multiplying the number by the percent written as a decimal or fraction. "30% of 80" means 0.30×80.

decimal method

Changing the percent to a decimal and then multiplying: 35% of $60 = 0.35 \times 60$.

fraction method

Changing the percent to a fraction and then multiplying: 25% of $64 = \frac{1}{4} \times 64$.

benchmark percent

An easy percent such as 10%, 50% or 25% that you can find in your head and use to build other percents.

CORE IDEAS

In Unit 3 you learned that "of" means times: $\frac{1}{2}$ of 10 is $\frac{1}{2} \times 10$. Percent works the same way. Since 30% means $\frac{30}{100}$, "30% of 80" means $\frac{30}{100} \times 80 = 24$. You are taking 30 parts out of every 100.

Decimal method. Change the percent to a decimal and multiply: 30% of 80 is $0.3 \times 80 = 24$. This method always works, and it is the one to use with a calculator.

Fraction method. Some percents are friendly fractions: $50\% = \frac{1}{2}$, $25\% = \frac{1}{4}$, $20\% = \frac{1}{5}$, $75\% = \frac{3}{4}$. Then the work is easy: 25% of 64 is $\frac{1}{4} \times 64 = 16$. Use the method that makes the numbers nicest.

Mental percents. 10% is $\frac{1}{10}$, so move the decimal point one place left: 10% of 230 is 23. 1% is $\frac{1}{100}$, so move it two places: 1% of 650 is 6.5. 5% is half of 10%. 50% is half, and 25% is half of half. Build other percents from these pieces: 15% = 10% + 5%, and 35% = 10% + 10% + 10% + 5%.

Why the pieces add up. Because of the distributive property, 35% of $60 = (30\% + 5\%)$ of $60 = 30\%$ of $60 + 5\%$ of 60. Splitting a percent into friendly pieces is just like splitting 7×98 into $7 \times 100 - 7 \times 2$ back in Unit 1.

Percents above 100%. 100% of a number is the number itself, so 150% of a number is more than the number. 150% of 40 is $1.5 \times 40 = 60$. If your percent is above 100%, your answer should be bigger than what you started with; if it is below 100%, your answer should be smaller.

KEY FORMULAS

$$n\% \text{ of } A = \frac{n}{100} \times A$$

$$10\% \text{ of } A = A \div 10 \qquad 1\% \text{ of } A = A \div 100$$

$$5\% = \tfrac{1}{2} \text{ of } 10\% \qquad 25\% = \tfrac{1}{4} \qquad 50\% = \tfrac{1}{2}$$

> **Practice habit.** Before you multiply, find 10% in your head and use it to estimate. For 35% of 60, 10% is 6, so the answer should be a little more than $3 \times 6 = 18$. If your calculator says 210, you know something went wrong.

WORKED EXAMPLES

Example 1. Find 35% of 60 in two ways.

Show solution

1. Decimal method: $35\% = 0.35$.
2. $0.35 \times 60 = 21$.
3. Mental method: 10% of 60 is 6, so 30% is $3 \times 6 = 18$.
4. 5% is half of 10%, so 5% of 60 is 3.
5. 35% = 30% + 5%, so 35% of 60 is $18 + 3 = 21$.
6. Both methods agree.

Answer: 21

Example 2. The Miller family's dairy farm produced 2,400 gallons of milk in June. In July the farm produced 115% of June's amount. How many gallons of milk did the farm produce in July?

Show solution

1. Estimate: 115% is a little more than 100%, so July should be a little more than 2,400 gallons.
2. Split the percent: $115\% = 100\% + 10\% + 5\%$.
3. 100% of 2,400 is 2,400.
4. 10% of 2,400 is 240.
5. 5% is half of 10%, so 5% of 2,400 is 120.
6. Add: $2{,}400 + 240 + 120 = 2{,}760$.
7. Check with the decimal method: $1.15 \times 2{,}400 = 2{,}760$.

Answer: 2,760 gallons

PRACTICE

1. Easy Find 10% of 230 in your head.

   Show answer

   23

2. Easy Find 50% of 84 in your head.

Show answer

42

3. Easy Find 25% of 48 in your head.

Show answer

12

4. Easy Find 1% of 650 in your head.

Show answer

6.5

5. Medium Find 15% of 80 by finding 10% and 5% and adding.

Show answer

12 ($8 + 4$)

6. Medium Find 40% of 75.

Show answer

30

7. Medium Mrs. Taylor's class has 28 students, and 75% of them went on the field trip to the history museum. How many students went on the field trip?

Show answer

21 students

8. Medium Find 6% of 350.

Show answer

21

9. Hard A farmer harvested 1,240 bushels of corn and sold 65% of it to the grain elevator. How many bushels did he sell?

Show answer

806 bushels

10. Hard Find 0.5% of 3,000.

Show answer

15

11. Hard Find 250% of 36.

Show answer

90

12. Hard Explain why you can find 5% of a number by finding 10% of it and then taking half.

Show answer

5% is half of 10%, because $\frac{5}{100}$ is half of $\frac{10}{100}$. So 5% of any number is half of 10% of that number. For example, 10% of 80 is 8, and 5% of 80 is 4.

13. Challenge A 2-pound bag of trail mix is 30% peanuts, 25% raisins and $\frac{1}{5}$ almonds by weight. The rest is chocolate pieces. How many ounces of chocolate pieces are in the bag? (1 pound = 16 ounces.)

Show answer

8 ounces (the chocolate is $100\% - 30\% - 25\% - 20\% = 25\%$ of 32 ounces)

COMMON MISTAKES

- Multiplying by the percent number itself, such as $35 \times 60 = 2{,}100$ for 35% of 60. Change the percent to a decimal or fraction first: $0.35 \times 60 = 21$. Estimate with 10% to catch this.
- Finding 10% correctly but then taking 5% as "10% minus 5" instead of half of 10%. 5% is always half of 10%.
- Expecting every answer to be smaller than the starting number. A percent over 100% gives an answer larger than the starting number.
- Writing small percents wrong, such as $0.5\% = 0.5$. Since $0.5\% = 0.005$, 0.5% of 3,000 is 15, not 1,500.

Make it a game · Benchmark Mental Math. Parent names a number and a percent, such as 35% of 80. You have 20 seconds to find it in your head using 10%, 5% and 50% as building blocks, then explain aloud. Write your steps on paper to check. Score two points for each right answer. Try to reach 14 points before the parent gets 10 on their own set.

8.3 Percent Equations and Proportions

OBJECTIVES

- Identify the part, the percent and the whole in a percent problem.
- Find a missing part, percent or whole using part = percent × whole or a percent proportion.
- Solve and check percent word problems.

WATCH

Math Antics - What Percent Is It?

mathantics · 8:28

https://www.youtube.com/watch?v=Uf-Rl1e2I4Q

> **WHEN TO WATCH**
> Before Example 1 (finding the percent).

BACKUP VIDEO

Math Antics - Percents Missing Total

mathantics · 5:07

https://www.youtube.com/watch?v=HxEQxS0QSwg

Before Example 2 (finding a missing whole).

KEY TERMS

whole

The total amount, the number that is 100%. It usually follows the word "of" in a percent question.

part

The piece of the whole being described. It usually comes next to the word "is" in a percent question.

percent equation

The rule part $=$ percent $\times$ whole, with the percent written as a decimal or fraction.

percent proportion

The proportion $\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$.

CORE IDEAS

Every percent problem has three numbers: the **part**, the **percent** and the **whole**. In "12 is 25% of 48," the part is 12, the percent is 25% and the whole is 48. A problem

gives you two of them and asks for the third. The word "of" usually points to the whole, and "is" usually points to the part.

In Lesson 8.2 you found the part: $\text{part} = \text{percent} \times \text{whole}$. This is the **percent equation**. If the part or the whole is the missing number, put a box in its place: "15 is 50% of what number?" becomes $15 = 0.5 \times \square$. Then ask what number times 0.5 gives 15, and divide: $\square = 15 \div 0.5 = 30$.

You can also use a **percent proportion**. A percent is a ratio out of 100, and the part compared to the whole is the same ratio. So $\dfrac{\text{part}}{\text{whole}} = \dfrac{\text{percent}}{100}$. Fill in the two numbers you know, put a box in for the missing one, and solve with cross products just as in Lesson 7.3.

Why the proportion works. Saying "12 is 25% of 48" means that 12 out of 48 is the same rate as 25 out of 100. Indeed $\dfrac{12}{48} = \dfrac{1}{4} = \dfrac{25}{100}$. The proportion simply says that the part-to-whole ratio is equal to the percent written as a fraction out of 100.

Always check. Put your answer back into the original sentence and see if it is true. If you found that 15 is 50% of 30, check: half of 30 is 15. Also ask whether the size makes sense: if the percent is less than 100%, the part must be smaller than the whole.

KEY FORMULAS

$$\text{part} = \text{percent} \times \text{whole}$$

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$$

$$\text{percent} = \frac{\text{part}}{\text{whole}} \times 100\% \qquad \text{whole} = \text{part} \div \text{percent}$$

Practice habit. Label before you solve: underline the whole (after "of"), circle the part (near "is"), and box the percent. Then check every answer by putting it back into the sentence.

WORKED EXAMPLES

Example 1. What percent of 40 is 18?

Show solution

1. The whole is 40 (after "of"). The part is 18. The percent is missing.
2. Set up the proportion: $\frac{18}{40} = \frac{\square}{100}$.
3. Cross products: $40 \times \square = 18 \times 100 = 1{,}800$.
4. Divide: $\square = 1{,}800 \div 40 = 45$.
5. Check: 45% of 40 is $0.45 \times 40 = 18$. True.

Answer: 45%

Example 2. Grace has saved \$54 for a new bicycle. That is 60% of the price of the bicycle. What is the price of the bicycle?

Show solution

1. The part is \$54, the percent is 60%, and the whole (the price) is missing.
2. Percent equation: $54 = 0.6 \times \square$.
3. Divide: $\square = 54 \div 0.6 = 540 \div 6 = 90$.
4. Or use a proportion: $\frac{54}{\square} = \frac{60}{100}$, so $60 \times \square = 5{,}400$ and $\square = 90$.
5. Size check: \$54 is less than the whole price, which makes sense for 60%.
6. Substitution check: 60% of \$90 is $0.6 \times 90 = 54$. True.

Answer: \$90

PRACTICE

1. Easy What number is 20% of 45?

   Show answer

   9

2. Easy 12 is what percent of 48?

   Show answer

   25%

3. Easy 15 is 50% of what number?

Show answer

30

4. Easy What percent of 50 is 17?

Show answer

34%

5. Medium 18 is 30% of what number?

Show answer

60

6. Medium 27 is what percent of 36?

Show answer

75%

7. Medium 42 is 120% of what number?

Show answer

35

8. Medium In basketball practice, Henry made 14 of his 20 free throws. What percent of his free throws did he make?

Show answer

70%

9. Hard The church choir has 9 tenors, and the tenors make up 15% of the choir. How many members does the choir have in all?

Show answer

60 members

10. Hard A seed packet promises that 85% of the seeds will sprout. Ruth planted 60 seeds from the packet, and 48 of them sprouted. What percent of Ruth's seeds sprouted?

Show answer

80% (a little less than the packet promised)

11. Hard 7 is what percent of 8?

Show answer

87.5%

12. Hard Explain why the proportion $\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$ is true. Use the fact that 9 is 75% of 12 as your example.

Show answer

A percent is a ratio out of 100, so 75% means 75 out of every 100. Saying 9 is 75% of 12 means the part 9 compares to the whole 12 at that same rate. Indeed $\frac{9}{12} = \frac{3}{4} = \frac{75}{100}$, so the two ratios are equal and form a proportion.

13. Challenge At the church potluck, 45 people chose the lasagna. This was 36% of the people who came. How many people at the potluck did **not** choose the lasagna?

Show answer

80 people (125 people came, and $125 - 45 = 80$)

COMMON MISTAKES

- Mixing up the part and the whole. The whole usually follows "of." In "27 is what percent of 36," the whole is 36, so the ratio is $\frac{27}{36}$, not $\frac{36}{27}$.
- Multiplying when you should divide. When the whole is missing, as in $18 = 0.3 \times \square$, you must divide: $18 \div 0.3 = 60$. A quick check catches this: 30% of 5.4 is not 18.
- Forgetting to write the answer as a percent. If $\frac{14}{20} = 0.7$, the answer to "what percent" is 70%, not 0.7%.
- Skipping the check. Put your answer back into the original sentence and make sure it is true.

> **Make it a game · Part, Percent, Whole.** Write 12 percent problems on cards with one piece missing. Draw a card and shout which piece is missing, part, percent or whole, then set up the equation or proportion and solve. Score a point for naming the missing piece and a point for the answer. Try to finish six cards within ten minutes.

8.4 Percent Increase and Decrease

OBJECTIVES

- Find the percent of increase or decrease from an original amount to a new amount.
- Explain why percent change is always measured against the original amount.
- Find a new amount after a given percent increase or decrease.

WATCH

Math Antics - Calculating Percent Change

mathantics · 12:51

https://www.youtube.com/watch?v=5nZEUpZX_P0

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Percentage Change (3 of 4: Combining Increase & Decrease)

Eddie Woo · 6:37

https://www.youtube.com/watch?v=UnCKunozt_o

After Example 2: why an increase followed by the same percent decrease does not bring you back to the start.

KEY TERMS

original amount

The starting amount, before the change. It is the whole (100%) in a percent change problem.

new amount

The amount after the change.

amount of change

How much the amount went up or down: the larger amount minus the smaller amount.

percent of increase

The amount of increase as a percent of the original amount.

percent of decrease

The amount of decrease as a percent of the original amount.

CORE IDEAS

When an amount goes up or down, we often want to know how big the change is *compared to where we started*. A price rising by \$2 is a big change for a \$4 loaf of bread but a tiny change for a \$400 bicycle. Percent change measures the change as a part of the original amount.

How to find percent change. First find the amount of change by subtracting. Then divide the change by the original amount and write the result as a percent. If the amount went up, it is a percent of increase; if it went down, it is a percent of decrease. For a loaf of bread that went from \$4 to \$5: the change is \$1, and $\frac{1}{4} = 25\%$, so the price increased by 25%.

Why the original amount is the base. "Percent change" answers the question "the change is what percent *of the starting amount*?" The starting amount is the whole, so it goes in the denominator. Watch what happens if you use the wrong base: from \$4 to \$5 is a 25% increase ($\frac{1}{4}$), but from \$5 back down to \$4 is only a 20% decrease ($\frac{1}{5}$). The change is the same \$1 both times, but the starting amounts are different, so the percents are different.

Finding the new amount. To increase 60 by 15%, find 15% of 60, which is 9, and add: $60 + 9 = 69$. A shortcut: the new amount is $100\% + 15\% = 115\%$ of the original, so $1.15 \times 60 = 69$. To decrease 60 by 15%, subtract: $60 - 9 = 51$, or use 85% of 60: $0.85 \times 60 = 51$.

A percent increase can be more than 100%. If a puppy grows from 8 pounds to 20 pounds, the increase is 12 pounds, and $\frac{12}{8} = 1.5 = 150\%$. A percent decrease, though, can never be more than 100%, because you cannot take away more than all of something.

KEY FORMULAS

$$\text{percent change} = \frac{\text{amount of change}}{\text{original amount}} \times 100\%$$

$$\text{new amount after an increase of } r\% = (100\% + r\%) \times \text{original}$$

$$\text{new amount after a decrease of } r\% = (100\% - r\%) \times \text{original}$$

Practice habit. Say the question out loud before you divide: "The change is what percent of the *original*?" Then estimate: a change of about half the original is about 50%, and a change about equal to the original is about 100%.

WORKED EXAMPLES

Example 1. Last year the youth group's car wash raised \$250. This year it raised \$310. What was the percent of increase?

Show solution

1. Original amount: \$250. New amount: \$310.
2. Amount of change: $310 - 250 = 60$ dollars.
3. Divide by the original: $\frac{60}{250} = \frac{24}{100} = 0.24$.
4. Write as a percent: $0.24 = 24\%$.
5. Check: 24% of 250 is $0.24 \times 250 = 60$, and $250 + 60 = 310$. True.

Answer: a 24% increase

Example 2. A hardware store marks a toolbox down from \$40 to \$34. A month later it marks the toolbox back up from \$34 to \$40. Find the percent of decrease and then the percent of increase. Round to the nearest tenth of a percent where needed.

Show solution

1. Markdown: original \$40, new \$34, change $40 - 34 = 6$ dollars.
2. Percent of decrease: $\frac{6}{40} = 0.15 = 15\%$.
3. Markup: now the original is \$34 and the new amount is \$40. The change is still 6 dollars.
4. Percent of increase: $\frac{6}{34} \approx 0.17647$.
5. Round: $0.17647 \approx 17.6\%$.
6. The same \$6 change gives different percents because the original amounts are different. The base is always the starting amount.

Answer: a 15% decrease, then about a 17.6% increase

PRACTICE

1. Easy A number increases from 20 to 25. What is the percent of increase?

   Show answer

   25%

2. Easy A number decreases from 50 to 40. What is the percent of decrease?

   Show answer

   20%

3. Easy Increase 80 by 10%. What is the new amount?

   Show answer

   88

4. Easy Decrease 200 by 30%. What is the new amount?

   Show answer

   140

5. **Medium** A number increases from 16 to 28. What is the percent of increase?

Show answer

75%

6. **Medium** The price of a dozen eggs went from \$3.00 to \$3.60. What was the percent of increase?

Show answer

20%

7. **Medium** Samuel's family used 900 kilowatt-hours of electricity in July and 720 kilowatt-hours in September. What was the percent of decrease?

Show answer

20%

8. **Medium** A puppy weighed 8 pounds when the Millers brought it home and 20 pounds three months later. What was the percent of increase in its weight?

Show answer

150%

9. **Hard** A school had 480 students last year. This year enrollment rose by 5%. How many students are enrolled this year?

Show answer

504 students

10. **Hard** A community garden produced 75 pounds of green beans last summer and 63 pounds this summer. What was the percent of decrease?

Show answer

16%

11. **Hard** The attendance at a Sunday school class went from 18 to 24. What was the percent of increase? Round to the nearest tenth of a percent.

Show answer

about 33.3% (exactly $33\frac{1}{3}\%$)

12. Hard Ruth says that going from 10 to 15 and going from 15 to 10 are both 50% changes, because the difference is 5 both times. Explain why she is wrong and give the correct percents.

Show answer

Percent change is measured against the original amount. From 10 to 15, the change of 5 is compared to 10: $\frac{5}{10} = 50\%$ increase. From 15 to 10, the change of 5 is compared to 15: $\frac{5}{15} = 33\frac{1}{3}\%$ decrease. Same change, different starting amounts, different percents.

13. Challenge A store raised the price of a \$50 jacket by 20%. Later it lowered the new price by 20%. What is the final price? Is it the same as the original \$50?

Show answer

\$48; it is not the same, because the 20% decrease is taken from the larger price of \$60 ($60 - 12 = 48$)

COMMON MISTAKES

- Dividing by the new amount instead of the original. Ask "the change is what percent of the *starting* amount?" and put the starting amount in the denominator.
- Giving the amount of change as the answer. A change from \$4 to \$5 is \$1, but the question asks for a percent: 25%.
- Thinking that a 20% increase followed by a 20% decrease brings you back to where you started. The two percents are taken from different amounts, so you end up lower.
- Forgetting to add or subtract after finding the percent of the original. 15% of 60 is 9, but 60 increased by 15% is $60 + 9 = 69$.

> **Make it a game · Original or Bust.** Parent gives two amounts, such as a plant growing from 20 to 26 inches or a jar of beans going from 50 to 35. You find the amount of change and then the percent increase or decrease, always dividing by the original. A parent sneaks in one that divides by the new amount, and you must catch it. Play ten rounds.

8.5 Discounts, Sales Tax, Tips and Markup

OBJECTIVES

- Find sale prices, sales tax, tips, markups and total costs.
- Find the percent of a discount or markup from two prices.
- Solve problems with successive discounts and explain why two discounts do not simply add.

WATCH

How to Calculate a Discount and Sale Price | Math with Mr. J

Math with Mr. J · 7:04

https://www.youtube.com/watch?v=5qxX222sLtE

> **WHEN TO WATCH**
> Before the discount example: finding the discount and the sale price.

Note: Tips work the same way as sales tax (a percent added on). Neither video covers markup, which works like a percent increase (Lesson 8.4).

BACKUP VIDEO

How to Calculate Sales Tax | Math with Mr. J

Math with Mr. J · 7:21

https://www.youtube.com/watch?v=nNhGLjHjNc8

Before the sales-tax example: finding the tax and the total cost.

KEY TERMS

discount

The amount taken off a regular price. A 25% discount takes 25% of the regular price away.

sale price

The price after the discount is subtracted from the regular price.

sales tax

Money a state or town adds to the price of many purchases. It is a percent of the price and is paid along with it.

tip

Extra money, usually a percent of the bill, given to a server or worker for good service. Also called a gratuity.

markup

The amount a store adds to what it paid for an item (its cost) to set its selling price.

successive discounts

Two or more discounts taken one after another, each on the price left by the one before.

CORE IDEAS

These are all percent-of-a-number problems from Lesson 8.2 and percent change problems from Lesson 8.4, dressed up in store clothes. The key question each time is: *what is the original amount, and do I add or subtract?*

Discounts subtract. A \$40 sweater at 25% off has a discount of $0.25 \times 40 = 10$ dollars, so the sale price is $40 - 10 = 30$ dollars. Shortcut: if you take 25% off, you pay $100\% - 25\% = 75\%$, and $0.75 \times 40 = 30$.

Sales tax, tips and markups add. A 6% sales tax on \$30 is $0.06 \times 30 = 1.80$ dollars, so you pay \$31.80 in all, or $1.06 \times 30 = 31.80$. A tip works the same way, figured on the bill. A store that buys a lamp for \$20 and marks it up 50% adds \$10 and sells it for \$30. For a markup, the original amount is the store's cost.

Tips in your head. To leave 15%, find 10% by moving the decimal point one place left, then add half of that. On a \$40 bill, 10% is \$4, 5% is \$2, so 15% is \$6. For 20%, double 10%: \$8.

Successive discounts. Sometimes a store takes an extra percent off the sale price. On a \$120 coat, 25% off and then an extra 10% off is *not* 35% off. After the first discount the price is \$90. The second discount is 10% of \$90, only \$9, so the final price is \$81. A single 35% discount would give \$78. Why the difference? The second percent is taken from a smaller amount, just as in Lesson 8.4, where the base for a percent is always the amount you start that step with.

When there is both a discount and sales tax, take the discount first and then figure the tax on the sale price, because tax is charged on what you actually pay for the item. When an amount does not come out to whole cents, round it to the nearest cent.

KEY FORMULAS

$$\text{sale price} = \text{regular price} - \text{discount} = (100\% - \text{discount rate}) \times \text{regular price}$$

$$\text{total} = \text{price} + \text{tax} = (100\% + \text{tax rate}) \times \text{price}$$

$$\text{selling price} = \text{cost} + \text{markup} = (100\% + \text{markup rate}) \times \text{cost}$$

Practice habit. Estimate every store total before you pay it. Round the price, find 10% in your head, and build the percent you need from 10% and 5%. If the register shows something far from your estimate, ask about it.

WORKED EXAMPLES

Example 1. A pair of work boots regularly costs \$45. They are on sale for 20% off, and the sales tax is 6%. What is the total cost?

Show solution

1. Discount: 20% of 45 is $0.2 \times 45 = 9$ dollars.
2. Sale price: $45 - 9 = 36$ dollars.
3. Tax on the sale price: $0.06 \times 36 = 2.16$ dollars.
4. Total: $36 + 2.16 = 38.16$ dollars.
5. Check with the shortcut: $0.8 \times 45 = 36$ and $1.06 \times 36 = 38.16$. They agree.

Answer: \$38.16

Example 2. A winter coat regularly costs \$120. It is marked 25% off, and on Saturday the store takes an extra 10% off the sale price. What is the final price? What single discount would give the same price?

Show solution

1. First discount: 25% of 120 is 30 dollars, so the sale price is $120 - 30 = 90$ dollars.
2. Second discount: 10% of the *sale price* 90 is 9 dollars.
3. Final price: $90 - 9 = 81$ dollars.
4. Total savings: $120 - 81 = 39$ dollars.
5. Single discount: $\dfrac{39}{120} = 0.325 = 32.5\%$.
6. Compare: a 35% discount would have given $0.65 \times 120 = 78$ dollars. The two discounts give less than 35% off because the second one is taken from a smaller price.

Answer: \$81, the same as a single 32.5% discount

PRACTICE

1. Easy A book that regularly costs \$16 is on sale for 25% off. What is the sale price?

   Show answer

   \$12

2. Easy The sales tax is 5%. How much tax is charged on a \$30 purchase?

   Show answer

   \$1.50

3. Easy Find a 15% tip on a \$40 restaurant bill in your head.

   Show answer

   \$6 (10% is \$4 and 5% is \$2)

4. Easy A store pays \$20 for a bird feeder and marks it up 50%. What is the selling price?

   Show answer

   \$30

5. Medium A hammer costs $24.00 and the sales tax is 7%. What is the total cost?

Show answer

$25.68

6. Medium The Miller family's dinner bill is $56. They leave a 20% tip. How much do they pay in all?

Show answer

$67.20

7. Medium A jacket that regularly costs $80 is on sale for $60. What is the percent of discount?

Show answer

25%

8. Medium A garden center buys rosebushes for $15 each and sells them for $21 each. What is the percent of markup?

Show answer

40%

9. Hard A bicycle regularly costs $180. It is on sale for 15% off, and the sales tax is 6%. What is the total cost?

Show answer

$162.18

10. Hard Grace's family has a lunch bill of $64.40 and wants to leave an 18% tip. How much is the tip? Round to the nearest cent. (You may use a calculator.)

Show answer

$11.59

11. Hard A pair of boots regularly priced at $60 is marked 30% off. At the register, an extra 20% is taken off the sale price. What is the final price?

Show answer

$33.60

12. Hard Henry thinks that 30% off followed by an extra 20% off is the same as 50% off. Explain why he is wrong.

Show answer

The extra 20% is taken from the sale price, which is smaller than the regular price, so it takes off less money than 20% of the regular price would. On a $100 item, 30% off leaves $70, and 20% of $70 is only $14, leaving $56. That is 44% off, not 50%.

13. Challenge After a 20% discount, Samuel paid $36 for a baseball glove (before tax). What was the regular price of the glove?

Show answer

$45 (he paid 80% of the regular price, and $36 \div 0.8 = 45$)

COMMON MISTAKES

- Adding successive discounts. 25% off and then 10% off is not 35% off; find the first sale price, then take the second percent of that new price.
- Stopping at the discount or the tax. If the question asks for the sale price or the total, you still must subtract the discount or add the tax.
- Finding the percent of discount or markup using the wrong base. A discount percent is based on the regular price; a markup percent is based on the store's cost.
- Rounding money too early or to the wrong place. Keep full values while you work and round to the nearest cent at the end.

> **Make it a game · The Pretend Shop.** Set up a pretend shop with eight household items and price tags. Parent is the customer and calls out a discount, tax or tip scenario. You, as clerk, compute the total in your head or on paper in 30 seconds. Use paper chips as play money. Each correct total earns a chip. Swap places when the chips run out.

8.6 Simple Interest and Saving

OBJECTIVES

- Use $I = Prt$ to find simple interest and the total amount for savings accounts and loans.
- Find a missing rate or time in a simple interest problem.
- Build a year-by-year table to find compound interest and compare it with simple interest.

WATCH

Interest (1 of 3: Calculating amount of Simple Interest earned over 4 years)

Eddie Woo · 8:52

https://www.youtube.com/watch?v=cDaz4_9Q1wY

> **WHEN TO WATCH**
> Before Example 1 (I = Prt).

Note: Eddie Woo teaches in Australia: he says 'per annum' (per year) and uses Australian dollars.

BACKUP VIDEO

Reviewing Simple and Compound Interest

Eddie Woo · 14:04

https://www.youtube.com/watch?v=1PQmoY4d6uE

Before the compound-interest table (objective 3).

KEY TERMS

interest

Money paid for the use of money. A bank pays you interest on your savings; you pay interest when you borrow.

principal

The amount of money first saved or borrowed, written P.

interest rate

The percent of the principal paid as interest each year, written r as a decimal.

simple interest

Interest paid only on the original principal, found with $I = Prt$.

compound interest

Interest paid on the principal *and* on the interest already earned, so the interest grows each year.

balance

The total amount in an account or owed on a loan: principal plus interest.

CORE IDEAS

When you put money in a savings account, the bank uses it and pays you **interest** for that use. When you borrow money, you pay interest to the lender. Either way, interest is a percent of the money, paid for each year the money is used.

Simple interest. If you save \$500 at 3% a year, each year you earn 3% of \$500, which is $0.03 \times 500 = 15$ dollars. In 4 years you earn $4 \times 15 = 60$ dollars. That is the formula $I = Prt$: interest equals **p**rincipal times **r**ate times **t**ime. Write the rate as a decimal and the time in years; 6 months is $t = 0.5$, and 18 months is $t = 1.5$.

The **balance** is the principal plus the interest: $500 + 60 = 560$ dollars. For a loan, this is how much you must pay back.

You can also work backward. If you know the interest and two of P, r and t, multiply the two you know and divide the interest by that product. For example, if \$400 earns \$100 at 5%, then each year it earns $400 \times 0.05 = 20$ dollars, so it takes $100 \div 20 = 5$ years.

Compound interest. Most savings accounts pay interest on the interest, too. Each year the interest is added to the balance, and next year's interest is figured on the new, larger balance. A table shows it clearly. For \$200 at 10% a year: year 1 earns \$20 (balance \$220); year 2 earns 10% of \$220, which is \$22 (balance \$242); year 3 earns \$24.20 (balance \$266.20). Simple interest would give only \$260. The extra comes from interest earning interest.

This is why saving early is wise. Compound interest grows a little faster every year, so money saved patiently over many years grows much more than the same money

saved for a short time. It also explains why carrying debt is costly: unpaid interest on a loan can compound against you.

KEY FORMULAS

$$I = Prt$$

$$\text{balance} = P + I$$

$$\text{compound interest, each year: new balance} = \text{old balance} + r \times \text{old balance}$$

Practice habit. Recite "$I = Prt$: interest is principal times rate times time," and before you multiply, always ask two questions: Is the rate a decimal? Is the time in years?

WORKED EXAMPLES

Example 1. Ruth deposits \$500 in a savings account that pays 3% simple interest per year. How much interest will she earn in 4 years, and what will her balance be?

Show solution

1. Write what you know: $P = 500$, $r = 0.03$, $t = 4$.
2. Use the formula: $I = Prt = 500 \times 0.03 \times 4$.
3. Multiply: $500 \times 0.03 = 15$, and $15 \times 4 = 60$. So $I = 60$ dollars.
4. Balance: $500 + 60 = 560$ dollars.
5. Check: 3% of \$500 is \$15 a year, and 4 years of \$15 is \$60.

Answer: \$60 in interest; balance \$560

Example 2. Samuel puts \$1,000 in a savings account that pays 5% interest compounded once a year. Make a year-by-year table to find his balance after 3 years. Round to the nearest cent. How much more is this than 5% simple interest would give?

Show solution

1. Year 1: interest $0.05 \times 1{,}000 = 50$; balance $1{,}000 + 50 = 1{,}050$.
2. Year 2: interest $0.05 \times 1{,}050 = 52.50$; balance $1{,}050 + 52.50 = 1{,}102.50$.
3. Year 3: interest $0.05 \times 1{,}102.50 = 55.125 \approx 55.13$; balance $1{,}102.50 + 55.13 = 1{,}157.63$.
4. Simple interest for comparison: $I = 1{,}000 \times 0.05 \times 3 = 150$, so the balance would be \$1,150.
5. Difference: $1{,}157.63 - 1{,}150 = 7.63$ dollars, earned because interest earned interest.

Answer: \$1,157.63, which is \$7.63 more than with simple interest

PRACTICE

1. Easy Find the simple interest on \$200 at 4% per year for 1 year.

   Show answer

   \$8

2. Easy Find the simple interest on \$600 at 5% per year for 3 years.

   Show answer

   \$90

3. Easy Write 6 months as a number of years, ready to use for t in $I = Prt$.

   Show answer

   $t = 0.5$ year

4. Easy Grace saves \$1,000 at 2% simple interest per year. What is her balance after 5 years?

   Show answer

   \$1,100

5. **Medium** Find the simple interest on $800 at 4.5% per year for 2 years.

Show answer

$72

6. **Medium** Henry borrows $300 to buy a used lawn mower for his mowing business. He agrees to pay 6% simple interest per year and to pay everything back in 18 months. How much will he pay back in all?

Show answer

$327 ($27 interest)

7. **Medium** Find the simple interest on $2,500 at 3% per year for 6 months.

Show answer

$37.50

8. **Medium** A deposit of $500 earned $45 in simple interest in 3 years. What was the yearly interest rate?

Show answer

3%

9. **Hard** How many years will it take $400 to earn $100 in simple interest at 5% per year?

Show answer

5 years

10. **Hard** The Miller family borrows $12,000 to buy a tractor for their farm. The loan has 7% simple interest per year for 4 years. What is the total amount they must repay?

Show answer

$15,360 ($3,360 interest)

11. **Hard** $200 is saved at 10% interest compounded once a year. Make a year-by-year table and find the balance after 3 years.

Show answer

$266.20 (balances $220, $242, $266.20)

12. Hard Explain why money saved with compound interest grows faster than money saved with simple interest at the same rate.

Show answer

With simple interest, the interest each year is figured only on the original principal, so it is the same every year. With compound interest, each year's interest is added to the balance, and the next year's interest is figured on that larger balance. The interest itself earns interest, so the amount earned grows every year.

13. Challenge Ruth's grandparents put $5,000 into a college savings account for her at 4% per year. How much more will the account hold after 2 years if the interest is compounded once a year instead of being simple interest?

Show answer

$8 more (compound: $5,200 then $5,408; simple: $5,400)

COMMON MISTAKES

- Using the percent as a whole number, such as $I = 500 \times 3 \times 4$. Write the rate as a decimal first: $3\% = 0.03$.
- Using months for time. t must be in years, so 18 months is $t = 1.5$, not $t = 18$.
- Giving the interest when the question asks for the balance or the total to repay. Add the interest to the principal.
- In a compound interest table, figuring each year's interest on the original principal. Each year's interest must be figured on the balance at the start of that year.

> **Make it a game · Interest Race.** Write I = Prt on a card. Parent deals a card with a principal, rate and time, such as $500 at 4% for 3 years. You find the interest and the total amount in 45 seconds. Then a missing-rate card for bonus points. Score one point for the interest and one for the total. Reach ten points before time runs out.

UNIT 8

Percent

Percent means "per hundred." It is one of the most useful ideas in everyday math: sale prices, sales tax, tips, test scores, harvests that grow from one year to the next, and the interest on savings and loans are all described with percents. In this unit you will learn what a percent really is, how to move easily among percents, fractions and decimals, and how to find a percent of a number in your head or on paper.

Then you will put percents to work. You will solve any percent problem by finding the part, the percent or the whole, measure how much an amount has grown or shrunk, figure store and restaurant bills, and see how money grows in a savings account with simple and compound interest.

This unit builds directly on fractions and decimals (Units 3 and 4) and on ratios and proportions (Unit 7), since a percent is simply a ratio out of 100. The habit of setting up a problem carefully, estimating first and checking at the end will serve you well in Algebra, where the percent equation becomes a true equation with a variable.

PREREQUISITES

- Multiply and divide decimals, including by powers of 10 (Lessons 4.3–4.4).
- Convert between fractions and decimals (Lesson 4.5).
- Simplify fractions and multiply a fraction by a whole number (Lessons 3.2 and 3.5).
- Write ratios and find unit rates (Lessons 7.1–7.2).
- Solve a proportion with cross products (Lesson 7.3).

MASTERY CHECKLIST

- ☐ I can explain that a percent is a ratio that compares a number to 100.
- ☐ I can convert among percents, fractions and decimals, including percents greater than 100% and less than 1%.
- ☐ I can find a percent of a number using the decimal method or the fraction method.
- ☐ I can find 10%, 5%, 1%, 25% and 50% of a number in my head and combine them.
- ☐ I can find a missing part, percent or whole using the percent equation or a percent proportion.
- ☐ I can find the percent of increase or decrease and explain why the original amount is the base.
- ☐ I can find a new amount after a percent increase or decrease.
- ☐ I can find sale prices, sales tax, tips, markups and total costs.
- ☐ I can work with successive discounts and explain why they do not simply add.
- ☐ I can use $I = Prt$ to find simple interest and the balance of a savings account or loan.
- ☐ I can make a year-by-year table to find compound interest and compare it with simple interest.

LESSONS IN THIS UNIT

8.1	**Understanding Percent**	45 minutes
8.2	**Finding a Percent of a Number**	2 sessions of 45 minutes
8.3	**Percent Equations and Proportions**	2 sessions of 45 minutes
8.4	**Percent Increase and Decrease**	2 sessions of 45 minutes
8.5	**Discounts, Sales Tax, Tips and Markup**	2 sessions of 45 minutes
8.6	**Simple Interest and Saving**	2 sessions of 45 minutes

Unit 8 test

You have 50 minutes. You may use a basic calculator, but do problems 1–4 by hand or in your head. Show your work for every problem. Round money to the nearest cent and percents to the nearest tenth of a percent unless the problem says otherwise.

1. **5 pts**
   Write 0.6% as a decimal.

2. **5 pts**
   Write $\frac{7}{20}$ as a percent.

3. **6 pts**
   Find 15% of 260 in your head. Show how you used 10% and 5%.

4. **5 pts**
   Find 125% of 48.

5. **7 pts**
   21 is what percent of 60?

6. **7 pts**
   24 is 40% of what number?

7. **7 pts**

   At the church bake sale, Grace's family baked 54 pies. That was 45% of all the pies sold at the sale. How many pies were sold at the bake sale in all?

8. **7 pts**

   A number increases from 40 to 52. What is the percent of increase?

9. **8 pts**

   The price of a bag of chicken feed dropped from $25 to $22. What was the percent of decrease?

10. **8 pts**

    A sweater regularly costs $64. It is on sale for 25% off, and the sales tax is 6%. What is the total cost?

11. **8 pts**

    Henry's family has a restaurant bill of $38 and leaves a 15% tip. How much do they pay in all?

12. **8 pts**

    A camping tent regularly costs $150. It is marked 20% off, and then an extra 10% is taken off the sale price. What is the final price?

13. **7 pts**

    Samuel deposits $900 in a savings account that pays 4% simple interest per year. How much interest will he earn in 3 years?

14. **7 pts**

    Ruth saves $400 in an account that pays 5% interest compounded once a year. Make a year-by-year table and find her balance after 2 years.

15. **5 pts**

    A $100 price is raised by 10%, and then the new price is lowered by 10%. Is the final price $100? Explain why or why not.

ANSWER KEY

1. 0.006
2. 35%
3. 39 (10% is 26 and 5% is 13)
4. 60
5. 35%
6. 60
7. 120 pies
8. 30%
9. 12%
10. $50.88
11. $43.70
12. $108
13. $108
14. $441 (balances $420, then $441)
15. No. The 10% increase is taken from $100, giving $110. The 10% decrease is taken from the larger amount $110, which takes off $11, so the final price is $99. The two percents are taken from different bases.

UNIT 9

Variables and Expressions

In this unit you take your first real step into algebra. You will learn to use letters, called variables, to stand for numbers, and to read and write algebraic expressions such as $3x + 5$. You will name the parts of an expression, find its value for given numbers, and translate English phrases and number patterns into the language of math.

You will also learn to simplify expressions by combining like terms and by using the distributive property, the same property you used for mental math in Unit 1. Along the way you will see *why* these rules work, using area models, groups of objects and the sign rules from Unit 5.

These skills are the tools of algebra. In Unit 10 you will use them to write and solve equations, so careful habits now (parentheses when substituting, carrying each sign with its term, checking by substitution) will pay off for the rest of the course and in Algebra I.

PREREQUISITES

- Use the order of operations, including exponents (Lesson 1.4).
- Use the distributive property for mental math (Lesson 1.3).
- Find the greatest common factor of two numbers (Lesson 2.4).
- Add, subtract, multiply and divide integers and rational numbers (Unit 5).
- Evaluate powers, including negative bases such as $(-3)^2$ and -3^2 (Lesson 6.1).

MASTERY CHECKLIST

- ☐ I can explain what a variable is and why we use letters to stand for numbers.
- ☐ I can name the terms, coefficients and constant of an expression, including negative coefficients and a coefficient of 1.
- ☐ I can evaluate an expression by substituting numbers, including negative numbers and exponents.
- ☐ I can use formulas for perimeter, distance and temperature conversion.
- ☐ I can translate word phrases into expressions, including "less than" and "times the sum of."
- ☐ I can write an expression for a number pattern and check it.
- ☐ I can identify like terms and combine them to simplify an expression.
- ☐ I can explain why $3x + 2x = 5x$ using the distributive property.
- ☐ I can expand expressions such as $-2(x + 3)$ using the distributive property and the sign rules.
- ☐ I can simplify expressions such as $3(x + 4) - 2(x - 1)$.
- ☐ I can factor the greatest common factor out of an expression such as $12n - 20$.

