

TRIGONOMETRY

First edition

9 units

48 lessons

About 9 hours of video

FIRST EDITION

COURSE GUIDE

This one-semester course teaches high-school trigonometry in 48 lessons. Students begin with similar right triangles and the six trigonometric ratios. They move on to exact values, radian measure and the unit circle, the graphs of the trigonometric functions, identities, inverse functions and equations, and the Laws of Sines and Cosines. The course ends with real applications in surveying, navigation and periodic motion. An optional ninth unit introduces polar coordinates and vectors.

The approach is classical. Formulas are derived and proved, not just handed over, so students learn why each one is true. Euclid's Elements is cited where it fits: the Pythagorean theorem (I.47), similar triangles (VI.4) and the geometric form of the Law of Cosines (II.12–13). Short history notes trace the subject from Hipparchus and Ptolemy's Almagest to Aryabhata, Regiomontanus and Euler. Students see trigonometry as part of a long human effort to measure the earth and the heavens by careful reasoning.

Most videos are by Eddie Woo, a veteran classroom teacher in Australia. He says “sine rule” and “cosine rule” for the Law of Sines and Law of Cosines, “related angle” for reference angle, and often works in degrees. Each lesson notes where his words differ from the course.

Start with Lesson 1.1 →

WHO IT IS FOR

Students ages 15–18 who have completed Geometry and Algebra II, together with the parents and teachers who guide them at home or in a classical school.

HOW TO USE THIS COURSE

1. Take one lesson at a time. Read the core ideas slowly, with pencil and paper, and follow each derivation line by line.

2. Study both worked examples. Cover the steps, try the problem yourself, then compare.
3. Do the practice set in order, from easy to challenge. Show every step and say how you rounded.
4. Check answers only after finishing the set. Rework every missed problem and write one sentence on what went wrong.
5. At the end of each unit, spend one day on review (the mastery checklist and the common mistakes) and one day on the unit test.
6. Take the midterm (Units 1–4) in week 9 and the final exam (Units 1–8) in week 18.
7. Unit 9 (polar coordinates and vectors) is optional. Families who skip it use those days for cumulative review.

SCOPE AND SEQUENCE

UNIT 1 • WEEKS 1–2

Right-Triangle Trigonometry

Define the trigonometric ratios from similar right triangles and use them to find missing sides and angles.

- 1.1 Similar Right Triangles and the Idea of a Ratio
- 1.2 Sine, Cosine and Tangent
- 1.3 Finding Unknown Sides
- 1.4 Finding Unknown Angles
- 1.5 The Reciprocal Ratios

UNIT 2 • WEEKS 2–3

Special Right Triangles and Exact Values

Derive exact values for 30° , 45° and 60° from the square and the equilateral triangle, and prove the cofunction identities.

- 2.1 The 45-45-90 Triangle
- 2.2 The 30-60-90 Triangle
- 2.3 Complementary Angles and Cofunctions

UNIT 3 • WEEKS 4–6

Angles, Radians and the Unit Circle

Extend trigonometry to angles of any size with radian measure and the unit circle, including arc length and angular speed.

- 3.1 Angles in Standard Position
- 3.2 Radian Measure
- 3.3 Converting Degrees and Radians
- 3.4 Arc Length and Sector Area
- 3.5 Linear and Angular Speed
- 3.6 The Unit Circle
- 3.7 Reference Angles and Signs by Quadrant
- 3.8 Exact Values Around the Unit Circle

UNIT 5 • WEEKS 9–11

Trigonometric Identities

Derive and prove the fundamental, Pythagorean, sum, difference, double-angle and half-angle identities.

- 5.1 Fundamental Identities
- 5.2 The Pythagorean Identities
- 5.3 Simplifying and Proving Identities
- 5.4 Sum and Difference Formulas
- 5.5 Double-Angle and Half-Angle Formulas

UNIT 4 • WEEKS 6–9

Graphs of the Trigonometric Functions

Graph sine, cosine, tangent and their reciprocals, and write sinusoidal equations for real periodic data.

- 4.1 The Graph of Sine
- 4.2 The Graph of Cosine
- 4.3 Amplitude and Period
- 4.4 Phase Shift and Vertical Shift
- 4.5 Writing Sinusoidal Equations
- 4.6 The Graph of Tangent
- 4.7 Graphs of Cosecant, Secant and Cotangent

UNIT 6 • WEEKS 11–13

Inverse Functions and Trigonometric Equations

Define the inverse trigonometric functions on restricted domains and solve trigonometric equations.

- 6.1 Inverse Sine and Restricted Domains
- 6.2 Inverse Cosine and Inverse Tangent
- 6.3 Compositions with Inverse Functions
- 6.4 Solving Basic Trigonometric Equations
- 6.5 Equations Using Factoring, Identities and Multiple Angles

UNIT 7 • WEEKS 13–15

Oblique Triangles: The Laws of Sines and Cosines

Prove and use the area formula, the Law of Sines and the Law of Cosines to solve any triangle.

- 7.1 The Area of a Triangle
- 7.2 The Law of Sines
- 7.3 The Ambiguous Case (SSA)
- 7.4 The Law of Cosines
- 7.5 Solving Any Triangle

UNIT 9 • WEEKS 17–18

Polar Coordinates and Vectors (optional)

Optional: work with polar coordinates, polar graphs and vectors.

- 9.1 Polar Coordinates
- 9.2 Converting Between Polar and Rectangular Form
- 9.3 Graphing Polar Equations
- 9.4 Vectors: Magnitude, Direction and Components
- 9.5 Vector Applications

UNIT 8 • WEEKS 15–16

Applications

Apply trigonometry to elevation, navigation, surveying, three-dimensional figures and periodic motion.

- 8.1 Angles of Elevation and Depression
- 8.2 Bearings and Navigation
- 8.3 Surveying and Triangulation
- 8.4 Problems in Three Dimensions
- 8.5 Modeling Periodic Motion

WEEKLY PACING CALENDAR

WEEK	PLAN
I	• Day 1: Lesson 1.1 Similar Right Triangles and the Idea of a Ratio • Day 2: Lesson 1.2 Sine, Cosine and Tangent • Day 3: Lesson 1.3 Finding Unknown Sides (day 1 of 2) • Day 4: Lesson 1.3 Finding Unknown Sides (day 2 of 2) • Day 5: Lesson 1.4 Finding Unknown Angles (day 1 of 2) <i>Days are numbered 1–90 through the semester. Four-day families: merge light days, such as a review day with the day before it, or the two days of a two-day lesson when the student is moving</i>

WEEK	PLAN
	<i>quickly. Keep every unit test.</i>
2	- Day 6: Lesson 1.4 Finding Unknown Angles (day 2 of 2) - Day 7: Lesson 1.5 The Reciprocal Ratios - Day 8: Unit 1 review - Day 9: Unit 1 test - Day 10: Lesson 2.1 The 45-45-90 Triangle
3	- Day 11: Lesson 2.2 The 30-60-90 Triangle - Day 12: Lesson 2.3 Complementary Angles and Cofunctions (day 1 of 2) - Day 13: Lesson 2.3 Complementary Angles and Cofunctions (day 2 of 2) - Day 14: Unit 2 review - Day 15: Unit 2 test
4	- Day 16: Lesson 3.1 Angles in Standard Position - Day 17: Lesson 3.2 Radian Measure - Day 18: Lesson 3.3 Converting Degrees and Radians - Day 19: Lesson 3.4 Arc Length and Sector Area (day 1 of 2) - Day 20: Lesson 3.4 Arc Length and Sector Area (day 2 of 2)
5	- Day 21: Lesson 3.5 Linear and Angular Speed (day 1 of 2) - Day 22: Lesson 3.5 Linear and Angular Speed (day 2 of 2) - Day 23: Lesson 3.6 The Unit Circle (day 1 of 2) - Day 24: Lesson 3.6 The Unit Circle (day 2 of 2) - Day 25: Lesson 3.7 Reference Angles and Signs by Quadrant
6	- Day 26: Lesson 3.8 Exact Values Around the Unit Circle (day 1 of 2) - Day 27: Lesson 3.8 Exact Values Around the Unit Circle (day 2 of 2) - Day 28: Unit 3 review - Day 29: Unit 3 test - Day 30: Lesson 4.1 The Graph of Sine
7	- Day 31: Lesson 4.2 The Graph of Cosine - Day 32: Lesson 4.3 Amplitude and Period (day 1 of 2) - Day 33: Lesson 4.3 Amplitude and Period (day 2 of 2) - Day 34: Lesson 4.4 Phase Shift and Vertical Shift (day 1 of 2) - Day 35: Lesson 4.4 Phase Shift and Vertical Shift (day 2 of 2)

WEEK	PLAN
8	- Day 36: Lesson 4.5 Writing Sinusoidal Equations (day 1 of 2) - Day 37: Lesson 4.5 Writing Sinusoidal Equations (day 2 of 2) - Day 38: Lesson 4.6 The Graph of Tangent - Day 39: Lesson 4.7 Graphs of Cosecant, Secant and Cotangent - Day 40: Unit 4 review
9	- Day 41: Unit 4 test - Day 42: Midterm review (Units 1–4) - Day 43: Midterm exam (Units 1–4) - Day 44: Lesson 5.1 Fundamental Identities - Day 45: Lesson 5.2 The Pythagorean Identities <i>Midterm week. The midterm covers Units 1–4. Unit 5 begins on Day 44.</i>
10	- Day 46: Lesson 5.3 Simplifying and Proving Identities (day 1 of 2) - Day 47: Lesson 5.3 Simplifying and Proving Identities (day 2 of 2) - Day 48: Lesson 5.4 Sum and Difference Formulas (day 1 of 2) - Day 49: Lesson 5.4 Sum and Difference Formulas (day 2 of 2) - Day 50: Lesson 5.5 Double-Angle and Half-Angle Formulas (day 1 of 2)
11	- Day 51: Lesson 5.5 Double-Angle and Half-Angle Formulas (day 2 of 2) - Day 52: Unit 5 review - Day 53: Unit 5 test - Day 54: Lesson 6.1 Inverse Sine and Restricted Domains - Day 55: Lesson 6.2 Inverse Cosine and Inverse Tangent
12	- Day 56: Lesson 6.3 Compositions with Inverse Functions - Day 57: Lesson 6.4 Solving Basic Trigonometric Equations (day 1 of 2) - Day 58: Lesson 6.4 Solving Basic Trigonometric Equations (day 2 of 2) - Day 59: Lesson 6.5 Equations Using Factoring, Identities and Multiple Angles (day 1 of 2) - Day 60: Lesson 6.5 Equations Using Factoring, Identities and Multiple Angles (day 2 of 2)
13	- Day 61: Unit 6 review - Day 62: Unit 6 test - Day 63: Lesson 7.1 The Area of a Triangle - Day 64: Lesson 7.2 The Law of Sines - Day 65: Lesson 7.3 The Ambiguous Case (SSA) (day 1 of 2)
14	Day 66: Lesson 7.3 The Ambiguous Case (SSA) (day 2 of 2)

WEEK	PLAN
	- Day 67: Lesson 7.4 The Law of Cosines (day 1 of 2) - Day 68: Lesson 7.4 The Law of Cosines (day 2 of 2) - Day 69: Lesson 7.5 Solving Any Triangle - Day 70: Unit 7 review
15	- Day 71: Unit 7 test - Day 72: Lesson 8.1 Angles of Elevation and Depression - Day 73: Lesson 8.2 Bearings and Navigation (day 1 of 2) - Day 74: Lesson 8.2 Bearings and Navigation (day 2 of 2) - Day 75: Lesson 8.3 Surveying and Triangulation (day 1 of 2)
16	- Day 76: Lesson 8.3 Surveying and Triangulation (day 2 of 2) - Day 77: Lesson 8.4 Problems in Three Dimensions - Day 78: Lesson 8.5 Modeling Periodic Motion - Day 79: Unit 8 review - Day 80: Unit 8 test <i>Units 1–8 are finished this week.</i>
17	- Day 81: Lesson 9.1 Polar Coordinates (optional Unit 9; families who skip it review Units 1–2 today) - Day 82: Lesson 9.2 Converting Between Polar and Rectangular Form (optional Unit 9; families who skip it review Unit 3 today) - Day 83: Lesson 9.3 Graphing Polar Equations (optional Unit 9; families who skip it review Unit 4 today) - Day 84: Lesson 9.4 Vectors: Magnitude, Direction and Components (optional Unit 9; families who skip it review Units 5–6 today) - Day 85: Lesson 9.5 Vector Applications (optional Unit 9; families who skip it review Units 7–8 today) <i>Unit 9 is optional. Families who skip it use these days for cumulative review and for reworking missed unit-test problems.</i>
18	- Day 86: Unit 9 review (optional; families who skip Unit 9 do a cumulative review of Units 1–8) - Day 87: Unit 9 test (optional; families who skip Unit 9 rework missed unit-test problems) - Day 88: Final exam review, Units 1–4 - Day 89: Final exam review, Units 5–8 - Day 90: Final exam (Units 1–8) <i>Final exam week. The final covers Units 1–8; Unit 9 is not tested.</i>

MATERIALS

Scientific calculator

Must have sin, cos, tan and their inverses, and a degree/radian mode switch. Check the mode before every problem.

Protractor

For drawing and measuring angles in standard position and checking triangle sketches.

Drawing compass

For circles, arcs and the unit circle.

Graph paper

For the unit circle and for graphing sine, cosine, tangent and their transformations.

Ruler or straightedge

For neat triangles and accurate axes.

Pencil and notebook

Keep all work, corrections and derivations in one notebook as a record of the course.

String and a round can (optional)

Wrap string around a can to see that one radian is the central angle that cuts off an arc as long as the radius, and to unroll the unit circle into the sine graph.

Homemade clinometer (optional)

A protractor with a drinking straw and a small weight on a string, for measuring angles of elevation outdoors in Units 1 and 8.

GRADING

Grades reward both careful daily work and mastery shown on tests. Corrected work counts, because learning from mistakes is the point of practice.

PART	%
Practice sets	20
Unit tests	40

PART	%
Midterm exam	15
Final exam	20
Notebook and corrections	5

Scale: A 90–100, B 80–89, C 70–79, D 60–69, below 60 redo

- Grade practice sets on completion and corrections: full credit when every problem is attempted and every missed problem is reworked.
- On tests, give partial credit for a correct setup and method even when the arithmetic slips.
- After a unit test below 70, review the unit and retake a set of similar problems before moving on.
- If your family skips Unit 9, compute the grade the same way; if you do Unit 9, its test counts as one more unit test.

A NOTE TO PARENTS

You do not need to be a math expert to teach this course well. Every lesson explains its ideas step by step, works two full examples, and gives answers for every practice problem. Your main job is to keep a steady rhythm, ask your student to explain their reasoning aloud, and make sure mistakes are corrected rather than skipped.

To check work, compare the final answer with the answer key, then look at the steps. Is the calculator in the right mode? Was rounding done only at the end? When your student is stuck, resist solving the problem for them. Ask: What is given? What is asked? Can you draw it? Which ratio or law connects them? The pacing is a guide, not a rule. Slow down when a topic needs more time, and use the review days to catch up.

Trigonometry has served surveyors, builders, navigators and astronomers for more than two thousand years, and it is used every day in engineering, physics, construction and mapping. Just as important, it trains careful reasoning: drawing a clear picture, choosing a method, proving that it works, and checking that the answer makes sense. These habits will serve your student well in every field.

UNIT 1

RIGHT-TRIANGLE TRIGONOMETRY

Trigonometry began as the art of measuring what cannot be reached: the height of a tower, the width of a river, the distance to the moon. This unit lays its foundation. We start from a fact of geometry, that right triangles sharing an acute angle are similar (Euclid I.32 and VI.4), and from it we see that the ratio of any two sides depends only on the angle. Those ratios get names: sine, cosine and tangent, and their reciprocals cosecant, secant and cotangent.

With these six ratios a student can solve any right triangle. Given an angle and a side, we find the other sides; given two sides, we find the angles with the inverse ratios. Along the way we meet the astronomers who built the first tables of chords and sines, Hipparchus and Ptolemy, and we trace the curious history of the word "sine." The problems come from the farm, the workshop and the sea: shadows of silos, ladders against barns, ramps, rafters and the course of a ship.

Everything later in the course rests on this unit. Unit 2 finds exact values for special angles, Unit 3 extends the ratios to angles of any size on the unit circle, and Units 7 and 8 use them to solve triangles that have no right angle and to model real measurements.

PREREQUISITES

- Solve linear equations and simple proportions, including equations with the unknown in the denominator.
- Use the Pythagorean theorem (Euclid I.47) to find a missing side of a right triangle.
- Know that the angles of a triangle sum to 180° (Euclid I.32).
- Recognize similar triangles and use proportional sides (AA similarity, Euclid VI.4).
- Simplify square roots and rationalize denominators.
- Use a scientific calculator and round to a stated place value.

MASTERY CHECKLIST

- ☐ I can explain why two right triangles that share an acute angle are similar and why their side ratios are equal.
- ☐ I can use similar right triangles, such as shadows, to find an unknown length.
- ☐ I can identify the opposite side, adjacent side and hypotenuse from a given acute angle.
- ☐ I can define sine, cosine and tangent as ratios and compute them from side lengths.
- ☐ I can choose the correct ratio and solve for a missing side, whether the unknown is in the numerator or the denominator.
- ☐ I can use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ to find an angle, with my

calculator.
in degree mode.

- ☐ I can solve a right triangle completely and check my answers for sense.
- ☐ I can define cosecant, secant and cotangent and evaluate them with a calculator.
- ☐ I can find all six trigonometric ratios of an acute angle when one ratio is given.
- ☐ I can prove simple relationships such as

$$\frac{\sin A}{\cos A} = \tan A$$

from the definitions.

$$(\sin A)^2 + (\cos A)^2 = 1$$

- ☐ I can set up and solve word problems involving ladders, ramps, roofs, shadows and navigation.

LESSONS IN THIS UNIT

1.1 Similar Right Triangles and the Idea of a Ratio

45 minutes

1.2 Sine, Cosine and Tangent

50 minutes

1.3 Finding Unknown Sides

50 minutes

1.4 Finding Unknown Angles

50 minutes

1.5 The Reciprocal Ratios

50 minutes

UNIT 1 TEST

Allow 60 minutes. A scientific calculator is permitted; set it to degree mode. Show your work and draw a labeled sketch for every triangle. Give exact answers where asked. Otherwise round lengths to the nearest tenth, angles to the nearest tenth of a degree, and ratios to four decimal places, rounding only at the final step. Total: 100 points.

1. **5 pts**

A right triangle has legs 7 and 24 and hypotenuse 25. A similar right triangle has a shorter leg of 21. Find its hypotenuse.

2. **6 pts**

A right triangle has legs 20 and 21 and hypotenuse 29. Let A be the angle opposite the leg of length 20. Find $\sin A$, $\cos A$ and $\tan A$ exactly and to four decimal places.

3. **5 pts**

Evaluate $\csc 23^\circ$ to four decimal places.

4. **6 pts**

In a right triangle, an acute angle measures 48° and the hypotenuse is 16. Find the side opposite the 48° angle, to the nearest tenth.

5. **6 pts**

In a right triangle, an acute angle measures 33° and the side adjacent to it is 21. Find the hypotenuse, to the nearest tenth.

6. **6 pts**

In a right triangle, the side opposite $\angle A$ is 4 and the hypotenuse is 16. Find $\angle A$ to the nearest tenth of a degree.

7. (6 pts)

On a sunny afternoon a 4-foot fence post casts a shadow 2.5 feet long. At the same moment a tree on Mr. Miller's farm casts a shadow 22 feet long. How tall is the tree? Explain why the method works.

8. (6 pts)

The boom of a crane is 60 feet long and is raised to an angle of 52° above the horizontal. How far above the boom's pivot is the tip of the boom? Round to the nearest tenth of a foot.

9. (9 pts)

If $\sin \theta = \frac{7}{25}$ for an acute angle θ , find $\cos \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$ and $\cot \theta$ exactly.

10. (9 pts)

Solve right triangle ABC (right angle C) with legs $a = 9$ and $b = 18$. Give c exactly and to the nearest tenth, and the angles to the nearest tenth of a degree.

11. (8 pts)

Samuel leans a 28-foot ladder against the side of a barn, with its foot 7 feet from the wall on level ground. Find the angle the ladder makes with the ground (nearest tenth of a degree) and how high up the wall it reaches (nearest tenth of a foot).

12. (7 pts)

A loading ramp must rise 2.5 feet and make a 4° angle with the level ground. How long must the ramp surface be? Round to the nearest tenth of a foot.

13. (10 pts)

Prove that for every acute angle A of a right triangle, $(\sin A)^2 + (\cos A)^2 = 1$. Then use it to find $\cos A$ exactly if $\sin A = \frac{3}{5}$.

14. (11 pts)

In right triangle ABC the right angle is at C , $AC = 12$ and $\angle B = 28^\circ$. Point D lies on side BC between B and C so that $\angle ADC = 55^\circ$. Find BD to the nearest tenth.

ANSWER KEY

1. The scale factor is 3, so the hypotenuse is 75.
2. $\sin A = \frac{20}{29} \approx 0.6897$, $\cos A = \frac{21}{29} \approx 0.7241$, $\tan A = \frac{20}{21} \approx 0.9524$
3. $\csc 23^\circ = \frac{1}{\sin 23^\circ} \approx 2.5593$
4. $16 \sin 48^\circ \approx 11.9$
5. $\frac{21}{\cos 33^\circ} \approx 25.0$
6. $A = \sin^{-1}(0.25) \approx 14.5^\circ$
7. The sun's rays make the same angle with the ground at both places, so the right triangles are similar and $\frac{h}{22} = \frac{4}{2.5}$. The tree is 35.2 feet tall.
8. $60 \sin 52^\circ \approx 47.3$ ft
9. Adjacent side $= \sqrt{625 - 49} = 24$. $\cos \theta = \frac{24}{25}$, $\tan \theta = \frac{7}{24}$, $\csc \theta = \frac{25}{7}$, $\sec \theta = \frac{25}{24}$,
 $\cot \theta = \frac{24}{7}$
10. $c = \sqrt{405} = 9\sqrt{5} \approx 20.1$; $A = \tan^{-1}(0.5) \approx 26.6^\circ$; $B \approx 63.4^\circ$
11. Angle $= \cos^{-1}\left(\frac{7}{28}\right) \approx 75.5^\circ$; height $= \sqrt{28^2 - 7^2} = \sqrt{735} \approx 27.1$ ft
12. $\frac{2.5}{\sin 4^\circ} \approx 35.8$ ft
13. With opposite a , adjacent b and hypotenuse c : $(\sin A)^2 + (\cos A)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$ by the Pythagorean theorem (Euclid I.47). Then $(\cos A)^2 = 1 - \frac{9}{25} = \frac{16}{25}$,
and since $\cos A > 0$, $\cos A = \frac{4}{5}$.
14. $BC = \frac{12}{\tan 28^\circ} \approx 22.5687$ and $DC = \frac{12}{\tan 55^\circ} \approx 8.4025$, so $BD \approx 14.2$.

1.1 SIMILAR RIGHT TRIANGLES AND THE IDEA OF A RATIO

OBJECTIVES

- Explain why two right triangles that share an acute angle are similar (Euclid I.32 and VI.4).
- Use the fixed ratio of two sides to find unknown lengths in similar right triangles.
- Describe why a single table of ratios can serve every right triangle with a given acute angle.

WATCH

Trigonometry (1 of 2: Exploring the connection between ratio of sides on different triangles)

Eddie Woo · 7:53

<https://www.youtube.com/watch?v=0AGBkR7q53k>

WHEN TO WATCH

After reading the Core ideas, before Example 1 (the shadow problem).

BACKUP VIDEO

Why is Trigonometry so important?

Eddie Woo · 5:20

<https://www.youtube.com/watch?v=VjdCDZ0f3s4>

Optional, any time: a 5-minute 'why trig matters' preview.

KEY TERMS

similar triangles

Triangles whose corresponding angles are equal. Their corresponding sides are proportional.

ratio

A comparison of two quantities by division, such as $\frac{3}{5}$ or 0.6.

hypotenuse

The side of a right triangle opposite the right angle. It is always the longest side.

legs

The two sides of a right triangle that form the right angle.

scale factor

The number by which every side of one figure is multiplied to get the corresponding side of a similar figure.

CORE IDEAS

Trigonometry means "triangle measure." Its whole foundation is one fact from geometry: in right triangles, the *shape* is fixed by a single acute angle, and the *size* does not change the ratios of the sides.

Here is why. Suppose $\triangle ABC$ and $\triangle DEF$ both have a right angle ($\angle C = \angle F = 90^\circ$) and share an acute angle, $\angle A = \angle D$. The angles of any triangle add to 180° (Euclid I.32), so the third angles are both $180^\circ - 90^\circ - \angle A$. They are equal too. All three pairs of angles match, so the triangles are equiangular, and by Euclid VI.4 their corresponding sides are proportional.

Proportional sides mean $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC} = k$, the scale factor. Now divide two sides within the same triangle: $\frac{EF}{DE} = \frac{k \cdot BC}{k \cdot AB} = \frac{BC}{AB}$. The k cancels. So the ratio of the side opposite the angle to the hypotenuse is the same in both triangles, however large or small they are. The same holds for any pair of sides.

This is the key idea of the whole course. Once we know the acute angle, every ratio of sides is fixed. A table of those ratios, made once, works for every right triangle that has that angle: a roof rafter, a ladder against a wall, or the shadow of a silo. In the next lesson we give the three most important ratios their names.

KEY FORMULAS

$$\angle A = \angle D, \angle C = \angle F = 90^\circ \implies \triangle ABC \sim \triangle DEF$$

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC} = k$$

$$\frac{EF}{DE} = \frac{BC}{AB}$$

From history. The Greek astronomer Hipparchus, in the second century BC, is credited with compiling an early table of chords: for a circle of fixed size, it listed the length of the chord cut off by each central angle. Such a table works only because the ratio depends on the angle and not on the size of the figure. About three centuries later, Ptolemy's *Almagest* gave a table of chords in half-degree steps, and astronomers used it for over a thousand years.

Euclid connection. Euclid I.32 shows that the angles of a triangle sum to two right angles, so two right triangles sharing one acute angle share all three angles. Euclid VI.4 then shows that equiangular triangles have proportional sides.

WORKED EXAMPLES

Example 1. A right triangle has legs 3 and 4 and hypotenuse 5. A similar right triangle has hypotenuse 15. Find the length of its shorter leg.

Show solution

1. The triangles are similar, so the scale factor is $k = \frac{15}{5} = 3$.
2. The shorter leg of the first triangle is 3, so the shorter leg of the second is $3 \cdot 3 = 9$.
3. Check with the ratio idea: in the first triangle, $\frac{\text{shorter leg}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$. In the second, $\frac{9}{15} = 0.6$. The ratio is the same, as it must be.

Answer: 9

Example 2. On Mr. Miller's farm, a 6-foot fence post casts a shadow 4 feet long. At the same moment, the grain silo casts a shadow 30 feet long. How tall is the silo? Explain why the method is valid.

Show solution

1. The sun's rays are so nearly parallel that they meet the level ground at the same angle at the post and at the silo.
 2. The post and its shadow form a right triangle, and so do the silo and its shadow. Both triangles have a right angle at the ground and the same acute angle where the sun's ray meets the ground. By Euclid I.32 and VI.4 they are similar.
 3. So the ratio $\frac{\text{height}}{\text{shadow}}$ is the same for both: $\frac{h}{30} = \frac{6}{4}$.
 4. Solve: $h = 30 \cdot \frac{6}{4} = 30 \cdot 1.5 = 45$.
-

Answer: The silo is 45 feet tall.

PRACTICE

1. **Easy** A right triangle has legs 5 and 12 and hypotenuse 13. Find the ratio of the shorter leg to the hypotenuse as a fraction and as a decimal rounded to the nearest thousandth.

Show answer

$$\frac{5}{13} \approx 0.385$$

2. **Easy** A right triangle has legs 6 and 8 and hypotenuse 10. A similar right triangle has hypotenuse 25. Find the length of its longer leg.

Show answer

$$20$$

3. **Easy** In right triangle ABC with right angle at C , the leg opposite $\angle A$ is 7 and the leg next to $\angle A$ is 24. Find the ratio $\frac{\text{opposite leg}}{\text{adjacent leg}}$ for $\angle A$ as a fraction and as a decimal rounded to the nearest thousandth.

Show answer

$$\frac{7}{24} \approx 0.292$$

4. **Medium** A right triangle has legs 8 and 15. Find the hypotenuse, then find the ratio of the shorter leg to the hypotenuse (exact fraction and decimal to the nearest thousandth).

Show answer

Hypotenuse 17; ratio $\frac{8}{17} \approx 0.471$

5. **Medium** Ruth is 5 feet tall and casts a shadow 3 feet long. At the same time, the barn casts a shadow 18 feet long. How tall is the barn?

Show answer

30 feet

6. **Medium** Two right triangles each have an acute angle of the same size, θ . In the first, the side opposite θ is 4.2 and the hypotenuse is 6.5. In the second, the hypotenuse is 13. Find the side opposite θ in the second triangle.

Show answer

8.4

7. **Medium** A ramp rises 2 feet over a horizontal run of 24 feet. Samuel builds a second ramp at the same angle with a horizontal run of 36 feet. How high does the second ramp rise?

Show answer

3 feet

8. **Hard** Explain why two right triangles that share one acute angle must be similar. Name the facts from Euclid you use.

Show answer

Both have a right angle and share the acute angle A . Since the angles of a triangle sum to 180° (Euclid I.32), the remaining angle in each is $90^\circ - A$, so all three angles match. Equiangular triangles have proportional sides (Euclid VI.4), so the triangles are similar.

9. **Hard** A right triangle has legs 9 and 40 and hypotenuse 41. A similar triangle has a perimeter of 180. Find the hypotenuse of the larger triangle.

Show answer

The first perimeter is $9 + 40 + 41 = 90$, so the scale factor is 2 and the hypotenuse is 82.

10. **Challenge** Right triangle ABC has legs $AC = 6$ and $BC = 8$ and hypotenuse $AB = 10$. The altitude CD is drawn from the right angle to the hypotenuse. It makes two smaller right triangles, each similar to $\triangle ABC$. Find CD , AD and DB .

Show answer

$$CD = 4.8, AD = 3.6, DB = 6.4$$

COMMON MISTAKES

- Matching the wrong sides. Before setting up a proportion, label each side by its position relative to the shared angle (opposite it, next to it, or the hypotenuse) and match sides with the same label.
- Assuming two right triangles are similar just because both have a right angle. They need a second equal angle as well; check that an acute angle is shared.
- Writing the proportion upside down on one side, such as $\frac{h}{30} = \frac{4}{6}$. Keep the same order in both ratios, for example height over shadow on both sides.

Make it a game • Shadow Stick Duel. On a sunny day, stand a meter stick or broom handle upright and measure its shadow. Measure a tree's or building's shadow at the same moment. Guess the height first, then compute it using similar right triangles and a ratio. Closest guess scores a point. Explain why equal sun angles make the triangles similar.

1.2 SINE, COSINE AND TANGENT

OBJECTIVES

- Define the sine, cosine and tangent of an acute angle as ratios of sides of a right triangle (SOH-CAH-TOA).
- Compute the sine, cosine and tangent of an acute angle from given side lengths, using the Pythagorean theorem when a side is missing.

WATCH

Trigonometry (2 of 2: Defining the ratios found as sine, cosine and tangent)

Eddie Woo · 9:15

<https://www.youtube.com/watch?v=zpG0tLvHVoQ>

WHEN TO WATCH

After reading the Core ideas (SOH-CAH-TOA), before Example 1.

BACKUP VIDEO

Defining the Trigonometric Ratios

Eddie Woo · 7:54

<https://www.youtube.com/watch?v=S113G9FDDXI>

If the labels opposite/adjacent/hypotenuse are still shaky, before Example 1.

KEY TERMS

opposite side

For an acute angle of a right triangle, the leg that does not touch the angle.

adjacent side

For an acute angle of a right triangle, the leg that forms the angle together with the hypotenuse.

sine

For an acute angle θ in a right triangle,

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

cosine

For an acute angle θ in a right triangle,

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

tangent

For an acute angle θ in a right triangle,

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}.$$

CORE IDEAS

In Lesson 1.1 we proved that every right triangle with a given acute angle has the same side ratios. Since these ratios depend only on the angle, we may give them names and treat them as properties of the angle itself.

Pick an acute angle θ in a right triangle. The hypotenuse is opposite the right angle. The **opposite** side is the leg across from θ . The **adjacent** side is the leg that touches θ . Which leg is opposite and which is adjacent depends on which acute angle you are looking at, so always label from the angle in question.

We define $\sin \theta = \frac{\text{opp}}{\text{hyp}}$, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ and $\tan \theta = \frac{\text{opp}}{\text{adj}}$. Many students remember these with the word **SOH-CAH-TOA**: Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, Tangent is Opposite over Adjacent.

Because the hypotenuse is the longest side, $\sin \theta$ and $\cos \theta$ are always between 0 and 1 for an acute angle. The tangent can be any positive number: it is small when the angle is small and grows very large as the angle nears 90° . Notice also that $\frac{\sin \theta}{\cos \theta} = \frac{\text{opp/hyp}}{\text{adj/hyp}} = \frac{\text{opp}}{\text{adj}} = \tan \theta$.

If only two sides are given, find the third with the Pythagorean theorem, $a^2 + b^2 = c^2$ (Euclid I.47), before writing the ratios. Give exact answers as fractions (with radicals if needed) and decimal answers rounded as the problem asks.

KEY FORMULAS

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

From history. Our word "sine" has a long journey behind it. The Indian astronomer Aryabhata (born AD 476) used the Sanskrit word *jya*, "bowstring," for the half-chord of a circle. Arabic scholars borrowed it as *jiba*; written without vowels, it was later read as *jaib*, a fold or pocket. Latin translators in the twelfth century rendered that as *sinus*, which also means a fold, and from *sinus* comes our "sine."

Euclid connection. The ratios are well defined because equiangular triangles have proportional sides (Euclid VI.4). When a side is missing, we find it with the Pythagorean theorem (Euclid I.47).

WORKED EXAMPLES

Example 1. Right triangle ABC has a right angle at C , with $a = BC = 5$, $b = AC = 12$ and $c = AB = 13$. Find $\sin A$, $\cos A$ and $\tan A$ exactly and to four decimal places.

Show solution

1. From $\angle A$: the opposite side is $BC = 5$, the adjacent side is $AC = 12$, and the hypotenuse is $AB = 13$.

$$2. \sin A = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13} \approx 0.3846.$$

$$3. \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13} \approx 0.9231.$$

$$4. \tan A = \frac{\text{opp}}{\text{adj}} = \frac{5}{12} \approx 0.4167.$$

Answer: $\sin A = \frac{5}{13} \approx 0.3846$, $\cos A = \frac{12}{13} \approx 0.9231$, $\tan A = \frac{5}{12} \approx 0.4167$

Example 2. A shed roof rafter rises 4 feet over a horizontal run of 9 feet. Let θ be the angle the rafter makes with the horizontal. Find $\sin \theta$, $\cos \theta$ and $\tan \theta$ exactly and to four decimal places.

Show solution

1. The rise, run and rafter form a right triangle. From θ , the rise (4 ft) is opposite and the run (9 ft) is adjacent.

2. The rafter is the hypotenuse: $c = \sqrt{4^2 + 9^2} = \sqrt{16 + 81} = \sqrt{97} \approx 9.8489$ ft.

$$3. \sin \theta = \frac{4}{\sqrt{97}} = \frac{4\sqrt{97}}{97} \approx 0.4061.$$

$$4. \cos \theta = \frac{9}{\sqrt{97}} = \frac{9\sqrt{97}}{97} \approx 0.9138.$$

$$5. \tan \theta = \frac{4}{9} \approx 0.4444.$$

$$6. \text{Check: } \frac{\sin \theta}{\cos \theta} = \frac{4/\sqrt{97}}{9/\sqrt{97}} = \frac{4}{9}, \text{ which matches } \tan \theta.$$

$$\text{Answer: } \sin \theta = \frac{4\sqrt{97}}{97} \approx 0.4061, \cos \theta = \frac{9\sqrt{97}}{97} \approx 0.9138, \tan \theta = \frac{4}{9} \approx 0.4444$$

PRACTICE

1. **Easy** In right triangle ABC with right angle C , the side opposite $\angle A$ is 3, the side adjacent to $\angle A$ is 4, and the hypotenuse is 5. Find $\sin A$ as an exact value.

Show answer

$$\sin A = \frac{3}{5} = 0.6$$

2. **Easy** A right triangle has legs 8 and 15 and hypotenuse 17. Let B be the angle opposite the leg of length 15. Find $\cos B$ exactly and to four decimal places.

Show answer

$$\cos B = \frac{8}{17} \approx 0.4706$$

3. **Easy** An acute angle of a right triangle has opposite side 9 and adjacent side 12. Find the tangent of the angle as an exact value.

Show answer

$$\frac{9}{12} = \frac{3}{4} = 0.75$$

4. **Medium** A right triangle has legs 2 and 3. Let A be the angle opposite the leg of length 2. Find $\sin A$ exactly (rationalize the denominator) and to four decimal places.

Show answer

The hypotenuse is $\sqrt{13}$, so $\sin A = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \approx 0.5547$.

5. **Medium** A right triangle has hypotenuse 10 and one leg 6. Let θ be the angle opposite the leg of length 6. Find $\cos \theta$ and $\tan \theta$ exactly.

Show answer

The other leg is $\sqrt{100 - 36} = 8$, so $\cos \theta = \frac{8}{10} = 0.8$ and $\tan \theta = \frac{6}{8} = 0.75$.

6. **Medium** A 13-foot ladder leans against a barn wall and reaches 12 feet up the wall. Let θ be the angle between the ladder and the level ground. Find $\sin \theta$, $\cos \theta$ and $\tan \theta$ exactly.

Show answer

The foot is $\sqrt{169 - 144} = 5$ ft from the wall. $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$.

7. **Medium** Explain why the sine and cosine of an acute angle are always less than 1, while the tangent can be greater than 1.

Show answer

Sine and cosine each divide a leg by the hypotenuse, and the hypotenuse is the longest side of a right triangle (it is opposite the largest angle; also $c^2 = a^2 + b^2 > a^2$). So these ratios are less than 1. Tangent divides one leg by the other, and the opposite leg can be longer than the adjacent leg, so tangent can exceed 1.

8. **Hard** Prove that for any acute angle A of a right triangle, $\tan A = \frac{\sin A}{\cos A}$.

Show answer

Let the opposite side be a , the adjacent side b and the hypotenuse c . Then
$$\frac{\sin A}{\cos A} = \frac{a/c}{b/c} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} = \tan A.$$

9. **Hard** Each step of a staircase rises 7 inches and runs 11 inches. Let θ be the angle the stair stringer makes with the floor. Find $\sin \theta$ and $\tan \theta$, each to four decimal places.

Show answer

The hypotenuse of one step is $\sqrt{49 + 121} = \sqrt{170} \approx 13.0384$ in. $\sin \theta = \frac{7}{\sqrt{170}} \approx 0.5369$ and $\tan \theta = \frac{7}{11} \approx 0.6364$.

10. **Challenge** In a right triangle, $\sin A = \cos A$ for one of the acute angles A . What must be true of the two legs? What is $\tan A$? What is the measure of A in degrees?

Show answer

$\frac{\text{opp}}{\text{hyp}} = \frac{\text{adj}}{\text{hyp}}$ forces $\text{opp} = \text{adj}$, so the legs are equal, $\tan A = 1$, and the triangle is isosceles, so $A = 45^\circ$.

11. **Challenge** Prove that for any acute angle A of a right triangle, $(\sin A)^2 + (\cos A)^2 = 1$.

Show answer

With opposite side a , adjacent side b and hypotenuse c , $(\sin A)^2 + (\cos A)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$, using $a^2 + b^2 = c^2$ (Euclid I.47).

COMMON MISTAKES

- Labeling opposite and adjacent from the wrong angle. Put your finger on the angle you are asked about first; the leg it does not touch is the opposite side.
- Using the hypotenuse as the adjacent side. The hypotenuse touches the angle too, but it is never called adjacent; adjacent always means the *leg* beside the angle.
- Forgetting to find the missing side first. If a ratio needs a side you were not given, use $a^2 + b^2 = c^2$ before writing the ratio.
- Rounding the hypotenuse early, for example using $\sqrt{97} \approx 9.8$. Keep the exact radical or full calculator value until the last step.

Make it a game • Triangle Memory Match. Make twelve cards: sine, cosine and tangent; their definitions in words; and their values for a 3-4-5 triangle, such as $3/5$, $4/5$ and $3/4$. Lay them face down and flip two at a time. Keep a pair that match. Then repeat with a 5-12-13 triangle, computing missing hypotenuses, and beat your record.

1.3 FINDING UNKNOWN SIDES

OBJECTIVES

- Choose the trigonometric ratio that connects a known angle, a known side and the unknown side.
- Solve for a missing side of a right triangle, whether the unknown appears in the numerator or the denominator.
- Apply these steps to practical problems with ladders, ramps, roofs and shadows.

WATCH

Finding Side Lengths with Trigonometry

Eddie Woo · 9:55

<https://www.youtube.com/watch?v=v-0aEduKB0c>

WHEN TO WATCH

Before Example 1; pause when the unknown lands in the denominator.

BACKUP VIDEO

Finding Sides in Triangles (Using Trigonometric Ratios to find the sides of a right angled triangle)

Eddie Woo · 11:10

https://www.youtube.com/watch?v=hg_uZAVI_U0

Before the practice set, for more worked examples.

KEY TERMS

given side

A side whose length is known at the start of the problem.

unknown side

The side whose length the problem asks you to find.

degree mode

The calculator setting in which angles are entered in degrees. All angles in this unit are in degrees.

rafter

A sloping roof beam running from the ridge to the wall. With the rise and the run it forms a right triangle.

CORE IDEAS

If we know one acute angle and one side of a right triangle, we can find every other side. The angle fixes the ratios (Lesson 1.1), and the one known side fixes the size.

To choose the ratio, label the sides from the known angle: opposite, adjacent, hypotenuse. Circle the side you know and the side you want. Then pick the ratio that uses exactly those two: opposite and hypotenuse means sine; adjacent and hypotenuse means cosine; opposite and adjacent means tangent.

Write the equation and solve it. If the unknown is in the numerator, multiply: from $\sin 40^\circ = \frac{x}{20}$ we get $x = 20 \sin 40^\circ$. If the unknown is in the denominator, it takes one more step: from $\sin 38^\circ = \frac{9}{x}$, multiply both sides by x to get $x \sin 38^\circ = 9$, then divide:

$$x = \frac{9}{\sin 38^\circ}.$$

Make sure your calculator is in **degree mode**. A quick test: $\sin 30^\circ$ should show 0.5. Do the whole computation in one calculator entry if you can, and round only at the end.

Check your answer for sense. A leg must be shorter than the hypotenuse. The side opposite the larger acute angle must be the longer leg. If your answer breaks one of these rules, look for a wrong ratio or a calculator in the wrong mode.

KEY FORMULAS

$$\text{opp} = \text{hyp} \cdot \sin \theta, \quad \text{adj} = \text{hyp} \cdot \cos \theta$$

$$\text{opp} = \text{adj} \cdot \tan \theta$$

$$\text{hyp} = \frac{\text{opp}}{\sin \theta} = \frac{\text{adj}}{\cos \theta}$$

WORKED EXAMPLES

Example 1. In right triangle ABC with right angle C , $\angle A = 32^\circ$ and the hypotenuse is $c = 15$. Find the legs a (opposite A) and b (adjacent to A), to the nearest tenth.

Show solution

1. From $\angle A$: a is opposite, b is adjacent, and $c = 15$ is the hypotenuse.
 2. Opposite and hypotenuse use sine: $\sin 32^\circ = \frac{a}{15}$, so $a = 15 \sin 32^\circ \approx 15(0.529919) \approx 7.9488$.
 3. Adjacent and hypotenuse use cosine: $\cos 32^\circ = \frac{b}{15}$, so $b = 15 \cos 32^\circ \approx 15(0.848048) \approx 12.7207$.
 4. Check: $7.9488^2 + 12.7207^2 \approx 63.18 + 161.82 = 225 = 15^2$. Also the leg opposite the smaller angle (32°) is the shorter leg, as it should be.
 5. Round: $a \approx 7.9$ and $b \approx 12.7$.
-

Answer: $a \approx 7.9$, $b \approx 12.7$

Example 2. Samuel is building a ramp for wheeling carts up to a porch that is 2 feet above the ground. The ramp is to make a 5° angle with the level ground. How long must the ramp surface be, and how far out from the porch will it reach along the ground? Give both to the nearest hundredth of a foot.

Show solution

1. The rise (2 ft), the horizontal run and the ramp surface form a right triangle with a 5° angle at the ground. From that angle, the rise is opposite, the run is adjacent and the ramp surface is the hypotenuse.

2. Ramp length L : opposite and hypotenuse use sine. $\sin 5^\circ = \frac{2}{L}$. The unknown is in the denominator, so multiply by L and divide by $\sin 5^\circ$: $L = \frac{2}{\sin 5^\circ} \approx \frac{2}{0.0871557} \approx 22.9474$ ft.

3. Run d : opposite and adjacent use tangent. $\tan 5^\circ = \frac{2}{d}$, so $d = \frac{2}{\tan 5^\circ} \approx \frac{2}{0.0874887} \approx 22.8601$ ft.

4. Check: the run is a leg, so it must be a little shorter than the ramp, and it is. Also $\sqrt{22.9474^2 - 2^2} \approx 22.8601$.

5. Round to the nearest hundredth.

Answer: Ramp length ≈ 22.95 ft; horizontal reach ≈ 22.86 ft

PRACTICE

1. **Easy** In a right triangle, an acute angle measures 40° and the hypotenuse is 20. Find the side opposite the 40° angle, to the nearest tenth.

Show answer

$$20 \sin 40^\circ \approx 12.9$$

2. **Easy** In a right triangle, an acute angle measures 55° and the side adjacent to it is 8. Find the side opposite the 55° angle, to the nearest tenth.

Show answer

$$8 \tan 55^\circ \approx 11.4$$

3. **Easy** In a right triangle, an acute angle measures 25° and the hypotenuse is 12. Find the side adjacent to the 25° angle, to the nearest tenth.

Show answer

$$12 \cos 25^\circ \approx 10.9$$

4. **Medium** In a right triangle, an acute angle measures 38° and the side opposite it is 9. Find the hypotenuse, to the nearest tenth.

Show answer

$$\frac{9}{\sin 38^\circ} \approx 14.6$$

5. **Medium** In a right triangle, an acute angle measures 62° and the side opposite it is 14. Find the side adjacent to the 62° angle, to the nearest tenth.

Show answer

$$\frac{14}{\tan 62^\circ} \approx 7.4$$

6. **Medium** A 24-foot ladder leans against the side of a barn and makes a 72° angle with the level ground. How high up the wall does the ladder reach? Round to the nearest tenth of a foot.

Show answer

$$24 \sin 72^\circ \approx 22.8 \text{ ft}$$

7. **Medium** Ruth flies a kite on 150 feet of taut, straight string. The string makes a 41° angle with the horizontal. Ignoring the height of her hand, how high is the kite? Round to the nearest tenth of a foot.

Show answer

$$150 \sin 41^\circ \approx 98.4 \text{ ft}$$

8. **Medium** Explain why, to find the hypotenuse x when you know the opposite side 9 and the angle 38° , you compute $\frac{9}{\sin 38^\circ}$ and not $9 \sin 38^\circ$. How can you tell at a glance that $9 \sin 38^\circ$ must be wrong?

Show answer

The equation is $\sin 38^\circ = \frac{9}{x}$. Multiplying by x gives $x \sin 38^\circ = 9$, and dividing by $\sin 38^\circ$ gives $x = \frac{9}{\sin 38^\circ}$. Since $\sin 38^\circ < 1$, $9 \sin 38^\circ$ is less than 9, but the hypotenuse must be longer than the leg 9, so that answer cannot be right.

9. **Hard** A roof truss spans 30 feet, and each side of the roof makes a 22° angle with the horizontal. The peak is directly above the center of the span. Ignoring overhang, find the length of each rafter and the height of the peak above the top of the walls, both to the nearest tenth of a foot.