LESSONS IN THIS UNIT

9.1	**Variables and Expressions**	45 minutes
9.2	**Evaluating Expressions**	2 sessions of 45 minutes
9.3	**Writing Expressions**	2 sessions of 45 minutes
9.4	**Combining Like Terms**	2 sessions of 45 minutes
9.5	**The Distributive Property with Variables**	2 sessions of 45 minutes

Unit 9 test

You have 50 minutes. A basic calculator is allowed but not needed for most problems. Show your work for every problem, including each substitution written in parentheses. Write expressions in simplest form. Include units in answers to word problems.

1. **6 pts**
   In the expression $5y^2 - y + 8$, what is the coefficient of y, and what is the constant?

2. **6 pts**
   A fitness club charges a one-time fee of \$25 to join plus \$30 per month. The total cost in dollars for m months is $30m + 25$. What do the coefficient 30 and the constant 25 mean?

3. **6 pts**
   Evaluate $4n - 7$ when $n = 9$.

4. **7 pts**
   Evaluate $x^2 - 3x$ when $x = -5$.

5. **7 pts**
   On a summer day in Rome the temperature is $30°\text{C}$. Use $F = \frac{9}{5}C + 32$ to find the temperature in degrees Fahrenheit.

6. **6 pts**

   Write an expression for "9 less than the product of 5 and a number n."

7. **7 pts**

   Henry has \$40 in his savings jar and adds \$6 every week. Write an expression for the amount in dollars in the jar after w weeks.

8. **7 pts**

   The sequence 3, 8, 13, 18, ... continues the same pattern. Write an expression for the nth number in the sequence.

9. **6 pts**

   Simplify $9a - 4 + 2a + 10$.

10. **8 pts**

    Simplify $2x^2 + 5x - 7x^2 + x - 3$.

11. **6 pts**

    Expand $-3(4k - 2)$.

12. **6 pts**

    Factor out the greatest common factor: $15x + 35$.

13. **8 pts**

    Simplify $4(2y + 3) - 5(y - 2)$.

14. **7 pts**

    A triangular garden has sides that measure $2x + 1$ feet, $3(x - 1)$ feet and $x + 4$ feet. Write a simplified expression for its perimeter.

15. **7 pts**

    Ruth simplified $4x + 3$ and got $7x$. Explain why she is wrong.

ANSWER KEY

1. Coefficient of y: -1; constant: 8
2. 30 is the cost per month ($30 each month); 25 is the one-time $25 joining fee, which does not change with the number of months.
3. 29
4. 40 ($(-5)^2 - 3(-5) = 25 + 15$)
5. $86°\text{F}$ ($54 + 32$)
6. $5n - 9$
7. $40 + 6w$ (or $6w + 40$) dollars
8. $5n - 2$
9. $11a + 6$
10. $-5x^2 + 6x - 3$
11. $-12k + 6$
12. $5(3x + 7)$
13. $3y + 22$ ($8y + 12 - 5y + 10$)
14. $6x + 2$ feet ($2x + 1 + 3x - 3 + x + 4$)
15. $4x$ and 3 are not like terms: one has the variable x and the other is a constant, so they cannot be added. $4x + 3$ is already in simplest form. (Check: if $x = 2$, $4(2) + 3 = 11$, but $7(2) = 14$.)

9.1 Variables and Expressions

OBJECTIVES

- Explain what a variable is and why we use letters to stand for numbers.
- Identify the terms, coefficients and constants in an expression.
- Describe what each part of an expression means in a real-life situation.

WATCH

Algebra Basics: What Is Algebra? - Math Antics

mathantics · 12:07

https://www.youtube.com/watch?v=NybHckSEQBI

> **WHEN TO WATCH**
> Before the Core ideas.

BACKUP VIDEO

Algebra Basics: What Are Polynomials? - Math Antics

mathantics · 11:09

https://www.youtube.com/watch?v=ffLLmV4mZwU

Before Example 1: terms, coefficients and constants. Math Antics calls an expression a 'polynomial'.

KEY TERMS

variable

A letter, such as x or n, that stands for a number. The number may change or may not be known yet.

algebraic expression

A math phrase made of numbers, variables and operation signs, such as $3x + 5$. An expression has no equals sign.

term

One piece of an expression that is added or subtracted. In $3x + 5$, the terms are $3x$ and 5.

coefficient

The number multiplied by a variable in a term. In $3x$, the coefficient is 3.

constant

A term with no variable. Its value never changes. In $3x + 5$, the constant is 5.

CORE IDEAS

A **variable** is a letter that stands for a number. Why use letters? Suppose apples cost \$2 per pound. For 3 pounds you pay 2×3; for 5 pounds you pay 2×5; for 10 pounds you pay 2×10. The pattern is always the same: 2 times the number of pounds. If we let p stand for the number of pounds, then $2p$ describes *every* case at once. One short expression does the work of a whole table.

When a number and a letter sit side by side, they are multiplied: $2p$ means $2 \times p$, and $5xy$ means $5 \times x \times y$. We avoid the $\times$ sign with letters because it looks too much like the letter x. We also write the number first: $2p$, not $p2$.

An **algebraic expression** is a math phrase such as $4x^2 - 7x + 9$. It has no equals sign, so it is not a statement to solve; it is a name for a number. The plus and minus signs split the expression into **terms**. Here the terms are $4x^2$, $-7x$ and 9.

Each sign belongs to the term that follows it. Remember from Unit 5 that subtracting is the same as adding the opposite: $4x^2 - 7x + 9 = 4x^2 + (-7x) + 9$. So the second term is $-7x$, and its **coefficient** is -7, not 7.

The **coefficient** is the number multiplying the variable part. When you see just x, the coefficient is 1, because $x = 1x$. When you see $-x$, the coefficient is -1. When you see $\frac{x}{4}$, the coefficient is $\frac{1}{4}$, because dividing by 4 is the same as multiplying by $\frac{1}{4}$. Do not confuse a coefficient with an exponent: in x^2 the coefficient is 1 and the exponent is 2.

A **constant** is a term with no variable, like the 9 in $4x^2 - 7x + 9$. In a real situation the constant is often a fixed amount, such as a one-time fee, while the coefficient is a rate, such as a price per item.

KEY FORMULAS

$$\underbrace{4}_{\text{coefficient}}\,x^2 - 7x + \underbrace{9}_{\text{constant}}$$

$$x = 1x \qquad -x = -1x \qquad \frac{x}{4} = \frac{1}{4}x$$

From history. The word *algebra* comes from a book written in Baghdad around 820 by the scholar Muhammad ibn Musa al-Khwarizmi. Its title includes the Arabic word *al-jabr*, meaning "restoring" or "completing," which described one of his steps for solving problems. Al-Khwarizmi wrote everything out in words, with no letters for unknowns; the letter symbols we use today came centuries later. His name also gives us the word *algorithm*.

Practice habit. Whenever you see an expression, point to each term and say its sign aloud with it: "four x squared, minus seven x, plus nine." Then name the coefficient of each term, remembering that a lone x has coefficient 1.

WORKED EXAMPLES

Example 1. In the expression $4x^2 - 7x + 9$, name the terms, the coefficient of each variable term, and the constant.

Show solution

1. Rewrite subtraction as adding the opposite: $4x^2 + (-7x) + 9$.
2. The plus signs split it into three terms: $4x^2$, $-7x$ and 9.
3. The coefficient of $4x^2$ is 4.
4. The coefficient of $-7x$ is -7 (the sign goes with the term).
5. The term 9 has no variable, so it is the constant.

Answer: Terms: $4x^2$, $-7x$, 9. Coefficients: 4 and -7. Constant: 9.

Example 2. Grace is selling muffins at the church bake sale for $2 each. Before the sale began, the bake sale table already had $15 in donations. The expression $2m + 15$ gives the total money, in dollars, at the table. Tell what the variable, the coefficient and the constant mean. Then find the total if Grace sells 30 muffins.

Show solution

1. The variable m stands for the number of muffins sold. It can change during the day.
2. The coefficient 2 is the price of one muffin: each muffin adds $2.
3. The constant 15 is the $15 in donations. It stays the same no matter how many muffins are sold.
4. For 30 muffins, put 30 in place of m: $2 \times 30 + 15$.
5. Multiply first: $2 \times 30 = 60$. Then add: $60 + 15 = 75$.
6. Check: 30 muffins at $2 is $60, and $60 plus $15 in donations is $75.

Answer: m is the number of muffins sold, 2 is the price per muffin in dollars, and 15 is the $15 in donations. For 30 muffins the total is $75.

PRACTICE

1. Easy How many terms are in the expression $5a + 3b - 8$?

   Show answer

   3 terms: $5a$, $3b$ and -8

2. Easy What is the coefficient of y in $9y + 4$?

   Show answer

   9

3. Easy What is the constant term in $6x - 11$?

   Show answer

   -11 (subtracting 11 is adding -11)

4. Easy What is the coefficient of x in $x + 7$?

   Show answer

   1, because $x = 1x$

5. **Medium** What is the coefficient of n in $3n^2 - n + 10$?

   Show answer

   -1, because $-n = -1n$

6. **Medium** What is the coefficient of k in $\frac{k}{4}$? Write it as a fraction.

   Show answer

   $\frac{1}{4}$, because $\frac{k}{4} = \frac{1}{4}k$

7. **Medium** True or false: In the expression $7 - 2x$, the coefficient of x is 2.

   Show answer

   False. The term is $-2x$, so the coefficient is -2.

8. **Medium** A bakery sells loaves of bread for \$4 each. Ruth buys b loaves and one \$3 jar of jam, so her total cost in dollars is $4b + 3$. What does the coefficient 4 mean? What does the constant 3 mean?

   Show answer

   The 4 is the price of each loaf (\$4 per loaf). The 3 is the fixed \$3 cost of the jam, which does not depend on how many loaves she buys.

9. **Hard** Samuel earns \$15 per hour doing yard work, plus a \$20 bonus for finishing the season. His total pay in dollars is $15h + 20$, where h is the number of hours. What is his total pay if he works 6 hours?

   Show answer

   \$110 ($15 \times 6 + 20 = 90 + 20$)

10. **Hard** A farm has c cows and h hens. The expression $4c + 2h$ gives the total number of legs. Explain what the coefficients 4 and 2 mean, then find the number of legs when there are 10 cows and 25 hens.

    Show answer

    Each cow has 4 legs and each hen has 2 legs. $4 \times 10 + 2 \times 25 = 40 + 50 = 90$ legs.

11. Hard Explain why mathematicians use a letter such as n instead of writing out many separate number examples.

Show answer

A letter can stand for any number, so one expression like $2n$ describes the same pattern for every case at once (2 times 1, 2 times 5, 2 times 100, and so on). It is shorter, it shows the rule clearly, and it lets us work with a number we do not know yet.

12. Challenge In the expression $\frac{3x}{5} - 2y + 7$, add together the coefficient of x, the coefficient of y and the constant. Give your answer as a decimal.

Show answer

$$\frac{3}{5} + (-2) + 7 = 0.6 - 2 + 7 = 5.6$$

COMMON MISTAKES

- Dropping the sign of a term. In $7 - 2x$ the term is $-2x$, so the coefficient is -2. Rewrite subtraction as adding the opposite if you are unsure.
- Saying the coefficient of x is 0 when no number is written. A lone x means $1x$, so its coefficient is 1 (and $-x$ has coefficient -1).
- Confusing the coefficient with the exponent. In x^2 the 2 is an exponent, not a coefficient; the coefficient is 1.
- Reading $3x$ as a two-digit number such as 30-something. A number next to a letter means multiplication: $3x = 3 \times x$.

> **Make it a game · Expression Anatomy.** Write ten expressions such as 4x + 7y - 3 on cards. Draw one and, in 20 seconds, point out each term, circle each coefficient and underline the constant, naming the variable too. Parent checks. Then you invent an expression and hide it behind your back for the parent to dissect. Score a point per correct label.

9.2 Evaluating Expressions

OBJECTIVES

- Evaluate expressions by substituting values, including negative numbers and exponents.
- Use the order of operations correctly after substituting.
- Use formulas for perimeter, distance and temperature to solve word problems.

WATCH

Algebraic Substitution (1 of 2: What does it mean?)

Eddie Woo · 6:16

https://www.youtube.com/watch?v=JMES1l4oAwk

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'pronumeral' for variable.

BACKUP VIDEO

Algebraic Substitution (2 of 2: Worked examples)

Eddie Woo · 10:03

https://www.youtube.com/watch?v=3pMrTCqel3I

Before the practice set, including substituting negative numbers.

KEY TERMS

evaluate

To find the value of an expression by putting in numbers for the variables and doing the arithmetic.

substitute

To replace a variable with a number.

formula

A rule, written with variables, that shows how quantities are related, such as $d = rt$.

value of an expression

The single number you get after you evaluate it.

CORE IDEAS

An expression such as $5n + 3$ is like a machine waiting for a number. To **evaluate** it, you **substitute** a number for the variable and then do the arithmetic. If $n = 7$, then $5n + 3 = 5(7) + 3 = 35 + 3 = 38$.

Always put the substituted number in parentheses. This keeps the multiplication visible and protects negative signs. If $y = -2$, then $12 - 3y$ becomes $12 - 3(-2) = 12 - (-6) = 12 + 6 = 18$. Without parentheses you might write $12 - 3 - 2$, which is a different problem.

After substituting, follow the order of operations from Lesson 1.4: grouping symbols first, then exponents, then multiplication and division from left to right, then addition and subtraction from left to right. Exponents come before multiplication, so in $3x^2$ you square x first, then multiply by 3.

Be careful with negatives and exponents. Recall from Lesson 6.1 that $(-5)^2 = 25$, but $-5^2 = -25$. So if $x = -5$, then $x^2 = (-5)^2 = 25$, while $-x^2$ means "the opposite of x^2," which is $-(25) = -25$. Parentheses around the substituted number make this clear.

A **formula** is an expression with a job. The perimeter of a rectangle is $P = 2l + 2w$. Distance traveled at a steady speed is $d = rt$ (rate times time). To change Celsius to Fahrenheit, use $F = \frac{9}{5}C + 32$; to change Fahrenheit to Celsius, use $C = \frac{5}{9}(F - 32)$. Why do these work? Water freezes at $0°$C and $32°$F and boils at $100°$C and $212°$F. That is 100 Celsius degrees for every 180 Fahrenheit degrees, and $\frac{180}{100} = \frac{9}{5}$. The "+ 32" lines up the two freezing points.

When you use a formula, check that the units match (miles per hour with hours, feet with feet) and write the unit in your answer.

KEY FORMULAS

$$P = 2l + 2w \quad \text{(perimeter of a rectangle)}$$

$$d = rt \quad \text{(distance = rate} \times \text{time)}$$

$$F = \frac{9}{5}C + 32 \qquad C = \frac{5}{9}(F - 32)$$

$$(-a)^2 = a^2 \qquad -a^2 = -(a^2)$$

Practice habit. Make it automatic: every time you substitute, write the number inside parentheses first, then simplify. Also memorize the temperature facts: water freezes at $0°\text{C} = 32°\text{F}$ and boils at $100°\text{C} = 212°\text{F}$, and use them to check whether a converted temperature is reasonable.

WORKED EXAMPLES

Example 1. Evaluate $3x^2 - 2x + 5$ when $x = -4$.

Show solution

1. Substitute with parentheses: $3(-4)^2 - 2(-4) + 5$.
2. Exponent first: $(-4)^2 = 16$, so we have $3(16) - 2(-4) + 5$.
3. Multiply: $3(16) = 48$ and $2(-4) = -8$, so we have $48 - (-8) + 5$.
4. Subtracting -8 is adding 8: $48 + 8 + 5$.
5. Add: $48 + 8 + 5 = 61$.
6. Check the signs: x^2 is positive, and $-2x$ is positive because x is negative, so every piece adds. A positive answer makes sense.

Answer: 61

Example 2. On a cold winter morning the Miller family's thermometer reads $-10°\text{C}$. Use the formula $F = \frac{9}{5}C + 32$ to find the temperature in degrees Fahrenheit.

Show solution

1. Substitute $C = -10$ with parentheses: $F = \frac{9}{5}(-10) + 32$.
2. Multiply: $\frac{9}{5} \times (-10) = \frac{-90}{5} = -18$.
3. Add: $-18 + 32 = 14$.
4. Check with the reverse formula: $C = \frac{5}{9}(14 - 32) = \frac{5}{9}(-18) = -10$. It matches.
5. Reasonableness: $-10°\text{C}$ is below freezing, and $14°\text{F}$ is below $32°\text{F}$, so the answer makes sense.

Answer: $14°\text{F}$

PRACTICE

1. Easy Evaluate $5n + 3$ when $n = 7$.

   Show answer

   38

2. Easy Evaluate $a^2 + 4$ when $a = 6$.

   Show answer

   40

3. Easy Evaluate $12 - 3y$ when $y = -2$.

   Show answer

   18 $(12 - 3(-2) = 12 + 6)$

4. Easy Evaluate $\frac{m}{4} + 9$ when $m = 20$.

   Show answer

   14

5. Medium Evaluate $2x^2$ when $x = -3$.

   Show answer

   18 $(2(-3)^2 = 2(9))$

6. Medium Evaluate $-x^2$ when $x = -5$.

   Show answer

   -25 $(-(-5)^2 = -(25))$

7. Medium Evaluate $3a - 2b$ when $a = 4$ and $b = -6$.

   Show answer

   24 $(3(4) - 2(-6) = 12 + 12)$

8. Medium Ruth's vegetable garden is a rectangle 18 feet long and 7.5 feet wide. Use $P = 2l + 2w$ to find how many feet of fencing she needs to go all the way around it.

   Show answer

   51 feet $(2(18) + 2(7.5) = 36 + 15)$

9. Hard The Miller family drives at an average speed of 62 miles per hour for 3.5 hours. Use $d = rt$ to find how far they travel.

   Show answer

   217 miles (62×3.5)

10. Hard A summer afternoon is $77°$F. Use $C = \frac{5}{9}(F - 32)$ to find the temperature in degrees Celsius.

    Show answer

    $25°$C ($\frac{5}{9}(45) = 25$)

11. Hard Evaluate $\frac{x^2 - y}{x + y}$ when $x = -3$ and $y = 5$.

    Show answer

    2 (top: $9 - 5 = 4$; bottom: $-3 + 5 = 2$; $4 \div 2 = 2$)

12. Hard Henry was asked to evaluate x^2 when $x = -4$. He wrote $-4^2 = -16$. Explain his mistake and give the correct value.

    Show answer

    Henry left out the parentheses. -4^2 means the opposite of 4^2, which is -16. He should have written $(-4)^2 = (-4)(-4) = 16$. The correct value is 16.

13. Challenge Evaluate $4t^3 - t$ when $t = -\frac{1}{2}$.

    Show answer

    0 ($4\left(-\frac{1}{8}\right) - \left(-\frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{2}$)

COMMON MISTAKES

- Substituting a negative number without parentheses, so $12 - 3y$ with $y = -2$ turns into $12 - 3 - 2$. Always write $12 - 3(-2)$ first.
- Treating $-x^2$ like $(-x)^2$. The exponent applies only to x; square first, then take the opposite.
- Multiplying before squaring. In $3x^2$ with $x = 4$, square first (16), then multiply (48), not $(12)^2 = 144$.
- Writing $3x$ with $x = 4$ as 34. A number next to a variable means multiplication: $3(4) = 12$.

Make it a game · Plug It In Relay. Write expressions such as 3x squared - 5 on cards. Parent calls a value for x, including negatives. You substitute and evaluate on paper, writing each order-of-operations step. Score a point for the answer and a point for correct steps. Run through eight cards, then try to beat your previous total by hitting a new best.

9.3 Writing Expressions

OBJECTIVES

- Translate word phrases into algebraic expressions using key words.
- Write expressions for real-life situations, such as costs and ages.
- Find an expression that describes a number pattern and check it.

WATCH

Translating Words to Algebra (2 of 2: Converting words into mathematical terms for calculations)

Eddie Woo · 8:25

https://www.youtube.com/watch?v=vv8_VyqOh6M

> **WHEN TO WATCH**
> Before Example 1 (translating words to algebra).

BACKUP VIDEO

Patterns & Rules (1 of 2: Building geometric shapes)

Eddie Woo · 12:55

https://www.youtube.com/watch?v=0HgX-KiTWvE

Before the number-pattern example (objective 3): toothpick-style patterns and rules.

KEY TERMS

sum

The answer to an addition. "The sum of a and b" is $a + b$.

difference

The answer to a subtraction. "The difference of a and b" is $a - b$.

product

The answer to a multiplication. "The product of a and b" is ab.

quotient

The answer to a division. "The quotient of a and b" is $\frac{a}{b}$.

sequence

A list of numbers that follows a pattern, such as 5, 9, 13, 17, ...

CORE IDEAS

Writing an expression is like translating a sentence from English into the language of math. First decide what the variable stands for. Then look for **key words** that tell you which operation to use.

Addition words: *sum, plus, more than, increased by, added to, total*. Subtraction words: *difference, minus, less than, decreased by, fewer than*. Multiplication words: *product, times, twice (times 2), of, multiplied by*. Division words: *quotient, divided by, per, split equally*.

Watch the order with "less than" and "more than." "7 less than a number n" means you start with n and take 7 away, so it is $n - 7$, not $7 - n$. Why? If you have 10 apples, "7 less than that" is $10 - 7 = 3$. The number that comes first in the sentence comes *second* in the expression. "Subtract 7 from n" works the same way: $n - 7$.

Use parentheses when a whole amount is acted on. "4 times the sum of n and 3" means find the sum first, then multiply, so it is $4(n + 3)$. Compare "4 times n, plus 3," which is $4n + 3$. These give different values: when $n = 1$, $4(1 + 3) = 16$ but $4(1) + 3 = 7$.

Expressions can also describe **patterns**. In the sequence 5, 9, 13, 17, ..., each number is 4 more than the one before, so the nth number has something to do with $4n$. Try $n = 1$: $4(1) = 4$, but we need 5, so add 1. The expression $4n + 1$ gives 5, 9, 13, 17 for $n = 1, 2, 3, 4$. The amount you add each time becomes the coefficient.

Always **check** an expression by trying a small number. Put in an easy value and see whether the expression matches what the words say.

KEY FORMULAS

$$\text{"}b \text{ less than } a\text{"} = a - b \qquad \text{"}b \text{ more than } a\text{"} = a + b$$

$$\text{"}k \text{ times the sum of } a \text{ and } b\text{"} = k(a + b)$$

Practice habit. Recite the four groups of key words until they are automatic: sum, plus, more than, increased by; difference, minus, less than, decreased by; product, times, twice, of; quotient, divided by, per. After writing any expression, test it with an easy number such as 10.

WORKED EXAMPLES

Example 1. Write an expression for "7 less than three times a number n."

Show solution

1. "Three times a number n" is $3n$.
2. "7 less than" something means start with that something and subtract 7.
3. So the expression is $3n - 7$.
4. Check with $n = 10$: three times 10 is 30, and 7 less than 30 is 23. The expression gives $3(10) - 7 = 23$. It matches.

Answer: $3n - 7$

Example 2. Samuel builds a row of squares out of toothpicks, with neighboring squares sharing a side. One square uses 4 toothpicks, two squares use 7 toothpicks, and three squares use 10 toothpicks. Write an expression for the number of toothpicks in a row of n squares.

Show solution

1. List the pattern: 4, 7, 10, ... Each new square adds 3 toothpicks (it shares one side with the square before it).
2. Because 3 are added each time, try $3n$. For $n = 1$, $3n = 3$, but we need 4.
3. Add 1 to fix it: $3n + 1$.
4. Check $n = 1$: $3(1) + 1 = 4$. Check $n = 2$: $3(2) + 1 = 7$. Check $n = 3$: $3(3) + 1 = 10$. All match.
5. Meaning: 1 toothpick starts the row on the left, and each square adds 3 more.

Answer: $3n + 1$ toothpicks

PRACTICE

1. Easy Write an expression for "the sum of a number x and 9."

   Show answer

   $x + 9$

2. Easy Write an expression for "the product of 6 and a number y."

Show answer

$6y$

3. Easy Write an expression for "a number n decreased by 5."

Show answer

$n - 5$

4. Easy Write an expression for "the quotient of a number k and 8."

Show answer

$\frac{k}{8}$

5. Medium Write an expression for "12 less than a number m."

Show answer

$m - 12$

6. Medium Write an expression for "twice a number x, increased by 11."

Show answer

$2x + 11$

7. Medium Write an expression for "4 times the sum of a number n and 3."

Show answer

$4(n + 3)$

8. Medium Grace buys p packs of pencils at \$3 per pack and one notebook for \$8. Write an expression for her total cost in dollars.

Show answer

$3p + 8$

9. Hard Henry is a years old. His father's age is 3 more than 4 times Henry's age. Write an expression for his father's age.

Show answer

$4a + 3$

10. Hard A family rents a moving van for \$45 per day plus \$0.20 per mile. Write an expression for the total cost in dollars for d days and m miles.

Show answer

$45d + 0.20m$

11. Hard Ruth wrote $7 - x$ for "7 less than a number x." Explain why this is wrong, using a number to show it.

Show answer

"7 less than x" means start with x and take away 7, so it is $x - 7$. For example, if $x = 10$, then 7 less than 10 is 3, and $10 - 7 = 3$, but Ruth's expression gives $7 - 10 = -3$.

12. Hard The sequence 5, 9, 13, 17, ... continues the same pattern. Write an expression for the nth number in the sequence.

Show answer

$4n + 1$

13. Challenge For a church potluck, tables are pushed together end to end in one long row. One table seats 6 people, two tables seat 10, three tables seat 14, and the pattern continues. Write an expression for the number of people who can sit at t tables, and tell what each number in your expression means.

Show answer

$4t + 2$: each table seats 4 people along its two long sides, and 2 more people sit at the two ends of the row.

COMMON MISTAKES

- Reversing the order with "less than." "5 less than n" is $n - 5$. Test with $n = 10$: 5 less than 10 is 5, and $10 - 5 = 5$.
- Leaving out parentheses. "3 times the sum of x and 2" is $3(x + 2)$, not $3x + 2$. If the words say "the sum" or "the difference" after "times," use parentheses.
- Mixing up "twice" and "squared." Twice a number is $2x$; a number squared is x^2.
- Writing a pattern expression without checking it. Always test your expression with $n = 1$, 2 and 3 against the actual pattern.

> **Make it a game · Translate It.** Parent reads a phrase, such as 'three less than twice a number' or 'the cost of n tickets at $8 each'. You write the algebraic expression in 15 seconds. Then you give the parent a real-life situation with your own invented phrase. Score a point per correct translation. Watch out for the less-than reversal. Play to twelve.

9.4 Combining Like Terms

OBJECTIVES

- Identify like terms and unlike terms in an expression.
- Explain, using the distributive property, why like terms can be combined by adding their coefficients.
- Simplify expressions by combining like terms, including terms with negative and fractional coefficients.

WATCH

Algebra Basics: Simplifying Polynomials - Math Antics

mathantics · 10:43

https://www.youtube.com/watch?v=DKC74YKJpNY

> **WHEN TO WATCH**
> Before Example 1. Math Antics says 'polynomial'; read it as 'expression'.

Note: Eddie Woo teaches in Australia: he says 'pronumeral' for variable.

BACKUP VIDEO

Introduction to Algebra (2 of 2: Outlining Abbreviations in Algebra, Simplifying algebraic terms)

Eddie Woo · 5:46

https://www.youtube.com/watch?v=U1yNskrHFv4

Before the practice set: simplifying algebraic terms.

KEY TERMS

like terms

Terms that have exactly the same variable part, with the same exponents. $3x$ and $-5x$ are like terms; so are 4 and 9.

unlike terms

Terms whose variable parts are different, such as $3x$ and $3y$, or x and x^2.

simplify

To rewrite an expression in a shorter, equal form, with no like terms left to combine.

simplest form (of an expression)

An expression with all like terms combined, usually written with the highest exponent first and the constant last.

CORE IDEAS

Like terms have exactly the same variable part. $4x$ and $9x$ are like terms. $2y^2$ and $-7y^2$ are like terms. Constants such as 3 and -8 are like terms with each other. But $5x$ and $5x^2$ are *not* like terms, and neither are $3a$ and $3b$. Only the coefficients may differ.

Why can we add like terms? Think of x as a box of pencils. Three boxes plus two boxes is five boxes: $3x + 2x = 5x$. The distributive property from Lesson 1.3 says the same thing exactly: $3x + 2x = (3 + 2)x = 5x$. We "pull out" the common factor x and add the coefficients.

The same reasoning shows why unlike terms cannot be combined. $3x + 2y$ has no common variable to pull out, just as 3 boxes of pencils plus 2 bags of marbles is not 5 of anything. And x and x^2 are different too: if $x = 4$, then $x = 4$ but $x^2 = 16$.

To combine like terms, add (or subtract) the coefficients and keep the variable part the same. The variable part does not change: $3x + 2x$ is $5x$, never $5x^2$. Subtracting works the same way: $8y - 3y = (8 - 3)y = 5y$, and $2a - 6a = (2 - 6)a = -4a$.

To simplify a longer expression: (1) rewrite subtraction as adding the opposite so each term keeps its sign; (2) group the like terms together (the commutative and associative properties let you move terms around); (3) add the coefficients in each group. For example, $5x + 3 - 2x + 8 = 5x + (-2x) + 3 + 8 = 3x + 11$.

You can check a simplification by substitution: pick a number, put it into the original and the simplified expression, and make sure both give the same value.

KEY FORMULAS

$$ax + bx = (a + b)x$$

$$ax - bx = (a - b)x$$

Practice habit. Before combining, underline or circle each group of like terms, and carry each term's sign with it. When you finish, check by substituting an easy number such as $x = 2$ into both the original and the simplified expression.

WORKED EXAMPLES

Example 1. Simplify $5x + 3 - 2x + 8$.

Show solution

1. Rewrite subtraction as adding the opposite: $5x + 3 + (-2x) + 8$.
2. Group like terms: $5x + (-2x) + 3 + 8$.
3. Combine the x terms: $(5 + (-2))x = 3x$.
4. Combine the constants: $3 + 8 = 11$.
5. Result: $3x + 11$.
6. Check with $x = 2$: original $5(2) + 3 - 2(2) + 8 = 10 + 3 - 4 + 8 = 17$; simplified $3(2) + 11 = 17$. They match.

Answer: $3x + 11$

Example 2. A carpenter builds a four-sided garden bed. Its sides measure $3x + 2$, $2x - 5$, $4x + 1$ and x feet. Write a simplified expression for the perimeter.

Show solution

1. Perimeter means add all the sides: $(3x + 2) + (2x - 5) + (4x + 1) + x$.
2. Group the x terms: $3x + 2x + 4x + x$. Remember $x = 1x$.
3. Add the coefficients: $3 + 2 + 4 + 1 = 10$, giving $10x$.
4. Group the constants: $2 + (-5) + 1 = -2$.
5. Result: $10x - 2$ feet.
6. Check with $x = 3$: the sides are 11, 1, 13 and 3, which add to 28; and $10(3) - 2 = 28$. They match.

Answer: $10x - 2$ feet

PRACTICE

1. **Easy** Simplify $4x + 9x$.

   Show answer

   $13x$

2. **Easy** Simplify $8y - 3y$.

   Show answer

   $5y$

3. **Easy** Are $3x^2$ and $3x$ like terms? Answer yes or no.

   Show answer

   No. The variable parts x^2 and x are different.

4. **Easy** Simplify $7a + 2 + 5a$.

   Show answer

   $12a + 2$

5. **Medium** Simplify $6m - 4 - 2m + 9$.

   Show answer

   $4m + 5$

6. **Medium** Simplify $x + x + x - 5x$.

   Show answer

   $-2x$

7. **Medium** Simplify $3a + 4b - a + 2b$.

   Show answer

   $2a + 6b$

8. **Medium** Henry rides his bike m miles on Monday, twice as far on Tuesday, and 3 miles less than Monday's distance on Wednesday. Write a simplified expression for the total number of miles he rides.

   Show answer

   $m + 2m + (m - 3) = 4m - 3$ miles

9. Hard Simplify $4k^2 + 3k - k^2 - 7k + 2$.

Show answer

$3k^2 - 4k + 2$

10. Hard Grace buys 3 notebooks for n dollars each on Monday. On Friday she buys 5 more notebooks at the same price and a pen for \$4. Write a simplified expression for the total she spends in dollars.

Show answer

$3n + 5n + 4 = 8n + 4$ dollars

11. Hard A rectangular sheep pen is $3x + 2$ yards long and $x - 1$ yards wide. Add all four sides to write a simplified expression for the perimeter.

Show answer

$(3x + 2) + (x - 1) + (3x + 2) + (x - 1) = 8x + 2$ yards

12. Hard Samuel says $3x + 2x = 5x^2$. Explain why he is wrong, using the distributive property.

Show answer

By the distributive property, $3x + 2x = (3 + 2)x = 5x$. Adding like terms adds the coefficients; the variable part stays x. (Check: if $x = 2$, $3(2) + 2(2) = 10$ and $5(2) = 10$, but $5(2)^2 = 20$.)

13. Challenge Simplify $\frac{1}{2}x + 5 + \frac{2}{3}x - \frac{3}{4}$.

Show answer

$\frac{7}{6}x + \frac{17}{4}$ (since $\frac{3}{6} + \frac{4}{6} = \frac{7}{6}$ and $\frac{20}{4} - \frac{3}{4} = \frac{17}{4}$)

COMMON MISTAKES

- Changing the variable part when combining, such as $3x + 2x = 5x^2$. Only the coefficients are added; x stays x.
- Combining unlike terms, such as $4x + 3 = 7x$ or $x^2 + x = x^3$. Only terms with exactly the same variable part combine.
- Losing a negative sign when terms are moved. In $6m - 4 - 2m$, the term is $-2m$. Rewrite subtraction as adding the opposite before grouping.
- Forgetting that a lone x has coefficient 1. In $x + 4x$, the answer is $5x$, not $4x$.

Make it a game · Like Terms Sort. Write 20 terms on slips of paper, such as 3x, -5, 2x squared, 7x and 4. Sort them into piles of like terms in 30 seconds, then combine each pile into one term. Write the final simplified expression. Parent checks by substituting x = 2 in both the original and simplified forms. Beat your sorting time next round.

9.5 The Distributive Property with Variables

OBJECTIVES

- Expand expressions such as $a(b+c)$ when they contain variables and negative numbers.
- Simplify expressions such as $3(x+4)-2(x-1)$ by distributing and then combining like terms.
- Factor a common number out of an expression, such as $6x+18=6(x+3)$.

WATCH

Algebra Basics: The Distributive Property - Math Antics

mathantics · 11:54

https://www.youtube.com/watch?v=v-6MShC82ow

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'expanding brackets' for distributing.

BACKUP VIDEO

Expanding Algebraic Expressions

Eddie Woo · 13:15

https://www.youtube.com/watch?v=aI9fAobtplA

Before the practice set: more expanding examples.

KEY TERMS

distributive property

Multiplying a sum (or difference) by a number is the same as multiplying each part and then adding (or subtracting): $a(b+c)=ab+ac$.

expand

To use the distributive property to remove parentheses, as in $4(x+6)=4x+24$.

factor (an expression)

To write an expression as a product, often by pulling out a common factor: $6x + 18 = 6(x + 3)$. Factoring undoes expanding.

greatest common factor (GCF)

The largest number that divides evenly into every coefficient and constant in the expression.

CORE IDEAS

In Lesson 1.3 you learned that $7 \times 98 = 7(100 - 2) = 700 - 14 = 686$. The same **distributive property** works when one of the numbers is a variable: $a(b + c) = ab + ac$. The number outside the parentheses multiplies *every* term inside.

Why is it true? Picture a rectangle 3 units tall and $x + 4$ units long. Split it into two smaller rectangles: one 3 by x and one 3 by 4. Their areas are $3x$ and 12. The whole area is the sum of the parts, so $3(x + 4) = 3x + 12$. Another way to see it: $3(x + 4)$ means three groups of $(x + 4)$, which is $x + x + x + 4 + 4 + 4 = 3x + 12$.

With negatives, multiply each term using the sign rules from Unit 5. Think of a subtraction inside the parentheses as adding a negative: $-2(x - 3) = -2(x + (-3)) = (-2)(x) + (-2)(-3) = -2x + 6$. The negative outside changes the sign of *every* term inside. A lone minus sign in front of parentheses means multiply by -1: $-(4m - 9) = -4m + 9$.

To simplify an expression like $3(x + 4) - 2(x - 1)$, first distribute each number, being careful to treat the -2 as negative: $3x + 12 - 2x + 2$. Then combine like terms as in Lesson 9.4: $x + 14$.

Factoring runs the distributive property backward. In $6x + 18$, both terms have a factor of 6 (the GCF of 6 and 18, from Lesson 2.4). Pull it out: $6x + 18 = 6(x) + 6(3) = 6(x + 3)$. Check any factoring by expanding again; you should get back what you started with. Always factor out the *greatest* common factor so nothing more can be pulled out.

KEY FORMULAS

$$a(b + c) = ab + ac \qquad a(b - c) = ab - ac$$

$$-(b + c) = -b - c \qquad -(b - c) = -b + c$$

$$ab + ac = a(b + c) \quad \text{(factoring)}$$

Practice habit. Draw an arrow from the number outside the parentheses to each term inside, and say each product aloud with its sign before writing it. Recite the sign rules as you go: same signs give a positive product, different signs give a negative product.

WORKED EXAMPLES

Example 1. Ruth buys 5 school kits. Each kit has a notebook that costs n dollars and a box of pens that costs \$2. Write an expression for the total cost with parentheses, then expand it.

Show solution

1. One kit costs $n + 2$ dollars.
2. Five kits cost 5 times that: $5(n + 2)$.
3. Distribute the 5 to each term: $5(n) + 5(2)$.
4. Multiply: $5n + 10$.
5. Meaning: $5n$ is the cost of the 5 notebooks, and 10 is the \$10 for the 5 boxes of pens.
6. Check with $n = 3$: $5(3 + 2) = 25$ and $5(3) + 10 = 25$. They match.

Answer: $5(n + 2) = 5n + 10$ dollars

Example 2. Simplify $3(x + 4) - 2(x - 1)$.

Show solution

1. Distribute the 3: $3(x + 4) = 3x + 12$.
2. Distribute the -2 to both terms: $(-2)(x) = -2x$ and $(-2)(-1) = +2$.
3. Write everything together: $3x + 12 - 2x + 2$.
4. Combine the x terms: $3x - 2x = x$.
5. Combine the constants: $12 + 2 = 14$.
6. Result: $x + 14$.
7. Check with $x = 1$: original $3(5) - 2(0) = 15$; simplified $1 + 14 = 15$. They match.

Answer: $x + 14$

PRACTICE

1. Easy Expand $4(x + 6)$.

   Show answer

   $4x + 24$

2. Easy Expand $7(2y - 3)$.

   Show answer

   $14y - 21$

3. Easy Expand $-2(a + 5)$.

   Show answer

   $-2a - 10$

4. Easy Factor out the greatest common factor: $6x + 18$.

   Show answer

   $6(x + 3)$

5. Medium Expand $-(4m - 9)$.

   Show answer

   $-4m + 9$

6. Medium Expand $-5(3 - 2k)$.

   Show answer

   $-15 + 10k$, or $10k - 15$

7. Medium Factor out the greatest common factor: $12n - 20$.

   Show answer

   $4(3n - 5)$

8. Medium Samuel orders 6 team shirts. Each shirt costs s dollars, plus \$3 to add a name on the back. Write an expression for the total cost using parentheses, then expand it.

   Show answer

   $6(s + 3) = 6s + 18$ dollars

9. Hard Simplify $5(2x - 1) - 3(x + 4)$.

Show answer

$7x - 17$ ($10x - 5 - 3x - 12$)

10. Hard Simplify $2(3a + 1) + 4(a - 2) - a$.

Show answer

$9a - 6$ ($6a + 2 + 4a - 8 - a$)

11. Hard A rectangular flower bed is w feet wide and $2w + 3$ feet long. Use $P = 2l + 2w$ to write a simplified expression for its perimeter.

Show answer

$2(2w + 3) + 2w = 4w + 6 + 2w = 6w + 6$ feet

12. Hard The Miller family buys train tickets for 4 adults at a dollars each and 4 children at c dollars each. Their total is $4a + 4c$ dollars. Factor this expression and explain what the factored form means.

Show answer

$4(a + c)$. It means 4 groups of one adult ticket plus one child ticket.

13. Hard Grace expanded $-2(x - 3)$ and got $-2x - 6$. Explain her mistake and give the correct answer.

Show answer

She multiplied -2 by x but did not use the sign rule on the second term. The -2 multiplies -3 too, and $(-2)(-3) = +6$. The correct answer is $-2x + 6$.

14. Challenge Simplify $-4(2x - 3) - (x - 5) + 3(x + 1)$.

Show answer

$-6x + 20$ ($-8x + 12 - x + 5 + 3x + 3$)

COMMON MISTAKES

- Multiplying only the first term inside the parentheses, as in $4(x + 6) = 4x + 6$. The outside number multiplies every term: $4x + 24$.
- Dropping the negative when distributing a subtraction, as in $-2(x - 1) = -2x - 2$. Treat the outside number as -2 and use the sign rules: $(-2)(-1) = +2$.

- Ignoring a lone minus sign before parentheses. $-(x - 5)$ means $-1(x - 5) = -x + 5$.
- Factoring out a common factor that is not the greatest, such as $12n - 20 = 2(6n - 10)$. Check whether the terms inside still share a factor; if so, use the GCF, 4, to get $4(3n - 5)$.

> **Make it a game · Distribute and Simplify.** Write twelve expressions, such as 3(x + 4) - 2(x - 1), on cards. Draw one, distribute and combine like terms aloud, writing each line. Parent checks by substituting x = 3 into both versions. A match means one point for you, a mismatch means find the error. Challenge: reach ten points in ten minutes.

UNIT 9

Variables and Expressions

In this unit you take your first real step into algebra. You will learn to use letters, called variables, to stand for numbers, and to read and write algebraic expressions such as $3x + 5$. You will name the parts of an expression, find its value for given numbers, and translate English phrases and number patterns into the language of math.

You will also learn to simplify expressions by combining like terms and by using the distributive property, the same property you used for mental math in Unit 1. Along the way you will see *why* these rules work, using area models, groups of objects and the sign rules from Unit 5.

These skills are the tools of algebra. In Unit 10 you will use them to write and solve equations, so careful habits now (parentheses when substituting, carrying each sign with its term, checking by substitution) will pay off for the rest of the course and in Algebra I.

PREREQUISITES

- Use the order of operations, including exponents (Lesson 1.4).
- Use the distributive property for mental math (Lesson 1.3).
- Find the greatest common factor of two numbers (Lesson 2.4).
- Add, subtract, multiply and divide integers and rational numbers (Unit 5).
- Evaluate powers, including negative bases such as $(-3)^2$ and -3^2 (Lesson 6.1).

MASTERY CHECKLIST

- ☐ I can explain what a variable is and why we use letters to stand for numbers.
- ☐ I can name the terms, coefficients and constant of an expression, including negative coefficients and a coefficient of 1.
- ☐ I can evaluate an expression by substituting numbers, including negative numbers and exponents.
- ☐ I can use formulas for perimeter, distance and temperature conversion.
- ☐ I can translate word phrases into expressions, including "less than" and "times the sum of."
- ☐ I can write an expression for a number pattern and check it.
- ☐ I can identify like terms and combine them to simplify an expression.
- ☐ I can explain why $3x + 2x = 5x$ using the distributive property.
- ☐ I can expand expressions such as $-2(x + 3)$ using the distributive property and the sign rules.
- ☐ I can simplify expressions such as $3(x + 4) - 2(x - 1)$.
- ☐ I can factor the greatest common factor out of an expression such as $12n - 20$.

LESSONS IN THIS UNIT

9.1	**Variables and Expressions**	45 minutes
9.2	**Evaluating Expressions**	2 sessions of 45 minutes
9.3	**Writing Expressions**	2 sessions of 45 minutes
9.4	**Combining Like Terms**	2 sessions of 45 minutes
9.5	**The Distributive Property with Variables**	2 sessions of 45 minutes

Unit 9 test

You have 50 minutes. A basic calculator is allowed but not needed for most problems. Show your work for every problem, including each substitution written in parentheses. Write expressions in simplest form. Include units in answers to word problems.

1. **6 pts**
   In the expression $5y^2 - y + 8$, what is the coefficient of y, and what is the constant?

2. **6 pts**
   A fitness club charges a one-time fee of \$25 to join plus \$30 per month. The total cost in dollars for m months is $30m + 25$. What do the coefficient 30 and the constant 25 mean?

3. **6 pts**
   Evaluate $4n - 7$ when $n = 9$.

4. **7 pts**
   Evaluate $x^2 - 3x$ when $x = -5$.

5. **7 pts**
   On a summer day in Rome the temperature is $30°\text{C}$. Use $F = \frac{9}{5}C + 32$ to find the temperature in degrees Fahrenheit.

6. **6 pts**

   Write an expression for "9 less than the product of 5 and a number n."

7. **7 pts**

   Henry has \$40 in his savings jar and adds \$6 every week. Write an expression for the amount in dollars in the jar after w weeks.

8. **7 pts**

   The sequence 3, 8, 13, 18, … continues the same pattern. Write an expression for the nth number in the sequence.

9. **6 pts**

   Simplify $9a - 4 + 2a + 10$.

10. **8 pts**

    Simplify $2x^2 + 5x - 7x^2 + x - 3$.

11. **6 pts**

    Expand $-3(4k - 2)$.

12. **6 pts**

    Factor out the greatest common factor: $15x + 35$.

13. **8 pts**

    Simplify $4(2y + 3) - 5(y - 2)$.

14. **7 pts**

    A triangular garden has sides that measure $2x + 1$ feet, $3(x - 1)$ feet and $x + 4$ feet. Write a simplified expression for its perimeter.

15. **7 pts**

    Ruth simplified $4x + 3$ and got $7x$. Explain why she is wrong.

ANSWER KEY

1. Coefficient of y: -1; constant: 8
2. 30 is the cost per month ($30 each month); 25 is the one-time $25 joining fee, which does not change with the number of months.
3. 29
4. 40 ($(-5)^2 - 3(-5) = 25 + 15$)
5. $86°\text{F}$ ($54 + 32$)
6. $5n - 9$
7. $40 + 6w$ (or $6w + 40$) dollars
8. $5n - 2$
9. $11a + 6$
10. $-5x^2 + 6x - 3$
11. $-12k + 6$
12. $5(3x + 7)$
13. $3y + 22$ ($8y + 12 - 5y + 10$)
14. $6x + 2$ feet ($2x + 1 + 3x - 3 + x + 4$)
15. $4x$ and 3 are not like terms: one has the variable x and the other is a constant, so they cannot be added. $4x + 3$ is already in simplest form. (Check: if $x = 2$, $4(2) + 3 = 11$, but $7(2) = 14$.)

UNIT 10

Equations and Inequalities

In Unit 9 you learned to write and simplify expressions. In this unit you put two expressions on opposite sides of an equal sign and ask, "What number makes this true?" You will picture every equation as a balance scale and solve it by doing the same thing to both sides, undoing what was done to the variable one step at a time.

You will start with one-step equations, move to two-step and multi-step equations (including the distributive property and variables on both sides), and then turn word problems into equations: consecutive numbers, ages, money and perimeter. The unit ends with inequalities, where the answer is a whole range of numbers and where multiplying or dividing by a negative number reverses the sign.

Solving equations is the heart of Algebra I. Parents can help by asking one question after every problem: "Did you check it?" Substituting the answer back into the original equation is the habit that matters most in this unit.

PREREQUISITES

- Add, subtract, multiply and divide integers, fractions and decimals (Units 3–5).
- Use the order of operations (Lesson 1.4).
- Multiply by a reciprocal to divide by a fraction (Lesson 3.6).
- Evaluate expressions by substitution (Lesson 9.2).
- Translate words into expressions (Lesson 9.3).
- Combine like terms and use the distributive property with variables (Lessons 9.4 and 9.5).

MASTERY CHECKLIST

- ☐ I can explain an equation using the picture of a balance scale.
- ☐ I can check whether a number is a solution of an equation by substituting it.
- ☐ I can solve one-step equations that use addition or subtraction.
- ☐ I can solve one-step equations that use multiplication or division, including fraction coefficients, by using the reciprocal.
- ☐ I can solve two-step equations by undoing the operations in reverse order.
- ☐ I can solve multi-step equations by combining like terms, distributing and moving variable terms to one side.
- ☐ I can define a variable, write an equation for a word problem, solve it, check it and answer in a sentence.
- ☐ I can solve word problems about consecutive integers, ages, money and perimeter.
- ☐ I can write and graph inequalities on a number line using open and closed circles.
- ☐ I can solve one- and two-step inequalities and explain why the sign reverses when I multiply or divide by a negative number.

LESSONS IN THIS UNIT

10.1	**Equations and Solutions**	45 minutes
10.2	**One-Step Equations with Addition and Subtraction**	45 minutes
10.3	**One-Step Equations with Multiplication and Division**	2 sessions of 45 minutes
10.4	**Two-Step Equations**	2 sessions of 45 minutes
10.5	**Multi-Step Equations**	2 sessions of 45 minutes
10.6	**Writing Equations for Word Problems**	2 sessions of 45 minutes
10.7	**Inequalities**	2 sessions of 45 minutes

Unit 10 test

You have 60 minutes. You may use a basic calculator for decimal arithmetic. Show every step of your work, and check each equation by substituting your answer. For word problems, define your variable, write an equation or inequality, and answer in a complete sentence with units. Give exact answers unless a problem says how to round.

1. **5 pts**
   Is $x = 3$ a solution of $5x - 4 = 11$? Answer yes or no and show your check.

2. **5 pts**
   Solve $x - 13 = -4$.

3. **7 pts**
   After spending $27.85 on groceries, Ruth had $42.15 left. Write and solve an equation to find how much money she started with.

4. **5 pts**
   Solve $-7y = 56$.

5. **6 pts**

   Solve $\frac{3}{4}x = 27$.

6. **6 pts**

   Solve $4x - 9 = 23$.

7. **8 pts**

   A plumber charges \$45 for a house call plus \$60 per hour. The Millers' bill was \$255. Write and solve an equation to find how many hours the plumber worked.

8. **7 pts**

   Solve $3(x - 2) + 4x = 29$.

9. **7 pts**

   Solve $6x - 5 = 2x + 19$.

10. **8 pts**

    The sum of three consecutive odd integers is 81. Find the integers.

11. **8 pts**

    A rectangular vegetable garden is 5 feet longer than it is wide. Its perimeter is 62 feet. Find the width and the length. Give the width first.

12. **6 pts**

    Solve $x + 4 > 1$ and describe its graph on a number line.

13. **7 pts**

    Solve $5 - 3x \le 20$.

14. **8 pts**

    Samuel wants to save at least \$250 for a mission trip. He has \$34 and earns \$9 per hour mowing lawns. Write and solve an inequality for the number of hours h he must work.

15. **7 pts**

    Solve $-2x < 10$. Then explain why the inequality sign must be reversed in your last step.

ANSWER KEY

1. Yes ($5 \cdot 3 - 4 = 15 - 4 = 11$)
2. $x = 9$
3. $m - 27.85 = 42.15$, so $m = 70$. Ruth started with \$70.00.
4. $y = -8$
5. $x = 36$
6. $x = 8$
7. $45 + 60h = 255$, so $60h = 210$ and $h = 3.5$. The plumber worked $3\frac{1}{2}$ hours.
8. $x = 5$
9. $x = 6$
10. $n + (n + 2) + (n + 4) = 81$, so $3n + 6 = 81$ and $n = 25$. The integers are 25, 27 and 29.
11. $2w + 2(w + 5) = 62$, so $4w + 10 = 62$ and $w = 13$. The garden is 13 feet wide and 18 feet long.
12. $x > -3$; an open circle at -3, shaded to the right.
13. $x \geq -5$
14. $34 + 9h \geq 250$, so $9h \geq 216$ and $h \geq 24$. He must work at least 24 hours.
15. $x > -5$. You divide both sides by -2. Dividing by a negative moves each number to the other side of 0 on the number line, which reverses their order (for example, $2 < 6$ but $-1 > -3$), so the sign must flip to keep the statement true. Check: $x = 0$ gives $0 < 10$, true; $x = -6$ gives $12 < 10$, false.

10.1 Equations and Solutions

OBJECTIVES

- Explain what an equation is using the picture of a balance scale.
- Check whether a given number is a solution of an equation by substituting it.
- Name the inverse operation that undoes adding, subtracting, multiplying or dividing.

WATCH

Equations & Inequalities (1 of 2: What are equations?)

Eddie Woo · 8:03

https://www.youtube.com/watch?v=xPx4xQxF1Bg

> **WHEN TO WATCH**
> Before the Core ideas.

BACKUP VIDEO

Introduction to Solving Equations

Eddie Woo · 4:24

https://www.youtube.com/watch?v=GoKrzNHrafk

Before Example 2: inverse operations and the balance idea.

KEY TERMS

equation

A math sentence that says two expressions are equal, such as $x + 7 = 11$. It always has an equal sign.

solution

A number that makes an equation true when you put it in place of the variable. The solution of $x + 7 = 11$ is 4, because $4 + 7 = 11$.

solve

To find the solution (or solutions) of an equation.

substitute

To put a number in place of a variable.

inverse operations

Two operations that undo each other. Addition and subtraction are inverse operations; so are multiplication and division.

CORE IDEAS

In Unit 9 you worked with **expressions** like $3x + 2$. An expression is like a phrase; it has no equal sign. An **equation** is a full sentence that says two expressions have the same value: $3x + 2 = 14$. The equation might be true or false, depending on what number x stands for.

Picture an equation as a **balance scale**. The left side of the equal sign sits in one pan and the right side sits in the other. When the equation is true, the two pans weigh exactly the same and the scale is level. The equal sign means "balances with."

A **solution** is a number that makes the scale balance. To check a number, **substitute** it for the variable and work out each side separately. If the two sides come out the same, the number is a solution. Check $x = 4$ in $3x + 2 = 14$: the left side is $3 \cdot 4 + 2 = 12 + 2 = 14$ and the right side is 14. They match, so 4 is a solution. Check $x = 5$: $3 \cdot 5 + 2 = 17$, and $17 \neq 14$, so 5 is not a solution.

Guessing and checking works, but it can be slow. In the next lessons you will **solve** equations directly by undoing what was done to the variable. For that you need **inverse operations**, operations that undo each other. Adding 8 is undone by subtracting 8. Multiplying by 5 is undone by dividing by 5. If you add 8 and then subtract 8, you are back where you started.

Here is the big rule that keeps the scale level: **whatever you do to one side, you must do to the other side.** If you take 3 pounds off only one pan, the scale tips. If you take 3 pounds off both pans, it stays level. Every method for solving equations in this unit is built on that one rule.

KEY FORMULAS

$$\text{If } a = b, \text{ then } a + c = b + c, \quad a - c = b - c, \quad a \cdot c = b \cdot c, \quad \frac{a}{c} = \frac{b}{c} \; (c \neq 0)$$

$$\text{Inverse pairs: } + \text{ and } - \qquad \times \text{ and } \div$$

From history. The equal sign was invented by the Welsh mathematician Robert Recorde in his book *The Whetstone of Witte* (1557). Before that, writers spelled out "is equal to" in words. Recorde chose two parallel lines of the same length because, as he wrote, no two things can be more equal.

Euclid connection. Euclid's first "common notion" in the Elements says that things equal to the same thing are equal to each other, and the next two say that if equals are added to equals, or taken from equals, the results are equal. That is exactly the balance rule we use to solve equations.

Practice habit. Always check by substitution: put your number into the original equation, work out each side separately, and see whether the two sides match. Say it as a habit: "Substitute, simplify each side, compare."

WORKED EXAMPLES

Example 1. Is $x = 6$ a solution of $4x - 5 = 19$?

Show solution

1. Substitute 6 for x on the left side: $4 \cdot 6 - 5$.
2. Multiply first: $24 - 5$.
3. Subtract: 19.
4. The right side is 19. The two sides match: $19 = 19$.
5. So $x = 6$ is a solution.

Answer: Yes, $x = 6$ is a solution.

Example 2. The Miller family bought 4 tickets to a county fair and paid \$6 for parking. They spent \$38 in all. The equation $4t + 6 = 38$ describes this, where t is the price of one ticket in dollars. Which of the prices \$7, \$8 or \$9 is the ticket price?

Show solution

1. Try $t = 7$: $4 \cdot 7 + 6 = 28 + 6 = 34$. Since $34 \neq 38$, 7 is not a solution.
2. Try $t = 8$: $4 \cdot 8 + 6 = 32 + 6 = 38$. Since $38 = 38$, 8 is a solution.
3. Try $t = 9$ to be sure: $4 \cdot 9 + 6 = 36 + 6 = 42$. Since $42 \neq 38$, 9 is not a solution.
4. Answer in a sentence: each ticket cost \$8.

Answer: Each ticket cost \$8.

PRACTICE

1. Easy Is $x = 4$ a solution of $x + 7 = 11$? Answer yes or no.

   Show answer

   Yes ($4 + 7 = 11$)

2. Easy Is $n = 6$ a solution of $3n = 21$? Answer yes or no.

   Show answer

   No ($3 \cdot 6 = 18$, not 21)

3. Easy Is $y = 9$ a solution of $y - 5 = 4$? Answer yes or no.

   Show answer

   Yes ($9 - 5 = 4$)

4. Easy What operation undoes "multiply by 7"?

   Show answer

   Divide by 7.

5. Medium Is $k = 20$ a solution of $\frac{k}{4} = 6$? Answer yes or no.

   Show answer

   No ($20 \div 4 = 5$, not 6)

6. **Medium** Is $x = -3$ a solution of $x + 10 = 7$? Answer yes or no.

Show answer

Yes ($-3 + 10 = 7$)

7. **Medium** Is $m = 5$ a solution of $2m + 3 = 12$? Answer yes or no.

Show answer

No ($2 \cdot 5 + 3 = 13$, not 12)

8. **Medium** Which of the numbers 3, 4 or 5 is a solution of $4x - 1 = 15$?

Show answer

$x = 4$ ($4 \cdot 4 - 1 = 15$)

9. **Medium** Ruth had some stickers. After she gave 12 stickers to her little brother, she had 30 left. This is described by $s - 12 = 30$. Which of the numbers 18, 38 or 42 is the number of stickers she started with?

Show answer

42 stickers ($42 - 12 = 30$)

10. **Hard** Samuel bought 3 identical notebooks for \$7.50 in all. The equation $3p = 7.50$ describes this, where p is the price of one notebook in dollars. Is $p = 2.50$ a solution? Answer yes or no.

Show answer

Yes ($3 \times 2.50 = 7.50$)

11. **Hard** Is $x = -2$ a solution of $5x + 4 = -6$? Answer yes or no.

Show answer

Yes ($5(-2) + 4 = -10 + 4 = -6$)

12. **Hard** Use the balance-scale picture to explain why you may subtract 6 from both sides of a true equation and still have a true equation, but you may not subtract 6 from only one side.

Show answer

A true equation is like a level scale: both pans weigh the same. Taking 6 off both pans leaves them still equal, so the scale stays level. Taking 6 off only one pan makes that side lighter, so the scale tips and the sides are no longer equal.

13. **Challenge** Find every whole number from 0 to 10 that is a solution of $x^2 = 4x$.

Show answer

0 and 4 ($0 = 0$ and $16 = 16$; no other whole number from 0 to 10 works)

COMMON MISTAKES

- Checking only part of the equation, such as computing $4 \cdot 6 = 24$ and stopping. Work out the whole side, $4 \cdot 6 - 5 = 19$, and then compare it with the other side.
- Ignoring the order of operations when substituting. In $2m + 3$ with $m = 5$, multiply first: $2 \cdot 5 + 3 = 13$, not $2 \cdot 8 = 16$.
- Writing $3x$ with $x = 4$ as 34. When you substitute next to a number, use a dot or parentheses: $3 \cdot 4$ or $3(4)$.
- Confusing an expression with an equation. An expression like $3x + 2$ has no equal sign, so it cannot be true or false; only an equation has solutions.

> **Make it a game · Balance Scale Game.** Use a ruler balanced on a pencil as a scale, with buttons for weights. Set up a simple equation like x + 3 = 7 by placing three buttons beside the unknown pile and seven on the other side. Find the unknown pile by removing the same amount from both sides. Check by substitution. Try six equations, then invent one for the parent.

10.2 One-Step Equations with Addition and Subtraction

OBJECTIVES

- Solve equations such as $x + 9 = 23$ and $x - 4 = 10$ by using the inverse operation on both sides.
- Check every solution by substituting it into the original equation.
- Write and solve one-step equations for simple word problems.

WATCH

Algebra Basics: Solving Basic Equations Part 1 - Math Antics

mathantics · 11:08

https://www.youtube.com/watch?v=l3XzepN03KQ

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'backtracking' for undoing operations in reverse order.

BACKUP VIDEO

Balancing & Backtracking: 2 Approaches for 1-Step Equations

Eddie Woo · 9:14

https://www.youtube.com/watch?v=tMkMJoSxbC8

Before the practice set.

KEY TERMS

isolate the variable

To get the variable all by itself on one side of the equation, such as $x = 14$.

inverse operation

The operation that undoes another one. Subtracting undoes adding, and adding undoes subtracting.

equivalent equations

Equations that have the same solution, such as $x + 9 = 23$ and $x = 14$.

CORE IDEAS

To **solve** an equation, you want the variable alone on one side: $x =$ some number. This is called **isolating the variable**. Ask, "What has been done to x?" Then do the opposite to get rid of it.

In $x + 9 = 23$, 9 has been **added** to x. Subtraction undoes addition, so subtract 9. To keep the balance level, subtract 9 from **both** sides: $x + 9 - 9 = 23 - 9$, which gives $x = 14$. Check: $14 + 9 = 23$. True.

In $x - 4 = 10$, 4 has been **subtracted** from x. Addition undoes subtraction, so add 4 to both sides: $x - 4 + 4 = 10 + 4$, which gives $x = 14$. Check: $14 - 4 = 10$. True.

Why does this work? Each step keeps the two sides equal, so each new equation has the same solution as the one before. These are called **equivalent equations**. The last one, $x = 14$, simply tells you the answer.

The variable can be on either side. $19 = n + 6$ means the same as $n + 6 = 19$. Subtract 6 from both sides to get $13 = n$, so $n = 13$. The same steps work with negative numbers, decimals and fractions; just use the rules you already know for adding and subtracting them.

KEY FORMULAS

$$x + a = b \;\Rightarrow\; x = b - a$$

$$x - a = b \;\Rightarrow\; x = b + a$$

From history. The word *algebra* comes from the title of a book written in Baghdad around 820 by al-Khwarizmi. The Arabic word *al-jabr* means "restoring." It described adding the same amount to both sides of an equation to restore a quantity that had been subtracted, which is exactly what you do when you solve $x - 4 = 10$ by adding 4 to both sides.

Practice habit. Before you move anything, say the undoing sentence out loud: "Plus 9 is undone by minus 9, on both sides." Then check mentally: put your answer back in and see that the two sides match.

WORKED EXAMPLES

Example 1. Solve $x + 9 = 23$.

Show solution

1. 9 is added to x, so subtract 9 from both sides: $x + 9 - 9 = 23 - 9$.
2. Simplify: $x = 14$.
3. Check: $14 + 9 = 23$. True.

Answer: $x = 14$

Example 2. Grace took some money to the craft fair. She spent \$14.75 on a gift for her grandmother and had \$23.40 left. Write and solve an equation to find how much money she brought.

Show solution

1. Let m be the number of dollars Grace brought.
2. Money brought minus money spent equals money left: $m - 14.75 = 23.40$.
3. 14.75 is subtracted from m, so add 14.75 to both sides: $m - 14.75 + 14.75 = 23.40 + 14.75$.
4. Simplify: $m = 38.15$.
5. Check: $38.15 - 14.75 = 23.40$. True.
6. Answer in a sentence: Grace brought \$38.15.

Answer: Grace brought \$38.15.

PRACTICE

1. Easy Solve $x + 5 = 12$.

   Show answer

   $x = 7$

2. Easy Solve $y - 8 = 15$.

   Show answer

   $y = 23$

3. Easy Solve $19 = n + 6$.

Show answer

$n = 13$

4. Easy After 17 new members joined, the church choir had 52 members. Write and solve an equation to find how many members the choir had before.

Show answer

$c + 17 = 52$, so $c = 35$. The choir had 35 members.

5. Medium Solve $x + 14 = 6$.

Show answer

$x = -8$

6. Medium Solve $p - 2.6 = 4.9$.

Show answer

$p = 7.5$

7. Medium Solve $t + \frac{3}{4} = 2$.

Show answer

$t = 1\frac{1}{4}$

8. Medium Henry's bean plant grew 9 centimeters this week. It is now 31 centimeters tall. Write and solve an equation to find how tall it was at the start of the week.

Show answer

$h + 9 = 31$, so $h = 22$. It was 22 centimeters tall.

9. Hard Solve $-7 + w = -20$.

Show answer

$w = -13$

10. Hard Overnight the temperature dropped 12 degrees Fahrenheit. In the morning it was $-5°$F. Write and solve an equation to find the temperature the evening before.

Show answer

$t - 12 = -5$, so $t = 7$. It was $7°$F.

11. Hard Ruth wants a bicycle that costs \$129.99. She has saved \$86.50. Write and solve an equation to find how much more she needs to save.

Show answer

$86.50 + s = 129.99$, so $s = 43.49$. She needs \$43.49 more.

12. Hard To solve $x - 4 = 10$, Samuel subtracted 4 from both sides and got $x = 6$. Explain what went wrong and give the correct solution.

Show answer

4 is being subtracted from x, so to undo it you must add 4, not subtract 4. Adding 4 to both sides gives $x = 14$. Check: $14 - 4 = 10$. (Samuel's answer fails the check: $6 - 4 = 2$, not 10.)

13. Challenge Solve $x - 2\frac{1}{3} = 5\frac{3}{4}$. Give your answer as a mixed number.

Show answer

$x = 8\frac{1}{12}$

COMMON MISTAKES

- Doing the same operation instead of the opposite one, such as subtracting 4 to solve $x - 4 = 10$. Ask what was done to x, then undo it.
- Changing only one side. Whatever you add or subtract on the left, do the same on the right, or the balance tips.
- Sign errors with negatives, such as solving $x + 14 = 6$ and getting $x = 8$. Compute $6 - 14 = -8$ carefully, then check: $-8 + 14 = 6$.
- Skipping the check. Substituting takes only a moment and catches nearly every mistake.

> **Make it a game · Undo It Quick.** Write 15 equations on cards, such as x + 9 = 23 and x - 4 = 10. Draw one, name the inverse operation, solve aloud and check by substituting. Parent times you with a stopwatch. Each correct solution with a checked answer earns two points. See if you can reach 20 points in under seven minutes.

10.3 One-Step Equations with Multiplication and Division

OBJECTIVES

- Solve equations such as $6x = 42$ and $\frac{n}{4} = 9$ by dividing or multiplying both sides.
- Solve equations with a fraction coefficient, such as $\frac{2}{3}x = 156$, by multiplying both sides by the reciprocal.
- Write, solve and check one-step equations for word problems.

WATCH

Algebra Basics: Solving Basic Equations Part 2 - Math Antics

mathantics · 9:35

https://www.youtube.com/watch?v=Qyd_v3DGzTM

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Solving Equations (1 of 2: Reviewing Algebraic Equations)

Eddie Woo · 9:36

https://www.youtube.com/watch?v=rSN9fNkR7fM

Before the fraction-coefficient example.

KEY TERMS

coefficient

The number multiplied by a variable. In $6x$, the coefficient is 6.

inverse operation

The operation that undoes another one. Dividing undoes multiplying, and multiplying undoes dividing.

reciprocal

The number you multiply by to get 1. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, because $\frac{2}{3} \times \frac{3}{2} = 1$.

CORE IDEAS

In $6x = 42$, x has been **multiplied** by 6. Division undoes multiplication, so divide **both sides** by 6: $\frac{6x}{6} = \frac{42}{6}$, which gives $x = 7$. Check: $6 \cdot 7 = 42$. True.

In $\frac{n}{4} = 9$, n has been **divided** by 4. Multiplication undoes division, so multiply both sides by 4: $4 \cdot \frac{n}{4} = 4 \cdot 9$, which gives $n = 36$. Check: $36 \div 4 = 9$. True.

Watch the signs. In $-3k = 27$, the coefficient is -3, so divide both sides by -3: $k = 27 \div (-3) = -9$. Check: $-3 \cdot (-9) = 27$. True. Divide by the whole coefficient, sign and all.

When the coefficient is a fraction, as in $\frac{2}{3}x = 156$, you could divide both sides by $\frac{2}{3}$. But dividing by a fraction is the same as multiplying by its **reciprocal** (Lesson 3.6), so multiply both sides by $\frac{3}{2}$. **Why does this work?** A number times its reciprocal is 1, so $\frac{3}{2} \cdot \frac{2}{3}x = 1x = x$. The left side becomes just x, and the right side becomes $\frac{3}{2} \cdot 156 = 234$.

Decimal coefficients work the same way. In $0.4m = 12$, divide both sides by 0.4: $m = 12 \div 0.4 = 120 \div 4 = 30$. You may use a calculator for messy decimals, but always check by substitution.

KEY FORMULAS

$$ax = b \;\Rightarrow\; x = \frac{b}{a} \quad (a \neq 0)$$

$$\frac{x}{a} = b \;\Rightarrow\; x = a \cdot b \quad (a \neq 0)$$

$$\frac{p}{q}x = b \;\Rightarrow\; x = \frac{q}{p} \cdot b$$

Practice habit. Recite reciprocal pairs until they are automatic: $\frac{2}{3}$ and $\frac{3}{2}$, $\frac{3}{4}$ and $\frac{4}{3}$, 5 and $\frac{1}{5}$, $-\frac{2}{5}$ and $-\frac{5}{2}$. A number and its reciprocal always have the same sign and multiply to 1.

WORKED EXAMPLES

Example 1. Solve $6x = 42$.

Show solution

1. x is multiplied by 6, so divide both sides by 6: $\frac{6x}{6} = \frac{42}{6}$.
2. Simplify: $x = 7$.
3. Check: $6 \cdot 7 = 42$. True.

Answer: $x = 7$

Example 2. Henry has read $\frac{2}{3}$ of a book. That is 156 pages. Write and solve an equation to find how many pages are in the whole book.

Show solution

1. Let p be the number of pages in the book.
2. Two-thirds of the pages is 156: $\frac{2}{3}p = 156$.
3. Multiply both sides by the reciprocal $\frac{3}{2}$: $\frac{3}{2} \cdot \frac{2}{3}p = \frac{3}{2} \cdot 156$.
4. The left side is $1p = p$. The right side is $156 \div 2 \times 3 = 78 \times 3 = 234$.
5. So $p = 234$.
6. Check: $\frac{2}{3} \cdot 234 = 234 \div 3 \times 2 = 78 \times 2 = 156$. True.
7. Answer in a sentence: the book has 234 pages.

Answer: The book has 234 pages.

PRACTICE

1. Easy Solve $5x = 35$.

   Show answer

   $x = 7$

2. Easy Solve $\frac{n}{4} = 9$.

   Show answer

   $n = 36$

3. Easy Solve $72 = 8y$.

Show answer

$y = 9$

4. Easy Solve $-3k = 27$.

Show answer

$k = -9$

5. Medium Solve $\frac{x}{-6} = 5$.

Show answer

$x = -30$

6. Medium Solve $0.4m = 12$.

Show answer

$m = 30$

7. Medium Solve $\frac{3}{5}x = 21$.

Show answer

$x = 35$

8. Medium Samuel paid \$13.50 for 6 pounds of apples. Write and solve an equation to find the price per pound.

Show answer

$6p = 13.50$, so $p = 2.25$. The apples cost \$2.25 per pound.

9. Hard Solve $-\frac{2}{3}y = 10$.

Show answer

$y = -15$

10. Hard A carpenter cut a board into 4 equal pieces. Each piece is $2\frac{1}{4}$ feet long. Write and solve an equation to find the length of the board before it was cut.

Show answer

$\frac{b}{4} = 2\frac{1}{4}$, so $b = 9$. The board was 9 feet long.

11. Hard At the bake sale, $\frac{3}{8}$ of the cookies were oatmeal cookies. There were 24 oatmeal cookies. Write and solve an equation to find the total number of cookies.

Show answer

$\frac{3}{8}c = 24$, so $c = 64$. There were 64 cookies.

12. Hard Explain why multiplying both sides of $\frac{3}{4}x = 18$ by $\frac{4}{3}$ leaves x by itself. Then solve the equation.

Show answer

$\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$, so $\frac{4}{3} \cdot \frac{3}{4} = 1$ and the left side becomes $1x = x$. Doing the same to the right side keeps the balance: $\frac{4}{3} \cdot 18 = 24$, so $x = 24$.

13. Challenge Ruth rode her bicycle at a steady speed for 2.5 hours and went 30 miles. Use $d = rt$ to write and solve an equation for her speed r in miles per hour.

Show answer

$2.5r = 30$, so $r = 12$. She rode 12 miles per hour.

14. Challenge Solve $\frac{5}{6}x = 2\frac{1}{2}$.

Show answer

$x = 3$

COMMON MISTAKES

- Dividing by the wrong number, such as solving $6x = 42$ by subtracting 6. The 6 is multiplied, so you divide by 6.
- Dropping the negative sign of the coefficient. In $-3k = 27$, divide by -3, not 3, and check: $-3 \cdot (-9) = 27$.
- Multiplying by the fraction itself instead of its reciprocal. For $\frac{2}{3}p = 156$, multiply by $\frac{3}{2}$, not by $\frac{2}{3}$. A quick estimate helps: if two-thirds of the book is 156 pages, the whole book must be more than 156.
- Treating $\frac{n}{4} = 9$ as $n = 9 \div 4$. The n is divided by 4, so undo it by multiplying: $n = 36$.

Make it a game · Isolate the Variable. Prepare cards with equations like 6x = 42, n/4 = 9 and 2/3 x = 156. Draw a card and stand when you have the answer. Explain which operation you used and why, including the reciprocal for fraction coefficients. Parent sneaks one wrong step into a solution for you to catch. Score a point for each right answer or caught mistake.

10.4 Two-Step Equations

OBJECTIVES

- Solve two-step equations such as $3x + 5 = 26$ and $\frac{x}{4} - 3 = 2$ by undoing the operations in reverse order.
- Explain why addition and subtraction are usually undone before multiplication and division.
- Check every solution by substituting it into the original equation.

WATCH

Algebra Basics: Solving 2-Step Equations - Math Antics

mathantics · 10:29

https://www.youtube.com/watch?v=LDIiYKYvvdA

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Solving Equations (2 of 2: Using two methods to solve the same algebraic equation)

Eddie Woo · 7:26

https://www.youtube.com/watch?v=ag4CZKjI6Hs

After Example 2: two ways to solve the same equation.

KEY TERMS

two-step equation

An equation that takes two inverse operations to solve, such as $3x + 5 = 26$ (subtract 5, then divide by 3).

reverse order

Undoing steps in the opposite order from how they were done: the last thing done is the first thing undone.

constant term

A number in an expression with no variable attached, such as the 5 in $3x + 5$.

CORE IDEAS

In $3x + 5 = 26$, two things were done to x. Following the order of operations, x was first **multiplied by 3**, and then **5 was added.** To get back to x, undo these steps in **reverse order**: first subtract 5, then divide by 3.

Think of putting on socks and then shoes. To take them off, you remove the shoes first and the socks second. The last thing done is the first thing undone. That is why, in most two-step equations, you undo the adding or subtracting **before** the multiplying or dividing.

Here are the steps for $3x + 5 = 26$. Subtract 5 from both sides: $3x = 21$. Divide both sides by 3: $x = 7$. Check in the **original** equation: $3 \cdot 7 + 5 = 21 + 5 = 26$. True.

Division works the same way. In $\frac{x}{4} - 3 = 2$, x was divided by 4 and then 3 was subtracted. Undo the subtraction first by adding 3: $\frac{x}{4} = 5$. Then undo the division by multiplying by 4: $x = 20$. Check: $20 \div 4 - 3 = 5 - 3 = 2$. True.

Be careful when the steps come in a different order. In $\frac{x + 4}{3} = 6$, the fraction bar groups $x + 4$, so 4 was added **first** and then the sum was divided by 3. Reverse order means multiply by 3 first ($x + 4 = 18$) and then subtract 4 ($x = 14$). Always ask, "What was done to x, and in what order?"

KEY FORMULAS

$$ax + b = c \;\Rightarrow\; ax = c - b \;\Rightarrow\; x = \frac{c - b}{a}$$

Practice habit. Before solving, say the story of x forward ("times 3, then plus 5") and then backward ("minus 5, then divide by 3"). After solving, always check in the original equation, not in one of your middle steps.

WORKED EXAMPLES

Example 1. Solve $3x + 5 = 26$.

Show solution

1. Story of x: times 3, then plus 5. Undo in reverse: minus 5, then divide by 3.
2. Subtract 5 from both sides: $3x + 5 - 5 = 26 - 5$, so $3x = 21$.
3. Divide both sides by 3: $x = 7$.
4. Check: $3 \cdot 7 + 5 = 21 + 5 = 26$. True.

Answer: $x = 7$

Example 2. Henry is saving for a telescope that costs $185. He already has $65 and saves $15 each week. Write and solve an equation to find how many weeks it will take him to have exactly $185.

Show solution

1. Let w be the number of weeks.
2. Money now plus money saved equals the cost: $65 + 15w = 185$.
3. Subtract 65 from both sides: $15w = 120$.
4. Divide both sides by 15: $w = 8$.
5. Check: $65 + 15 \cdot 8 = 65 + 120 = 185$. True.
6. Answer in a sentence: it will take Henry 8 weeks.

Answer: It will take Henry 8 weeks.

PRACTICE

1. Easy Solve $2x + 3 = 11$.

   Show answer

   $x = 4$

2. Easy Solve $5n - 4 = 31$.

   Show answer

   $n = 7$

3. Easy Solve $\frac{x}{3} + 2 = 8$.

Show answer

$x = 18$

4. Easy Solve $4y + 9 = 1$.

Show answer

$y = -2$

5. Medium Solve $-2x + 7 = 19$.

Show answer

$x = -6$

6. Medium Solve $3 = \frac{x}{5} - 4$.

Show answer

$x = 35$

7. Medium Solve $0.5m + 1.2 = 4.7$.

Show answer

$m = 7$

8. Medium Grace ordered 3 identical scarves online and paid \$4 for shipping. Her total was \$46. Write and solve an equation to find the price of one scarf.

Show answer

$3s + 4 = 46$, so $s = 14$. Each scarf cost \$14.

9. Hard Solve $\frac{2}{3}x - 5 = 7$.

Show answer

$x = 18$

10. Hard A moving truck rents for \$29 plus \$0.75 per mile. The Miller family's bill was \$65.75. Write and solve an equation to find how many miles they drove.

Show answer

$29 + 0.75m = 65.75$, so $m = 49$. They drove 49 miles.

11. Hard A rectangular garden is 18 feet long. It takes 58 feet of fence to go all the way around it. Write and solve an equation to find the width of the garden.

Show answer

$2w + 2 \cdot 18 = 58$, so $2w + 36 = 58$, $2w = 22$, $w = 11$. The garden is 11 feet wide.

12. Hard To solve $3x + 5 = 26$, explain why you subtract 5 before you divide by 3.

Show answer

In $3x + 5$, x was multiplied by 3 first and then 5 was added. To undo steps you reverse their order, like taking off shoes before socks, so the last step (adding 5) is undone first by subtracting 5. Then $3x = 21$, and dividing by 3 gives $x = 7$.

13. Challenge Solve $\frac{x + 4}{3} = 6$.

Show answer

$x = 14$ (multiply by 3 first: $x + 4 = 18$; then subtract 4)

14. Challenge Solve $7 - 2x = 15$.

Show answer

$x = -4$

COMMON MISTAKES

- Dividing first and forgetting to divide every term. If you divide $3x + 5 = 26$ by 3, you must divide the 5 too. It is simpler to subtract 5 first.
- Losing the negative sign on the coefficient. In $-2x + 7 = 19$, after subtracting 7 you have $-2x = 12$, so divide by -2 to get $x = -6$.
- In $7 - 2x = 15$, subtracting 7 and writing $2x = 8$. The term is $-2x$, so you get $-2x = 8$ and $x = -4$.
- Checking in a middle step instead of the original equation. A mistake in the first step will not show up unless you check in the equation you started with.

> **Make it a game · Reverse Order Relay.** Write 12 two-step equations on cards, such as 3x + 5 = 26. For each, say the order in which x was built, then undo it in reverse order, writing each line. Check by substituting. Race the clock, scoring two points for every correct solution with a check. Switch with the parent and have them solve your set.

10.5 Multi-Step Equations

OBJECTIVES

- Simplify each side of an equation by combining like terms and using the distributive property before solving.
- Solve simple equations with the variable on both sides by first moving the variable terms to one side.
- Check every solution by substituting it into both sides of the original equation.

WATCH

Solving Equations w/ Variables on Both Sides

Eddie Woo · 8:34

https://www.youtube.com/watch?v=dLRPfAh4SP8

> **WHEN TO WATCH**
> Before the variables-on-both-sides example.

Note: Eddie Woo teaches in Australia: he says 'brackets' for parentheses.

BACKUP VIDEO

Solving Equations with Brackets (1 of 2: Review & introduction)

Eddie Woo · 12:25

https://www.youtube.com/watch?v=_MMeaIxS8dA

Before Example 1: equations with brackets (parentheses) that must be distributed first.

KEY TERMS

multi-step equation

An equation that needs more than two steps to solve, often because a side must be simplified first.

like terms

Terms with exactly the same variable part, such as $4x$ and $3x$, or 5 and -2. Only like terms can be combined.

variables on both sides

An equation such as $5x + 2 = 2x + 14$, where the variable appears on the left side and on the right side.

CORE IDEAS

Many equations look complicated at first but become two-step equations once you tidy them up. The plan has three parts: **(1) simplify each side**, **(2) get the variable terms on one side**, **(3) solve the two-step equation that is left.** Then check.