Show answer

$$\text{Each half-span is 15 ft. Rafter} = \frac{15}{\cos 22^\circ} \approx 16.2 \text{ ft; height} = 15 \tan 22^\circ \approx 6.1 \text{ ft.}$$

10. **Hard** When the sun is 48° above the horizon, an oak tree on level ground casts a shadow 35 feet long. How tall is the tree? Round to the nearest tenth of a foot.

Show answer

$$35 \tan 48^\circ \approx 38.9 \text{ ft}$$

11. **Challenge** In right triangle ABC the right angle is at C , $AC = 10$ and $\angle B = 32^\circ$. Point D lies on side BC between B and C so that $\angle ADC = 50^\circ$. Find BD to the nearest tenth.

Show answer

$$BC = \frac{10}{\tan 32^\circ} \approx 16.0033 \text{ and } DC = \frac{10}{\tan 50^\circ} \approx 8.3910, \text{ so}$$
$$BD = BC - DC \approx 7.6.$$

COMMON MISTAKES

- Calculator in radian mode. Then $\sin 40$ gives about 0.745 instead of 0.643. Test with $\sin 30^\circ = 0.5$ before you start.
- Multiplying when you should divide. If the unknown is in the denominator, as in $\cos \theta = \frac{8}{x}$, the answer is $x = \frac{8}{\cos \theta}$. Write the equation first, then solve it step by step.
- Rounding the trig value too early, such as using $\sin 5^\circ \approx 0.09$. That changes the ramp length from about 22.95 ft to 22.22 ft. Keep the full calculator value until the end.
- Choosing the ratio by habit. Always label opposite, adjacent and hypotenuse from the given angle, then pick the ratio that uses the known side and the unknown side.

Make it a game • Set-Up Sprint. Parent reads scenarios: a 12-foot ladder at 70 degrees, a rafter rising from a 20-foot wall at 35 degrees. In 20 seconds, name the ratio and write the equation, without solving. Bonus points for solving when the unknown is in the denominator. Ten scenarios per round; record your score and try to beat it.

1.4 FINDING UNKNOWN ANGLES

OBJECTIVES

- Use the inverse ratios $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to find an acute angle from two sides of a right triangle.
- Solve a right triangle completely: find all three sides and both acute angles from the given data.

WATCH

Finding Angles with Trigonometry (1 of 2: What's an Inverse?)

Eddie Woo · 4:50

https://www.youtube.com/watch?v=7N_x5Bk4gs4

WHEN TO WATCH

After reading the Core ideas on inverse ratios, before Example 1.

Note: Woo uses a Casio calculator ('shift' key for $\sin^{-1}$); keys differ on other calculators.

BACKUP VIDEO

Finding Angles with Trigonometry (2 of 2: Using the Calculator)

Eddie Woo · 7:55

<https://www.youtube.com/watch?v=zSx4TsEjAM8>

Before the practice set, to check calculator keystrokes (degree mode).

KEY TERMS

inverse sine

$\sin^{-1} x$ (also written $\arcsin x$) is the acute angle whose sine is x , for $0 < x < 1$.

inverse tangent

$\tan^{-1} x$ (also written $\arctan x$) is the acute angle whose tangent is x , for $x > 0$.

inverse cosine

$\cos^{-1} x$ (also written $\arccos x$) is the acute angle whose cosine is x , for $0 < x < 1$.

solving a triangle

Finding all unknown sides and angles of a triangle from the parts that are given.

CORE IDEAS

In Lesson 1.3 we knew an angle and found a side. Now we turn the question around: we know two sides and want the angle. The converse of Lesson 1.1 also holds: if two right triangles have the same ratio of two corresponding sides, the Pythagorean theorem (Euclid I.47) makes all three sides proportional, and triangles with proportional sides are equiangular (Euclid VI.5). So the ratio fixes the acute angle. So for each ratio there is exactly one acute angle that produces it.

The calculator finds that angle with the inverse keys. If $\sin A = \frac{5}{15}$, then $A = \sin^{-1}\left(\frac{5}{15}\right)$, read "the angle whose sine is $\frac{5}{15}$." In the same way $\cos^{-1}$ undoes cosine and $\tan^{-1}$ undoes tangent. Choose the inverse by the same rule as before: label the two known sides from the angle you want, and pick the ratio that uses those two sides.

Be careful with notation. The -1 in $\sin^{-1} x$ means "inverse," not "reciprocal." It is *not* $\frac{1}{\sin x}$. The reciprocal ratios get their own names in Lesson 1.5.

To **solve a right triangle**, you need the right angle plus two more parts, at least one of them a side. Find a missing side with the Pythagorean theorem or a trig ratio, find one acute angle with an inverse ratio, and find the other acute angle by subtracting from 90° , since the two acute angles of a right triangle add to 90° (Euclid I.32).

Whenever you can, compute each part from the *given* data rather than from parts you have already rounded.

KEY FORMULAS

$$\sin A = x \iff A = \sin^{-1} x \quad (0^\circ < A < 90^\circ)$$

$$A = \cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right), \quad A = \tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right)$$

$$A + B = 90^\circ$$

From history. Before electronic calculators, people found angles by reading a printed table of sines or tangents backward: they looked down the column of values until they found the nearest one, then read off the angle. Surveyors, navigators and students did this for centuries. The inverse key does the same search in an instant.

Euclid connection. Because the angles of a triangle sum to two right angles (Euclid I.32), the two acute angles of a right triangle add to 90° . Once you find one, the other follows by subtraction.

WORKED EXAMPLES

Example 1. In right triangle ABC with right angle C , the side opposite $\angle A$ is $a = 5$ and the hypotenuse is $c = 15$. Find $\angle A$ and $\angle B$ to the nearest tenth of a degree.

Show solution

1. From $\angle A$ we know the opposite side and the hypotenuse, so use sine: $\sin A = \frac{5}{15} = \frac{1}{3}$.
2. Use the inverse: $A = \sin^{-1}\left(\frac{1}{3}\right) \approx 19.4712^\circ$ (calculator in degree mode).
3. The acute angles add to 90° : $B = 90^\circ - 19.4712^\circ = 70.5288^\circ$.
4. Check: $\sin 19.4712^\circ \approx 0.3333$. Round to the nearest tenth.

Answer: $A \approx 19.5^\circ$, $B \approx 70.5^\circ$

Example 2. Mr. Miller's rectangular hayfield measures 250 meters by 100 meters. He mows a straight path along a diagonal from one corner to the opposite corner. Find the length of the path to the nearest tenth of a meter, and the angles the path makes with the longer side and with the shorter side, to the nearest tenth of a degree.

Show solution

1. The two sides of the field and the diagonal form a right triangle with legs 250 m and 100 m.
2. Diagonal: $d = \sqrt{250^2 + 100^2} = \sqrt{62,500 + 10,000} = \sqrt{72,500} = 50\sqrt{29} \approx 269.2582$ m.
3. Let θ be the angle between the path and the longer side. From θ , the 100 m side is opposite and the 250 m side is adjacent, so use tangent: $\tan \theta = \frac{100}{250} = 0.4$.
4. $\theta = \tan^{-1}(0.4) \approx 21.8014^\circ$.
5. The angle with the shorter side is $90^\circ - 21.8014^\circ = 68.1986^\circ$.
6. Check with a second ratio: $\cos \theta = \frac{250}{269.2582} \approx 0.92848$, and $\cos^{-1}(0.92848) \approx 21.80^\circ$. It agrees.

Answer: Path ≈ 269.3 m; angle with longer side $\approx 21.8^\circ$; angle with shorter side $\approx 68.2^\circ$

PRACTICE

1. **Easy** Find the acute angle A if $\sin A = 0.5$. Give the answer in degrees.

Show answer

$$A = 30^\circ$$

2. **Easy** Find the acute angle A if $\tan A = 1.2$, to the nearest tenth of a degree.

Show answer

$$A \approx 50.2^\circ$$

3. **Easy** Find the acute angle A if $\cos A = 0.35$, to the nearest tenth of a degree.

Show answer

$$A \approx 69.5^\circ$$

4. **Medium** In a right triangle, the side opposite $\angle A$ is 8 and the side adjacent to $\angle A$ is 15. Find $\angle A$ to the nearest tenth of a degree.

Show answer

$$A = \tan^{-1}\left(\frac{8}{15}\right) \approx 28.1^\circ$$

5. **Medium** In a right triangle, the hypotenuse is 20 and the side adjacent to $\angle A$ is 15. Find $\angle A$ to the nearest tenth of a degree.

Show answer

$$A = \cos^{-1}(0.75) \approx 41.4^\circ$$

6. **Medium** A common rule for wheelchair ramps is a rise of 1 foot for every 12 feet of horizontal run. What angle does such a ramp make with the level ground? Round to the nearest tenth of a degree.

Show answer

$$\tan^{-1}\left(\frac{1}{12}\right) \approx 4.8^\circ$$

7. **Medium** A 20-foot ladder leans against a wall with its foot 5 feet from the base of the wall on level ground. What angle does the ladder make with the ground? Round to the nearest tenth of a degree.

Show answer

$$\cos^{-1}\left(\frac{5}{20}\right) \approx 75.5^\circ$$

8. **Hard** Solve right triangle ABC (right angle C) with hypotenuse $c = 30$ and leg $a = 18$. Give b exactly and the angles to the nearest tenth of a degree.

Show answer

$$b = \sqrt{900 - 324} = 24; A = \sin^{-1}(0.6) \approx 36.9^\circ; B \approx 53.1^\circ$$

9. **Hard** Solve right triangle ABC (right angle C) with $\angle A = 27^\circ$ and $b = 15$ (the leg adjacent to A). Give sides to the nearest tenth.

Show answer

$$B = 63^\circ; a = 15 \tan 27^\circ \approx 7.6; c = \frac{15}{\cos 27^\circ} \approx 16.8$$

10. **Hard** A student types $\sin^{-1}(1.3)$ into a calculator and gets an error. Explain why. Then explain why $\tan^{-1}(1.3)$ does not give an error.

Show answer

The sine of an acute angle is a leg divided by the hypotenuse, and the hypotenuse is the longest side, so a sine is always less than 1. No angle has sine 1.3, so $\sin^{-1}(1.3)$ is undefined. Tangent is one leg divided by the other, which can be any positive number, so $\tan^{-1}(1.3)$ exists (about 52.4°).

11. **Challenge** A fishing boat leaves the harbor and sails 21 miles due east, then turns and sails 20 miles due north. How far is it from the harbor, and at what angle north of due east does the harbor-to-boat line point? Give the angle to the nearest tenth of a degree.

Show answer

Distance $\sqrt{21^2 + 20^2} = \sqrt{841} = 29$ miles; angle $\tan^{-1}\left(\frac{20}{21}\right) \approx 43.6^\circ$ north of east.

COMMON MISTAKES

- Reading $\sin^{-1} x$ as $\frac{1}{\sin x}$. The -1 means inverse: $\sin^{-1}$ takes a ratio and gives back an angle.
- Forgetting degree mode. In radian mode, $\tan^{-1}(1.2)$ shows 0.876 instead of 50.2. Check the mode first.
- Using a rounded side or angle to find the next part. Compute each part from the given numbers when possible, so rounding errors do not pile up.
- Pressing the plain trig key instead of the inverse key. If your "angle" is a number less than 1, you probably pressed $\sin$ instead of $\sin^{-1}$.

Make it a game • Stairs and Ramps. Measure the rise and run of a stair step, a ramp or a book propped on another. Compute the angle using inverse tangent, then check it with a protractor. Which is steeper? Then parent gives two sides of a right triangle and you solve it completely: all three sides and both acute angles, in two minutes.

1.5 THE RECIPROCAL RATIOS

OBJECTIVES

- Define the cosecant, secant and cotangent of an acute angle as the reciprocals of sine, cosine and tangent.
- Evaluate all six trigonometric ratios of an acute angle when one ratio is given, by sketching a right triangle.
- Use a calculator to evaluate the reciprocal ratios and to find an angle from a reciprocal ratio.

WATCH

The Basic Trig Ratios & their Reciprocals

Eddie Woo · 6:09

<https://www.youtube.com/watch?v=pJPMCcWDLuY>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo says 'cosec' for csc.

BACKUP VIDEO

Meanings of the Trigonometric Ratios (3 of 4: Why is one of them called "secant"?)

Eddie Woo · 6:59

<https://www.youtube.com/watch?v=B3w5JhaJQiA>

Optional, after the lesson: why one ratio is called 'secant'.

KEY TERMS

cosecant

For an acute angle θ , $\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin \theta}$.

secant

For an acute angle θ , $\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos \theta}$.

cotangent

For an acute angle θ , $\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan \theta}$.

reciprocal

The reciprocal of a nonzero number x is $\frac{1}{x}$. A fraction's reciprocal is the fraction turned upside down.

CORE IDEAS

A right triangle has three sides, so from one acute angle there are six ways to divide one side by another. We have named three: sine, cosine and tangent. The other three are their upside-down forms. Flip $\frac{\text{opp}}{\text{hyp}}$ and you get the **cosecant**; flip $\frac{\text{adj}}{\text{hyp}}$ and you get the **secant**; flip $\frac{\text{opp}}{\text{adj}}$ and you get the **cotangent**.

Since each new ratio is the reciprocal of an old one, $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$ and $\cot \theta = \frac{1}{\tan \theta}$. Pair them by the letters: sine with cosecant, cosine with secant, tangent with cotangent. (Notice that the "co" names do *not* pair with each other.) Also, since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, flipping gives $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

Most calculators have no keys for csc, sec and cot. Compute them as reciprocals:

$\sec 35^\circ = \frac{1}{\cos 35^\circ}$. To find an angle from a reciprocal ratio, flip it first: if $\sec \theta = 2.5$, then $\cos \theta = \frac{1}{2.5} = 0.4$ and $\theta = \cos^{-1}(0.4)$. Do not type $\sec^{-1}$ as $\frac{1}{\cos^{-1}}$.

All six from one. If you know one ratio of an acute angle, you know them all. Draw a right triangle and label two sides to match the given ratio; any triangle with that ratio has the right shape (Euclid I.47 and VI.5; see Lesson 1.4). Find the third side with the Pythagorean theorem (Euclid I.47), then read off all six ratios. For example, $\sin \theta = \frac{3}{7}$ means we may take opposite = 3 and hypotenuse = 7; the adjacent side is then $\sqrt{49 - 9} = \sqrt{40} = 2\sqrt{10}$.

KEY FORMULAS

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

From history. The words "tangent" and "secant" were introduced by the Danish mathematician Thomas Fincke in his book *Geometria Rotundi* (1583). In a circle, the tangent line touches the circle, from the Latin *tangere*, "to touch," and the secant line cuts through it, from *secare*, "to cut." The trigonometric tangent and secant are lengths along such lines drawn in a circle of radius 1.

Euclid connection. Finding all six ratios from one depends on the Pythagorean theorem, Euclid I.47, to supply the missing side, and on Euclid VI.5, which guarantees that triangles with proportional sides are equiangular, so any triangle we draw with the given ratio has the same angles.

WORKED EXAMPLES

Example 1. Right triangle ABC has right angle C , with $a = 5$, $b = 12$ and $c = 13$. Find $\csc A$, $\sec A$ and $\cot A$ exactly and to four decimal places.

Show solution

1. From $\angle A$: opposite = 5, adjacent = 12, hypotenuse = 13.

$$2. \csc A = \frac{\text{hyp}}{\text{opp}} = \frac{13}{5} = 2.6.$$

$$3. \sec A = \frac{\text{hyp}}{\text{adj}} = \frac{13}{12} \approx 1.0833.$$

$$4. \cot A = \frac{\text{adj}}{\text{opp}} = \frac{12}{5} = 2.4.$$

5. Check: $\sin A = \frac{5}{13}$, and its reciprocal is $\frac{13}{5}$, which matches $\csc A$.

Answer: $\csc A = \frac{13}{5} = 2.6$, $\sec A = \frac{13}{12} \approx 1.0833$, $\cot A = \frac{12}{5} = 2.4$

Example 2. Let θ be an acute angle with $\sin \theta = \frac{3}{7}$. Find the exact values of the other five trigonometric ratios of θ , with denominators rationalized.

Show solution

1. Sketch a right triangle with an acute angle θ . Since $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{7}$, label the opposite side 3 and the hypotenuse 7.

2. Find the adjacent side: $\text{adj} = \sqrt{7^2 - 3^2} = \sqrt{40} = 2\sqrt{10}$.

3. $\cos \theta = \frac{2\sqrt{10}}{7} \approx 0.9035$.

4. $\tan \theta = \frac{3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20} \approx 0.4743$.

5. $\csc \theta = \frac{7}{3} \approx 2.3333$.

6. $\sec \theta = \frac{7}{2\sqrt{10}} = \frac{7\sqrt{10}}{20} \approx 1.1068$.

7. $\cot \theta = \frac{2\sqrt{10}}{3} \approx 2.1082$.

8. Check: $\tan \theta \cdot \cot \theta = \frac{3}{2\sqrt{10}} \cdot \frac{2\sqrt{10}}{3} = 1$, as reciprocals must.

Answer: $\cos \theta = \frac{2\sqrt{10}}{7}$, $\tan \theta = \frac{3\sqrt{10}}{20}$, $\csc \theta = \frac{7}{3}$, $\sec \theta = \frac{7\sqrt{10}}{20}$, $\cot \theta = \frac{2\sqrt{10}}{3}$

PRACTICE

1. Easy If $\sin A = 0.4$ for an acute angle A , find $\csc A$ exactly.

Show answer

$$\csc A = \frac{1}{0.4} = 2.5$$

2. Easy A right triangle has legs 8 and 15 and hypotenuse 17. Let A be the angle opposite the leg of length 8. Find $\sec A$ exactly and to four decimal places.

Show answer

$$\sec A = \frac{17}{15} \approx 1.1333$$

3. Easy If $\tan A = \frac{4}{3}$ for an acute angle A , find $\cot A$ exactly.

Show answer

$$\cot A = \frac{3}{4} = 0.75$$

4. **Medium** If $\cos \theta = \frac{5}{13}$ for an acute angle θ , find $\tan \theta$, $\csc \theta$ and $\cot \theta$ exactly.

Show answer

$$\text{Opposite side} = \sqrt{169 - 25} = 12. \tan \theta = \frac{12}{5}, \csc \theta = \frac{13}{12}, \cot \theta = \frac{5}{12}.$$

5. **Medium** Use a calculator to evaluate $\sec 35^\circ$ to four decimal places.

Show answer

$$\sec 35^\circ = \frac{1}{\cos 35^\circ} \approx 1.2208$$

6. **Medium** Use a calculator to evaluate $\cot 62^\circ$ to four decimal places.

Show answer

$$\cot 62^\circ = \frac{1}{\tan 62^\circ} \approx 0.5317$$

7. **Medium** A guy wire runs from the top of a 30-foot radio mast to a stake in the level ground. The wire makes a 58° angle with the ground. Show that the wire's length is $30 \csc 58^\circ$ and find it to the nearest tenth of a foot.

Show answer

From the 58° angle, the mast is opposite and the wire is the hypotenuse, so $\frac{\text{wire}}{30} = \csc 58^\circ$ and the wire is $30 \csc 58^\circ = \frac{30}{\sin 58^\circ} \approx 35.4$ ft.

8. **Medium** If $\tan \theta = 2$ for an acute angle θ , find all six trigonometric ratios of θ exactly, with denominators rationalized.

Show answer

Opposite 2, adjacent 1, hypotenuse $\sqrt{5}$. $\sin \theta = \frac{2\sqrt{5}}{5}$, $\cos \theta = \frac{\sqrt{5}}{5}$, $\tan \theta = 2$,
 $\csc \theta = \frac{\sqrt{5}}{2}$, $\sec \theta = \sqrt{5}$, $\cot \theta = \frac{1}{2}$.

9. **Hard** Find the acute angle θ if $\sec \theta = 2.5$. Round to the nearest tenth of a degree.

Show answer

$$\cos \theta = \frac{1}{2.5} = 0.4, \text{ so } \theta = \cos^{-1}(0.4) \approx 66.4^\circ.$$

10. **Hard** The rafters of a barn roof make an angle θ with the horizontal, where $\sec \theta = 1.25$. Each rafter covers a horizontal run of 16 feet. Find the length of each rafter and the rise of the roof, exactly.

Show answer

Rafter = $16 \sec \theta = 20$ ft. Then the rise is $\sqrt{20^2 - 16^2} = 12$ ft.

11. **Hard** Explain why $\csc A > 1$ and $\sec A > 1$ for every acute angle A , but $\cot A$ can be any positive number.

Show answer

$\csc A$ and $\sec A$ divide the hypotenuse by a leg. The hypotenuse is the longest side, so each quotient is greater than 1. (Equivalently, they are reciprocals of $\sin A$ and $\cos A$, which are between 0 and 1.) $\cot A$ divides one leg by the other, and either leg may be the longer, so $\cot A$ can be any positive number.

12. **Challenge** Prove that $1 + (\tan A)^2 = (\sec A)^2$ for every acute angle A . Then use it to find $\sec A$ if $\tan A = 0.75$.

Show answer

With opposite a , adjacent b , hypotenuse c : $1 + \frac{a^2}{b^2} = \frac{b^2 + a^2}{b^2} = \frac{c^2}{b^2} = (\sec A)^2$ by Euclid I.47. If $\tan A = 0.75$, then $(\sec A)^2 = 1.5625$, and since $\sec A > 0$, $\sec A = 1.25$.

COMMON MISTAKES

- Pairing the wrong functions, such as thinking $\csc \theta = \frac{1}{\cos \theta}$. Each reciprocal pair has exactly one "co": sine with cosecant, cosine with secant, tangent with cotangent.
- Using the inverse key for a reciprocal. $\cos^{-1}(35)$ is not $\sec 35^\circ$; it is an error. Compute $\sec 35^\circ$ as $1 \div \cos 35^\circ$.
- Finding an angle from a reciprocal ratio without flipping it first. If $\csc \theta = 3$, use $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, not $\sin^{-1}(3)$.
- Mislabeling the sketch in an all-six problem. Write the given ratio as a fraction of named sides first (for example $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}$), then label those two sides before using the Pythagorean theorem.

Make it a game · Six in a Row. Parent names one ratio, such as $\sin A = 5/13$. In 30 seconds, sketch the right triangle, find the missing side, and write all six ratios: sine, cosine, tangent, cosecant, secant and cotangent. Check by multiplying each reciprocal pair; it should equal one. Try ten parent-given ratios and record your fastest time.

UNIT 1

RIGHT-TRIANGLE TRIGONOMETRY

Trigonometry began as the art of measuring what cannot be reached: the height of a tower, the width of a river, the distance to the moon. This unit lays its foundation. We start from a fact of geometry, that right triangles sharing an acute angle are similar (Euclid I.32 and VI.4), and from it we see that the ratio of any two sides depends only on the angle. Those ratios get names: sine, cosine and tangent, and their reciprocals cosecant, secant and cotangent.

With these six ratios a student can solve any right triangle. Given an angle and a side, we find the other sides; given two sides, we find the angles with the inverse ratios. Along the way we meet the astronomers who built the first tables of chords and sines, Hipparchus and Ptolemy, and we trace the curious history of the word "sine." The problems come from the farm, the workshop and the sea: shadows of silos, ladders against barns, ramps, rafters and the course of a ship.

Everything later in the course rests on this unit. Unit 2 finds exact values for special angles, Unit 3 extends the ratios to angles of any size on the unit circle, and Units 7 and 8 use them to solve triangles that have no right angle and to model real measurements.

PREREQUISITES

- Solve linear equations and simple proportions, including equations with the unknown in the denominator.
- Use the Pythagorean theorem (Euclid I.47) to find a missing side of a right triangle.
- Know that the angles of a triangle sum to 180° (Euclid I.32).
- Recognize similar triangles and use proportional sides (AA similarity, Euclid VI.4).
- Simplify square roots and rationalize denominators.
- Use a scientific calculator and round to a stated place value.

MASTERY CHECKLIST

- ☐ I can explain why two right triangles that share an acute angle are similar and why their side ratios are equal.
- ☐ I can use similar right triangles, such as shadows, to find an unknown length.
- ☐ I can identify the opposite side, adjacent side and hypotenuse from a given acute angle.
- ☐ I can define sine, cosine and tangent as ratios and compute them from side lengths.
- ☐ I can choose the correct ratio and solve for a missing side, whether the unknown is in the numerator or the denominator.
- ☐ I can use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ to find an angle, with my

calculator.
in degree mode.

- ☐ I can solve a right triangle completely and check my answers for sense.
- ☐ I can define cosecant, secant and cotangent and evaluate them with a calculator.
- ☐ I can find all six trigonometric ratios of an acute angle when one ratio is given.
- ☐ I can prove simple relationships such as

$$\frac{\tan A}{\sin A} = \frac{1}{\cos A}$$

from the definitions.

$$(\sin A)^2 + (\cos A)^2 = 1$$

- ☐ I can set up and solve word problems involving ladders, ramps, roofs, shadows and navigation.

LESSONS IN THIS UNIT

1.1 Similar Right Triangles and the Idea of a Ratio

45 minutes

1.2 Sine, Cosine and Tangent

50 minutes

1.3 Finding Unknown Sides

50 minutes

1.4 Finding Unknown Angles

50 minutes

1.5 The Reciprocal Ratios

50 minutes

UNIT 1 TEST

Allow 60 minutes. A scientific calculator is permitted; set it to degree mode. Show your work and draw a labeled sketch for every triangle. Give exact answers where asked. Otherwise round lengths to the nearest tenth, angles to the nearest tenth of a degree, and ratios to four decimal places, rounding only at the final step. Total: 100 points.

1. **5 pts**

A right triangle has legs 7 and 24 and hypotenuse 25. A similar right triangle has a shorter leg of 21. Find its hypotenuse.

2. **6 pts**

A right triangle has legs 20 and 21 and hypotenuse 29. Let A be the angle opposite the leg of length 20. Find $\sin A$, $\cos A$ and $\tan A$ exactly and to four decimal places.

3. **5 pts**

Evaluate $\csc 23^\circ$ to four decimal places.

4. **6 pts**

In a right triangle, an acute angle measures 48° and the hypotenuse is 16. Find the side opposite the 48° angle, to the nearest tenth.

5. **6 pts**

In a right triangle, an acute angle measures 33° and the side adjacent to it is 21. Find the hypotenuse, to the nearest tenth.

6. **6 pts**

In a right triangle, the side opposite $\angle A$ is 4 and the hypotenuse is 16. Find $\angle A$ to the nearest tenth of a degree.

7. (6 pts)

On a sunny afternoon a 4-foot fence post casts a shadow 2.5 feet long. At the same moment a tree on Mr. Miller's farm casts a shadow 22 feet long. How tall is the tree? Explain why the method works.

8. (6 pts)

The boom of a crane is 60 feet long and is raised to an angle of 52° above the horizontal. How far above the boom's pivot is the tip of the boom? Round to the nearest tenth of a foot.

9. (9 pts)

If $\sin \theta = \frac{7}{25}$ for an acute angle θ , find $\cos \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$ and $\cot \theta$ exactly.

10. (9 pts)

Solve right triangle ABC (right angle C) with legs $a = 9$ and $b = 18$. Give c exactly and to the nearest tenth, and the angles to the nearest tenth of a degree.

11. (8 pts)

Samuel leans a 28-foot ladder against the side of a barn, with its foot 7 feet from the wall on level ground. Find the angle the ladder makes with the ground (nearest tenth of a degree) and how high up the wall it reaches (nearest tenth of a foot).

12. (7 pts)

A loading ramp must rise 2.5 feet and make a 4° angle with the level ground. How long must the ramp surface be? Round to the nearest tenth of a foot.

13. (10 pts)

Prove that for every acute angle A of a right triangle, $(\sin A)^2 + (\cos A)^2 = 1$. Then use it to find $\cos A$ exactly if $\sin A = \frac{3}{5}$.

14. (11 pts)

In right triangle ABC the right angle is at C , $AC = 12$ and $\angle B = 28^\circ$. Point D lies on side BC between B and C so that $\angle ADC = 55^\circ$. Find BD to the nearest tenth.

ANSWER KEY

1. The scale factor is 3, so the hypotenuse is 75.
2. $\sin A = \frac{20}{29} \approx 0.6897$, $\cos A = \frac{21}{29} \approx 0.7241$, $\tan A = \frac{20}{21} \approx 0.9524$
3. $\csc 23^\circ = \frac{1}{\sin 23^\circ} \approx 2.5593$
4. $16 \sin 48^\circ \approx 11.9$
5. $\frac{21}{\cos 33^\circ} \approx 25.0$
6. $A = \sin^{-1}(0.25) \approx 14.5^\circ$
7. The sun's rays make the same angle with the ground at both places, so the right triangles are similar and $\frac{h}{22} = \frac{4}{2.5}$. The tree is 35.2 feet tall.
8. $60 \sin 52^\circ \approx 47.3$ ft
9. Adjacent side $= \sqrt{625 - 49} = 24$. $\cos \theta = \frac{24}{25}$, $\tan \theta = \frac{7}{24}$, $\csc \theta = \frac{25}{7}$, $\sec \theta = \frac{25}{24}$,
 $\cot \theta = \frac{24}{7}$
10. $c = \sqrt{405} = 9\sqrt{5} \approx 20.1$; $A = \tan^{-1}(0.5) \approx 26.6^\circ$; $B \approx 63.4^\circ$
11. Angle $= \cos^{-1}\left(\frac{7}{28}\right) \approx 75.5^\circ$; height $= \sqrt{28^2 - 7^2} = \sqrt{735} \approx 27.1$ ft
12. $\frac{2.5}{\sin 4^\circ} \approx 35.8$ ft
13. With opposite a , adjacent b and hypotenuse c : $(\sin A)^2 + (\cos A)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$ by the Pythagorean theorem (Euclid I.47). Then $(\cos A)^2 = 1 - \frac{9}{25} = \frac{16}{25}$,
and since $\cos A > 0$, $\cos A = \frac{4}{5}$.
14. $BC = \frac{12}{\tan 28^\circ} \approx 22.5687$ and $DC = \frac{12}{\tan 55^\circ} \approx 8.4025$, so $BD \approx 14.2$.

SPECIAL RIGHT TRIANGLES AND EXACT VALUES

In Unit 1 you found trigonometric ratios with a calculator, as rounded decimals. In this unit you will find some of them exactly. Two triangles come straight from the simplest figures in geometry. Half of a square is the 45-45-90 triangle, with sides in the ratio $1 : 1 : \sqrt{2}$. Half of an equilateral triangle is the 30-60-90 triangle, with sides in the ratio $1 : \sqrt{3} : 2$. With the Pythagorean theorem (Euclid I.47), these two figures give exact values such as $\sin 30^\circ = \frac{1}{2}$ and $\cos 45^\circ = \frac{\sqrt{2}}{2}$, and you will prove each one rather than memorize it.

The unit closes with the cofunction identities. The two acute angles of a right triangle are complements, and the side opposite one is adjacent to the other. So the sine of an angle is the cosine of its complement, and the same holds for tangent and cotangent, secant and cosecant. This explains the *co-* in the names and ties the exact values together in one small table.

These exact values will be used throughout the rest of the course. In Unit 3 they become the coordinates of key points on the unit circle. They also give the heights and zeros of the graphs in Unit 4, and they are the starting values for the identities in Unit 5 and the equations in Unit 6.

PREREQUISITES

- Define and compute sine, cosine and tangent of an acute angle from side lengths (Lesson 1.2).
- Define cosecant, secant and cotangent as reciprocal ratios (Lesson 1.5).
- Explain why similar right triangles have equal side ratios (Lesson 1.1).
- Use the Pythagorean theorem and the angle sum of a triangle from Geometry.
- Simplify square roots and rationalize a denominator, such as $\frac{6}{\sqrt{3}} = 2\sqrt{3}$.

MASTERY CHECKLIST

- ☐ I can derive the
1 : ratio of a 45-45-90 triangle from the
1 : diagonal of a square.
 $\sqrt{2}$
- ☐ I can find the missing sides of a 45-45-90 triangle from either a leg or the hypotenuse, in exact form and as a rounded decimal.
- ☐ I can derive the
1 : ratio of a 30-60-90 triangle from
 $\sqrt{3}$ half of an equilateral triangle.
2
- ☐ I can find the missing sides of a 30-60-90 triangle from any one side, starting with the short leg.
- ☐ I can state and explain the exact values of all six trigonometric ratios of
 $30^\circ, 45^\circ$
and 60° .
d
- ☐ I can prove that
 $\sin \theta = \cos(90^\circ - \theta)$ and the other cofunction identities
using one right triangle.
 θ)
- ☐ I can rewrite a trigonometric ratio as its cofunction of the complementary angle.
- ☐ I can solve equations such as
 $\sin(2x) = \cos(3x - 15^\circ)$ when both angles are acute.

- I can use special right triangles to solve practical problems about roofs, ladders, shadows, fields and ships.

LESSONS IN THIS UNIT

2.1 The 45-45-90 Triangle	45 minutes
2.2 The 30-60-90 Triangle	50 minutes
2.3 Complementary Angles and Cofunctions	50 minutes

UNIT 2 TEST

Allow 50 minutes. Problems 1 through 7, 11 and 12 must be done without a calculator; give exact values with denominators rationalized. A scientific calculator may be used on Problems 8, 9, 10 and 13, but only to round after you have written the exact answer. Round decimal answers to the nearest tenth unless a problem says otherwise. Show your work; a sketch of each triangle earns partial credit. Total: 100 points.

1. 6 pts

Give the exact values of $\sin 45^\circ$, $\cos 45^\circ$ and $\tan 45^\circ$.

2. 6 pts

The hypotenuse of an isosceles right triangle is 14 cm. Find the exact length of each leg.

3. 6 pts

Give the exact values of $\sin 60^\circ$, $\cos 60^\circ$ and $\tan 60^\circ$.

4. 6 pts

In a 30-60-90 triangle the long leg is 12 in. Find the exact lengths of the short leg and the hypotenuse.

5. 6 pts

Fill in the blank with an acute angle in degrees: $\cos 47^\circ = \sin \underline{\hspace{1cm}}$.

6. (6 pts)

Find the exact value of $\sin 30^\circ \cos 60^\circ + \tan 45^\circ$.

7. (6 pts)

Find the exact value of $\csc 30^\circ + \sec 45^\circ \cos 45^\circ$.

8. (8 pts)

Ruth's square vegetable garden is 18 ft on each side. She lays a straight stone path from one corner to the opposite corner. How long is the path? Give the exact value and round to the nearest tenth of a foot.

9. (8 pts)

When the sun is 30° above the horizon, how long is the shadow cast on level ground by a barn wall 24 ft tall? Give the exact value and round to the nearest tenth of a foot.

10. (8 pts)

The front of an A-frame cabin is an equilateral triangle that stands 15 ft tall from the middle of its base to its peak. How long is each side of the triangle? Give the exact value and round to the nearest tenth of a foot.

11. (8 pts)

Find x in degrees if $\sin(3x - 4^\circ) = \cos(2x + 9^\circ)$ and both angles are acute. Check your answer.

12. (12 pts)

Let A and B be the acute angles of a right triangle. Prove that $A + B = 90^\circ$ and that $\sin A = \cos B$. Then explain why this shows $\sin \theta = \cos(90^\circ - \theta)$ for every acute angle θ .

13. (14 pts)

In triangle ABC , $\angle B = 60^\circ$ and $\angle C = 45^\circ$, and the altitude from A to side BC has length 6. Find the exact length of BC and the exact area of the triangle. Round each to the nearest hundredth.

ANSWER KEY

1. $\sin 45^\circ = \frac{\sqrt{2}}{2}$, $\cos 45^\circ = \frac{\sqrt{2}}{2}$, $\tan 45^\circ = 1$
2. $\frac{14}{\sqrt{2}} = 7\sqrt{2}$ cm
3. $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$
4. Short leg $= \frac{12}{\sqrt{3}} = 4\sqrt{3}$ in; hypotenuse $= 8\sqrt{3}$ in
5. 43°
6. $\frac{1}{2} \cdot \frac{1}{2} + 1 = \frac{5}{4}$
7. $2 + \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2 + 1 = 3$
8. $18\sqrt{2}$ ft ≈ 25.5 ft
9. $\tan 30^\circ = \frac{24}{x}$, so $x = \frac{24}{\sqrt{3}/3} = 24\sqrt{3}$ ft ≈ 41.6 ft
10. The height is the long leg of a 30-60-90 triangle, so the short leg is $\frac{15}{\sqrt{3}} = 5\sqrt{3}$ and each side (the hypotenuse) is $10\sqrt{3}$ ft ≈ 17.3 ft
11. $(3x - 4^\circ) + (2x + 9^\circ) = 90^\circ$, so $5x = 85^\circ$ and $x = 17^\circ$. Check: the angles are 47° and 43° , both acute, with sum 90° .
12. The angles of a triangle add to 180° (Euclid I.32), and the right angle uses 90° , so $A + B = 90^\circ$. Let a be the leg opposite A and c the hypotenuse. Then $\sin A = \frac{a}{c}$. The same leg a is adjacent to B , so $\cos B = \frac{a}{c}$. Hence $\sin A = \cos B$. Given any acute angle θ , build a right triangle with $A = \theta$; then $B = 90^\circ - \theta$, and $\sin \theta = \cos(90^\circ - \theta)$.
13. Let D be the foot of the altitude. In triangle ABD , $AD = 6$ is the long leg (opposite 60°), so $BD = \frac{6}{\sqrt{3}} = 2\sqrt{3}$. Triangle ADC is 45-45-90, so $DC = 6$. Then $BC = 6 + 2\sqrt{3} \approx 9.46$, and the area is $\frac{1}{2} \cdot 6 \cdot (6 + 2\sqrt{3}) = 18 + 6\sqrt{3} \approx 28.39$ square units.

2.1 THE 45-45-90 TRIANGLE

OBJECTIVES

- Derive the side ratio $1 : 1 : \sqrt{2}$ of a 45-45-90 triangle from the diagonal of a square using the Pythagorean theorem (Euclid I.47).
- State and explain the exact values of all six trigonometric ratios of 45° .
- Find the missing sides of any 45-45-90 triangle in exact form and as a rounded decimal.

WATCH

Exact Values of Trigonometric Ratios (2 of 2: 60° and 45°)

Eddie Woo · 5:11

<https://www.youtube.com/watch?v=VkgH1X2JBuM>

WHEN TO WATCH

After reading the Core ideas (the square's diagonal), before Example 1. The 45° half is what this lesson needs.

Note: Covers 60° and 45° together; the 60° part previews Lesson 2.2.

BACKUP VIDEO

Exact Values for Trigonometric Ratios

Eddie Woo · 10:43

<https://www.youtube.com/watch?v=wzHorwUAe7w>

Before the practice set, for another pass at the exact values.

KEY TERMS

45-45-90 triangle

A right triangle whose two acute angles are both 45° ; it is also called an isosceles right triangle because its two legs are equal.

diagonal of a square

The segment joining two opposite corners of a square; it cuts the square into two congruent 45-45-90 triangles.

exact value

A value written with whole numbers, fractions and radicals, such as $\frac{\sqrt{2}}{2}$, rather than a rounded decimal such as 0.707.

rationalizing the denominator

Rewriting a fraction so that no radical remains in the denominator, for example $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

CORE IDEAS

Start with a square of side s and draw one diagonal d . The diagonal cuts the square into two triangles whose three sides match (s, s, d) , so the triangles are congruent. Each has a right angle at the corner of the square. Its other two angles are equal, because they sit opposite the two equal sides s , and by Euclid I.32 they add to $180^\circ - 90^\circ = 90^\circ$. So each is 45° . Every 45-45-90 triangle is half of a square.

Now find the diagonal. By the Pythagorean theorem (Euclid I.47), $d^2 = s^2 + s^2 = 2s^2$, so $d = \sqrt{2s^2} = s\sqrt{2}$. The sides are therefore in the ratio $s : s : s\sqrt{2}$, or $1 : 1 : \sqrt{2}$. All 45-45-90 triangles are similar (Lesson 1.1), so this ratio holds for every one of them, large or small.

Read the ratios straight off a triangle with legs 1 and hypotenuse $\sqrt{2}$. For the 45° angle, $\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}}$. Multiply the top and bottom by $\sqrt{2}$ to get $\frac{\sqrt{2}}{2}$. The adjacent leg is also 1, so $\cos 45^\circ = \frac{\sqrt{2}}{2}$ too, and $\tan 45^\circ = \frac{1}{1} = 1$. The reciprocals are $\csc 45^\circ = \sec 45^\circ = \sqrt{2}$ and $\cot 45^\circ = 1$. As a decimal, $\frac{\sqrt{2}}{2} \approx 0.707107$; your calculator's value of $\sin 45^\circ$ should agree.

Use the ratio in both directions. From a leg to the hypotenuse, multiply by $\sqrt{2}$. From the hypotenuse to a leg, divide by $\sqrt{2}$, which is the same as multiplying by $\frac{\sqrt{2}}{2}$. A quick check: the hypotenuse is always the longest side, about 1.41 times a leg. Keep $\sqrt{2}$ exact until the last step, and round only the final answer.

KEY FORMULAS

$$\text{leg} : \text{leg} : \text{hypotenuse} = 1 : 1 : \sqrt{2}$$

$$d = s\sqrt{2} \quad s = \frac{d}{\sqrt{2}} = \frac{d\sqrt{2}}{2}$$

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}, \quad \tan 45^\circ = 1$$

$$\csc 45^\circ = \sec 45^\circ = \sqrt{2}, \quad \cot 45^\circ = 1$$

From history. An Old Babylonian clay tablet in the Yale collection, known as YBC 7289, shows a square with its diagonals drawn. Written in base 60, it gives the ratio of diagonal to side as 1;24,51,10, which is about 1.414213 and agrees with $\sqrt{2}$ to five decimal places. Greek mathematicians later proved something deeper: the diagonal and side of a square are incommensurable, meaning no ratio of whole numbers equals $\sqrt{2}$ exactly.

Euclid connection. Euclid I.34 shows that a diagonal bisects a parallelogram, so the two halves of the square are equal. Euclid I.47, the Pythagorean theorem, then gives $d^2 = s^2 + s^2$, the whole source of the $\sqrt{2}$.

WORKED EXAMPLES

Example 1. An isosceles right triangle has legs of 7 cm. Find the exact length of the hypotenuse, then round it to the nearest hundredth of a centimeter.

Show solution

1. An isosceles right triangle is a 45-45-90 triangle, so its sides are in the ratio $1 : 1 : \sqrt{2}$.
2. Going from a leg to the hypotenuse, multiply by $\sqrt{2}$: hypotenuse = $7\sqrt{2}$ cm.
3. Check with Euclid I.47: $7^2 + 7^2 = 98$ and $(7\sqrt{2})^2 = 49 \cdot 2 = 98$. It agrees.
4. Round only at the end: $7\sqrt{2} \approx 9.899495 \approx 9.90$ cm.

Answer: $7\sqrt{2}$ cm $\approx$ 9.90 cm

Example 2. Mr. Miller's pasture is a square. A straight path runs from one corner to the opposite corner and is 200 ft long. Find the exact length of a side of the pasture and round it to the nearest tenth of a foot. Then find the area of the pasture.

Show solution

1. The path is a diagonal, so it is the hypotenuse of a 45-45-90 triangle whose legs are the sides s of the pasture.
2. Hypotenuse = $s\sqrt{2}$, so $s\sqrt{2} = 200$ and $s = \frac{200}{\sqrt{2}}$.
3. Rationalize: $s = \frac{200}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{200\sqrt{2}}{2} = 100\sqrt{2}$ ft.
4. As a decimal, $100\sqrt{2} \approx 141.421356 \approx 141.4$ ft.
5. Area = $s^2 = (100\sqrt{2})^2 = 10,000 \cdot 2 = 20,000$ square feet. Use the exact side here; squaring the rounded 141.4 would give 19,993.96, which is slightly off.
6. Notice the shortcut: since $d^2 = 2s^2$, the area of a square is $s^2 = \frac{d^2}{2} = \frac{200^2}{2} = 20,000$.

Answer: Side = $100\sqrt{2}$ ft ≈ 141.4 ft; area = 20,000 square feet

PRACTICE

1. **Easy** An isosceles right triangle has legs of 5 in. Find the exact length of the hypotenuse, then round to the nearest tenth of an inch.

Show answer

$$5\sqrt{2} \text{ in} \approx 7.1 \text{ in}$$

2. **Easy** The hypotenuse of a 45-45-90 triangle is 10 m. Find the exact length of each leg, then round to the nearest tenth of a meter.

Show answer

$$\frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ m} \approx 7.1 \text{ m}$$

3. **Easy** Find the exact value of $\sin 45^\circ + \cos 45^\circ$.

Show answer

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.414$$

4. **Medium** The hypotenuse of an isosceles right triangle is $6\sqrt{2}$ ft. Find the exact length of each leg.

Show answer

$$\frac{6\sqrt{2}}{\sqrt{2}} = 6 \text{ ft}$$

5. **Medium** Find the exact value of $\sec^2 45^\circ - \tan^2 45^\circ$. (Here $\sec^2 45^\circ$ means $(\sec 45^\circ)^2$.)

Show answer

$$(\sqrt{2})^2 - 1^2 = 2 - 1 = 1$$

6. **Medium** A roof with a 12-in-12 pitch rises 12 in for every 12 in of horizontal run, so each rafter makes a 45° angle with the horizontal. Samuel is cutting a rafter with a horizontal run of 14 ft. Ignoring any overhang, how long is the rafter? Give the exact value and round to the nearest tenth of a foot.

Show answer

$$14\sqrt{2} \text{ ft} \approx 19.8 \text{ ft}$$

7. **Medium** A square has a perimeter of 40 cm. Find the exact length of its diagonal, then round to the nearest hundredth of a centimeter.

Show answer

$$\text{Side} = 10 \text{ cm, so diagonal} = 10\sqrt{2} \text{ cm} \approx 14.14 \text{ cm}$$

8. **Hard** A ship sails 50 miles in a straight line on a course exactly halfway between due north and due east, so its path makes a 45° angle with due north. How far north and how far east of its starting point is the ship? Give exact values and round to the nearest tenth of a mile.

Show answer

$$\text{North: } 25\sqrt{2} \approx 35.4 \text{ mi; east: } 25\sqrt{2} \approx 35.4 \text{ mi}$$

9. **Hard** The hypotenuse of an isosceles right triangle is 9 in. Find the exact area of the triangle.

Show answer

$$\text{Each leg is } \frac{9}{\sqrt{2}}, \text{ so the area is } \frac{1}{2} \cdot \frac{9}{\sqrt{2}} \cdot \frac{9}{\sqrt{2}} = \frac{81}{4} = 20.25 \text{ square inches}$$

10. **Hard** Explain why $\sin 45^\circ = \cos 45^\circ$ and $\tan 45^\circ = 1$ in every 45-45-90 triangle, no matter how large it is.

Show answer

In a right triangle with a 45° angle, the other acute angle is also 45° (Euclid I.32), so the two legs are opposite equal angles and are equal. For either 45° angle, the opposite leg and the adjacent leg have the same length, so $\frac{\text{opp}}{\text{hyp}} = \frac{\text{adj}}{\text{hyp}}$, which says $\sin 45^\circ = \cos 45^\circ$, and $\frac{\text{opp}}{\text{adj}} = 1$. Size does not matter because all 45-45-90 triangles are similar, and similar triangles have the same side ratios.

11. **Challenge** Ruth's father is making an octagonal tabletop from a square board 40 in on a side. He cuts off four congruent isosceles right triangles, one at each corner, so that all eight sides of the octagon are equal. How far from each corner, measured along the edge, should he make each cut, and how long is each side of the octagon? Give exact values and round to the nearest hundredth of an inch.

Show answer

Let x be the distance from the corner. Each cut edge is $x\sqrt{2}$, and the leftover part of each side of the square is $40 - 2x$. Setting $40 - 2x = x\sqrt{2}$ gives $x = \frac{40}{2 + \sqrt{2}} = 40 - 20\sqrt{2} \approx 11.72$ in. Each octagon side is $x\sqrt{2} = 40\sqrt{2} - 40 \approx 16.57$ in.

COMMON MISTAKES

- Multiplying by $\sqrt{2}$ when going from the hypotenuse to a leg. You should divide by $\sqrt{2}$. Always check that the hypotenuse comes out as the longest side.
- Rounding $\sqrt{2}$ to 1.4 too early. For a 14 ft run, $14 \times 1.4 = 19.6$ ft, but the true rafter length is $14\sqrt{2} \approx 19.8$ ft. Keep $\sqrt{2}$ exact and round only the final answer.
- Leaving a radical in the denominator or rationalizing it incorrectly. To simplify $\frac{10}{\sqrt{2}}$, multiply the top and bottom by $\sqrt{2}$: $\frac{10\sqrt{2}}{2} = 5\sqrt{2}$.
- Using the $1 : 1 : \sqrt{2}$ ratio on a right triangle whose legs are not equal. The ratio applies only when both acute angles are 45° .