Simplify each side. Combine like terms (Lesson 9.4): in $4x + 3x - 5 = 23$, the left side becomes $7x - 5$, so $7x - 5 = 23$. Use the distributive property (Lesson 9.5) to clear parentheses: in $3(x + 4) = 27$, the left side becomes $3x + 12$, so $3x + 12 = 27$, $3x = 15$ and $x = 5$.

Variables on both sides. In $5x + 2 = 2x + 14$, there are x's in both pans of the balance. Take $2x$ away from both sides, just as you would take the same weight off both pans: $5x - 2x + 2 = 2x - 2x + 14$, which gives $3x + 2 = 14$. Now it is a two-step equation: $3x = 12$, so $x = 4$.

Why is it fair to subtract $2x$ from both sides? Whatever number x stands for, $2x$ is the same amount on each side. Removing the same amount from both pans keeps the balance level, exactly as removing 2 or 14 would. It is usually easiest to subtract the **smaller** variable term so the coefficient that is left is positive.

Check in the original equation, and work out each side separately. For $5x + 2 = 2x + 14$ with $x = 4$: the left side is $5 \cdot 4 + 2 = 22$ and the right side is $2 \cdot 4 + 14 = 22$. They match, so $x = 4$ is correct.

KEY FORMULAS

1. Simplify each side 2. Variable terms to one side 3. Undo $+$ / $-$ 4. Undo $\times$ / $\div$ 5. Check

$$a(b + c) = ab + ac$$

> **Practice habit.** Recite the five-step plan until it is automatic: "Simplify each side, variables to one side, undo plus or minus, undo times or divide, check." Before you start, glance at each side and ask, "Can this side be simplified first?"

WORKED EXAMPLES

Example 1. Solve $4x + 3x - 5 = 23$.

Show solution

1. Combine like terms on the left: $4x + 3x = 7x$, so $7x - 5 = 23$.
2. Add 5 to both sides: $7x = 28$.
3. Divide both sides by 7: $x = 4$.
4. Check: $4 \cdot 4 + 3 \cdot 4 - 5 = 16 + 12 - 5 = 23$. True.

Answer: $x = 4$

Example 2. Ruth has \$50 and saves \$6 each week. Her brother Henry has \$86 and saves \$3 each week. The equation $50 + 6w = 86 + 3w$ tells when they will have the same amount of money. Solve it and say how much each will have then.

Show solution

1. Subtract $3w$ from both sides: $50 + 3w = 86$.
2. Subtract 50 from both sides: $3w = 36$.
3. Divide both sides by 3: $w = 12$.
4. Check the left side: $50 + 6 \cdot 12 = 50 + 72 = 122$.
5. Check the right side: $86 + 3 \cdot 12 = 86 + 36 = 122$. Both sides match.
6. Answer in a sentence: after 12 weeks, Ruth and Henry will each have \$122.

Answer: After 12 weeks; each will have \$122.

PRACTICE

1. Easy Solve $2x + 5x = 42$.

   Show answer

   $x = 6$

2. Easy Solve $3(x + 2) = 21$.

   Show answer

   $x = 5$

3. Easy Solve $6y - 2y + 1 = 17$.

Show answer

$y = 4$

4. Easy Solve $5x = 2x + 18$.

Show answer

$x = 6$

5. Medium Solve $4(n - 3) = 20$.

Show answer

$n = 8$

6. Medium Solve $7x + 3 = 4x + 24$.

Show answer

$x = 7$

7. Medium Solve $2(x + 5) + 3x = 35$.

Show answer

$x = 5$

8. Medium The perimeter of a picture frame is given by $2(w + 4) + 2w = 40$, where w is the width in inches. Solve for the width.

Show answer

$4w + 8 = 40$, so $w = 8$. The frame is 8 inches wide.

9. Medium Grace bought 4 bags of flour. Ruth bought 2 bags of the same flour and a \$9 bag of sugar. They spent the same amount, so $4f = 2f + 9$, where f is the price of one bag of flour in dollars. Solve for the price of a bag of flour.

Show answer

$2f = 9$, so $f = 4.50$. A bag of flour costs \$4.50.

10. Hard Solve $3(2x - 1) = 2(x + 5) + 7$.

Show answer

$x = 5$

11. Hard Solve $5 - 2(x - 4) = 3x - 2$.

Show answer

$x = 3$

12. Hard For a church dinner, Hall A charges \$150 plus \$8 per guest, and Hall B charges \$90 plus \$11 per guest. Solve $150 + 8g = 90 + 11g$ to find the number of guests for which the two halls cost the same.

Show answer

$60 = 3g$, so $g = 20$. At 20 guests both halls cost \$310.

13. Hard Henry solved $4(x - 3) = 20$ by first writing $4x - 3 = 20$. Explain his mistake and solve the equation correctly.

Show answer

The 4 must multiply both terms inside the parentheses: $4(x - 3) = 4x - 12$, not $4x - 3$. Then $4x - 12 = 20$, $4x = 32$, and $x = 8$. Check: $4(8 - 3) = 4 \cdot 5 = 20$.

14. Challenge Solve $0.2(x + 10) = 0.5x - 1$.

Show answer

$x = 10$

COMMON MISTAKES

- Distributing to only the first term: $3(x + 4)$ is $3x + 12$, not $3x + 4$. Draw arrows from the outside number to every term inside.
- Losing a sign when distributing a negative. In $5 - 2(x - 4)$, the -2 multiplies both terms: $-2x + 8$. So the side becomes $13 - 2x$.
- Combining terms across the equal sign, such as adding $5x$ on the left to $2x$ on the right. Terms on different sides can only be moved by doing the same operation to both sides.
- Checking in the simplified equation instead of the original one. Substitute into the very first equation, and work out each side separately.

Make it a game · Both Sides Showdown. Write ten multi-step equations, some with the variable on both sides. Take turns drawing a card and solving it on paper, simplifying each side first. The first correct finisher with a checked solution takes the card. Parent judges. Whoever has the most cards after the pile is gone is the Equation Champion.

10.6 Writing Equations for Word Problems

OBJECTIVES

- Define a variable, write an equation from a word problem, solve it, check it and answer in a complete sentence.
- Write expressions for related amounts, such as consecutive integers, ages now and later, and the sides of a rectangle.
- Solve word problems about consecutive numbers, ages, money and perimeter.

WATCH

Equations: Interpreting & Solving Word Problems

Eddie Woo · 8:36

https://www.youtube.com/watch?v=cjzlZuqP5cU

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Solving Equation Problems (3 of 3: Translating to an equation)

Eddie Woo · 6:01

https://www.youtube.com/watch?v=C65vdjQEl_c

Before the practice set: translating sentences into equations.

KEY TERMS

define a variable

To say in words exactly what your letter stands for, including units, such as "Let w = the width in feet."

consecutive integers

Integers that follow one after another, such as 23, 24, 25. If the first is n, the next ones are $n + 1$ and $n + 2$.

consecutive even (or odd) integers

Even (or odd) integers in a row, such as 26, 28, 30 or 7, 9, 11. Each is 2 more than the one before: n, $n + 2$, $n + 4$.

CORE IDEAS

Word problems become manageable when you follow the same five steps every time: **(1) Define** a variable in words. **(2) Write** the other unknown amounts in terms of that variable. **(3) Write an equation** that says two things are equal. **(4) Solve** it. **(5) Check** your answer in the *story*, not just the equation, and **answer in a sentence** with units.

Choose the variable for the amount that the other amounts are built from. If "the length is 3 feet more than twice the width," let w be the width; then the length is $2w + 3$. Use the translation skills from Lesson 9.3: "more than" means add, "times" or "twice" means multiply, "is" means equals.

Consecutive integers. If the first integer is n, the next two are $n + 1$ and $n + 2$, because each is 1 more than the one before. Consecutive *even* integers, like 26, 28, 30, go up by 2, so they are n, $n + 2$, $n + 4$. Consecutive *odd* integers, like 7, 9, 11, also go up by 2, so they are written the same way: n, $n + 2$, $n + 4$.

Ages. Everyone gets older at the same rate. If Samuel is s years old now, in 12 years he will be $s + 12$, and 5 years ago he was $s - 5$. Add the same number of years to every person in the problem.

Checking in the story catches mistakes that checking in the equation cannot, such as an equation set up wrong. Also ask whether the answer makes sense: a number of people or coins must be a whole number, and a length must be positive.

KEY FORMULAS

Consecutive integers: $n,\ n + 1,\ n + 2$ Consecutive even or odd integers: $n,\ n + 2,\ n + 4$

$$\text{Perimeter of a rectangle: } P = 2l + 2w$$

$$\text{Value of coins} = (\text{number of coins}) \times (\text{value of one coin})$$

From history. Word problems are very old. Al-Khwarizmi's book on algebra, written in Baghdad around 820, used its methods on practical problems of trade and inheritance, and Fibonacci's *Liber Abaci* (1202) filled many chapters with problems about buying, selling and sharing. Like you, those readers turned a story into an equation and then solved it.

Practice habit. Recite the five steps: "Define, write, solve, check in the story, answer in a sentence." Before solving, make a quick estimate so you can tell if your answer is reasonable.

WORKED EXAMPLES

Example 1. The sum of three consecutive integers is 72. Find the integers.

Show solution

1. Define: let n = the first integer. The next two are $n + 1$ and $n + 2$.
2. Write the equation: $n + (n + 1) + (n + 2) = 72$.
3. Combine like terms: $3n + 3 = 72$.
4. Subtract 3 from both sides: $3n = 69$.
5. Divide both sides by 3: $n = 23$. The integers are 23, 24 and 25.
6. Check in the story: they are consecutive, and $23 + 24 + 25 = 72$. True.
7. Answer: the integers are 23, 24 and 25.

Answer: 23, 24 and 25

Example 2. Samuel's father is 3 times as old as Samuel. In 12 years, his father will be twice as old as Samuel. How old are they now?

Show solution

1. Define: let s = Samuel's age now, in years. His father's age now is $3s$.
2. In 12 years: Samuel will be $s + 12$ and his father will be $3s + 12$.
3. Write the equation (father then = twice Samuel then): $3s + 12 = 2(s + 12)$.
4. Distribute: $3s + 12 = 2s + 24$.
5. Subtract $2s$ from both sides: $s + 12 = 24$.
6. Subtract 12 from both sides: $s = 12$. His father is $3 \cdot 12 = 36$.
7. Check in the story: 36 is 3 times 12. In 12 years they will be 24 and 48, and 48 is twice 24. True.
8. Answer in a sentence: Samuel is 12 years old and his father is 36 years old.

Answer: Samuel is 12 years old and his father is 36.

PRACTICE

1. Easy A number increased by 15 is 42. Write and solve an equation to find the number.

   Show answer

   $n + 15 = 42$, so $n = 27$.

2. Easy Five times a number, minus 8, is 37. Write and solve an equation to find the number.

   Show answer

   $5n - 8 = 37$, so $n = 9$.

3. Easy The sum of two consecutive integers is 55. Find the integers.

   Show answer

   $n + (n + 1) = 55$, so $n = 27$. The integers are 27 and 28.

4. Easy Grace bought a notebook for \$3.50 and some pens for \$1.25 each. She spent \$8.50 in all. Write and solve an equation to find how many pens she bought.

   Show answer

   $3.50 + 1.25p = 8.50$, so $1.25p = 5$ and $p = 4$. She bought 4 pens.

5. Medium The sum of three consecutive even integers is 84. Find the integers.

   Show answer

   $n + (n + 2) + (n + 4) = 84$, so $3n + 6 = 84$ and $n = 26$. The integers are 26, 28 and 30.

6. Medium Ruth is 4 years older than her brother Henry. The sum of their ages is 30. How old is each? Give Henry's age first.

   Show answer

   $h + (h + 4) = 30$, so $h = 13$. Henry is 13 and Ruth is 17.

7. Medium The youth group raised \$480 for a food pantry. They received \$120 in donations and charged \$8 for each car they washed. Write and solve an equation to find how many cars they washed.

   Show answer

   $8c + 120 = 480$, so $8c = 360$ and $c = 45$. They washed 45 cars.

8. Medium A rectangular pen is 3 feet longer than twice its width. Its perimeter is 54 feet. Find the width and the length.

Show answer

$2w + 2(2w + 3) = 54$, so $6w + 6 = 54$ and $w = 8$. The width is 8 feet and the length is 19 feet.

9. Hard On a two-day trip, the Miller family drove 85 miles farther on the second day than on the first. They drove 645 miles in all. How far did they drive each day? Give the first day first.

Show answer

$d + (d + 85) = 645$, so $2d = 560$ and $d = 280$. They drove 280 miles the first day and 365 miles the second day.

10. Hard Samuel has only quarters and dimes, worth \$2.75 in all. He has 3 more dimes than quarters. How many of each coin does he have? Give the quarters first.

Show answer

$0.25q + 0.10(q + 3) = 2.75$, so $0.35q + 0.30 = 2.75$, $0.35q = 2.45$ and $q = 7$. He has 7 quarters and 10 dimes.

11. Hard A 12-foot board is cut into two pieces. The longer piece is 2 feet more than 3 times the shorter piece. How long is each piece? Give the shorter piece first.

Show answer

$x + (3x + 2) = 12$, so $4x = 10$ and $x = 2.5$. The pieces are 2.5 feet and 9.5 feet long.

12. Hard Explain why three consecutive odd integers are written n, $n + 2$, $n + 4$ and not n, $n + 1$, $n + 3$.

Show answer

Odd integers in a row, like 7, 9, 11, are 2 apart, so each one is 2 more than the one before: n, $n + 2$, $n + 4$. Adding 1 to an odd number gives an even number, so $n + 1$ would not be odd.

13. **Challenge** Grace's mother is 26 years older than Grace. In 4 years, her mother will be 3 times as old as Grace. How old is each now? Give Grace's age first.

Show answer

$(g + 26) + 4 = 3(g + 4)$, so $g + 30 = 3g + 12$, $18 = 2g$ and $g = 9$. Grace is 9 and her mother is 35. (In 4 years: 13 and 39, and $39 = 3 \cdot 13$.)

COMMON MISTAKES

- Not defining the variable, then answering the wrong question. Write "Let w = the width in feet" first, and reread the question before you write the final sentence.
- Adding the years to only one person in an age problem. If 4 years pass, add 4 to *every* person's age.
- Writing consecutive even or odd integers as n, $n + 1$, $n + 2$. Even and odd integers in a row are 2 apart: n, $n + 2$, $n + 4$.
- Mixing up number of coins with value of coins. Seven quarters is worth $7 \times 0.25 = 1.75$ dollars, not 7 dollars.

> **Make it a game · Word Problem Theater.** Parent reads a short word problem, such as consecutive integers adding to 57. You stand, declare 'Let x equal...', write the equation, solve and check, then answer in a complete sentence, like a stage announcement. Parent invents the next problem. Score a point for each fully correct sequence of define, write, solve, check and answer.

10.7 Inequalities

OBJECTIVES

- Write inequalities from words and graph them on a number line using open and closed circles.
- Solve one-step and two-step inequalities and check the solution.
- Explain why multiplying or dividing both sides by a negative number reverses the inequality sign.

WATCH

Algebra Basics: Inequalities In Algebra - Math Antics

mathantics · 14:11

https://www.youtube.com/watch?v=RyesLifeUBw

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Math Antics - Basic Inequalities

mathantics · 12:32

https://www.youtube.com/watch?v=mgHO-bsCDrA

Before the Core ideas, if the inequality symbols are new.

KEY TERMS

inequality

A math sentence that compares two amounts with $<$, $>$, $\leq$ or $\geq$, such as $x + 7 \leq 12$.

inequality symbols

$<$ means "is less than," $>$ means "is greater than," $\leq$ means "is less than or equal to" (at most), and $\geq$ means "is greater than or equal to" (at least).

solution of an inequality

Any number that makes the inequality true. An inequality usually has infinitely many solutions, such as every number less than or equal to 5.

open circle / closed circle

On a number-line graph, an open (hollow) circle means the endpoint is *not* included ($<$ or $>$); a closed (filled) circle means it *is* included ($\leq$ or $\geq$).

CORE IDEAS

An **inequality** compares two amounts that may not be equal. "You must be at least 48 inches tall to ride" is $h \geq 48$. "A bag may weigh at most 50 pounds" is $w \leq 50$. Unlike an equation, an inequality usually has many solutions: every height of 48 inches or more works.

To **graph** an inequality on a number line, put a circle at the boundary number and shade the side that makes it true. For $x < 3$, use an **open circle** at 3 (3 itself is not a solution) and shade to the left. For $x \geq -2$, use a **closed circle** at -2 (it is included) and shade to the right. Tip: when the variable is on the left, the symbol points the way you shade.

You **solve** an inequality almost exactly like an equation. Adding or subtracting the same number on both sides keeps it true: from $x + 7 \leq 12$, subtract 7 to get $x \leq 5$. Multiplying or dividing both sides by the same **positive** number also keeps it true: from $4x \geq 28$, divide by 4 to get $x \geq 7$.

The one new rule: when you multiply or divide both sides by a **negative** number, you must **reverse** the inequality sign. **Why?** Start with the true statement $2 < 5$. Multiply both sides by -1: you get -2 and -5. On the number line, -2 is to the right of -5, so $-2 > -5$. Multiplying by a negative flips every number to the other side of 0, which reverses their order. In debts and fortunes: a fortune of \$5 is better than a fortune of \$2, but a debt of \$5 is worse than a debt of \$2.

You can also see the rule without multiplying by a negative at all. Take $-3x < 12$. Add $3x$ to both sides: $0 < 12 + 3x$. Subtract 12 from both sides: $-12 < 3x$. Divide by positive 3: $-4 < x$, which is the same as $x > -4$. Dividing $-3x < 12$ by -3 and flipping the sign gives the same answer, $x > -4$.

Check an inequality by testing a number from your answer and a number that is not. For $x > -4$, test $x = 0$: $-3 \cdot 0 = 0 < 12$, true. Test $x = -5$: $-3 \cdot (-5) = 15 < 12$, false. Good, so the shading is on the correct side.

KEY FORMULAS

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}$$

$$\text{If } a < b \text{ and } c < 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}$$

$$\text{at least: } \geq \qquad \text{at most: } \leq \qquad \text{more than: } > \qquad \text{fewer than: } <$$

Practice habit. Say the flip rule as you work: "Add or subtract anything, no flip. Multiply or divide by a positive, no flip. Multiply or divide by a negative, flip." Then test one number from the shaded side and one from the other side.

WORKED EXAMPLES

Example 1. Grace needs at least 50 points to earn a merit badge. She already has 32 points. Write and solve an inequality for the number of points p she still needs, and describe the graph.

Show solution

1. Points she has plus points she needs must be at least 50: $32 + p \geq 50$.
2. Subtract 32 from both sides: $p \geq 18$.
3. Graph: a closed circle at 18 (18 points is enough), shaded to the right.
4. Check: test $p = 20$: $32 + 20 = 52 \geq 50$, true. Test $p = 10$: $32 + 10 = 42 \geq 50$, false. The shading is correct.
5. Answer in a sentence: Grace needs 18 or more points.

Answer: $p \geq 18$; she needs at least 18 more points.

Example 2. Solve $-3x + 4 > 19$ and describe the graph.

Show solution

1. Subtract 4 from both sides: $-3x > 15$.
2. Divide both sides by -3. Dividing by a negative, so reverse the sign: $x < -5$.
3. Graph: an open circle at -5 (not included), shaded to the left.
4. Check: test $x = -6$: $-3 \cdot (-6) + 4 = 18 + 4 = 22 > 19$, true.
5. Test $x = 0$: $-3 \cdot 0 + 4 = 4 > 19$, false. The solution is correct.

Answer: $x < -5$

PRACTICE

1. Easy Write an inequality for "x is at most 10."

   Show answer

   $x \le 10$

2. Easy Is $x = 4$ a solution of $2x + 1 > 9$? Answer yes or no.

   Show answer

   No ($2 \cdot 4 + 1 = 9$, and 9 is not greater than 9)

3. Easy Solve $x - 6 < 3$.

   Show answer

   $x < 9$

4. Easy Describe the number-line graph of $x \ge -2$.

   Show answer

   A closed circle at -2, shaded to the right.

5. Medium Solve $4x \ge 28$.

   Show answer

   $x \ge 7$

6. Medium Solve $-5x < 30$.

   Show answer

   $x > -6$

7. Medium Solve $\frac{n}{-2} \le 3$.

   Show answer

   $n \ge -6$

8. Medium A farm truck can safely carry at most 2,000 pounds. Each sack of feed weighs 50 pounds. Write and solve an inequality for the number of sacks s the truck can carry.

   Show answer

   $50s \le 2000$, so $s \le 40$. The truck can carry at most 40 sacks.

9. Hard Solve $3 - 2x \geq 11$.

Show answer

$x \leq -4$

10. Hard Henry has \$40. He wants to buy a book for \$12 and some packs of baseball cards for \$3.50 each. Write and solve an inequality for the number of packs c he can buy.

Show answer

$12 + 3.50c \leq 40$, so $3.50c \leq 28$ and $c \leq 8$. He can buy at most 8 packs.

11. Hard The church choir needs at least \$600 for a trip. They have \$180 and are selling candles for \$6 each. Write and solve an inequality for the number of candles c they must sell.

Show answer

$180 + 6c \geq 600$, so $6c \geq 420$ and $c \geq 70$. They must sell at least 70 candles.

12. Hard Use the true statement $2 < 6$ to explain why dividing both sides of an inequality by a negative number reverses the sign.

Show answer

Divide both sides of $2 < 6$ by -2: you get -1 and -3. On the number line -1 is to the right of -3, so $-1 > -3$. Dividing by a negative moves each number to the other side of 0, which reverses their order, so the sign must flip to keep the statement true.

13. Challenge Solve $\frac{x}{-3} + 4 > 1$.

Show answer

$x < 9$ (subtract 4: $\frac{x}{-3} > -3$; multiply by -3 and reverse the sign)

COMMON MISTAKES

- Forgetting to reverse the sign when multiplying or dividing by a negative. Circle the negative number before you divide, and test a number at the end.
- Reversing the sign just because a negative number appears. In $x + 6 < 2$, subtracting 6 gives $x < -4$ with no flip; you added and subtracted, but you did not multiply or divide by a negative.

- Using the wrong circle. Use an open circle for $<$ and $>$, and a closed circle for $\leq$ and $\geq$.
- Mixing up "at least" and "at most." "At least 70" means 70 or more ($\geq$); "at most 40" means 40 or fewer ($\leq$).

> **Make it a game · Open or Closed.** Draw a number line on a paper strip. Parent calls an inequality in words, such as 'x is at least 3'. You write it with symbols and mark the line with an open or closed circle, and shade the correct direction. Then solve two-step inequalities from cards. Score a point for each correct graph. Beat ten points.

UNIT 10

Equations and Inequalities

In Unit 9 you learned to write and simplify expressions. In this unit you put two expressions on opposite sides of an equal sign and ask, "What number makes this true?" You will picture every equation as a balance scale and solve it by doing the same thing to both sides, undoing what was done to the variable one step at a time.

You will start with one-step equations, move to two-step and multi-step equations (including the distributive property and variables on both sides), and then turn word problems into equations: consecutive numbers, ages, money and perimeter. The unit ends with inequalities, where the answer is a whole range of numbers and where multiplying or dividing by a negative number reverses the sign.

Solving equations is the heart of Algebra I. Parents can help by asking one question after every problem: "Did you check it?" Substituting the answer back into the original equation is the habit that matters most in this unit.

PREREQUISITES

- Add, subtract, multiply and divide integers, fractions and decimals (Units 3–5).
- Use the order of operations (Lesson 1.4).
- Multiply by a reciprocal to divide by a fraction (Lesson 3.6).
- Evaluate expressions by substitution (Lesson 9.2).
- Translate words into expressions (Lesson 9.3).
- Combine like terms and use the distributive property with variables (Lessons 9.4 and 9.5).

MASTERY CHECKLIST

- ☐ I can explain an equation using the picture of a balance scale.
- ☐ I can check whether a number is a solution of an equation by substituting it.
- ☐ I can solve one-step equations that use addition or subtraction.
- ☐ I can solve one-step equations that use multiplication or division, including fraction coefficients, by using the reciprocal.
- ☐ I can solve two-step equations by undoing the operations in reverse order.
- ☐ I can solve multi-step equations by combining like terms, distributing and moving variable terms to one side.
- ☐ I can define a variable, write an equation for a word problem, solve it, check it and answer in a sentence.
- ☐ I can solve word problems about consecutive integers, ages, money and perimeter.
- ☐ I can write and graph inequalities on a number line using open and closed circles.
- ☐ I can solve one- and two-step inequalities and explain why the sign reverses when I multiply or divide by a negative number.

LESSONS IN THIS UNIT

10.1	**Equations and Solutions**	45 minutes
10.2	**One-Step Equations with Addition and Subtraction**	45 minutes
10.3	**One-Step Equations with Multiplication and Division**	2 sessions of 45 minutes
10.4	**Two-Step Equations**	2 sessions of 45 minutes
10.5	**Multi-Step Equations**	2 sessions of 45 minutes
10.6	**Writing Equations for Word Problems**	2 sessions of 45 minutes
10.7	**Inequalities**	2 sessions of 45 minutes

Unit 10 test

You have 60 minutes. You may use a basic calculator for decimal arithmetic. Show every step of your work, and check each equation by substituting your answer. For word problems, define your variable, write an equation or inequality, and answer in a complete sentence with units. Give exact answers unless a problem says how to round.

1. **5 pts**
   Is $x = 3$ a solution of $5x - 4 = 11$? Answer yes or no and show your check.

2. **5 pts**
   Solve $x - 13 = -4$.

3. **7 pts**
   After spending $27.85 on groceries, Ruth had $42.15 left. Write and solve an equation to find how much money she started with.

4. **5 pts**
   Solve $-7y = 56$.

5. **6 pts**

   Solve $\frac{3}{4}x = 27$.

6. **6 pts**

   Solve $4x - 9 = 23$.

7. **8 pts**

   A plumber charges \$45 for a house call plus \$60 per hour. The Millers' bill was \$255. Write and solve an equation to find how many hours the plumber worked.

8. **7 pts**

   Solve $3(x - 2) + 4x = 29$.

9. **7 pts**

   Solve $6x - 5 = 2x + 19$.

10. **8 pts**

    The sum of three consecutive odd integers is 81. Find the integers.

11. **8 pts**

    A rectangular vegetable garden is 5 feet longer than it is wide. Its perimeter is 62 feet. Find the width and the length. Give the width first.

12. **6 pts**

    Solve $x + 4 > 1$ and describe its graph on a number line.

13. **7 pts**

    Solve $5 - 3x \le 20$.

14. **8 pts**

    Samuel wants to save at least \$250 for a mission trip. He has \$34 and earns \$9 per hour mowing lawns. Write and solve an inequality for the number of hours h he must work.

15. **7 pts**

    Solve $-2x < 10$. Then explain why the inequality sign must be reversed in your last step.

ANSWER KEY

1. Yes ($5 \cdot 3 - 4 = 15 - 4 = 11$)
2. $x = 9$
3. $m - 27.85 = 42.15$, so $m = 70$. Ruth started with \$70.00.
4. $y = -8$
5. $x = 36$
6. $x = 8$
7. $45 + 60h = 255$, so $60h = 210$ and $h = 3.5$. The plumber worked $3\frac{1}{2}$ hours.
8. $x = 5$
9. $x = 6$
10. $n + (n + 2) + (n + 4) = 81$, so $3n + 6 = 81$ and $n = 25$. The integers are 25, 27 and 29.
11. $2w + 2(w + 5) = 62$, so $4w + 10 = 62$ and $w = 13$. The garden is 13 feet wide and 18 feet long.
12. $x > -3$; an open circle at -3, shaded to the right.
13. $x \geq -5$
14. $34 + 9h \geq 250$, so $9h \geq 216$ and $h \geq 24$. He must work at least 24 hours.
15. $x > -5$. You divide both sides by -2. Dividing by a negative moves each number to the other side of 0 on the number line, which reverses their order (for example, $2 < 6$ but $-1 > -3$), so the sign must flip to keep the statement true. Check: $x = 0$ gives $0 < 10$, true; $x = -6$ gives $12 < 10$, false.

UNIT 11

The Coordinate Plane and Graphing

In this unit you join numbers and pictures. Two number lines crossing at right angles make the coordinate plane, where every point has an address, an ordered pair (x, y). You will plot points, find distances along grid lines, turn input-output tables into rules, and graph equations as straight lines.

Next you will measure how steep a line is with its slope, rise over run, and see that slope is a rate of change: dollars per week, miles per hour, gallons per minute. Finally you will read the slope and starting value right out of an equation $y = mx + b$, and meet direct variation, $y = kx$.

This unit uses the integers of Unit 5 and the equations of Unit 10, and it leads straight into Algebra I, where graphing lines is a central skill. In Unit 12 you will use the coordinate plane again to find distances with the Pythagorean theorem.

PREREQUISITES

- Compare and order integers on a number line (Lesson 5.1).
- Add, subtract, multiply and divide integers (Lessons 5.2–5.5).
- Evaluate expressions by substitution, including negative values (Lesson 9.2).
- Solve one- and two-step equations and check by substitution (Lessons 10.2–10.4).
- Write and simplify ratios and rates (Lessons 7.1–7.2).

MASTERY CHECKLIST

- ☐ I can plot a point from its ordered pair and name the coordinates of a plotted point.
- ☐ I can tell which quadrant or axis a point lies in from the signs of its coordinates.
- ☐ I can find the distance between two points on the same horizontal or vertical line.
- ☐ I can complete an input-output table from a rule and plot its points.
- ☐ I can write a rule from a table and check it with every pair.
- ☐ I can graph a linear equation by making a table of at least three points.
- ☐ I can decide whether a point is on a line by substitution.
- ☐ I can find the slope of a line from a graph or from two points.
- ☐ I can tell positive, negative, zero and undefined slopes apart and explain why a vertical line's slope is undefined.
- ☐ I can explain slope as a rate of change with units.
- ☐ I can read the slope and y-intercept from

$y =$ and graph the line from them.
$mx +$
b

- ☐ I can write equations for fee-plus-rate problems and direct variation

problems and use them to answer questions.

LESSONS IN THIS UNIT

11.1 The Coordinate Plane	45 minutes
11.2 Tables, Rules and Graphs	2 sessions of 45 minutes
11.3 Graphing Linear Equations	2 sessions of 45 minutes
11.4 Slope	2 sessions of 45 minutes
11.5 Slope-Intercept Form and Direct Variation	2 sessions of 45 minutes

Unit 11 test

You have 50 minutes. Have graph paper, a pencil and a ruler ready. A basic calculator is allowed but should rarely be needed. Show your work for every problem: tables, substitutions and the rise and run you used. Write slopes that are not whole numbers as fractions in lowest terms or as exact decimals, and give money to the nearest cent. Include units in word-problem answers.

1. **5 pts**
   In which quadrant is the point $(-7, -2)$?

2. **4 pts**
   Find the distance between $(-3, 6)$ and $(-3, -4)$.

3. **6 pts**
   On a county map, each grid unit stands for 1 mile. Henry's house is at $(-5, 2)$ and the farm stand is at $(7, 2)$, on the same straight road. How many miles apart are they?

4. **6 pts**
   Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 1, 4, 7, 10.

5. **7 pts**

   For a church potluck, square tables are pushed together in a long row. One table seats 6 people, two tables seat 10, and three tables seat 14. How many people can sit at a row of 12 tables?

6. **6 pts**

   Is the point $(-3, 11)$ on the line $y = -4x - 1$? Answer yes or no and show your check.

7. **7 pts**

   Make a table for $y = 2x - 3$ using the x-values -1, 0, 1 and 2, then graph the line.

8. **7 pts**

   At what point does the line $y = -3x + 12$ cross the x-axis?

9. **7 pts**

   Find the slope of the line through $(-2, 7)$ and $(4, -5)$.

10. **7 pts**

    Write the equation of the line through $(0, -4)$ and $(2, 2)$ in slope-intercept form.

11. **7 pts**

    On a family trip, the car travels at a steady speed. After 2 hours it has gone 110 miles, and after 5 hours it has gone 275 miles. Find the rate of change of the distance.

12. **7 pts**

    Give the slope and the y-intercept b of $y = -\dfrac{3}{4}x + 2$.

13. **8 pts**

    A garden tiller rents for a $15 fee plus $8 per hour. Write an equation for the cost y of renting it for x hours, and find the cost of renting it for 5 hours.

14. **8 pts**

    The cost of apples varies directly with their weight. Four pounds cost $6. Write a direct variation equation for the cost y of x pounds, and find the cost of 11 pounds.

15. **8 pts**

Explain why the slope of every horizontal line is 0, while the slope of every vertical line is undefined.

ANSWER KEY

1. Quadrant III
2. 10 units
3. 12 miles
4. $y = 3x - 2$
5. 50 people (rule $y = 4x + 2$)
6. Yes ($-4 \times (-3) - 1 = 12 - 1 = 11$)
7. -5, -3, -1, 1; points $(-1, -5)$, $(0, -3)$, $(1, -1)$, $(2, 1)$
8. $(4, 0)$
9. -2
10. $y = 3x - 4$
11. 55 miles per hour
12. Slope $-\frac{3}{4}$, $b = 2$
13. $y = 8x + 15$; \$55
14. $y = 1.5x$; \$16.50
15. Between two points on a horizontal line the rise is 0, and 0 divided by a nonzero run is 0. Between two points on a vertical line the run is 0, so the slope would mean dividing by 0, which has no answer. So the slope is undefined.

11.1 The Coordinate Plane

OBJECTIVES

- Plot points and name the coordinates of points on the coordinate plane.
- Identify the four quadrants and tell which quadrant or axis a point lies in from the signs of its coordinates.
- Find the distance between two points that lie on the same horizontal or vertical line.

WATCH

Algebra Basics: Graphing On The Coordinate Plane - Math Antics

mathantics · 10:14

https://www.youtube.com/watch?v=9Uc62CuQjc4

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'Cartesian plane' or 'number plane' for the coordinate plane.

BACKUP VIDEO

Introduction to the Cartesian Plane

Eddie Woo · 6:01

https://www.youtube.com/watch?v=qcvLjX5Npyk

Before the practice set.

KEY TERMS

coordinate plane

A flat surface made by two number lines that cross at right angles. Every point on it can be named by a pair of numbers.

x-axis and y-axis

The x-axis is the horizontal (left-right) number line. The y-axis is the vertical (up-down) number line.

origin

The point where the two axes cross. Its coordinates are $(0, 0)$.

ordered pair

A pair of numbers (x, y) that gives the location of a point. The first number, the x-coordinate, tells how far to move left or right; the second, the y-coordinate, tells how far to move up or down.

quadrant

One of the four regions the axes divide the plane into. They are numbered I, II, III and IV, going counterclockwise and starting at the upper right.

CORE IDEAS

You already know the number line: one line with zero in the middle, positive numbers to the right and negative numbers to the left. Now take a second number line, stand it straight up, and let it cross the first one at zero. Up is positive and down is negative. These two lines are the **axes**, and together they make the **coordinate plane**. The point where they cross is the **origin**, $(0, 0)$.

Every point has an address called an **ordered pair**, written (x, y). To plot $(4, -3)$, start at the origin, move 4 units right (because $x = 4$), then 3 units down (because $y = -3$). Always move along the x-axis first, then up or down. The order matters: $(4, -3)$ and $(-3, 4)$ are two different points, which is why we call it an *ordered* pair.

The axes split the plane into four **quadrants**. The signs of the coordinates tell you which one a point is in. Quadrant I (upper right): $(+, +)$. Quadrant II (upper left): $(-, +)$. Quadrant III (lower left): $(-, -)$. Quadrant IV (lower right): $(+, -)$. A point with a 0 coordinate, such as $(0, -6)$ or $(5, 0)$, lies on an axis and is not in any quadrant.

Distance on a horizontal or vertical line. If two points have the same y-coordinate, they sit on the same horizontal line, and the distance between them is just the distance between their x-coordinates on a number line. For $(-4, 3)$ and $(5, 3)$, that is $5 - (-4) = 9$ units. You can also count: 4 units from -4 to 0, then 5 more to 5, for 9 in

all. If two points have the same x-coordinate, they are on a vertical line, so subtract the y-coordinates instead.

Why subtract? Subtracting finds how far apart two numbers are on a number line. Subtract the smaller number from the larger, or take the absolute value of the difference, so a distance is never negative. Distance is a length, and lengths are always positive or zero.

KEY FORMULAS

$$\text{Same } y\text{-coordinate (horizontal line): distance} = |x_2 - x_1|$$

$$\text{Same } x\text{-coordinate (vertical line): distance} = |y_2 - y_1|$$

$$\text{I} : (+, +) \qquad \text{II} : (-, +) \qquad \text{III} : (-, -) \qquad \text{IV} : (+, -)$$

From history. In 1637 the French mathematician and philosopher René Descartes published *La Géométrie*. In it he showed how points and curves can be described with numbers and equations, so that problems in geometry could be solved with algebra. This joining of algebra and geometry is why the coordinate plane is often called the *Cartesian* plane, after Descartes.

Practice habit. Recite the quadrant signs until they are automatic: "One, plus-plus; two, minus-plus; three, minus-minus; four, plus-minus." When plotting, say "x first: walk, then climb" so you always move left or right before up or down.

WORKED EXAMPLES

Example 1. Describe how to plot the point $P(-4, 3)$, starting at the origin, and name the quadrant it is in.

Show solution

1. The x-coordinate is -4, so move 4 units to the left along the x-axis.
2. The y-coordinate is 3, so from there move 3 units up.
3. Mark the point and label it P.
4. The signs are $(-, +)$: left of the y-axis and above the x-axis. That is Quadrant II.

Answer: Move 4 left and 3 up; P is in Quadrant II.

Example 2. On a map of the Miller farm, each unit on the grid stands for 10 feet. The barn is at $(-3, -2)$ and the well is at $(-3, 6)$. How far apart are the barn and the well?

Show solution

1. The x-coordinates are both -3, so the two points lie on the same vertical line.
2. Subtract the y-coordinates: $6 - (-2) = 6 + 2 = 8$ units.
3. Check by counting: 2 units up from -2 to the x-axis, then 6 more units up to 6. That is $2 + 6 = 8$ units.
4. Each unit is 10 feet, so the distance is $8 \times 10 = 80$ feet.

Answer: 80 feet

PRACTICE

1. Easy In which quadrant is the point $(-4, 7)$?

   Show answer

   Quadrant II

2. Easy In which quadrant is the point $(5, -3)$?

   Show answer

   Quadrant IV

3. Easy Start at the origin. Move 3 units left and then 2 units up. What ordered pair names the point where you end?

   Show answer

   $(-3, 2)$

4. Easy The point $(0, -6)$ is not in any quadrant. On which axis does it lie?

   Show answer

   On the y-axis, 6 units below the origin.

5. Medium Find the distance between $(2, 5)$ and $(9, 5)$.

   Show answer

   7 units

6. **Medium** Find the distance between $(-4, 3)$ and $(-4, -8)$.

Show answer

11 units

7. **Medium** Find the distance between $(-6, -2)$ and $(3, -2)$.

Show answer

9 units

8. **Medium** Point A is at $(3, 4)$. Grace moves it 5 units to the left and 6 units down. What are the new coordinates of A?

Show answer

$(-2, -2)$

9. **Hard** On a town map, each grid unit is one city block. The church is at $(-3, 4)$ and the library is at $(-3, -6)$, on the same street. How many blocks apart are they?

Show answer

10 blocks

10. **Hard** The Miller family is fencing a rectangular goat pen. On their plan, each unit is 1 meter, and the corners of the pen are at $(-2, 3)$, $(5, 3)$, $(5, -1)$ and $(-2, -1)$. How many meters of fence do they need to go all the way around?

Show answer

22 meters (the pen is 7 m by 4 m)

11. **Hard** Explain why the order of the numbers in an ordered pair matters. Use the points $(2, 5)$ and $(5, 2)$ in your answer.

Show answer

The first number always tells how far to move left or right, and the second tells how far to move up or down. $(2, 5)$ means 2 right and 5 up, but $(5, 2)$ means 5 right and 2 up, so they are two different points. If we did not agree on the order, one pair of numbers could name two places.

12. **Challenge** Ruth is planning a rectangular vegetable garden on a grid where each unit is 1 yard. Three corners are at $(-4, -2)$, $(6, -2)$ and $(6, 5)$. Find (a) the coordinates of the fourth corner and (b) the area of the garden in square yards.

    Show answer

    (a) $(-4, 5)$; (b) 70 square yards (10 yd by 7 yd)

13. **Challenge** Point P is at $(2, -3)$. Point Q is on the same vertical line as P and is 8 units away from it. Give both possible locations of Q.

    Show answer

    $(2, 5)$ or $(2, -11)$

COMMON MISTAKES

- Switching the coordinates, for example plotting $(4, -3)$ by going 4 up instead of 4 right. The x-coordinate always comes first and always means left or right.
- Naming the quadrants in the wrong order. Start at the upper right with I and go counterclockwise: I, II, III, IV.
- Subtracting coordinates without minding the signs, such as getting $6 - 2 = 4$ for the distance from -2 to 6. Write $6 - (-2) = 8$, or count across zero, to check.
- Saying a point like $(0, 4)$ is in Quadrant I. A point with a zero coordinate is on an axis, not in a quadrant.

> **Make it a game · Living Coordinate Plane.** Tape a large coordinate plane on the floor with tape or chalk, labeled from -5 to 5. Parent calls an ordered pair and you stand on it. Then name the quadrant. Parent plays too, and anyone who stands on the wrong point is out of the round. Try 15 calls, then call points for the parent to find.

11.2 Tables, Rules and Graphs

OBJECTIVES

- Complete an input-output table from a rule such as $y = 2x - 1$.
- Write a rule that fits every pair in a table and check it with each pair.
- Turn the pairs in a table into ordered pairs and plot them on the coordinate plane.

WATCH

Algebra Basics: What Are Functions? - Math Antics

mathantics · 11:34

https://www.youtube.com/watch?v=52tpYl2tTqk

> **WHEN TO WATCH**
> Before Example 1: input-output rules and function tables. It also shows f(x) notation, which is not used yet.

BACKUP VIDEO

Math Antics - Number Patterns

mathantics · 13:53

https://www.youtube.com/watch?v=vV7C7bXm4VI

Before Example 2: finding the rule for a number pattern.

KEY TERMS

input

The number you start with and put into a rule. In this lesson the input is usually called x.

output

The number that comes out after you apply the rule to the input. In this lesson the output is usually called y.

input-output table

A table that lists inputs and their matching outputs side by side.

rule

An equation that tells how to get the output from the input, such as $y = 3x + 2$.

CORE IDEAS

Think of a rule as a machine. You drop in an **input**, the machine does the same steps every time, and an **output** comes out. For the rule $y = 2x - 1$, an input of

4 gives $2 \times 4 - 1 = 7$. An **input-output table** keeps a record of several inputs and their outputs.

To **complete a table** from a rule, substitute each input for x and evaluate, just as you did in Unit 9. Use the order of operations, and be careful with negative inputs: for $y = -2x + 3$, the input -1 gives $-2 \times (-1) + 3 = 2 + 3 = 5$.

To **find a rule** from a table, look at how the outputs change when the input goes up by 1. If the outputs go up by 3 each time, the rule multiplies by 3, so it starts with $3x$. Then compare $3x$ to the real output to see what else must happen. If $x = 1$ gives an output of 5 and $3 \times 1 = 3$, you must add 2, so try $y = 3x + 2$.

Why check every pair? A rule can work for one pair by accident. The rule $y = x + 2$ turns 1 into 3, but if the table also says 2 gives 6, then $y = x + 2$ fails, because $2 + 2 = 4$, not 6. A rule is correct only when it works for *every* pair in the table.

Each row of a table is an ordered pair (x, y), so a table is really a list of points. **Plot the points** on the coordinate plane. You will often see them line up, which is a first look at the graphs of the next lesson. Plotting lets you *see* a pattern: points rising to the right mean the outputs grow as the inputs grow.

KEY FORMULAS

$$\text{input } x \longrightarrow \boxed{\text{rule}} \longrightarrow \text{output } y$$

If y goes up by m each time x goes up by 1, the rule begins $y = m \cdot x + \dots$

Practice habit. Check every rule by substitution: before you say a rule is right, put each input from the table into it and confirm you get each output. One pair that fails means the rule is wrong.

WORKED EXAMPLES

Example 1. Find a rule for this table, then list the points to plot. Inputs x: 0, 1, 2, 3. Outputs y: 2, 5, 8, 11.

Show solution

1. Each time x goes up by 1, y goes up by 3 ($5 - 2 = 3$, $8 - 5 = 3$, $11 - 8 = 3$). So the rule begins with $3x$.
2. Compare: at $x = 0$, $3x = 0$, but the output is 2. So add 2. Try $y = 3x + 2$.
3. Check $x = 1$: $3 \times 1 + 2 = 5$. Correct.
4. Check $x = 2$: $3 \times 2 + 2 = 8$. Correct.
5. Check $x = 3$: $3 \times 3 + 2 = 11$. Correct.
6. The points to plot are $(0, 2)$, $(1, 5)$, $(2, 8)$ and $(3, 11)$.

Answer: $y = 3x + 2$; points $(0, 2)$, $(1, 5)$, $(2, 8)$, $(3, 11)$

Example 2. Ruth has \$12 in her savings jar and adds \$5 every week. Make a table for weeks 0 through 4, write a rule for the amount y after x weeks, and use the rule to find how much she will have after 10 weeks.

Show solution

1. Week 0: \$12. Week 1: \$17. Week 2: \$22. Week 3: \$27. Week 4: \$32.
2. The amount goes up by 5 each week, so the rule begins with $5x$.
3. At $x = 0$ the amount is 12, so add 12: $y = 5x + 12$.
4. Check week 3: $5 \times 3 + 12 = 15 + 12 = 27$. Correct.
5. After 10 weeks: $y = 5 \times 10 + 12 = 50 + 12 = 62$.
6. Ruth will have \$62.

Answer: $y = 5x + 12$; \$62 after 10 weeks

PRACTICE

1. Easy The rule is $y = x + 4$. What is the output when the input is 7?

   Show answer

   11

2. Easy The rule is $y = 3x$. Find the outputs for the inputs 0, 1, 2 and 3.

Show answer

0, 3, 6, 9

3. Easy The rule is $y = 2x - 1$. Find the outputs for the inputs 1, 2, 3 and 4, and write the four ordered pairs you would plot.

Show answer

1, 3, 5, 7; points $(1, 1)$, $(2, 3)$, $(3, 5)$, $(4, 7)$

4. Easy Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 8, 9, 10, 11.

Show answer

$y = x + 7$

5. Medium Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 4, 8, 12, 16.

Show answer

$y = 4x$

6. Medium Find a rule for this table. Inputs x: 0, 1, 2, 3. Outputs y: 1, 4, 7, 10.

Show answer

$y = 3x + 1$

7. Medium Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 3, 5, 7, 9.

Show answer

$y = 2x + 1$

8. Medium At the church bake sale, each loaf of bread costs \$4. Write a rule for the cost y in dollars of x loaves, and use it to find the cost of 9 loaves.

Show answer

$y = 4x$; \$36

9. Hard Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 2, 7, 12, 17.

Show answer

$y = 5x - 3$

10. Hard The rule is $y = -2x + 3$. Find the outputs for the inputs -1, 0, 1 and 2.

Show answer

5, 3, 1, -1; points $(-1, 5)$, $(0, 3)$, $(1, 1)$, $(2, -1)$

11. Hard A candle is 12 inches tall and burns down 2 inches every hour. Write a rule for its height y in inches after x hours, and find its height after 4 hours.

Show answer

$y = 12 - 2x$; 4 inches

12. Hard For the table with inputs 1, 2, 3 and outputs 3, 6, 9, Grace says the rule is $y = x + 2$ because $1 + 2 = 3$. Explain what is wrong and give the correct rule.

Show answer

Her rule works only for the first pair. For $x = 2$ it gives $2 + 2 = 4$, but the table says 6. Each time x goes up by 1, y goes up by 3, and $y = 3x$ works for every pair: $3, 6, 9$.

13. Challenge Find a rule for this table, where the inputs go up by 2. Inputs x: 2, 4, 6, 8. Outputs y: 7, 13, 19, 25.

Show answer

$y = 3x + 1$ (the outputs go up 6 for every 2 steps in x, which is 3 per step)

14. Challenge Henry makes a row of squares out of toothpicks, with neighboring squares sharing a side. One square takes 4 toothpicks, two squares take 7, and three squares take 10. (a) How many toothpicks does a row of 25 squares take? (b) How many squares are in a row that uses exactly 100 toothpicks?

Show answer

(a) 76 toothpicks (rule $y = 3x + 1$); (b) 33 squares

COMMON MISTAKES

- Checking a rule with only the first pair. A rule must work for every row of the table, so test them all.
- Using the change in y as the multiplier when the inputs do not go up by 1. If x goes up by 2 and y goes up by 6, the multiplier is $6 \div 2 = 3$, not 6.
- Forgetting the starting amount in word problems. If Ruth starts with \$12 and adds \$5 a week, the rule is $y = 5x + 12$, not $y = 5x$.
- Plotting (y, x) instead of (x, y). The input always goes first in the ordered pair.

Make it a game · Mystery Rule Machine. One player is the machine, hiding a rule such as y = 2x - 1 on paper. The other calls out inputs, and the machine says the output. After four inputs, the guesser must write the rule and test it on every pair. Score a point if the rule fits all pairs. Swap roles and play four rounds.

11.3 Graphing Linear Equations

OBJECTIVES

- Graph a linear equation by making a table of at least three points, plotting them and drawing the line.
- Decide whether a point is on the graph of an equation by substituting its coordinates.
- Use the graph of an equation to answer questions about a real situation.

WATCH

Graphing Straight Lines (1 of 2: Interpreting equations as graphs)

Eddie Woo · 6:41

https://www.youtube.com/watch?v=0jlrgJrlt_w

> **WHEN TO WATCH**
> Before Example 1: making a table and plotting a straight line.

BACKUP VIDEO

Basic Linear Functions - Math Antics

mathantics · 13:24

https://www.youtube.com/watch?v=MXV65i9g1Xg

Preview for Lessons 11.4–11.5, any time after Example 2.

KEY TERMS

solution of an equation in two variables

An ordered pair (x, y) that makes the equation true when you substitute it. For $y = x + 2$, the pair $(3, 5)$ is a solution because $5 = 3 + 2$.

linear equation

An equation whose graph is a straight line, such as $y = 2x - 1$ or $x + y = 5$.

graph of an equation

The picture made by plotting every solution of the equation. For a linear equation the picture is a line.

x-intercept

The point where a graph crosses the x-axis. Its y-coordinate is 0.

CORE IDEAS

An equation like $y = 2x - 1$ has many solutions, not just one. Each solution is an ordered pair: $(0, -1)$, $(1, 1)$, $(2, 3)$, and so on. Each of these makes the equation true. When you plot all of the solutions, they form a straight line, so the equation is called a **linear equation**.

To graph a linear equation: (1) make a table with at least three x-values, (2) find each y-value by substitution, (3) plot the ordered pairs, (4) draw a straight line through them with a ruler and put arrows on both ends. The arrows show that the line keeps going forever, because there are always more solutions.

Why three points when two decide a line? Two points are enough to draw a line, but the third point is your check. If the three points do not line up, you have made an arithmetic mistake somewhere. Choose easy x-values such as -1, 0, 1 and 2. If the equation has a fraction like $\frac{1}{2}x$, choose even x-values so the y-values come out whole.

Is a point on the line? Substitute its coordinates into the equation. If the equation comes out true, the point is a solution and lies on the line. If not, it is off the line. For $y = 3x + 1$, the point $(2, 7)$ is on the line because $3 \times 2 + 1 = 7$, but $(4, 9)$ is not, because $3 \times 4 + 1 = 13$, not 9.

Some lines are special. The equation $y = 3$ means "y is always 3, whatever x is," so its graph is a **horizontal** line 3 units above the x-axis. The equation $x = -2$ means "x is always -2," so its graph is a **vertical** line 2 units left of the y-axis.

Graphs help with real situations. Where a line crosses the x-axis, $y = 0$. If y is the gallons left in a tank, the **x-intercept** tells you when the tank is empty. To find it without a graph, put 0 in for y and solve the equation.

KEY FORMULAS

$$\text{Point } (a, b) \text{ is on the line } \iff \text{ substituting } x = a,\ y = b \text{ makes the equation true}$$

$$y = c \text{ is a horizontal line;} \qquad x = c \text{ is a vertical line}$$

From history. René Descartes's idea of joining algebra and geometry is what makes this lesson possible: an equation in x and y can be drawn as a picture, and a picture can be described by an equation. Mathematicians had studied lines and curves for centuries, but Descartes's 1637 work made it routine to study them with algebra.

Euclid connection. Euclid's first postulate says that a straight line can be drawn from any point to any point. That is why two plotted points are enough to draw a line, and a third point serves as a check.

Practice habit. Always plot at least three points and look to see that they line up before you draw the line. Then check one more point by substitution.

WORKED EXAMPLES

Example 1. Graph $y = 2x - 1$. Then decide whether the point $(3, 5)$ is on the line.

Show solution

1. Choose x-values -1, 0, 1, 2.
2. $x = -1$: $y = 2 \times (-1) - 1 = -3$. Point $(-1, -3)$.
3. $x = 0$: $y = 2 \times 0 - 1 = -1$. Point $(0, -1)$.
4. $x = 1$: $y = 2 \times 1 - 1 = 1$. Point $(1, 1)$.
5. $x = 2$: $y = 2 \times 2 - 1 = 3$. Point $(2, 3)$.
6. Plot the four points. They line up, so draw a straight line through them with arrows on both ends.
7. Test $(3, 5)$: $2 \times 3 - 1 = 5$. The equation is true, so $(3, 5)$ is on the line.

Answer: Yes, $(3, 5)$ is on the line $y = 2x - 1$.

Example 2. A rain barrel holds 40 gallons and drains at 5 gallons per minute. The gallons left after x minutes is $y = 40 - 5x$. Make a table, graph the line, and find when the barrel will be empty.

Show solution

1. Choose x-values 0, 2, 4, 6.
2. $x = 0$: $y = 40 - 0 = 40$. $x = 2$: $y = 40 - 10 = 30$. $x = 4$: $y = 40 - 20 = 20$. $x = 6$: $y = 40 - 30 = 10$.
3. Plot $(0, 40)$, $(2, 30)$, $(4, 20)$, $(6, 10)$ and draw the line. It slants down to the right because the water goes down as time goes on.
4. The barrel is empty when $y = 0$, where the line crosses the x-axis.
5. Solve $0 = 40 - 5x$: add $5x$ to both sides to get $5x = 40$, so $x = 8$.
6. Check: $40 - 5 \times 8 = 40 - 40 = 0$. Correct.

Answer: The barrel is empty after 8 minutes.

PRACTICE

1. Easy Is the point $(2, 7)$ on the line $y = 3x + 1$? Answer yes or no.

   Show answer

   Yes ($3 \times 2 + 1 = 7$)

2. Easy Is the point $(4, 9)$ on the line $y = 2x + 3$? Answer yes or no.

   Show answer

   No ($2 \times 4 + 3 = 11$, not 9)

3. Easy Complete the table for $y = x - 4$ using the x-values 0, 2 and 4.

   Show answer

   -4, -2, 0; points $(0, -4)$, $(2, -2)$, $(4, 0)$

4. Easy For the equation $y = -x + 2$, find y when $x = -3$.

   Show answer

   $y = 5$

5. Medium Is the point $(-1, -5)$ on the line $y = 4x - 1$? Answer yes or no.

   Show answer

   Yes ($4 \times (-1) - 1 = -5$)

6. Medium Is $(-2, 10)$ a solution of $x + y = 8$? Answer yes or no.

Show answer

Yes ($-2 + 10 = 8$)

7. Medium Make a table for $y = \frac{1}{2}x + 1$ using the x-values -2, 0, 2 and 4.

Show answer

0, 1, 2, 3; points $(-2, 0)$, $(0, 1)$, $(2, 2)$, $(4, 3)$

8. Medium At what point does the line $y = 2x - 6$ cross the x-axis?

Show answer

$(3, 0)$

9. Hard Which one of these points is NOT on the line $y = 3x - 2$: $(1, 1)$, $(2, 5)$, $(3, 7)$, $(-1, -5)$?

Show answer

$(2, 5)$, because $3 \times 2 - 2 = 4$, not 5

10. Hard Renting a bicycle costs \$5 plus \$3 per hour, so the cost is $y = 3x + 5$ dollars for x hours. Is the point $(4, 17)$ on the graph? Answer yes or no, and say what the point means.

Show answer

Yes ($3 \times 4 + 5 = 17$). It means renting for 4 hours costs \$17.

11. Hard Grace walks at a steady 3 miles per hour, so the distance she has walked is $y = 3x$ miles after x hours. How many hours does it take her to walk 7.5 miles?

Show answer

2.5 hours

12. Hard Samuel graphs a line using only two points and draws it. Explain why it is wise to plot a third point.

Show answer

Two points are enough to draw a line, but if either one has an arithmetic mistake the whole line is wrong and nothing shows it. A third point acts as a check: if all three line up, the work is very likely correct; if not, one of the points must be fixed.

13. **Challenge** Samuel has \$1 and saves \$2 each week, so he has $y = 2x + 1$ dollars after x weeks. Ruth has \$7 and spends \$1 each week, so she has $y = 7 - x$ dollars. Make a table for both for weeks 0 through 4 and find the week when they have the same amount. How much do they each have then?

Show answer

Week 2; \$5 each. The lines cross at $(2, 5)$.

COMMON MISTAKES

- Drawing a line through only two points with no check. Plot a third point; if it is off the line, look for an arithmetic mistake.
- Mixing up horizontal and vertical lines. $y = 3$ is horizontal (every point has y-coordinate 3); $x = 3$ is vertical (every point has x-coordinate 3).
- Substituting the coordinates in the wrong places when testing a point. In $(2, 5)$, 2 goes in for x and 5 goes in for y.
- Making sign mistakes with negative x-values, such as writing $2 \times (-1) - 1 = 1$. Write the parentheses and work slowly: $-2 - 1 = -3$.

> **Make it a game · Is It On the Line?.** Write a linear equation, such as y = 2x + 1, on paper. Parent calls out points, and you test each by substituting before shouting on the line or off the line. Then pick three points of your own and sketch the graph on grid paper. Score a point for each correct test. Play ten points and then switch equations.

11.4 Slope

OBJECTIVES

- Find the slope of a line as rise over run, from a graph or from two points.
- Tell whether a slope is positive, negative, zero or undefined, and explain why a vertical line has undefined slope.
- Interpret slope as a rate of change in word problems.

WATCH

Algebra Basics: Slope And Distance - Math Antics

mathantics · 12:00

https://www.youtube.com/watch?v=rpMu98yRk40

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'gradient' for slope.

BACKUP VIDEO

Introduction to Gradient (1 of 2: Definition & Meaning)

Eddie Woo · 11:36

https://www.youtube.com/watch?v=jY25uxFoLG0

Before the practice set.

KEY TERMS

slope

A number that tells how steep a line is and which way it tilts. It equals the rise divided by the run.

rise

The vertical change between two points: how far up (positive) or down (negative) you move.

run

The horizontal change between two points: how far right (positive) or left (negative) you move.

rate of change

How much one quantity changes for each 1 unit of change in another, such as dollars per week or miles per hour. On a graph it is the slope.

undefined

Having no value. A slope is undefined when the run is 0, because we cannot divide by 0.

CORE IDEAS

Some lines are steep and some are gentle. Some go uphill from left to right and some go downhill. **Slope** is a single number that describes both. To find it, pick two points on the line and move from one to the other. The up-or-down change is the **rise**; the left-or-right change is the **run**. Slope is $\frac{\text{rise}}{\text{run}}$.

With two points (x_1, y_1) and (x_2, y_2), the rise is $y_2 - y_1$ and the run is $x_2 - x_1$. For $(1, 2)$ and $(4, 8)$: rise $= 8 - 2 = 6$, run $= 4 - 1 = 3$, slope $= \frac{6}{3} = 2$. Subtract in the **same order** on top and bottom. If you start with the second point on top, start with it on the bottom too. Either order gives the same answer, because $\frac{-6}{-3}$ is also 2.

Four kinds of slope. A line going *up* from left to right has a **positive** slope. A line going *down* from left to right has a **negative** slope. A **horizontal** line has a rise of 0, and $\frac{0}{\text{run}} = 0$, so its slope is **zero**. A **vertical** line has a run of 0, and division by 0 has no answer (Lesson 5.5), so its slope is **undefined**. A bigger number (ignoring the sign) means a steeper line: slope 5 is steeper than slope $\frac{1}{2}$.

Why is the slope the same anywhere on a line? A straight line rises at a steady rate. If it rises 2 for every 1 step to the right, it rises 4 for every 2 steps and 6 for every 3 steps. The fractions $\frac{2}{1}$, $\frac{4}{2}$ and $\frac{6}{3}$ are all equal, so any two points on the line give the same slope.

Slope is a rate of change. If a graph shows dollars saved (y) against weeks (x), the slope is dollars per week. If it shows miles against hours, the slope is miles per hour. A negative rate of change means the quantity is going down, like water draining from a tank. Always give the units of a rate.

KEY FORMULAS

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_2 \neq x_1)$$

uphill: positive downhill: negative horizontal: 0 vertical: undefined

Practice habit. Recite "rise over run, y over x, same order top and bottom."
Before you compute, look at the two points and predict the sign: uphill positive, downhill negative.

WORKED EXAMPLES

Example 1. Find the slope of the line through $(1, 2)$ and $(4, 8)$.

Show solution

1. Predict: going from $x = 1$ to $x = 4$, y goes from 2 up to 8, so the slope is positive.
2. Rise: $8 - 2 = 6$.
3. Run: $4 - 1 = 3$.
4. Slope $= \dfrac{6}{3} = 2$.
5. Check the other order: $\dfrac{2 - 8}{1 - 4} = \dfrac{-6}{-3} = 2$. Same answer.

Answer: 2

Example 2. Water is draining from a tank at a steady rate. After 2 minutes there are 70 gallons left, and after 6 minutes there are 50 gallons left. Find the slope of the line through $(2, 70)$ and $(6, 50)$ and tell what it means.

Show solution

1. Predict: the amount of water goes down, so the slope is negative.
2. Rise (change in gallons): $50 - 70 = -20$.
3. Run (change in minutes): $6 - 2 = 4$.
4. Slope $= \dfrac{-20}{4} = -5$.
5. Units: gallons per minute. The tank loses 5 gallons every minute.
6. Check: from 2 to 6 minutes is 4 minutes, and $4 \times 5 = 20$ gallons lost; $70 - 20 = 50$. Correct.

Answer: -5 gallons per minute; the tank loses 5 gallons each minute.

PRACTICE

1. Easy A line rises 6 units for every 2 units it runs to the right. What is its slope?

   Show answer

   3

2. Easy Find the slope of the line through $(0, 0)$ and $(3, 12)$.

   Show answer

   4

3. Easy Find the slope of the line through $(2, 3)$ and $(6, 5)$.

   Show answer

   $\dfrac{1}{2}$

4. Easy What is the slope of the horizontal line $y = 4$?

   Show answer

   0

5. Medium Find the slope of the line through $(-1, 4)$ and $(3, -4)$.

   Show answer

   -2

6. **Medium** Find the slope of the line through $(-3, -2)$ and $(3, 2)$.

   Show answer

   $\frac{2}{3}$

7. **Medium** Find the slope of the line through $(5, 1)$ and $(5, 9)$. What kind of line is it?

   Show answer

   Undefined; it is a vertical line (the run is $5 - 5 = 0$).

8. **Medium** A wheelchair ramp at the church entrance rises 2 feet over a horizontal distance of 24 feet. What is the slope of the ramp? Write it as a fraction in lowest terms.

   Show answer

   $\frac{1}{12}$

9. **Hard** Ruth saves the same amount each week. After 3 weeks she has \$41, and after 8 weeks she has \$66. Find the rate of change of her savings.

   Show answer

   \$5 per week

10. **Hard** A hiking family walks steadily downhill. After 1 hour they are at an elevation of 3,200 feet, and after 4 hours they are at 2,300 feet. Find the rate of change of their elevation.

    Show answer

    -300 feet per hour (they go down 300 feet each hour)

11. **Hard** Explain why a horizontal line has a slope of 0 but a vertical line has an undefined slope.

    Show answer

    On a horizontal line the rise between any two points is 0, and 0 divided by any nonzero run is 0. On a vertical line the run is 0, so the slope would be the rise divided by 0. No number times 0 gives a nonzero rise, so division by 0 has no answer, and the slope is undefined.

12. Hard Find two points on the line $y = 3x - 2$ by using $x = 0$ and $x = 1$, and then find the slope of the line.

Show answer

Points $(0, -2)$ and $(1, 1)$; slope 3

13. Challenge A line passes through $(2, 1)$ and has a slope of 3. The point $(6, \square)$ is also on the line. What number goes in the box?

Show answer

13 (the run from 2 to 6 is 4, so the rise is $3 \times 4 = 12$, and $1 + 12 = 13$)

COMMON MISTAKES

- Putting run over rise, as in $\dfrac{x_2 - x_1}{y_2 - y_1}$. Slope is always the change in y on top: rise over run.
- Subtracting in different orders on top and bottom, such as $\dfrac{8 - 2}{1 - 4}$. This gives the wrong sign. Start with the same point on top and bottom.
- Saying a vertical line has slope 0. A vertical line has run 0, so its slope is undefined; it is the horizontal line that has slope 0.
- Leaving off units in a rate-of-change problem. Say \$5 per week or -300 feet per hour, not just 5 or -300.

> **Make it a game · Slope Stairs.** Use masking tape on the floor or draw a grid. Parent names two points and you walk the rise and then the run between them, saying the slope as a fraction. Then call out a vertical or horizontal line and name its slope, positive, negative, zero or undefined. One point per correct call. Race to twelve.

11.5 Slope-Intercept Form and Direct Variation

OBJECTIVES

- Read the slope and y-intercept from an equation in the form $y = mx + b$ and graph the line by starting at b and using m.
- Write an equation in slope-intercept form for a word problem such as cost = fee + rate × hours.
- Recognize direct variation $y = kx$, find the constant of variation k, and use it to solve problems.

WATCH

Basic Linear Functions - Math Antics

mathantics · 13:24

https://www.youtube.com/watch?v=MXV65i9g1Xg

> **WHEN TO WATCH**
> Before Example 1: y = mx + b, slope and y-intercept, and lines through the origin (y = mx).

Note: Shared video: also the backup for Lesson 11.3. Eddie Woo teaches in Australia: the backup writes direct variation with k and may say 'gradient'.

BACKUP VIDEO

Direct Variation (1 of 3: Introduction)

Eddie Woo · 6:06

https://www.youtube.com/watch?v=tEvRN7Dxajk

Before the direct-variation section (objective 3).

KEY TERMS

y-intercept

The point where a line crosses the y-axis. Its x-coordinate is 0. In $y = mx + b$, the y-intercept is $(0, b)$.

slope-intercept form

The form $y = mx + b$ of a linear equation, where m is the slope and b is the y-intercept.

direct variation

A relationship $y = kx$ in which y is always the same number k times x. Its graph is a line through the origin.

constant of variation

The fixed number k in a direct variation $y = kx$. You can find it as $k = \frac{y}{x}$ for any pair with $x \neq 0$.

CORE IDEAS

Look again at a rule like $y = 2x + 3$. When $x = 0$, $y = 2 \times 0 + 3 = 3$, so the line crosses the y-axis at $(0, 3)$. That point is the **y-intercept**. Each time x goes up by 1, y goes up by 2, so the rise for a run of 1 is 2. That is the **slope**. In general, in $y = mx + b$, the number multiplying x is the slope m and the number added on is the y-intercept b. This is called **slope-intercept form**.

Watch the signs. In $y = -2x + 5$, $m = -2$ and $b = 5$. In $y = 4x - 7$, think of it as $y = 4x + (-7)$, so $b = -7$. In $y = 8 - 3x$, reorder it as $y = -3x + 8$, so $m = -3$ and $b = 8$. In $y = x$, the slope is 1 and $b = 0$.

Graphing from b and m. Plot the y-intercept $(0, b)$ first. Then use the slope as rise over run to find more points. For $y = \frac{2}{3}x - 1$, start at $(0, -1)$, go up 2 and right 3 to reach $(3, 1)$, then up 2 and right 3 again to reach $(6, 3)$. For a negative slope such as -2, think $\frac{-2}{1}$: go down 2 and right 1. Check one point by substitution, then draw the line.

Word problems. Many costs are a fixed fee plus a rate: cost = fee + rate × hours. If a plumber charges a $60 visit fee plus $45 per hour, the cost is $y = 45x + 60$. The rate, $45 per hour, is the slope; the fee, $60, is the y-intercept, the cost for 0 hours of work.

Direct variation is the special case where $b = 0$: $y = kx$. Then doubling x doubles y, tripling x triples y, and the ratio $\frac{y}{x}$ is always k. Buying apples at $1.50 per pound is direct variation: $y = 1.5x$. Because $y = k \times 0 = 0$ when $x = 0$, every direct variation graph passes through the origin. A fee such as the plumber's $60 breaks direct variation, since 0 hours still costs $60.

KEY FORMULAS

$$y = mx + b \qquad m = \text{slope}, \quad (0, b) = \text{y-intercept}$$

$$\text{cost} = \text{fee} + \text{rate} \times \text{hours}$$

$$\text{Direct variation: } y = kx, \qquad k = \frac{y}{x}$$

Practice habit. Say the form aloud as you read each equation: "y equals m x plus b, slope then start." For every word problem, name the slope (the rate, with its units) and the y-intercept (the starting amount) before you write the equation.

WORKED EXAMPLES

Example 1. Find the slope and y-intercept of $y = -2x + 5$, and describe how to graph it.

Show solution

1. The equation is in the form $y = mx + b$ with $m = -2$ and $b = 5$.
2. Plot the y-intercept $(0, 5)$.
3. Slope $-2 = \dfrac{-2}{1}$: from $(0, 5)$ go down 2 and right 1 to $(1, 3)$.
4. Again: down 2 and right 1 to $(2, 1)$.
5. Check $(2, 1)$: $-2 \times 2 + 5 = -4 + 5 = 1$. Correct.
6. Draw the line through the points. It goes downhill, as a negative slope should.

Answer: Slope -2, y-intercept $(0, 5)$

Example 2. A plumber charges a \$60 visit fee plus \$45 per hour. Write an equation for the cost y of x hours of work, tell what the slope and y-intercept mean, and find the cost of a job that takes 3.5 hours.

Show solution

1. The rate is \$45 per hour, so $m = 45$. The fee is \$60, so $b = 60$.
2. Equation: $y = 45x + 60$.
3. Meaning: the slope 45 is the cost of each hour; the y-intercept $(0, 60)$ is the cost before any work is done.
4. Substitute $x = 3.5$: $45 \times 3.5 = 157.5$.
5. Add the fee: $157.5 + 60 = 217.5$.
6. Estimate check: about 4 hours at \$45 is \$180, plus \$60 is \$240; our answer is a little less, which fits 3.5 hours.

Answer: $y = 45x + 60$; \$217.50

PRACTICE

1. Easy Give the slope and the y-intercept b of $y = 4x + 7$.

   Show answer

   Slope 4, $b = 7$

2. Easy Give the slope and the y-intercept b of $y = -x + 3$.

   Show answer

   Slope -1, $b = 3$

3. Easy Give the slope and the y-intercept b of $y = \frac{1}{2}x - 6$.

   Show answer

   Slope $\frac{1}{2}$, $b = -6$

4. Easy Write the equation of the line with slope 3 and y-intercept $(0, -2)$.

   Show answer

   $y = 3x - 2$

5. **Medium** Give the slope and the y-intercept b of $y = 8 - 3x$.

Show answer

Slope -3, $b = 8$

6. **Medium** Does $y = 2x + 3$ show direct variation? Answer yes or no.

Show answer

No; its graph crosses the y-axis at $(0, 3)$, not at the origin.

7. **Medium** y varies directly with x, and $y = 24$ when $x = 6$. Find (a) the constant of variation k and (b) y when $x = 10$.

Show answer

(a) $k = 4$; (b) $y = 40$

8. **Medium** A line crosses the y-axis at $(0, -1)$ and has slope $\frac{2}{3}$. Write its equation in slope-intercept form.

Show answer

$y = \frac{2}{3}x - 1$ (it also passes through $(3, 1)$)

9. **Hard** The church fellowship hall rents for a \$50 fee plus \$25 per hour. Write an equation for the cost y of renting it for x hours, and find the cost of renting it for 6 hours.

Show answer

$y = 25x + 50$; \$200

10. **Hard** Grace's babysitting pay varies directly with the hours she works. She earned \$57 for 6 hours. How much will she earn for 10 hours?

Show answer

\$95 ($k = 57 \div 6 = 9.5$ dollars per hour)

11. **Hard** Write the equation of the line through $(0, 3)$ and $(2, 11)$ in slope-intercept form.

Show answer

$y = 4x + 3$

12. Hard Explain why the graph of every direct variation $y = kx$ passes through the origin.

Show answer

When $x = 0$, $y = k \times 0 = 0$, so the point $(0, 0)$ is always a solution. In words: zero of the first quantity always gives zero of the second, such as 0 pounds of apples costing \$0.

13. Challenge Write the equation of the line through $(2, 7)$ and $(6, 15)$ in slope-intercept form.

Show answer

$y = 2x + 3$ (slope $\dfrac{15 - 7}{6 - 2} = 2$; then $7 = 2 \times 2 + b$ gives $b = 3$)

14. Challenge Plumber A charges a \$40 fee plus \$50 per hour. Plumber B charges an \$80 fee plus \$40 per hour. For how many hours of work do the two plumbers charge the same amount, and what is that amount?

Show answer

4 hours; \$240 (solve $40 + 50x = 80 + 40x$)

COMMON MISTAKES

- Dropping the sign of b. In $y = 4x - 7$, the y-intercept is $(0, -7)$, not $(0, 7)$.
- Taking the first number as the slope when the equation is written in a different order. In $y = 8 - 3x$, the slope is -3 (the number with x), and $b = 8$.
- Graphing the slope from the origin instead of from the y-intercept. Plot $(0, b)$ first, then count rise and run from there.
- Calling every line direct variation. Only lines through the origin ($b = 0$) are direct variation; a starting fee makes b not zero.

> **Make it a game · Graph From Equation.** Write ten equations in slope-intercept form on cards. Draw one, name the slope and y-intercept, then plot the intercept and use the slope to find two more points on grid paper. Parent checks with one extra point by substitution. Add a word-problem card like fee plus rate times hours. Score a point per correct graph.

UNIT 11

The Coordinate Plane and Graphing

In this unit you join numbers and pictures. Two number lines crossing at right angles make the coordinate plane, where every point has an address, an ordered pair (x, y). You will plot points, find distances along grid lines, turn input-output tables into rules, and graph equations as straight lines.

Next you will measure how steep a line is with its slope, rise over run, and see that slope is a rate of change: dollars per week, miles per hour, gallons per minute. Finally you will read the slope and starting value right out of an equation $y = mx + b$, and meet direct variation, $y = kx$.

This unit uses the integers of Unit 5 and the equations of Unit 10, and it leads straight into Algebra I, where graphing lines is a central skill. In Unit 12 you will use the coordinate plane again to find distances with the Pythagorean theorem.

PREREQUISITES

- Compare and order integers on a number line (Lesson 5.1).
- Add, subtract, multiply and divide integers (Lessons 5.2–5.5).
- Evaluate expressions by substitution, including negative values (Lesson 9.2).
- Solve one- and two-step equations and check by substitution (Lessons 10.2–10.4).
- Write and simplify ratios and rates (Lessons 7.1–7.2).

MASTERY CHECKLIST

- ☐ I can plot a point from its ordered pair and name the coordinates of a plotted point.
- ☐ I can tell which quadrant or axis a point lies in from the signs of its coordinates.
- ☐ I can find the distance between two points on the same horizontal or vertical line.
- ☐ I can complete an input-output table from a rule and plot its points.
- ☐ I can write a rule from a table and check it with every pair.
- ☐ I can graph a linear equation by making a table of at least three points.
- ☐ I can decide whether a point is on a line by substitution.
- ☐ I can find the slope of a line from a graph or from two points.
- ☐ I can tell positive, negative, zero and undefined slopes apart and explain why a vertical line's slope is undefined.
- ☐ I can explain slope as a rate of change with units.
- ☐ I can read the slope and y-intercept from

$y =$ and graph the line from them.
$mx +$
b

- ☐ I can write equations for fee-plus-rate problems and direct variation

problems and use them to answer questions.

LESSONS IN THIS UNIT

11.1 The Coordinate Plane	45 minutes
11.2 Tables, Rules and Graphs	2 sessions of 45 minutes
11.3 Graphing Linear Equations	2 sessions of 45 minutes
11.4 Slope	2 sessions of 45 minutes
11.5 Slope-Intercept Form and Direct Variation	2 sessions of 45 minutes

Unit 11 test

You have 50 minutes. Have graph paper, a pencil and a ruler ready. A basic calculator is allowed but should rarely be needed. Show your work for every problem: tables, substitutions and the rise and run you used. Write slopes that are not whole numbers as fractions in lowest terms or as exact decimals, and give money to the nearest cent. Include units in word-problem answers.

1. **5 pts**
   In which quadrant is the point $(-7, -2)$?

2. **4 pts**
   Find the distance between $(-3, 6)$ and $(-3, -4)$.

3. **6 pts**
   On a county map, each grid unit stands for 1 mile. Henry's house is at $(-5, 2)$ and the farm stand is at $(7, 2)$, on the same straight road. How many miles apart are they?

4. **6 pts**
   Find a rule for this table. Inputs x: 1, 2, 3, 4. Outputs y: 1, 4, 7, 10.

5. **7 pts**

   For a church potluck, square tables are pushed together in a long row. One table seats 6 people, two tables seat 10, and three tables seat 14. How many people can sit at a row of 12 tables?

6. **6 pts**

   Is the point $(-3, 11)$ on the line $y = -4x - 1$? Answer yes or no and show your check.

7. **7 pts**

   Make a table for $y = 2x - 3$ using the x-values -1, 0, 1 and 2, then graph the line.

8. **7 pts**

   At what point does the line $y = -3x + 12$ cross the x-axis?

9. **7 pts**

   Find the slope of the line through $(-2, 7)$ and $(4, -5)$.

10. **7 pts**

    Write the equation of the line through $(0, -4)$ and $(2, 2)$ in slope-intercept form.

11. **7 pts**

    On a family trip, the car travels at a steady speed. After 2 hours it has gone 110 miles, and after 5 hours it has gone 275 miles. Find the rate of change of the distance.

12. **7 pts**

    Give the slope and the y-intercept b of $y = -\dfrac{3}{4}x + 2$.

13. **8 pts**

    A garden tiller rents for a $15 fee plus $8 per hour. Write an equation for the cost y of renting it for x hours, and find the cost of renting it for 5 hours.

14. **8 pts**

    The cost of apples varies directly with their weight. Four pounds cost $6. Write a direct variation equation for the cost y of x pounds, and find the cost of 11 pounds.

15. **8 pts**

Explain why the slope of every horizontal line is 0, while the slope of every vertical line is undefined.

ANSWER KEY

1. Quadrant III
2. 10 units
3. 12 miles
4. $y = 3x - 2$
5. 50 people (rule $y = 4x + 2$)
6. Yes ($-4 \times (-3) - 1 = 12 - 1 = 11$)
7. $-5, -3, -1, 1$; points $(-1, -5)$, $(0, -3)$, $(1, -1)$, $(2, 1)$
8. $(4, 0)$
9. -2
10. $y = 3x - 4$
11. 55 miles per hour
12. Slope $-\frac{3}{4}$, $b = 2$
13. $y = 8x + 15$; \$55
14. $y = 1.5x$; \$16.50
15. Between two points on a horizontal line the rise is 0, and 0 divided by a nonzero run is 0. Between two points on a vertical line the run is 0, so the slope would mean dividing by 0, which has no answer. So the slope is undefined.

UNIT 12

Geometry

In this unit you use your arithmetic and algebra to measure the world around you. You will find unknown angles where lines cross, learn why every triangle's angles add to $180°$, find the perimeter and area of flat shapes and circles, find how much a box or a can holds and how much material covers it, and finally meet the famous Pythagorean theorem about right triangles.

Every formula in this unit is explained, not just given. The area of a parallelogram, triangle and trapezoid all come from the rectangle; the area of a circle comes from cutting it into wedges; volume comes from stacking layers; and the Pythagorean theorem comes from rearranging four triangles inside a square. Along the way you will meet Euclid's Elements, Archimedes' estimate of π, and Pythagoras.

Parents: this unit draws on fractions, decimals, square roots (Unit 6), solving simple equations (Unit 10) and the coordinate plane (Unit 11). A basic calculator is allowed for messy arithmetic and square roots. Many problems come from building, gardening and the home, so a tape measure, a protractor and a few cans and boxes make good hands-on helpers.

PREREQUISITES

- Multiply and divide whole numbers, fractions and decimals (Units 3 and 4).
- Find squares and square roots and estimate square roots (Lessons 6.4 and 6.5).
- Evaluate formulas by substitution (Lesson 9.2).
- Solve one- and two-step equations (Lessons 10.2 to 10.4).
- Plot points and find horizontal and vertical distances on the coordinate plane (Lesson 11.1).

MASTERY CHECKLIST

- ☐ I can classify angles and find complements, supplements, vertical angles and adjacent angles on a line.
- ☐ I can find the angles formed when a transversal crosses two parallel lines.
- ☐ I can explain why the angles of a triangle add to

$180°$ and find a missing angle.

- ☐ I can classify triangles by sides and angles and find the angle sum of any polygon with

$(n-2) \times 180°$.

- ☐ I can find the perimeter and area of rectangles, parallelograms, triangles, trapezoids and composite figures, and explain where the area formulas come from.
- ☐ I can find the circumference and area of a circle in terms of

π or using an approximation.

- ☐ I can find the volume and surface area of prisms and cylinders.
- ☐ I can state the Pythagorean theorem, explain a picture proof of it and find the hypotenuse of a right triangle.
- ☐ I can find a missing leg, find the distance between two points and use the converse to test for a right angle.