Make it a game • Diagonal Guess. Guess the length of the diagonal of a square floor tile or a sheet of paper before measuring. Then compute it with the $1 : 1 : \text{square root } 2$ ratio, and measure to check. Closest guess scores. Next, parent calls any of the six ratios of 45 degrees and you answer with exact, rationalized values.

2.2 THE 30-60-90 TRIANGLE

OBJECTIVES

- Derive the side ratio $1 : \sqrt{3} : 2$ of a 30-60-90 triangle by cutting an equilateral triangle (Euclid I.1) in half.
- State and explain the exact values of the six trigonometric ratios of 30° and 60° .
- Find the missing sides of any 30-60-90 triangle from any one side, in exact form and as a rounded decimal.

WATCH

Exact Values of Trigonometric Ratios (1 of 2: 30°)

Eddie Woo · 7:31

<https://www.youtube.com/watch?v=SiA-WTnKNgs>

WHEN TO WATCH

After reading the Core ideas (halving the equilateral triangle), before Example 1.

BACKUP VIDEO

Exact Values for Trigonometric Ratios

Eddie Woo · 10:31

<https://www.youtube.com/watch?v=tDMBtXmxrr4>

Before the practice set, for more exact-value practice.

KEY TERMS

30-60-90 triangle

A right triangle whose acute angles are 30° and 60° ; it is exactly half of an equilateral triangle.

short leg

In a 30-60-90 triangle, the leg opposite the 30° angle; it is half the hypotenuse.

long leg

In a 30-60-90 triangle, the leg opposite the 60° angle; it is $\sqrt{3}$ times the short leg.

altitude

The perpendicular segment from a vertex of a triangle to the line containing the opposite side.

CORE IDEAS

Begin with an equilateral triangle of side 2, the figure Euclid constructs in I.1. Its three angles are equal, and by Euclid I.32 they add to 180° , so each is 60° . Drop the

altitude from the top vertex to the base. The two small triangles share the altitude and have equal hypotenuses of 2, so they are congruent (hypotenuse-leg). Therefore the altitude cuts the base into two halves of length 1 and splits the top 60° angle into two angles of 30° . Each half is a right triangle with angles 30° , 60° and 90° .

Find the altitude h with Euclid I.47: $1^2 + h^2 = 2^2$, so $h^2 = 3$ and $h = \sqrt{3}$. The sides are short leg 1, long leg $\sqrt{3}$, hypotenuse 2. Since all 30-60-90 triangles are similar, every one has sides in the ratio $1 : \sqrt{3} : 2$. In words: the hypotenuse is twice the short leg, and the long leg is $\sqrt{3}$ times the short leg. The short leg is always opposite the 30° angle, and the long leg is opposite the 60° angle.

Read the ratios from the triangle with sides 1, $\sqrt{3}$, 2. At the 30° angle the opposite side is 1 and the adjacent side is $\sqrt{3}$, so $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. At the 60° angle the roles swap: $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$ and $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$. The reciprocals follow: $\csc 30^\circ = 2$, $\sec 30^\circ = \frac{2\sqrt{3}}{3}$, $\cot 30^\circ = \sqrt{3}$, and $\csc 60^\circ = \frac{2\sqrt{3}}{3}$, $\sec 60^\circ = 2$, $\cot 60^\circ = \frac{\sqrt{3}}{3}$.

To solve a 30-60-90 triangle, find the short leg first; then every other side is easy. If you know the hypotenuse, halve it. If you know the long leg, divide by $\sqrt{3}$ and rationalize. Check your work: the sides must satisfy short < long < hypotenuse, since $1 < \sqrt{3} \approx 1.732 < 2$.

KEY FORMULAS

$$\text{short leg} : \text{long leg} : \text{hypotenuse} = 1 : \sqrt{3} : 2$$

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3}$$

$$\csc 30^\circ = 2, \quad \sec 30^\circ = \frac{2\sqrt{3}}{3}, \quad \cot 30^\circ = \sqrt{3}$$

$$\csc 60^\circ = \frac{2\sqrt{3}}{3}, \quad \sec 60^\circ = 2, \quad \cot 60^\circ = \frac{\sqrt{3}}{3}$$

$$\text{Area of an equilateral triangle of side } s = \frac{\sqrt{3}}{4}s^2$$

From history. In the *Almagest*, Ptolemy built his table of chords by starting from chords he could find exactly from regular figures inscribed in a circle. The side of the inscribed regular hexagon is the chord of 60° , and it equals the radius, because the hexagon is made of six equilateral triangles. The side of the inscribed square is the chord of 90° . From such exact starting values he worked out the rest of the table.

Euclid connection. Euclid I.1 constructs an equilateral triangle on a given segment, and I.32 shows its angles are each 60° . Halving it and applying I.47 gives the $1 : \sqrt{3} : 2$ ratio.

WORKED EXAMPLES

Example 1. In a 30-60-90 triangle the short leg is 5 in. Find the exact lengths of the hypotenuse and the long leg, and round the long leg to the nearest tenth of an inch.

Show solution

1. The sides are in the ratio $1 : \sqrt{3} : 2$, so multiply each part by 5.
2. Hypotenuse $= 2 \cdot 5 = 10$ in.
3. Long leg $= 5\sqrt{3}$ in $\approx 8.660254 \approx 8.7$ in.
4. Check with Euclid I.47: $5^2 + (5\sqrt{3})^2 = 25 + 75 = 100 = 10^2$. It agrees.

Answer: Hypotenuse $= 10$ in; long leg $= 5\sqrt{3}$ in ≈ 8.7 in

Example 2. Samuel leans a ladder against the side of a barn so that it makes a 60° angle with the level ground. The top of the ladder touches the wall 15 ft above the ground. How long is the ladder, and how far is its foot from the wall? Give exact values and round to the nearest tenth of a foot.

Show solution

1. The wall, the ground and the ladder form a 30-60-90 triangle. The ladder is the hypotenuse. The 60° angle is at the foot of the ladder, so the height on the wall (15 ft) is opposite 60° : it is the long leg. The distance along the ground is the short leg.

2. Find the short leg first: long leg $= \sqrt{3} \cdot$ short leg, so short leg $= \frac{15}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$ ft.

3. Hypotenuse $= 2 \cdot$ short leg $= 10\sqrt{3}$ ft.

4. Check with a ratio: $\sin 60^\circ = \frac{15}{10\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$. It agrees.

5. Round at the end: ladder $10\sqrt{3} \approx 17.320508 \approx 17.3$ ft; foot of ladder $5\sqrt{3} \approx 8.660254 \approx 8.7$ ft from the wall.

Answer: Ladder $= 10\sqrt{3}$ ft ≈ 17.3 ft; foot of the ladder $= 5\sqrt{3}$ ft ≈ 8.7 ft from the wall

PRACTICE

1. **Easy** In a 30-60-90 triangle the short leg is 4 cm. Find the exact lengths of the hypotenuse and the long leg, and round the long leg to the nearest tenth.

Show answer

$$\text{Hypotenuse} = 8 \text{ cm; long leg} = 4\sqrt{3} \text{ cm} \approx 6.9 \text{ cm}$$

2. **Easy** In a 30-60-90 triangle the hypotenuse is 12 m. Find the exact lengths of the short leg and the long leg, and round the long leg to the nearest tenth.

Show answer

$$\text{Short leg} = 6 \text{ m; long leg} = 6\sqrt{3} \text{ m} \approx 10.4 \text{ m}$$

3. Easy Find the exact value of $\sin 30^\circ + \cos 60^\circ$.

Show answer

$$\frac{1}{2} + \frac{1}{2} = 1$$

4. Medium In a 30-60-90 triangle the long leg is 9 ft. Find the exact lengths of the short leg and the hypotenuse, and round each to the nearest tenth.

Show answer

$$\text{Short leg} = \frac{9}{\sqrt{3}} = 3\sqrt{3} \text{ ft} \approx 5.2 \text{ ft}; \text{ hypotenuse} = 6\sqrt{3} \text{ ft} \approx 10.4 \text{ ft}$$

5. Medium Find the exact value of $2 \sin 60^\circ \cos 60^\circ$.

Show answer

$$2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2} \approx 0.866$$

6. Medium A roof truss is shaped like an equilateral triangle with sides of 18 ft. How tall is the truss from the middle of its base to its peak? Give the exact value and round to the nearest tenth of a foot.

Show answer

$$\text{Half the truss is a 30-60-90 triangle with short leg 9 ft, so the height is } 9\sqrt{3} \text{ ft} \approx 15.6 \text{ ft}$$

7. Medium When the sun is 60° above the horizon, a grain silo casts a shadow 25 ft long on level ground. How tall is the silo? Give the exact value and round to the nearest tenth of a foot.

Show answer

$$\tan 60^\circ = \frac{h}{25}, \text{ so } h = 25\sqrt{3} \text{ ft} \approx 43.3 \text{ ft}$$

8. **Hard** Show that an equilateral triangle with side s has area $\frac{\sqrt{3}}{4}s^2$. Then find the exact area of an equilateral triangle with sides of 10 cm, and round to the nearest tenth.

Show answer

The altitude is the long leg of a 30-60-90 triangle with short leg $\frac{s}{2}$, so $h = \frac{\sqrt{3}}{2}s$ and the area is $\frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$. For $s = 10$: area = $25\sqrt{3} \approx 43.3$ square centimeters

9. **Hard** The floor of a garden gazebo is a regular hexagon with sides of 6 ft. Lines from the center to each corner divide it into six equilateral triangles. Find the exact area of the floor and round to the nearest tenth of a square foot.

Show answer

$$6 \cdot \frac{\sqrt{3}}{4} \cdot 6^2 = 54\sqrt{3} \approx 93.5 \text{ square feet}$$

10. **Hard** Explain why, in every 30-60-90 triangle, the side opposite the 30° angle is exactly half the hypotenuse.

Show answer

Place a copy of the triangle beside the original, reflected across the long leg. The two 60° angles become the base angles of a larger triangle, and the two 30° angles combine at the top to make 60° . All three angles of the large triangle are 60° , so it is equilateral, and all its sides equal the hypotenuse. Its base is made of two copies of the short leg, so twice the short leg equals the hypotenuse. That is why $\sin 30^\circ = \frac{1}{2}$.

11. **Challenge** In triangle ABC , $\angle B = 45^\circ$ and $\angle C = 30^\circ$, and the altitude from A to side BC has length 6. Find the exact length of BC and the exact perimeter of the triangle. Round each to the nearest hundredth.

Show answer

Let D be the foot of the altitude. Triangle ABD is 45-45-90, so $BD = 6$ and $AB = 6\sqrt{2}$. Triangle ACD is 30-60-90 with short leg $AD = 6$, so $DC = 6\sqrt{3}$ and $AC = 12$. Then $BC = 6 + 6\sqrt{3} \approx 16.39$ and the perimeter is $18 + 6\sqrt{2} + 6\sqrt{3} \approx 36.88$.

COMMON MISTAKES

- Putting $\sqrt{3}$ on the wrong leg. The short leg is opposite the 30° angle, and the long leg ($\sqrt{3}$ times as long) is opposite the 60° angle. Mark the angles on your sketch before you label the sides.
- Mixing up the values for 30° and 60° , such as writing $\sin 30^\circ = \frac{\sqrt{3}}{2}$. Remember that $\sin 30^\circ$ is the small value $\frac{1}{2}$, because the side opposite the smaller angle is the shorter side.
- Multiplying by $\sqrt{3}$ when the long leg is given. To go from the long leg back to the short leg, divide by $\sqrt{3}$; then rationalize, as in $\frac{15}{\sqrt{3}} = 5\sqrt{3}$.
- Thinking the hypotenuse is $\sqrt{3}$ times a leg. The hypotenuse is exactly twice the short leg. Check that your sides satisfy $\text{short} < \text{long} < \text{hypotenuse}$.

Make it a game • Fill the Grid. Draw a blank grid with 30, 45 and 60 degrees down the side and sine, cosine, tangent, cosecant, secant and cotangent across the top. Fill in all eighteen exact values from memory in 3 minutes. Check against a cut equilateral triangle. Count correct squares; try again tomorrow to beat your score.

2.3 COMPLEMENTARY ANGLES AND COFUNCTIONS

OBJECTIVES

- Prove that $\sin \theta = \cos(90^\circ - \theta)$ and the other cofunction identities for acute angles, using one right triangle.
- Use the cofunction identities to rewrite ratios and to solve equations such as $\sin(2x + 10^\circ) = \cos(3x - 15^\circ)$.
- Build and use the table of exact values for 30° , 45° and 60° .

WATCH

Trigonometric Ratios of Complementary Angles (1 of 3: Noticing the pattern)

Eddie Woo · 8:29

<https://www.youtube.com/watch?v=jAj-JHX6JTs>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

Trigonometric Ratios of Complementary Angles (2 of 3: Basic questions)

Eddie Woo · 5:45

<https://www.youtube.com/watch?v=C4Owlh6p2Gw>

Right before Example 2: equations like $\sin(2x+10^\circ) = \cos(3x-15^\circ)$.

KEY TERMS

complementary angles

Two angles whose measures add to 90° ; each is the complement of the other.

cofunctions

The pairs sine and cosine, tangent and cotangent, secant and cosecant. Each function of an angle equals its cofunction of the complement.

cofunction identity

An equation such as $\sin \theta = \cos(90^\circ - \theta)$ that is true for every acute angle θ .

CORE IDEAS

In right triangle ABC with the right angle at C , the acute angles A and B add to 90° , because all three angles add to 180° (Euclid I.32). So $B = 90^\circ - A$: the two acute angles of every right triangle are complementary.

Now look at the sides. Call the side opposite A by a , the side opposite B by b , and the hypotenuse c . The leg a is opposite A , but it is adjacent to B . So $\sin A = \frac{a}{c}$ and $\cos B = \frac{a}{c}$. They are the same ratio. Since $B = 90^\circ - A$, we have proved $\sin A = \cos(90^\circ - A)$. In the same way, $\tan A = \frac{a}{b} = \cot B$ and $\sec A = \frac{c}{b} = \csc B$, and reading each pair backward gives the other three identities.

This explains the prefix *co-*. The cosine of an angle is the sine of its complement, the cotangent is the tangent of the complement, and the cosecant is the secant of the complement. Each function equals its cofunction of the complement. For example, $\sin 20^\circ = \cos 70^\circ$ and $\tan 38^\circ = \cot 52^\circ$. You can check these on a calculator.

The identities also explain the pattern in the exact-value table. Since 30° and 60° are complements, $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$ and $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, and 45° is its own complement, so $\sin 45^\circ = \cos 45^\circ$. Reading down the table from 30° to 60° , the sines are $\frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, and the cosines are the same list in reverse. Tangents are $\frac{\sqrt{3}}{3}$, 1, $\sqrt{3}$, and cotangents run in reverse. Learn the sine row and the pattern, and the whole table follows.

To solve an equation like $\sin(\text{one angle}) = \cos(\text{another angle})$ when both angles are acute, set the two angles to add to 90° . Then check that each angle really is acute. (Why must they be complements? If $\sin P = \cos Q$ with P and Q acute, then $\sin P = \sin(90^\circ - Q)$. As an acute angle grows, the opposite side grows while the hypotenuse stays fixed, so its sine grows; two different acute angles never share a sine. So $P = 90^\circ - Q$.)

KEY FORMULAS

$$\sin \theta = \cos(90^\circ - \theta) \quad \cos \theta = \sin(90^\circ - \theta)$$

$$\tan \theta = \cot(90^\circ - \theta) \quad \cot \theta = \tan(90^\circ - \theta)$$

$$\sec \theta = \csc(90^\circ - \theta) \quad \csc \theta = \sec(90^\circ - \theta)$$

$$\sin 30^\circ, \sin 45^\circ, \sin 60^\circ = \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2} \quad \cos 30^\circ, \cos 45^\circ, \cos 60^\circ = \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{1}}{2}$$

$$\tan 30^\circ, \tan 45^\circ, \tan 60^\circ = \frac{\sqrt{3}}{3}, 1, \sqrt{3}$$

From history. The *co-* in cosine, cotangent and cosecant stands for "complement." Cosine began as the "sine of the complement" (in Latin, *sinus complementi*), and the shortened names came into use in the 1600s. Because each function of an angle equals its cofunction of the complement, printed tables often listed angles only from 0° to 45° down the page; the values for 45° to 90° were read from the same columns, with the sine and cosine headings swapped. This halved the labor of the table makers.

Euclid connection. Euclid I.32 shows that the angles of a triangle add to two right angles. In a right triangle, this leaves exactly one right angle for the two acute angles to share, which is why they are complementary.

WORKED EXAMPLES

Example 1. Ruth leans a 16 ft ladder against a barn wall so that it makes a 62° angle with the level ground. (a) What angle does the ladder make with the wall? (b) Write the height the ladder reaches on the wall using sine, then using cosine, and find it to the nearest tenth of a foot.

Show solution

1. The ground, the wall and the ladder form a right triangle with the right angle at the base of the wall. The two acute angles are complementary, so the angle at the wall is $90^\circ - 62^\circ = 28^\circ$.
2. The height h on the wall is opposite the 62° angle, so $\sin 62^\circ = \frac{h}{16}$ and $h = 16 \sin 62^\circ$.
3. The same side is adjacent to the 28° angle at the wall, so $\cos 28^\circ = \frac{h}{16}$ and $h = 16 \cos 28^\circ$.
4. The two forms must agree because $\sin 62^\circ = \cos(90^\circ - 62^\circ) = \cos 28^\circ$. A calculator gives $\sin 62^\circ \approx 0.882948$ and $\cos 28^\circ \approx 0.882948$.
5. So $h = 16 \sin 62^\circ \approx 14.127162 \approx 14.1$ ft.

Answer: (a) 28° ; (b) $h = 16 \sin 62^\circ = 16 \cos 28^\circ \approx 14.1$ ft

Example 2. Find x in degrees if $\sin(2x + 10^\circ) = \cos(3x - 15^\circ)$ and both angles are acute.

Show solution

1. When the sine of one acute angle equals the cosine of another, the two angles are complementary.
2. So $(2x + 10^\circ) + (3x - 15^\circ) = 90^\circ$.
3. Simplify: $5x - 5^\circ = 90^\circ$, so $5x = 95^\circ$ and $x = 19^\circ$.
4. Check: $2(19^\circ) + 10^\circ = 48^\circ$ and $3(19^\circ) - 15^\circ = 42^\circ$. Both are acute and $48^\circ + 42^\circ = 90^\circ$, so $\sin 48^\circ = \cos 42^\circ$. It checks.

Answer: $x = 19^\circ$

PRACTICE

1. Easy Fill in the blank with an acute angle in degrees: $\sin 20^\circ = \cos \underline{\hspace{1cm}}$.

Show answer

$$70^\circ$$

2. Easy Fill in the blank with an acute angle in degrees: $\tan 38^\circ = \cot \underline{\hspace{1cm}}$.

Show answer

$$52^\circ$$

3. Easy Fill in the blank with an acute angle in degrees: $\sec 81^\circ = \csc \underline{\hspace{1cm}}$.

Show answer

$$9^\circ$$

4. Medium Without a calculator, find the exact value of $\frac{\sin 25^\circ}{\cos 65^\circ}$.

Show answer

$$1, \text{ since } \sin 25^\circ = \cos 65^\circ$$

5. Medium Find the exact value of $\cos 30^\circ \cdot \sec 60^\circ - \tan 45^\circ$.

Show answer

$$\frac{\sqrt{3}}{2} \cdot 2 - 1 = \sqrt{3} - 1 \approx 0.732$$

6. Medium A 20 ft ladder leans against a wall and makes a 70° angle with the level ground. (a) What angle does the ladder make with the wall? (b) Find the height the ladder reaches on the wall to the nearest tenth of a foot.

Show answer

$$(a) 20^\circ; (b) h = 20 \sin 70^\circ = 20 \cos 20^\circ \approx 18.8 \text{ ft}$$

7. Medium A 10 ft fence post stands straight up on level ground. The sun's rays make a 34° angle with the post. (a) Find the altitude of the sun, the angle of the rays above the horizon. (b) Find the length of the post's shadow to the nearest tenth of a foot.

Show answer

$$(a) 90^\circ - 34^\circ = 56^\circ; (b) \text{ shadow} = 10 \tan 34^\circ = 10 \cot 56^\circ \approx 6.7 \text{ ft}$$

8. **Hard** Find x in degrees if $\cos(x + 12^\circ) = \sin(2x + 9^\circ)$ and both angles are acute.

Show answer

$(x + 12^\circ) + (2x + 9^\circ) = 90^\circ$, so $3x = 69^\circ$ and $x = 23^\circ$ (the angles are 35° and 55°)

9. **Hard** Find x in degrees if $\cot(x + 40^\circ) = \tan(2x - 10^\circ)$ and both angles are acute.

Show answer

$(x + 40^\circ) + (2x - 10^\circ) = 90^\circ$, so $3x = 60^\circ$ and $x = 20^\circ$ (the angles are 60° and 30°)

10. **Hard** Prove that $\tan \theta = \cot(90^\circ - \theta)$ for every acute angle θ .

Show answer

Draw right triangle ABC with the right angle at C and $\angle A = \theta$. By Euclid I.32, $\angle B = 90^\circ - \theta$. Let a and b be the legs opposite A and B . Then $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$. At angle B , the leg a is adjacent and b is opposite, so $\cot(90^\circ - \theta) = \cot B = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$. The two ratios are the same, so $\tan \theta = \cot(90^\circ - \theta)$.

11. **Challenge** Without a calculator, find the exact value of the product $\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ$. Explain your reasoning.

Show answer

1. Pair each $\tan k^\circ$ with $\tan(90^\circ - k^\circ) = \cot k^\circ$ for $k = 1, 2, \dots, 44$. Each pair is $\tan k^\circ \cot k^\circ = 1$. The leftover middle factor is $\tan 45^\circ = 1$. So the whole product is 1.

COMMON MISTAKES

- Pairing a function with the wrong partner, such as writing $\sin 20^\circ = \sin 70^\circ$ or $\tan 38^\circ = \cos 52^\circ$. The function must change to its cofunction (sine to cosine, tangent to cotangent, secant to cosecant) and the angle must change to its complement.
- Setting the two angles equal instead of making them add to 90° . In $\sin(2x + 10^\circ) = \cos(3x - 15^\circ)$, the angles are complements, so their sum is 90° .
- Forgetting to check that the angles are acute. After solving, substitute x back in and confirm that both angles lie between 0° and 90° .

- Mixing up the 30° and 60° entries of the table. Remember that sine increases from 30° to 60° ($\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$), and cosine runs through the same values in reverse.

Make it a game · Cofunction Dominoes. Make twelve dominoes. Each end shows a ratio, such as $\sin 20$ degrees or $\cos 70$ degrees. Chain them so each touching pair are cofunctions of complementary angles. Then solve $\sin(2x + 10) = \cos(3x - 15)$ in two minutes: set the angles to add to 90 degrees. Write your own equation for your parent.

SPECIAL RIGHT TRIANGLES AND EXACT VALUES

In Unit 1 you found trigonometric ratios with a calculator, as rounded decimals. In this unit you will find some of them exactly. Two triangles come straight from the simplest figures in geometry. Half of a square is the 45-45-90 triangle, with sides in the ratio $1 : 1 : \sqrt{2}$. Half of an equilateral triangle is the 30-60-90 triangle, with sides in the ratio $1 : \sqrt{3} : 2$. With the Pythagorean theorem (Euclid I.47), these two figures give exact values such as $\sin 30^\circ = \frac{1}{2}$ and $\cos 45^\circ = \frac{\sqrt{2}}{2}$, and you will prove each one rather than memorize it.

The unit closes with the cofunction identities. The two acute angles of a right triangle are complements, and the side opposite one is adjacent to the other. So the sine of an angle is the cosine of its complement, and the same holds for tangent and cotangent, secant and cosecant. This explains the *co-* in the names and ties the exact values together in one small table.

These exact values will be used throughout the rest of the course. In Unit 3 they become the coordinates of key points on the unit circle. They also give the heights and zeros of the graphs in Unit 4, and they are the starting values for the identities in Unit 5 and the equations in Unit 6.

PREREQUISITES

- Define and compute sine, cosine and tangent of an acute angle from side lengths (Lesson 1.2).
- Define cosecant, secant and cotangent as reciprocal ratios (Lesson 1.5).
- Explain why similar right triangles have equal side ratios (Lesson 1.1).
- Use the Pythagorean theorem and the angle sum of a triangle from Geometry.
- Simplify square roots and rationalize a denominator, such as $\frac{6}{\sqrt{3}} = 2\sqrt{3}$.

MASTERY CHECKLIST

- ☐ I can derive the
1 : ratio of a 45-45-90 triangle from the
1 : diagonal of a square.
 $\sqrt{2}$
- ☐ I can find the missing sides of a 45-45-90 triangle from either a leg or the hypotenuse, in exact form and as a rounded decimal.
- ☐ I can derive the
1 : ratio of a 30-60-90 triangle from
 $\sqrt{3}$ half of an equilateral triangle.
2
- ☐ I can find the missing sides of a 30-60-90 triangle from any one side, starting with the short leg.
- ☐ I can state and explain the exact values of all six trigonometric ratios of
 $30^\circ, 45^\circ$
and 60° .
d
- ☐ I can prove that
 $\sin \theta = \cos(90^\circ - \theta)$ and the other cofunction identities
using one right triangle.
- ☐ I can rewrite a trigonometric ratio as its cofunction of the complementary angle.
- ☐ I can solve equations such as
 $\sin(2x) = \cos(3x - 15^\circ)$ when both angles are acute.

- I can use special right triangles to solve practical problems about roofs, ladders, shadows, fields and ships.

LESSONS IN THIS UNIT

2.1 The 45-45-90 Triangle	45 minutes
2.2 The 30-60-90 Triangle	50 minutes
2.3 Complementary Angles and Cofunctions	50 minutes

UNIT 2 TEST

Allow 50 minutes. Problems 1 through 7, 11 and 12 must be done without a calculator; give exact values with denominators rationalized. A scientific calculator may be used on Problems 8, 9, 10 and 13, but only to round after you have written the exact answer. Round decimal answers to the nearest tenth unless a problem says otherwise. Show your work; a sketch of each triangle earns partial credit. Total: 100 points.

1. 6 pts

Give the exact values of $\sin 45^\circ$, $\cos 45^\circ$ and $\tan 45^\circ$.

2. 6 pts

The hypotenuse of an isosceles right triangle is 14 cm. Find the exact length of each leg.

3. 6 pts

Give the exact values of $\sin 60^\circ$, $\cos 60^\circ$ and $\tan 60^\circ$.

4. 6 pts

In a 30-60-90 triangle the long leg is 12 in. Find the exact lengths of the short leg and the hypotenuse.

5. 6 pts

Fill in the blank with an acute angle in degrees: $\cos 47^\circ = \sin \underline{\hspace{1cm}}$.

6. **6 pts**

Find the exact value of $\sin 30^\circ \cos 60^\circ + \tan 45^\circ$.

7. **6 pts**

Find the exact value of $\csc 30^\circ + \sec 45^\circ \cos 45^\circ$.

8. **8 pts**

Ruth's square vegetable garden is 18 ft on each side. She lays a straight stone path from one corner to the opposite corner. How long is the path? Give the exact value and round to the nearest tenth of a foot.

9. **8 pts**

When the sun is 30° above the horizon, how long is the shadow cast on level ground by a barn wall 24 ft tall? Give the exact value and round to the nearest tenth of a foot.

10. **8 pts**

The front of an A-frame cabin is an equilateral triangle that stands 15 ft tall from the middle of its base to its peak. How long is each side of the triangle? Give the exact value and round to the nearest tenth of a foot.

11. **8 pts**

Find x in degrees if $\sin(3x - 4^\circ) = \cos(2x + 9^\circ)$ and both angles are acute. Check your answer.

12. **12 pts**

Let A and B be the acute angles of a right triangle. Prove that $A + B = 90^\circ$ and that $\sin A = \cos B$. Then explain why this shows $\sin \theta = \cos(90^\circ - \theta)$ for every acute angle θ .

13. **14 pts**

In triangle ABC , $\angle B = 60^\circ$ and $\angle C = 45^\circ$, and the altitude from A to side BC has length 6. Find the exact length of BC and the exact area of the triangle. Round each to the nearest hundredth.

ANSWER KEY

1. $\sin 45^\circ = \frac{\sqrt{2}}{2}$, $\cos 45^\circ = \frac{\sqrt{2}}{2}$, $\tan 45^\circ = 1$
2. $\frac{14}{\sqrt{2}} = 7\sqrt{2}$ cm
3. $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$
4. Short leg $= \frac{12}{\sqrt{3}} = 4\sqrt{3}$ in; hypotenuse $= 8\sqrt{3}$ in
5. 43°
6. $\frac{1}{2} \cdot \frac{1}{2} + 1 = \frac{5}{4}$
7. $2 + \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2 + 1 = 3$
8. $18\sqrt{2}$ ft ≈ 25.5 ft
9. $\tan 30^\circ = \frac{24}{x}$, so $x = \frac{24}{\sqrt{3}/3} = 24\sqrt{3}$ ft ≈ 41.6 ft
10. The height is the long leg of a 30-60-90 triangle, so the short leg is $\frac{15}{\sqrt{3}} = 5\sqrt{3}$ and each side (the hypotenuse) is $10\sqrt{3}$ ft ≈ 17.3 ft
11. $(3x - 4^\circ) + (2x + 9^\circ) = 90^\circ$, so $5x = 85^\circ$ and $x = 17^\circ$. Check: the angles are 47° and 43° , both acute, with sum 90° .
12. The angles of a triangle add to 180° (Euclid I.32), and the right angle uses 90° , so $A + B = 90^\circ$. Let a be the leg opposite A and c the hypotenuse. Then $\sin A = \frac{a}{c}$. The same leg a is adjacent to B , so $\cos B = \frac{a}{c}$. Hence $\sin A = \cos B$. Given any acute angle θ , build a right triangle with $A = \theta$; then $B = 90^\circ - \theta$, and $\sin \theta = \cos(90^\circ - \theta)$.
13. Let D be the foot of the altitude. In triangle ABD , $AD = 6$ is the long leg (opposite 60°), so $BD = \frac{6}{\sqrt{3}} = 2\sqrt{3}$. Triangle ADC is 45-45-90, so $DC = 6$. Then $BC = 6 + 2\sqrt{3} \approx 9.46$, and the area is $\frac{1}{2} \cdot 6 \cdot (6 + 2\sqrt{3}) = 18 + 6\sqrt{3} \approx 28.39$ square units.

UNIT 3

ANGLES, RADIANS AND THE UNIT CIRCLE

Units 1 and 2 defined sine, cosine and tangent as ratios of sides in a right triangle, which limits them to acute angles. This unit sets them free. We first widen the idea of an angle to a rotation of any size, positive or negative, and learn a second, more natural unit for measuring it: the radian, which measures an angle by the arc it cuts off, counted in radii.

Radians make the geometry of circles simple. Arc length becomes $s = r\theta$, sector area becomes $A = \frac{1}{2}r^2\theta$, and the speed of a point on a turning wheel becomes $v = r\omega$, formulas we put to work on irrigation pivots, pulleys, gears, tractors and millstones.

Finally, building on Euler's idea of sine and cosine as functions of an arc on a circle of radius 1, we define the sine and cosine of any angle as the coordinates of a point on the unit circle. Reference angles and the signs of the four quadrants let us evaluate every trigonometric function of every angle, exactly for the special angles. This unit is the foundation for everything that follows: the graphs of Unit 4, the identities of Unit 5 and the equations of Unit 6 all live on the unit circle.

PREREQUISITES

- Define and evaluate the six trigonometric ratios of an acute angle in a right triangle (Lessons 1.2 and 1.5).
- Use inverse ratios such as $\sin^{-1}$ to find an acute angle (Lesson 1.4).
- Know the exact values of sine, cosine and tangent for 30° , 45° and 60° (Lessons 2.1 to 2.3).
- Use the Pythagorean theorem and the distance formula in the coordinate plane.
- Know the circumference $2\pi r$ and area πr^2 of a circle, and convert units of length and time.

MASTERY CHECKLIST

- ☐ I can draw any angle in standard position and find coterminal angles.
- ☐ I can explain what a radian is and why radian measure does not depend on the size of the circle.
- ☐ I can convert angles between degrees and radians, giving exact answers in terms of π .
- ☐ I can derive and use $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.
- ☐ I can derive $v = r\omega$ and solve problems with wheels, pulleys and gears.
- ☐ I can define the sine and cosine of any angle using the unit circle.
- ☐ I can find all six trigonometric functions from a point (x, y) on the terminal side.
- ☐ I can find the reference angle of any angle and tell the sign of each function in each quadrant.
- ☐ I can give exact values of all six functions for every multiple of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including quadrantal angles.
- ☐ I can find the remaining trigonometric functions given one

value and the quadrant.

LESSONS IN THIS UNIT

3.1	Angles in Standard Position	45 minutes
3.2	Radian Measure	45 minutes
3.3	Converting Degrees and Radians	40 minutes
3.4	Arc Length and Sector Area	50 minutes
3.5	Linear and Angular Speed	50 minutes
3.6	The Unit Circle	50 minutes
3.7	Reference Angles and Signs by Quadrant	50 minutes
3.8	Exact Values Around the Unit Circle	2 sessions of 45 minutes

UNIT 3 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but give exact answers wherever a problem asks for them, and show your work. Unless a problem says otherwise, round lengths and other decimal answers to the nearest hundredth and angles in degrees to the nearest tenth of a degree. State whether each angle is in degrees or radians. Total: 100 points.

1. 5 pts

Find the angle between 0° and 360° that is coterminal with -790° , and name its quadrant.

2. 5 pts

A central angle in a circle of radius 5 cm cuts off an arc 8 cm long. Find the angle in radians.

3. 5 pts

Convert 300° to radians. Give the exact value.

4. (5 pts)

Convert $\frac{7\pi}{12}$ radians to degrees.

5. (8 pts)

A fan-shaped window above a barn door is a sector of a circle with radius 1.5 m and central angle 120° . Find the length of trim needed for the curved edge and the area of glass, exactly and to the nearest hundredth.

6. (8 pts)

A tractor's rear wheel has radius 0.75 m and turns at 40 rpm. How fast is the tractor moving, in kilometers per hour, to the nearest tenth?

7. (7 pts)

A motor pulley of radius 3 in. turns at 1200 rpm and drives, by a belt, a pulley of radius 8 in. on a corn sheller. How fast does the larger pulley turn? Explain why the belt makes $r_1\omega_1 = r_2\omega_2$ true.

8. (8 pts)

The point $(-7, 24)$ lies on the terminal side of θ . Find the exact values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

9. (8 pts)

Use a reference angle to find $\tan 1000^\circ$, to the nearest thousandth. Show the coterminal angle, quadrant and reference angle.

10. (9 pts)

Find the exact values of $\sin \frac{4\pi}{3}$, $\cos \frac{7\pi}{4}$, $\tan \frac{5\pi}{6}$ and $\sec \pi$.

11. (8 pts)

Given $\sin \theta = -\frac{8}{17}$ with θ in Quadrant IV, find the exact values of $\cos \theta$, $\tan \theta$ and $\csc \theta$.

12. (10 pts)

Prove that the area of a sector of a circle of radius r with central angle θ (in radians) is $A = \frac{1}{2}r^2\theta$. Then explain why the formula would be wrong if θ were in degrees.

13. 14 pts

A Ferris wheel of radius 40 ft has its center 45 ft above the ground and turns counterclockwise once every 2 minutes. A rider starts level with the center on the right side (the 3 o'clock position). (a) Through what angle, in radians, has the rider turned after 50 seconds? (b) How high above the ground is the rider then? (c) What is the rider's linear speed, exactly and to the nearest hundredth of a foot per second?

ANSWER KEY

1. $-790^\circ + 1080^\circ = 290^\circ$, Quadrant IV
2. $\theta = \frac{8}{5} = 1.6$ radians
3. $\frac{5\pi}{3}$
4. 105°
5. $\theta = \frac{2\pi}{3}$. Curved edge: $s = 1.5 \cdot \frac{2\pi}{3} = \pi \approx 3.14$ m. Area: $A = \frac{1}{2}(1.5)^2 \cdot \frac{2\pi}{3} = 0.75\pi \approx 2.36$ m²
6. $\omega = 80\pi$ rad/min; $v = 0.75 \cdot 80\pi = 60\pi$ m/min; $60\pi \cdot \frac{60}{1000} = 3.6\pi \approx 11.3$ km/h
7. $3 \cdot 1200 = 8\omega_2$, so $\omega_2 = 450$ rpm. The belt does not slip, so both rims move at the belt's linear speed, and $v = r\omega$ for each pulley.
8. $r = 25$; $\sin \theta = \frac{24}{25}$, $\cos \theta = -\frac{7}{25}$, $\tan \theta = -\frac{24}{7}$
9. $1000^\circ - 720^\circ = 280^\circ$, Quadrant IV, reference angle 80° ; tangent is negative in Quadrant IV, so $\tan 1000^\circ = -\tan 80^\circ \approx -5.671$
10. $-\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{3}, -1$
11. $y = -8, r = 17, x = 15$: $\cos \theta = \frac{15}{17}$, $\tan \theta = -\frac{8}{15}$, $\csc \theta = -\frac{17}{8}$
12. The sector is the same fraction of the disk as its angle is of a full turn, and a full turn is 2π radians: $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$. Multiplying by πr^2 gives $A = \frac{\pi r^2 \theta}{2\pi} = \frac{1}{2}r^2\theta$. The proof used the fact that a full turn is 2π ; in degrees a full turn is 360° , so the formula would become $A = \frac{\pi r^2 \theta}{360}$ instead.
13. (a) $\omega = \frac{2\pi}{120} = \frac{\pi}{60}$ rad/s, so $\theta = 50 \cdot \frac{\pi}{60} = \frac{5\pi}{6}$. (b) Height = $45 + 40 \sin \frac{5\pi}{6} = 45 + 20 = 65$ ft. (c) $v = 40 \cdot \frac{\pi}{60} = \frac{2\pi}{3} \approx 2.09$ ft/s

3.1 ANGLES IN STANDARD POSITION

OBJECTIVES

- Draw angles of any size, including negative angles and angles greater than 360° , in standard position.
- Find coterminal angles and name the quadrant in which an angle's terminal side lies.

WATCH

Introduction to Angles (Precalculus - Trigonometry 1)

Professor Leonard · 26:59

<https://www.youtube.com/watch?v=c41QejoWnb4>

WHEN TO WATCH

After reading the Core ideas, before Example 1. Long (27 min); the coterminal-angle part is what matters most.

Note: Woo says 'anticlockwise' for counterclockwise.

BACKUP VIDEO

Intro to Trigonometric Functions (1 of 2: Angles of any magnitude)

Eddie Woo · 9:44

<https://www.youtube.com/watch?v=FTTkXGJ7E3Y>

Short alternative before Example 1: angles past 360° on the unit circle.

KEY TERMS

standard position

An angle whose vertex is at the origin and whose initial side lies along the positive x -axis.

initial side

The ray where the rotation that forms an angle begins.

terminal side

The ray where the rotation that forms an angle ends.

coterminal angles

Angles in standard position that share the same terminal side; their measures differ by an integer multiple of 360° .

quadrantal angle

An angle in standard position whose terminal side lies on an axis, such as 0° , 90° , 180° or 270° .

CORE IDEAS

In Geometry an angle was a figure: two rays with a common vertex, measuring between 0° and 180° . In trigonometry we think of an angle as a **rotation**. A ray starts in one position (the initial side) and turns to a final position (the terminal side). The measure of the angle tells how far, and in which direction, the ray turned.

An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive x -axis. Counterclockwise rotation gives a **positive** angle; clockwise rotation gives a **negative** angle. One full turn is 360° , so an angle of 450° is one full turn plus 90° more, and an angle of -90° is a quarter turn clockwise.

The axes divide the plane into four quadrants, numbered I, II, III, IV counterclockwise starting from the upper right. For an angle θ between 0° and 360° : Quadrant I if $0^\circ < \theta < 90^\circ$, Quadrant II if $90^\circ < \theta < 180^\circ$, Quadrant III if $180^\circ < \theta < 270^\circ$, Quadrant IV if $270^\circ < \theta < 360^\circ$. An angle whose terminal side lies on an axis is **quadrantal** and belongs to no quadrant.

Why do coterminal angles differ by multiples of 360° ? Adding one full turn brings the terminal side back to exactly where it was. So θ , $\theta + 360^\circ$, $\theta - 360^\circ$, $\theta + 720^\circ$ and so on all share a terminal side. Conversely, if two angles share a terminal side, the extra rotation from one to the other must be a whole number of full turns. Thus every angle coterminal with θ has the form $\theta + 360^\circ k$ for some integer k .

To find the angle between 0° and 360° that is coterminal with a given angle, add or subtract 360° as many times as needed. For a large angle, divide by 360 to see how many full turns to remove. This one angle tells you the quadrant and, later, the values of every trigonometric function.

KEY FORMULAS

Coterminal angles: $\theta + 360^\circ k$, $k = 0, \pm 1, \pm 2, \dots$

One revolution = 360°

From history. The division of the circle into 360 degrees comes to us from Babylonian astronomy, whose arithmetic used base 60 (sexagesimal) numbers. Greek astronomers such as Hipparchus and Ptolemy adopted the Babylonian system, and Ptolemy's *Almagest* divides each degree into sixtieths. Our minutes and seconds of arc, like the minutes and seconds of the hour, are survivors of that base-60 system. Exactly why 360 was chosen is not known for certain, but it is a convenient number: it can be divided evenly by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15 and many more.

Euclid connection. In Euclid's *Elements* an angle is the inclination of two lines that meet, and the angles he works with are always less than two right angles. Trigonometry keeps Euclid's angle but adds direction and rotation, so that angles of any size, positive or negative, can be measured.

WORKED EXAMPLES

Example 1. Find one positive angle and one negative angle that are coterminal with 140° .

Show solution

1. Coterminal angles have the form $140^\circ + 360^\circ k$.
2. Take $k = 1$: $140^\circ + 360^\circ = 500^\circ$. This is one more full counterclockwise turn.
3. Take $k = -1$: $140^\circ - 360^\circ = -220^\circ$. Turning 220° clockwise from the positive x -axis lands in Quadrant II, the same place as 140° .
4. Many other answers are possible, such as 860° or -580° .

Answer: 500° and -220° (other answers possible)

Example 2. The auger on Mr. Miller's grain wagon turns clockwise through $3\frac{1}{4}$ revolutions. A paint mark on its shaft starts on the positive x -axis. Express the rotation as an angle in degrees, find the coterminal angle between 0° and 360° , and tell where the mark ends up.

Show solution

1. Clockwise rotation is negative. One revolution is 360° , so the angle is $-3.25 \times 360^\circ = -1170^\circ$.
2. Add full turns until the angle lies between 0° and 360° . Since $1170 \div 360 = 3.25$, adding 4 turns is enough: $-1170^\circ + 4(360^\circ) = -1170^\circ + 1440^\circ = 270^\circ$.
3. An angle of 270° is quadrantal; its terminal side is the negative y -axis.
4. Check: three full clockwise turns return the mark to the start, and the extra quarter turn clockwise moves it from the positive x -axis down to the negative y -axis.

Answer: -1170° , coterminal with 270° ; the mark ends on the negative y -axis (pointing straight down).

PRACTICE

1. **Easy** In which quadrant does the terminal side of 200° lie?

Show answer

Quadrant III

2. **Easy** Find one positive and one negative angle coterminal with 75° .

Show answer

435° and -285° (other answers possible)

3. **Easy** Find the angle between 0° and 360° that is coterminal with 400° .

Show answer

40°

4. **Medium** Find the angle between 0° and 360° that is coterminal with -500° , and name its quadrant.

Show answer

$-500^\circ + 720^\circ = 220^\circ$, Quadrant III

5. **Medium** Find the angle between 0° and 360° that is coterminal with 1000° , and name its quadrant.

Show answer

$$1000^\circ - 720^\circ = 280^\circ, \text{ Quadrant IV}$$

6. **Medium** Sketch -135° in standard position. Name its quadrant and give the positive coterminal angle less than 360° .

Show answer

A 135° clockwise turn ends in Quadrant III; coterminal with 225° .

7. **Medium** A Ferris wheel at the county fair turns counterclockwise through $2\frac{1}{3}$ revolutions. Express the rotation in degrees, and find the coterminal angle between 0° and 360° .

Show answer

$$\frac{7}{3} \times 360^\circ = 840^\circ, \text{ coterminal with } 120^\circ$$

8. **Hard** List every angle coterminal with 50° that lies between -1000° and 1000° , in increasing order.

Show answer

$$-670^\circ, -310^\circ, 50^\circ, 410^\circ, 770^\circ$$

9. **Hard** Ruth watches the minute hand of the kitchen clock from 12:00 to 3:20. Through what angle, in degrees, does the minute hand turn? Give a signed angle (remember that clock hands turn clockwise).

Show answer

$$3\frac{1}{3} \text{ revolutions clockwise: } -\frac{10}{3} \times 360^\circ = -1200^\circ$$

10. **Hard** The terminal side of θ lies in Quadrant II, with $0^\circ < \theta < 360^\circ$. In which quadrant does the terminal side of $\theta + 180^\circ$ lie? Of $-\theta$?

Show answer

$\theta + 180^\circ$ is in Quadrant IV; $-\theta$ is in Quadrant III.

11. **Challenge** Explain why two angles α and β in standard position are coterminal exactly when $\alpha - \beta$ is an integer multiple of 360° .

Show answer

If $\alpha - \beta = 360^\circ k$, then α is β followed by k full turns, and each full turn returns the terminal side to the same ray, so they are coterminal. Conversely, if they share a terminal side, the rotation from β to α starts and ends on the same ray, which happens only after a whole number of full turns; so $\alpha - \beta = 360^\circ k$.

COMMON MISTAKES

- Treating a negative angle as if it were positive. A negative angle turns **clockwise**; sketch the direction before naming the quadrant.
- Subtracting 360° only once from a large angle. Keep adding or subtracting 360° until the result lies between 0° and 360° ; dividing by 360 first tells you how many turns to remove.
- Placing a quadrantal angle such as 180° or 270° in a quadrant. Its terminal side lies on an axis, so it is in no quadrant.
- Counting quadrants clockwise. Quadrants are numbered I, II, III, IV **counterclockwise**, starting at the upper right.

Make it a game • Human Compass Spin. Stand at a taped origin facing the positive x direction, which you decide. Parent calls an angle: negative 135 degrees, 450, 225, 720. Turn your body to show it, then name the quadrant and a coterminal angle between 0 and 360. Ten correct turns in a row earns a trophy.

3.2 RADIAN MEASURE

OBJECTIVES

- Define the radian as the central angle that subtends an arc equal in length to the radius.
- Find the radian measure of a central angle from $\theta = \frac{s}{r}$ and express common fractions of a turn in radians.
- Explain why radian measure does not depend on the size of the circle and why it is a natural way to measure angles.

WATCH

Introduction to Radians (2 of 3: Defining a better way)

Eddie Woo · 9:19

<https://www.youtube.com/watch?v=EZhCDaiTahM>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

Introduction to Radians (Precalculus - Trigonometry 3)

Professor Leonard · 17:26

https://www.youtube.com/watch?v=EWd_FtOfyPc

Before the practice set, for a slower American walkthrough.

KEY TERMS

radian

The measure of a central angle that subtends (cuts off) an arc whose length equals the radius of the circle.

subtend

To cut off: a central angle subtends the arc that lies between its two sides.

central angle

An angle whose vertex is at the center of a circle.

radian measure

The ratio $\theta = \frac{s}{r}$ of the arc length s to the radius r ; it is a pure number with no units of length.

CORE IDEAS

Degrees are a human choice: someone decided to cut the circle into 360 parts. The **radian** comes from the circle itself. Lay the radius of a circle along its edge, bending it to fit the curve. The central angle that cuts off that arc is **one radian**.

More generally, if a central angle cuts off an arc of length s on a circle of radius r , its radian measure is $\theta = \frac{s}{r}$, the number of radii that fit along the arc. Because it is length divided by length, a radian measure has no units of length; we often write "rad" only to make clear that it is not degrees, and an angle written with no unit is understood to be in radians.