LESSONS IN THIS UNIT

12.1 Angles and Angle Relationships	45 minutes
12.2 Triangles and Polygons	2 sessions of 45 minutes
12.3 Perimeter and Area	2 sessions of 45 minutes
12.4 Circles	2 sessions of 45 minutes
12.5 Volume and Surface Area	2 sessions of 45 minutes
12.6 The Pythagorean Theorem	2 sessions of 45 minutes
12.7 Using the Pythagorean Theorem	2 sessions of 45 minutes

Unit 12 test

You have 60 minutes. You may use a basic calculator for messy arithmetic and square roots. For problems with π, give the answer in terms of π or use $\pi \approx 3.14$, exactly as each problem says. Show your work, include units in every answer, and round only where the problem tells you to.

1. **5 pts**
   Find the complement and the supplement of a 27° angle.

2. **6 pts**
   A transversal crosses two parallel lines. One of the interior angles is 112°. Find (a) its alternate interior angle and (b) the angle adjacent to that alternate interior angle along the same parallel line.

3. **6 pts**
   Two angles of a triangle measure 38° and 79°. Find the third angle.

4. **6 pts**
   Find the measure of each interior angle of a regular pentagon.

5. **7 pts**

   A trapezoid has bases of 9 cm and 15 cm and a height of 6 cm. Find its area.

6. **7 pts**

   The Miller family's living room is a rectangle 14 ft by 10 ft with a 4 ft by 6 ft rectangular corner cut out for a closet. Carpet costs $3 per square foot. How much will it cost to carpet the room?

7. **6 pts**

   A circle has a diameter of 20 cm. Find its circumference and its area in terms of π.

8. **7 pts**

   Ruth is sewing lace around the edge of a round tablecloth with a radius of 3 feet for the church potluck. How many feet of lace does she need? Use $\pi \approx 3.14$.

9. **7 pts**

   A cylinder has a radius of 4 inches and a height of 9 inches. Find its volume. Use $\pi \approx 3.14$.

10. **7 pts**

    Henry is painting all six outside faces of a wooden box 10 in long, 6 in wide and 4 in high. How many square inches will he paint?

11. **6 pts**

    A right triangle has legs of 8 m and 15 m. Find the hypotenuse.

12. **7 pts**

    A 25-foot ladder leans against a wall on level ground. Its foot is 7 feet from the wall. How high up the wall does it reach?

13. **7 pts**

    Find the distance between the points $(-1, -2)$ and $(5, 6)$.

14. **7 pts**

    A triangle has sides of 7 in, 9 in and 12 in. Is it a right triangle? Answer yes or no, and show why.

15. **9 pts**

Explain why the three angles of any triangle add to 180°.

ANSWER KEY

1. Complement 63°; supplement 153°
2. (a) 112° (b) 68°
3. 63°
4. 108° ($540^\circ \div 5$)
5. 72 square centimeters
6. \$348 (116 square feet)
7. $C = 20\pi$ cm; $A = 100\pi$ square cm
8. 18.84 feet
9. 452.16 cubic inches
10. 248 square inches
11. 17 meters
12. 24 feet ($625 - 49 = 576$)
13. 10 units (legs 6 and 8)
14. No; $49 + 81 = 130$, but $12^2 = 144$
15. Draw a line through one corner parallel to the opposite side. The two angles on either side of the triangle's corner along that line are alternate interior angles with the triangle's other two angles, so they equal them. Together the three angles make a straight line, which is 180°.

12.1 Angles and Angle Relationships

OBJECTIVES

- Measure and classify angles as acute, right, obtuse or straight.
- Find unknown angles using complementary, supplementary, vertical and adjacent angle relationships.
- Find the angles formed when a line crosses two parallel lines, using corresponding and alternate interior angles.

WATCH

Math Antics - Angles & Degrees

mathantics · 9:17

https://www.youtube.com/watch?v=_n3KZR1DSEo

> **WHEN TO WATCH**
> Before Example 1: degrees, protractors, and complementary and supplementary angles.

Note: Eddie Woo teaches in Australia: he says 'co-interior' for same-side interior angles and 'vertically opposite' for vertical angles.

BACKUP VIDEO

Angles on Parallel Lines (2 of 2: Naming the relationships)

Eddie Woo · 8:05

https://www.youtube.com/watch?v=2Ei_-6Lmm8E

Before the parallel-lines example (objective 3): corresponding, alternate and co-interior angles.

KEY TERMS

angle

The figure made by two rays that start at the same point, called the vertex. We measure how far apart the rays are opened in degrees (°).

acute, right, obtuse, straight

An acute angle is less than 90°. A right angle is exactly 90° (a square corner). An obtuse angle is between 90° and 180°. A straight angle is exactly 180° (a straight line).

complementary and supplementary angles

Two angles are complementary if they add to $90°$, and supplementary if they add to $180°$.

adjacent angles

Two angles that share a vertex and a side but do not overlap, like two slices of pie next to each other.

vertical angles

The two angles across from each other when two lines cross. Vertical angles are always equal.

transversal

A line that crosses two other lines. When it crosses two parallel lines, it makes pairs of equal angles called corresponding angles and alternate interior angles.

CORE IDEAS

An **angle** is made by two rays that start at one point, the vertex. We measure an angle in degrees: a full turn all the way around is $360°$, half a turn (a straight line) is $180°$, and a quarter turn (a square corner) is $90°$. To measure with a protractor, put its center hole on the vertex, line up the zero line with one ray, and read where the other ray crosses the scale. Use the scale that starts at 0 on your first ray, and ask yourself whether the angle looks smaller or larger than a square corner so you pick the right number.

We sort angles by size. **Acute** angles are less than $90°$, **right** angles are exactly $90°$, **obtuse** angles are between $90°$ and $180°$, and a **straight** angle is $180°$. Two angles that add to $90°$ are **complementary**; two that add to $180°$ are **supplementary**. So the complement of $34°$ is $90° - 34° = 56°$, and its supplement is $180° - 34° = 146°$.

Adjacent angles on a line. When a ray stands on a straight line, it splits the straight angle into two adjacent angles. Together they make the whole straight line, so they add to $180°$. If one is $70°$, the other must be $110°$.

Why vertical angles are equal. When two lines cross, they make four angles. Call them 1, 2, 3, 4 going around, so angles 1 and 3 are across from each other. Angles 1 and 2 lie along one line, so $\angle 1 + \angle 2 = 180°$. Angles 2 and 3 lie along the other line, so $\angle 2 + \angle 3 = 180°$. Both angle 1 and angle 3 are "whatever is left after taking angle 2 from $180°$", so they must be equal. If one angle is $72°$, the four angles are $72°$, $108°$, $72°$, $108°$.

Parallel lines and a transversal. Parallel lines run in the same direction and never meet, like the two rails of a train track. A **transversal** is a line that crosses both. Because the parallel lines point the same way, the transversal crosses each of them at the same tilt, so the four angles at one crossing match the four angles at the other. Angles in the same position at each crossing (for example, both above the line and to the right of the transversal) are **corresponding angles**, and they are equal. Angles between the parallel lines on opposite sides of the transversal are **alternate interior angles**, and they are equal too.

This means that when a transversal crosses two parallel lines, there are only two different angle sizes among all eight angles, and those two sizes are supplementary. If one angle is 65°, every angle is either 65° or $180^\circ - 65^\circ = 115^\circ$.

KEY FORMULAS

$$\text{complementary: } \angle A + \angle B = 90^\circ \qquad \text{supplementary: } \angle A + \angle B = 180^\circ$$

$$\text{Vertical angles are equal.}$$

Parallel lines and a transversal: corresponding angles are equal; alternate interior angles are equal.

Euclid connection. Euclid proves that vertical angles are equal in Elements I.15, with the same "both add to a straight line with the angle between them" reasoning used here. In Elements I.29 he shows that a line crossing two parallel lines makes the alternate angles equal.

Practice habit. Recite until automatic: "Complementary makes 90, supplementary makes 180, vertical angles are equal, and angles on a straight line add to 180." Before finding an angle, glance at it and say whether it should be acute or obtuse, so you can catch a wrong answer.

WORKED EXAMPLES

Example 1. Find the complement and the supplement of a 34° angle.

Show solution

1. Complementary angles add to 90°: $90^\circ - 34^\circ = 56^\circ$.
2. Supplementary angles add to 180°: $180^\circ - 34^\circ = 146^\circ$.
3. Check: $34 + 56 = 90$ ✓ and $34 + 146 = 180$ ✓.

Answer: Complement 56°, supplement 146°

Example 2. Samuel builds a farm gate with two parallel horizontal rails. A straight diagonal brace crosses both rails. At the top rail, one of the angles between the rails (an interior angle) measures 40°. Find (a) its alternate interior angle at the bottom rail and (b) the angle next to that one along the bottom rail (adjacent to it, on the other side of the brace).

Show solution

1. The rails are parallel and the brace is a transversal.
2. Alternate interior angles are equal, so the angle at the bottom rail is 40°.
3. The other interior angle at the bottom rail sits next to it along the straight rail, so the two add to 180°.
4. $180^\circ - 40^\circ = 140^\circ$.
5. Check: at a crossing with parallel lines, only two sizes appear, and $40 + 140 = 180$ ✓.

Answer: (a) 40° (b) 140°

PRACTICE

1. Easy Find the complement of a 38° angle.

   Show answer

   52°

2. Easy Find the supplement of a 115° angle.

   Show answer

   65°

3. Easy Classify each angle as acute, right, obtuse or straight: (a) 135° (b) 90° (c) 17° (d) 180°.

   Show answer

   (a) obtuse (b) right (c) acute (d) straight

4. Easy Two straight lines cross. One of the four angles is 72°. What is the angle vertical to it (directly across from it)?

   Show answer

   72°

5. Medium Two straight lines cross. One of the four angles is $72°$. What is the measure of an angle adjacent to it?

Show answer

$108°$ (adjacent angles on a line add to $180°$)

6. Medium Henry's kitchen has a square corner (a right angle). He puts in a corner shelf whose front edge splits the corner into two angles. One of them is $28°$. What is the other?

Show answer

$62°$

7. Medium A transversal crosses two parallel lines. One angle is $58°$. What is the measure of the angle corresponding to it at the other parallel line?

Show answer

$58°$ (corresponding angles are equal)

8. Medium A transversal crosses two parallel lines. One of the interior angles is $123°$. What is the measure of its alternate interior angle?

Show answer

$123°$ (alternate interior angles are equal)

9. Hard Ruth's father saws across a straight board at a slant. On one side of the cut, the cut makes a $35°$ angle with the long straight edge of the board. What angle does the cut make with that same edge on the other side?

Show answer

$145°$ (the two angles lie along a straight edge, so they add to $180°$)

10. Hard Elm Street and Oak Street are parallel. Church Street crosses both of them in a straight line. At Elm Street, one of the angles Church Street makes is $65°$. At Oak Street, what two different angle sizes are formed?

Show answer

$65°$ and $115°$

11. Hard Two angles are supplementary. The larger one is 4 times the smaller one. Find both angles.

Show answer

36° and 144° (since $5 \times 36 = 180$)

12. Challenge Two lines cross. Two vertical angles measure $(2x + 10)^\circ$ and $(3x - 15)^\circ$. Find the measure of each of these angles.

Show answer

60° each ($2x + 10 = 3x - 15$ gives $x = 25$)

13. Challenge Explain why vertical angles are always equal.

Show answer

Each of the two vertical angles forms a straight line (180°) together with the same angle between them. So each one equals 180° minus that same angle, which means they are equal.

COMMON MISTAKES

- Mixing up complementary and supplementary. Remember: "C" comes before "S" just as 90 comes before 180, so complementary makes 90° and supplementary makes 180°.
- Reading the wrong scale on a protractor (getting 130° for an angle that is clearly acute). Decide first whether the angle is smaller or larger than a square corner, then pick the matching number.
- Saying adjacent angles are equal. Adjacent angles on a straight line add to 180°; it is the vertical angles (across from each other) that are equal.
- Using corresponding or alternate interior angle rules when the lines are not parallel. Those angles are equal only when the two lines crossed by the transversal are parallel.

> **Make it a game · Angle Charades.** Use your arms as the rays of an angle. Parent calls out an angle name, such as acute, right, obtuse, straight, or a measure, and you show it with your arms. Then solve missing angles from drawings: complementary, supplementary and vertical. Score a point per correct pose and a point per found angle. Switch roles at ten points.

12.2 Triangles and Polygons

OBJECTIVES

- Explain why the angles of every triangle add to 180° and use this to find a missing angle.
- Classify triangles by their sides and by their angles.
- Find the angle sum of any polygon with $(n-2)\times 180^\circ$ and each angle of a regular polygon.

WATCH

Math Antics - Triangles

mathantics · 7:40

https://www.youtube.com/watch?v=mLeNaZcy-hE

> **WHEN TO WATCH**
> Before Example 1: classifying triangles and the 180° angle sum.

BACKUP VIDEO

Angle Sums of a Polygon (1 of 2: Interior angles)

Eddie Woo · 9:21

https://www.youtube.com/watch?v=frjiwP32PC0

Before the polygon angle-sum example.

KEY TERMS

polygon
A closed flat figure made of straight sides that meet only at their ends. Triangles, rectangles and stop-sign octagons are polygons; a circle is not.

equilateral, isosceles, scalene
Names for triangles by their sides: an equilateral triangle has all three sides equal, an isosceles triangle has at least two equal sides, and a scalene triangle has no equal sides.

acute, right, obtuse triangle

Names for triangles by their angles: an acute triangle has three acute angles, a right triangle has one $90°$ angle, and an obtuse triangle has one angle greater than $90°$.

interior angle

An angle inside a polygon, formed where two of its sides meet.

regular polygon

A polygon whose sides are all the same length and whose angles are all the same size, like a square or a stop sign.

CORE IDEAS

The angles of a triangle add to $180°$. Tear the three corners off a paper triangle and set them side by side: they always fit together into a straight line. Here is why. Take a triangle with corners A, B and C, with AB along the bottom. Through the top corner C, draw a line parallel to the bottom side. Along this new line, three angles sit side by side at C: the triangle's own angle at C in the middle, and one angle on each side. The angle on the left is an alternate interior angle with the triangle's angle at A (the side AC is a transversal of the two parallel lines), so it equals angle A. In the same way, the angle on the right equals angle B. The three angles together make a straight line, so $\angle A + \angle B + \angle C = 180°$.

So, to find a missing angle in a triangle, subtract the two known angles from $180°$. If two angles are $47°$ and $68°$, the third is $180° - 47° - 68° = 65°$. This also tells you a triangle can have at most one right angle or one obtuse angle, since two of them would already add to $180°$ or more. In a right triangle the two acute angles add to $90°$, so they are complementary.

Classifying triangles. By sides, a triangle is **equilateral** (three equal sides), **isosceles** (at least two equal sides) or **scalene** (no equal sides). By angles, it is **acute**, **right** or **obtuse**. In an isosceles triangle, the two angles across from the equal sides (the base angles) are equal. In an equilateral triangle all three angles are equal, so each is $180° \div 3 = 60°$.

Angle sum of a polygon. A polygon with n sides can be cut into triangles by drawing diagonals from one corner to every other corner it is not already joined to. A

pentagon (5 sides) splits into 3 triangles, a hexagon (6 sides) into 4 triangles; in general you get $n - 2$ triangles. The triangles' angles together fill exactly the polygon's angles, so the angle sum is $(n - 2) \times 180^\circ$. For a quadrilateral that is $2 \times 180^\circ = 360^\circ$; for a pentagon, $3 \times 180^\circ = 540^\circ$.

In a **regular polygon** all the angles are equal, so divide the angle sum by the number of angles. A regular hexagon has angle sum $4 \times 180^\circ = 720^\circ$, so each angle is $720^\circ \div 6 = 120^\circ$.

KEY FORMULAS

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\text{angle sum of an } n\text{-sided polygon} = (n - 2) \times 180^\circ$$

$$\text{each angle of a regular } n\text{-gon} = \frac{(n - 2) \times 180^\circ}{n}$$

From history. Euclid of Alexandria wrote the Elements around 300 BC, a book that was used to teach geometry for more than two thousand years. He builds each fact on earlier ones, just as we used parallel lines from Lesson 12.1 to prove the triangle angle sum.

Euclid connection. Euclid proves in Elements I.32 that the three angles of any triangle together equal two right angles, that is, 180°. His proof uses a line through one corner parallel to the opposite side, as in this lesson.

Practice habit. Recite the polygon angle sums until automatic: triangle 180, quadrilateral 360, pentagon 540, hexagon 720, octagon 1080. Notice each new side adds another 180°.

WORKED EXAMPLES

Example 1. Two angles of a triangle measure $47°$ and $68°$. Find the third angle and classify the triangle by its angles.

Show solution

1. The three angles add to $180°$.
2. Add the known angles: $47° + 68° = 115°$.
3. Subtract: $180° - 115° = 65°$.
4. All three angles ($47°$, $68°$, $65°$) are less than $90°$, so the triangle is acute.
5. Check: $47 + 68 + 65 = 180$ ✓.

Answer: $65°$; an acute triangle

Example 2. The Miller family is building a gazebo with a floor shaped like a regular hexagon. Find the sum of the floor's interior angles and the measure of each corner angle, so they know how to cut the boards at the corners.

Show solution

1. A hexagon has $n = 6$ sides.
2. Angle sum: $(6 - 2) \times 180° = 4 \times 180° = 720°$.
3. A regular hexagon has 6 equal angles: $720° \div 6 = 120°$.
4. Check: $6 \times 120 = 720$ ✓, and $120°$ is obtuse, which matches the wide corners of a hexagon.

Answer: Angle sum $720°$; each corner $120°$

PRACTICE

1. Easy Two angles of a triangle are $50°$ and $60°$. Find the third angle.

   Show answer

   $70°$

2. Easy One acute angle of a right triangle is $35°$. Find the other acute angle.

   Show answer

   $55°$

3. Easy A triangle has angles of $40°$, $50°$ and $90°$, and no two of its sides are the same length. Classify it by its angles and by its sides.

Show answer

A right scalene triangle

4. Easy Find the sum of the interior angles of a pentagon.

Show answer

$540°$

5. Medium Find the sum of the interior angles of an octagon (8 sides).

Show answer

$1080°$

6. Medium The top angle of an isosceles triangle is $40°$, and its two base angles are equal. Find each base angle.

Show answer

$70°$ each

7. Medium Grace plants a triangular flower bed. Two of its corners measure $48°$ and $77°$. What is the angle at the third corner?

Show answer

$55°$

8. Medium Three angles of a quadrilateral are $85°$, $95°$ and $110°$. Find the fourth angle.

Show answer

$70°$

9. Hard A stop sign is a regular octagon. What is the measure of each of its interior angles?

Show answer

$135°$ ($1080° \div 8$)

10. Hard A carpenter builds a roof frame shaped like an isosceles triangle. Each base angle (where the rafters meet the floor beam) is $28°$. What is the angle at the peak of the roof?

Show answer

$124°$

11. Hard The angles of a triangle measure $x°$, $2x°$ and $3x°$. Find all three angles.

Show answer

$30°$, $60°$, $90°$ ($6x = 180$, so $x = 30$)

12. Challenge The interior angles of a polygon add to $1440°$. How many sides does the polygon have?

Show answer

10 sides ($1440 \div 180 = 8 = n - 2$)

13. Challenge Explain why the three angles of any triangle add to $180°$.

Show answer

Draw a line through one corner parallel to the opposite side. The two angles beside the triangle's corner on that line are alternate interior angles with the other two angles of the triangle, so they are equal to them. The three angles together form a straight line, which is $180°$.

COMMON MISTAKES

- Using $(n - 2) \times 180°$ with n as the number of triangles instead of the number of sides. Count the sides first, then subtract 2.
- Dividing the angle sum by the number of sides for a polygon that is not regular. Only a regular polygon has all angles equal.
- Forgetting that an isosceles triangle has two equal base angles. If the top angle is $40°$, split the remaining $140°$ evenly: $70°$ each.
- Saying a triangle can have two right angles. Two right angles already add to $180°$, leaving nothing for the third angle.

Make it a game · Triangle Sort. Draw or cut out twelve triangles of different shapes on paper. Parent calls a description, like isosceles and obtuse, and you grab the matching triangle in five seconds. Then pick any triangle, measure two angles and compute the third from the 180 degree rule. Score a point per correct grab. Try to reach fifteen points.

12.3 Perimeter and Area

OBJECTIVES

- Find the perimeter and area of rectangles, parallelograms, triangles and trapezoids.
- Explain how the parallelogram, triangle and trapezoid area formulas come from the rectangle.
- Find the area and perimeter of composite figures by adding and subtracting simpler shapes.

WATCH

Math Antics - Area

mathantics · 10:26

https://www.youtube.com/watch?v=xCdxURXMdFY

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'trapezium', which is the Australian/British name for a trapezoid.

BACKUP VIDEO

Area of a Trapezium

Eddie Woo · 12:29

https://www.youtube.com/watch?v=G_bdnIIT9No

Before the trapezoid example.

KEY TERMS

perimeter

The distance all the way around a flat figure. It is a length, so it is measured in units like feet or centimeters.

area

The amount of flat surface a figure covers, measured in square units like square feet (ft^2) or square centimeters (cm^2).

base and height

The base is the side you choose to measure along the bottom. The height is the straight-up (perpendicular) distance from the base to the opposite side or corner, measured at a right angle to the base, not along a slanted side.

parallelogram

A four-sided figure whose opposite sides are parallel, like a rectangle that has been pushed over.

trapezoid

A four-sided figure with exactly one pair of parallel sides, called its two bases.

composite figure

A figure made by joining or cutting simpler shapes, such as an L-shaped room.

CORE IDEAS

Perimeter is the distance around: add up the lengths of all the sides. For a rectangle, $P = 2l + 2w$. **Area** counts how many unit squares cover a figure. A rectangle 9 ft long and 4 ft wide holds 4 rows of 9 one-foot squares, so its area is $9 \times 4 = 36$ square feet. That is why $A = lw$ for a rectangle. Perimeter is in plain units (feet); area is in square units (square feet).

Parallelogram. Cut a right triangle off one slanted end of a parallelogram and slide it to the other end. The pieces fit together into a rectangle with the same base and the same height, and no area was gained or lost. So a parallelogram's area is $A = bh$. The height must be the straight-up distance between the two bases, not the slanted side.

Triangle. Two copies of the same triangle, with one turned around, fit together to make a parallelogram with the same base and height. The triangle is exactly half of it, so $A = \frac{1}{2}bh$. A triangle with base 12 in and height 7 in has area $\frac{1}{2} \times 12 \times 7 = 42$ square inches.

Trapezoid. Two copies of the same trapezoid, with one turned upside down, fit together into a parallelogram. Its base is the two bases joined, $b_1 + b_2$, and its height is the same h. The trapezoid is half of it, so $A = \frac{1}{2}(b_1 + b_2)h$. You can also think of it as the average of the two bases times the height.

Composite figures. Break a complicated shape into rectangles, triangles and trapezoids, find each area, and add. Or find the area of a larger simple shape and subtract the part that is missing. For the perimeter, walk all the way around the outside edge and add each length; be careful not to count lines you drew inside the figure.

KEY FORMULAS

$$\text{Rectangle: } A = lw, \quad P = 2l + 2w$$

$$\text{Parallelogram: } A = bh$$

$$\text{Triangle: } A = \frac{1}{2}bh$$

$$\text{Trapezoid: } A = \frac{1}{2}(b_1 + b_2)h$$

Euclid connection. In Elements I.35 and I.41, Euclid shows that parallelograms on the same base and between the same parallels have equal area, and that a triangle is half of a parallelogram with the same base and height. This is the same cutting-and-rearranging idea used in this lesson.

Practice habit. Before computing, say the formula out loud and say the units: "Area of a triangle is one-half base times height, in square feet." Then estimate: the answer for a triangle should be about half of the rectangle around it.

WORKED EXAMPLES

Example 1. A trapezoid has parallel bases of 8 inches and 14 inches, and a height of 5 inches. Find its area.

Show solution

1. Use $A = \frac{1}{2}(b_1 + b_2)h$.
2. Add the bases: $8 + 14 = 22$.
3. Multiply by the height: $22 \times 5 = 110$.
4. Take half: $\frac{1}{2} \times 110 = 55$.
5. Check: the average base is 11 in, and $11 \times 5 = 55$ ✓.

Answer: 55 square inches

Example 2. The Miller family's patio is a rectangle 12 ft long and 5 ft wide with a 3 ft by 2 ft rectangular notch cut out of one corner (the notch is 3 ft along the long side and 2 ft along the short side). Find the area and the perimeter of the patio.

Show solution

1. Area of the whole rectangle: $12 \times 5 = 60$ square feet.
2. Area of the notch: $3 \times 2 = 6$ square feet.
3. Patio area: $60 - 6 = 54$ square feet.
4. Perimeter: walk around. The six edges are 12 ft, 5 ft, $12 - 3 = 9$ ft, 2 ft, 3 ft and $5 - 2 = 3$ ft.
5. Add: $12 + 5 + 9 + 2 + 3 + 3 = 34$ feet.
6. Check: this equals the perimeter of the full rectangle, $2 \times 12 + 2 \times 5 = 34$ ✓. A corner notch swaps two outside edges for two inside edges of the same lengths.

Answer: Area 54 square feet; perimeter 34 feet

PRACTICE

1. Easy Find the perimeter of a rectangle 9 ft long and 4 ft wide.

   Show answer

   26 feet

2. Easy A parallelogram has a base of 10 cm and a height of 6 cm. Find its area.

   Show answer

   60 square centimeters

3. Easy A triangle has a base of 12 in and a height of 7 in. Find its area.

   Show answer

   42 square inches

4. Easy Find the perimeter of a square whose sides are 7.5 m long.

   Show answer

   30 meters

5. **Medium** A trapezoid has bases of 6 ft and 10 ft and a height of 4 ft. Find its area.

   Show answer

   32 square feet

6. **Medium** A right triangle has legs of 5 cm and 12 cm (the two sides that form the right angle). Find its area.

   Show answer

   30 square centimeters

7. **Medium** Grace's rectangular vegetable garden is 15 ft long and 8 ft wide. How many feet of fencing does she need to go all the way around it?

   Show answer

   46 feet

8. **Medium** The church lawn is a rectangle 20 ft by 30 ft. Sod costs $0.45 per square foot. How much will it cost to cover the lawn with sod?

   Show answer

   $270.00 (600 square feet)

9. **Hard** Grace decides to leave a 3-foot opening in her 15 ft by 8 ft garden fence for a gate, with no fencing across the opening. Fencing costs $3.50 per foot. How much will the fencing cost?

   Show answer

   $150.50 (43 feet of fencing)

10. **Hard** A parallelogram has an area of 84 square inches and a base of 12 inches. What is its height?

    Show answer

    7 inches

11. **Hard** The end wall of Henry's barn is a rectangle 24 ft wide and 10 ft tall with a triangle on top. The triangle has a base of 24 ft and a height of 6 ft. Find the area of the whole wall.

    Show answer

    312 square feet ($240 + 72$)

12. **Challenge** A trapezoid has an area of 45 square cm and a height of 5 cm. One base is 7 cm. How long is the other base?

    Show answer

    11 cm ($\frac{1}{2}(7 + b) \times 5 = 45$ gives $7 + b = 18$)

13. **Challenge** Explain why the area of a triangle is $\frac{1}{2}bh$.

    Show answer

    Two copies of the same triangle, one turned around, fit together to form a parallelogram with the same base and height. The parallelogram's area is bh, and the triangle is exactly half of it, so its area is $\frac{1}{2}bh$.

COMMON MISTAKES

- Using the slanted side as the height of a parallelogram or triangle. The height is always measured straight up, at a right angle to the base.
- Forgetting the $\frac{1}{2}$ in the triangle and trapezoid formulas. Estimate: a triangle should cover about half of the rectangle drawn around it.
- Mixing up perimeter and area, or their units. Perimeter is a length (feet); area is in square units (square feet).
- Counting inside lines in the perimeter of a composite figure. Only the outside edges count.

> **Make it a game · Area Builder.** Cut out paper shapes: a rectangle, parallelogram, triangle and trapezoid. Prove each formula by cutting and rearranging into a rectangle, then explain aloud. Parent draws a composite figure with side lengths and you find the perimeter and area in two minutes. Score a point per proof and a point per correct answer. Play to ten.

12.4 Circles

OBJECTIVES

- Identify the radius, diameter and circumference of a circle and explain what π means.
- Find the circumference and area of a circle, giving answers in terms of π or as decimals using $\pi \approx 3.14$ or the calculator's π key.
- Explain why the area of a circle is πr^2 by cutting the circle into wedges.

WATCH

Math Antics - Circles, Circumference And Area

mathantics · 7:56

https://www.youtube.com/watch?v=O-cawByg2aA

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

The Area of a Circle (1 of 2: Seeing the Composite Figure)

Eddie Woo · 5:56

https://www.youtube.com/watch?v=J2P4x_HTKkU

After the 'cut the circle into wedges' paragraph (objective 3).

KEY TERMS

circle
All the points in a flat surface that are the same distance from one point, called the center.

radius
The distance from the center of a circle to the circle itself (plural: radii). We write it r.

diameter
The distance straight across a circle through its center. It is twice the radius: $d = 2r$.

circumference
The distance around a circle; the circle's perimeter.

pi (π)

The number you get when you divide any circle's circumference by its diameter. It is about 3.14, but its decimal never ends and never repeats.

CORE IDEAS

A **circle** is every point at the same distance, the **radius** r, from a center point. The **diameter** d goes straight across through the center, so it is two radii: $d = 2r$, and $r = \dfrac{d}{2}$.

Wrap a string around a can and compare it to the distance across the top: the string is a little more than 3 diameters long. For every circle, big or small, the circumference divided by the diameter is the same number, called π (the Greek letter pi). That is why $C = \pi d$, or, since $d = 2r$, $C = 2\pi r$. The number π is irrational (Lesson 6.5): $\pi = 3.14159\ldots$ Common approximations are 3.14 and $\dfrac{22}{7}$.

Why the area is πr^2. Cut a circle into many thin wedges, like slices of a pie. Lay them in a row, pointing up and down by turns. The row looks almost like a parallelogram. Its height is the radius r. Half of the wedges' curved edges are on the bottom and half are on the top, so the bottom is half the circumference: $\dfrac{1}{2} \times 2\pi r = \pi r$. The thinner the wedges, the more the shape becomes a true rectangle with base πr and height r. So the area is $\pi r \times r = \pi r^2$.

Exact or approximate? An answer "in terms of π" leaves π as a symbol and is exact: a circle with radius 5 cm has area $\pi \times 5^2 = 25\pi$ square centimeters. If a problem says to use $\pi \approx 3.14$, multiply by 3.14: $25 \times 3.14 = 78.5$ square centimeters. If it says to use the calculator's π key, you get $78.539\ldots$ and round as told. Always read which one the problem asks for.

Remember to square only the radius: πr^2 means $\pi \times r \times r$. If you are given the diameter, find the radius first.

KEY FORMULAS

$$d = 2r$$

$$C = \pi d = 2\pi r$$

$$A = \pi r^2$$

$$\pi \approx 3.14 \approx \frac{22}{7}$$

From history. Archimedes of Syracuse, a Greek mathematician of the 200s BC, trapped the circle between polygons with 96 sides, one inside the circle and one outside. By working out their perimeters, he showed that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$. That is why $\frac{22}{7}$, which equals $3\frac{1}{7}$, is still used as an approximation of π.

Practice habit. Recite the two circle formulas, "Circumference is pi d; area is pi r squared," and learn the first multiples of 3.14: 3.14, 6.28, 9.42, 12.56, 15.7. Estimate with $\pi \approx 3$ first: a circle's area is a little more than 3 times r^2.

WORKED EXAMPLES

Example 1. A circle has a radius of 5 cm. Find its circumference and its area. Give both answers in terms of π.

Show solution

1. Circumference: $C = 2\pi r = 2 \times \pi \times 5 = 10\pi$ cm.
2. Area: $A = \pi r^2 = \pi \times 5^2 = 25\pi$ square cm.
3. Check by estimating with $\pi \approx 3$: $C \approx 30$ cm and $A \approx 75$ square cm, which are reasonable for a circle 10 cm across.

Answer: $C = 10\pi$ cm (about 31.4 cm); $A = 25\pi$ square cm (about 78.5 square cm)

Example 2. Ruth is making a circular flower garden 14 feet across. She will put plastic edging all the way around it and cover the inside with mulch. Using $\pi \approx 3.14$, how many feet of edging does she need, and how many square feet will the mulch cover?

Show solution

1. "14 feet across" is the diameter: $d = 14$ ft, so $r = 7$ ft.
2. Edging is the circumference: $C = \pi d \approx 3.14 \times 14 = 43.96$ feet.
3. Mulch covers the area: $A = \pi r^2 \approx 3.14 \times 7^2 = 3.14 \times 49 = 153.86$ square feet.
4. Check with $\pi \approx 3$: $3 \times 14 = 42$ and $3 \times 49 = 147$, both close ✓.

Answer: About 43.96 feet of edging; about 153.86 square feet of mulch

PRACTICE

1. Easy A circle has a radius of 9 inches. What is its diameter?

   Show answer

   18 inches

2. Easy A circle has a diameter of 15 ft. What is its radius?

   Show answer

   7.5 feet

3. Easy Find the circumference of a circle with a diameter of 10 m. Give your answer in terms of π.

   Show answer

   10π meters

4. Easy Find the area of a circle with a radius of 3 cm. Give your answer in terms of π.

   Show answer

   9π square centimeters

5. Medium Find the circumference of a circle with a radius of 6 inches. Use $\pi \approx 3.14$.

   Show answer

   37.68 inches

6. **Medium** Find the area of a circle with a radius of 10 ft. Use $\pi \approx 3.14$.

   Show answer

   314 square feet

7. **Medium** Samuel's bicycle wheel is 26 inches across. How far does the bicycle roll forward when the wheel turns around once? Use the calculator's π key and round to the nearest tenth of an inch.

   Show answer

   About 81.7 inches (26π)

8. **Medium** A round pizza is 12 inches across. Find its area. Use $\pi \approx 3.14$.

   Show answer

   113.04 square inches

9. **Hard** A lawn sprinkler waters a circle with a radius of 15 feet. How many square feet of lawn does it water? Use the calculator's π key and round to the nearest square foot.

   Show answer

   About 707 square feet (225π)

10. **Hard** The circumference of a circle is 50.24 cm. Using $\pi \approx 3.14$, find its radius.

    Show answer

    8 cm (the diameter is $50.24 \div 3.14 = 16$ cm)

11. **Hard** Find the area of a half circle (semicircle) whose diameter is 8 inches. Give your answer in terms of π.

    Show answer

    8π square inches (half of 16π)

12. **Challenge** A church window is a rectangle 4 ft wide and 6 ft tall, with a semicircle on top whose diameter is the 4 ft top edge of the rectangle. Find the total area of the window. Use $\pi \approx 3.14$.

    Show answer

    30.28 square feet ($24 + 6.28$)

13. **Challenge** A circle has an area of 78.5 square feet. Using $\pi \approx 3.14$, find its radius.

Show answer

5 feet ($r^2 = 78.5 \div 3.14 = 25$)

14. **Challenge** Explain why the area of a circle is πr^2.

Show answer

Cut the circle into many thin wedges and lay them side by side, alternating up and down. They form almost a rectangle whose height is the radius r and whose base is half the circumference, πr. Its area is $\pi r \times r = \pi r^2$.

COMMON MISTAKES

- Using the diameter in $A = \pi r^2$. If you are given the diameter, divide by 2 to get the radius first.
- Squaring π along with the radius, or computing $(2r)^2$ for r^2. Only the radius is squared: $\pi \times r \times r$.
- Mixing up the formulas. Circumference is a length ($C = \pi d$, in feet); area is in square units ($A = \pi r^2$, in square feet).
- Ignoring the problem's directions about π. Give the answer in terms of π, with 3.14, or with the π key, exactly as the problem says.

> **Make it a game · Circle Measure Hunt.** Find five round things in the house: a plate, a can, a clock, a cup and a coin. Measure the diameter of each with a ruler, then compute the radius, circumference and area using pi as 3.14. Check circumference by wrapping string around the circle. Score a point per correct measurement. Whoever finds the closest match wins.

12.5 Volume and Surface Area

OBJECTIVES

- Find the volume of prisms and cylinders with $V = Bh$ and explain why it works by stacking layers.
- Use a net to find the surface area of rectangular prisms, triangular prisms and cylinders.
- Solve word problems about filling and covering boxes, tanks and cans, with the correct cubic or square units.

WATCH

Math Antics - Volume

mathantics · 12:36

https://www.youtube.com/watch?v=qJwecTgce6c

> **WHEN TO WATCH**
> Before Example 1 (volume).

BACKUP VIDEO

Surface Area of a Prism (1 of 2: Introduction)

Eddie Woo · 5:53

https://www.youtube.com/watch?v=tFuBZvoBqE0

Before the surface-area example: using a net.

KEY TERMS

prism

A solid with two matching, parallel ends (the bases) that are polygons, joined by rectangles. A box is a rectangular prism; a tent shape is a triangular prism.

cylinder

A solid with two matching, parallel circle ends joined by one curved side, like a soup can.

volume

The amount of space inside a solid, measured in cubic units such as cubic inches (in^3).

surface area

The total area of all the outside surfaces of a solid, measured in square units such as square inches (in^2).

net

A flat pattern that folds up to make a solid, like a cardboard box cut open and laid flat.

lateral area

The area of the sides of a prism or cylinder, not counting the two bases. For a can, it is the area of the label.

CORE IDEAS

Volume counts cubes. A box 6 in long, 4 in wide and 3 in high has a bottom layer of $6 \times 4 = 24$ one-inch cubes. There are 3 such layers, so it holds $24 \times 3 = 72$ cubic inches. The bottom layer's count is just the area of the base, B, and the number of layers is the height, h. So for any prism, **volume = base area × height**: $V = Bh$. For a rectangular prism, $B = lw$, so $V = lwh$.

The same layer idea works for any solid whose cross sections are all the same as its base. A triangular prism is a stack of triangles, so $V = Bh$ with $B = \frac{1}{2}bh$ of the triangle. A cylinder is a stack of circles, so $B = \pi r^2$ and $V = \pi r^2 h$. Be careful: the "h" in $V = Bh$ is the height (or length) of the solid, which is different from the height of a triangle base.

Surface area from a net. Imagine cutting a box along its edges and laying it flat. You get six rectangles: top and bottom ($l \times w$), front and back ($l \times h$), and two ends ($w \times h$). Add their areas: $SA = 2lw + 2lh + 2wh$. For the 6 in by 4 in by 3 in box, $SA = 2(24) + 2(18) + 2(12) = 48 + 36 + 24 = 108$ square inches.

Surface area of a cylinder. Peel the label off a can and flatten it: it is a rectangle. Its height is the can's height h, and its length is the distance around the can, the circumference $2\pi r$. So the label (lateral area) is $2\pi rh$. Add the two circle ends, $2\pi r^2$, to get $SA = 2\pi r^2 + 2\pi rh$.

Which one do I need? Volume answers "how much does it hold?" (water in a tank, sand in a sandbox) and is in cubic units. Surface area answers "how much covers it?" (paint, wrapping paper, a label) and is in square units.

KEY FORMULAS

$$\text{Prism or cylinder: } V = Bh$$

$$\text{Rectangular prism: } V = lwh, \quad SA = 2lw + 2lh + 2wh$$

$$\text{Cylinder: } V = \pi r^2 h, \quad SA = 2\pi r^2 + 2\pi rh$$

> **Practice habit.** Ask yourself "fill or cover?" before every problem: fill means volume in cubic units, cover means surface area in square units. Recite "Volume is base times height."

WORKED EXAMPLES

Example 1. A box is 6 in long, 4 in wide and 3 in high. Find its volume and its surface area.

Show solution

1. Base area: $B = 6 \times 4 = 24$ square inches.
2. Volume: $V = Bh = 24 \times 3 = 72$ cubic inches.
3. Faces of the net: top and bottom $6 \times 4 = 24$ each; front and back $6 \times 3 = 18$ each; ends $4 \times 3 = 12$ each.
4. Surface area: $2(24) + 2(18) + 2(12) = 48 + 36 + 24 = 108$ square inches.
5. Check: $6 \times 4 \times 3 = 72$ ✓.

Answer: Volume 72 cubic inches; surface area 108 square inches

Example 2. A farmer has a closed cylindrical water tank with a radius of 3 ft and a height of 5 ft. Using $\pi \approx 3.14$, find (a) how many cubic feet of water it holds and (b) how many square feet of metal make up its outside, including the top and bottom.

Show solution

1. (a) Base area: $\pi r^2 \approx 3.14 \times 3^2 = 3.14 \times 9 = 28.26$ square feet.
2. Volume: $V = Bh \approx 28.26 \times 5 = 141.3$ cubic feet.
3. (b) Top and bottom: $2 \times 28.26 = 56.52$ square feet.
4. Curved side: $2\pi rh \approx 2 \times 3.14 \times 3 \times 5 = 94.2$ square feet.
5. Surface area: $56.52 + 94.2 = 150.72$ square feet.
6. Check in terms of π: $V = 45\pi$ and $SA = 18\pi + 30\pi = 48\pi$; $45 \times 3.14 = 141.3$ ✓ and $48 \times 3.14 = 150.72$ ✓.