Why does this not depend on the circle? Enlarge a circle by a scale factor k . Every length, the radius and the arc alike, is multiplied by k , so the ratio $\frac{ks}{kr} = \frac{s}{r}$ is unchanged. A given angle therefore has the same radian measure in every circle, just as the trigonometric ratios of an acute angle are the same in every similar right triangle.

A full turn cuts off the whole circumference, $2\pi r$. So one full revolution measures $\frac{2\pi r}{r} = 2\pi$ radians, about 6.28 radians. It follows that a half turn is π radians, a quarter turn is $\frac{\pi}{2}$, and in general a fraction f of a turn is $2\pi f$ radians. One radian is a bit less than one sixth of a turn.

Radians are natural because they make the arithmetic of circles simple. Since $\theta = \frac{s}{r}$, the length of an arc is just $s = r\theta$, with no stray conversion factor. In later courses, especially calculus, the formulas for the rates of change of sine and cosine are clean only when angles are measured in radians. For this reason mathematicians and scientists use radians as the standard angle measure.

KEY FORMULAS

$$\theta = \frac{s}{r} \quad (\theta \text{ in radians})$$

$$\text{One revolution} = 2\pi \text{ radians}$$

$$\text{Half revolution} = \pi \text{ radians}, \quad \text{Quarter revolution} = \frac{\pi}{2} \text{ radians}$$

From history. In the early 1700s the English mathematician Roger Cotes recognized that measuring an angle by the arc it cuts off on a circle of radius 1 is especially natural. The word "radian" itself is much younger: it first appeared in print in 1873, in examination questions set by James Thomson, the brother of Lord Kelvin. Long before either man, Eratosthenes used the same idea of comparing an arc to an angle to estimate the size of the earth.

Euclid connection. Euclid VI.33 states that in equal circles, angles at the center have the same ratio as the arcs on which they stand. That is exactly why the arc a central angle cuts off measures the angle, and why the ratio $\frac{s}{r}$ is a fair measure of it.

WORKED EXAMPLES

Example 1. A central angle in a circle of radius 5 cm cuts off an arc 12 cm long. Find the angle in radians.

Show solution

1. Radian measure is the number of radii that fit along the arc: $\theta = \frac{s}{r}$.
2. Substitute: $\theta = \frac{12 \text{ cm}}{5 \text{ cm}} = 2.4$.
3. The centimeters cancel, so the answer is a pure number of radians.

Answer: $\theta = 2.4$ radians

Example 2. A center-pivot irrigation system on a farm has an arm 400 m long that sweeps around a central tower. In one afternoon the end of the arm travels 1000 m along its circular path. Through what angle, in radians, did the arm turn? Is that more or less than half a revolution, and about what fraction of a full revolution is it (to the nearest hundredth)?

Show solution

1. The end of the arm moves along a circle of radius $r = 400$ m, and the arc length is $s = 1000$ m.
 2. $\theta = \frac{s}{r} = \frac{1000}{400} = 2.5$ radians.
 3. Half a revolution is $\pi \approx 3.14159$ radians. Since $2.5 < \pi$, the arm turned less than half a revolution.
 4. A full revolution is 2π radians, so the fraction is $\frac{2.5}{2\pi} \approx 0.397887$, about 0.40 of a revolution.
-

Answer: 2.5 radians, less than half a revolution (about 0.40 of a full turn)

PRACTICE

1. **Easy** A central angle in a circle of radius 4 in. cuts off an arc of 10 in. Find the angle in radians.

Show answer

2.5 radians

2. **Easy** How many radians are in one full revolution? Give the exact value and a decimal to the nearest hundredth.

Show answer

$2\pi \approx 6.28$ radians

3. **Easy** Give the exact radian measure of a quarter turn.

Show answer

$\frac{\pi}{2} \approx 1.57$ radians

4. **Medium** In a circle of radius 3 ft, a central angle measures 2 radians. How long is the arc it cuts off?

Show answer

$$s = r\theta = 3 \times 2 = 6 \text{ ft}$$

5. **Medium** Give the exact radian measure of $\frac{1}{3}$ of a turn, $\frac{3}{4}$ of a turn and $\frac{1}{6}$ of a turn.

Show answer

$$\frac{2\pi}{3}, \frac{3\pi}{2}, \frac{\pi}{3}$$

6. **Medium** An angle of 3 radians is drawn in standard position. In which quadrant does its terminal side lie? Explain using π .

Show answer

Quadrant II, because $\frac{\pi}{2} \approx 1.571 < 3 < \pi \approx 3.142$.

7. **Medium** The blade of a farm windmill is 4 m long. As the blade turns, its tip travels 9 m along its circular path. Through how many radians did the blade turn?

Show answer

$$\theta = \frac{9}{4} = 2.25 \text{ radians}$$

8. **Hard** In which quadrant does each angle, in radians, terminate: 2, 4, 5, 6? Use $\frac{\pi}{2} \approx 1.571$, $\pi \approx 3.142$, $\frac{3\pi}{2} \approx 4.712$, $2\pi \approx 6.283$.

Show answer

2: Quadrant II; 4: Quadrant III; 5: Quadrant IV; 6: Quadrant IV

9. **Hard** A wagon wheel of radius 0.5 m rolls 10 m along a level lane without slipping. Through how many radians does the wheel turn? How many revolutions is that, to the nearest hundredth?

Show answer

$$\theta = \frac{10}{0.5} = 20 \text{ radians, which is } \frac{20}{2\pi} \approx 3.18 \text{ revolutions}$$

10. **Hard** Samuel says that a central angle in a large circle must have a bigger radian measure than the same angle in a small circle, because the arc is longer. Explain why he is wrong.

Show answer

A larger circle is a scaled copy of the smaller one. If the scale factor is k , both the arc and the radius are multiplied by k , so $\frac{ks}{kr} = \frac{s}{r}$. The arc is longer, but so is the radius, and their ratio, the radian measure, is the same.

11. **Challenge** According to the ancient account, Eratosthenes found that the arc from Syene to Alexandria was $\frac{1}{50}$ of a full circle around the earth and measured about 5000 stadia. Find the central angle in radians (exact and to the nearest thousandth), then use $r = \frac{s}{\theta}$ to estimate the earth's radius in stadia, to the nearest hundred stadia.

Show answer

$$\theta = \frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126 \text{ radians}; r = \frac{5000}{\pi/25} = \frac{125000}{\pi} \approx 39,800 \text{ stadia}$$

COMMON MISTAKES

- Forgetting that $\theta = \frac{s}{r}$ gives radians only, never degrees. Label the answer "radians" so it is not mistaken for degrees.
- Using the diameter instead of the radius. Radian measure counts **radii** along the arc; halve a diameter before dividing.
- Thinking π radians is a full turn. A full turn is 2π radians; π radians is only a half turn.
- Using arc length and radius in different units, such as feet and inches. Convert both to the same unit first so they cancel.

Make it a game • String Radian. Cut a string equal to the radius of a plate. Lay it along the plate's rim as many times as it fits and mark the spot. How many radii go around? Draw lines from the center to the string's ends and measure the angle. You have found one radian. Explain $\theta = s$ over r .

3.3 CONVERTING DEGREES AND RADIANS

OBJECTIVES

- Derive the conversion factors between degrees and radians from $180^\circ = \pi$ radians.
- Convert angle measures between degrees and radians, giving exact answers in terms of π or decimal answers rounded as directed.

WATCH

Introduction to Radians (3 of 3: Formal definition & conversion)

Eddie Woo • 6:38

<https://www.youtube.com/watch?v=INdiZqlclYc>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Converting Radians and Degrees (Precalculus - Trigonometry 4)

Professor Leonard • 38:45

<https://www.youtube.com/watch?v=sCwcbZSZL5o>

Before the practice set, for many more conversions (long; skip around).

KEY TERMS

conversion factor

A fraction equal to 1, such as $\frac{\pi \text{ rad}}{180^\circ}$, used to change units without changing the size of a quantity.

exact value

An answer written without rounding, for example $\frac{5\pi}{6}$ rather than 2.618.

minute of arc

One sixtieth of a degree, written $1'$; one second of arc ($1''$) is one sixtieth of a minute.

CORE IDEAS

Everything rests on one fact: a full turn is both 360° and 2π radians. Dividing by 2, a half turn gives

$$180^\circ = \pi \text{ radians.}$$

From this, two conversion factors follow, each equal to 1: $\frac{\pi \text{ rad}}{180^\circ}$ and $\frac{180^\circ}{\pi \text{ rad}}$. To change degrees to radians, multiply by $\frac{\pi}{180^\circ}$ so the degrees cancel. To change radians to degrees, multiply by $\frac{180^\circ}{\pi}$ so the radians cancel. If you forget which is which, ask which unit must cancel.

The same rule can be seen as a proportion. An angle is the same fraction of a full turn in either unit, so $\frac{\theta_{\text{deg}}}{360^\circ} = \frac{\theta_{\text{rad}}}{2\pi}$. Solving for θ_{rad} gives $\theta_{\text{rad}} = \theta_{\text{deg}} \cdot \frac{\pi}{180^\circ}$.

Learn the common ones by heart, since you will use them constantly: $30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$, $270^\circ = \frac{3\pi}{2}$, $360^\circ = 2\pi$. Many others are multiples: $150^\circ = 5 \cdot 30^\circ = \frac{5\pi}{6}$.

One radian is $\frac{180^\circ}{\pi} \approx 57.2958^\circ$, and one degree is $\frac{\pi}{180} \approx 0.017453$ radians. When an answer must be a decimal, keep π in the calculation until the last step and round only at the end. Check your calculator's mode: it must be set to the unit you are entering.

KEY FORMULAS

$$180^\circ = \pi \text{ radians}$$

$$\theta_{\text{rad}} = \theta_{\text{deg}} \cdot \frac{\pi}{180^\circ}$$

$$\theta_{\text{deg}} = \theta_{\text{rad}} \cdot \frac{180^\circ}{\pi}$$

$$1 \text{ radian} \approx 57.2958^\circ, \quad 1^\circ \approx 0.017453 \text{ radians}$$

WORKED EXAMPLES

Example 1. Convert 150° to radians. Give the exact value and a decimal to the nearest hundredth.

Show solution

1. Multiply by $\frac{\pi}{180^\circ}$ so the degrees cancel: $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{150\pi}{180}$.
2. Simplify the fraction by dividing top and bottom by 30: $\frac{150\pi}{180} = \frac{5\pi}{6}$.
3. Check: $150^\circ = 5 \cdot 30^\circ$ and $30^\circ = \frac{\pi}{6}$, so $150^\circ = \frac{5\pi}{6}$.
4. As a decimal, $\frac{5\pi}{6} \approx 2.617994$, which rounds to 2.62.

Answer: $\frac{5\pi}{6} \approx 2.62$ radians

Example 2. The arm of the center-pivot irrigation system on the Miller farm turned through 2.5 radians one afternoon. Convert this angle to degrees, to the nearest tenth of a degree. Then express $-\frac{7\pi}{4}$ radians in degrees.

Show solution

1. Multiply by $\frac{180^\circ}{\pi}$ so the radians cancel: $2.5 \cdot \frac{180^\circ}{\pi} = \frac{450^\circ}{\pi}$.
2. Evaluate without rounding π early: $\frac{450^\circ}{\pi} \approx 143.2394^\circ$, which rounds to 143.2° .
3. Sanity check: one radian is about 57.3° , and $2.5 \times 57.3 \approx 143$. Good.
4. For $-\frac{7\pi}{4}$: $-\frac{7\pi}{4} \cdot \frac{180^\circ}{\pi} = -\frac{7 \cdot 180^\circ}{4} = -315^\circ$. Here π cancels, so the answer is exact.

Answer: 2.5 radians $\approx 143.2^\circ$; $-\frac{7\pi}{4} = -315^\circ$

PRACTICE

1. Easy Convert 60° to radians. Give the exact value.

Show answer

$$\frac{\pi}{3}$$

2. Easy Convert $\frac{3\pi}{4}$ radians to degrees.

Show answer

$$135^\circ$$

3. Easy Convert 225° to radians. Give the exact value.

Show answer

$$\frac{5\pi}{4}$$

4. Medium Convert -210° to radians. Give the exact value.

Show answer

$$-\frac{7\pi}{6}$$

5. Medium Convert $\frac{11\pi}{6}$ radians to degrees.

Show answer

$$330^\circ$$

6. Medium Convert 1 radian to degrees, to the nearest tenth of a degree.

Show answer

$$\frac{180^\circ}{\pi} \approx 57.3^\circ$$

7. Medium Convert 40° to radians, to the nearest hundredth.

Show answer

$$\frac{2\pi}{9} \approx 0.70 \text{ radians}$$

8. **Hard** A surveyor's instrument reports an angle of 0.75 radians between two fence lines. Convert it to degrees, to the nearest tenth of a degree.

Show answer

$$0.75 \cdot \frac{180^\circ}{\pi} \approx 43.0^\circ$$

9. **Hard** An irrigation pivot turns through 252° in one day. Express this angle in radians, exactly and to the nearest hundredth.

Show answer

$$\frac{7\pi}{5} \approx 4.40 \text{ radians}$$

10. **Hard** Convert $32^\circ 15'$ (32 degrees, 15 minutes) to radians, to the nearest thousandth.

Show answer

$$32^\circ 15' = 32.25^\circ, \text{ and } 32.25 \cdot \frac{\pi}{180} \approx 0.563 \text{ radians}$$

11. **Challenge** Find the angle in $[0, 2\pi)$ that is coterminal with $\frac{17\pi}{4}$, and give its measure in degrees as well.

Show answer

$$\frac{17\pi}{4} - 4\pi = \frac{\pi}{4}, \text{ which is } 45^\circ$$

12. **Challenge** Through how many radians does the hour hand of a clock turn in one minute? Give the exact value, and explain using a proportion why multiplying degrees by $\frac{\pi}{180}$ gives the same answer.

Show answer

The hour hand makes one turn in 12 hours = 720 minutes, so it turns $\frac{2\pi}{720} = \frac{\pi}{360}$ radians per minute. In degrees that is $\frac{360^\circ}{720} = 0.5^\circ$, and $0.5 \cdot \frac{\pi}{180} = \frac{\pi}{360}$ because both units describe the same fraction of a turn: $\frac{\theta_{\text{deg}}}{360} = \frac{\theta_{\text{rad}}}{2\pi}$.

COMMON MISTAKES

- Multiplying by the wrong factor. Write the units: to cancel degrees, degrees must be in the denominator, so use $\frac{\pi}{180^\circ}$.
- Rounding π to 3.14 in the middle of a problem. Keep π exact (or use the calculator's π key) and round only at the end.
- Leaving a fraction like $\frac{150\pi}{180}$ unsimplified. Reduce it to lowest terms: $\frac{5\pi}{6}$.
- Treating $32^\circ 15'$ as 32.15° . Minutes are sixtieths, so $15' = \frac{15}{60} = 0.25^\circ$.

Make it a game • Clock Face Radians. Use a clock face. Parent calls a time, such as 4:00 or 10:00. State the hour hand's angle from 3:00 going counterclockwise, in degrees and in radians. Then convert any degree measure parent names, such as 135, 210 or 330, giving exact answers in terms of pi. Beat your time over ten rounds.

3.4 ARC LENGTH AND SECTOR AREA

OBJECTIVES

- Derive the arc length formula $s = r\theta$ and the sector area formula $A = \frac{1}{2}r^2\theta$ from proportions of a full circle.
- Use the formulas to find arc lengths, sector areas, radii and central angles, converting degrees to radians when needed.
- Solve practical problems involving arcs and sectors, such as irrigation pivots, sprinklers and distances on the earth.

WATCH

Circular Measure: Arc Length, Sector & Segment Area Formulas

Eddie Woo · 11:56

<https://www.youtube.com/watch?v=OJEwU7YnAE8>

WHEN TO WATCH

After reading the Core ideas (where $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ are derived), before Example 1.

Note: Woo also covers segment area, which this lesson does not need.

BACKUP VIDEO

Area of a Sector, Angular Velocity, Applications (Precalculus - Trigonometry 5)

Professor Leonard · 36:57

<https://www.youtube.com/watch?v=x6wnYbOBCic>

Before the practice set; the first two-thirds cover arc length on the earth and sector area.

KEY TERMS

arc length

The distance along a circle between the two points where the sides of a central angle meet it.

sector

The region of a disk bounded by two radii and the arc between them, shaped like a slice of pie.

segment of a circle

The region between a chord and the arc it cuts off.

proportional

Two quantities are proportional when one is always the same multiple of the other; here, arc length and sector area are each proportional to the central angle.

CORE IDEAS

A central angle θ (in radians) is the fraction $\frac{\theta}{2\pi}$ of a full turn. Its arc is the same fraction of the circumference, and its sector is the same fraction of the area of the disk. That single idea gives both formulas.

Arc length. $\frac{s}{2\pi r} = \frac{\theta}{2\pi}$. Multiply both sides by $2\pi r$: $s = r\theta$. This is just the definition of the radian, $\theta = \frac{s}{r}$, solved for s .

Sector area. $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$. Multiply both sides by πr^2 : $A = \frac{\pi r^2 \theta}{2\pi} = \frac{1}{2} r^2 \theta$. Since $s = r\theta$, this can also be written $A = \frac{1}{2} r s$, which looks like the area of a triangle with base s and height r . That is no accident: cut a sector into many thin slices, and each slice is nearly a thin triangle whose height is r .

Both formulas require θ in **radians**. If the angle is given in degrees, convert it first, or use the degree forms $s = \frac{\pi r \theta}{180}$ and $A = \frac{\pi r^2 \theta}{360}$, which are the same formulas with the conversion built in. The simpler radian forms are one more reason radians are natural.

The formulas can be run backward. Given s and r , find $\theta = \frac{s}{r}$. Given A and r , find $\theta = \frac{2A}{r^2}$. Given A and θ , find $r = \sqrt{\frac{2A}{\theta}}$.

KEY FORMULAS

$$s = r\theta \quad (\theta \text{ in radians})$$

$$A = \frac{1}{2}r^2\theta \quad (\theta \text{ in radians})$$

$$A = \frac{1}{2}rs$$

$$s = \frac{\pi r \theta}{180}, \quad A = \frac{\pi r^2 \theta}{360} \quad (\theta \text{ in degrees})$$

From history. Archimedes showed that the area of a circle equals the area of a right triangle whose legs are the radius and the circumference, that is, $\frac{1}{2} \cdot r \cdot 2\pi r = \pi r^2$. The sector formula $A = \frac{1}{2}rs$ is the same idea applied to part of a circle.

Euclid connection. Euclid VI.33 shows that in equal circles, central angles are in the same ratio as the arcs they stand on; the same reasoning shows that sectors are in that ratio too. This proportionality is the whole basis of the formulas $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.

WORKED EXAMPLES

Example 1. A circle has radius 8 cm. For a central angle of $\frac{3\pi}{4}$ radians, find the arc length and the sector area. Give exact answers and decimals to the nearest tenth.

Show solution

1. The angle is already in radians, so use the formulas directly.
2. Arc length: $s = r\theta = 8 \cdot \frac{3\pi}{4} = 6\pi \approx 18.849556$, which rounds to 18.8 cm.
3. Sector area: $A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 64 \cdot \frac{3\pi}{4} = 24\pi \approx 75.398224$, which rounds to 75.4 cm².
4. Check with $A = \frac{1}{2}rs = \frac{1}{2} \cdot 8 \cdot 6\pi = 24\pi$. It agrees.

Answer: $s = 6\pi \approx 18.8$ cm; $A = 24\pi \approx 75.4$ cm²

Example 2. A center-pivot irrigation arm 400 m long is set to sweep through 150° to water one part of a field. How far does the end of the arm travel, and how much land does it water? Round the distance to the nearest meter and the area to the nearest hundred square meters.

Show solution

1. Convert to radians: $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{5\pi}{6}$.
2. Distance traveled by the end of the arm (an arc): $s = r\theta = 400 \cdot \frac{5\pi}{6} = \frac{1000\pi}{3} \approx 1047.1976$ m, which rounds to 1047 m.
3. Area watered (a sector): $A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 160,000 \cdot \frac{5\pi}{6} = \frac{200,000\pi}{3} \approx 209,439.5$ m², which rounds to 209,400 m².
4. Since one hectare is 10,000 m², that is about 20.9 hectares.

Answer: The end travels $\frac{1000\pi}{3} \approx 1047$ m; the area watered is $\frac{200,000\pi}{3} \approx 209,400$ m² (about 20.9 hectares).

PRACTICE

1. **Easy** Find the arc length cut off by a central angle of 2 radians in a circle of radius 10 m.

Show answer

$$s = 20 \text{ m}$$

2. **Easy** Find the exact arc length cut off by a central angle of $\frac{\pi}{3}$ in a circle of radius 6 in.

Show answer

$$s = 2\pi \approx 6.28 \text{ in.}$$

3. **Easy** Find the area of a sector with radius 4 ft and central angle 1.5 radians.

Show answer

$$A = \frac{1}{2} \cdot 16 \cdot 1.5 = 12 \text{ ft}^2$$

4. **Medium** A circle has radius 9 in. For a central angle of 40° , find the exact arc length and sector area.

Show answer

$$\theta = \frac{2\pi}{9}; s = 2\pi \approx 6.28 \text{ in.}; A = 9\pi \approx 28.27 \text{ in.}^2$$

5. **Medium** An arc of length 15 cm lies on a circle of radius 6 cm. Find the central angle in radians and the area of the sector.

Show answer

$$\theta = 2.5 \text{ radians}; A = \frac{1}{2} \cdot 36 \cdot 2.5 = 45 \text{ cm}^2$$

6. **Medium** The pendulum of a grandfather clock is 1.2 m long and swings through an angle of 10° . How far does the tip travel in one swing? Give your answer in centimeters, to the nearest centimeter.

Show answer

$$s = 1.2 \cdot \frac{\pi}{18} \approx 0.2094 \text{ m} \approx 21 \text{ cm}$$

7. **Medium** Ruth cuts an apple pie 10 in. across into 8 equal slices. Find the exact area of the top of one slice and the length of its outer crust edge, then round each to the nearest hundredth.

Show answer

$$r = 5, \theta = \frac{\pi}{4}. \text{ Area} = \frac{25\pi}{8} \approx 9.82 \text{ in.}^2; \text{ crust edge} = \frac{5\pi}{4} \approx 3.93 \text{ in.}$$

8. **Hard** A sector of a circle of radius 10 m has area 50 m^2 . Find its central angle in radians.

Show answer

$$\theta = \frac{2A}{r^2} = \frac{100}{100} = 1 \text{ radian}$$

9. **Hard** A lawn sprinkler throws water 15 ft and turns back and forth through 120° . How much lawn does it water? Give the exact area and the area to the nearest square foot.

Show answer

$$A = \frac{1}{2} \cdot 225 \cdot \frac{2\pi}{3} = 75\pi \approx 236 \text{ ft}^2$$

10. **Hard** Two towns lie on the same north-south line (meridian), one at latitude 38° N and the other at 44° N. Taking the earth's radius as 3960 miles, find the distance between them along the earth's surface, to the nearest mile.

Show answer

$$\theta = 6^\circ = \frac{\pi}{30}; s = 3960 \cdot \frac{\pi}{30} = 132\pi \approx 415 \text{ miles}$$

11. **Challenge** Prove that the area of a sector is $A = \frac{1}{2}rs$, where s is its arc length, starting from the fact that the sector is the same fraction of the disk as its arc is of the circumference.

Show answer

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}. \text{ Multiply both sides by } \pi r^2: A = \frac{\pi r^2 s}{2\pi r} = \frac{1}{2}rs.$$

12. **Challenge** A chord cuts off an arc of 60° in a circle of radius 10 cm. Find the area of the segment between the chord and the arc, exactly and to the nearest hundredth. (Hint: the two radii and the chord form an equilateral triangle.)

Show answer

$$\text{Sector: } \frac{1}{2} \cdot 100 \cdot \frac{\pi}{3} = \frac{50\pi}{3}. \text{ Equilateral triangle with side 10 and height } 5\sqrt{3}: \\ \text{area } 25\sqrt{3}. \text{ Segment} = \frac{50\pi}{3} - 25\sqrt{3} \approx 9.06 \text{ cm}^2.$$

COMMON MISTAKES

- Using degrees in $s = r\theta$ or $A = \frac{1}{2}r^2\theta$. Convert the angle to radians first; otherwise the answer is about 57 times too large.
- Using the diameter for r . Halve a diameter before substituting.
- Forgetting to square the radius in the area formula, or forgetting the factor $\frac{1}{2}$. Check with $A = \frac{1}{2}rs$.
- Confusing a sector with a segment. A sector includes the triangle formed by the two radii; a segment is what remains after that triangle is removed.

Make it a game • String Arcs. Trace a lid on paper and mark its center. Parent names an angle in radians, such as π over 3 or 2. Predict the arc length and sector area before checking: lay a string along the arc and measure it. Within a millimeter earns a point. Then find a radius when given an arc and an angle.

3.5 LINEAR AND ANGULAR SPEED

OBJECTIVES

- Define angular speed and linear speed, and derive $v = r\omega$ from $s = r\theta$.
- Convert between revolutions per minute and radians per unit of time.
- Solve problems with wheels, pulleys, gears, tractors and millstones, including belt-connected pulleys and meshed gears.

WATCH

Introduction to Circular Motion: Angular & Linear Velocity

Eddie Woo • 12:39

<https://www.youtube.com/watch?v=Le6hV9ANr9Q>

WHEN TO WATCH

After reading the Core ideas (where $v = r\omega$ is derived), before Example 1 (the tractor wheel).

Note: Woo teaches this in a mechanics unit: he writes ω as $d\theta/dt$ and 'theta-dot' and briefly mentions angular acceleration. Skip that calculus notation; $v = r\omega$ is the takeaway. Neither video covers pulleys or gears.

BACKUP VIDEO

Area of a Sector, Angular Velocity, Applications (Precalculus - Trigonometry 5)

Professor Leonard • 36:57

<https://www.youtube.com/watch?v=x6wnYbOBCic>

Before Example 1; the angular-speed part is the last third (how fast the earth spins you).

KEY TERMS

angular speed

The angle turned per unit of time, $\omega = \frac{\theta}{t}$, usually in radians per second or radians per minute. The symbol ω is the Greek letter omega.

linear speed

The distance traveled along the circular path per unit of time, $v = \frac{s}{t}$.

revolutions per minute (rpm)

The number of full turns made in one minute; each revolution is 2π radians.

belt drive

Two pulleys joined by a belt that does not slip, so the rims of both pulleys move at the same linear speed.

CORE IDEAS

A turning wheel has two kinds of speed. Its **angular speed** ω tells how fast it turns: how many radians per second or per minute. A point on the wheel has a **linear speed** v : how many feet or meters per second it travels along its circular path. Every point on a rigid wheel has the same angular speed, but points farther from the axle move faster.

Deriving $v = r\omega$. In time t the wheel turns through angle θ , so a point at distance r from the center travels the arc $s = r\theta$. Divide both sides by t : $\frac{s}{t} = r \cdot \frac{\theta}{t}$, that is, $v = r\omega$. As with arc length, ω must be in **radians** per unit time.

Machines are usually rated in revolutions per minute. Since one revolution is 2π radians, $\omega = 2\pi \cdot (\text{rpm})$ radians per minute. Watch the units of time: to change radians per minute into radians per second, divide by 60.

Pulleys and gears. When a belt joins two pulleys without slipping, every point of the belt moves at one speed, and so do the rims of both pulleys. Therefore $r_1\omega_1 = r_2\omega_2$: the smaller pulley turns faster, in inverse proportion to its radius. Meshed gears behave the same way, because the teeth at the point of contact move together. Since the number of teeth is proportional to the radius, $n_1 \cdot (\text{rpm})_1 = n_2 \cdot (\text{rpm})_2$.

A wheel that rolls without slipping moves forward exactly the length of arc that touches the ground. So the speed of a tractor or wagon equals the linear speed of a point on the rim of its wheel: $v = r\omega$.

KEY FORMULAS

$$\omega = \frac{\theta}{t}, \quad v = \frac{s}{t}$$

$$v = r\omega \quad (\omega \text{ in radians per unit time})$$

$$\omega = 2\pi \times (\text{revolutions per unit time})$$

$$r_1\omega_1 = r_2\omega_2 \quad (\text{belt-connected pulleys or meshed gears})$$

From history. For centuries, mills turned grain into flour by gearing a water wheel or windmill to a heavy millstone. Millwrights chose wheels and gears of different sizes to make a slow, powerful water wheel drive a millstone at a faster speed, exactly the trade-off described by $r_1\omega_1 = r_2\omega_2$.

WORKED EXAMPLES

Example 1. A tractor's rear wheel has a diameter of 1.6 m and turns at 30 revolutions per minute. Find the wheel's angular speed in radians per minute and the tractor's speed in kilometers per hour, to the nearest tenth.

Show solution

1. Angular speed: $\omega = 30 \cdot 2\pi = 60\pi \approx 188.5$ radians per minute.
2. The radius is half the diameter: $r = 0.8$ m.
3. Linear speed: $v = r\omega = 0.8 \cdot 60\pi = 48\pi$ m/min ≈ 150.796 m/min.
4. Convert: multiply by 60 minutes per hour and divide by 1000 meters per kilometer. $48\pi \cdot \frac{60}{1000} = 2.88\pi \approx 9.047787$ km/h, which rounds to 9.0 km/h.
5. The wheel rolls without slipping, so this is also the tractor's speed.

Answer: $\omega = 60\pi \approx 188.5$ rad/min; $v \approx 9.0$ km/h

Example 2. In a small grain mill, an electric motor turns a pulley of radius 2 in. at 1800 rpm. A belt joins it to a pulley of radius 6 in. that drives the millstone. How fast does the large pulley turn, in rpm? How fast does the belt move, in feet per minute, to the nearest foot per minute?

Show solution

1. The belt does not slip, so both rims have the same linear speed: $r_1\omega_1 = r_2\omega_2$.
2. Using revolutions per minute on both sides (the factor 2π cancels): $2 \cdot 1800 = 6 \cdot \omega_2$, so $\omega_2 = 600$ rpm.
3. Belt speed equals the rim speed of the small pulley: $v = r\omega = 2 \cdot (1800 \cdot 2\pi) = 7200\pi$ in./min.
4. Convert to feet: $\frac{7200\pi}{12} = 600\pi \approx 1884.96$ ft/min, which rounds to 1885 ft/min.
5. Check with the large pulley: $6 \cdot (600 \cdot 2\pi) = 7200\pi$ in./min. The same.

Answer: The large pulley turns at 600 rpm; the belt moves at $600\pi \approx 1885$ ft/min.

PRACTICE

1. **Easy** A wheel turns at 3 revolutions per second. Find its angular speed in radians per second, exactly and to the nearest tenth.

Show answer

$$6\pi \approx 18.8 \text{ rad/s}$$

2. **Easy** A point on the rim of a wheel of radius 0.5 m turns with angular speed 4 rad/s. Find its linear speed.

Show answer

$$v = 0.5 \cdot 4 = 2 \text{ m/s}$$

3. **Easy** A gear turns at a steady $\frac{\pi}{5}$ radians per second. Through what angle does it turn in 10 seconds?

Show answer

$$\theta = \omega t = 2\pi \text{ radians (one full turn)}$$

4. **Medium** A millstone of radius 0.6 m turns at 12 rpm. How fast does a point on its outer edge move, in meters per minute, to the nearest tenth?

Show answer

$$\omega = 24\pi \text{ rad/min}; v = 0.6 \cdot 24\pi = 14.4\pi \approx 45.2 \text{ m/min}$$

5. **Medium** The earth turns once every 24 hours. Find its angular speed in radians per hour, exactly, and the linear speed of a point on the equator in miles per hour, to the nearest mile per hour. Use 3960 miles for the earth's radius.

Show answer

$$\omega = \frac{\pi}{12} \approx 0.2618 \text{ rad/h}; v = 3960 \cdot \frac{\pi}{12} = 330\pi \approx 1037 \text{ mph}$$

6. **Medium** A gear with 40 teeth turns at 90 rpm and drives a smaller gear with 16 teeth. How fast does the smaller gear turn?

Show answer

$$40 \cdot 90 = 16 \cdot \omega, \text{ so } \omega = 225 \text{ rpm}$$

7. **Medium** A Ferris wheel of radius 25 ft makes one revolution every 40 seconds. Find its angular speed in radians per second and the linear speed of a rider in feet per second, to the nearest hundredth.

Show answer

$$\omega = \frac{2\pi}{40} = \frac{\pi}{20} \approx 0.16 \text{ rad/s}; v = 25 \cdot \frac{\pi}{20} = \frac{5\pi}{4} \approx 3.93 \text{ ft/s}$$

8. **Hard** Samuel rides a bicycle with wheels 26 in. in diameter at 15 miles per hour. How many revolutions per minute do the wheels make, to the nearest rpm? (1 mile = 63,360 in.)

Show answer

$$v = 15 \cdot 63,360 \div 60 = 15,840 \text{ in./min}; \omega = \frac{15,840}{13} \approx 1218.46 \text{ rad/min}; \text{rpm} = \frac{1218.46}{2\pi} \approx 194$$

9. **Hard** A bucket is raised from a well by a rope that winds onto a drum of radius 4 in. The drum is cranked at 30 rpm. How fast does the bucket rise, in feet per minute, to the nearest tenth? How long does it take to raise the bucket 50 ft, to the nearest second?

Show answer

$$v = 4 \cdot 60\pi = 240\pi \text{ in./min} = 20\pi \approx 62.8 \text{ ft/min}; \text{time} = \frac{50}{20\pi} \text{ min} \approx 48 \text{ s}$$

10. **Challenge** Explain why two pulleys joined by a belt that does not slip satisfy $r_1\omega_1 = r_2\omega_2$, and why the smaller pulley must turn faster.

Show answer

In any time interval, the length of belt that passes over one pulley equals the length that passes over the other, because the belt is one unbroken loop that does not stretch or slip. So the rims have equal linear speeds $v_1 = v_2$. Since $v = r\omega$, $r_1\omega_1 = r_2\omega_2$. If $r_1 < r_2$, then $\omega_1 = \frac{r_2}{r_1}\omega_2 > \omega_2$: the smaller pulley must turn more radians to feed out the same length of belt.

11. **Challenge** A bicycle has a front chainring with 48 teeth and a rear sprocket with 16 teeth. The rear wheel is 27 in. in diameter. If the rider pedals at 80 rpm, how fast does the bicycle travel, in miles per hour, to the nearest tenth?

Show answer

$$\begin{aligned} \text{Rear wheel: } \frac{48 \cdot 80}{16} &= 240 \text{ rpm. } v = 13.5 \cdot 240 \cdot 2\pi = 6480\pi \text{ in./min. In mph:} \\ \frac{6480\pi \cdot 60}{63,360} &\approx 19.3 \text{ mph} \end{aligned}$$

COMMON MISTAKES

- Using revolutions per minute directly in $v = r\omega$. First multiply by 2π to get radians per minute.
- Mixing units of time or length, such as inches with feet or minutes with hours. Write units at every step and convert before the final answer.
- Assuming connected pulleys have the same angular speed. Belt-connected pulleys share **linear** speed; their angular speeds differ.
- Using the diameter for r . Halve the diameter first.

Make it a game • Circle Walk Speed. Walk around a chair at a measured radius for three laps while your parent times you. Compute your angular speed in revolutions per minute, then radians per second, then your linear speed $v = r$ times omega. Then walk again at double the pace and predict the new speeds before computing.

3.6 THE UNIT CIRCLE

OBJECTIVES

- Define the sine and cosine of any angle as the coordinates of a point on the unit circle.
- Define all six trigonometric functions for a point (x, y) at distance r from the origin on the terminal side of an angle, and evaluate them.
- Explain why these definitions agree with the right-triangle ratios for acute angles and do not depend on the point chosen.

WATCH

Trigonometry on the Unit Circle (1 of 3: Introduction)

Eddie Woo · 11:04

<https://www.youtube.com/watch?v=ILRrqqhJzdE>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

Redefining the Trig Functions on the Unit Circle (1 of 2: The Basic Concept)

Eddie Woo · 10:53

<https://www.youtube.com/watch?v=uu2nJEbap80>

Before Example 1, if the 'coordinates as sine and cosine' idea needs a second explanation.

KEY TERMS

unit circle

The circle of radius 1 centered at the origin, with equation $x^2 + y^2 = 1$.

terminal point

The point where the terminal side of an angle in standard position meets the unit circle.

trigonometric function

A function that assigns to each angle θ a number such as $\sin \theta$ or $\cos \theta$, defined from a point on the terminal side of θ .

undefined

A function value that does not exist because its definition would require dividing by zero, such as $\tan 90^\circ$.

CORE IDEAS

The ratios SOH-CAH-TOA work only for acute angles, because only acute angles fit inside a right triangle. To give meaning to $\sin 150^\circ$ or $\cos(-2)$, we need a definition that works for any angle. The unit circle supplies it.

Draw an angle θ in standard position. Its terminal side meets the unit circle at a point $P(x, y)$. We **define**

$$\cos \theta = x, \quad \sin \theta = y.$$

So cosine is the horizontal coordinate and sine is the vertical coordinate of the terminal point. For an acute angle, drop a perpendicular from P to the x -axis: you get a right triangle with hypotenuse 1, adjacent side x and opposite side y , so $\cos \theta = \frac{x}{1}$ and $\sin \theta = \frac{y}{1}$, the same as before. The new definition extends the old one.

More generally, let (x, y) be **any** point other than the origin on the terminal side, and let $r = \sqrt{x^2 + y^2}$ be its distance from the origin (Pythagorean theorem). Then

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}, \quad \csc \theta = \frac{r}{y}, \quad \sec \theta = \frac{r}{x}, \quad \cot \theta = \frac{x}{y}.$$

Note that r is always positive, while x and y may be negative. That is how trigonometric values become negative.

Why doesn't the choice of point matter? If (x, y) and (x', y') are two points on the same terminal side, at distances r and r' , then $(x', y') = k(x, y)$ with $k = \frac{r'}{r} > 0$, and $r' = kr$. Every ratio is unchanged: for example $\frac{y'}{r'} = \frac{ky}{kr} = \frac{y}{r}$. These are similar triangles once more. Taking the point with $r = 1$ gives the unit-circle definition.

A function is undefined when its denominator is zero. Since $r > 0$, sine and cosine are defined for every angle. But $\tan \theta$ and $\sec \theta$ are undefined when $x = 0$ (terminal side on the y -axis), and $\cot \theta$ and $\csc \theta$ are undefined when $y = 0$ (terminal side on the x -axis). Also, because the coordinates of a point on the unit circle lie between -1 and 1 , we always have $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$.

KEY FORMULAS

$$\cos \theta = x, \quad \sin \theta = y \quad \text{for } (x, y) \text{ on the unit circle}$$

$$r = \sqrt{x^2 + y^2}$$

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc \theta = \frac{r}{y} \quad (y \neq 0), \quad \sec \theta = \frac{r}{x} \quad (x \neq 0), \quad \cot \theta = \frac{x}{y} \quad (y \neq 0)$$

From history. For many centuries a sine was not a ratio but a length: the half-chord in a circle of some chosen radius, and tables used a large radius so the entries could be whole numbers. In his *Introductio in analysin infinitorum* (1748), Leonhard Euler treated sine and cosine as functions of an arc on a circle of radius 1. That is the point of view of this lesson, and it is the one used throughout modern mathematics.

Euclid connection. The distance $r = \sqrt{x^2 + y^2}$ comes straight from the Pythagorean theorem, Euclid I.47, applied to the right triangle with legs $|x|$ and $|y|$. Applied to the unit circle, the same proposition says $x^2 + y^2 = 1$ for every terminal point.

WORKED EXAMPLES

Example 1. The point $(-3, 4)$ lies on the terminal side of an angle θ in standard position. Find the exact values of all six trigonometric functions of θ .

Show solution

1. Find r : $r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.
2. With $x = -3$, $y = 4$, $r = 5$: $\sin \theta = \frac{y}{r} = \frac{4}{5}$, $\cos \theta = \frac{x}{r} = -\frac{3}{5}$, $\tan \theta = \frac{y}{x} = -\frac{4}{3}$.
3. The reciprocals: $\csc \theta = \frac{5}{4}$, $\sec \theta = -\frac{5}{3}$, $\cot \theta = -\frac{3}{4}$.
4. Check: the point is in Quadrant II, where $x < 0$ and $y > 0$, so sine is positive and cosine is negative, as found.

Answer: $\sin \theta = \frac{4}{5}$, $\cos \theta = -\frac{3}{5}$, $\tan \theta = -\frac{4}{3}$, $\csc \theta = \frac{5}{4}$, $\sec \theta = -\frac{5}{3}$, $\cot \theta = -\frac{3}{4}$

Example 2. The point $(2, -5)$ lies on the terminal side of θ . Find the exact values of all six trigonometric functions of θ , rationalizing denominators, and give $\sin \theta$ and $\cos \theta$ to the nearest thousandth.

Show solution

$$1. r = \sqrt{2^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}.$$

$$2. \sin \theta = \frac{-5}{\sqrt{29}} = -\frac{5\sqrt{29}}{29} \approx -0.928 \text{ and } \cos \theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29} \approx 0.371.$$

$$3. \tan \theta = \frac{-5}{2} = -\frac{5}{2}.$$

$$4. \csc \theta = \frac{\sqrt{29}}{-5} = -\frac{\sqrt{29}}{5}, \sec \theta = \frac{\sqrt{29}}{2}, \cot \theta = \frac{2}{-5} = -\frac{2}{5}.$$

$$5. \text{Check: } \sin^2 \theta + \cos^2 \theta = \frac{25}{29} + \frac{4}{29} = 1, \text{ as it must be for the point } \left(\frac{x}{r}, \frac{y}{r}\right) \text{ on the unit circle.}$$

$$\text{Answer: } \sin \theta = -\frac{5\sqrt{29}}{29} \approx -0.928, \cos \theta = \frac{2\sqrt{29}}{29} \approx 0.371, \tan \theta = -\frac{5}{2}, \csc \theta = -\frac{\sqrt{29}}{5}, \sec \theta = \frac{\sqrt{29}}{2}, \cot \theta = -\frac{2}{5}$$

PRACTICE

1. **Easy** The point $(3, 4)$ lies on the terminal side of θ . Find $\sin \theta$, $\cos \theta$ and $\tan \theta$.

Show answer

$$r = 5; \sin \theta = \frac{4}{5}, \cos \theta = \frac{3}{5}, \tan \theta = \frac{4}{3}$$

2. **Easy** The terminal side of θ meets the unit circle at $(-0.28, 0.96)$. Find $\cos \theta$ and $\sin \theta$, and check that the point really is on the unit circle.

Show answer

$$\cos \theta = -0.28, \sin \theta = 0.96; (-0.28)^2 + 0.96^2 = 0.0784 + 0.9216 = 1$$

3. **Easy** The point $(-8, -15)$ lies on the terminal side of θ . Find r , $\sin \theta$, $\cos \theta$ and $\tan \theta$.

Show answer

$$r = 17; \sin \theta = -\frac{15}{17}, \cos \theta = -\frac{8}{17}, \tan \theta = \frac{15}{8}$$

4. **Medium** The point $(-5, 12)$ lies on the terminal side of θ . Find the exact values of all six trigonometric functions.

Show answer

$$r = 13; \sin \theta = \frac{12}{13}, \cos \theta = -\frac{5}{13}, \tan \theta = -\frac{12}{5}, \csc \theta = \frac{13}{12}, \sec \theta = -\frac{13}{5}, \cot \theta = -\frac{5}{12}$$

5. **Medium** The terminal point of θ on the unit circle lies in Quadrant II and has y -coordinate 0.6. Find its x -coordinate, $\cos \theta$ and $\tan \theta$.

Show answer

$$x^2 = 1 - 0.36 = 0.64, \text{ and } x < 0 \text{ in Quadrant II, so } x = \cos \theta = -0.8; \tan \theta = \frac{0.6}{-0.8} = -0.75$$

6. **Medium** The point $(1, -\sqrt{3})$ lies on the terminal side of θ . Find the exact values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

Show answer

$$r = 2; \sin \theta = -\frac{\sqrt{3}}{2}, \cos \theta = \frac{1}{2}, \tan \theta = -\sqrt{3}$$

7. **Medium** A Ferris wheel of radius 30 ft has its center at the origin. A rider is at the point $(18, -24)$, measured in feet from the center. If θ is the angle in standard position to the rider, find $\sin \theta$ and $\cos \theta$.

Show answer

$$r = 30; \sin \theta = -\frac{24}{30} = -\frac{4}{5}, \cos \theta = \frac{18}{30} = \frac{3}{5}$$

8. **Hard** The point $(0, -7)$ lies on the terminal side of θ . Find all six trigonometric functions, stating which are undefined and why.

Show answer

$$r = 7; \sin \theta = -1, \cos \theta = 0, \csc \theta = -1, \cot \theta = 0. \tan \theta \text{ and } \sec \theta \text{ are undefined because } x = 0 \text{ would be a denominator.}$$

9. **Hard** Using the definitions $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$ and $\tan \theta = \frac{y}{x}$, prove that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ whenever $\cos \theta \neq 0$.

Show answer

$\frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} = \tan \theta$. The condition $\cos \theta \neq 0$ means $x \neq 0$, so every step is defined.

10. **Challenge** Explain why the values of the six trigonometric functions of θ do not depend on which point of the terminal side is used.

Show answer

Any two points (x, y) and (x', y') on the same terminal side (a ray from the origin) satisfy $x' = kx$, $y' = ky$ with $k > 0$, and then $r' = \sqrt{k^2x^2 + k^2y^2} = kr$. Each ratio, such as $\frac{y'}{r'} = \frac{ky}{kr} = \frac{y}{r}$, has the factor k cancel, so all six values are the same. Geometrically, the triangles formed with the x -axis are similar.

11. **Challenge** The terminal side of θ lies along the line $y = -2x$ in Quadrant IV. Find the exact values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

Show answer

In Quadrant IV, $x > 0$; take $x = 1$, $y = -2$, $r = \sqrt{5}$. $\sin \theta = -\frac{2\sqrt{5}}{5}$, $\cos \theta = \frac{\sqrt{5}}{5}$, $\tan \theta = -2$

COMMON MISTAKES

- Giving r a negative sign. The distance $r = \sqrt{x^2 + y^2}$ is always positive; the signs of the trigonometric values come only from x and y .
- Swapping sine and cosine. On the unit circle, cosine is the x -coordinate (first) and sine is the y -coordinate (second), in alphabetical order: $(\cos \theta, \sin \theta)$.
- Writing $\tan \theta = 0$ when $x = 0$. Division by zero is undefined, not zero; it is when $y = 0$ that $\tan \theta = 0$.
- Forgetting the sign when taking a square root. If you find $x^2 = 0.64$, use the quadrant to decide whether $x = 0.8$ or $x = -0.8$.

Make it a game • Blank Circle Challenge. With a blank sheet and 3 minutes, draw the unit circle from memory: angles at every multiple of 30 and 45 degrees, with coordinates for each. Parent checks against the lesson. Score one point per correct coordinate. Then parent names a point such as $(-3, 4)$ and you give all six trigonometric values.

3.7 REFERENCE ANGLES AND SIGNS BY QUADRANT

OBJECTIVES

- Find the reference angle of any angle given in degrees or radians.
- Determine the sign of each trigonometric function in each quadrant and explain why.
- Evaluate trigonometric functions of any angle using its reference angle and the sign of its quadrant.

WATCH

Related Trigonometric Ratios (1 of 2: Basic Examples)

Eddie Woo · 12:29

<https://www.youtube.com/watch?v=DOS0ZaFgD9s>

WHEN TO WATCH

After reading the Core ideas on reference angles, before Example 1.

Note: Woo says 'related angle' for reference angle and teaches the ASTC ('All Stations To Central') sign rule.

BACKUP VIDEO

Evaluating the Sign of Trigonometric Ratios

Eddie Woo · 9:48

<https://www.youtube.com/watch?v=W11ClfgCJjw>

Before Example 2: signs of the functions by quadrant.

KEY TERMS

reference angle

The acute angle θ' between the terminal side of an angle in standard position and the x -axis.

ASTC

A memory aid ("All Students Take Calculus") for the functions that are positive in Quadrants I, II, III, IV: All, Sine, Tangent, Cosine.

reflection symmetry

The unit circle is unchanged by reflection in either axis, so points in different quadrants with the same reference angle have coordinates that differ only in sign.

CORE IDEAS

The **reference angle** θ' of an angle θ is the acute angle its terminal side makes with the x -axis (always the x -axis, never the y -axis). First find the coterminal angle between 0° and 360° ; then, if θ is in Quadrant II, $\theta' = 180^\circ - \theta$; in Quadrant III, $\theta' = \theta - 180^\circ$; in Quadrant IV, $\theta' = 360^\circ - \theta$. In radians, replace 180° by π and 360° by 2π .

Why reference angles work. Reflect the unit circle across the y -axis, the x -axis, or both. The terminal point of θ lands on the terminal point of θ' in Quadrant I. Reflection does not change distances, so the coordinates of the two points are the same except for sign: if θ' has terminal point (a, b) , then θ has terminal point $(\pm a, \pm b)$. Therefore every trigonometric function of θ equals the same function of θ' , up to sign:

$$f(\theta) = \pm f(\theta').$$

Signs by quadrant. The sign comes from the signs of x and y (remember $r > 0$). In Quadrant I, $x > 0$ and $y > 0$, so all six functions are positive. In Quadrant II, $x < 0$ and $y > 0$, so only sine (and its reciprocal cosecant) is positive. In Quadrant III, both are negative, so only tangent $\frac{y}{x}$ (and cotangent) is positive. In Quadrant IV, $x > 0$ and $y < 0$, so only cosine (and secant) is positive. All, Sine, Tangent, Cosine.

The method. To evaluate $f(\theta)$: (1) find a coterminal angle between 0° and 360° ; (2) note its quadrant; (3) find the reference angle θ' ; (4) evaluate $f(\theta')$, exactly if θ' is 30° , 45° or 60° , otherwise with a calculator; (5) attach the sign for that quadrant.

KEY FORMULAS

$$\text{QII: } \theta' = 180^\circ - \theta \ (\pi - \theta), \quad \text{QIII: } \theta' = \theta - 180^\circ \ (\theta - \pi), \quad \text{QIV: } \theta' = 360^\circ - \theta \ (2\pi - \theta)$$

$$f(\theta) = \pm f(\theta'), \text{ with the sign determined by the quadrant of } \theta$$

Positive: QI all, QII sin, csc, QIII tan, cot, QIV cos, sec

Euclid connection. Reference angles rest on congruent triangles: the right triangle formed by the terminal point, the origin and the x -axis in any quadrant is congruent to the one formed by the reference angle in Quadrant I, since both have a right angle, the same acute angle and hypotenuse 1 (two angles and a side, Euclid I.26).

WORKED EXAMPLES

Example 1. Find the exact value of $\sin 240^\circ$.

Show solution

1. 240° lies between 180° and 270° , so it is in Quadrant III.

2. Reference angle: $\theta' = 240^\circ - 180^\circ = 60^\circ$.

3. From the 30-60-90 triangle, $\sin 60^\circ = \frac{\sqrt{3}}{2}$.

4. In Quadrant III, $y < 0$, so sine is negative: $\sin 240^\circ = -\frac{\sqrt{3}}{2}$.

Answer: $\sin 240^\circ = -\frac{\sqrt{3}}{2} \approx -0.866$

Example 2. Use a reference angle to find $\cos 1300^\circ$, to the nearest thousandth. Then check with a calculator in degree mode.

Show solution

1. Remove full turns: $1300 \div 360 \approx 3.6$, so subtract $3 \cdot 360^\circ = 1080^\circ$: $1300^\circ - 1080^\circ = 220^\circ$.

2. 220° is in Quadrant III, where cosine is negative.

3. Reference angle: $\theta' = 220^\circ - 180^\circ = 40^\circ$.

4. $\cos 40^\circ \approx 0.766044$, so $\cos 1300^\circ = -\cos 40^\circ \approx -0.766044$, which rounds to -0.766 .

5. A calculator in degree mode gives $\cos 1300^\circ \approx -0.766044$. It agrees.

Answer: $\cos 1300^\circ = -\cos 40^\circ \approx -0.766$

PRACTICE

1. Easy Find the reference angle of 160° .

Show answer

20°

2. **Easy** Find the reference angle of $\frac{5\pi}{4}$.

Show answer

$$\frac{5\pi}{4} - \pi = \frac{\pi}{4}$$

3. **Easy** In which quadrant is sine negative and cosine positive?

Show answer

Quadrant IV

4. **Medium** Find the reference angle of -250° .

Show answer

Coterminal with 110° (Quadrant II); reference angle 70°

5. **Medium** Find the exact value of $\cos 135^\circ$.

Show answer

$$-\frac{\sqrt{2}}{2}$$

6. **Medium** Find the exact value of $\tan 330^\circ$.

Show answer

Reference angle 30° , Quadrant IV: $-\frac{\sqrt{3}}{3}$

7. **Medium** A Ferris wheel has radius 20 ft and its center is 25 ft above the ground. A rider's position is described by the angle θ measured counterclockwise from the horizontal line through the center, pointing right. The rider's height above the ground is $25 + 20 \sin \theta$ feet. Find her height when $\theta = 210^\circ$.

Show answer

$$\sin 210^\circ = -\frac{1}{2}, \text{ so the height is } 25 - 10 = 15 \text{ ft}$$

8. **Hard** Find the reference angle of 5 radians, exactly and to the nearest thousandth.

Show answer

$$\frac{3\pi}{2} \approx 4.712 < 5 < 2\pi, \text{ so Quadrant IV; } \theta' = 2\pi - 5 \approx 1.283$$

9. **Hard** Find both angles θ with $0^\circ \leq \theta < 360^\circ$ for which $\sin \theta = -0.4$, to the nearest tenth of a degree.

Show answer

Reference angle $\sin^{-1}(0.4) \approx 23.58^\circ$. Sine is negative in Quadrants III and IV:
 $\theta \approx 180^\circ + 23.58^\circ \approx 203.6^\circ$ and $\theta \approx 360^\circ - 23.58^\circ \approx 336.4^\circ$

10. **Hard** The hub of a farm windmill is 10 m above the ground and each blade is 4 m long. The height of a blade tip is $10 + 4 \sin \theta$ meters, where θ is measured counterclockwise from the horizontal. Find the exact height of the tip when $\theta = 300^\circ$, and round to the nearest hundredth.

Show answer

$$\sin 300^\circ = -\frac{\sqrt{3}}{2}, \text{ so the height is } 10 - 2\sqrt{3} \approx 6.54 \text{ m}$$

11. **Challenge** Use the symmetry of the unit circle to explain why $\sin(180^\circ - \theta) = \sin \theta$ and $\cos(180^\circ - \theta) = -\cos \theta$ for an acute angle θ .

Show answer

Let θ have terminal point (a, b) in Quadrant I. The angle $180^\circ - \theta$ makes the same angle θ with the negative x -axis, so its terminal point is the reflection of (a, b) across the y -axis, namely $(-a, b)$. Reading off the coordinates, $\sin(180^\circ - \theta) = b = \sin \theta$ and $\cos(180^\circ - \theta) = -a = -\cos \theta$.

12. **Challenge** Find the exact value of $\sin 2025^\circ \cdot \cos(-480^\circ)$.

Show answer

$$2025^\circ - 1800^\circ = 225^\circ, \text{ so } \sin 2025^\circ = \frac{\sqrt{2}}{2}. -480^\circ + 720^\circ = 240^\circ, \text{ so}$$
$$\cos(-480^\circ) = -\frac{1}{2}. \text{ Product: } \frac{\sqrt{2}}{4} \approx 0.354$$

COMMON MISTAKES

- Measuring the reference angle from the y -axis. The reference angle is always measured to the x -axis; for 120° it is 60° , not 30° .
- Finding the reference angle correctly but forgetting the sign. After evaluating $f(\theta')$, always ask which quadrant θ is in and apply ASTC.
- Using the sign of the reference angle's quadrant instead of the original angle's. The reference angle is always in Quadrant I; the sign comes from where θ itself terminates.

- Finding the reference angle of a large or negative angle without first reducing it to a coterminal angle between 0° and 360° .

Make it a game • Four Corners. Label the four room corners as quadrants I to IV. Parent calls a function and a sign, such as 'tangent negative.' Run to the correct corner or corners and shout the pattern word that helps you remember. Then parent calls an angle like 210 degrees; run to its quadrant and state the reference angle.

3.8 EXACT VALUES AROUND THE UNIT CIRCLE

OBJECTIVES

- Find exact values of the trigonometric functions for every multiple of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including the quadrantal angles.
- Find the remaining trigonometric functions of θ given one function value and the quadrant of θ .

WATCH

Exact Values (3 of 4: Using the unit circle for trigonometry).

Eddie Woo · 7:22

<https://www.youtube.com/watch?v=YN5qlkbsGg0>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

How to Use the Unit Circle in Trigonometry (Precalculus - Trigonometry 7).

Professor Leonard · 42:57

<https://www.youtube.com/watch?v=GdRZfemxTFI>

Before the practice set; long (43 min) but builds the whole unit circle step by step.

KEY TERMS

special angles

Angles whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$ (30° , 45° , 60°), together with the quadrantal angles; their trigonometric values can be written exactly.

quadrantal angle

An angle whose terminal side lies on an axis, such as 0 , $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$.

terminal point

The point $(\cos \theta, \sin \theta)$ where the terminal side of θ meets the unit circle.

CORE IDEAS

Quadrant I. From the special right triangles of Unit 2, the terminal points in the first quadrant are: $0 \rightarrow (1, 0)$; $\frac{\pi}{6} \rightarrow \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$; $\frac{\pi}{4} \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$; $\frac{\pi}{3} \rightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; $\frac{\pi}{2} \rightarrow (0, 1)$. Remember each point as $(\cos \theta, \sin \theta)$. A helpful pattern: as the angle grows from 0 to $\frac{\pi}{2}$, the sines are $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$, and the cosines are the same list in reverse.

The rest of the circle. By reference angles (Lesson 3.7), every multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ has a terminal point that is a reflection of one of these, so its coordinates are the same numbers with signs from its quadrant. For example, $\frac{5\pi}{6}$ has reference angle $\frac{\pi}{6}$ and lies in Quadrant II, so its terminal point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Angles with reference angle $\frac{\pi}{6}$ have denominators of 6 ($\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$); reference angle $\frac{\pi}{4}$ gives denominators of 4; reference angle $\frac{\pi}{3}$ gives denominators of 3.

Quadrantal angles. These terminal points sit on the axes: $0 \rightarrow (1, 0)$, $\frac{\pi}{2} \rightarrow (0, 1)$, $\pi \rightarrow (-1, 0)$, $\frac{3\pi}{2} \rightarrow (0, -1)$. Read sine and cosine straight from the coordinates. Tangent and secant are undefined where $x = 0$ (at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$); cotangent and cosecant are undefined where $y = 0$ (at 0 and π).

Given one value and the quadrant. Suppose $\cos \theta = -\frac{5}{13}$ and θ is in Quadrant III. Since $\cos \theta = \frac{x}{r}$, choose $x = -5$ and $r = 13$ (remember $r > 0$, so the sign goes on x). The Pythagorean theorem gives $y^2 = r^2 - x^2 = 169 - 25 = 144$, so $y = \pm 12$; the quadrant decides the sign, here $y = -12$. Now all six functions follow from x , y and r . The same method works for any given value.

KEY FORMULAS

$$\left(\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \left(\cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$(\cos 0, \sin 0) = (1, 0), \quad \left(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}\right) = (0, 1), \quad (\cos \pi, \sin \pi) = (-1, 0), \quad \left(\cos \frac{3\pi}{2}, \sin \frac{3\pi}{2}\right) = (0, -1)$$

$$x^2 + y^2 = r^2 \quad (\text{to find the missing coordinate})$$

From history. Ptolemy's *Almagest* contains a table of chords for arcs in steps of half a degree, built up from a few exactly known values, such as the chords of 60° and 90° , by geometric theorems. The exact values you are learning play the same role today: they are fixed points from which other values can be found.

Euclid connection. Every "given one value, find the rest" problem uses Euclid I.47: $x^2 + y^2 = r^2$ for the right triangle formed by the point (x, y) , the origin and the x -axis.

WORKED EXAMPLES

Example 1. Find the exact values of $\sin \frac{7\pi}{6}$, $\cos \frac{3\pi}{4}$ and $\tan \frac{5\pi}{3}$.

Show solution

1. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$: Quadrant III, reference angle $\frac{\pi}{6}$. Terminal point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$,
so $\sin \frac{7\pi}{6} = -\frac{1}{2}$.
2. $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$: Quadrant II, reference angle $\frac{\pi}{4}$. Terminal point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, so
 $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$.
3. $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$: Quadrant IV, reference angle $\frac{\pi}{3}$. Terminal point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, so
 $\tan \frac{5\pi}{3} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$.

Answer: $\sin \frac{7\pi}{6} = -\frac{1}{2}$, $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$, $\tan \frac{5\pi}{3} = -\sqrt{3}$

Example 2. Given $\cos \theta = -\frac{5}{13}$ with θ in Quadrant III, find the exact values of the other five trigonometric functions.

Show solution

1. Since $\cos \theta = \frac{x}{r}$ and $r > 0$, take $x = -5$, $r = 13$.
2. By the Pythagorean theorem, $y^2 = 13^2 - (-5)^2 = 169 - 25 = 144$, so $y = \pm 12$.
In Quadrant III, $y < 0$, so $y = -12$.
3. $\sin \theta = \frac{y}{r} = -\frac{12}{13}$, $\tan \theta = \frac{y}{x} = \frac{-12}{-5} = \frac{12}{5}$.
4. Reciprocals: $\csc \theta = -\frac{13}{12}$, $\sec \theta = -\frac{13}{5}$, $\cot \theta = \frac{5}{12}$.
5. Check the signs with ASTC: in Quadrant III only tangent and cotangent are positive. Correct.

Answer: $\sin \theta = -\frac{12}{13}$, $\tan \theta = \frac{12}{5}$, $\csc \theta = -\frac{13}{12}$, $\sec \theta = -\frac{13}{5}$, $\cot \theta = \frac{5}{12}$

PRACTICE

1. Easy Find the exact value of $\sin \frac{5\pi}{6}$.

Show answer

$$\frac{1}{2}$$

2. Easy Find the exact value of $\cos \frac{4\pi}{3}$.

Show answer

$$-\frac{1}{2}$$

3. Easy Find the exact value of $\tan \pi$.

Show answer

$$\frac{0}{-1} = 0$$

4. Medium Find the exact value of $\sec \frac{7\pi}{4}$.

Show answer

$$\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2}, \text{ so } \sec \frac{7\pi}{4} = \sqrt{2}$$

5. **Medium** Find all six trigonometric functions of π , stating which are undefined.

Show answer

Terminal point $(-1, 0)$: $\sin \pi = 0$, $\cos \pi = -1$, $\tan \pi = 0$, $\sec \pi = -1$; $\csc \pi$ and $\cot \pi$ are undefined.

6. **Medium** Find the exact value of $\csc \frac{11\pi}{6}$.

Show answer

$$\sin \frac{11\pi}{6} = -\frac{1}{2}, \text{ so } \csc \frac{11\pi}{6} = -2$$

7. **Medium** Given $\sin \theta = \frac{3}{5}$ with θ in Quadrant II, find $\cos \theta$ and $\tan \theta$.

Show answer

$$y = 3, r = 5, x = -4: \cos \theta = -\frac{4}{5}, \tan \theta = -\frac{3}{4}$$

8. **Hard** Given $\tan \theta = -2$ and $\sin \theta > 0$, find the exact values of $\sin \theta$, $\cos \theta$ and $\sec \theta$.

Show answer

Tangent negative and sine positive means Quadrant II, so $x = -1$, $y = 2$,
 $r = \sqrt{5}$. $\sin \theta = \frac{2\sqrt{5}}{5}$, $\cos \theta = -\frac{\sqrt{5}}{5}$, $\sec \theta = -\sqrt{5}$

9. **Hard** Find every angle θ in $[0, 2\pi)$ with $\sin \theta = -\frac{\sqrt{3}}{2}$. Give exact answers.

Show answer

Reference angle $\frac{\pi}{3}$; sine is negative in Quadrants III and IV: $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$

10. **Hard** An old mill's water wheel has radius 4 m, and its axle is 3 m above the surface of the stream. A bucket starts level with the axle on the right side (the 3 o'clock position) and the wheel turns counterclockwise through $\frac{7\pi}{6}$ radians. Using the axle as the origin, the bucket is then at $\left(4 \cos \frac{7\pi}{6}, 4 \sin \frac{7\pi}{6}\right)$. How high above the water is the bucket, and how far is it horizontally from the axle (exact, then to the nearest hundredth)?

Show answer

$4 \sin \frac{7\pi}{6} = -2$, so the height is $3 - 2 = 1$ m. $4 \cos \frac{7\pi}{6} = -2\sqrt{3}$, so the bucket is $2\sqrt{3} \approx 3.46$ m to the left of the axle.

11. **Challenge** Given $\sec \theta = -3$ and $\tan \theta > 0$, find the exact values of $\sin \theta$ and $\cot \theta$.

Show answer

$$\cos \theta = -\frac{1}{3} \text{ and tangent positive means Quadrant III: } x = -1, r = 3, y = -\sqrt{9-1} = -2\sqrt{2}. \sin \theta = -\frac{2\sqrt{2}}{3}, \cot \theta = \frac{-1}{-2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

12. **Challenge** How many different angles in $[0, 2\pi)$ are multiples of $\frac{\pi}{6}$ or of $\frac{\pi}{4}$ (or both)? Explain your count.

Show answer

There are 12 multiples of $\frac{\pi}{6}$ (0 through $\frac{11\pi}{6}$) and 8 multiples of $\frac{\pi}{4}$. An angle is a multiple of both exactly when it is a multiple of $\frac{\pi}{2}$, and there are 4 of those, counted twice. So $12 + 8 - 4 = 16$ angles: the 16 points of the familiar unit-circle diagram.

COMMON MISTAKES

- Mixing up $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. Picture the angle: $\frac{\pi}{6}$ is close to the x -axis, so its point is far right and low; its x -coordinate (cosine) is the larger value $\frac{\sqrt{3}}{2}$.
- Putting the negative sign on r when a value like $\cos \theta = -\frac{5}{13}$ is given. r is always positive; the negative belongs to x or y .
- Choosing the wrong sign for the missing coordinate after a square root. Always use the quadrant to choose $+$ or $-$.
- Writing that $\tan \frac{\pi}{2} = 0$ or $\csc \pi = 0$. When the denominator is 0 the value is undefined.

Make it a game • Unit Circle Riddles. Parent reads riddles: 'I am a point on the unit circle with y -coordinate negative one half, in Quadrant III. What is my angle in radians? What are my other five functions?' Answer in 45 seconds. Then write three riddles of your own, each with a single correct answer, and stump your parent.

ANGLES, RADIANS AND THE UNIT CIRCLE

Units 1 and 2 defined sine, cosine and tangent as ratios of sides in a right triangle, which limits them to acute angles. This unit sets them free. We first widen the idea of an angle to a rotation of any size, positive or negative, and learn a second, more natural unit for measuring it: the radian, which measures an angle by the arc it cuts off, counted in radii.

Radians make the geometry of circles simple. Arc length becomes $s = r\theta$, sector area becomes $A = \frac{1}{2}r^2\theta$, and the speed of a point on a turning wheel becomes $v = r\omega$, formulas we put to work on irrigation pivots, pulleys, gears, tractors and millstones.

Finally, building on Euler's idea of sine and cosine as functions of an arc on a circle of radius 1, we define the sine and cosine of any angle as the coordinates of a point on the unit circle. Reference angles and the signs of the four quadrants let us evaluate every trigonometric function of every angle, exactly for the special angles. This unit is the foundation for everything that follows: the graphs of Unit 4, the identities of Unit 5 and the equations of Unit 6 all live on the unit circle.

PREREQUISITES

- Define and evaluate the six trigonometric ratios of an acute angle in a right triangle (Lessons 1.2 and 1.5).
- Use inverse ratios such as $\sin^{-1}$ to find an acute angle (Lesson 1.4).
- Know the exact values of sine, cosine and tangent for 30° , 45° and 60° (Lessons 2.1 to 2.3).
- Use the Pythagorean theorem and the distance formula in the coordinate plane.
- Know the circumference $2\pi r$ and area πr^2 of a circle, and convert units of length and time.

MASTERY CHECKLIST

- ☐ I can draw any angle in standard position and find coterminal angles.
- ☐ I can explain what a radian is and why radian measure does not depend on the size of the circle.
- ☐ I can convert angles between degrees and radians, giving exact answers in terms of π .
- ☐ I can derive and use $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.
- ☐ I can derive $v = r\omega$ and solve problems with wheels, pulleys and gears.
- ☐ I can define the sine and cosine of any angle using the unit circle.
- ☐ I can find all six trigonometric functions from a point (x, y) on the terminal side.
- ☐ I can find the reference angle of any angle and tell the sign of each function in each quadrant.
- ☐ I can give exact values of all six functions for every multiple of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including quadrantal angles.
- ☐ I can find the remaining trigonometric functions given one

value and the quadrant.

LESSONS IN THIS UNIT

3.1	Angles in Standard Position	45 minutes
3.2	Radian Measure	45 minutes
3.3	Converting Degrees and Radians	40 minutes
3.4	Arc Length and Sector Area	50 minutes
3.5	Linear and Angular Speed	50 minutes
3.6	The Unit Circle	50 minutes
3.7	Reference Angles and Signs by Quadrant	50 minutes
3.8	Exact Values Around the Unit Circle	2 sessions of 45 minutes

UNIT 3 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but give exact answers wherever a problem asks for them, and show your work. Unless a problem says otherwise, round lengths and other decimal answers to the nearest hundredth and angles in degrees to the nearest tenth of a degree. State whether each angle is in degrees or radians. Total: 100 points.

1. 5 pts

Find the angle between 0° and 360° that is coterminal with -790° , and name its quadrant.

2. 5 pts

A central angle in a circle of radius 5 cm cuts off an arc 8 cm long. Find the angle in radians.

3. 5 pts

Convert 300° to radians. Give the exact value.

4. (5 pts)

Convert $\frac{7\pi}{12}$ radians to degrees.

5. (8 pts)

A fan-shaped window above a barn door is a sector of a circle with radius 1.5 m and central angle 120° . Find the length of trim needed for the curved edge and the area of glass, exactly and to the nearest hundredth.

6. (8 pts)

A tractor's rear wheel has radius 0.75 m and turns at 40 rpm. How fast is the tractor moving, in kilometers per hour, to the nearest tenth?

7. (7 pts)

A motor pulley of radius 3 in. turns at 1200 rpm and drives, by a belt, a pulley of radius 8 in. on a corn sheller. How fast does the larger pulley turn? Explain why the belt makes $r_1\omega_1 = r_2\omega_2$ true.

8. (8 pts)

The point $(-7, 24)$ lies on the terminal side of θ . Find the exact values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

9. (8 pts)

Use a reference angle to find $\tan 1000^\circ$, to the nearest thousandth. Show the coterminal angle, quadrant and reference angle.

10. (9 pts)

Find the exact values of $\sin \frac{4\pi}{3}$, $\cos \frac{7\pi}{4}$, $\tan \frac{5\pi}{6}$ and $\sec \pi$.

11. (8 pts)

Given $\sin \theta = -\frac{8}{17}$ with θ in Quadrant IV, find the exact values of $\cos \theta$, $\tan \theta$ and $\csc \theta$.

12. (10 pts)

Prove that the area of a sector of a circle of radius r with central angle θ (in radians) is $A = \frac{1}{2}r^2\theta$. Then explain why the formula would be wrong if θ were in degrees.

13. 14 pts

A Ferris wheel of radius 40 ft has its center 45 ft above the ground and turns counterclockwise once every 2 minutes. A rider starts level with the center on the right side (the 3 o'clock position). (a) Through what angle, in radians, has the rider turned after 50 seconds? (b) How high above the ground is the rider then? (c) What is the rider's linear speed, exactly and to the nearest hundredth of a foot per second?

ANSWER KEY

1. $-790^\circ + 1080^\circ = 290^\circ$, Quadrant IV
2. $\theta = \frac{8}{5} = 1.6$ radians
3. $\frac{5\pi}{3}$
4. 105°
5. $\theta = \frac{2\pi}{3}$. Curved edge: $s = 1.5 \cdot \frac{2\pi}{3} = \pi \approx 3.14$ m. Area: $A = \frac{1}{2}(1.5)^2 \cdot \frac{2\pi}{3} = 0.75\pi \approx 2.36$ m²
6. $\omega = 80\pi$ rad/min; $v = 0.75 \cdot 80\pi = 60\pi$ m/min; $60\pi \cdot \frac{60}{1000} = 3.6\pi \approx 11.3$ km/h
7. $3 \cdot 1200 = 8\omega_2$, so $\omega_2 = 450$ rpm. The belt does not slip, so both rims move at the belt's linear speed, and $v = r\omega$ for each pulley.
8. $r = 25$; $\sin \theta = \frac{24}{25}$, $\cos \theta = -\frac{7}{25}$, $\tan \theta = -\frac{24}{7}$
9. $1000^\circ - 720^\circ = 280^\circ$, Quadrant IV, reference angle 80° ; tangent is negative in Quadrant IV, so $\tan 1000^\circ = -\tan 80^\circ \approx -5.671$
10. $-\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{3}, -1$
11. $y = -8, r = 17, x = 15$: $\cos \theta = \frac{15}{17}$, $\tan \theta = -\frac{8}{15}$, $\csc \theta = -\frac{17}{8}$
12. The sector is the same fraction of the disk as its angle is of a full turn, and a full turn is 2π radians: $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$. Multiplying by πr^2 gives $A = \frac{\pi r^2 \theta}{2\pi} = \frac{1}{2} r^2 \theta$. The proof used the fact that a full turn is 2π ; in degrees a full turn is 360° , so the formula would become $A = \frac{\pi r^2 \theta}{360}$ instead.
13. (a) $\omega = \frac{2\pi}{120} = \frac{\pi}{60}$ rad/s, so $\theta = 50 \cdot \frac{\pi}{60} = \frac{5\pi}{6}$. (b) Height = $45 + 40 \sin \frac{5\pi}{6} = 45 + 20 = 65$ ft. (c) $v = 40 \cdot \frac{\pi}{60} = \frac{2\pi}{3} \approx 2.09$ ft/s

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

In Unit 3 the trigonometric functions became functions of any angle, defined by a point moving around the unit circle. In this unit we unroll that circle onto a number line and draw the six functions as graphs. Sine and cosine become smooth waves that repeat every 2π ; tangent and cotangent become branches separated by asymptotes that repeat every π ; cosecant and secant become U-shaped branches built on the sine and cosine waves. Every feature of every graph (its zeros, peaks, asymptotes, symmetry and period) is explained from the circle, not memorized.

We then learn to stretch, squeeze and slide the sine wave. The four numbers in $y = a \sin(b(x - c)) + d$ control the amplitude, period, phase shift and midline, and reading them quickly is the central skill of the unit. Working backward, we write equations for waves from a graph or from data about maxima and minima. This is how hours of daylight, tide heights, Ferris wheels, water wheels and vibrating strings are described mathematically.

These graphs prepare the way for what follows. Restricting the graphs to one branch gives the inverse functions of Unit 6, reading solutions off a graph prepares for trigonometric equations, and the sinusoidal models return in full in the Applications unit.

PREREQUISITES

- Convert between degrees and radians (Lessons 3.2–3.3).
- Define sine, cosine and tangent of any angle from the unit circle (Lesson 3.6).
- Use reference angles and quadrant signs (Lesson 3.7).
- Give exact values of all six functions at multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including quadrantal angles (Lesson 3.8).
- Recognize vertical and horizontal shifts and stretches of a graph, such as $y = f(x - h) + k$ (Algebra II).
- Solve simple linear equations and work with fractions involving π .

MASTERY CHECKLIST

- ☐ I can build the graph of $y = \sin x$ from the unit circle and state its domain, range, period, zeros, maxima, minima and odd symmetry.
- ☐ I can graph $y = \cos x$ and explain why cosine is even, and prove that the cosine graph is the sine graph shifted $\frac{\pi}{2}$ to the left.
- ☐ I can use periodicity and symmetry to find every solution of $\sin x = k$ or $\cos x = k$ on a given interval.
- ☐ I can find the amplitude $|a|$ and period $\frac{2\pi}{|b|}$ of $y = a \sin(bx)$ and explain why these formulas hold.
- ☐ I can factor the inside of $y = a \sin(b(x - d)) + k$ to find the phase shift, and find the a , b , d and k of any sinusoid.
- ☐ I can graph one cycle of a shifted sinusoid from its five key points.
- ☐ I can write a sine or cosine equation for a wave from its graph

or from the location and value of a maximum and a minimum.

- ☐ I can build and use sinusoidal models for daylight, tides, wheels and other periodic situations.

- ☐ I can graph

$y = a \sin(b(x - c)) + d$, explain its asymptotes and its period

π , and find the period and asymptotes of

$$y = a \tan(b(x - c)) + d.$$

- ☐ I can sketch

$$y = \csc x \text{ and } y = \sec x$$

$$\text{and } y = \cot x$$

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LESSONS IN THIS UNIT

4.1 The Graph of Sine	50 minutes
4.2 The Graph of Cosine	45 minutes
4.3 Amplitude and Period	50 minutes
4.4 Phase Shift and Vertical Shift	2 sessions of 45 minutes
4.5 Writing Sinusoidal Equations	2 sessions of 45 minutes
4.6 The Graph of Tangent	50 minutes
4.7 Graphs of Cosecant, Secant and Cotangent	50 minutes

UNIT 4 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; a graphing calculator is not. Show your work, including sketches where they help. Give exact values when the problem asks for them; otherwise round final answers to the nearest hundredth.

Angles and x -values are in radians unless a problem says degrees. Total: 100 points.

1. (5 pts)

Find every solution of $\sin x = \frac{\sqrt{3}}{2}$ on $[0, 2\pi)$. Give exact values in radians.

2. (5 pts)

Use the symmetry and period of the cosine graph to find the exact value of $\cos\left(-\frac{11\pi}{6}\right)$.

3. (6 pts)

Find the amplitude and period of $y = -5 \cos\left(\frac{\pi}{4}x\right)$, and give the value of y at $x = 0$.

4. (6 pts)

Write an equation $y = a \sin(bx)$ with $a > 0$ and $b > 0$ that has amplitude 1.5 and period 4π .

5. (6 pts)

For $y = \tan\left(\frac{x}{2}\right)$, find the period and the two vertical asymptotes nearest the origin.

6. (6 pts)

Sketch $y = -2 \csc x$ on $(0, 2\pi)$. Give the asymptote inside the interval, the highest value of the downward branch and the lowest value of the upward branch.

7. (6 pts)

As Samuel rides his bicycle at a steady pace, the height of one pedal above the road is $h(t) = 6 \sin(2\pi(t - 0.25)) + 11$ inches, t seconds after he starts timing. Find the greatest height, the least height, and the time for one full turn of the pedal.

8. 8 pts

For $y = 3 \sin \left(2x - \frac{\pi}{3} \right) + 4$, find the amplitude, period, phase shift (with direction), vertical shift, maximum and minimum.

9. 8 pts

A sinusoid has a maximum at $(1, 9)$, and the next minimum is at $(4, 3)$. Write a cosine equation $y = a \cos(b(x - c)) + d$ with $a > 0$ and $b > 0$.

10. 8 pts

At a harbor, high tide of 3.2 m occurs at 2:00 a.m., and the next low tide of 0.8 m occurs at 8:00 a.m. Let t be hours after midnight. (a) Write a cosine model $h(t) = a \cos(b(t - c)) + d$ with $a > 0$, $b > 0$. (b) At what time after the 2:00 a.m. high tide does the water first fall to 2.6 m?

11. 8 pts

For $y = 3 \tan \left(x - \frac{\pi}{6} \right) - 1$, find the center point of the branch nearest the origin, the asymptotes on either side of it, and the two halfway key points.

12. 8 pts

A lighthouse stands 3 km from a straight shoreline. When its beam makes angle θ with the line to the nearest shore point P , the lit spot is $d = 3 \tan \theta$ km from P and the beam's length is $L = 3 \sec \theta$ km. Find d and L when $\theta = 45^\circ$. Give exact values and values to the nearest hundredth of a kilometer.

13. 12 pts

A mill's water wheel is 20 feet in diameter, and its axle is 8 feet above the water. It turns once every 30 seconds. A bucket is at the very top at $t = 0$. (a) Write a model $h(t) = a \cos(bt) + d$ for the bucket's height above the water, in feet, after t seconds. (b) How far below the water surface does the bucket go? (c) Find its height after 10 seconds.

14. 8 pts

Prove that $\sin \left(x + \frac{\pi}{2} \right) = \cos x$ for every x , using the unit circle. Then explain what this identity says about the graphs of sine and cosine.

ANSWER KEY

1. $x = \frac{\pi}{3}, \frac{2\pi}{3}$
2. Cosine is even, so $\cos\left(-\frac{11\pi}{6}\right) = \cos \frac{11\pi}{6} = \cos\left(2\pi - \frac{\pi}{6}\right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx 0.87$
3. Amplitude 5; period $\frac{2\pi}{\pi/4} = 8$; at $x = 0, y = -5$ (the graph starts at its minimum)
4. $b = \frac{2\pi}{4\pi} = \frac{1}{2}$; $y = 1.5 \sin\left(\frac{x}{2}\right)$
5. Period $\frac{\pi}{1/2} = 2\pi$; asymptotes $x = -\pi$ and $x = \pi$
6. Asymptote $x = \pi$. On $(0, \pi)$ the branch opens downward with highest point $\left(\frac{\pi}{2}, -2\right)$; on $(\pi, 2\pi)$ it opens upward with lowest point $\left(\frac{3\pi}{2}, 2\right)$.
7. Greatest height $11 + 6 = 17$ in; least height $11 - 6 = 5$ in; one turn takes $\frac{2\pi}{2\pi} = 1$ second
8. $2x - \frac{\pi}{3} = 2\left(x - \frac{\pi}{6}\right)$. Amplitude 3, period π , phase shift $\frac{\pi}{6}$ to the right, vertical shift 4 up, maximum 7, minimum 1
9. $d = 6, a = 3$, half-period 3 so period 6 and $b = \frac{\pi}{3}, c = 1$: $y = 3 \cos\left(\frac{\pi}{3}(x - 1)\right) + 6$
10. (a) $d = 2, a = 1.2$, period 12 h so $b = \frac{\pi}{6}, c = 2$: $h(t) = 1.2 \cos\left(\frac{\pi}{6}(t - 2)\right) + 2$. (b) $1.2 \cos\left(\frac{\pi}{6}(t - 2)\right) = 0.6$, so $\cos(\dots) = \frac{1}{2}$, $\frac{\pi}{6}(t - 2) = \frac{\pi}{3}, t = 4$: at 4:00 a.m.
11. Center $\left(\frac{\pi}{6}, -1\right)$; asymptotes $x = \frac{\pi}{6} \pm \frac{\pi}{2}$, that is $x = -\frac{\pi}{3}$ and $x = \frac{2\pi}{3}$; halfway points $\left(-\frac{\pi}{12}, -4\right)$ and $\left(\frac{5\pi}{12}, 2\right)$
12. $d = 3 \tan 45^\circ = 3$ km; $L = 3 \sec 45^\circ = 3\sqrt{2} \approx 4.24$ km
13. (a) $a = 10, d = 8, b = \frac{2\pi}{30} = \frac{\pi}{15}$: $h(t) = 10 \cos\left(\frac{\pi}{15}t\right) + 8$. (b) Minimum $8 - 10 = -2$: 2 feet below the surface. (c) $h(10) = 10 \cos \frac{2\pi}{3} + 8 = -5 + 8 = 3$ feet above the water.
14. Let $P = (\cos x, \sin x)$ be the point for angle x , and Q the point for $x + \frac{\pi}{2}$, a quarter turn further. A quarter turn counterclockwise about the origin sends every point (u, v) to $(-v, u)$: it turns the horizontal leg of the right triangle under P into a vertical leg of the same signed length, and the vertical leg into a horizontal leg pointing the opposite way. So $Q = (-\sin x, \cos x)$ in every quadrant. The y -coordinate of Q is $\sin\left(x + \frac{\pi}{2}\right)$, so $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. Graphically, the cosine

graph is the sine graph shifted $\frac{\pi}{2}$ units to the left: each sine peak at $\frac{\pi}{2} + 2n\pi$ becomes a cosine peak at $2n\pi$.

4.1 THE GRAPH OF SINE

OBJECTIVES

- Build the graph of $y = \sin x$ by unwinding the unit circle, with x measured in radians.
- State the domain, range, period, zeros, maxima, minima and symmetry of $y = \sin x$.
- Use the graph and its periodicity to evaluate sines and to find every solution of $\sin x = k$ on an interval.

WATCH

Graphs of Basic Trigonometric Functions (1 of 3: Investigating the unit circle with spaghetti)

Eddie Woo · 8:49

<https://www.youtube.com/watch?v=n2elzauaDjQ>

WHEN TO WATCH

After reading the Core ideas, before Example 1 (unwinding the unit circle).

Note: Woo may label the x -axis in degrees; this course uses radians ($90^\circ = \pi/2$).

BACKUP VIDEO

How to Graph Trigonometric Functions (1 of 2: Sine)

Eddie Woo · 13:15

https://www.youtube.com/watch?v=Oq4mYV1d_p0

Before the practice set, for a step-by-step sketch of $y = \sin x$.

KEY TERMS

periodic function

A function f for which some positive number p satisfies $f(x + p) = f(x)$ for every x in its domain.

period

The smallest positive p with $f(x + p) = f(x)$ for all x . For $y = \sin x$ the period is 2π .

cycle

One complete pattern of a periodic graph, covering an interval whose length is one period.

zero

An input x where $f(x) = 0$; on a graph, a place where the curve meets the x -axis.

odd function

A function with $f(-x) = -f(x)$ for all x ; its graph is symmetric about the origin.

sinusoid

The wave-shaped curve of the sine function, or any stretch or shift of it.

CORE IDEAS

In Lesson 3.6 we defined $\sin x$ as the y -coordinate of the point where the terminal side of angle x meets the unit circle. Now we make x the horizontal axis and plot that height as y . Picture a point walking counterclockwise around the unit circle. Because the radius is 1, the arc it walks equals the angle in radians, so we can "unroll" the arc onto the x -axis and plot the point's height above it. That is why we graph trig functions in radians: one unit on the x -axis is the same length as one unit on the y -axis.

Follow the height through one turn. At $x = 0$ the point is at $(1, 0)$, so $\sin 0 = 0$. It rises to the top of the circle at $x = \frac{\pi}{2}$, where $\sin x = 1$. It falls back to height 0 at $x = \pi$, down to -1 at $x = \frac{3\pi}{2}$, and back to 0 at $x = 2\pi$. These five **key points** $(0, 0)$, $(\frac{\pi}{2}, 1)$, $(\pi, 0)$, $(\frac{3\pi}{2}, -1)$, $(2\pi, 0)$ divide one cycle into quarters. Filling in values such as $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{\pi}{3} \approx 0.866$ shows the smooth, rounded shape: steep near the zeros and flat near the top and bottom.

After 2π the point is back where it started, so the heights repeat: $\sin(x + 2\pi) = \sin x$. The graph is the same cycle copied forever to the left and right. No shorter shift works, since the height 1 occurs only once per turn, so the period is exactly 2π . From the circle we can read every feature. The domain is all real numbers, because every angle has a terminal side. The range is $[-1, 1]$, because a point on a circle of radius 1 is never higher than 1 or lower than -1 . The zeros are $x = n\pi$ for every integer n . The maximum value 1 occurs at $x = \frac{\pi}{2} + 2n\pi$, and the minimum value -1 at $x = \frac{3\pi}{2} + 2n\pi$.

Two symmetries come straight from the circle. Reflecting angle x across the x -axis gives angle $-x$; the point $(\cos x, \sin x)$ goes to $(\cos x, -\sin x)$. So $\sin(-x) = -\sin x$: sine is an **odd function**, and its graph is symmetric about the origin. Reflecting across the y -axis sends angle x to angle $\pi - x$ and keeps the height, so $\sin(\pi - x) = \sin x$. On the graph this means each hump is symmetric about its peak. That is exactly why the equation $\sin x = k$ has two solutions in each cycle $[x_0, x_0 + 2\pi)$ whenever $-1 < k < 1$.

KEY FORMULAS

$$\sin(x + 2\pi) = \sin x$$

$$\sin(-x) = -\sin x$$

$$\sin(\pi - x) = \sin x$$

$$-1 \leq \sin x \leq 1$$

$$\sin x = 0 \iff x = n\pi \quad (n \text{ an integer})$$

From history. For many centuries the sine was not a function on a graph but a length in a circle: Ptolemy tabulated whole chords, and the Indian astronomers used the half-chord. Drawing a curve of y against x only became natural after Descartes published his coordinate geometry in *La Géométrie* (1637). Euler's *Introductio* (1748) then treated sine as a function of a number, which is the point of view behind the graph in this lesson.

WORKED EXAMPLES

Example 1. Find all zeros of $y = \sin x$ on the interval $[0, 4\pi]$. Give exact values in radians.

Show solution

1. The zeros are the angles whose point on the unit circle has height 0. Those points are $(1, 0)$ and $(-1, 0)$.
2. The point $(1, 0)$ is reached at $x = 0, 2\pi, 4\pi, \dots$ and the point $(-1, 0)$ at $x = \pi, 3\pi, \dots$ So the zeros are the integer multiples of π : $x = n\pi$.
3. List the multiples of π from 0 to 4π : $0, \pi, 2\pi, 3\pi, 4\pi$.
4. Check on the graph: two full cycles fit in $[0, 4\pi]$, and each cycle crosses the axis at its start, middle and end. Shared endpoints give $3 + 3 - 1 = 5$ zeros.

Answer: $x = 0, \pi, 2\pi, 3\pi, 4\pi$ (radians)

Example 2. Use the graph of $y = \sin x$ and its period to find every solution of $\sin x = \frac{1}{2}$ on the interval $[0, 4\pi]$. Give exact values in radians.

Show solution

1. Draw the horizontal line $y = \frac{1}{2}$ across two cycles of the sine graph. In each cycle it cuts the first hump twice and misses the trough, so we expect 2 solutions per cycle, 4 in all.
2. First cycle: from the unit circle (Lesson 3.8), $\sin \frac{\pi}{6} = \frac{1}{2}$. By the symmetry $\sin(\pi - x) = \sin x$, also $\sin \frac{5\pi}{6} = \frac{1}{2}$.
3. Second cycle: add the period 2π to each. $\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ and $\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$.
4. Check the interval: $4\pi = \frac{24\pi}{6}$, so $\frac{17\pi}{6}$ is inside, while the next candidate $\frac{25\pi}{6}$ is outside.
5. Decimal check: $\frac{\pi}{6} \approx 0.524$, $\frac{5\pi}{6} \approx 2.618$, $\frac{13\pi}{6} \approx 6.807$, $\frac{17\pi}{6} \approx 8.901$, all between 0 and $4\pi \approx 12.566$.

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ (radians)

PRACTICE

1. Easy State the domain and the range of $y = \sin x$.

Show answer

Domain: all real numbers. Range: $[-1, 1]$.

2. Easy What is the period of $y = \sin x$, in radians? Give the exact value.

Show answer

2π

3. Easy Use the five key points of the sine graph to give the exact value of $\sin \frac{3\pi}{2}$.

Show answer

-1

4. **Medium** Use the period of sine to find the exact value of $\sin \frac{9\pi}{4}$.

Show answer

$$\sin \frac{9\pi}{4} = \sin \left(\frac{\pi}{4} + 2\pi \right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.707$$

5. **Medium** Sine is an odd function. Use this fact to find the exact value of $\sin \left(-\frac{\pi}{3} \right)$.

Show answer

$$\sin \left(-\frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} \approx -0.866$$

6. **Medium** Find every solution of $\sin x = -\frac{1}{2}$ on $[0, 2\pi)$. Give exact values in radians.

Show answer

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

7. **Medium** A small water wheel has radius 1 meter. A paddle tip starts level with the axle on the right (the 3 o'clock position) and the wheel turns counterclockwise through $\frac{25\pi}{6}$ radians. Its height above the axle is $\sin \theta$ meters, where θ is the angle turned. How high above the axle is the tip? Give an exact value.

Show answer

$$\sin \frac{25\pi}{6} = \sin \left(\frac{\pi}{6} + 4\pi \right) = \sin \frac{\pi}{6} = \frac{1}{2} \text{ meter}$$

8. **Hard** On the same water wheel (height above the axle $h = \sin \theta$ meters), for what fraction of each full turn is the paddle tip more than $\frac{1}{2}$ meter above the axle?

Show answer

$\sin \theta > \frac{1}{2}$ for $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$, an interval of length $\frac{2\pi}{3}$. As a fraction of 2π that is $\frac{1}{3}$ of each turn.

9. **Hard** How many solutions does $\sin x = 0.3$ have on the interval $[0, 6\pi]$? Explain using the graph.

Show answer

6. The interval holds 3 full cycles, and the line $y = 0.3$ crosses the hump of each cycle twice (and never touches the troughs). The endpoints are zeros, not solutions.

10. **Challenge** Explain why $\sin(x + \pi) = -\sin x$ for every x , using the unit circle. What does this say about the graph?

Show answer

Adding π turns the terminal side half a turn, to the point diametrically opposite. The opposite of $(\cos x, \sin x)$ on the unit circle is $(-\cos x, -\sin x)$, so the new height is $-\sin x$. On the graph, shifting the curve π units horizontally gives its mirror image across the x -axis: each hump becomes a trough.

11. **Challenge** Find the smallest positive x , other than $\frac{2\pi}{7}$, for which $\sin x = \sin \frac{2\pi}{7}$. Give an exact value in radians.

Show answer

$$\text{By } \sin(\pi - x) = \sin x: x = \pi - \frac{2\pi}{7} = \frac{5\pi}{7} \approx 2.244$$

COMMON MISTAKES

- Graphing in degrees on one axis and radians on another, or labeling the x -axis 1, 2, 3 as if they were $\frac{\pi}{2}, \pi, \dots$. Mark the axis in multiples of $\frac{\pi}{2}$ or $\frac{\pi}{6}$, and remember $\pi \approx 3.14$, so the first zero after 0 is a little past 3.
- Drawing the sine curve with sharp corners at the top and bottom, like a zigzag. Near $x = \frac{\pi}{2}$ the point on the circle is moving sideways, so the height barely changes: the curve is rounded and flat at every peak and trough.
- Stopping after one solution of $\sin x = k$. Each full cycle normally gives two solutions, x and $\pi - x$; then add multiples of 2π to fill the whole interval.
- Starting the sine graph at its maximum. Sine starts at the origin, $(0, 0)$, rising; it is cosine that starts at its maximum.

Make it a game · Unwind the Circle. Draw a unit circle on the left of graph paper. Plot the heights of the sine at every multiple of π over six, moving across to the right to build the wave. Parent names features: zero, maximum, minimum, odd symmetry. Place sticky notes on each feature of one cycle in 60 seconds.

4.2 THE GRAPH OF COSINE

OBJECTIVES

- Graph $y = \cos x$ from the unit circle and state its domain, range, period, zeros, maxima, minima and symmetry.
- Prove that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ and explain why the cosine graph is the sine graph shifted $\frac{\pi}{2}$ to the left.
- Use the cosine graph to evaluate cosines and solve $\cos x = k$ on an interval.

WATCH

Understanding the Cosine Curve through the Unit Circle

Eddie Woo · 8:23

<https://www.youtube.com/watch?v=U-jPpBtcVxU>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo may label the x-axis in degrees; this course uses radians.

BACKUP VIDEO

Graphing the Cosine Curve

Eddie Woo · 6:50

<https://www.youtube.com/watch?v=JGsnZmjO3aI>

Before the practice set, for a quick sketching walkthrough.

KEY TERMS

even function

A function with $f(-x) = f(x)$ for all x ; its graph is symmetric about the y -axis.

horizontal shift

Sliding a graph left or right without changing its shape. The graph of $y = f(x + c)$ with $c > 0$ is the graph of $y = f(x)$ moved c units to the left.

cofunction

Cosine is the sine of the complementary angle: the name comes from *complementi sinus*, "sine of the complement."

key points

The five points that divide one cycle into quarters; for $y = \cos x$ they are $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, $(2\pi, 1)$.