Answer: (a) about 141.3 cubic feet (b) about 150.72 square feet

PRACTICE

1. Easy Find the volume of a rectangular prism 5 cm long, 4 cm wide and 2 cm high.

   Show answer

   40 cubic centimeters

2. Easy Find the volume of a cube whose edges are 3 inches long.

   Show answer

   27 cubic inches

3. Easy Find the surface area of a cube whose edges are 3 inches long.

   Show answer

   54 square inches (6×9)

4. Easy A triangular prism has a triangle base with an area of 12 square feet. The prism is 10 feet long. Find its volume.

   Show answer

   120 cubic feet

5. Medium A cylinder has a radius of 2 inches and a height of 10 inches. Find its volume in terms of π.

   Show answer

   40π cubic inches

6. Medium Find the surface area of a rectangular prism 10 cm long, 5 cm wide and 4 cm high.

   Show answer

   220 square centimeters

7. Medium Henry builds a sandbox 6 ft long and 4 ft wide. How many cubic feet of sand does he need to fill it 1.5 ft deep?

   Show answer

   36 cubic feet

8. Medium Grace wraps a gift box 12 in long, 8 in wide and 3 in high. If the paper does not overlap, how many square inches of wrapping paper cover the box exactly?

Show answer

312 square inches

9. Medium A box has a volume of 180 cubic inches. Its base is 6 in by 5 in. How tall is the box?

Show answer

6 inches ($180 \div 30$)

10. Hard A grain silo is a cylinder with a radius of 10 ft and a height of 40 ft. How many cubic feet of grain can it hold? Use the calculator's π key and round to the nearest cubic foot.

Show answer

About 12,566 cubic feet (4000π)

11. Hard A soup can has a radius of 1.5 in and a height of 4 in. How many square inches of paper does the label use? (The label covers the curved side exactly, with no overlap.) Use $\pi \approx 3.14$.

Show answer

37.68 square inches

12. Hard A triangular prism is 10 cm long. Each triangle end is a right triangle with sides of 3 cm, 4 cm and 5 cm (the 3 cm and 4 cm sides form the right angle). Find the surface area of the prism.

Show answer

132 square centimeters (two triangles of 6 plus rectangles of 30, 40 and 50)

13. Challenge Samuel's fish tank is 20 in long, 10 in wide and 12 in high. One gallon is 231 cubic inches. How many gallons does the tank hold when full? Round to the nearest tenth of a gallon.

Show answer

About 10.4 gallons (2400 cubic inches)

14. **Challenge** Explain why the volume of a prism is the area of its base times its height.

Show answer

The base area tells how many unit cubes fit in one layer one unit thick on the bottom. The prism is a stack of identical layers, and the height tells how many layers there are. So the number of cubes is base area times height.

COMMON MISTAKES

- Giving volume in square units or surface area in cubic units. Volume fills (cubic units); surface area covers (square units).
- Forgetting faces when finding surface area. Sketch the net and check that you have every face: six for a box, five for a triangular prism, three pieces for a cylinder.
- Using the diameter as the radius in $\pi r^2 h$. Halve the diameter first.
- Confusing the height of a triangle base with the length of the prism. In $V = Bh$, first find the triangle's area B, then multiply by the prism's length.

> **Make it a game · Box and Net Challenge.** Take a small cereal box. Unfold it into a net, measure the faces, and compute surface area. Then compute volume by length times width times height and explain it as stacked layers. Estimate before you measure, and score a point for each estimate within ten percent. Race the clock for the whole sequence.

12.6 The Pythagorean Theorem

OBJECTIVES

- State the Pythagorean theorem and name the legs and hypotenuse of a right triangle.
- Explain a picture proof of the theorem by rearranging four triangles inside a square.
- Find the hypotenuse of a right triangle and recognize Pythagorean triples.

WATCH

Math Antics - The Pythagorean Theorem

mathantics · 12:55

https://www.youtube.com/watch?v=WqhlG3Vakw8

> **WHEN TO WATCH**
> Before Example 1.

Note: Eddie Woo teaches in Australia: he says 'Pythagoras' theorem'.

BACKUP VIDEO

Pythagoras' Theorem (2 of 3: Dissection Proof)

Eddie Woo · 9:43

https://www.youtube.com/watch?v=EzCWuNZpBaQ

After the picture-proof paragraph (objective 2).

KEY TERMS

legs

The two sides of a right triangle that form the right angle.

hypotenuse

The side across from the right angle in a right triangle. It is always the longest side.

Pythagorean theorem

In every right triangle, the square of the hypotenuse equals the sum of the squares of the legs: $a^2 + b^2 = c^2$.

Pythagorean triple

Three whole numbers that fit $a^2 + b^2 = c^2$, like 3, 4, 5.

CORE IDEAS

In a right triangle, the two sides that make the right angle are the **legs**, usually called a and b. The side across from the right angle is the **hypotenuse**, c; it is always the longest side. The **Pythagorean theorem** says: if you build a square on each side, the two smaller squares together have exactly the same area as the square on the hypotenuse. In symbols, $a^2 + b^2 = c^2$.

Try it with legs 3 and 4 and hypotenuse 5: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. ✓ The theorem works only for right triangles. For a triangle without a right angle, $a^2 + b^2$ will not equal c^2.

A picture proof. Make four copies of a right triangle with legs a and b and hypotenuse c. Draw a big square whose sides are each $a + b$ long. *Arrangement 1:* put one triangle in each corner of the big square, so that each side of the big square is made of one leg a and one leg b. The hypotenuses form a tilted square in the middle, with area c^2. *Arrangement 2:* slide the same four triangles into two rectangles, each a by b, in two opposite corners of the big square. Now the uncovered space is two squares, one a by a and one b by b, with total area $a^2 + b^2$.

In both arrangements the big square is the same and the four triangles are the same, so the uncovered space must be the same too. The uncovered space was c^2 the first time and $a^2 + b^2$ the second time. Therefore $a^2 + b^2 = c^2$.

Finding the hypotenuse. Square the legs, add, then take the square root. With legs 6 and 8: $6^2 + 8^2 = 36 + 64 = 100$, so $c = \sqrt{100} = 10$. Often the sum of the squares is not a perfect square, so the hypotenuse is irrational; then use the square root key on a calculator and round as told.

Pythagorean triples are whole-number answers. Learn 3-4-5, 5-12-13, 8-15-17 and 7-24-25. Any whole-number multiple of a triple is also a triple: doubling 3-4-5 gives 6-8-10, since every square gets multiplied by 4.

KEY FORMULAS

$$a^2 + b^2 = c^2$$

$$c = \sqrt{a^2 + b^2}$$

Pythagorean triples: 3-4-5, 5-12-13, 8-15-17, 7-24-25, and their multiples

From history. The theorem is named for Pythagoras, a Greek thinker of the 500s BC whose followers studied numbers and shapes. People knew the rule earlier: Babylonian clay tablets from long before list sets of numbers that fit $a^2 + b^2 = c^2$. The Greeks are remembered for giving a proof that it is true for every right triangle.

Euclid connection. Euclid proves the Pythagorean theorem in Elements I.47: in a right triangle, the square on the side across from the right angle equals the squares on the other two sides. Its converse, which you will use in Lesson 12.7, is I.48.

Practice habit. Recite the perfect squares from 1^2 to 25^2 and the triples 3-4-5, 5-12-13, 8-15-17, 7-24-25. Before computing a hypotenuse, check whether the legs are part of a triple you already know.

WORKED EXAMPLES

Example 1. A right triangle has legs of 6 cm and 8 cm. Find the length of the hypotenuse.

Show solution

1. Write the theorem: $a^2 + b^2 = c^2$.
2. Substitute: $6^2 + 8^2 = c^2$.
3. Square and add: $36 + 64 = 100$, so $c^2 = 100$.
4. Take the square root: $c = \sqrt{100} = 10$.
5. Check: 6-8-10 is double the triple 3-4-5 ✓, and 10 is the longest side ✓.

Answer: 10 cm

Example 2. A baseball diamond is a square with sides of 90 feet. The catcher throws the ball straight from home plate to second base, which is the opposite corner of the square. How far is the throw? Use a calculator and round to the nearest tenth of a foot.

Show solution

1. The diagonal of the square splits it into two right triangles. The legs are two sides of the square, 90 ft and 90 ft; the throw is the hypotenuse.
2. $c^2 = 90^2 + 90^2 = 8100 + 8100 = 16{,}200$.
3. $c = \sqrt{16{,}200} \approx 127.279$.
4. Round: $c \approx 127.3$ feet.
5. Check: the hypotenuse must be longer than 90 but shorter than $90 + 90 = 180$ ✓.

Answer: About 127.3 feet

PRACTICE

1. Easy A right triangle has legs of 3 in and 4 in. Find the hypotenuse.

   Show answer

   5 inches

2. Easy A right triangle has legs of 5 m and 12 m. Find the hypotenuse.

   Show answer

   13 meters

3. Easy A right triangle has sides of 8 ft, 15 ft and 17 ft. How long is its hypotenuse?

   Show answer

   17 feet (the longest side)

4. Easy Is 6, 8, 10 a Pythagorean triple? Answer yes or no.

   Show answer

   Yes; $36 + 64 = 100$

5. **Medium** Is 4, 5, 6 a Pythagorean triple? Answer yes or no.

Show answer

No; $16 + 25 = 41$, but $6^2 = 36$

6. **Medium** A right triangle has legs of 9 cm and 12 cm. Find the hypotenuse.

Show answer

15 cm ($81 + 144 = 225$)

7. **Medium** A rectangular pasture is 40 m long and 30 m wide. How long is a straight path from one corner to the opposite corner?

Show answer

50 meters

8. **Medium** A right triangle has legs of 4 in and 7 in. Find the hypotenuse to the nearest tenth of an inch. You may use a calculator for the square root.

Show answer

About 8.1 inches ($\sqrt{65}$)

9. **Hard** A flat-screen TV is 48 inches wide and 27 inches tall. TVs are measured by the diagonal of the screen. How long is the diagonal, to the nearest tenth of an inch? Use a calculator.

Show answer

About 55.1 inches ($\sqrt{3033}$)

10. **Hard** Ruth rides her bike 7 miles due south and then 24 miles due west. How far is she, in a straight line, from where she started?

Show answer

25 miles ($49 + 576 = 625$)

11. **Hard** Multiply each number of the triple 8, 15, 17 by 3. What new triple do you get? (Check that it works.)

Show answer

24, 45, 51 ($576 + 2025 = 2601 = 51^2$)

12. **Challenge** A square tile is 10 cm on each side. How long is its diagonal, to the nearest tenth of a centimeter? Use a calculator.

    Show answer

    About 14.1 cm ($\sqrt{200}$)

13. **Challenge** Explain how four copies of a right triangle placed inside a square with side $a + b$ show that $a^2 + b^2 = c^2$.

    Show answer

    Placed one way, the four triangles leave a tilted square of area c^2 uncovered. Rearranged into two rectangles, the same four triangles leave two squares of areas a^2 and b^2 uncovered. The big square and the triangles are the same both times, so the uncovered areas are equal: $a^2 + b^2 = c^2$.

COMMON MISTAKES

- Adding the legs instead of their squares: $3 + 4 \neq 5^2$. Square each leg first, add, then take the square root.
- Forgetting the last step and giving c^2 as the answer (100 instead of 10). The hypotenuse is the square root of the sum.
- Writing $\sqrt{a^2 + b^2} = a + b$. The square root of a sum is not the sum of the square roots: $\sqrt{9 + 16} = 5$, not 7.
- Using the theorem on a triangle that is not a right triangle, or putting a leg in place of c. The hypotenuse c is always across from the right angle and is the longest side.

> **Make it a game · Name That Side.** Cut four identical right triangles from paper and arrange them inside a square, as in the picture proof. Name the legs and hypotenuse aloud as parent points to each. Then state the theorem from memory and explain why the pieces prove it. Parent sneaks in one wrong statement for you to catch. Score a point per catch.

12.7 Using the Pythagorean Theorem

OBJECTIVES

- Find a missing leg or hypotenuse of a right triangle in word problems.
- Find the distance between two points on the coordinate plane by drawing a right triangle.
- Use the converse of the Pythagorean theorem to decide whether a triangle is a right triangle, as in the carpenter's 3-4-5 check.

WATCH

Review: Pythagoras' Theorem & its Converse

Eddie Woo · 11:16

https://www.youtube.com/watch?v=pzTAtqIdqFc

> **WHEN TO WATCH**
> Before Example 1: the theorem and its converse.

Note: Eddie Woo teaches in Australia: he says 'Pythagoras' theorem'.

BACKUP VIDEO

Calculating Distance (2 of 5: Using Pythagoras w/ Coordinates)

Eddie Woo · 5:56

https://www.youtube.com/watch?v=98nRnimn7G4

Before the distance-on-a-grid example (objective 2).

KEY TERMS

missing leg

An unknown side that forms the right angle. Find it with $a^2 = c^2 - b^2$.

distance between two points

The length of the straight segment joining two points on the coordinate plane.

converse

A statement made by switching the "if" part and the "then" part of another statement. The converse of a true statement is not always true, so it needs its own reason.

square corner

A builder's name for a right angle ($90°$).

CORE IDEAS

Finding a missing leg. If you know the hypotenuse c and one leg b, start from $a^2 + b^2 = c^2$ and subtract b^2 from both sides: $a^2 = c^2 - b^2$. Subtract, do not add, because the hypotenuse's square is the largest. A 13-foot ladder with its foot 5 feet from a wall reaches $\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$ feet up the wall. Check: a leg must be shorter than the hypotenuse.

In word problems, draw a sketch and mark the right angle first. Walls and the ground, the sides of a rectangle, north and east, and up and across are all at right angles. The side across from the right angle is c.

Distance on the coordinate plane. To find the distance between $(-2, 1)$ and $(4, 9)$, draw a right triangle whose hypotenuse is the segment between them. The horizontal leg is the change in x: $4 - (-2) = 6$. The vertical leg is the change in y: $9 - 1 = 8$. So the distance is $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$. (This is the same counting of units along a horizontal or vertical line that you did in Lesson 11.1.)

The converse. The Pythagorean theorem says: if a triangle has a right angle, then $a^2 + b^2 = c^2$. The converse says: if the sides of a triangle fit $a^2 + b^2 = c^2$ (with c the longest side), then the triangle has a right angle. The converse is also true. Here is why: build a right triangle with legs a and b. By the theorem its hypotenuse is $\sqrt{a^2 + b^2} = c$. So it has the same three side lengths as our triangle, and two triangles with the same three sides are exactly the same shape, so our triangle has a right angle too.

The carpenter's 3-4-5 check. To make sure a corner is square, a carpenter measures 3 feet along one board and 4 feet along the other and marks both spots. If the marks are exactly 5 feet apart, the corner is a right angle, because $3^2 + 4^2 = 5^2$. For bigger projects, use 6-8-10 or 9-12-15. If the longest side squared is not equal to the sum of the other two squares, the triangle is not a right triangle.

KEY FORMULAS

$$a^2 = c^2 - b^2, \quad a = \sqrt{c^2 - b^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If $a^2 + b^2 = c^2$, the triangle is a right triangle.

Euclid connection. Euclid proves the converse in Elements I.48, right after the theorem itself in I.47: if the square on one side of a triangle equals the squares on the other two sides, the angle between those two sides is a right angle.

Practice habit. Before you compute, ask "Am I finding the hypotenuse or a leg?" Say "hypotenuse: add the squares; leg: subtract the squares." Then check that the hypotenuse came out as the longest side.

WORKED EXAMPLES

Example 1. Henry leans a 13-foot ladder against the side of the barn. The foot of the ladder is 5 feet from the wall, and the ground is level. How high up the wall does the ladder reach?

Show solution

1. The wall and the ground meet at a right angle. The ladder is the hypotenuse: $c = 13$. One leg is $b = 5$.
2. Use $a^2 = c^2 - b^2$: $a^2 = 13^2 - 5^2 = 169 - 25 = 144$.
3. Take the square root: $a = \sqrt{144} = 12$.
4. Check: $5^2 + 12^2 = 25 + 144 = 169 = 13^2$ ✓, and 12 is less than 13 ✓.

Answer: 12 feet

Example 2. Find the distance between the points $(-2, 1)$ and $(4, 9)$ on the coordinate plane.

Show solution

1. Draw a right triangle with the segment as its hypotenuse and legs that are horizontal and vertical.
2. Horizontal leg: $4 - (-2) = 6$ units.
3. Vertical leg: $9 - 1 = 8$ units.
4. Distance: $\sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.
5. Check: 6-8-10 is double the triple 3-4-5 ✓.

Answer: 10 units

PRACTICE

1. Easy A right triangle has a hypotenuse of 10 cm and one leg of 6 cm. Find the other leg.

   Show answer

   8 cm

2. Easy A right triangle has a hypotenuse of 13 in and one leg of 12 in. Find the other leg.

   Show answer

   5 inches

3. Easy A triangle has sides of 5, 12 and 13 inches. Is it a right triangle? Answer yes or no.

   Show answer

   Yes; $25 + 144 = 169 = 13^2$

4. Easy A triangle has sides of 6, 7 and 9 cm. Is it a right triangle? Answer yes or no.

   Show answer

   No; $36 + 49 = 85$, but $9^2 = 81$

5. **Medium** Find the distance between $(1, 2)$ and $(4, 6)$.

   Show answer

   5 units

6. **Medium** Find the distance between $(-3, -1)$ and $(5, 5)$.

   Show answer

   10 units (legs 8 and 6)

7. **Medium** A 17-foot ladder leans against a house, with its foot 8 feet from the wall on level ground. How high up the wall does the ladder reach?

   Show answer

   15 feet ($289 - 64 = 225$)

8. **Medium** Samuel is building a deck. To check a corner, he measures 6 ft along one edge and 8 ft along the other and marks both spots. The marks are exactly 10 ft apart. Is the corner square (a right angle)? Answer yes or no.

   Show answer

   Yes; $36 + 64 = 100 = 10^2$

9. **Hard** Grace flies a kite on a 50-foot string pulled straight. The kite is directly above a spot on the ground 30 feet from where she holds the string. Ignoring Grace's height, how high is the kite?

   Show answer

   40 feet ($2500 - 900 = 1600$)

10. **Hard** A right triangle has a hypotenuse of 12 m and one leg of 7 m. Find the other leg to the nearest tenth of a meter. Use a calculator for the square root.

    Show answer

    About 9.7 meters ($\sqrt{95}$)

11. **Hard** Find the distance between $(-4, 3)$ and $(2, -1)$ to the nearest tenth. Use a calculator for the square root.

    Show answer

    About 7.2 units ($\sqrt{36 + 16} = \sqrt{52}$)

12. Hard A carpenter checks a door frame by measuring 3 ft up one side and 4 ft across the bottom. The two marks are 4 ft 10 in apart. Is the corner a right angle? Answer yes or no.

Show answer

No; for a right angle the marks would be exactly 5 ft apart

13. Challenge A storage crate is 3 ft long, 4 ft wide and 12 ft tall. What is the longest straight pole that fits inside, reaching from a bottom corner to the opposite top corner?

Show answer

13 feet (the floor diagonal is 5 ft, and $\sqrt{5^2 + 12^2} = 13$)

14. Challenge Explain why a carpenter can be sure a corner is square when the 3-foot mark and the 4-foot mark are exactly 5 feet apart.

Show answer

The marks and the corner form a triangle with sides 3, 4 and 5, and $3^2 + 4^2 = 5^2$. By the converse of the Pythagorean theorem, a triangle whose sides fit $a^2 + b^2 = c^2$ is a right triangle, so the angle at the corner is 90°.

COMMON MISTAKES

- Adding the squares when finding a leg. For a missing leg, subtract: $a^2 = c^2 - b^2$. Check that the leg is shorter than the hypotenuse.
- Using the wrong side as c when testing for a right triangle. Always let c be the longest side.
- Getting a negative or wrong leg on the coordinate plane by subtracting coordinates carelessly. Subtract with care, $4 - (-2) = 6$, and remember a length is never negative.
- Forgetting to take the square root at the end, giving 144 instead of 12.

> **Make it a game · Distance Detective.** Plot two points on grid paper and draw the right triangle connecting them. Find the distance with the Pythagorean theorem, then check by measuring with a ruler. Parent hides a word problem such as a ladder against a wall. Race to solve it within a minute. Score a point per correct answer. Play to twelve points.

UNIT 12

Geometry

In this unit you use your arithmetic and algebra to measure the world around you. You will find unknown angles where lines cross, learn why every triangle's angles add to $180°$, find the perimeter and area of flat shapes and circles, find how much a box or a can holds and how much material covers it, and finally meet the famous Pythagorean theorem about right triangles.

Every formula in this unit is explained, not just given. The area of a parallelogram, triangle and trapezoid all come from the rectangle; the area of a circle comes from cutting it into wedges; volume comes from stacking layers; and the Pythagorean theorem comes from rearranging four triangles inside a square. Along the way you will meet Euclid's Elements, Archimedes' estimate of π, and Pythagoras.

Parents: this unit draws on fractions, decimals, square roots (Unit 6), solving simple equations (Unit 10) and the coordinate plane (Unit 11). A basic calculator is allowed for messy arithmetic and square roots. Many problems come from building, gardening and the home, so a tape measure, a protractor and a few cans and boxes make good hands-on helpers.

PREREQUISITES

- Multiply and divide whole numbers, fractions and decimals (Units 3 and 4).
- Find squares and square roots and estimate square roots (Lessons 6.4 and 6.5).
- Evaluate formulas by substitution (Lesson 9.2).
- Solve one- and two-step equations (Lessons 10.2 to 10.4).
- Plot points and find horizontal and vertical distances on the coordinate plane (Lesson 11.1).

MASTERY CHECKLIST

- ☐ I can classify angles and find complements, supplements, vertical angles and adjacent angles on a line.
- ☐ I can find the angles formed when a transversal crosses two parallel lines.
- ☐ I can explain why the angles of a triangle add to

$180°$ and find a missing angle.

- ☐ I can classify triangles by sides and angles and find the angle sum of any polygon with

$(n-2) \times 180°$.

- ☐ I can find the perimeter and area of rectangles, parallelograms, triangles, trapezoids and composite figures, and explain where the area formulas come from.
- ☐ I can find the circumference and area of a circle in terms of

π or using an approximation.

- ☐ I can find the volume and surface area of prisms and cylinders.
- ☐ I can state the Pythagorean theorem, explain a picture proof of it and find the hypotenuse of a right triangle.
- ☐ I can find a missing leg, find the distance between two points and use the converse to test for a right angle.

LESSONS IN THIS UNIT

12.1 Angles and Angle Relationships	45 minutes
12.2 Triangles and Polygons	2 sessions of 45 minutes
12.3 Perimeter and Area	2 sessions of 45 minutes
12.4 Circles	2 sessions of 45 minutes
12.5 Volume and Surface Area	2 sessions of 45 minutes
12.6 The Pythagorean Theorem	2 sessions of 45 minutes
12.7 Using the Pythagorean Theorem	2 sessions of 45 minutes

Unit 12 test

You have 60 minutes. You may use a basic calculator for messy arithmetic and square roots. For problems with π, give the answer in terms of π or use $\pi \approx 3.14$, exactly as each problem says. Show your work, include units in every answer, and round only where the problem tells you to.

1. **5 pts**
   Find the complement and the supplement of a 27° angle.

2. **6 pts**
   A transversal crosses two parallel lines. One of the interior angles is 112°. Find (a) its alternate interior angle and (b) the angle adjacent to that alternate interior angle along the same parallel line.

3. **6 pts**
   Two angles of a triangle measure 38° and 79°. Find the third angle.

4. **6 pts**
   Find the measure of each interior angle of a regular pentagon.

5. **7 pts**

   A trapezoid has bases of 9 cm and 15 cm and a height of 6 cm. Find its area.

6. **7 pts**

   The Miller family's living room is a rectangle 14 ft by 10 ft with a 4 ft by 6 ft rectangular corner cut out for a closet. Carpet costs $3 per square foot. How much will it cost to carpet the room?

7. **6 pts**

   A circle has a diameter of 20 cm. Find its circumference and its area in terms of π.

8. **7 pts**

   Ruth is sewing lace around the edge of a round tablecloth with a radius of 3 feet for the church potluck. How many feet of lace does she need? Use $\pi \approx 3.14$.

9. **7 pts**

   A cylinder has a radius of 4 inches and a height of 9 inches. Find its volume. Use $\pi \approx 3.14$.

10. **7 pts**

    Henry is painting all six outside faces of a wooden box 10 in long, 6 in wide and 4 in high. How many square inches will he paint?

11. **6 pts**

    A right triangle has legs of 8 m and 15 m. Find the hypotenuse.

12. **7 pts**

    A 25-foot ladder leans against a wall on level ground. Its foot is 7 feet from the wall. How high up the wall does it reach?

13. **7 pts**

    Find the distance between the points $(-1, -2)$ and $(5, 6)$.

14. **7 pts**

    A triangle has sides of 7 in, 9 in and 12 in. Is it a right triangle? Answer yes or no, and show why.

15. **9 pts**

Explain why the three angles of any triangle add to 180°.

ANSWER KEY

1. Complement 63°; supplement 153°
2. (a) 112° (b) 68°
3. 63°
4. 108° ($540^\circ \div 5$)
5. 72 square centimeters
6. \$348 (116 square feet)
7. $C = 20\pi$ cm; $A = 100\pi$ square cm
8. 18.84 feet
9. 452.16 cubic inches
10. 248 square inches
11. 17 meters
12. 24 feet ($625 - 49 = 576$)
13. 10 units (legs 6 and 8)
14. No; $49 + 81 = 130$, but $12^2 = 144$
15. Draw a line through one corner parallel to the opposite side. The two angles on either side of the triangle's corner along that line are alternate interior angles with the triangle's other two angles, so they equal them. Together the three angles make a straight line, which is 180°.

UNIT 13

Data and Probability

This last unit turns arithmetic toward two practical questions: "What is typical?" and "How likely is it?" In the first two lessons you will describe a list of numbers with the mean, median, mode and range, see how one unusual value (an outlier) can mislead, and learn to choose the measure that tells the truth about the data. You will also read data given as frequency counts.

In the last two lessons you will measure chance with probability, a number from 0 to 1. You will find probabilities of simple events, use complements, compare what should happen in theory with what actually happens in an experiment, count outcomes with organized lists, tree diagrams and the counting principle, and find the probability of independent events.

Everything here uses skills you already have: fractions, decimals and percents (Units 3, 4 and 8), ratios and proportions (Unit 7), and working backward with equations (Unit 10). Statistics and probability come back in Algebra I and in everyday life, from batting averages to weather forecasts.

PREREQUISITES

- Add, subtract, multiply and divide whole numbers and decimals (Units 1 and 4).
- Simplify fractions and multiply fractions (Lessons 3.2 and 3.5).
- Convert among fractions, decimals and percents (Lessons 4.5 and 8.1).
- Use proportions and unit rates to make predictions (Lessons 7.2 and 7.3).
- Round decimals to a given place (Lesson 4.1).

MASTERY CHECKLIST

- ☐ I can find the mean, median, mode and range of a data set.
- ☐ I can find the median when there is an even number of values.
- ☐ I can find a missing value that gives a data set a desired mean.
- ☐ I can identify an outlier and explain how it changes the mean, median and range.
- ☐ I can choose the mean, median or mode to describe a data set and explain my choice.
- ☐ I can find the mean, median and mode of data given as frequency counts.
- ☐ I can find the probability of a simple event and write it as a fraction, a decimal and a percent.
- ☐ I can use the complement to find the probability that an event does not happen.
- ☐ I can find an experimental probability and use a probability to predict results.
- ☐ I can list a sample space with an organized list or a tree diagram.
- ☐ I can use the counting principle to count outcomes.
- ☐ I can find the probability of two independent events by multiplying.

LESSONS IN THIS UNIT

13.1 Mean, Median, Mode and Range	45 minutes
13.2 Choosing a Measure and Outliers	2 sessions of 45 minutes
13.3 Probability of Simple Events	45 minutes
13.4 Counting and Compound Events	2 sessions of 45 minutes

Unit 13 test

You have 50 minutes. A basic calculator is allowed. Show your work for every problem. Write probabilities as fractions in lowest terms unless the problem asks for a decimal or percent. Round to the nearest hundredth only when a problem says to round, and give money to the nearest cent.

1. **6 pts**
   Find the mean, median, mode and range of 7, 3, 9, 3 and 8.

2. **6 pts**
   Find the median of 14, 22, 9, 17, 30 and 12.

3. **8 pts**
   Samuel's first four math test scores are 84, 90, 78 and 88. What score does he need on the fifth test to have a mean of 86?

4. **6 pts**
   Identify the outlier in the data set 18, 21, 19, 20, 52, 22. Then find the mean of the data without the outlier.

5. **8 pts**
   Ten students reported how many books they read over the summer: 2 books: 3 students; 3 books: 4 students; 4 books: 2 students; 6 books: 1 student. Find the mean, median and mode.

6. **8 pts**

   Six used bicycles at a yard sale are priced $80, $95, $75, $90, $85 and $400. Find the mean price and the median price. Which better describes a typical price?

7. **6 pts**

   You roll a fair number cube with faces 1 through 6. What is the probability of rolling a number less than 3?

8. **6 pts**

   A bag holds 5 red, 3 green and 2 yellow marbles. You pick one without looking. What is the probability that it is not green? Write it as a fraction, a decimal and a percent.

9. **8 pts**

   Ruth planted 60 bean seeds and 51 of them sprouted. What is the experimental probability that a seed sprouts, as a decimal? About how many of 200 seeds should she expect to sprout?

10. **6 pts**

    For the church potluck, Henry is making sandwiches with one kind of bread (3 choices), one meat (4 choices) and one cheese (2 choices). How many different sandwiches are possible?

11. **8 pts**

    You flip a fair coin and spin a spinner with 5 equal sections numbered 1 through 5. What is the probability of tails and an odd number?

12. **8 pts**

    You roll two fair number cubes. What is the probability that the sum is 9?

13. **8 pts**

    Explain why the median often describes a typical value better than the mean when a data set has an outlier.

14. **8 pts**

Grace's five piano practice sessions this week had a mean length of 14 minutes. If one session is left out, the mean of the other four is 12 minutes. How long was the session that was left out?

ANSWER KEY

1. Mean 6, median 7, mode 3, range 6
2. 15.5
3. 90
4. Outlier 52; mean without it 20
5. Mean 3.2, median 3, mode 3
6. Mean \$137.50, median \$87.50; the median, because \$400 is an outlier.
7. $\frac{1}{3}$
8. $\frac{7}{10} = 0.7 = 70\%$
9. 0.85; about 170 seeds
10. 24 sandwiches
11. $\frac{3}{10}$
12. $\frac{4}{36} = \frac{1}{9}$ (the pairs 3-6, 4-5, 5-4, 6-3)
13. The mean adds every value, so one very large or very small value pulls it far from where most of the data are. The median is the middle value in order, so an outlier at one end barely changes it, and it stays close to the typical values.
14. 22 minutes

13.1 Mean, Median, Mode and Range

OBJECTIVES

- Compute the mean, median, mode and range of a list of numbers.
- Find the median of a data set with an even number of values.
- Find a missing value that gives a data set a desired mean.

WATCH

Math Antics - Mean, Median and Mode

mathantics · 11:04

https://www.youtube.com/watch?v=B1HEzNTGeZ4

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Using the Mean (1 of 2: Finding an extra value)

Eddie Woo · 6:40

https://www.youtube.com/watch?v=UG-19DEAbJU

Before the missing-value example (objective 3): finding the value that gives a target mean.

KEY TERMS

data

A collection of facts or numbers, such as test scores or daily temperatures. Each number in the list is called a value.

mean

The sum of all the values divided by how many values there are. It is also called the average.

median

The middle value when the data are put in order from least to greatest. If there are two middle values, the median is the mean of those two.

mode

The value that appears most often. A data set can have one mode, more than one mode, or no mode.

range

The largest value minus the smallest value. It tells how spread out the data are.

CORE IDEAS

When you have a list of numbers, such as your quiz scores, it helps to have one number that describes the whole list. The mean, median and mode are three ways to describe the *center* of the data, the value that is typical. The range describes something different: how spread out the data are.

The **mean** shares everything out equally. Add all the values, then divide by how many values there are. If four friends have 3, 5, 6 and 10 cookies, they have $3 + 5 + 6 + 10 = 24$ cookies together. Shared equally, each friend gets $24 \div 4 = 6$ cookies, so the mean is 6. That is why the mean is sometimes called the *fair share*.

The **median** is the middle value. First put the values in order from least to greatest; this step is required, because the middle of an unordered list means nothing. With an odd number of values, exactly one value sits in the middle. With an even number of values, two values share the middle, so the median is their mean. For 2, 4, 7, 9 the middle values are 4 and 7, and the median is $\frac{4 + 7}{2} = 5.5$.

The **mode** is the value that appears most often. In 3, 5, 5, 8, 5, 2 the mode is 5. If two values tie for most often, both are modes; if every value appears the same number of times, there is no mode. The **range** is the largest value minus the smallest value. A small range means the values are close together; a large range means they are spread out.

Because the mean is the sum divided by the count, the sum equals the mean times the count. This lets you find a missing value. If you want 5 scores to have a mean of 80, the 5 scores must add up to $5 \times 80 = 400$. Subtract the scores you already have from 400 to find the score you still need.

KEY FORMULAS

$$\text{mean} = \frac{\text{sum of the values}}{\text{number of values}}$$

$$\text{sum of the values} = \text{mean} \times \text{number of values}$$

$$\text{range} = \text{largest value} - \text{smallest value}$$

Practice habit. Estimate the mean before you compute it: it must lie between the smallest and largest values, and usually near the middle of the bunch. If your computed mean falls outside that range, you made a mistake.

WORKED EXAMPLES

Example 1. Grace's scores on six 10-point quizzes were 8, 10, 7, 9, 10 and 4. Find the mean, median, mode and range.

Show solution

1. Mean: add the values. $8 + 10 + 7 + 9 + 10 + 4 = 48$.
2. Divide by the number of values, 6. $48 \div 6 = 8$, so the mean is 8.
3. Median: put the values in order: 4, 7, 8, 9, 10, 10.
4. There are 6 values, so the two middle values are 8 and 9. The median is $\frac{8 + 9}{2} = 8.5$.
5. Mode: 10 appears twice and every other value once, so the mode is 10.
6. Range: $10 - 4 = 6$.
7. Check: the mean 8 is between the smallest value 4 and the largest value 10, so it is reasonable.

Answer: Mean 8, median 8.5, mode 10, range 6

Example 2. Samuel bowled four games with scores of 112, 125, 108 and 131. What must he score in his fifth game so that the mean of all five games is 120?

Show solution

1. For a mean of 120 over 5 games, the five scores must add up to $5 \times 120 = 600$.
2. Add the four scores he has: $112 + 125 + 108 + 131 = 476$.
3. Subtract to find the missing score: $600 - 476 = 124$.
4. Check: $476 + 124 = 600$ and $600 \div 5 = 120$. Correct.

Answer: He must score 124 in the fifth game.

PRACTICE

1. Easy Find the mean of 4, 7, 9, 12 and 8.

   Show answer

   8

2. Easy Find the median of 15, 3, 9, 21 and 11.

   Show answer

   11

3. Easy Find the mode of 2, 5, 3, 5, 7, 2, 5, 8.

   Show answer

   5

4. Easy Find the range of 34, 18, 51, 27 and 45.

   Show answer

   33

5. Medium Find the median of 6, 14, 9, 3, 11 and 8.

   Show answer

   8.5

6. Medium Henry recorded the high temperature each day for a week: 68°F, 72°F, 75°F, 70°F, 68°F, 74°F and 77°F. What was the mean high temperature?

   Show answer

   72°F

7. Medium Find the mean, median, mode and range of 5, 9, 5, 12, 6 and 11.

Show answer

Mean 8, median 7.5, mode 5, range 7

8. Medium The Miller family's grocery bills for four weeks were $142.50, $128.75, $155.25 and $133.50. What was the mean weekly grocery bill?

Show answer

$140.00

9. Hard Ruth's first four test scores are 88, 92, 79 and 85. What score does she need on her fifth test to have a mean of 87?

Show answer

91

10. Hard The mean of six numbers is 15. Five of the numbers are 12, 18, 10, 20 and 14. What is the sixth number?

Show answer

16

11. Hard Explain why you must put the values in order before finding the median.

Show answer

The median is the middle value of the data from least to greatest. In an unordered list the number in the middle position could be the largest or smallest value, so it would not describe the center. Ordering the values puts the true middle value in the middle position.

12. Challenge The mean weight of 5 pumpkins from Samuel's garden is 20 pounds. After he picks a sixth pumpkin, the mean weight of all six pumpkins is 22 pounds. How much does the sixth pumpkin weigh?

Show answer

32 pounds

COMMON MISTAKES

- Finding the median without first putting the values in order. Always write the values from least to greatest before you look for the middle.

- Dividing by the wrong number when finding the mean. Count the values carefully, including any repeated values and any zeros.
- With an even number of values, picking one of the two middle values as the median. Take the mean of the two middle values instead.
- Giving the number of times a value appears as the mode. The mode is the value itself (in 4, 6, 6, 6 the mode is 6, not 3).

> **Make it a game · Data Dash.** Collect ten real numbers from your home, such as the number of books on each shelf or minutes you spend on chores. Write them down, then race the clock to find the mean, median, mode and range. Parent checks. Then remove one number and see which value changes most. Beat your time on a second set.

13.2 Choosing a Measure and Outliers

OBJECTIVES

- Identify an outlier and describe how it changes the mean, median and range.
- Choose the mean, median or mode as the best description of a data set and explain the choice.
- Find the mean, median and mode of data given as frequency counts.

WATCH

Comparing Measures of Central Tendency (1 of 2: Mean & Mode)

Eddie Woo · 8:33

https://www.youtube.com/watch?v=sjjnFxDX_QE

> **WHEN TO WATCH**
> Before the 'choosing a measure' section.

BACKUP VIDEO

Characterising a Dataset (1 of 3: Discussing outliers)

Eddie Woo · 8:41

https://www.youtube.com/watch?v=tuJdiQz0X9w

Before Example 1 (outliers).

KEY TERMS

measure of center

A single number that describes a typical value of a data set. The mean, median and mode are measures of center.

outlier

A value that is much larger or much smaller than the other values in a data set.

frequency

The number of times a value appears in a data set.

frequency count

A way of listing data by giving each value together with how many times it appears, such as "score 8: 5 students".

CORE IDEAS

In Lesson 13.1 you learned three measures of center. They often give different answers, so you must decide which one tells the truth about the data best. The deciding question is: which number describes a *typical* member of the group?

An **outlier** is a value far away from the rest. Suppose five children in a piano class are 9, 10, 10, 11 and 12 years old, and their teacher, who is 68, is added to the list. The mean jumps from 10.4 to 20, but nobody in the room is about 20 years old. The mean uses every value, so one extreme value pulls it toward itself. The median only moves from 10 to 10.5, because it depends on the middle position, not on how big the largest value is. The range grows from 3 to 59, so an outlier makes the data look much more spread out.

Here is a simple guide. Use the **mean** when the values are fairly close together with no outliers, because it uses every value. Use the **median** when there is an outlier, such as a list of house prices with one mansion, because it is not pulled by extreme values. Use the **mode** when you want the most common choice, especially for data that are not numbers at all, such as favorite pie flavors or shoe sizes a store should stock most.

Data are often given as **frequency counts** instead of a long list. "Score 9: 6 students" means the value 9 appears 6 times. To find the total number of values, add the counts. To find the sum of the values, multiply each value by its count and add the products. Then the mean is that sum divided by the total count.

To find the median from frequency counts, first find the position of the middle value. With 20 values, the middle values are the 10th and 11th. Then count through the values from smallest to largest, adding up the counts, until you reach those positions. The mode is simply the value with the largest count.

KEY FORMULAS

$$\text{mean from counts} = \frac{\text{sum of (value} \times \text{count)}}{\text{sum of the counts}}$$

> **Practice habit.** When you meet a data set, scan it for outliers before you compute anything. Ask yourself: "Is any value far from the rest? If so, the median will describe the group better than the mean."

WORKED EXAMPLES

Example 1. The people at a children's piano lesson are 9, 10, 10, 11 and 12 years old, plus the teacher, who is 68. Identify the outlier. Find the mean age with and without the outlier, and find the median with the outlier. Which measure best describes a typical age?

Show solution

1. The value 68 is far from all the others, so 68 is the outlier.
2. Mean with the outlier: $9 + 10 + 10 + 11 + 12 + 68 = 120$, and $120 \div 6 = 20$.
3. Mean without the outlier: $9 + 10 + 10 + 11 + 12 = 52$, and $52 \div 5 = 10.4$.
4. Median with the outlier: in order the ages are 9, 10, 10, 11, 12, 68. The middle values are 10 and 11, so the median is $\frac{10 + 11}{2} = 10.5$.
5. No one in the group is near 20, but most people are near 10.5. The median describes a typical age best.

Answer: Outlier 68; mean 20 with it and 10.4 without it; median 10.5. The median best describes a typical age.

Example 2. On a 10-point spelling quiz, the students in Mrs. Miller's class scored as follows: 6 points: 2 students; 7 points: 3 students; 8 points: 5 students; 9 points: 6 students; 10 points: 4 students. Find the mean, median, mode and range of the scores.

Show solution

1. Number of students: $2 + 3 + 5 + 6 + 4 = 20$.
2. Sum of the scores: $6 \times 2 + 7 \times 3 + 8 \times 5 + 9 \times 6 + 10 \times 4 = 12 + 21 + 40 + 54 + 40 = 167$.
3. Mean: $167 \div 20 = 8.35$.
4. Median: with 20 values, the middle values are the 10th and 11th. The 6s fill positions 1–2, the 7s positions 3–5, the 8s positions 6–10, and the 9s positions 11–16.
5. So the 10th value is 8 and the 11th value is 9. The median is $\frac{8 + 9}{2} = 8.5$.
6. Mode: 9 has the largest count (6 students), so the mode is 9.
7. Range: $10 - 6 = 4$.
8. Check: the mean 8.35 lies between 6 and 10 and close to the median. Reasonable.

Answer: Mean 8.35, median 8.5, mode 9, range 4

PRACTICE

1. Easy Identify the outlier in this data set: 23, 25, 22, 24, 61, 26.

   Show answer

   61

2. Easy A data set is given as frequency counts: the value 2 appears 3 times, the value 4 appears 1 time, and the value 5 appears 2 times. Write the data as a list. How many values are there?

   Show answer

   2, 2, 2, 4, 5, 5; there are 6 values.

3. Easy The Miller children asked everyone at the church potluck to name their favorite kind of pie. Which measure should they use to report the most popular pie: the mean, the median or the mode? Why?

Show answer

The mode, because it gives the most common choice, and pie flavors are not numbers, so the mean and median cannot be found.

4. Easy Find the mean and the median of 4, 5, 5, 6 and 30.

Show answer

Mean 10, median 5

5. Medium Remove the outlier 30 from the data set 4, 5, 5, 6, 30. Find the new mean and the new median. Which measure changed?

Show answer

New mean 5, new median 5. The mean changed (from 10 to 5); the median stayed 5.

6. Medium A soccer team recorded its goals in 8 games: 0 goals in 1 game, 1 goal in 3 games, 2 goals in 2 games, 3 goals in 1 game and 4 goals in 1 game. Find the mean number of goals per game.

Show answer

1.75 goals per game

7. Medium Use the soccer data from Problem 6 (0 goals in 1 game, 1 goal in 3 games, 2 goals in 2 games, 3 goals in 1 game, 4 goals in 1 game). Find the median and the mode.

Show answer

Median 1.5, mode 1

8. Medium Find the range of the data set 12, 15, 14, 13, 16, 14. Then find the range after the value 49 is added.

Show answer

4 before, 37 after

9. Hard Five houses on a street sold for $210,000, $225,000, $198,000, $232,000 and $1,150,000. Find the mean price and the median price. Which better describes a typical house on the street?

Show answer

Mean $403,000, median $225,000. The median is better, because the $1,150,000 house is an outlier that pulls the mean above every other price.

10. Hard Grace counted the children in each family at the church potluck: 1 child: 3 families; 2 children: 5 families; 3 children: 4 families; 4 children: 2 families; 8 children: 1 family. Find the mean number of children per family, rounded to the nearest tenth, and the median. A calculator may be used.

Show answer

Mean about 2.7, median 2

11. Hard Explain why one outlier changes the mean a lot but changes the median very little.

Show answer

The mean adds up every value, so a very large or very small value changes the sum and pulls the mean toward it. The median depends only on which value is in the middle position. An outlier sits at one end of the ordered list, so it shifts the middle position by at most one place and the median barely moves.

12. Challenge A data set of 5 values has a mean of 12 and a median of 10. The largest value is changed from 20 to 45. What are the new mean and the new median?

Show answer

New mean 17, median still 10

COMMON MISTAKES

- Treating a frequency count as a data value, for example averaging the counts 2, 3, 5, 6, 4 instead of the scores. Multiply each value by its count, then divide by the total count.
- Using the mean to describe data with an outlier. Look for outliers first; if there is one, the median usually describes a typical value better.
- Finding the median from frequency counts by picking the middle value in the list of different values. Count positions using the counts to find where the true middle value falls.

- Thinking an outlier must be removed. An outlier is real data; note it and choose a measure that is not misled by it.

> **Make it a game · Outlier Argument.** Write a list of seven numbers with one extreme value. Calculate the mean, median and range with and without the outlier. Then argue to the parent which measure best describes the data and why. Parent plays devil's advocate. Score a point for every sound explanation. Create a new data set with an outlier for the parent to analyze.

13.3 Probability of Simple Events

OBJECTIVES

- Find the probability of a simple event and write it as a fraction, a decimal and a percent.
- Use complementary events to find the probability that an event does not happen.
- Compare experimental and theoretical probability and use probability to make predictions.

WATCH

Math Antics - Basic Probability

mathantics · 11:28

https://www.youtube.com/watch?v=KzfWUEJjG18

> **WHEN TO WATCH**
> Before Example 1.

BACKUP VIDEO

Theoretical vs. Experimental Probability

Eddie Woo · 4:24

https://www.youtube.com/watch?v=NWi6r-c4c-w

Before the experimental-probability section.

KEY TERMS

outcome

One possible result of an experiment. Rolling a number cube has six outcomes: 1, 2, 3, 4, 5 and 6.

event

One outcome or a group of outcomes, such as "rolling an even number".

probability

A number from 0 to 1 that tells how likely an event is. 0 means impossible and 1 means certain.

complement

The event that a given event does not happen. The complement of "rolling a 6" is "not rolling a 6".

theoretical probability

The probability found by reasoning about equally likely outcomes, without doing the experiment.

experimental probability

The probability found by actually doing an experiment many times and counting how often the event happens.

CORE IDEAS

Probability measures how likely something is. When all the outcomes are **equally likely**, such as the six faces of a fair number cube, the probability of an event is the number of outcomes in the event divided by the total number of outcomes. The chance of rolling a 5 is $\frac{1}{6}$, because 1 of the 6 equally likely faces is a 5.

A probability is always between 0 and 1. The event can include no outcomes (probability 0, impossible), all of the outcomes (probability 1, certain), or something in between. So a probability can never be negative or greater than 1. You may write it as a fraction, a decimal or a percent: $\frac{1}{4} = 0.25 = 25\%$. A probability of $\frac{1}{2}$ means the event is as likely as not.

Every outcome is either in an event or in its **complement**, never both. So the two probabilities together make the whole, 1. That gives a shortcut: $P(\text{not } A) = 1 - P(A)$. If the chance of rain is 30%, the chance of no rain is $100\% - 30\% = 70\%$.

Theoretical probability comes from reasoning: a fair coin has 2 equally likely sides, so $P(\text{heads}) = \frac{1}{2}$. **Experimental probability** comes from trying it: if you flip a coin 20 times and get 12 heads, the experimental probability of heads is $\frac{12}{20} = 0.6$. The two often differ in a short experiment, but as you do more and more trials, the experimental probability usually gets closer to the theoretical probability.

Probability helps you predict. If an event has probability p and you repeat the experiment n times, you can expect the event about $p \times n$ times. This is a prediction, not a promise: the real count may be a little more or a little less.

KEY FORMULAS

$$P(\text{event}) = \frac{\text{number of outcomes in the event}}{\text{total number of equally likely outcomes}}$$

$$P(\text{not } A) = 1 - P(A)$$

$$\text{experimental probability} = \frac{\text{number of times the event happened}}{\text{number of trials}}$$

$$\text{expected number of times} = P(\text{event}) \times \text{number of trials}$$

From history. Probability became a branch of mathematics in 1654, when the French mathematicians Blaise Pascal and Pierre de Fermat wrote letters to each other about a puzzle: how to divide the prize fairly when a game of chance is stopped before it is finished. Their careful reasoning about equally likely outcomes is the same method you use in this lesson.

Practice habit. Recite the common fraction–decimal–percent equivalents until they are automatic: $\frac{1}{2} = 0.5 = 50\%$, $\frac{1}{4} = 0.25 = 25\%$, $\frac{3}{4} = 0.75 = 75\%$, $\frac{1}{5} = 0.2 = 20\%$, $\frac{1}{10} = 0.1 = 10\%$, $\frac{1}{3} = 0.\overline{3}$, $\frac{1}{8} = 0.125 = 12.5\%$. Also check every probability you find: it must be between 0 and 1.

Example 1. A bag holds 4 red, 5 blue and 3 green marbles. You pick one marble without looking. Find the probability that it is blue, and the probability that it is not blue. Write each as a fraction in lowest terms.

Show solution

1. Total number of marbles: $4 + 5 + 3 = 12$. Each marble is equally likely to be picked.
2. There are 5 blue marbles, so $P(\text{blue}) = \dfrac{5}{12}$.
3. As a decimal, $5 \div 12 \approx 0.417$, or about 41.7%.
4. Not blue is the complement: $P(\text{not blue}) = 1 - \dfrac{5}{12} = \dfrac{7}{12}$.
5. Check by counting: there are $4 + 3 = 7$ marbles that are not blue, and $\dfrac{7}{12}$ matches.

Answer: $P(\text{blue}) = \dfrac{5}{12}$ (about 41.7%); $P(\text{not blue}) = \dfrac{7}{12}$

Example 2. Henry tossed a paper cup 50 times. It landed on its side 32 times, upside down 11 times and right side up 7 times. Find the experimental probability that the cup lands on its side, as a decimal and a percent. Then predict how many times it would land on its side in 200 tosses.

Show solution

1. Check the trials: $32 + 11 + 7 = 50$. Good.
2. Experimental probability of landing on its side: $\dfrac{32}{50}$.
3. Change to a decimal: $\dfrac{32}{50} = \dfrac{64}{100} = 0.64$, which is 64%.
4. Prediction for 200 tosses: $0.64 \times 200 = 128$.
5. Check: 200 is 4 times 50, and $32 \times 4 = 128$. The answers agree.

Answer: 0.64, or 64%; about 128 times in 200 tosses

PRACTICE

1. Easy You roll a fair number cube with faces 1 through 6. What is the probability of rolling a 4? Write it as a fraction.

   Show answer

   $\frac{1}{6}$

2. Easy You roll a fair number cube with faces 1 through 6. What is the probability of rolling an even number? Write it as a fraction in lowest terms and as a decimal.

   Show answer

   $\frac{1}{2} = 0.5$

3. Easy True or false: an event can have a probability of 1.2.

   Show answer

   False. A probability is never greater than 1.

4. Easy A spinner has 8 equal sections numbered 1 through 8. What is the probability of spinning a number greater than 5? Write it as a fraction and as a decimal.

   Show answer

   $\frac{3}{8} = 0.375$

5. Medium A bag has 3 red and 7 yellow apples. You pick one without looking. What is the probability that it is not red? Write it as a fraction and as a percent.

   Show answer

   $\frac{7}{10}$, or 70%

6. Medium For a church raffle, 250 tickets were sold and one winning ticket will be drawn. Samuel bought 5 tickets. What is the probability that Samuel wins? Write it as a fraction in lowest terms and as a percent.

   Show answer

   $\frac{1}{50}$, or 2%

7. Medium The letters of the word MISSISSIPPI are written on 11 cards, one letter per card. You pick one card without looking. What is the probability of picking an S?

Show answer

$\frac{4}{11}$

8. Medium Ruth shot 40 free throws at basketball practice and made 26 of them. What is the experimental probability that she makes a free throw? Write it as a fraction in lowest terms and as a percent.

Show answer

$\frac{13}{20}$, or 65%

9. Hard Use Ruth's experimental probability from Problem 8 (she made 26 of 40 free throws). About how many free throws should she expect to make if she shoots 60?

Show answer

About 39 free throws

10. Hard The weather forecast says there is a 30% chance of rain on the day of the Miller family picnic. What is the probability that it does not rain? Write it as a percent and as a decimal.

Show answer

70%, or 0.7

11. Hard Grace flipped a coin 10 times and got 7 heads. She says, "The probability of heads for this coin must be $\frac{7}{10}$, so the coin is unfair." Explain why she should not be so sure.

Show answer

$\frac{7}{10}$ is only the experimental probability from 10 flips. In a small number of trials the results often differ from the theoretical probability of $\frac{1}{2}$ just by chance. She should flip the coin many more times; if the coin is fair, the fraction of heads should get closer to $\frac{1}{2}$.

12. **Challenge** A bag holds only red and blue marbles. The probability of picking a red marble is $\frac{2}{5}$. If there are 18 blue marbles, how many red marbles are in the bag?

Show answer

12 red marbles

COMMON MISTAKES

- Dividing the number of outcomes in the event by the number of outcomes not in the event (for 4 red and 6 blue, writing $\frac{4}{6}$ for red). Always divide by the total number of outcomes: $\frac{4}{10}$.
- Forgetting that the outcomes must be equally likely. A spinner with one large red section and two small sections does not have $P(\text{red}) = \frac{1}{3}$.
- Expecting experimental and theoretical probability to match exactly. They usually differ in a few trials and get closer with many trials.
- Writing a probability larger than 1 or a percent larger than 100%. If you get one, you probably divided in the wrong order.

> **Make it a game · Paper Chip Probability.** Put colored paper chips in a bag: say five red, three blue, two yellow. Before each draw, you state the probability of each color as a fraction, a decimal and a percent. Then draw 20 times with replacement and compare the experimental results to the theoretical numbers. Find the probability of not drawing red. Score a point per correct form.

13.4 Counting and Compound Events

OBJECTIVES

- List all outcomes of a compound event with an organized list or a tree diagram.
- Use the counting principle to find the number of possible outcomes.
- Find the probability of independent events by multiplying their probabilities.

WATCH

Counting Outcomes (Size of Sample Space)

Eddie Woo · 5:10

https://www.youtube.com/watch?v=CsXL6OvrIfY

> **WHEN TO WATCH**
> Before Example 1: listing outcomes and the size of the sample space.

Note: Eddie Woo teaches in Australia: he says 'dot diagram' for a two-dice outcome grid.

BACKUP VIDEO

Two-Dice Questions & Dot Diagrams (1 of 2: Introduction)

Eddie Woo · 5:46

https://www.youtube.com/watch?v=UN4yu3TdxDI

Before the practice set: the two-dice table.

KEY TERMS

compound event

An event made of two or more simple events, such as flipping a coin and then rolling a number cube.

sample space

The list of all possible outcomes of an experiment.

tree diagram

A drawing that shows every outcome by branching: one set of branches for the first choice, then branches from each of those for the next choice.

counting principle

If one choice can be made in m ways and a second choice can be made in n ways, the two choices together can be made in $m \times n$ ways.

independent events

Events where the result of one does not change the probability of the other, such as two separate coin flips.

CORE IDEAS

A **compound event** combines two or more simple events. To find its probability, you first need the whole **sample space**, the list of all outcomes. An **organized list** keeps you from missing or repeating any. For flipping two coins, fix the first coin and run through the second: HH, HT, then TH, TT. There are 4 outcomes. Notice that HT and TH are different outcomes, because the coins are flipped one after the other.

A **tree diagram** shows the same thing as a picture. Suppose lunch is one sandwich (ham, turkey or cheese) and one drink (milk or juice). Draw 3 branches for the sandwiches. From the end of each sandwich branch, draw 2 branches for the drinks. Each path from start to finish is one lunch, and there are 6 paths: ham-milk, ham-juice, turkey-milk, turkey-juice, cheese-milk, cheese-juice.

The tree shows why the **counting principle** works. Every one of the 3 sandwich branches splits into the same 2 drink branches, so there are 3 groups of 2 paths, or $3 \times 2 = 6$ paths. If a third choice is added, such as 2 kinds of fruit, every one of those 6 paths splits again into 2, giving $3 \times 2 \times 2 = 12$. Multiply the number of ways to make each choice.

Two events are **independent** when one does not affect the other. A coin does not remember what a number cube did. For independent events, $P(A \text{ and } B) = P(A) \times P(B)$. Why multiply? Flip a coin and roll a number cube: there are $2 \times 6 = 12$ equally likely outcomes. Heads happens in half of them, 6 outcomes, and of those, a 5 or 6 happens in $\frac{2}{6}$, which is 2 outcomes. So $P(\text{heads and 5 or 6}) = \frac{2}{12} = \frac{1}{2} \times \frac{2}{6}$. Taking a fraction of a fraction means multiplying.

When you draw an object from a bag twice, putting the first object **back** before the second draw keeps the draws independent, because the bag is the same both times. If you do not put it back, the second draw depends on the first, which is a topic for later courses. In this lesson, every repeated draw is made *with replacement*.

KEY FORMULAS

$$\text{number of outcomes} = m \times n \quad \text{(counting principle)}$$

$$P(A \text{ and } B) = P(A) \times P(B) \quad \text{for independent events}$$

Practice habit. Memorize the sizes of the common sample spaces: one coin has 2 outcomes, two coins 4, three coins 8; one number cube has 6 outcomes, two number cubes 36. Before you start a list or tree, use the counting principle to know how many outcomes you should end up with.

WORKED EXAMPLES

Example 1. For a picnic lunch, Ruth can choose one sandwich (ham, turkey or cheese), one drink (milk or juice) and one fruit (apple or banana). How many different lunches are possible?

Show solution

1. Tree diagram idea: 3 branches for the sandwiches, each splitting into 2 drink branches, giving 6 paths.
2. Each of those 6 paths splits into 2 fruit branches, giving 12 paths.
3. Counting principle: $3 \times 2 \times 2 = 12$.
4. Check with part of an organized list: the ham lunches are ham-milk-apple, ham-milk-banana, ham-juice-apple, ham-juice-banana. That is 4 for each sandwich, and $3 \times 4 = 12$.

Answer: 12 different lunches

Example 2. Samuel flips a fair coin and rolls a fair number cube. What is the probability that he gets heads and a number greater than 4?

Show solution

1. The coin and the number cube do not affect each other, so the events are independent.
2. $P(\text{heads}) = \frac{1}{2}$.
3. Numbers greater than 4 are 5 and 6, so $P(\text{greater than } 4) = \frac{2}{6} = \frac{1}{3}$.
4. Multiply: $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
5. Check by counting: there are $2 \times 6 = 12$ outcomes, and only H5 and H6 work, so the probability is $\frac{2}{12} = \frac{1}{6}$. It matches.

Answer: $\frac{1}{6}$

PRACTICE

1. Easy You flip a coin two times. List the sample space. How many outcomes are there?

   Show answer

   HH, HT, TH, TT; 4 outcomes

2. Easy Henry has 4 shirts and 3 pairs of pants. How many different outfits of one shirt and one pair of pants can he make?

   Show answer

   12 outfits

3. Easy You flip a fair coin two times. What is the probability of getting heads both times? Write it as a fraction and as a decimal.

   Show answer

   $\frac{1}{4} = 0.25$

4. Easy You roll two number cubes, one red and one blue. How many outcomes are possible?

Show answer

36

5. Medium A spinner has 4 equal sections colored red, blue, green and yellow. You spin it and flip a coin. What is the probability of red and tails?

Show answer

$\frac{1}{8}$

6. Medium An ice cream shop offers 5 flavors, 2 kinds of cone and 3 toppings. How many different cones with one flavor, one kind of cone and one topping can Grace order?

Show answer

30 different cones

7. Medium You roll two fair number cubes. Use an organized list to find the probability that the sum is 7. Write it as a fraction in lowest terms.

Show answer

$\frac{6}{36} = \frac{1}{6}$ (the pairs 1-6, 2-5, 3-4, 4-3, 5-2, 6-1)

8. Medium A bicycle lock uses a 3-digit code. Each digit can be any of 0 through 9, and digits may repeat. How many different codes are possible?

Show answer

1,000 codes

9. Hard You roll two fair number cubes. What is the probability that the sum is 8?

Show answer

$\frac{5}{36}$ (the pairs 2-6, 3-5, 4-4, 5-3, 6-2)

10. Hard A bag holds 3 red and 2 blue marbles. Samuel draws a marble, puts it back, and draws again. What is the probability that both marbles are red? Write it as a fraction and as a decimal.

Show answer

$\frac{9}{25} = 0.36$

11. Hard Grace flips a fair coin 3 times. What is the probability that she gets at least one tails? (Hint: use the complement.)

Show answer

$\frac{7}{8}$

12. Hard Explain, using a tree diagram, why 3 choices of sandwich and 4 choices of drink give 3×4 possible lunches instead of $3 + 4$.

Show answer

In the tree, each of the 3 sandwich branches splits into 4 drink branches. So there are 3 groups of 4 paths, and 3 groups of 4 is $3 \times 4 = 12$. Adding would count each choice only once, not every pair of choices.

13. Challenge The 5 members of a youth choir committee will choose a president and a secretary. The same person cannot hold both jobs. In how many different ways can the two jobs be filled?

Show answer

20 ways

COMMON MISTAKES

- Adding the numbers of choices instead of multiplying. Picture the tree: every branch of the first choice splits into all the branches of the next choice.
- Treating outcomes such as HT and TH, or 2-5 and 5-2 on two number cubes, as the same outcome. They are different outcomes; listing them both keeps all outcomes equally likely.
- Missing outcomes in a list. Work in order: fix the first choice, run through every second choice, then move to the next first choice. Check the count with the counting principle.
- Multiplying probabilities of events that are not independent, such as drawing twice without putting the marble back. Only multiply when one event does not

change the other.

> **Make it a game · Outfit Tree.** Invent a pretend wardrobe with three shirts, two hats and four scarves on paper. Draw a tree diagram or organized list of all outfits, then predict the total by the counting principle before you finish. Parent checks. Then invent a compound event, like flipping a coin and spinning a paper spinner, and count its outcomes.

UNIT 13

Data and Probability

This last unit turns arithmetic toward two practical questions: "What is typical?" and "How likely is it?" In the first two lessons you will describe a list of numbers with the mean, median, mode and range, see how one unusual value (an outlier) can mislead, and learn to choose the measure that tells the truth about the data. You will also read data given as frequency counts.

In the last two lessons you will measure chance with probability, a number from 0 to 1. You will find probabilities of simple events, use complements, compare what should happen in theory with what actually happens in an experiment, count outcomes with organized lists, tree diagrams and the counting principle, and find the probability of independent events.

Everything here uses skills you already have: fractions, decimals and percents (Units 3, 4 and 8), ratios and proportions (Unit 7), and working backward with equations (Unit 10). Statistics and probability come back in Algebra I and in everyday life, from batting averages to weather forecasts.

PREREQUISITES

- Add, subtract, multiply and divide whole numbers and decimals (Units 1 and 4).
- Simplify fractions and multiply fractions (Lessons 3.2 and 3.5).
- Convert among fractions, decimals and percents (Lessons 4.5 and 8.1).
- Use proportions and unit rates to make predictions (Lessons 7.2 and 7.3).
- Round decimals to a given place (Lesson 4.1).

MASTERY CHECKLIST

- ☐ I can find the mean, median, mode and range of a data set.
- ☐ I can find the median when there is an even number of values.
- ☐ I can find a missing value that gives a data set a desired mean.
- ☐ I can identify an outlier and explain how it changes the mean, median and range.
- ☐ I can choose the mean, median or mode to describe a data set and explain my choice.
- ☐ I can find the mean, median and mode of data given as frequency counts.
- ☐ I can find the probability of a simple event and write it as a fraction, a decimal and a percent.
- ☐ I can use the complement to find the probability that an event does not happen.
- ☐ I can find an experimental probability and use a probability to predict results.
- ☐ I can list a sample space with an organized list or a tree diagram.
- ☐ I can use the counting principle to count outcomes.
- ☐ I can find the probability of two independent events by multiplying.

LESSONS IN THIS UNIT

13.1 Mean, Median, Mode and Range	45 minutes
13.2 Choosing a Measure and Outliers	2 sessions of 45 minutes
13.3 Probability of Simple Events	45 minutes
13.4 Counting and Compound Events	2 sessions of 45 minutes

Unit 13 test

You have 50 minutes. A basic calculator is allowed. Show your work for every problem. Write probabilities as fractions in lowest terms unless the problem asks for a decimal or percent. Round to the nearest hundredth only when a problem says to round, and give money to the nearest cent.

1. **6 pts**
   Find the mean, median, mode and range of 7, 3, 9, 3 and 8.

2. **6 pts**
   Find the median of 14, 22, 9, 17, 30 and 12.

3. **8 pts**
   Samuel's first four math test scores are 84, 90, 78 and 88. What score does he need on the fifth test to have a mean of 86?

4. **6 pts**
   Identify the outlier in the data set 18, 21, 19, 20, 52, 22. Then find the mean of the data without the outlier.

5. **8 pts**
   Ten students reported how many books they read over the summer: 2 books: 3 students; 3 books: 4 students; 4 books: 2 students; 6 books: 1 student. Find the mean, median and mode.

6. **8 pts**

   Six used bicycles at a yard sale are priced $80, $95, $75, $90, $85 and $400. Find the mean price and the median price. Which better describes a typical price?

7. **6 pts**

   You roll a fair number cube with faces 1 through 6. What is the probability of rolling a number less than 3?

8. **6 pts**

   A bag holds 5 red, 3 green and 2 yellow marbles. You pick one without looking. What is the probability that it is not green? Write it as a fraction, a decimal and a percent.

9. **8 pts**

   Ruth planted 60 bean seeds and 51 of them sprouted. What is the experimental probability that a seed sprouts, as a decimal? About how many of 200 seeds should she expect to sprout?

10. **6 pts**

    For the church potluck, Henry is making sandwiches with one kind of bread (3 choices), one meat (4 choices) and one cheese (2 choices). How many different sandwiches are possible?

11. **8 pts**

    You flip a fair coin and spin a spinner with 5 equal sections numbered 1 through 5. What is the probability of tails and an odd number?

12. **8 pts**

    You roll two fair number cubes. What is the probability that the sum is 9?

13. **8 pts**

    Explain why the median often describes a typical value better than the mean when a data set has an outlier.

14. **8 pts**

Grace's five piano practice sessions this week had a mean length of 14 minutes. If one session is left out, the mean of the other four is 12 minutes. How long was the session that was left out?

ANSWER KEY

1. Mean 6, median 7, mode 3, range 6
2. 15.5
3. 90
4. Outlier 52; mean without it 20
5. Mean 3.2, median 3, mode 3
6. Mean \$137.50, median \$87.50; the median, because \$400 is an outlier.
7. $\frac{1}{3}$
8. $\frac{7}{10} = 0.7 = 70\%$
9. 0.85; about 170 seeds
10. 24 sandwiches
11. $\frac{3}{10}$
12. $\frac{4}{36} = \frac{1}{9}$ (the pairs 3-6, 4-5, 5-4, 6-3)
13. The mean adds every value, so one very large or very small value pulls it far from where most of the data are. The median is the middle value in order, so an outlier at one end barely changes it, and it stays close to the typical values.
14. 22 minutes

COURSE EXAM

Midterm Exam

Covers Units 1–6. Time: 75 minutes.

You have 75 minutes. No calculator is allowed on this exam; do all work by hand. Show your work for every problem, since partial credit is given for correct steps. Write fractions in lowest terms, and write improper fractions as mixed numbers when the problem is about a measurement. Write repeating decimals with a bar. When an answer is not exact, round as the problem tells you. Include units (feet, dollars, cups, miles) in answers to word problems.

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1. **4 pts** **Lesson 1.4**

   Evaluate $18 - 2 \times 3^2 + 12 \div 4$.

2. **4 pts** **Lesson 1.3**

   Use the distributive property to find 8×99 mentally. Show how you split the problem.

3. **4 pts** **Lesson 1.5**

   On a three-day trip, the Miller family drove 387 miles, 412 miles and 296 miles. Round each distance to the nearest hundred miles and estimate the total distance.

4. **3 pts** **Lesson 2.1**

   Use the divisibility rule for 9 to decide whether 7,452 is divisible by 9. Answer yes or no and show the rule.

5. **4 pts** **Lesson 2.3**

   Write the prime factorization of 360 in exponent form.

6. **4 pts** **Lesson 2.5**

   Use the Euclidean algorithm to find the greatest common factor of 252 and 198. Show each division.

7. **4 pts** **Lesson 2.6**

   Henry mows the lawn every 6 days and washes the family car every 8 days. He did both jobs today. In how many days will he next do both jobs on the same day?

8. **4 pts** **Lesson 3.2**

   Simplify $\frac{42}{56}$ to lowest terms.

9. **4 pts** **Lesson 3.3**

   Add: $\frac{5}{6} + \frac{3}{8}$. Write the answer as a mixed number.

10. **4 pts** **Lesson 3.4**

    Subtract: $6\frac{1}{3} - 2\frac{3}{4}$.

11. **4 pts** **Lesson 3.5**

    Multiply: $2\frac{2}{5} \times 1\frac{7}{8}$. Write the answer in simplest form.

12. **4 pts** **Lesson 3.6**

    A carpenter has a board $7\frac{1}{2}$ feet long. He cuts it into pieces that are each $\frac{5}{8}$ foot long. How many pieces does he get?

13. **4 pts** **Lesson 3.7**

    A cookie recipe uses $\frac{3}{4}$ cup of sugar to make 24 cookies. Grace wants to make 40 cookies for the church bake sale. How much sugar does she need?

14. **4 pts** **Lesson 4.1**

    Order from least to greatest: 0.605, 0.65, 0.6, 0.056.

15. **4 pts** **Lesson 4.2**

    Samuel buys 3 notebooks at $2.45 each and one pen for $1.29. He pays with a $10 bill. How much change should he receive?

16. **4 pts** **Lesson 4.4**

    Divide: $7.56 \div 0.18$.

17. **4 pts** **Lesson 4.5**

    Write $\frac{11}{12}$ as a decimal. Use a bar for any repeating digits.

18. **4 pts** **Lesson 5.3**

    Evaluate $-8 - (-15) + (-4)$.

19. (4 pts) **Lesson 5.2**

At dawn the temperature was $-7°\text{F}$. By noon it had risen 18 degrees, and by midnight it had fallen 13 degrees from the noon temperature. What was the temperature at midnight?

20. (4 pts) **Lesson 5.6**

Evaluate $(-2.5)(-4) + \left(-\frac{3}{4}\right) \div \frac{3}{8}$.

21. (5 pts) **Lesson 5.4**

Explain why $(-3)(-4) = 12$. Use the distributive property and the fact that $4 + (-4) = 0$.

22. (3 pts) **Lesson 6.4**

Find $\sqrt{\frac{64}{121}}$.

23. (4 pts) **Lesson 6.2**

Use the laws of exponents to write $\frac{5^7 \cdot 5^3}{5^8}$ as a single power of 5, then evaluate it.

24. (4 pts) **Lesson 6.3**

Multiply and write the answer in scientific notation: $(4 \times 10^5)(6 \times 10^3)$.

25. (5 pts) **Lesson 6.5**

Challenge. A square garden has an area of 58 square yards. Without a calculator, estimate the side length to the nearest tenth of a yard. Show your reasoning.

ANSWER KEY

1. 3
2. $8 \times 99 = 8 \times (100 - 1) = 800 - 8 = 792$
3. $400 + 400 + 300 = 1{,}100$ miles
4. Yes. The digit sum is $7 + 4 + 5 + 2 = 18$, and 18 is divisible by 9.
5. $2^3 \cdot 3^2 \cdot 5$
6. $252 = 1 \times 198 + 54$; $198 = 3 \times 54 + 36$; $54 = 1 \times 36 + 18$; $36 = 2 \times 18 + 0$. The GCF is 18.
7. In 24 days (the LCM of 6 and 8)
8. $\dfrac{3}{4}$
9. $\dfrac{20}{24} + \dfrac{9}{24} = \dfrac{29}{24} = 1\frac{5}{24}$
10. $6\frac{4}{12} - 2\frac{9}{12} = 5\frac{16}{12} - 2\frac{9}{12} = 3\frac{7}{12}$
11. $\dfrac{12}{5} \times \dfrac{15}{8} = \dfrac{9}{2} = 4\frac{1}{2}$
12. $\dfrac{15}{2} \div \dfrac{5}{8} = \dfrac{15}{2} \times \dfrac{8}{5} = 12$ pieces
13. $\dfrac{3}{4} \times \dfrac{40}{24} = \dfrac{3}{4} \times \dfrac{5}{3} = 1\frac{1}{4}$ cups
14. 0.056, 0.6, 0.605, 0.65
15. $\$1.36$
16. $756 \div 18 = 42$
17. $0.91\overline{6}$
18. $-8 + 15 - 4 = 3$
19. $-7 + 18 - 13 = -2$, so $-2°\text{F}$
20. $10 + (-2) = 8$
21. Since $4 + (-4) = 0$, we have $(-3)(4 + (-4)) = (-3)(0) = 0$. By the distributive property this also equals $(-3)(4) + (-3)(-4) = -12 + (-3)(-4)$. So $-12 + (-3)(-4) = 0$, which means $(-3)(-4)$ must be the opposite of -12, which is 12.
22. $\dfrac{8}{11}$
23. $5^{7+3-8} = 5^2 = 25$
24. $24 \times 10^8 = 2.4 \times 10^9$

25. $\sqrt{58}$ is between 7 and 8 because $7^2 = 49$ and $8^2 = 64$. Since $7.6^2 = 57.76$ and $7.7^2 = 59.29$, and $7.65^2 = 58.5225$ is more than 58, the side is about 7.6 yards.

COURSE EXAM

Final Exam

Covers Units 1–13. Time: 2 hours.

You have 2 hours. A basic calculator is allowed, except on problems marked "by hand." Show your work for every problem, since partial credit is given for correct steps. Write fractions in lowest terms, and write improper fractions as mixed numbers when the problem is about a measurement. Round money to the nearest cent, and round other answers as each problem tells you. For problems with π, follow the directions in the problem. Check equation answers by substituting, and include units in answers to word problems.

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1. **3 pts** **Lesson 1.4**

   By hand, evaluate $4 + 6 \times (9 - 5)^2 \div 3$.

2. **3 pts** **Lesson 2.6**

   By hand, use prime factorization to find the GCF and the LCM of 45 and 75.

3. **4 pts** **Lesson 3.7**

   By hand: Hannah has $3\frac{3}{4}$ yards of fabric. She uses $1\frac{1}{2}$ yards for an apron and $\frac{5}{8}$ yard for a pillow. How much fabric is left?

4. **3 pts** **Lesson 4.5**

   By hand, write $\frac{7}{40}$ as a decimal.

5. **3 pts** **Lesson 5.6**

   By hand, evaluate $-3(4 - 9) + (-12) \div 4$.

6. **4 pts** **Lesson 5.5**

   Explain why $8 \div 0$ is undefined. Use the fact that division undoes multiplication.

7. **3 pts** **Lesson 6.3**

   Write 0.00052 in scientific notation.

8. **3 pts** **Lesson 7.2**

   An 18-ounce box of oats costs $3.42, and a 28-ounce box costs $5.04. Which box is the better buy, and what is its unit price?

9. **3 pts** **Lesson 7.3**

   Solve the proportion $\frac{6}{15} = \frac{n}{40}$.

10. **3 pts** **Lesson 7.4**

    On a map, 1 inch represents 25 miles. Two towns are $3\frac{3}{4}$ inches apart on the map. What is the actual distance between them?

11. **4 pts** **Lesson 7.5**

    Challenge. A truck travels at 45 miles per hour. How many feet per second is that? (1 mile = 5,280 feet and 1 hour = 3,600 seconds.)

12. **3 pts** **Lesson 8.3**

    18 is what percent of 72?

13. **3 pts** **Lesson 8.4**

    The price of a bushel of apples at the farm stand rose from \$40 to \$46. What was the percent increase?

14. **4 pts** **Lesson 8.5**

    Challenge. A pair of work boots is priced at \$140. It is marked 25% off, and then an extra 10% is taken off the sale price at the register. Sales tax of 6% is added to the final sale price. What is the total cost?

15. **3 pts** **Lesson 8.6**

    Ruth deposits \$1,200 in a savings account that pays 3.5% simple interest per year. How much interest does she earn in 4 years?

16. **3 pts** **Lesson 9.2**

    Evaluate $3x^2 - 2x + 5$ when $x = -2$.

17. **3 pts** **Lesson 9.5**

    Simplify $4(x + 3) - 3(x - 2)$.

18. **3 pts** **Lesson 10.4**

    Solve $5x - 7 = 38$ and check your solution.

19. **4 pts** **Lesson 10.5**

    Solve $4(x - 3) = 2x + 10$ and check your solution.

20. **4 pts** **Lesson 10.6**

The perimeter of a rectangular garden is 86 feet. Its length is 7 feet more than its width. Write and solve an equation to find the width and the length.

21. **3 pts** **Lesson 10.7**

Solve $-4x + 7 \leq 31$.

22. **3 pts** **Lesson 11.4**

Find the slope of the line through $(-2, 5)$ and $(4, -7)$.

23. **4 pts** **Lesson 11.5**

An electrician charges a \$75 visit fee plus \$55 per hour. Write an equation in slope-intercept form for the cost C of a job lasting h hours, and use it to find the cost of a 2.5-hour job.

24. **3 pts** **Lesson 12.2**

Find the measure of each interior angle of a regular nonagon (9 sides).

25. **3 pts** **Lesson 12.4**

A round tablecloth has a diameter of 60 inches. Using $\pi \approx 3.14$, find its area.

26. **4 pts** **Lesson 12.5**

A cylindrical can has a radius of 3 inches and a height of 10 inches. Find its volume in terms of π, then use the calculator's π key to round it to the nearest tenth.

27. **4 pts** **Lesson 12.7**

A 26-foot ladder leans against a wall. The bottom of the ladder is 10 feet from the base of the wall. How high up the wall does the ladder reach?

28. **3 pts** **Lesson 13.1**

Grace's first four test scores are 82, 91, 76 and 87. What score does she need on the fifth test to have a mean of 85?

29. **3 pts** **Lesson 13.4**

You roll a number cube (1 to 6) and flip a coin. What is the probability of rolling an even number and getting heads? Give the answer as a fraction.

30. **4 pts** **Lesson 13.2**

Samuel recorded the hours he practiced piano each week for six weeks: 4, 5, 5, 6, 7, 30. (The 30 was the week before a recital.) Find the mean and the median, and explain which one better describes a typical week.

ANSWER KEY

1. $4 + 6 \times 16 \div 3 = 4 + 96 \div 3 = 4 + 32 = 36$
2. $45 = 3^2 \cdot 5$ and $75 = 3 \cdot 5^2$, so GCF $3 \cdot 5 = 15$, LCM $3^2 \cdot 5^2 = 225$
3. $\frac{30}{8} - \frac{12}{8} - \frac{5}{8} = \frac{13}{8} = 1\frac{5}{8}$ yards
4. 0.175 (it terminates because $40 = 2^3 \cdot 5$)
5. $-3(-5) + (-3) = 15 - 3 = 12$
6. If $8 \div 0$ equaled some number q, then $q \times 0$ would have to equal 8. But any number times 0 is 0, never 8. No number works, so $8 \div 0$ is undefined.
7. 5.2×10^{-4}
8. The 28-ounce box, at \$0.18 per ounce (the 18-ounce box is \$0.19 per ounce)
9. $15n = 240$, so $n = 16$
10. $3.75 \times 25 = 93.75$ miles
11. $\frac{45 \times 5280}{3600} = \frac{237{,}600}{3600} = 66$ feet per second
12. $18 \div 72 = 0.25 = 25\%$
13. $\frac{6}{40} = 0.15 = 15\%$
14. $140 \times 0.75 = 105$; $105 \times 0.90 = 94.50$; $94.50 \times 1.06 = 100.17$, so \$100.17
15. $I = 1200 \times 0.035 \times 4 = 168$, so \$168
16. $3(4) - 2(-2) + 5 = 12 + 4 + 5 = 21$
17. $4x + 12 - 3x + 6 = x + 18$
18. $x = 9$ (check: $5(9) - 7 = 38$)
19. $4x - 12 = 2x + 10$, $2x = 22$, $x = 11$ (check: $4(8) = 32$ and $22 + 10 = 32$)
20. $2(w + w + 7) = 86$, so $4w + 14 = 86$ and $w = 18$. Width 18 feet, length 25 feet.
21. $-4x \le 24$; dividing by -4 reverses the sign, so $x \ge -6$
22. $\frac{-7 - 5}{4 - (-2)} = \frac{-12}{6} = -2$
23. $C = 55h + 75$; $C = 55(2.5) + 75 = 137.50 + 75 = 212.50$, so \$212.50
24. $(9 - 2) \times 180^\circ = 1260^\circ$, and $1260^\circ \div 9 = 140^\circ$
25. The radius is 30 inches, so $A \approx 3.14 \times 30^2 = 3.14 \times 900 = 2826$ square inches
26. $\pi \times 3^2 \times 10 = 90\pi$ cubic inches, about 282.7 cubic inches
27. $\sqrt{26^2 - 10^2} = \sqrt{676 - 100} = \sqrt{576} = 24$ feet

28. The five scores must total $85 \times 5 = 425$. The first four total 336, and $425 - 336 = 89$, so a score of 89
29. $\frac{3}{6} \times \frac{1}{2} = \frac{1}{4}$
30. The mean is $57 \div 6 = 9.5$ hours and the median is 5.5 hours. The median is better, because the outlier 30 pulls the mean up above every other week, while the median stays in the middle of the usual weeks.