CORE IDEAS

Cosine is the x -coordinate of the point on the unit circle. To graph it we again unroll the angle onto the horizontal axis, but now we plot how far the point is to the right (positive) or left (negative) of the center. At $x = 0$ the point is $(1, 0)$, so $\cos 0 = 1$. A quarter turn later it is at the top, $(0, 1)$, so $\cos \frac{\pi}{2} = 0$. At $x = \pi$ it is at $(-1, 0)$, so $\cos \pi = -1$; at $\frac{3\pi}{2}$ it is at the bottom, so the cosine is 0 again; and at 2π it is back to 1. The five key points are $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, $(2\pi, 1)$.

The features match those of sine. The domain is all real numbers, the range is $[-1, 1]$, and the period is 2π , since $\cos(x + 2\pi) = \cos x$. The zeros are $x = \frac{\pi}{2} + n\pi$. The maximum 1 occurs at $x = 2n\pi$ and the minimum -1 at $x = \pi + 2n\pi$. The symmetry is different. Reflecting angle x across the x -axis gives angle $-x$ and keeps the x -coordinate, so $\cos(-x) = \cos x$. Cosine is an **even function**, and its graph is a mirror image of itself across the y -axis.

Why does the cosine graph look exactly like the sine graph? Rotate the point $P = (\cos x, \sin x)$ a quarter turn counterclockwise to the point Q for angle $x + \frac{\pi}{2}$. Drop perpendiculars from P and Q to the x -axis. A quarter turn counterclockwise about the origin carries each point (u, v) to $(-v, u)$: the horizontal leg of the triangle under P becomes a vertical segment of the same signed length, and the vertical leg becomes a horizontal segment pointing the opposite way. Applying this to $P = (\cos x, \sin x)$ gives $Q = (-\sin x, \cos x)$ for every x , in any quadrant. Reading off the second coordinate gives $\sin(x + \frac{\pi}{2}) = \cos x$, and the first gives $\cos(x + \frac{\pi}{2}) = -\sin x$. (Checking one case: $x = 0$ gives $Q = (0, 1)$, as it should.)

The identity $\cos x = \sin(x + \frac{\pi}{2})$ says that cosine does at x what sine does $\frac{\pi}{2}$ later. So the cosine graph is the sine graph slid $\frac{\pi}{2}$ units to the **left**: the sine peak at $\frac{\pi}{2}$ moves to the cosine peak at 0. Equivalently, $\sin x = \cos(x - \frac{\pi}{2})$: sine is cosine slid $\frac{\pi}{2}$ to the

right. This also agrees with the cofunction identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ from Lesson 2.3, since $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\pi - \left(\frac{\pi}{2} - x\right)\right) = \sin\left(x + \frac{\pi}{2}\right)$. Sine and cosine are one curve seen from two starting points.

KEY FORMULAS

$$\cos(x + 2\pi) = \cos x$$

$$\cos(-x) = \cos x$$

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\sin x = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos x = 0 \iff x = \frac{\pi}{2} + n\pi \quad (n \text{ an integer})$$

From history. The word cosine is a contraction of the Latin *complementi sinus*, "the sine of the complement," because the cosine of an acute angle is the sine of the angle that completes it to a right angle. For this reason a table of sines from 0° to 90° already serves as a table of cosines: just read it from the other end.

Euclid connection. In a right triangle the two acute angles add to a right angle, because all three angles add to two right angles (Euclid I.32). That is why the side adjacent to one acute angle is opposite the other, and why the cosine of an angle equals the sine of its complement.

WORKED EXAMPLES

Example 1. On the interval $[0, 2\pi]$, find the zeros of $y = \cos x$, the x -values where it reaches its maximum, and the x -value where it reaches its minimum. Give exact values in radians.

Show solution

1. Zeros: the point on the unit circle has x -coordinate 0 at the top and bottom, $(0, 1)$ and $(0, -1)$. These occur at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
2. Maximum: the x -coordinate is largest, 1, at the point $(1, 0)$, reached at $x = 0$ and again at $x = 2\pi$.
3. Minimum: the x -coordinate is smallest, -1 , at the point $(-1, 0)$, reached at $x = \pi$.
4. These are the five key points of one cycle: $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, $(2\pi, 1)$.

Answer: Zeros $x = \frac{\pi}{2}, \frac{3\pi}{2}$; maximum value 1 at $x = 0$ and $x = 2\pi$; minimum value -1 at $x = \pi$.

Example 2. On Mr. Miller's farm an old hand pump is driven by a crank. The crank pin travels on a circle of radius 1 foot around the axle, and its horizontal position relative to the axle is $x = \cos \theta$ feet, where θ is the angle the crank has turned from the 3 o'clock position. During one turn, $0 \leq \theta < 2\pi$, at what angles is the pin exactly $\frac{1}{2}$ foot to the **left** of the axle? Give exact values in radians.

Show solution

1. Left of the axle means a negative horizontal position, so we solve $\cos \theta = -\frac{1}{2}$.
2. On the graph, the line $y = -\frac{1}{2}$ crosses the cosine curve twice in one cycle, once on the way down to the trough at π and once on the way back up. The two solutions are symmetric about $\theta = \pi$.
3. The reference angle is $\frac{\pi}{3}$, since $\cos \frac{\pi}{3} = \frac{1}{2}$. Cosine is negative in Quadrants II and III.
4. Quadrant II: $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Quadrant III: $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.
5. Check the symmetry: both are $\frac{\pi}{3}$ away from π . Decimals: $\frac{2\pi}{3} \approx 2.094$ and $\frac{4\pi}{3} \approx 4.189$.

Answer: $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ radians

PRACTICE

1. **Easy** Use the key points of the cosine graph to give the exact value of $\cos \pi$.

Show answer

$$-1$$

2. **Easy** State the range and the period of $y = \cos x$.

Show answer

Range $[-1, 1]$; period 2π

3. **Easy** Cosine is an even function. Use this to find the exact value of $\cos\left(-\frac{\pi}{4}\right)$.

Show answer

$$\cos\left(-\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.707$$

4. **Medium** Use the period of cosine to find the exact value of $\cos \frac{7\pi}{3}$.

Show answer

$$\cos \frac{7\pi}{3} = \cos \left(\frac{\pi}{3} + 2\pi \right) = \cos \frac{\pi}{3} = \frac{1}{2}$$

5. **Medium** Find all zeros of $y = \cos x$ on $[0, 4\pi]$. Give exact values in radians.

Show answer

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

6. **Medium** Find the smallest positive number c such that $\cos x = \sin(x + c)$ for every x . In which direction, and how far, must the sine graph be shifted to land on the cosine graph?

Show answer

$$c = \frac{\pi}{2}; \text{ shift the sine graph } \frac{\pi}{2} \text{ units to the left.}$$

7. **Medium** Find every solution of $\cos x = \frac{\sqrt{2}}{2}$ on $[0, 2\pi)$. Give exact values in radians.

Show answer

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

8. **Hard** A windmill drives a pump through a crank whose pin moves on a circle of radius 1 foot, with horizontal position $x = \cos \theta$ feet from the axle. For what fraction of each full turn is the pin more than $\frac{\sqrt{3}}{2}$ foot to the right of the axle?

Show answer

$$\cos \theta > \frac{\sqrt{3}}{2} \text{ for } -\frac{\pi}{6} < \theta < \frac{\pi}{6} \text{ (in one turn, } 0 \leq \theta < \frac{\pi}{6} \text{ and } \frac{11\pi}{6} < \theta < 2\pi). \text{ Total length } \frac{\pi}{3}, \text{ which is } \frac{1}{6} \text{ of a turn.}$$

9. **Hard** Sketch $y = \sin x$ and $y = \cos x$ on the same axes. At what values of x in $[0, 2\pi]$ do the graphs cross? Give exact values in radians.

Show answer

$$\text{Where } \sin x = \cos x: x = \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}$$

10. **Hard** A point on the rim of a wheel of radius 1 meter starts at the 3 o'clock position, and the wheel turns counterclockwise through 1000° . Its horizontal position relative to the hub is $\cos \theta$ meters. Find that position to the nearest thousandth of a meter.

Show answer

$1000^\circ - 2(360^\circ) = 280^\circ$, and $\cos 280^\circ = \cos 80^\circ \approx 0.174$ m (to the right of the hub)

11. **Challenge** Explain, using the unit circle, why $\cos(\pi - x) = -\cos x$ for every x . What symmetry of the cosine graph does this describe?

Show answer

The angles x and $\pi - x$ are mirror images across the y -axis, so their points are $(\cos x, \sin x)$ and $(-\cos x, \sin x)$: the x -coordinates are opposites. On the graph, x and $\pi - x$ are equally far from $\frac{\pi}{2}$, on opposite sides: values the same distance to the left and right of $x = \frac{\pi}{2}$ are opposites, so the curve is symmetric about the point $(\frac{\pi}{2}, 0)$.

12. **Challenge** Without a calculator, use the graph of cosine to put $\cos 1$, $\cos 2$ and $\cos 3$ (radians) in order from least to greatest. Then check with a calculator, rounding to the nearest thousandth.

Show answer

Cosine decreases on $[0, \pi]$ and $1 < 2 < 3 < \pi$, so $\cos 3 < \cos 2 < \cos 1$. Check: $-0.990 < -0.416 < 0.540$.

COMMON MISTAKES

- Shifting the wrong way. Since $\cos x = \sin(x + \frac{\pi}{2})$, the cosine graph is the sine graph moved **left**, not right. Check with a point: the sine peak at $\frac{\pi}{2}$ must move to the cosine peak at 0.
- Starting the cosine graph at the origin. Cosine starts at its maximum, $(0, 1)$; it is sine that passes through the origin.
- Confusing even and odd. Cosine is even, $\cos(-x) = \cos x$, because reflecting across the x -axis keeps the x -coordinate. Sine is odd. Test with $x = \frac{\pi}{3}$ if unsure.
- Giving only one solution of $\cos x = k$. In each cycle there are usually two, symmetric about the trough at π (or about the peak at 0): x and $2\pi - x$.

Make it a game · Slide the Sine. Trace a sine graph onto waxed paper or parchment. Slide it left by π over 2 on top of a cosine graph and see the match. Parent proposes shifts: right π over 2, left 3π over 2, right 3π over 2. Predict which ones also give cosine, then test. State the two key differences.

4.3 AMPLITUDE AND PERIOD

OBJECTIVES

- Find the amplitude $|a|$ and the period $\frac{2\pi}{|b|}$ of $y = a \sin(bx)$ and $y = a \cos(bx)$, and explain why these formulas hold.
- Graph one cycle of $y = a \sin(bx)$ or $y = a \cos(bx)$ by marking five key points at quarter-period steps.
- Write an equation $y = a \sin(bx)$ or $y = a \cos(bx)$ with a given amplitude and period.

WATCH

Modifications of Trigonometric Graphs (1 of 5: Frequency/Period)

Eddie Woo · 11:04

<https://www.youtube.com/watch?v=3dSta0NHomY>

WHEN TO WATCH

After reading the Core ideas on period, before Example 1.

Note: Woo works in degrees, so his period is $360^\circ/n$; ours is $2\pi/|b|$. He also says 'frequency' for the effect of b .

BACKUP VIDEO

Modifications of Trigonometric Graphs (2 of 5: Amplitude)

Eddie Woo · 5:42

<https://www.youtube.com/watch?v=1aWgEUIGDDU>

Before Example 1 too: the 5-minute amplitude companion to the primary video.

KEY TERMS

amplitude

Half the distance between the maximum and minimum values of a sinusoid. For $y = a \sin(bx)$ or $y = a \cos(bx)$ it is $|a|$.

period

The horizontal length of one cycle. For $y = a \sin(bx)$ or $y = a \cos(bx)$ it is $\frac{2\pi}{|b|}$.

vertical stretch

Multiplying the output of a function by a constant, which stretches the graph away from (or shrinks it toward) the x -axis.

horizontal compression

Multiplying the input by a constant b with $|b| > 1$, which squeezes the graph toward the y -axis; $|b| < 1$ stretches it.

frequency

The number of cycles per unit of input,

$$\text{equal to } \frac{1}{\text{period}} = \frac{|b|}{2\pi}.$$

CORE IDEAS

The number a in front changes the height. In $y = a \sin x$ every output of $\sin x$ is multiplied by a . Since $-1 \leq \sin x \leq 1$, the new outputs run from $-|a|$ to $|a|$. The zeros do not move (because $a \cdot 0 = 0$), but the peaks rise to $|a|$ and the troughs fall to $-|a|$. The **amplitude** is $|a|$, half the distance from the lowest point to the highest. If a is negative, every output changes sign, so the graph is also reflected across the x -axis: $y = -2 \sin x$ starts at 0 going **down**. Amplitude is a distance, so it is never negative.

The number b inside changes the width. In $y = \sin(bx)$ with $b > 0$, a full cycle happens while the input bx runs from 0 to 2π . Solving $0 \leq bx \leq 2\pi$ gives $0 \leq x \leq \frac{2\pi}{b}$. So the period is $\frac{2\pi}{b}$. A large b makes the inside angle grow quickly, so the cycles are squeezed together; a b between 0 and 1 spreads them out. If $b < 0$, we use $\sin(-u) = -\sin u$ and $\cos(-u) = \cos u$ to make b positive first, so in every case the period is $\frac{2\pi}{|b|}$. Exactly the same reasoning works for cosine.

Five key points make a quick, accurate sketch. Find the period P , then divide one cycle into quarters: $x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$. For $y = a \sin(bx)$ with $a > 0$ the heights at these points are $0, a, 0, -a, 0$ (the sine pattern). For $y = a \cos(bx)$ with $a > 0$ they are $a, 0, -a, 0, a$ (the cosine pattern). If $a < 0$, flip the signs. Plot the five points, join them with a smooth wave, and copy the cycle to the left and right as needed.

Working backward. Given an amplitude A and period P , choose $a = A$ (or $-A$ for a reflected graph) and $b = \frac{2\pi}{P}$. For example, amplitude 4 and period $\frac{\pi}{3}$ give $b = \frac{2\pi}{\pi/3} = 6$, so $y = 4 \sin(6x)$ is one answer. In practical settings the input is often time; then the

period is the time for one full cycle, and the frequency $\frac{1}{P}$ is the number of cycles per unit of time.

KEY FORMULAS

$$\text{amplitude} = |a|$$

$$\text{period } P = \frac{2\pi}{|b|}$$

$$b = \frac{2\pi}{P} \quad (b > 0)$$

$$\text{frequency} = \frac{1}{P} = \frac{|b|}{2\pi}$$

$$\text{key points at } x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$$

From history. In *The Analytical Theory of Heat* (1822), Joseph Fourier showed that a great many periodic patterns can be built by adding together sine and cosine waves with different amplitudes and periods. The two numbers studied in this lesson are therefore the basic "dials" behind the study of sound, light, vibrating strings and much of modern engineering.

WORKED EXAMPLES

Example 1. Find the amplitude and period of $y = 3 \sin(2x)$, and list the five key points of the cycle that starts at $x = 0$.

Show solution

1. Amplitude: $a = 3$, so the amplitude is $|3| = 3$. The graph runs from $y = -3$ to $y = 3$.
2. Period: $b = 2$, so $P = \frac{2\pi}{2} = \pi$. One cycle fits between $x = 0$ and $x = \pi$.
3. Quarter-period steps: $\frac{P}{4} = \frac{\pi}{4}$, so the key x -values are $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.
4. Sine pattern times 3: heights $0, 3, 0, -3, 0$. Check one: at $x = \frac{\pi}{4}$,
 $3 \sin \left(2 \cdot \frac{\pi}{4} \right) = 3 \sin \frac{\pi}{2} = 3$.
5. Key points: $(0, 0), \left(\frac{\pi}{4}, 3 \right), \left(\frac{\pi}{2}, 0 \right), \left(\frac{3\pi}{4}, -3 \right), (\pi, 0)$. Join them with a smooth wave.

Answer: Amplitude 3, period π ; key points $(0, 0), \left(\frac{\pi}{4}, 3 \right), \left(\frac{\pi}{2}, 0 \right), \left(\frac{3\pi}{4}, -3 \right), (\pi, 0)$

Example 2. Ruth watches her fishing bobber ride the ripples on a pond. Its height above its resting level is $h = -4 \cos\left(\frac{\pi}{3}t\right)$ centimeters, t seconds after she starts watching. Find the amplitude, the period, and the first time $t > 0$ when the bobber is at its highest point.

Show solution

1. Amplitude: $|a| = |-4| = 4$ cm. The bobber moves between 4 cm below and 4 cm above its resting level.
2. Period: $b = \frac{\pi}{3}$, so $P = \frac{2\pi}{\pi/3} = 2\pi \cdot \frac{3}{\pi} = 6$ seconds for one full bob.
3. Because $a = -4$ is negative, the cosine pattern is flipped: at $t = 0$, $h = -4 \cos 0 = -4$, the lowest point.
4. Flipped cosine key heights at $t = 0, 1.5, 3, 4.5, 6$ are $-4, 0, 4, 0, -4$. The highest point comes half a period after the start.
5. Check: at $t = 3$, $h = -4 \cos\left(\frac{\pi}{3} \cdot 3\right) = -4 \cos \pi = -4(-1) = 4$. Correct.

Answer: Amplitude 4 cm, period 6 seconds; first highest point at $t = 3$ seconds

PRACTICE

1. **Easy** Find the amplitude and period of $y = 5 \sin x$.

Show answer

Amplitude 5, period 2π

2. **Easy** Find the amplitude and period of $y = \cos(4x)$.

Show answer

Amplitude 1, period $\frac{2\pi}{4} = \frac{\pi}{2}$

3. **Easy** Find the amplitude and period of $y = -3 \sin x$, and describe how its graph differs from $y = 3 \sin x$.

Show answer

Amplitude 3 (not -3), period 2π ; it is the graph of $y = 3 \sin x$ reflected across the x -axis.

4. **Medium** Find the amplitude and period of $y = \frac{1}{2} \cos\left(\frac{x}{3}\right)$.

Show answer

Amplitude $\frac{1}{2}$, period $\frac{2\pi}{1/3} = 6\pi$

5. **Medium** Find the amplitude and period of $y = 2 \sin(\pi x)$.

Show answer

Amplitude 2, period $\frac{2\pi}{\pi} = 2$

6. **Medium** Write an equation of the form $y = a \sin(bx)$ with $a > 0$ and $b > 0$ that has amplitude 4 and period $\frac{\pi}{3}$.

Show answer

$b = \frac{2\pi}{\pi/3} = 6$, so $y = 4 \sin(6x)$

7. **Medium** For $y = -2 \cos(3x)$, give the maximum value, the minimum value, and the smallest positive x (in radians) where the maximum occurs.

Show answer

Maximum 2, minimum -2 . The maximum occurs where $\cos(3x) = -1$, first at $3x = \pi$, so $x = \frac{\pi}{3}$.

8. **Hard** When the A string of a violin is in tune, a point on the string vibrates with displacement $y = 0.02 \sin(880\pi t)$ centimeters, where t is in seconds. Find the period (exact, and in seconds to three significant figures) and the frequency in vibrations per second.

Show answer

Period $\frac{2\pi}{880\pi} = \frac{1}{440} \approx 0.00227$ second; frequency 440 vibrations per second

9. **Hard** A graph of the form $y = a \cos(bx)$, with $a > 0$ and $b > 0$, has a maximum at $(0, 6)$, and its next maximum is at $(8, 6)$. Find a and b (give b exactly).

Show answer

$a = 6$; the period is 8, so $b = \frac{2\pi}{8} = \frac{\pi}{4}$. The equation is $y = 6 \cos\left(\frac{\pi}{4}x\right)$.

10. **Hard** A harbor buoy rises and falls on the swells. Its height above its average level is $y = 1.5 \sin\left(\frac{\pi}{4}t\right)$ meters, t seconds after it passes its average level going up. How far does it travel from its highest point to its lowest point, and how many seconds does that take?

Show answer

It travels twice the amplitude, $2(1.5) = 3$ meters. The period is $\frac{2\pi}{\pi/4} = 8$ seconds, and highest to lowest is half a period: 4 seconds.

11. **Challenge** Explain why replacing x by $2x$ in $y = \sin x$ cuts the period in half, rather than doubling it.

Show answer

In $y = \sin(2x)$ the inside angle $2x$ grows twice as fast as x . The sine completes a cycle when its inside angle has grown by 2π , and $2x$ grows by 2π while x grows by only π . So each cycle takes half as much horizontal distance: the period is π , and the graph is squeezed toward the y -axis.

12. **Challenge** Find every x in $\left[0, \frac{2\pi}{3}\right]$ with $2 \sin(3x) = 1$. Give exact values in radians.

Show answer

$\sin(3x) = \frac{1}{2}$. As x runs over $\left[0, \frac{2\pi}{3}\right]$, $3x$ runs over $[0, 2\pi]$, so $3x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, giving $x = \frac{\pi}{18}, \frac{5\pi}{18}$.

COMMON MISTAKES

- Giving a negative amplitude. For $y = -3 \sin x$ the amplitude is 3; the negative sign only means a reflection across the x -axis.
- Using b itself as the period, or computing $\frac{b}{2\pi}$. The period is $\frac{2\pi}{|b|}$. Check: a bigger b must give a **shorter** period.
- Mixing up the patterns. Sine key heights are $0, a, 0, -a, 0$; cosine key heights are $a, 0, -a, 0, a$. Plug $x = 0$ into the equation to see where the graph starts.
- Spacing the key points by the whole period instead of a quarter of it. Divide the period by 4 to get the step between key points.

Make it a game • Graph Dictation. Sit back to back. Parent describes a graph in words without naming the equation: 'Tallest at 3, lowest at negative 3, one cycle every π .' Draw one cycle with five key points, then write the equation. Reveal and compare. Swap roles with your own mystery wave. Closest match wins.

4.4 PHASE SHIFT AND VERTICAL SHIFT

OBJECTIVES

- Identify the amplitude, period, phase shift and vertical shift of $y = a \sin(b(x - c)) + d$ and $y = a \cos(b(x - c)) + d$, including when the inside is written as $bx - k$.
- Find the midline, maximum value and minimum value of a sinusoid.
- Graph one cycle of a shifted sinusoid from its five key points.

WATCH

Trigonometric Graphs with Multiple Modifications (1 of 2: Phase & Frequency)

Eddie Woo · 12:34

<https://www.youtube.com/watch?v=IRab2gO2arE>

WHEN TO WATCH

Before Example 2: why $\sin(2x - 90^\circ)$ shifts only 45° (factor out b first).

Note: Woo works in degrees and calls b the 'frequency' change.

BACKUP VIDEO

Trigonometric Graphs with Multiple Modifications (2 of 2: Amplitude & Vertical Shift)

Eddie Woo · 9:20

<https://www.youtube.com/watch?v=CZaLNk6m5ik>

Before Example 1: amplitude and vertical shift together.

KEY TERMS

phase shift

The horizontal shift c in

$y = a \sin(b(x - c)) + d$. If $c > 0$ the graph moves c units right; if $c < 0$ it moves $|c|$ units left.

vertical shift

The number d in

$y = a \sin(b(x - c)) + d$, which moves the whole graph up ($d > 0$) or down ($d < 0$).

midline

The horizontal line $y = d$ halfway between the maximum and minimum values of a sinusoid.

maximum and minimum values

For a sinusoid with amplitude $|a|$ and midline $y = d$, the maximum is $d + |a|$ and the minimum is $d - |a|$.

CORE IDEAS

Vertical shift. Adding d to every output lifts the whole graph d units (or lowers it if $d < 0$). The wave now rocks about the line $y = d$ instead of the x -axis. This line is the **midline**. The amplitude still measures how far the wave goes above and below it, so the maximum value is $d + |a|$ and the minimum value is $d - |a|$. For example, $y = 4 \sin x - 2$ has midline $y = -2$, maximum 2 and minimum -6 .

Phase shift. In $y = \sin(x - c)$, the output at x is what $\sin$ gave at $x - c$. So the value $\sin 0 = 0$ (the start of a rising cycle) now happens at $x = c$, and every feature of the graph arrives c units later: the graph moves c units to the **right**. With $x + c$ instead, it moves left. This horizontal shift is called the **phase shift**. When a period change is also present, the form $y = a \sin(b(x - c)) + d$ shows the shift directly: a cycle starts when $b(x - c) = 0$, that is, at $x = c$, and ends when $b(x - c) = 2\pi$, that is, at $x = c + \frac{2\pi}{b}$.

Factor out b first. Equations are often written as $y = a \sin(bx - k) + d$. The phase shift is **not** k . Factor: $bx - k = b\left(x - \frac{k}{b}\right)$, so $c = \frac{k}{b}$. For instance, $y = \sin(2x - \pi)$ equals $\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$: its phase shift is $\frac{\pi}{2}$ to the right, not π . A quick check: the cycle starts where the inside is 0, and $2x - \pi = 0$ gives $x = \frac{\pi}{2}$.

Graphing in five steps. (1) Write the equation as $y = a \sin(b(x - c)) + d$ and read the four numbers. (2) Draw the midline $y = d$ and light lines at $y = d + |a|$ and $y = d - |a|$. (3) Find the period $P = \frac{2\pi}{|b|}$. (4) Start at $x = c$ and step by $\frac{P}{4}$ to get five key x -values ending at $x = c + P$. (5) Use the sine pattern (midline, max, midline, min, midline) or the cosine pattern (max, midline, min, midline, max), flipped if $a < 0$, and join the points smoothly. Every shift, stretch and reflection can be read straight from the factored form.

KEY FORMULAS

$$y = a \sin(b(x - c)) + d \quad y = a \cos(b(x - c)) + d$$

$$\text{amplitude } |a|, \quad \text{period } \frac{2\pi}{|b|}, \quad \text{phase shift } c, \quad \text{midline } y = d$$

$$\text{maximum} = d + |a|, \quad \text{minimum} = d - |a|$$

$$a \sin(bx - k) + d = a \sin\left(b\left(x - \frac{k}{b}\right)\right) + d$$

WORKED EXAMPLES

Example 1. For $y = 2 \sin\left(x - \frac{\pi}{3}\right) + 1$, find the amplitude, period, phase shift, vertical shift, maximum and minimum, and list the five key points of the cycle that starts at the phase shift.

Show solution

1. Read the form $y = a \sin(b(x - c)) + d$: $a = 2$, $b = 1$, $c = \frac{\pi}{3}$, $d = 1$.
2. Amplitude 2; period $\frac{2\pi}{1} = 2\pi$; phase shift $\frac{\pi}{3}$ to the right; vertical shift 1 up, so the midline is $y = 1$.
3. Maximum $= 1 + 2 = 3$; minimum $= 1 - 2 = -1$.
4. Key x -values: start at $\frac{\pi}{3}$ and step by $\frac{2\pi}{4} = \frac{\pi}{2}$: $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}$.
5. Sine pattern on the midline $y = 1$: heights 1, 3, 1, -1, 1. Check: at $x = \frac{5\pi}{6}$,
 $2 \sin\left(\frac{5\pi}{6} - \frac{\pi}{3}\right) + 1 = 2 \sin \frac{\pi}{2} + 1 = 3$.
6. Key points: $\left(\frac{\pi}{3}, 1\right), \left(\frac{5\pi}{6}, 3\right), \left(\frac{4\pi}{3}, 1\right), \left(\frac{11\pi}{6}, -1\right), \left(\frac{7\pi}{3}, 1\right)$.

Answer: Amplitude 2, period 2π , phase shift $\frac{\pi}{3}$ right, vertical shift 1 up (midline $y = 1$), maximum 3, minimum -1; key points $\left(\frac{\pi}{3}, 1\right), \left(\frac{5\pi}{6}, 3\right), \left(\frac{4\pi}{3}, 1\right), \left(\frac{11\pi}{6}, -1\right), \left(\frac{7\pi}{3}, 1\right)$

Example 2. For $y = -3 \cos \left(2x + \frac{\pi}{2} \right) - 1$, find the amplitude, period, phase shift, vertical shift, maximum and minimum, and the smallest $x \geq 0$ where the maximum occurs.

Show solution

- Factor out $b = 2$ inside: $2x + \frac{\pi}{2} = 2 \left(x + \frac{\pi}{4} \right) = 2 \left(x - \left(-\frac{\pi}{4} \right) \right)$. So $c = -\frac{\pi}{4}$.
- Read: $a = -3$, $b = 2$, $c = -\frac{\pi}{4}$, $d = -1$.
- Amplitude $|-3| = 3$; period $\frac{2\pi}{2} = \pi$; phase shift $\frac{\pi}{4}$ to the **left**; vertical shift 1 down, midline $y = -1$.
- Maximum $= -1 + 3 = 2$; minimum $= -1 - 3 = -4$.
- Because $a < 0$ the cosine pattern is flipped: the cycle starting at $x = -\frac{\pi}{4}$ begins at the **minimum** -4 . The maximum comes half a period later, at $x = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$.
- Check: at $x = \frac{\pi}{4}$, the inside is $2 \cdot \frac{\pi}{4} + \frac{\pi}{2} = \pi$, and $-3 \cos \pi - 1 = 3 - 1 = 2$.
Since the previous maximum is a full period earlier, at $-\frac{3\pi}{4}$, the value $\frac{\pi}{4}$ is the smallest $x \geq 0$.

Answer: Amplitude 3, period π , phase shift $\frac{\pi}{4}$ left (that is, $c = -\frac{\pi}{4}$), vertical shift 1 down (midline $y = -1$), maximum 2, minimum -4 ; first maximum for $x \geq 0$ at $x = \frac{\pi}{4}$

PRACTICE

- Easy** For $y = \sin x + 3$, give the midline, the maximum value and the minimum value.

Show answer

Midline $y = 3$; maximum 4; minimum 2

- Easy** Give the phase shift of $y = \cos \left(x - \frac{\pi}{4} \right)$ and its direction.

Show answer

$\frac{\pi}{4}$ to the right

3. **Easy** For $y = 4 \sin x - 2$, give the amplitude, vertical shift, maximum value and minimum value.

Show answer

Amplitude 4; vertical shift 2 down ($d = -2$); maximum 2; minimum -6

4. **Medium** Find the amplitude, period, phase shift and vertical shift of $y = 3 \sin \left(2 \left(x - \frac{\pi}{6} \right) \right) + 1$.

Show answer

Amplitude 3, period π , phase shift $\frac{\pi}{6}$ right, vertical shift 1 up

5. **Medium** Find the period and the phase shift of $y = \sin(3x - \pi)$.

Show answer

$3x - \pi = 3 \left(x - \frac{\pi}{3} \right)$: period $\frac{2\pi}{3}$, phase shift $\frac{\pi}{3}$ to the right

6. **Medium** For $y = -2 \cos(x + \pi) + 5$, give the amplitude, phase shift, vertical shift, maximum and minimum.

Show answer

Amplitude 2; phase shift π to the left ($c = -\pi$); vertical shift 5 up; maximum 7; minimum 3

7. **Medium** At the county fair, a rider's height above the ground on the Ferris wheel is $h(t) = 30 \sin \left(\frac{\pi}{15}(t - 7.5) \right) + 35$ feet, t seconds after boarding. Find the greatest height, the least height, and the time for one full revolution.

Show answer

Greatest height $35 + 30 = 65$ ft; least height $35 - 30 = 5$ ft; one revolution takes $\frac{2\pi}{\pi/15} = 30$ seconds

8. **Hard** Find the amplitude, period, phase shift, vertical shift, maximum and minimum of $y = 4 \cos \left(\frac{\pi}{2}x + \pi \right) - 3$.

Show answer

$\frac{\pi}{2}x + \pi = \frac{\pi}{2}(x + 2)$. Amplitude 4, period $\frac{2\pi}{\pi/2} = 4$, phase shift 2 to the left ($c = -2$), vertical shift 3 down, maximum 1, minimum -7

9. **Hard** The depth of water at a fishing pier is $D(t) = 1.8 \sin\left(\frac{\pi}{6}(t - 2)\right) + 4.5$ meters, t hours after midnight. Find the time and depth of the first high tide after midnight, and the time and depth of the next low tide. Give depths to the nearest tenth of a meter.

Show answer

High tide when $\frac{\pi}{6}(t - 2) = \frac{\pi}{2}: t = 5$ (5:00 a.m.), depth $4.5 + 1.8 = 6.3$ m. Low tide when $\frac{\pi}{6}(t - 2) = \frac{3\pi}{2}: t = 11$ (11:00 a.m.), depth $4.5 - 1.8 = 2.7$ m.

10. **Challenge** Show that $y = \sin\left(x - \frac{\pi}{2}\right) + 2$ and $y = -\cos x + 2$ have the same graph. Explain both algebraically and by comparing key points.

Show answer

Algebra: $\sin x = \cos\left(x - \frac{\pi}{2}\right)$ from Lesson 4.2, so replacing x with $x - \frac{\pi}{2}$ gives $\sin\left(x - \frac{\pi}{2}\right) = \cos(x - \pi) = -\cos x$, since moving half a turn changes the sign of the cosine. Key points: the first graph's cycle starting at $x = \frac{\pi}{2}$ has heights 2, 3, 2, 1, 2 at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$, and $-\cos x + 2$ gives exactly the same values there (for example $-\cos \pi + 2 = 3$). Both have period 2π , so the graphs agree everywhere.

11. **Challenge** Find every x in $[0, 2\pi)$ where $y = 2 \sin\left(x - \frac{\pi}{3}\right) + 1$ equals 2. Give exact values in radians.

Show answer

$\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$, so $x - \frac{\pi}{3} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (these are the only choices with $x - \frac{\pi}{3}$ in $\left[-\frac{\pi}{3}, \frac{5\pi}{3}\right)$). So $x = \frac{\pi}{2}$ or $x = \frac{7\pi}{6}$.

COMMON MISTAKES

- Reading the phase shift before factoring. In $y = \sin(2x - \pi)$ the shift is $\frac{\pi}{2}$, not π . Always rewrite the inside as $b(x - c)$, or set the inside equal to 0 and solve for x .
- Getting the direction backward. $x - c$ shifts **right** by c ; $x + c$ shifts **left**. Test a point: in $y = \sin\left(x - \frac{\pi}{3}\right)$ the rising zero moves from 0 to $\frac{\pi}{3}$, to the right.
- Forgetting that the vertical shift moves the maximum and minimum too. The maximum is $d + |a|$, not $|a|$.

- Starting the key points at $x = 0$ instead of $x = c$. The shifted cycle begins at the phase shift and ends one period later.

Make it a game • Four Facts Flash. Parent writes an equation on a card, such as $y = 2 \sin(3(x - \pi/6)) + 1$. You have 20 seconds to state the amplitude, period, phase shift and vertical shift, plus midline and extremes. Trap card: $y = \sin(2x - \pi/3)$, where the shift is $\pi/6$, not $\pi/3$. Score five equations.

4.5 WRITING SINUSOIDAL EQUATIONS

OBJECTIVES

- Find a , b , c and d for a sinusoid from a graph or from the location and value of a maximum and a minimum.
- Write both a cosine model and an equivalent sine model for the same periodic data.
- Build and use sinusoidal models for real data such as hours of daylight, tide heights and Ferris wheel rides.

WATCH

Simple Harmonic Not-Motion: Fluctuating Temperature

Eddie Woo · 17:23

<https://www.youtube.com/watch?v=m2sfAb28PSY>

WHEN TO WATCH

Before Example 2: builds $T = -5\cos(\pi t/12) + 18$ from a daily high and low, then solves it.

Note: Woo frames this as 'simple harmonic' behavior and writes n for b ; ignore that label. Temperatures are in $^{\circ}\text{C}$.

BACKUP VIDEO

The Ferris Wheel - Trigonometric Function Model (1 of 3: Setting up the equation)

Eddie Woo · 9:59

<https://www.youtube.com/watch?v=Cu1jg7wLOKk>

Before the practice set: a Ferris wheel model built from radius, period and lowest point.

KEY TERMS

sinusoidal model

An equation $y = a \sin(b(x - c)) + d$ or $y = a \cos(b(x - c)) + d$ chosen to describe a quantity that repeats regularly.

half-period

The horizontal distance from a maximum to the next minimum (or minimum to the next maximum). It is half of the period.

quarter-period

The horizontal distance from a maximum or minimum to the next midline crossing; one fourth of the period.

midline value

The average of the maximum and minimum values, $d = \frac{M + m}{2}$.

CORE IDEAS

Writing an equation is graphing in reverse. A sinusoid is completely determined by four facts: how high its maximum is, how low its minimum is, how long a cycle takes, and where one particular feature (a maximum, say) sits. Let M be the maximum value and m the minimum value. The midline lies halfway between them, so $d = \frac{M + m}{2}$. The amplitude is half the distance between them, so $|a| = \frac{M - m}{2}$. For example, with $M = 15$ and $m = 9$ we get $d = 12$ and $|a| = 3$.

Next find the period P . The distance between two consecutive maxima is one period; the distance from a maximum to the next minimum is half a period. Then $b = \frac{2\pi}{P}$.

Finally choose the phase shift by matching the start of the basic pattern to a feature of the data. **Cosine** starts its cycle at a maximum, so for $y = a \cos(b(x - c)) + d$ with $a > 0$, take $c =$ the x -value of any maximum. (With $a < 0$, take c at a minimum.) **Sine** starts its cycle on the midline going up, so for $y = a \sin(b(x - c)) + d$ with $a > 0$, take $c =$ the x -value of a rising midline crossing, which is one quarter-period before a maximum.

A sinusoid has many correct equations. Any maximum can serve as c , so adding a whole period to c changes nothing. A cosine model and a sine model can describe the same curve, and so can a model with $a < 0$ started at a minimum. That is not a flaw; it is the periodicity at work. What must agree is the graph, so always **check** your equation: substitute the x -value of the maximum and of the minimum and confirm you get M and m .

Real-world data come with units and a time origin, so state them clearly: " $t =$ days after January 1" or " $t =$ hours after midnight." The period comes from the situation (365 days for the seasons, about 12.4 hours between high tides on many coasts, the time for one turn of a wheel). Once the model is built, it answers questions: plug in a time to predict a value, or set the model equal to a value and solve using the exact values from Unit 3.

KEY FORMULAS

$$d = \frac{M + m}{2}$$

$$|a| = \frac{M - m}{2}$$

$$P = 2 \times (\text{distance from a maximum to the next minimum}), \quad b = \frac{2\pi}{P}$$

cosine model ($a > 0$) : $c = x\text{-value of a maximum}$

sine model ($a > 0$) : $c = (x\text{-value of a maximum}) - \frac{P}{4}$

From history. Predicting tides was a matter of life and death for sailors entering harbors. In the 1870s William Thomson, later Lord Kelvin, designed a tide-predicting machine: a set of gears and pulleys that added together many sinusoids, each with its own amplitude and period, and drew a harbor's tide curve for a whole year ahead. Machines of this kind were used to print tide tables well into the twentieth century.

WORKED EXAMPLES

Example 1. A sinusoid has a maximum at $\left(\frac{\pi}{4}, 5\right)$, and the next minimum is at $\left(\frac{3\pi}{4}, -1\right)$. Write an equation of the form $y = a \cos(b(x - c)) + d$ with $a > 0$ and $b > 0$.

Show solution

1. Midline: $d = \frac{5 + (-1)}{2} = 2$. Amplitude: $a = \frac{5 - (-1)}{2} = 3$.
2. Half-period: $\frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$, so $P = \pi$ and $b = \frac{2\pi}{\pi} = 2$.
3. Cosine starts at a maximum, so $c = \frac{\pi}{4}$.
4. Equation: $y = 3 \cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 2$.
5. Check the minimum: at $x = \frac{3\pi}{4}$, $2\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \pi$, and $3 \cos \pi + 2 = -3 + 2 = -1$. Correct.
6. Note: a quarter-period before the maximum is $\frac{\pi}{4} - \frac{\pi}{4} = 0$, so the same graph is also $y = 3 \sin(2x) + 2$.

Answer: $y = 3 \cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 2$ (equivalently $y = 3 \sin(2x) + 2$)

Example 2. In Samuel's town the longest day of the year, day 172 (about June 21), has 15 hours of daylight, and the shortest day, half a year later, has 9 hours. Using a period of 365 days, write a cosine model $D(t)$ for the hours of daylight on day t of the year. Then find, to the nearest day, the two days on which the model gives exactly 13.5 hours of daylight.

Show solution

1. Midline: $d = \frac{15 + 9}{2} = 12$. Amplitude: $a = \frac{15 - 9}{2} = 3$.
2. Period 365 days, so $b = \frac{2\pi}{365} \approx 0.0172142$.
3. The maximum is on day 172, and cosine starts at a maximum, so $c = 172$.
4. Model: $D(t) = 3 \cos \left(\frac{2\pi}{365}(t - 172) \right) + 12$. Check: $D(172) = 3 + 12 = 15$, and half a year later the cosine is -1 , giving 9.
5. Set $D(t) = 13.5$: $3 \cos \left(\frac{2\pi}{365}(t - 172) \right) = 1.5$, so $\cos \left(\frac{2\pi}{365}(t - 172) \right) = \frac{1}{2}$.
6. Within one year around day 172 this means $\frac{2\pi}{365}(t - 172) = \pm \frac{\pi}{3}$. Solve: $t - 172 = \pm \frac{365}{6} \approx \pm 60.833$.
7. So $t \approx 172 - 60.833 = 111.167$ and $t \approx 172 + 60.833 = 232.833$. To the nearest day: day 111 (about April 21) and day 233 (about August 21).

Answer: $D(t) = 3 \cos \left(\frac{2\pi}{365}(t - 172) \right) + 12$; 13.5 hours of daylight on about day 111 and day 233

PRACTICE

1. **Easy** A sinusoid has maximum value 10 and minimum value 2. Find its amplitude and the equation of its midline.

Show answer

$$\text{Amplitude } \frac{10 - 2}{2} = 4; \text{ midline } y = \frac{10 + 2}{2} = 6$$

2. **Easy** A sinusoid has consecutive maxima at $x = 1$ and $x = 9$. Find its period and the value of $b > 0$ (exact).

Show answer

$$\text{Period } 8; b = \frac{2\pi}{8} = \frac{\pi}{4}$$

3. **Easy** A graph of the form $y = a \cos(bx)$ has a maximum at $(0, 3)$, its next maximum at $(\pi, 3)$, and minimum value -3 . Write its equation with $a > 0, b > 0$.

Show answer

$$a = 3, \text{ period } \pi, b = 2: y = 3 \cos(2x)$$

4. **Medium** A sinusoid has a maximum at $(2, 8)$, and the next minimum is at $(5, 2)$. Write a cosine equation $y = a \cos(b(x - c)) + d$ with $a > 0, b > 0$.

Show answer

$$d = 5, a = 3, \text{ half-period } 3 \text{ so } P = 6 \text{ and } b = \frac{\pi}{3}, c = 2: y = 3 \cos\left(\frac{\pi}{3}(x - 2)\right) + 5$$

5. **Medium** Write a sine equation $y = a \sin(b(x - c)) + d$ with $a > 0, b > 0$ for a sinusoid with amplitude 2, period π and midline $y = -1$ that crosses its midline going up at $x = \frac{\pi}{6}$.

Show answer

$$b = 2: y = 2 \sin\left(2\left(x - \frac{\pi}{6}\right)\right) - 1$$

6. **Medium** Rewrite your answer to Problem 4 as a sine model $y = a \sin(b(x - c)) + d$ with $a > 0, b > 0$, and c between 0 and 6.

Show answer

The rising midline crossing is a quarter-period (1.5) before the maximum at 2, so $c = 0.5$: $y = 3 \sin\left(\frac{\pi}{3}(x - 0.5)\right) + 5$

7. **Medium** At a harbor, high tide of 2.6 m occurs at 4:00 a.m., and the next low tide of 0.4 m occurs at 10:12 a.m. Let t be hours after midnight. Write a cosine model $h(t) = a \cos(b(t - c)) + d$ for the water height, with $a > 0$ and $b > 0$. Give b exactly and to six decimal places.

Show answer

$$d = 1.5, a = 1.1; \text{ half-period } 6.2 \text{ h so } P = 12.4 \text{ h and } b = \frac{2\pi}{12.4} = \frac{\pi}{6.2} \approx 0.506708$$

$$; c = 4. h(t) = 1.1 \cos\left(\frac{\pi}{6.2}(t - 4)\right) + 1.5$$

8. **Hard** Using the tide model from Problem 7, at what time after the 4:00 a.m. high tide does the water first fall to 2.05 m? Give the time to the nearest minute.

Show answer

$$1.1 \cos\left(\frac{\pi}{6.2}(t-4)\right) + 1.5 = 2.05 \text{ gives } \cos(\dots) = \frac{1}{2}, \text{ so } \frac{\pi}{6.2}(t-4) = \frac{\pi}{3} \text{ and } t - 4 = \frac{6.2}{3} \approx 2.0667 \text{ h. } t \approx 6.0667 \text{ h, which is about 6:04 a.m.}$$

9. **Hard** A Ferris wheel 50 feet in diameter has its lowest point 5 feet above the ground and turns once every 2 minutes. Ruth boards at the bottom at $t = 0$. Write a model $h(t) = a \cos(bt) + d$ for her height in feet after t minutes, and find her height after 20 seconds.

Show answer

$$\text{Center height } d = 5 + 25 = 30; \text{ she starts at the bottom, so } a = -25; b = \frac{2\pi}{2} = \pi. h(t) = -25 \cos(\pi t) + 30. \text{ After 20 s, } t = \frac{1}{3}: h = -25 \cos \frac{\pi}{3} + 30 = -12.5 + 30 = 17.5 \text{ ft}$$

10. **Hard** On Mr. Miller's farm the average daily high temperature peaks at 88°F in July (month $t = 7$) and is lowest, 40°F, in January ($t = 1$). Write a cosine model $T(t)$ with a period of 12 months, and use it to predict the average high in May ($t = 5$).

Show answer

$$d = 64, a = 24, b = \frac{2\pi}{12} = \frac{\pi}{6}, c = 7: T(t) = 24 \cos\left(\frac{\pi}{6}(t-7)\right) + 64. T(5) = 24 \cos\left(-\frac{\pi}{3}\right) + 64 = 12 + 64 = 76^\circ\text{F}$$

11. **Challenge** Two students model the same data from Worked Example 1. Hannah writes $y = 3 \cos \left(2 \left(x - \frac{\pi}{4} \right) \right) + 2$ and Peter writes $y = -3 \cos \left(2 \left(x - \frac{3\pi}{4} \right) \right) + 2$. Is Peter's equation also correct? Explain why sinusoidal equations are not unique.

Show answer

Yes. Peter used $a < 0$, so his cycle starts at a minimum, and he placed it at the minimum $x = \frac{3\pi}{4}$. His graph has the same midline, amplitude, period and a minimum at $\left(\frac{3\pi}{4}, -1 \right)$, so it is the same curve; check $x = \frac{\pi}{4}$: $-3 \cos(-\pi) + 2 = 5$. Equations are not unique because the graph repeats: any maximum (or minimum, with $a < 0$) can serve as the starting point, and sine and cosine are shifts of each other.

12. **Challenge** A sinusoid has a minimum at $(-1, -7)$, and the next maximum is at $(3, 1)$. Write a sine equation $y = a \sin(b(x - c)) + d$ with $a > 0$, $b > 0$ and the smallest possible $c \geq 0$.

Show answer

$d = -3$, $a = 4$, half-period 4 so $P = 8$ and $b = \frac{\pi}{4}$. The rising midline crossing is a quarter-period (2) after the minimum: $c = 1$. $y = 4 \sin \left(\frac{\pi}{4}(x - 1) \right) - 3$

COMMON MISTAKES

- Using the max-to-min distance as the whole period. From a maximum to the next minimum is only **half** a period; double it.
- Taking the amplitude as $M - m$ or the midline as $\frac{M - m}{2}$. The amplitude is **half** the difference, and the midline is the **average** $\frac{M + m}{2}$.
- Using a maximum as the starting point of a sine model. Sine starts on the midline going up, a quarter-period **before** a maximum. Cosine (with $a > 0$) starts at the maximum.
- Not checking. Substitute the x -values of the given maximum and minimum into your equation; if they do not return M and m , find the error before going on.

Make it a game • Two Clues, One Wave. Parent writes a maximum point and the next minimum point on a slip, such as $(2, 9)$ and $(8, 1)$. In 90 seconds, write a cosine model, then an equivalent sine model. Test both by plugging in the points. Swap roles with your own clues. Most correct tests wins.

4.6 THE GRAPH OF TANGENT

OBJECTIVES

- Graph $y = \tan x$, explaining its vertical asymptotes, zeros, period π and odd symmetry from the unit circle.
- Find the period, asymptotes, zeros and key points of $y = a \tan(b(x - c)) + d$.
- Use the tangent graph to solve problems involving slopes and distances, such as a lighthouse beam sweeping a shore.

WATCH

Understanding the graph of $\tan(x)$ (1 of 2: Thinking Visually)

Eddie Woo · 12:26

<https://www.youtube.com/watch?v=LYKZJbbpir8>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo may label the x-axis in degrees; this course uses radians.

BACKUP VIDEO

How to Sketch $y = \tan(x)$ Quickly & Accurately

Eddie Woo · 4:48

<https://www.youtube.com/watch?v=yUaiUPeoTaM>

Before Example 2, for a quick sketching method.

KEY TERMS

vertical asymptote

A vertical line $x = k$ that a graph approaches but never touches, with $|y|$ growing without bound as x nears k .

tangent line of the unit circle

The vertical line $x = 1$, which touches the unit circle only at $(1, 0)$; the value $\tan \theta$ is the height at which the terminal side (extended) meets this line.

branch

One connected piece of the tangent graph, lying between two consecutive asymptotes.

slope of the terminal side

For an angle θ in standard position, the slope of its terminal side is $\frac{\sin \theta}{\cos \theta} = \tan \theta$.

CORE IDEAS

Where the name comes from. Draw the vertical line $x = 1$, which touches the unit circle at $(1, 0)$. Extend the terminal side of an acute angle θ until it meets this line at T . The triangle with vertices O , $(1, 0)$, T is a right triangle (a tangent line is perpendicular to the radius), and it is similar to the triangle with vertices O , $(\cos \theta, 0)$, $(\cos \theta, \sin \theta)$. Matching sides, the height of T is $\frac{\sin \theta}{\cos \theta} = \tan \theta$. So the tangent is the length cut off on the *tangent line* — hence its name — and it is also the **slope** of the terminal side.

Asymptotes and zeros. Since $\tan x = \frac{\sin x}{\cos x}$, it is undefined wherever $\cos x = 0$, that is, at $x = \frac{\pi}{2} + n\pi$. As x climbs toward $\frac{\pi}{2}$, the terminal side becomes nearly vertical: its slope grows without bound, and the point T runs off up the tangent line. Just past $\frac{\pi}{2}$, the slope is hugely negative. So the graph has **vertical asymptotes** at $x = \frac{\pi}{2} + n\pi$. The zeros are where $\sin x = 0$: $x = n\pi$. The range is all real numbers, since every slope occurs.

Period π , not 2π . Adding π moves the point on the unit circle to the opposite point, $(-\cos x, -\sin x)$. Then $\tan(x + \pi) = \frac{-\sin x}{-\cos x} = \tan x$: the terminal side points the other way along the **same line**, so it has the same slope. No smaller shift works, since $\tan x = 0$ only at multiples of π . Also $\tan(-x) = \frac{-\sin x}{\cos x} = -\tan x$, so tangent is **odd**. One branch on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ passes through the key points $\left(-\frac{\pi}{4}, -1\right)$, $(0, 0)$, $\left(\frac{\pi}{4}, 1\right)$, always increasing; copies of it repeat every π .

Transformations. For $y = a \tan(b(x - c)) + d$ there is no amplitude, since the graph has no maximum. Instead: the period is $\frac{\pi}{|b|}$; the center of a branch is the point (c, d) ; and the asymptotes of that branch are where $b(x - c) = \pm \frac{\pi}{2}$, that is, at $x = c \pm \frac{\pi}{2|b|}$, half a period on either side of the center. The halfway points between center and asymptotes have heights $d - a$ and $d + a$ (left and right). If $a < 0$, each branch falls instead of rising.

KEY FORMULAS

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan(x + \pi) = \tan x, \quad \tan(-x) = -\tan x$$

$$\text{asymptotes of } y = \tan x : x = \frac{\pi}{2} + n\pi$$

$$y = a \tan(b(x - c)) + d : \text{period } \frac{\pi}{|b|}$$

$$\text{asymptotes beside the center: } x = c \pm \frac{\pi}{2|b|}$$

From history. Long before the name existed, astronomers tabulated tangent and cotangent values as shadow lengths: a stick of fixed length casts a shadow whose length depends on the sun's altitude. Al-Biruni (around the year 1000) wrote a whole treatise on shadows. The modern name *tangent*, from the Latin *tangere*, "to touch," was introduced by Thomas Fincke in 1583, after the touching line in the figure above.

Euclid connection. Euclid III.18 proves that a line touching a circle is perpendicular to the radius drawn to the point of contact; that is why the triangle with legs 1 and $\tan \theta$ has a right angle. The comparison with the triangle of legs $\cos \theta$ and $\sin \theta$ is the similar-triangle proportion of Euclid VI.4.

WORKED EXAMPLES

Example 1. For $y = \tan(2x)$, find the period, and list the vertical asymptotes and the zeros on the interval $[0, \pi]$. Give exact values in radians.

Show solution

1. Period: $\frac{\pi}{|b|} = \frac{\pi}{2}$.

2. Asymptotes occur where the inside is an odd multiple of $\frac{\pi}{2}$: $2x = \frac{\pi}{2} + n\pi$, so $x = \frac{\pi}{4} + \frac{n\pi}{2}$. In $[0, \pi]$: $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

3. Zeros occur where the inside is a multiple of π : $2x = n\pi$, so $x = \frac{n\pi}{2}$. In $[0, \pi]$: $x = 0, \frac{\pi}{2}, \pi$.

4. Check: zeros and asymptotes alternate, half a period ($\frac{\pi}{4}$) apart, as they should.
A full branch runs from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, centered at the zero $\frac{\pi}{2}$.

Answer: Period $\frac{\pi}{2}$; asymptotes $x = \frac{\pi}{4}, \frac{3\pi}{4}$; zeros $x = 0, \frac{\pi}{2}, \pi$

Example 2. Graph one branch of $y = 2 \tan \left(\frac{1}{2} \left(x - \frac{\pi}{2} \right) \right) + 1$. Give the period, the asymptotes beside the center, the center point, and the two halfway key points.

Show solution

1. Read $a = 2$, $b = \frac{1}{2}$, $c = \frac{\pi}{2}$, $d = 1$.
2. Period: $\frac{\pi}{1/2} = 2\pi$.
3. Center of the branch: $(c, d) = \left(\frac{\pi}{2}, 1 \right)$.
4. Asymptotes: $\frac{1}{2} \left(x - \frac{\pi}{2} \right) = \pm \frac{\pi}{2}$ gives $x - \frac{\pi}{2} = \pm \pi$, so $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
(These are half a period, π , on each side of the center.)
5. Halfway points: $\frac{1}{2} \left(x - \frac{\pi}{2} \right) = \pm \frac{\pi}{4}$ gives $x = 0$ and $x = \pi$. At $x = 0$:
 $2 \tan \left(-\frac{\pi}{4} \right) + 1 = -2 + 1 = -1$. At $x = \pi$: $2 \tan \frac{\pi}{4} + 1 = 3$.
6. Sketch: plot $(0, -1)$, $\left(\frac{\pi}{2}, 1 \right)$, $(\pi, 3)$, draw dashed asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, and draw a rising branch through the points that bends toward the asymptotes.

Answer: Period 2π ; asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$; center $\left(\frac{\pi}{2}, 1 \right)$; key points $(0, -1)$ and $(\pi, 3)$

PRACTICE

1. Easy What is the period of $y = \tan x$? Give the exact value in radians.

Show answer

$$\pi$$

2. Easy List the vertical asymptotes of $y = \tan x$ on the interval $[0, 2\pi]$.

Show answer

$$x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2}$$

3. Easy Tangent is an odd function. Use this to find the exact value of $\tan \left(-\frac{\pi}{3} \right)$.

Show answer

$$-\tan \frac{\pi}{3} = -\sqrt{3} \approx -1.732$$

4. **Medium** For $y = \tan(3x)$, find the period and the two asymptotes nearest the origin.

Show answer

Period $\frac{\pi}{3}$; asymptotes $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$

5. **Medium** For $y = \tan\left(\frac{x}{4}\right)$, find the period and the two asymptotes nearest the origin.

Show answer

Period $\frac{\pi}{1/4} = 4\pi$; asymptotes $x = -2\pi$ and $x = 2\pi$

6. **Medium** For $y = \tan\left(x - \frac{\pi}{4}\right)$, find the zero of the branch centered nearest the origin and the two asymptotes beside it.

Show answer

Zero $x = \frac{\pi}{4}$; asymptotes $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

7. **Medium** A lighthouse stands 2 km from a straight shoreline. Its beam makes an angle θ with the line from the lighthouse to the nearest point P of the shore, and the lit spot on the shore is $d = 2 \tan \theta$ km from P . How far from P is the spot when $\theta = 60^\circ$? Give the exact value and the value to the nearest hundredth of a kilometer.

Show answer

$$d = 2 \tan 60^\circ = 2\sqrt{3} \approx 3.46 \text{ km}$$

8. **Hard** For the lighthouse in Problem 7, with $-90^\circ < \theta < 90^\circ$, for which angles is the lit spot within 2 km of P ? Explain using the tangent graph.

Show answer

Need $|2 \tan \theta| \leq 2$, i.e. $-1 \leq \tan \theta \leq 1$. Tangent is increasing on $(-90^\circ, 90^\circ)$ and equals ± 1 at $\pm 45^\circ$, so $-45^\circ \leq \theta \leq 45^\circ$.

9. **Hard** Find every solution of $\tan x = \sqrt{3}$ on $[0, 2\pi)$. Explain why the two solutions are exactly π apart.

Show answer

$x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$. Tangent has period π , and each branch takes every value exactly once, so solutions repeat every π .

10. **Hard** A carpenter describes roof pitch as rise per 12 inches of run; a roof making angle θ with the horizontal rises $\tan \theta$ inches per inch of run. Find the angle of an 8-in-12 roof to the nearest tenth of a degree, and explain from the tangent graph why there is exactly one such acute angle.

Show answer

$\tan \theta = \frac{8}{12} = \frac{2}{3}$, so $\theta = \tan^{-1} \frac{2}{3} \approx 33.7^\circ$. On $(0^\circ, 90^\circ)$ tangent increases steadily from 0 toward infinity, so it takes the value $\frac{2}{3}$ exactly once.

11. **Challenge** Prove that $\tan(x + \pi) = \tan x$, and explain why the period of tangent is π even though sine and cosine have period 2π .

Show answer

Adding π sends the unit-circle point $(\cos x, \sin x)$ to the opposite point

$(-\cos x, -\sin x)$, so $\tan(x + \pi) = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$. Sine and cosine each change sign after half a turn, but the signs cancel in the quotient.

Geometrically, the terminal sides of x and $x + \pi$ lie on the same line through the origin, so they have the same slope. No smaller positive shift works because $\tan x = 0$ only at multiples of π , so the period is exactly π .

12. **Challenge** For $y = -\tan\left(2x + \frac{\pi}{2}\right)$, find the period, the phase shift, and the asymptotes of the branch centered at the phase shift. Does each branch rise or fall?

Show answer

$2x + \frac{\pi}{2} = 2\left(x + \frac{\pi}{4}\right)$: period $\frac{\pi}{2}$, phase shift $\frac{\pi}{4}$ left ($c = -\frac{\pi}{4}$), asymptotes at $x = -\frac{\pi}{4} \pm \frac{\pi}{4}$, that is $x = -\frac{\pi}{2}$ and $x = 0$. Since $a = -1 < 0$, each branch falls.

COMMON MISTAKES

- Using 2π for the period of tangent. The period of $y = \tan(bx)$ is $\frac{\pi}{|b|}$, not $\frac{2\pi}{|b|}$.
- Drawing the graph crossing its asymptotes, or joining two branches across an asymptote. Draw the asymptotes first as dashed lines; each branch stays between its own pair.
- Talking about the "amplitude" of tangent. The tangent graph has no maximum or minimum, so it has no amplitude; the number a is a vertical stretch that sets the heights of the halfway points.

- Putting the asymptotes at the zeros of sine. $\tan x = \frac{\sin x}{\cos x}$ is undefined where the denominator $\cos x$ is 0, at $\frac{\pi}{2} + n\pi$; where $\sin x = 0$ the tangent is 0.

Make it a game · Slope Stick. Pin a pencil by its eraser to a coin at the origin of a sheet of paper. Parent turns the pencil to angles. At each, call the tangent: zero, small positive, large, or undefined (vertical). Explain what the asymptote means. Then predict tan at 45, 90 and 135 degrees before rotating.

4.7 GRAPHS OF COSECANT, SECANT AND COTANGENT

OBJECTIVES

- Sketch $y = \csc x$, $y = \sec x$ and $y = \cot x$ from the graphs of sine, cosine and tangent using the reciprocal rule.
- State the domain, range, period and asymptotes of each reciprocal function.
- Graph transformed reciprocal functions such as $y = a \sec(b(x - c)) + d$ by first drawing the guide sinusoid.

WATCH

Graphing Reciprocal Trigonometric Functions (1 of 2: Cosecant)

Eddie Woo · 3:05

<https://www.youtube.com/watch?v=tjF5CKQe4is>

WHEN TO WATCH

After reading the Core ideas, before Example 1 (short: cosecant from sine).

Note: Woo says 'cosec'. Cotangent is in his follow-up video wkWn3JPcLSE (not assigned).

BACKUP VIDEO

How to Graph Cosecant and Secant (Precalculus - Trigonometry 15)

Professor Leonard · 29:55

https://www.youtube.com/watch?v=kYIKNTXaf_E

Before Example 2: graphing a transformed secant from its guide curve.

KEY TERMS

reciprocal function

The function $\frac{1}{f(x)}$, defined wherever $f(x) \neq 0$. Cosecant, secant and cotangent are the reciprocals of sine, cosine and tangent.

guide graph

The sine or cosine curve drawn lightly first, from which the cosecant or secant graph is sketched.

U-shaped branch

One piece of a cosecant or secant graph: it touches the guide graph at a peak or trough and rises (or falls) toward two asymptotes.

cotangent

$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$, defined where $\sin x \neq 0$.

CORE IDEAS

The reciprocal rule. To graph $y = \frac{1}{f(x)}$ from $y = f(x)$, use four facts. (1) Where $f(x) = \pm 1$, the reciprocal is the same number, so the graphs touch. (2) Where $f(x)$ is near 0, $\frac{1}{f(x)}$ is huge, and where $f(x) = 0$ the reciprocal is undefined: a vertical asymptote. (3) The reciprocal has the same sign as $f(x)$. (4) When $|f(x)|$ grows, $\left|\frac{1}{f(x)}\right|$ shrinks, and the reverse.

Cosecant. Draw $y = \sin x$ lightly. Its zeros $x = n\pi$ become the asymptotes of $y = \csc x$. At each peak $\left(\frac{\pi}{2} + 2n\pi, 1\right)$ the cosecant graph touches the sine graph and opens **upward**, rising toward the asymptotes on either side; at each trough $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$ it touches and opens **downward**. Because $|\sin x| \leq 1$, we get $|\csc x| \geq 1$: the range is $(-\infty, -1] \cup [1, \infty)$. The period is 2π , and cosecant is odd, like sine.

Secant. In the same way, $y = \sec x$ is built on $y = \cos x$. The asymptotes are at the zeros of cosine, $x = \frac{\pi}{2} + n\pi$; upward branches touch the cosine peaks at $x = 2n\pi$ and downward branches touch the troughs at $x = \pi + 2n\pi$. The range is again $(-\infty, -1] \cup [1, \infty)$, the period 2π , and secant is even. In the tangent-line figure of Lesson 4.6, $\sec \theta$ is the length of the segment from the origin to T on the line $x = 1$: by similar triangles it is $\frac{1}{\cos \theta}$. That segment **cuts** the circle, which is the origin of the name.

Cotangent. Since $\cot x = \frac{\cos x}{\sin x}$, its asymptotes are at the zeros of sine, $x = n\pi$, and its zeros are at the zeros of cosine, $x = \frac{\pi}{2} + n\pi$. Like tangent it has period π and range all real numbers, but each branch **falls** from left to right: as $\tan x$ rises through 1, $\cot x$ falls through 1. The key points of the branch on $(0, \pi)$ are $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{4}, -1\right)$.

Transformations. To graph $y = a \csc(b(x - c)) + d$ or $y = a \sec(b(x - c)) + d$, first draw the guide graph $y = a \sin(b(x - c)) + d$ (or the cosine version) exactly as in Lesson 4.4. Draw asymptotes where the guide crosses its midline $y = d$. Then draw U-shaped branches that touch the guide at each maximum and minimum and open away from the midline. The branches never enter the strip between $y = d - |a|$ and $y = d + |a|$.

KEY FORMULAS

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

$\csc x$ and $\cot x$: asymptotes at $x = n\pi$

$\sec x$: asymptotes at $x = \frac{\pi}{2} + n\pi$

$$|\csc x| \geq 1, \quad |\sec x| \geq 1$$

periods: $\csc, \sec : 2\pi$; $\cot : \pi$

From history. The name *secant* comes from the Latin *secare*, "to cut," because the segment it measures cuts through the circle; Thomas Fincke introduced both *tangent* and *secant* as names in 1583. The "co-" functions are the functions of the complement: the cotangent of an angle is the tangent of its complement, and the cosecant is the secant of its complement.

Euclid connection. In the tangent-line figure, the right triangle with legs 1 and $\tan \theta$ has hypotenuse $\sec \theta$. The Pythagorean theorem (Euclid I.47) then gives $\sec^2 \theta = 1 + \tan^2 \theta$, an identity we will prove carefully later in the course.

WORKED EXAMPLES

Example 1. Sketch $y = \csc x$ on $(0, 2\pi)$. Give the asymptote inside the interval, the lowest point of the upward branch and the highest point of the downward branch.

Show solution

1. Draw $y = \sin x$ lightly on $[0, 2\pi]$. Its zeros are $0, \pi, 2\pi$, so $y = \csc x$ has asymptotes there; the one inside the open interval is $x = \pi$.
2. On $(0, \pi)$, $\sin x > 0$ with peak $\left(\frac{\pi}{2}, 1\right)$. The cosecant is positive, touches the sine curve at $\left(\frac{\pi}{2}, 1\right)$, and climbs toward the asymptotes $x = 0$ and $x = \pi$. So $\left(\frac{\pi}{2}, 1\right)$ is the lowest point of this upward branch.
3. On $(\pi, 2\pi)$, $\sin x < 0$ with trough $\left(\frac{3\pi}{2}, -1\right)$. The cosecant is negative, touches there and falls toward the asymptotes. So $\left(\frac{3\pi}{2}, -1\right)$ is the highest point of this downward branch.
4. Check a value: $\csc \frac{\pi}{6} = \frac{1}{1/2} = 2$, above the sine value $\frac{1}{2}$, as the reciprocal rule predicts.

Answer: Asymptote $x = \pi$ (also $x = 0$ and $x = 2\pi$ at the ends); upward branch lowest point $\left(\frac{\pi}{2}, 1\right)$; downward branch highest point $\left(\frac{3\pi}{2}, -1\right)$

Example 2. Graph $y = 2 \sec(2x) + 1$ on $[0, \pi]$. Find the period, the asymptotes in the interval, the lowest value of the upward branches, the highest value of the downward branch, and the range.

Show solution

1. Guide graph: $y = 2 \cos(2x) + 1$, with amplitude 2, period $\frac{2\pi}{2} = \pi$ and midline $y = 1$. It runs from -1 to 3 .
2. The secant graph has the same period, π .
3. Asymptotes where $\cos(2x) = 0$ (where the guide crosses its midline): $2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, so $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.
4. Guide maxima: at $x = 0$ and $x = \pi$, $2(1) + 1 = 3$. Upward branches touch there, so their lowest value is 3 .
5. Guide minimum: at $x = \frac{\pi}{2}$, $2(-1) + 1 = -1$. The downward branch touches there, so its highest value is -1 .
6. Range: the branches never enter the strip between -1 and 3 , so the range is $(-\infty, -1] \cup [3, \infty)$.

Answer: Period π ; asymptotes $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$; upward branches bottom out at $y = 3$ (at $x = 0, \pi$); downward branch tops out at $y = -1$ (at $x = \frac{\pi}{2}$); range $(-\infty, -1] \cup [3, \infty)$

PRACTICE

1. Easy Find the exact value of $\csc \frac{\pi}{6}$.

Show answer

$$\frac{1}{\sin(\pi/6)} = \frac{1}{1/2} = 2$$

2. Easy List the vertical asymptotes of $y = \sec x$ on the interval $[0, 2\pi]$.

Show answer

$$x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2}$$

3. Easy State the range of $y = \csc x$.

Show answer

$$(-\infty, -1] \cup [1, \infty)$$

4. Medium On the interval $[0, 2\pi]$, list the zeros and the vertical asymptotes of $y = \cot x$.

Show answer

$$\text{Zeros } x = \frac{\pi}{2}, \frac{3\pi}{2}; \text{ asymptotes } x = 0, \pi, 2\pi$$

5. Medium Find the period of $y = \sec(3x)$ and the period of $y = \cot(3x)$.

Show answer

$$\sec(3x): \frac{2\pi}{3}; \cot(3x): \frac{\pi}{3}$$

6. Medium For $y = 3 \csc x$, give the lowest value of the upward branches, the highest value of the downward branches, and the range.

Show answer

$$\text{Upward branches bottom out at } y = 3 \text{ (at } x = \frac{\pi}{2} + 2n\pi\text{); downward branches top out at } y = -3 \text{ (at } x = \frac{3\pi}{2} + 2n\pi\text{); range } (-\infty, -3] \cup [3, \infty)$$

7. Medium A lighthouse stands 2 km from a straight shoreline. When its beam makes angle θ with the line to the nearest point of shore, the beam travels $L = 2 \sec \theta$ km to reach the shore. Find L when $\theta = 60^\circ$.

Show answer

$$L = 2 \sec 60^\circ = \frac{2}{1/2} = 4 \text{ km}$$

8. Hard For $y = \csc\left(x - \frac{\pi}{4}\right) + 2$ on $[0, 2\pi]$, find the asymptotes, the lowest point of the upward branch, and the highest point of the downward branch.

Show answer

$$\text{Asymptotes where } \sin\left(x - \frac{\pi}{4}\right) = 0: x = \frac{\pi}{4}, \frac{5\pi}{4}. \text{ Upward branch lowest point } \left(\frac{3\pi}{4}, 3\right); \text{ downward branch highest point } \left(\frac{7\pi}{4}, 1\right).$$

9. **Hard** For the lighthouse in Problem 7, with $-90^\circ < \theta < 90^\circ$, for which angles does the beam travel at most 4 km to reach the shore? Explain using the secant graph.

Show answer

Need $2 \sec \theta \leq 4$, so $\sec \theta \leq 2$, which means $\cos \theta \geq \frac{1}{2}$ (cosine is positive here).

The secant branch on $(-90^\circ, 90^\circ)$ has its lowest point 1 at 0° and equals 2 at $\pm 60^\circ$, so $-60^\circ \leq \theta \leq 60^\circ$.

10. **Hard** A vertical fence post 10 feet tall casts a shadow of length $s = 10 \cot \alpha$ feet on level ground when the sun's altitude is α . Find the shadow length when $\alpha = 30^\circ$, to the nearest tenth of a foot, and use the cotangent graph to describe what happens to the shadow as α increases from near 0° to 90° .

Show answer

$s = 10 \cot 30^\circ = 10\sqrt{3} \approx 17.3$ ft. Cotangent falls steadily on $(0^\circ, 90^\circ)$: near sunrise the shadow is extremely long (the asymptote at 0°), and it shrinks to 0 when the sun is straight overhead ($\cot 90^\circ = 0$).

11. **Challenge** Explain why $y = \sec x$ never takes a value strictly between -1 and 1 .

Show answer

For every x in its domain, $0 < |\cos x| \leq 1$. Taking reciprocals of positive numbers reverses the inequality, so $|\sec x| = \frac{1}{|\cos x|} \geq 1$. So $\sec x \geq 1$ or $\sec x \leq -1$, never in between. On the graph, the secant branches only touch the cosine curve at $y = \pm 1$ and then move away from the x -axis.

12. **Challenge** Find every solution of $\csc x = -2$ on $[0, 2\pi)$. Give exact values in radians.

Show answer

$$\sin x = -\frac{1}{2}, \text{ so } x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

COMMON MISTAKES

- Putting the asymptotes at the peaks of the guide graph. Asymptotes of $\csc$ and $\sec$ go where the guide sine or cosine is **zero** (crosses its midline); the peaks and troughs are where the branches **touch**.
- Drawing the reciprocal branch on the wrong side. A branch opens away from the midline: above a peak it opens up, below a trough it opens down. Check the sign:

$\csc x$ is negative wherever $\sin x$ is.

- Treating $\cot x$ as $\tan^{-1} x$. The cotangent is the **reciprocal** $\frac{1}{\tan x}$, not the inverse function; on a calculator compute $\cot x$ as $1 \div \tan x$ (or $\cos x \div \sin x$ when $\tan x$ is undefined).
- Giving cotangent period 2π or drawing its branches rising. Cotangent has period π , like tangent, but each branch falls from left to right.

Make it a game • Draw the Flip. Parent draws a sine curve lightly. In 2 minutes, sketch its cosecant right on top, putting asymptotes wherever the sine is zero and U-shaped branches touching the peaks. Count the asymptotes in one period, then state the domain and range. Repeat with a tangent graph to sketch the cotangent.

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

In Unit 3 the trigonometric functions became functions of any angle, defined by a point moving around the unit circle. In this unit we unroll that circle onto a number line and draw the six functions as graphs. Sine and cosine become smooth waves that repeat every 2π ; tangent and cotangent become branches separated by asymptotes that repeat every π ; cosecant and secant become U-shaped branches built on the sine and cosine waves. Every feature of every graph (its zeros, peaks, asymptotes, symmetry and period) is explained from the circle, not memorized.

We then learn to stretch, squeeze and slide the sine wave. The four numbers in $y = a \sin(b(x - c)) + d$ control the amplitude, period, phase shift and midline, and reading them quickly is the central skill of the unit. Working backward, we write equations for waves from a graph or from data about maxima and minima. This is how hours of daylight, tide heights, Ferris wheels, water wheels and vibrating strings are described mathematically.

These graphs prepare the way for what follows. Restricting the graphs to one branch gives the inverse functions of Unit 6, reading solutions off a graph prepares for trigonometric equations, and the sinusoidal models return in full in the Applications unit.

PREREQUISITES

- Convert between degrees and radians (Lessons 3.2–3.3).
- Define sine, cosine and tangent of any angle from the unit circle (Lesson 3.6).
- Use reference angles and quadrant signs (Lesson 3.7).
- Give exact values of all six functions at multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including quadrantal angles (Lesson 3.8).
- Recognize vertical and horizontal shifts and stretches of a graph, such as $y = f(x - h) + k$ (Algebra II).
- Solve simple linear equations and work with fractions involving π .

MASTERY CHECKLIST

- ☐ I can build the graph of $y = \sin x$ from the unit circle and state its domain, range, period, zeros, maxima, minima and odd symmetry.
- ☐ I can graph $y = \cos x$ and explain why cosine is even, and prove that the cosine graph is the sine graph shifted $\frac{\pi}{2}$ to the left.
- ☐ I can use periodicity and symmetry to find every solution of $\sin x = k$ or $\cos x = k$ on a given interval.
- ☐ I can find the amplitude $|a|$ and period $\frac{2\pi}{|b|}$ of $y = a \sin(bx)$ and explain why these formulas hold.
- ☐ I can factor the inside of $y = a \sin(b(x - d)) + k$ to find the phase shift, and find the amplitude, maximum and minimum of any sinusoid.
- ☐ I can graph one cycle of a shifted sinusoid from its five key points.
- ☐ I can write a sine or cosine equation for a wave from its graph

or from the location and value of a maximum and a minimum.

- ☐ I can build and use sinusoidal models for daylight, tides, wheels and other periodic situations.

- ☐ I can graph

$y = a \sin(b(x - c)) + d$, explain its asymptotes and its period

π , and find the period and asymptotes of

$$y = a \tan(b(x - c)) + d.$$

- ☐ I can sketch

$$y = \csc x \text{ or } y = \sec x$$

$$\text{and } y = \cot x$$

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LESSONS IN THIS UNIT

4.1	The Graph of Sine	50 minutes
4.2	The Graph of Cosine	45 minutes
4.3	Amplitude and Period	50 minutes
4.4	Phase Shift and Vertical Shift	2 sessions of 45 minutes
4.5	Writing Sinusoidal Equations	2 sessions of 45 minutes
4.6	The Graph of Tangent	50 minutes
4.7	Graphs of Cosecant, Secant and Cotangent	50 minutes

UNIT 4 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; a graphing calculator is not. Show your work, including sketches where they help. Give exact values when the problem asks for them; otherwise round final answers to the nearest hundredth.

Angles and x -values are in radians unless a problem says degrees. Total: 100 points.

1. (5 pts)

Find every solution of $\sin x = \frac{\sqrt{3}}{2}$ on $[0, 2\pi)$. Give exact values in radians.

2. (5 pts)

Use the symmetry and period of the cosine graph to find the exact value of $\cos\left(-\frac{11\pi}{6}\right)$.

3. (6 pts)

Find the amplitude and period of $y = -5 \cos\left(\frac{\pi}{4}x\right)$, and give the value of y at $x = 0$.

4. (6 pts)

Write an equation $y = a \sin(bx)$ with $a > 0$ and $b > 0$ that has amplitude 1.5 and period 4π .

5. (6 pts)

For $y = \tan\left(\frac{x}{2}\right)$, find the period and the two vertical asymptotes nearest the origin.

6. (6 pts)

Sketch $y = -2 \csc x$ on $(0, 2\pi)$. Give the asymptote inside the interval, the highest value of the downward branch and the lowest value of the upward branch.

7. (6 pts)

As Samuel rides his bicycle at a steady pace, the height of one pedal above the road is $h(t) = 6 \sin(2\pi(t - 0.25)) + 11$ inches, t seconds after he starts timing. Find the greatest height, the least height, and the time for one full turn of the pedal.

8. **8 pts**

For $y = 3 \sin \left(2x - \frac{\pi}{3} \right) + 4$, find the amplitude, period, phase shift (with direction), vertical shift, maximum and minimum.

9. **8 pts**

A sinusoid has a maximum at $(1, 9)$, and the next minimum is at $(4, 3)$. Write a cosine equation $y = a \cos(b(x - c)) + d$ with $a > 0$ and $b > 0$.

10. **8 pts**

At a harbor, high tide of 3.2 m occurs at 2:00 a.m., and the next low tide of 0.8 m occurs at 8:00 a.m. Let t be hours after midnight. (a) Write a cosine model $h(t) = a \cos(b(t - c)) + d$ with $a > 0$, $b > 0$. (b) At what time after the 2:00 a.m. high tide does the water first fall to 2.6 m?

11. **8 pts**

For $y = 3 \tan \left(x - \frac{\pi}{6} \right) - 1$, find the center point of the branch nearest the origin, the asymptotes on either side of it, and the two halfway key points.

12. **8 pts**

A lighthouse stands 3 km from a straight shoreline. When its beam makes angle θ with the line to the nearest shore point P , the lit spot is $d = 3 \tan \theta$ km from P and the beam's length is $L = 3 \sec \theta$ km. Find d and L when $\theta = 45^\circ$. Give exact values and values to the nearest hundredth of a kilometer.

13. **12 pts**

A mill's water wheel is 20 feet in diameter, and its axle is 8 feet above the water. It turns once every 30 seconds. A bucket is at the very top at $t = 0$. (a) Write a model $h(t) = a \cos(bt) + d$ for the bucket's height above the water, in feet, after t seconds. (b) How far below the water surface does the bucket go? (c) Find its height after 10 seconds.

14. **8 pts**

Prove that $\sin \left(x + \frac{\pi}{2} \right) = \cos x$ for every x , using the unit circle. Then explain what this identity says about the graphs of sine and cosine.

ANSWER KEY

1. $x = \frac{\pi}{3}, \frac{2\pi}{3}$
2. Cosine is even, so $\cos\left(-\frac{11\pi}{6}\right) = \cos\frac{11\pi}{6} = \cos\left(2\pi - \frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx 0.87$
3. Amplitude 5; period $\frac{2\pi}{\pi/4} = 8$; at $x = 0, y = -5$ (the graph starts at its minimum)
4. $b = \frac{2\pi}{4\pi} = \frac{1}{2}$; $y = 1.5 \sin\left(\frac{x}{2}\right)$
5. Period $\frac{\pi}{1/2} = 2\pi$; asymptotes $x = -\pi$ and $x = \pi$
6. Asymptote $x = \pi$. On $(0, \pi)$ the branch opens downward with highest point $\left(\frac{\pi}{2}, -2\right)$; on $(\pi, 2\pi)$ it opens upward with lowest point $\left(\frac{3\pi}{2}, 2\right)$.
7. Greatest height $11 + 6 = 17$ in; least height $11 - 6 = 5$ in; one turn takes $\frac{2\pi}{2\pi} = 1$ second
8. $2x - \frac{\pi}{3} = 2\left(x - \frac{\pi}{6}\right)$. Amplitude 3, period π , phase shift $\frac{\pi}{6}$ to the right, vertical shift 4 up, maximum 7, minimum 1
9. $d = 6, a = 3$, half-period 3 so period 6 and $b = \frac{\pi}{3}, c = 1$: $y = 3 \cos\left(\frac{\pi}{3}(x - 1)\right) + 6$
10. (a) $d = 2, a = 1.2$, period 12 h so $b = \frac{\pi}{6}, c = 2$: $h(t) = 1.2 \cos\left(\frac{\pi}{6}(t - 2)\right) + 2$. (b) $1.2 \cos\left(\frac{\pi}{6}(t - 2)\right) = 0.6$, so $\cos(\dots) = \frac{1}{2}$, $\frac{\pi}{6}(t - 2) = \frac{\pi}{3}, t = 4$: at 4:00 a.m.
11. Center $\left(\frac{\pi}{6}, -1\right)$; asymptotes $x = \frac{\pi}{6} \pm \frac{\pi}{2}$, that is $x = -\frac{\pi}{3}$ and $x = \frac{2\pi}{3}$; halfway points $\left(-\frac{\pi}{12}, -4\right)$ and $\left(\frac{5\pi}{12}, 2\right)$
12. $d = 3 \tan 45^\circ = 3$ km; $L = 3 \sec 45^\circ = 3\sqrt{2} \approx 4.24$ km
13. (a) $a = 10, d = 8, b = \frac{2\pi}{30} = \frac{\pi}{15}$: $h(t) = 10 \cos\left(\frac{\pi}{15}t\right) + 8$. (b) Minimum $8 - 10 = -2$: 2 feet below the surface. (c) $h(10) = 10 \cos\frac{2\pi}{3} + 8 = -5 + 8 = 3$ feet above the water.
14. Let $P = (\cos x, \sin x)$ be the point for angle x , and Q the point for $x + \frac{\pi}{2}$, a quarter turn further. A quarter turn counterclockwise about the origin sends every point (u, v) to $(-v, u)$: it turns the horizontal leg of the right triangle under P into a vertical leg of the same signed length, and the vertical leg into a horizontal leg pointing the opposite way. So $Q = (-\sin x, \cos x)$ in every quadrant. The y -coordinate of Q is $\sin\left(x + \frac{\pi}{2}\right)$, so $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. Graphically, the cosine

graph is the sine graph shifted $\frac{\pi}{2}$ units to the left: each sine peak at $\frac{\pi}{2} + 2n\pi$ becomes a cosine peak at $2n\pi$.

TRIGONOMETRIC IDENTITIES

An identity is an equation that is true for every allowed angle. In this unit we collect the basic identities (reciprocal, quotient, cofunction and even–odd), prove the Pythagorean identities from Euclid I.47 on the unit circle, and then derive the sum, difference, double-angle and half-angle formulas. Each new formula is proved from the ones before it, so the unit is a small chain of reasoning in the spirit of Euclid's *Elements*.

Along the way you will learn to prove identities properly: start with one side, justify each step, and end with the other side. You will also learn why some tempting steps, such as squaring both sides, can make a false statement look true, and how a single counterexample settles a false claim.

These identities are the working tools for the rest of the course. They let us find exact values like $\sin 15^\circ$ and $\cos 22.5^\circ$, and they are needed to solve trigonometric equations in Unit 6. Ptolemy used the same ideas, in the language of chords, to build the trigonometric table in his *Almagest*.

PREREQUISITES

- Define all six trigonometric functions for a point (x, y) at distance r on the terminal side of an angle (Lesson 3.6).
- Use reference angles and quadrant signs to evaluate trigonometric functions of any angle (Lesson 3.7).
- Know the exact values for multiples of 30° and 45° ($\frac{\pi}{6}$ and $\frac{\pi}{4}$) and the quadrantal angles (Lessons 2.1–2.3 and 3.8).
- Recognize even and odd symmetry in the graphs of sine, cosine and tangent (Lessons 4.1, 4.2 and 4.6).
- Algebra II skills: factoring (including differences of squares), adding rational expressions, and simplifying radicals.
- Use the distance formula and the Pythagorean theorem.

MASTERY CHECKLIST

- ☐ I can state the reciprocal, quotient, cofunction and even–odd identities and explain where each comes from.
- ☐ I can rewrite an expression in sines and cosines and simplify it.
- ☐ I can derive $\sin^2 \theta$ from the Pythagorean theorem and $\cos^2 \theta$ derive the two companion identities from it.
- ☐ I can find all trigonometric values from one value and the quadrant, choosing the correct signs.
- ☐ I can prove an identity by transforming one side into the other, using strategies such as common denominators, factoring and conjugates.
- ☐ I can explain why working on both sides or squaring both sides is not a valid proof, and disprove a false identity with a counterexample.
- ☐ I can derive the formula for $\cos(A \pm B)$ with the distance formula and derive the other sum and difference formulas from it.
- ☐ I can use sum and difference formulas to find exact values such as $\sin \frac{\pi}{6}$ and $\cos \frac{7\pi}{12}$.
- ☐ I can derive the double-angle formulas and the three forms of $\cos 2\theta$.

- ☐ I can derive and use the half-angle formulas, choosing the sign from the quadrant of

$$\frac{\theta}{2}.$$

- ☐ I can use identities to solve practical problems involving roofs, cranes, slopes and sprinklers.

LESSONS IN THIS UNIT

5.1 Fundamental Identities	50 minutes
5.2 The Pythagorean Identities	45 minutes
5.3 Simplifying and Proving Identities	2 sessions of 45 minutes
5.4 Sum and Difference Formulas	2 sessions of 45 minutes
5.5 Double-Angle and Half-Angle Formulas	2 sessions of 45 minutes

UNIT 5 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but when a problem asks for an exact value, give it in simplified radical or fraction form; you may add a decimal check. Round decimal answers as each problem directs. For proofs, start with one side and show every step. Total: 100 points.

1. **5 pts**

If $\cos \theta = 0.25$, find $\sec \theta$.

2. **5 pts**

Simplify $\sin(-\theta) \csc \theta$ to a number.

3. **6 pts**

Given $\sin \theta = -\frac{8}{17}$ with θ in Quadrant III, find the exact values of $\cos \theta$ and $\tan \theta$.

4. (8 pts)

A path climbs a hillside on Mr. Miller's farm at an acute angle θ with $\tan \theta = \frac{5}{12}$. Ruth walks 65 meters along the path. Use the Pythagorean identities to find $\cos \theta$ and $\sin \theta$, then find the horizontal distance $65 \cos \theta$ and the height gained $65 \sin \theta$, in meters.

5. (6 pts)

Simplify $(1 - \sin^2 \theta)(1 + \tan^2 \theta)$ to a number.

6. (8 pts)

Prove: $\frac{\sin \theta}{1 - \cos \theta} = \frac{1 + \cos \theta}{\sin \theta}$.

7. (6 pts)

Samuel claims that $(\sin \theta - \cos \theta)^2 = 1$ is an identity. Find the exact value of the left side at $\theta = 45^\circ$, and explain whether one such check settles the question.

8. (6 pts)

Find the exact value of $\sin 105^\circ$.

9. (6 pts)

Find the exact value of $\tan 75^\circ$ in simplest form.

10. (8 pts)

Suppose $\sin A = \frac{5}{13}$ and $\cos B = \frac{3}{5}$, with A and B both in Quadrant I. Find the exact values of $\sin(A + B)$ and $\cos(A - B)$.

11. (8 pts)

A crane's 30-meter boom points 60° above horizontal. The operator lowers it by 45° . The height of the tip above the pivot is then $30 \sin(60^\circ - 45^\circ)$. Use a difference formula to find the height exactly, then to the nearest hundredth of a meter.

12. (8 pts)

Given $\cos \theta = \frac{5}{13}$ with θ in Quadrant IV, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

13. (6 pts)

Use a half-angle formula to find the exact value of $\sin 22.5^\circ$.

14. (6 pts)

A garden sprinkler throws water at 14 meters per second at an angle θ above level ground. Ignoring air resistance, the water lands $R = \frac{v^2}{g} \cdot 2 \sin \theta \cos \theta$ meters away, with $v = 14$ and $g = 9.8$. Rewrite R with a double-angle formula and find R when $\theta = 30^\circ$, exactly and to the nearest tenth of a meter.

15. (8 pts)

Prove: $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$.

ANSWER KEY

$$1. \sec \theta = \frac{1}{0.25} = 4$$

$$2. -\sin \theta \cdot \frac{1}{\sin \theta} = -1 \text{ (where } \sin \theta \neq 0 \text{)}$$

$$3. \cos \theta = -\frac{15}{17}, \tan \theta = \frac{8}{15}$$

$$4. \sec^2 \theta = 1 + \frac{25}{144} = \frac{169}{144}, \text{ so } \cos \theta = \frac{12}{13} \text{ and } \sin \theta = \tan \theta \cos \theta = \frac{5}{13}. \text{ Horizontal distance 60 m; height gained 25 m.}$$

$$5. \cos^2 \theta \cdot \sec^2 \theta = 1 \text{ (where } \cos \theta \neq 0 \text{)}$$

$$6. \text{ Multiply the left side by } \frac{1 + \cos \theta}{1 + \cos \theta}: \frac{\sin \theta(1 + \cos \theta)}{1 - \cos^2 \theta} = \frac{\sin \theta(1 + \cos \theta)}{\sin^2 \theta} = \frac{1 + \cos \theta}{\sin \theta}.$$

$$7. \text{ At } 45^\circ \text{ the left side is } \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)^2 = 0 \neq 1. \text{ One counterexample is enough to show it is not an identity. (The correct identity is } (\sin \theta - \cos \theta)^2 = 1 - 2 \sin \theta \cos \theta \text{.)}$$

$$8. \sin(60^\circ + 45^\circ) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.965926$$

$$9. \tan(45^\circ + 30^\circ) = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3} \approx 3.732051$$

$$10. \cos A = \frac{12}{13}, \sin B = \frac{4}{5}, \sin(A + B) = \frac{5}{13} \cdot \frac{3}{5} + \frac{12}{13} \cdot \frac{4}{5} = \frac{63}{65}; \cos(A - B) = \frac{12}{13} \cdot \frac{3}{5} - \frac{4}{5} \cdot \frac{12}{13} = \frac{56}{65}$$

$$11. \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}, \text{ so the height is } \frac{15(\sqrt{6} - \sqrt{2})}{2} \approx 7.76 \text{ meters}$$

$$12. \sin \theta = -\frac{12}{13}, \sin 2\theta = 2 \left(-\frac{12}{13} \right) \frac{5}{13} = -\frac{120}{169}; \cos 2\theta = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$$

$$13. \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2} \approx 0.382683$$

$$14. R = 20 \sin 2\theta = 20 \sin 60^\circ = 10\sqrt{3} \approx 17.3 \text{ meters}$$

$$15. \text{ Use } \sin 2\theta = 2 \sin \theta \cos \theta \text{ and } \cos 2\theta = 2 \cos^2 \theta - 1: \frac{2 \sin \theta \cos \theta}{1 + 2 \cos^2 \theta - 1} = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

5.1 FUNDAMENTAL IDENTITIES

OBJECTIVES

- State and explain the reciprocal, quotient, cofunction and even–odd identities.
- Use the fundamental identities to rewrite and simplify trigonometric expressions.
- Find the values of several trigonometric functions from the values of sine and cosine.

WATCH

Trigonometric Identities (1 of 3: Reciprocals, Ratios & Complements)

Eddie Woo · 6:44

<https://www.youtube.com/watch?v=UmP1h3jgpAs>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

Trigonometric Identities & the Magic Hexagon (1 of 2: Reciprocals and Quotients)

Eddie Woo · 6:58

<https://www.youtube.com/watch?v=BUapOC8lQU4>

After the lesson, as a memory aid for all the identities.

KEY TERMS

identity

An equation that is true for every value of the variable for which both sides are defined.

quotient identities

The identities $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

reciprocal identities

The identities $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$ and $\cot \theta = \frac{1}{\tan \theta}$.

cofunction identities

Identities such as $\sin(90^\circ - \theta) = \cos \theta$ that relate a function of an angle to its cofunction of the complementary angle.

even function

A function with $f(-x) = f(x)$ for every x in its domain; its graph is symmetric about the y -axis. Cosine and secant are even.

odd function

A function with $f(-x) = -f(x)$ for every x in its domain; its graph is symmetric about the origin. Sine, cosecant, tangent and cotangent are odd.

CORE IDEAS

An **identity** is an equation that holds for every allowed value of the variable. The equation $\sin \theta = 0.5$ is not an identity; it is true only for certain angles. But $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is true for every angle where $\cos \theta \neq 0$. Identities let us replace one expression by another that is equal to it, just as in algebra we replace $2(x + 3)$ by $2x + 6$.

The **reciprocal** and **quotient identities** come straight from the definitions in Lesson 3.6. If the point (x, y) lies on the terminal side of θ at distance r from the origin, then $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$, $\csc \theta = \frac{r}{y}$, $\sec \theta = \frac{r}{x}$ and $\cot \theta = \frac{x}{y}$. So $\csc \theta$ is the reciprocal of $\sin \theta$, and so on. Also $\frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan \theta$. Every trigonometric function can therefore be written using only sine and cosine. That is the first and most useful strategy for simplifying.

The **cofunction identities** were proved for acute angles in Lesson 2.3: in a right triangle the two acute angles add to 90° , and the side opposite one angle is adjacent to the other. So $\sin(90^\circ - \theta) = \cos \theta$ and $\cos(90^\circ - \theta) = \sin \theta$. Dividing one by the other gives $\tan(90^\circ - \theta) = \cot \theta$, and taking reciprocals gives the rest. In radians, 90° is $\frac{\pi}{2}$. (In Lesson 5.4 we will see that these identities hold for every angle, not just acute ones.)

The **even-odd identities** come from a reflection. The angle $-\theta$ is the mirror image of θ across the x -axis. Reflecting the point (x, y) across the x -axis gives $(x, -y)$: the x -coordinate stays and the y -coordinate changes sign. Hence $\cos(-\theta) = \cos \theta$ (cosine is **even**) and $\sin(-\theta) = -\sin \theta$ (sine is **odd**). Then $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$, so tangent is odd. Reciprocals keep the same parity: secant is even; cosecant and cotangent are odd. You saw this symmetry already in the graphs of Unit 4.

To simplify an expression: (1) remove negative angles and complements with the even-odd and cofunction identities; (2) rewrite everything in sines and cosines; (3)

multiply, cancel and combine fractions as in algebra. Always remember that an identity holds only where both sides are defined. For example, $\sec \theta \cos \theta = 1$ is true only when $\cos \theta \neq 0$.

KEY FORMULAS

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta, \quad \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta, \quad \tan \left(\frac{\pi}{2} - \theta \right) = \cot \theta$$

$$\csc \left(\frac{\pi}{2} - \theta \right) = \sec \theta, \quad \sec \left(\frac{\pi}{2} - \theta \right) = \csc \theta, \quad \cot \left(\frac{\pi}{2} - \theta \right) = \tan \theta$$

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta, \quad \tan(-\theta) = -\tan \theta$$

$$\csc(-\theta) = -\csc \theta, \quad \sec(-\theta) = \sec \theta, \quad \cot(-\theta) = -\cot \theta$$

From history. The names of the cofunctions record the cofunction identities. "Cosine" is short for the Latin *complementi sinus*, "the sine of the complement," because the cosine of an angle is the sine of its complement. In the same way the cotangent and cosecant are the tangent and secant of the complementary angle.

Euclid connection. Euclid I.32 shows that the angles of a triangle add to two right angles, so the two acute angles of a right triangle add to one right angle. That is exactly why the sine of one acute angle equals the cosine of the other.

WORKED EXAMPLES

Example 1. An angle θ has $\sin \theta = \frac{5}{13}$ and $\cos \theta = \frac{12}{13}$. Use the reciprocal and quotient identities to find the exact values of $\tan \theta$, $\csc \theta$, $\sec \theta$ and $\cot \theta$.

Show solution

1. Quotient identity: $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{5/13}{12/13} = \frac{5}{12}$. The 13s cancel.
2. Reciprocal identity: $\csc \theta = \frac{1}{\sin \theta} = \frac{13}{5}$.
3. Reciprocal identity: $\sec \theta = \frac{1}{\cos \theta} = \frac{13}{12}$.
4. Reciprocal identity: $\cot \theta = \frac{1}{\tan \theta} = \frac{12}{5}$. (Check with the other quotient identity: $\frac{\cos \theta}{\sin \theta} = \frac{12/13}{5/13} = \frac{12}{5}$. It agrees.)

Answer: $\tan \theta = \frac{5}{12}$, $\csc \theta = \frac{13}{5}$, $\sec \theta = \frac{13}{12}$, $\cot \theta = \frac{12}{5}$

Example 2. Simplify $\sec(-\theta) \sin(90^\circ - \theta) \tan(-\theta) \cot(90^\circ - \theta)$ to a single term. Then give its exact value when $\theta = 30^\circ$.

Show solution

1. Remove the negative angles with the even-odd identities: $\sec(-\theta) = \sec \theta$ (secant is even) and $\tan(-\theta) = -\tan \theta$ (tangent is odd).
2. Remove the complements with the cofunction identities: $\sin(90^\circ - \theta) = \cos \theta$ and $\cot(90^\circ - \theta) = \tan \theta$.
3. The expression becomes $\sec \theta \cdot \cos \theta \cdot (-\tan \theta) \cdot \tan \theta$.
4. Since $\sec \theta = \frac{1}{\cos \theta}$, the product $\sec \theta \cos \theta = 1$. What remains is $-\tan \theta \cdot \tan \theta = -\tan^2 \theta$.
5. At $\theta = 30^\circ$: $\tan 30^\circ = \frac{1}{\sqrt{3}}$, so $-\tan^2 30^\circ = -\frac{1}{3}$.

Answer: $-\tan^2 \theta$; at $\theta = 30^\circ$ its value is $-\frac{1}{3}$

PRACTICE

1. Easy If $\cos \theta = 0.4$, find $\sec \theta$.

Show answer

$$\sec \theta = \frac{1}{0.4} = 2.5$$

2. Easy If $\sin \theta = 0.28$ and $\cos \theta = 0.96$, find the exact value of $\tan \theta$ as a fraction in lowest terms.

Show answer

$$\tan \theta = \frac{0.28}{0.96} = \frac{7}{24}$$

3. Easy Use a cofunction identity to write $\cos 65^\circ$ as the sine of an acute angle. Then give its value to six decimal places.

Show answer

$$\cos 65^\circ = \sin 25^\circ \approx 0.422618$$

4. Medium Simplify $\tan \theta \cos \theta$ to a single trigonometric function.

Show answer

$$\frac{\sin \theta}{\cos \theta} \cdot \cos \theta = \sin \theta \text{ (where } \cos \theta \neq 0 \text{)}$$

5. Medium An angle α has $\sin \alpha = 0.6$ and $\cos \alpha = 0.8$. Find $\sin(-\alpha)$, $\cos(-\alpha)$ and $\tan(-\alpha)$.

Show answer

$$\sin(-\alpha) = -0.6, \cos(-\alpha) = 0.8, \tan(-\alpha) = -0.75$$

6. Medium Simplify $\csc \theta \tan \theta$ to a single trigonometric function.

Show answer

$$\frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{1}{\cos \theta} = \sec \theta$$

7. **Medium** Mr. Miller is framing a barn roof with a 5-in-12 pitch: the roof rises 5 inches for every 12 inches of horizontal run, and the rafter makes angle θ with the horizontal, so $\sin \theta = \frac{5}{13}$ and $\cos \theta = \frac{12}{13}$. The length of a rafter equals its horizontal run times $\sec \theta$. Find the length of a rafter whose horizontal run is 18 feet (exact, in feet).

Show answer

$$\sec \theta = \frac{13}{12}, \text{ so the rafter is } 18 \cdot \frac{13}{12} = 19.5 \text{ feet}$$

8. **Hard** An angle θ has $\sin \theta = -\frac{12}{13}$ and $\cos \theta = \frac{5}{13}$. Find the exact values of $\tan(-\theta)$, $\sec(-\theta)$ and $\cot(90^\circ - \theta)$.

Show answer

$$\tan \theta = -\frac{12}{5}, \text{ so } \tan(-\theta) = \frac{12}{5}; \sec(-\theta) = \sec \theta = \frac{13}{5}; \cot(90^\circ - \theta) = \tan \theta = -\frac{12}{5}$$

9. **Hard** Simplify completely:

$$\frac{\cos(90^\circ - \theta) \csc(-\theta)}{\cot(-\theta) \tan \theta}$$

Show answer

The numerator is $\sin \theta \cdot (-\csc \theta) = -1$ and the denominator is $(-\cot \theta) \tan \theta = -1$, so the expression equals 1 (wherever it is defined).

10. **Hard** A wheelchair ramp at a church entrance has a horizontal run of 24 feet and makes angle θ with the ground, where $\cos \theta = \frac{12}{\sqrt{145}}$. The ramp's sloped length equals the run times $\sec \theta$. Find the sloped length exactly and to the nearest tenth of a foot.

Show answer

$$\sec \theta = \frac{\sqrt{145}}{12}, \text{ so the length is } 24 \cdot \frac{\sqrt{145}}{12} = 2\sqrt{145} \approx 24.1 \text{ feet}$$

11. **Challenge** Explain, using the unit circle, why $\cos(-\theta) = \cos \theta$ for every angle θ , and why $\sin(-\theta) = -\sin \theta$.

Show answer

The terminal side of $-\theta$ is the reflection of the terminal side of θ across the x -axis. Reflection across the x -axis sends the point $(\cos \theta, \sin \theta)$ to $(\cos \theta, -\sin \theta)$. That reflected point is $(\cos(-\theta), \sin(-\theta))$, so $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$.

12. **Challenge** Starting from $\sin(90^\circ - \theta) = \cos \theta$ and $\cos(90^\circ - \theta) = \sin \theta$, prove that $\tan(90^\circ - \theta) = \cot \theta$ and $\sec(90^\circ - \theta) = \csc \theta$ wherever both sides are defined.

Show answer

$$\tan(90^\circ - \theta) = \frac{\sin(90^\circ - \theta)}{\cos(90^\circ - \theta)} = \frac{\cos \theta}{\sin \theta} = \cot \theta, \text{ and } \sec(90^\circ - \theta) = \frac{1}{\cos(90^\circ - \theta)} = \frac{1}{\sin \theta} = \csc \theta.$$

COMMON MISTAKES

- Treating $\sin^{-1} \theta$ as $\csc \theta$. The reciprocal of sine is $\csc \theta = \frac{1}{\sin \theta}$; the notation $\sin^{-1}$ means the inverse function (Lesson 1.4). Write $(\sin \theta)^{-1}$ or $\csc \theta$ when you mean the reciprocal.
- Mixing up which functions are even. Only cosine and secant are even; the other four are odd. Picture the reflection $(x, y) \rightarrow (x, -y)$: only quantities built from x and r alone stay the same.
- Cancelling across a sum, such as turning $\frac{\sin \theta + 1}{\sin \theta}$ into just 1 by crossing out the sines. You may cancel only common factors, never terms: $\frac{\sin \theta + 1}{\sin \theta} = 1 + \csc \theta$.
- Forgetting where an identity holds. $\sec \theta \cos \theta = 1$ fails when $\cos \theta = 0$, because $\sec \theta$ is then undefined. Note excluded angles when they matter.

Make it a game • Identity Tall Tales. Write three identities on a card: two true, one false, such as 'sec theta equals one over sine theta.' Your parent finds the tall tale and gives the correct version. Use reciprocal, quotient, cofunction and even-odd identities. Swap roles, then simplify tan theta times cos theta to a single function.

5.2 THE PYTHAGOREAN IDENTITIES

OBJECTIVES

- Derive $\sin^2 \theta + \cos^2 \theta = 1$ from the Pythagorean theorem on the unit circle, and derive the two companion identities from it.
- Use the Pythagorean identities and the quadrant to find all trigonometric values from one given value.
- Use the Pythagorean identities to simplify trigonometric expressions.

WATCH

Trigonometric Identities (2 of 3: Pythagorean Relationships)

Eddie Woo · 6:35

<https://www.youtube.com/watch?v=cW-Trdxdjzk>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

Pythagorean Identities for Trigonometric Functions (Precalculus - Trigonometry 10)

Professor Leonard · 37:54

<https://www.youtube.com/watch?v=huP3v7iZuLM>

Before the practice set; long (38 min), for more worked examples.

KEY TERMS

Pythagorean identities

The three identities $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$.

$\sin^2 \theta$

Shorthand for $(\sin \theta)^2$, the square of the sine. It is not $\sin(\theta^2)$.

quadrant sign

Whether a trigonometric value is positive or negative, decided by the quadrant of the terminal side (Lesson 3.7). A square root gives only a size; the quadrant gives the sign.

CORE IDEAS

Let the terminal side of θ meet the unit circle at P . By the definition in Lesson 3.6, $P = (\cos \theta, \sin \theta)$. Drop a perpendicular from P to the x -axis. This makes a right triangle with legs $|\cos \theta|$ and $|\sin \theta|$ and hypotenuse 1 (the radius). By the Pythagorean theorem (Euclid I.47), $|\cos \theta|^2 + |\sin \theta|^2 = 1^2$. Squaring removes the absolute values, so

$$\sin^2 \theta + \cos^2 \theta = 1.$$

If P lies on an axis there is no triangle, but the identity still holds: for example at 90° , $1^2 + 0^2 = 1$. In every case it simply says that P is at distance 1 from the origin.

Divide both sides of $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$ (allowed when $\cos \theta \neq 0$):

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}, \quad \text{so} \quad \tan^2 \theta + 1 = \sec^2 \theta.$$

Divide instead by $\sin^2 \theta$ (when $\sin \theta \neq 0$) to get $1 + \cot^2 \theta = \csc^2 \theta$. These are the two companion identities. They are not new facts; they are the same Pythagorean fact seen through different ratios.

Each identity can be rearranged: $\cos^2 \theta = 1 - \sin^2 \theta$, $\sin^2 \theta = 1 - \cos^2 \theta$, $\tan^2 \theta = \sec^2 \theta - 1$, and so on. Learn to spot these forms. Whenever you see $1 - \sin^2 \theta$, think $\cos^2 \theta$; whenever you see $\sec^2 \theta - 1$, think $\tan^2 \theta$.

To find a value from another, solve the identity and take a square root. A square root gives two signs, and the identity alone cannot choose between them. The quadrant chooses. For example, if $\sin \theta = \frac{3}{5}$, then $\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos \theta = \pm \frac{4}{5}$. In Quadrant I it is $\frac{4}{5}$; in Quadrant II it is $-\frac{4}{5}$.

KEY FORMULAS

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}, \quad \sin \theta = \pm \sqrt{1 - \cos^2 \theta} \quad (\text{sign from the quadrant})$$

From history. Ptolemy used this same fact in the *Almagest* to build his table of chords. Knowing the chord of an arc in a circle, he found the chord of the remaining arc of the semicircle by the Pythagorean theorem, because the two chords and the diameter form a right triangle. In modern language that step is $\sin^2 \theta + \cos^2 \theta = 1$.

Euclid connection. The identity $\sin^2 \theta + \cos^2 \theta = 1$ is Euclid I.47, the Pythagorean theorem, applied to a right triangle whose hypotenuse is the radius of the unit circle. Euclid III.31 (the angle in a semicircle is right) is what made it usable for Ptolemy's chords.

WORKED EXAMPLES

Example 1. Given $\sin \theta = \frac{3}{5}$ and θ in Quadrant II, find the exact values of $\cos \theta$ and $\tan \theta$.

Show solution

1. Start from $\sin^2 \theta + \cos^2 \theta = 1$: $\cos^2 \theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$.
2. So $\cos \theta = \pm \frac{4}{5}$. In Quadrant II the x -coordinate is negative, so $\cos \theta = -\frac{4}{5}$.
3. Quotient identity: $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{3/5}{-4/5} = -\frac{3}{4}$. (Tangent is negative in Quadrant II, as expected.)

Answer: $\cos \theta = -\frac{4}{5}$, $\tan \theta = -\frac{3}{4}$

Example 2. Given $\tan \theta = 2$ and θ in Quadrant III, find the exact values of $\sec \theta$, $\cos \theta$ and $\sin \theta$.

Show solution

1. Use $1 + \tan^2 \theta = \sec^2 \theta$: $\sec^2 \theta = 1 + 4 = 5$, so $\sec \theta = \pm\sqrt{5}$.
2. In Quadrant III cosine is negative, so its reciprocal $\sec \theta$ is negative too:
 $\sec \theta = -\sqrt{5}$.
3. Then $\cos \theta = \frac{1}{\sec \theta} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$.
4. Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, we have $\sin \theta = \tan \theta \cos \theta = 2 \cdot \left(-\frac{\sqrt{5}}{5}\right) = -\frac{2\sqrt{5}}{5}$. It is negative, as it must be in Quadrant III.
5. Check: $\sin^2 \theta + \cos^2 \theta = \frac{20}{25} + \frac{5}{25} = 1$. Correct.

Answer: $\sec \theta = -\sqrt{5} \approx -2.236$, $\cos \theta = -\frac{\sqrt{5}}{5} \approx -0.447$, $\sin \theta = -\frac{2\sqrt{5}}{5} \approx -0.894$

PRACTICE

1. **Easy** Without a calculator, find the exact value of $\sin^2 37^\circ + \cos^2 37^\circ$.

Show answer

1

2. **Easy** If $\cos \theta = 0.6$ and θ is in Quadrant I, find $\sin \theta$.

Show answer

$$\sin \theta = \sqrt{1 - 0.36} = \sqrt{0.64} = 0.8$$

3. **Easy** Find the exact value of $\sec^2 \theta - \tan^2 \theta$ for any angle where both are defined.

Show answer

$$1, \text{ since } \sec^2 \theta = 1 + \tan^2 \theta$$

4. **Medium** If $\sin \theta = -\frac{5}{13}$ and θ is in Quadrant IV, find the exact values of $\cos \theta$ and $\tan \theta$.

Show answer

$$\cos \theta = \frac{12}{13}, \tan \theta = -\frac{5}{12}$$

5. Medium Simplify $(1 - \cos^2 \theta) \csc \theta$ to a single trigonometric function.

Show answer

$$\sin^2 \theta \cdot \frac{1}{\sin \theta} = \sin \theta$$

6. Medium If $\csc \theta = -3$ and θ is in Quadrant III, find the exact values of $\cot \theta$ and $\cos \theta$.

Show answer

$$\cot^2 \theta = 9 - 1 = 8 \text{ and cotangent is positive in Quadrant III, so } \cot \theta = 2\sqrt{2}.$$

$$\text{Then } \sin \theta = -\frac{1}{3} \text{ and } \cos \theta = \cot \theta \sin \theta = -\frac{2\sqrt{2}}{3}.$$

7. Medium Samuel is building a shed roof. Each 15-foot rafter makes an acute angle θ with the horizontal, where $\sin \theta = 0.6$. Find $\cos \theta$, then find the horizontal run of one rafter, $15 \cos \theta$, in feet.

Show answer

$$\cos \theta = \sqrt{1 - 0.36} = 0.8, \text{ so the run is } 15(0.8) = 12 \text{ feet}$$

8. **Hard** A farm lane climbs a hill at an acute angle θ with $\tan \theta = 0.75$. Ruth walks 200 meters along the lane. Use $1 + \tan^2 \theta = \sec^2 \theta$ to find $\cos \theta$, then find the horizontal distance she covered, $200 \cos \theta$, in meters.

Show answer

$$\sec^2 \theta = 1.5625, \text{ so } \sec \theta = 1.25 \text{ and } \cos \theta = 0.8. \text{ The horizontal distance is } 200(0.8) = 160 \text{ meters.}$$

9. **Hard** Simplify $(\sec \theta - 1)(\sec \theta + 1)$ to a single term.

Show answer

$$\sec^2 \theta - 1 = \tan^2 \theta$$

10. **Hard** Simplify $\sin^4 \theta - \cos^4 \theta$ so that it has no fourth powers.

Show answer

$$(\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta) = \sin^2 \theta - \cos^2 \theta, \text{ which also equals } 2 \sin^2 \theta - 1$$

11. **Challenge** Starting from $\sin^2 \theta + \cos^2 \theta = 1$, prove that $1 + \cot^2 \theta = \csc^2 \theta$. State for which angles your proof is valid.

Show answer

For $\sin \theta \neq 0$, divide both sides by $\sin^2 \theta$: $\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$, that is, $1 + \cot^2 \theta = \csc^2 \theta$. It is valid for all angles except integer multiples of 180° (π radians), where $\cot$ and $\csc$ are undefined.

12. **Challenge** A classmate says an angle has $\sin \theta = 0.8$ and $\cos \theta = 0.7$. Explain why this is impossible.

Show answer

Every angle satisfies $\sin^2 \theta + \cos^2 \theta = 1$, but $0.8^2 + 0.7^2 = 0.64 + 0.49 = 1.13 \neq 1$. The point $(0.7, 0.8)$ is not on the unit circle, so no angle has these values.

COMMON MISTAKES

- Forgetting the sign. $\sqrt{1 - \sin^2 \theta}$ is never negative, but $\cos \theta$ can be. Always write $\pm$ first, then use the quadrant to choose.
- Writing $\sin^2 \theta + \cos^2 \theta = 1$ as $\sin \theta + \cos \theta = 1$. Squares do not distribute away; for $\theta = 45^\circ$, $\sin \theta + \cos \theta = \sqrt{2}$, not 1.
- Misreading $\sin^2 \theta$ as $\sin(\theta^2)$. The exponent applies to the value of the sine: $\sin^2 \theta = (\sin \theta)^2$.
- Getting the companion identities backwards, such as $\tan^2 \theta + \sec^2 \theta = 1$. Rederive them in one line by dividing $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$ or by $\sin^2 \theta$.

Make it a game • Complete the Identity. Parent starts each sentence and you finish it in 5 seconds: 'sin squared plus cos squared equals..., 'one plus tan squared equals..., 'one plus cot squared equals...' Then derive the first identity by drawing the unit circle and a right triangle. Finally, given $\cos \theta = -4/5$ in Quadrant II, find $\sin \theta$.

5.3 SIMPLIFYING AND PROVING IDENTITIES

OBJECTIVES

- Prove a trigonometric identity by transforming one side step by step until it matches the other.
- Use standard strategies: rewrite in sines and cosines, combine fractions, factor, and multiply by a conjugate.
- Recognize invalid steps in a proposed proof and disprove a false identity with a counterexample.

WATCH

How to Prove Trigonometric Identities (and how not to)

Eddie Woo · 7:37

https://www.youtube.com/watch?v=rS1T_b4WZzA

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Trigonometric Identity Proofs (1 of 2: Overall strategies)

Eddie Woo · 4:20

<https://www.youtube.com/watch?v=2t4XRAhiQpk>

Before the practice set, for strategy tips.

KEY TERMS

proof of an identity

A chain of equalities, each justified by algebra or a known identity, that begins with one side of the equation and ends with the other.

conjugate

For an expression $a + b$, the expression $a - b$ (and the reverse). Their product $a^2 - b^2$ often turns into a single term by a Pythagorean identity.

counterexample

A single value of the variable, where both sides are defined, for which the two sides of a proposed identity are not equal. One counterexample proves the equation is not an identity.

CORE IDEAS

To prove an identity, we may **not** assume it is true. So we do not treat it like an equation to solve by doing the same thing to both sides. Instead we start with one side and transform it, one justified step at a time, until it becomes the other side. Each step must be an algebra rule or an identity already proved. This is the same discipline Euclid used: every line rests on something earlier.

Useful strategies, in rough order: (1) Start with the more complicated side; it has more to simplify. (2) Rewrite everything in sines and cosines. (3) Combine fractions over a common denominator, or split one fraction into two. (4) Factor, including differences of squares such as $1 - \sin^2 \theta = (1 - \sin \theta)(1 + \sin \theta)$. (5) Look for Pythagorean forms such as $1 - \cos^2 \theta$ or $\sec^2 \theta - 1$. (6) If a denominator is $1 \pm \sin \theta$ or $1 \pm \cos \theta$, multiply top and bottom by its conjugate. (7) Keep your eye on the target side and ask which functions it uses.

Why the rule against working on both sides? Some steps cannot be reversed, and they can make a false statement look true. Start with the false equation $1 = -1$. Square both sides: $1 = 1$, which is true. Reaching a true statement proved nothing, because squaring is not reversible. The same thing happens with trigonometry. If you square both sides of the false claim $\sin \theta = \sqrt{1 - \cos^2 \theta}$, you get the true identity $\sin^2 \theta = 1 - \cos^2 \theta$, yet the original fails whenever $\sin \theta < 0$. Multiplying both sides by an expression that could be zero is unsafe in the same way.

Checking a few angles on a calculator is a good way to **test** a claim, but it can never **prove** an identity, because there are infinitely many angles. However, one angle where the two sides differ is enough to **disprove** it. For example, $(\sin \theta + \cos \theta)^2 = 1$ fails at $\theta = 45^\circ$, where the left side is $(\sqrt{2})^2 = 2$.

KEY FORMULAS

$$(1 - \sin \theta)(1 + \sin \theta) = 1 - \sin^2 \theta = \cos^2 \theta$$

$$(1 - \cos \theta)(1 + \cos \theta) = 1 - \cos^2 \theta = \sin^2 \theta$$

$$(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = \sec^2 \theta - \tan^2 \theta = 1$$

Euclid connection. Euclid's *Elements* is the classic model of proof: each proposition is derived only from definitions, postulates, common notions and propositions already proved. A proof of an identity follows the same pattern, with each step resting on algebra or an identity proved earlier in this unit.

WORKED EXAMPLES

Example 1. Prove the identity $\sec \theta - \cos \theta = \sin \theta \tan \theta$.

Show solution

1. Start with the left side, which can be rewritten in sines and cosines: $\sec \theta - \cos \theta = \frac{1}{\cos \theta} - \cos \theta$.

2. Use the common denominator $\cos \theta$: $= \frac{1}{\cos \theta} - \frac{\cos^2 \theta}{\cos \theta} = \frac{1 - \cos^2 \theta}{\cos \theta}$.

3. Pythagorean identity: $1 - \cos^2 \theta = \sin^2 \theta$, so this is $\frac{\sin^2 \theta}{\cos \theta}$.

4. Split the square: $= \sin \theta \cdot \frac{\sin \theta}{\cos \theta} = \sin \theta \tan \theta$, which is the right side.

5. Quick check (not a proof): at $\theta = 60^\circ$ the left side is $2 - \frac{1}{2} = \frac{3}{2}$ and the right side is $\frac{\sqrt{3}}{2} \cdot \sqrt{3} = \frac{3}{2}$.

Answer: Left side $= \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta} = \sin \theta \tan \theta =$ right side (for $\cos \theta \neq 0$).

Example 2. Prove the identity $\frac{\cos \theta}{1 - \sin \theta} = \frac{1 + \sin \theta}{\cos \theta}$.

Show solution

1. Start with the left side. Its denominator is $1 - \sin \theta$, so multiply top and bottom by the conjugate $1 + \sin \theta$ (this is multiplying by 1, which is always allowed where $1 + \sin \theta \neq 0$).
2.
$$\frac{\cos \theta}{1 - \sin \theta} \cdot \frac{1 + \sin \theta}{1 + \sin \theta} = \frac{\cos \theta(1 + \sin \theta)}{1 - \sin^2 \theta}$$
3. Pythagorean identity: $1 - \sin^2 \theta = \cos^2 \theta$, so we have $\frac{\cos \theta(1 + \sin \theta)}{\cos^2 \theta}$.
4. Cancel the common factor $\cos \theta$: $= \frac{1 + \sin \theta}{\cos \theta}$, which is the right side.
5. Notice we never wrote the two sides with an equals sign between them and cross-multiplied. We transformed one side only.

Answer: Multiply the left side by $\frac{1 + \sin \theta}{1 + \sin \theta}$: $\frac{\cos \theta(1 + \sin \theta)}{\cos^2 \theta} = \frac{1 + \sin \theta}{\cos \theta} =$ right side.

PRACTICE

1. Easy Simplify $\cos \theta \tan \theta \csc \theta$ to a number.

Show answer

$$\cos \theta \cdot \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta} = 1$$

2. Easy Simplify $\frac{\sin^2 \theta}{1 - \cos^2 \theta}$ to a number.

Show answer

$$\frac{\sin^2 \theta}{\sin^2 \theta} = 1 \text{ (where } \sin \theta \neq 0 \text{)}$$

3. Easy Simplify $\sec \theta \cot \theta$ to a single trigonometric function.

Show answer

$$\frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} = \frac{1}{\sin \theta} = \csc \theta$$

4. Medium Prove: $\tan \theta + \cot \theta = \sec \theta \csc \theta$.

Show answer

$$\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \sec \theta \csc \theta.$$

5. Medium Prove: $(1 + \tan^2 \theta) \cos^2 \theta = 1$.

Show answer

$$(1 + \tan^2 \theta) \cos^2 \theta = \sec^2 \theta \cos^2 \theta = \frac{1}{\cos^2 \theta} \cdot \cos^2 \theta = 1 \text{ (where } \cos \theta \neq 0\text{)}.$$

6. Medium Ruth claims that $(\sin \theta + \cos \theta)^2 = 1$ is an identity. Find the exact value of the left side at $\theta = 45^\circ$, and decide whether she is right.

Show answer

$$\text{At } 45^\circ \text{ the left side is } \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right)^2 = (\sqrt{2})^2 = 2 \neq 1, \text{ so it is not an identity.}$$

The correct identity is $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$.

7. Medium Prove: $\frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} = 2 \sec \theta$.

Show answer

$$\begin{aligned} \text{Common denominator: } \frac{\cos^2 \theta + (1 + \sin \theta)^2}{\cos \theta(1 + \sin \theta)} &= \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{\cos \theta(1 + \sin \theta)} = \\ \frac{2 + 2 \sin \theta}{\cos \theta(1 + \sin \theta)} &= \frac{2(1 + \sin \theta)}{\cos \theta(1 + \sin \theta)} = \frac{2}{\cos \theta} = 2 \sec \theta. \end{aligned}$$

8. **Hard** Prove: $\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$.

Show answer

$$\begin{aligned} \text{Multiply the left side by } \frac{1 + \cos \theta}{1 + \cos \theta}: \frac{1 - \cos^2 \theta}{\sin \theta(1 + \cos \theta)} &= \frac{\sin^2 \theta}{\sin \theta(1 + \cos \theta)} = \\ \frac{\sin \theta}{1 + \cos \theta}. \end{aligned}$$

9. **Hard** Samuel claims that $\sqrt{1 - \sin^2 \theta} = \cos \theta$ for every angle. Evaluate the left side and the right side exactly at $\theta = 180^\circ$, and state the correct identity.

Show answer

Left side: $\sqrt{1 - 0} = 1$. Right side: $\cos 180^\circ = -1$. They differ, so the claim is false. The correct identity is $\sqrt{1 - \sin^2 \theta} = |\cos \theta|$.

10. **Hard** Prove: $\sec^4 \theta - \tan^4 \theta = \sec^2 \theta + \tan^2 \theta$.

Show answer

Factor the left side as a difference of squares: $(\sec^2 \theta - \tan^2 \theta)(\sec^2 \theta + \tan^2 \theta)$. Since $\sec^2 \theta - \tan^2 \theta = 1$, this equals $\sec^2 \theta + \tan^2 \theta$.

11. **Challenge** Ruth tries to prove $\sin \theta = \sqrt{1 - \cos^2 \theta}$ like this: she squares both sides to get $\sin^2 \theta = 1 - \cos^2 \theta$, which is a true identity, and concludes the original is true. Explain the flaw and give an angle where the original statement fails.

Show answer

She assumed the statement and squared both sides, and squaring is not reversible: $a^2 = b^2$ only gives $a = \pm b$. The square root is never negative, so the original fails whenever $\sin \theta < 0$. For example, at $\theta = 270^\circ$ the left side is -1 and the right side is 1 .

12. **Challenge** Prove: $\frac{\tan \theta - \sin \theta}{\sin^3 \theta} = \frac{\sec \theta}{1 + \cos \theta}$.

Show answer

$$\begin{aligned} \text{Numerator: } \tan \theta - \sin \theta &= \frac{\sin \theta}{\cos \theta} - \sin \theta = \frac{\sin \theta(1 - \cos \theta)}{\cos \theta} \\ \frac{1 - \cos \theta}{\cos \theta \sin^2 \theta} &= \frac{1 - \cos \theta}{\cos \theta(1 - \cos \theta)(1 + \cos \theta)} = \frac{1}{\cos \theta(1 + \cos \theta)} = \frac{\sec \theta}{1 + \cos \theta}. \end{aligned}$$

COMMON MISTAKES

- Working on both sides at once, or cross-multiplying, and then declaring victory when you reach a true statement. Transform one side only, and write a single chain of equalities from one side to the other.
- Squaring or taking square roots carelessly. $\sqrt{\cos^2 \theta} = |\cos \theta|$, not $\cos \theta$. Watch signs whenever a square root appears.
- Adding fractions by adding denominators, such as $\frac{1}{\sin \theta} + \frac{1}{\cos \theta} = \frac{1}{\sin \theta + \cos \theta}$.
Use a common denominator: $\frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta}$.
- Thinking a calculator check proves an identity. Agreement at a few angles is only evidence; disagreement at one angle is a disproof.

Make it a game • Real or Fake Identity. Parent writes two equations, one a true identity and one false, such as $\sin x + \cos x = 1$. Disprove the false one with a counterexample, like $x = 45$ degrees. Then prove the true one by transforming one side, naming each strategy: rewrite in sines and cosines, combine fractions, factor or use a conjugate.

5.4 SUM AND DIFFERENCE FORMULAS

OBJECTIVES

- Derive the formula for $\cos(A - B)$ from the distance formula on the unit circle, and derive the other sum and difference formulas from it.
- Use the sum and difference formulas to find exact values such as $\sin 15^\circ$ and $\cos \frac{7\pi}{12}$.
- Use the formulas to evaluate $\sin(A \pm B)$ and $\cos(A \pm B)$ from given values, and to prove related identities.

WATCH

Compound Angle Identities (1 of 3: Proving $\sin(a+b)$ geometrically).

Eddie Woo · 13:20

<https://www.youtube.com/watch?v=zpyPTBgUqkQ>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo calls these 'compound angle' identities and proves $\sin(a+b)$ with a geometric diagram, not the distance-formula proof of $\cos(A-B)$ used here.

BACKUP VIDEO

Compound Angle Identities (2 of 4: Cosine).

Eddie Woo · 7:03

<https://www.youtube.com/watch?v=Y9Sw8spGbAU>

Before Example 2: deriving the difference and cosine formulas from the sum formula.

KEY TERMS

sum formula

A formula that expresses a function of $A + B$, such as $\sin(A + B)$, in terms of functions of A and of B .

difference formula

A formula that expresses a function of $A - B$, such as $\cos(A - B)$, in terms of functions of A and of B .

Ptolemy's theorem

In a quadrilateral inscribed in a circle,
the product of the diagonals equals the
sum of the products of the two pairs of
opposite sides.

CORE IDEAS

A warning first: $\cos(A - B)$ is **not** $\cos A - \cos B$. For example, $\cos(90^\circ - 30^\circ) = \cos 60^\circ = \frac{1}{2}$, but $\cos 90^\circ - \cos 30^\circ = -\frac{\sqrt{3}}{2}$. We need a true formula, and we will prove it.

Proof of the difference formula for cosine. Put $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$ on the unit circle. The angle from OQ to OP is $A - B$. By the distance formula,

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2.$$

Expanding and using $\cos^2 A + \sin^2 A = 1$ and $\cos^2 B + \sin^2 B = 1$ gives $PQ^2 = 2 - 2(\cos A \cos B + \sin A \sin B)$. Now rotate the whole picture about the origin by $-B$. Rotation does not change distances. Q moves to $(1, 0)$ and P moves to $(\cos(A - B), \sin(A - B))$. Using the distance formula again, $PQ^2 = (\cos(A - B) - 1)^2 + \sin^2(A - B) = 2 - 2\cos(A - B)$. Setting the two expressions for PQ^2 equal gives

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

The other formulas follow. Replace B by $-B$ and use the even-odd identities: $\cos(A + B) = \cos A \cos(-B) + \sin A \sin(-B) = \cos A \cos B - \sin A \sin B$. Put $A = 90^\circ$ in the difference formula: $\cos(90^\circ - B) = 0 \cdot \cos B + 1 \cdot \sin B = \sin B$, so the cofunction identities hold for **every** angle, not just acute ones. Replacing B by $90^\circ - B$ in this result gives $\cos B = \sin(90^\circ - B)$, the other cofunction identity, also for every angle. Then

$$\sin(A + B) = \cos((90^\circ - A) - B) = \cos(90^\circ - A) \cos B + \sin(90^\circ - A) \sin B = \sin A \cos B + \cos A \sin B.$$

Replacing B by $-B$ gives $\sin(A - B) = \sin A \cos B - \cos A \sin B$. Finally, $\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$; divide top and bottom by $\cos A \cos B$ to get $\frac{\tan A + \tan B}{1 - \tan A \tan B}$.

Using them. Write an unfamiliar angle as a sum or difference of special angles:

$$15^\circ = 45^\circ - 30^\circ, 75^\circ = 45^\circ + 30^\circ, 105^\circ = 60^\circ + 45^\circ, \text{ and in radians } \frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4},$$

$\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$. When given $\sin A$ and $\cos B$ instead of angles, first find $\cos A$ and $\sin B$ with the Pythagorean identity and the quadrants (Lesson 5.2), then substitute.

Memory aid: cosine formulas keep cosines together and sines together, and the sign **flips** ($\cos(A + B)$ has a minus). Sine formulas mix (sin cos then cos sin), and the sign **matches**.

KEY FORMULAS

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

From history. Ptolemy (2nd century AD) needed a table of chords for astronomy in the *Almagest*. Using his theorem on quadrilaterals inscribed in a circle, he showed how to find the chord of the sum or difference of two arcs from the chords of the arcs themselves. In modern notation, those results are the sum and difference formulas for sine. With them, and with the chords of a few known arcs, he built his whole table.

Euclid connection. The distance formula used in the proof is Euclid I.47 (the Pythagorean theorem) placed on a coordinate grid. The step "rotation does not change distance" is the same idea as Euclid's congruent triangles: moving a figure rigidly does not change its sides.

WORKED EXAMPLES

Example 1. Find the exact value of $\sin 15^\circ$. Then give a decimal approximation to six decimal places.

Show solution

1. Write $15^\circ = 45^\circ - 30^\circ$, a difference of special angles.
2. Difference formula: $\sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$.
3. Substitute exact values: $= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$.
4. Decimal: $\frac{2.449490 - 1.414214}{4} \approx 0.258819$. A calculator gives $\sin 15^\circ \approx 0.258819$, which agrees.

Answer: $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.258819$

Example 2. Suppose $\sin A = \frac{3}{5}$ with A in Quadrant I, and $\cos B = -\frac{5}{13}$ with B in Quadrant II. Find the exact values of $\sin(A + B)$ and $\cos(A + B)$, and name the quadrant of $A + B$.

Show solution

1. Find the missing values. $\cos^2 A = 1 - \frac{9}{25} = \frac{16}{25}$, and cosine is positive in Quadrant I, so $\cos A = \frac{4}{5}$.

2. $\sin^2 B = 1 - \frac{25}{169} = \frac{144}{169}$, and sine is positive in Quadrant II, so $\sin B = \frac{12}{13}$.

3. $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \cdot \left(-\frac{5}{13}\right) + \frac{4}{5} \cdot \frac{12}{13} = -\frac{15}{65} + \frac{48}{65} = \frac{33}{65}$.

4. $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \cdot \left(-\frac{5}{13}\right) - \frac{3}{5} \cdot \frac{12}{13} = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$.

5. Sine positive and cosine negative, so $A + B$ is in Quadrant II.

6. Check: $33^2 + 56^2 = 1089 + 3136 = 4225 = 65^2$, so $\sin^2(A + B) + \cos^2(A + B) = 1$. Correct.

Answer: $\sin(A + B) = \frac{33}{65} \approx 0.5077$, $\cos(A + B) = -\frac{56}{65} \approx -0.8615$; $A + B$ is in Quadrant II

PRACTICE

1. Easy Find the exact value of $\cos 15^\circ$ using $15^\circ = 45^\circ - 30^\circ$.

Show answer

$$\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.965926$$

2. Easy Write $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$ as the sine of one angle, and give its exact value.

Show answer

$$\sin(40^\circ + 20^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2} \approx 0.866025$$

3. Easy Find the exact value of $\cos 75^\circ$ using $75^\circ = 45^\circ + 30^\circ$.

Show answer

$$\cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.258819$$

4. Medium Find the exact value of $\tan 15^\circ$ in simplest form.

Show answer

$$\tan(45^\circ - 30^\circ) = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3} \approx 0.267949$$

5. Medium Find the exact value of $\cos \frac{7\pi}{12}$ using $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$.

Show answer

$$\cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4} \approx -0.258819$$

6. Medium The boom of a crane is 20 meters long. It starts at 30° above horizontal, and the operator raises it another 45° . The height of the boom's tip above the pivot is $20 \sin(30^\circ + 45^\circ)$. Find this height exactly using a sum formula, and to the nearest hundredth of a meter.

Show answer

$$\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}, \text{ so the height is } 5(\sqrt{6} + \sqrt{2}) \approx 19.32 \text{ meters}$$

7. Medium Use a sum formula to simplify $\sin(x + \pi)$.

Show answer

$$\sin x \cos \pi + \cos x \sin \pi = \sin x \cdot (-1) + \cos x \cdot 0 = -\sin x$$

8. **Hard** Suppose $\sin A = \frac{4}{5}$ with A in Quadrant II, and $\cos B = \frac{5}{13}$ with B in Quadrant IV. Find the exact values of $\cos(A - B)$ and $\sin(A - B)$, and name the quadrant of $A - B$.

Show answer

$$\begin{aligned} \cos A &= -\frac{3}{5}, \sin B = -\frac{12}{13}, \cos(A - B) = \left(-\frac{3}{5}\right) \frac{5}{13} + \frac{4}{5} \left(-\frac{12}{13}\right) = -\frac{63}{65}; \\ \sin(A - B) &= \frac{4}{5} \cdot \frac{5}{13} - \left(-\frac{3}{5}\right) \left(-\frac{12}{13}\right) = -\frac{16}{65}. \end{aligned}$$

Both negative, so $A - B$ is in Quadrant III.

9. **Hard** Mr. Miller's barn has a gambrel roof. The steep lower section rises 24 inches per 12 inches of run (slope $\tan A = 2$) and the upper section rises 6 inches per 12 inches of run (slope $\tan B = \frac{1}{2}$). The roof bends through the angle $A - B$ where the sections meet. Use the tangent difference formula to find $\tan(A - B)$, then find the angle $A - B$ to the nearest tenth of a degree.

Show answer

$$\tan(A - B) = \frac{2 - \frac{1}{2}}{1 + 2 \cdot \frac{1}{2}} = \frac{3/2}{2} = \frac{3}{4}, \text{ so } A - B = \tan^{-1}(0.75) \approx 36.9^\circ$$

10. **Hard** Prove: $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$.

Show answer

$(\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B) = 2 \sin A \cos B$; the $\cos A \sin B$ terms cancel.

11. **Challenge** Derive the formula $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ from the sine and cosine difference formulas. State when it is valid.

Show answer

$\tan(A - B) = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$. Divide every term of the numerator and denominator by $\cos A \cos B$: $\frac{\tan A - \tan B}{1 + \tan A \tan B}$. Valid when $\cos A \neq 0$, $\cos B \neq 0$ and $\cos(A - B) \neq 0$.

12. **Challenge** Prove: $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B$.

Show answer

The product is $(\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$. Replace $\cos^2 B = 1 - \sin^2 B$ and $\sin^2 A = 1 - \cos^2 A$: $\cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B = \cos^2 A - \sin^2 B$.

COMMON MISTAKES

- Distributing the function: writing $\sin(A + B) = \sin A + \sin B$. Test it: $\sin(30^\circ + 60^\circ) = 1$, but $\sin 30^\circ + \sin 60^\circ \approx 1.366$.
- Getting the sign wrong in the cosine formulas. In $\cos(A + B)$ the middle sign is minus; in $\cos(A - B)$ it is plus. Sine keeps the same sign as the angle.

- Using the wrong signs for the missing values. When $\sin A$ is given, find $\cos A$ with the Pythagorean identity and then choose its sign from the quadrant of A before substituting.
- Mixing degrees and radians in one problem, such as $\frac{\pi}{3} + 45^\circ$. Convert first so both angles use the same unit.

Make it a game · Angle Sum Draw. Make cards with 30, 45, 60, 90, 120 and 135 degrees. Draw two, then add or subtract to make an angle, such as 45 minus 30 = 15. Write the sum or difference formula and compute the exact value. Check against known values. Ten draws; one point for each correct exact answer.

5.5 DOUBLE-ANGLE AND HALF-ANGLE FORMULAS

OBJECTIVES

- Derive the double-angle formulas from the sum formulas, and the half-angle formulas from the double-angle formulas for cosine.
- Use double-angle and half-angle formulas to find exact values, choosing the correct sign from the quadrant.
- Use the formulas to simplify expressions and prove identities.

WATCH

Proving the Double and Half Angle Formulas for Trigonometry (Precalculus - Trigonometry 27)

Professor Leonard · 27:17

<https://www.youtube.com/watch?v=Epa2mi8hzBA>

WHEN TO WATCH

After reading the Core ideas, before Example 1; proves both the double-angle and half-angle formulas.

BACKUP VIDEO

Simplifying a Trigonometric Expansion with Double Angle Identities

Eddie Woo · 8:00

<https://www.youtube.com/watch?v=jFXBttiASGw>

Before the practice set: simplifying with double-angle identities.

KEY TERMS

double-angle formulas

Formulas for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$ in terms of functions of θ .

half-angle formulas

Formulas for $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$ and $\tan \frac{\theta}{2}$ in terms of functions of θ .

power-reducing formulas

The formulas $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$, which replace a square by a first power of cosine.

CORE IDEAS

Double angles. Put $B = A = \theta$ in the sum formulas. From $\sin(A + B) = \sin A \cos B + \cos A \sin B$ we get $\sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta$. From $\cos(A + B) = \cos A \cos B - \sin A \sin B$ we get $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$. From the tangent sum formula, $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$. Notice that $\sin 2\theta$ is **not** $2 \sin \theta$: doubling the angle does not double the sine.

The cosine formula has two other forms. Replace $\sin^2 \theta$ by $1 - \cos^2 \theta$: $\cos 2\theta = 2 \cos^2 \theta - 1$. Replace $\cos^2 \theta$ by $1 - \sin^2 \theta$: $\cos 2\theta = 1 - 2 \sin^2 \theta$. Choose the form that matches what you know or what you want.

Half angles. Solve $\cos 2x = 1 - 2 \sin^2 x$ for $\sin^2 x$: $\sin^2 x = \frac{1 - \cos 2x}{2}$. Likewise $\cos^2 x = \frac{1 + \cos 2x}{2}$. Now let $x = \frac{\theta}{2}$, so $2x = \theta$, and take square roots:

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}.$$

The sign is decided by the quadrant of $\frac{\theta}{2}$, **not** the quadrant of θ . For tangent there is a form with no sign question: multiply $\tan \frac{\theta}{2} = \frac{\sin(\theta/2)}{\cos(\theta/2)}$ top and bottom by $2 \cos \frac{\theta}{2}$ to get $\frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \cos^2(\theta/2)} = \frac{\sin \theta}{1 + \cos \theta}$.

With half angles we can reach new exact values: 22.5° is half of 45° , 15° is half of 30° , and $\frac{\pi}{8}$ is half of $\frac{\pi}{4}$. Half-angle and sum-difference methods must agree. For example,

the half-angle formula gives $\sin 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{2}$, while Lesson 5.4 gave $\frac{\sqrt{6} - \sqrt{2}}{4}$.

Both are about 0.258819; squaring each gives $\frac{2 - \sqrt{3}}{4}$.

KEY FORMULAS

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad (\text{sign from the quadrant of } \frac{\theta}{2})$$

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

From history. Ptolemy's *Almagest* also shows how to find the chord of half an arc from the chord of the whole arc, which is a half-angle formula in chord language. By halving again and again, starting from arcs whose chords he already knew, he reached chords of very small arcs. Those small chords were the starting point for filling in his table.

Euclid connection. In Worked Example 2 the A-frame is an isosceles triangle. The line from the peak straight down to the middle of the base splits it into two congruent right triangles, so it bisects the peak angle. That is why the half angle appears.

WORKED EXAMPLES

Example 1. Given $\sin \theta = \frac{3}{5}$ with θ in Quadrant II, find the exact values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

Show solution

1. First find $\cos \theta$: $\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$, and cosine is negative in Quadrant II, so $\cos \theta = -\frac{4}{5}$.

2. $\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) = -\frac{24}{25}$.

3. $\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2 \cdot \frac{9}{25} = 1 - \frac{18}{25} = \frac{7}{25}$. (Check with the other form: $\cos^2 \theta - \sin^2 \theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$.)

4. $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24/25}{7/25} = -\frac{24}{7}$.

5. Sense check: θ is between 90° and 180° (about 143.1°), so 2θ is about 286.3° , in Quadrant IV, where sine is negative and cosine positive. That matches.

Answer: $\sin 2\theta = -\frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$, $\tan 2\theta = -\frac{24}{7}$

Example 2. Samuel is building an A-frame play tent. Two 10-foot poles meet at the peak, and the angle θ between them satisfies $\cos \theta = \frac{1}{8}$. The height of the peak is $10 \cos \frac{\theta}{2}$ and the width of the base is $2 \cdot 10 \sin \frac{\theta}{2}$. Find the exact height and base width, and give the width to the nearest tenth of a foot.

Show solution

1. The vertical line from the peak bisects the peak angle, making two right triangles with hypotenuse 10 and angle $\frac{\theta}{2}$ at the top. Since $\theta < 180^\circ$, $\frac{\theta}{2}$ is acute, so both half-angle values are positive.

$$2. \cos \frac{\theta}{2} = \sqrt{\frac{1 + \frac{1}{8}}{2}} = \sqrt{\frac{9/8}{2}} = \sqrt{\frac{9}{16}} = \frac{3}{4}. \text{ Height} = 10 \cdot \frac{3}{4} = 7.5 \text{ feet.}$$

$$3. \sin \frac{\theta}{2} = \sqrt{\frac{1 - \frac{1}{8}}{2}} = \sqrt{\frac{7/8}{2}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}.$$

$$4. \text{ Width} = 20 \cdot \frac{\sqrt{7}}{4} = 5\sqrt{7} \approx 13.2 \text{ feet.}$$

$$5. \text{ Check with the Pythagorean theorem on one half: } 7.5^2 + \left(\frac{5\sqrt{7}}{2}\right)^2 = 56.25 + 43.75 = 100 = 10^2. \text{ Correct.}$$

Answer: Height 7.5 feet; base width $5\sqrt{7} \approx 13.2$ feet

PRACTICE

1. Easy If $\sin \theta = 0.6$ and $\cos \theta = 0.8$, find $\sin 2\theta$.

Show answer

$$2(0.6)(0.8) = 0.96$$

2. Easy Find the exact value of $2 \sin 15^\circ \cos 15^\circ$.

Show answer

$$\sin 30^\circ = \frac{1}{2}$$

3. Easy Find the exact value of $\cos^2 \frac{\pi}{8} - \sin^2 \frac{\pi}{8}$.

Show answer

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.707107$$

4. Medium Use a half-angle formula to find the exact value of $\cos 22.5^\circ$.

Show answer

$$\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2} \approx 0.923880 \text{ (positive, since } 22.5^\circ \text{ is in Quadrant I)}$$

5. Medium Use a half-angle formula to find the exact value of $\sin 15^\circ$.

Show answer

$$\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2} \approx 0.258819$$

6. Medium If $\tan \theta = 2$, find the exact value of $\tan 2\theta$.

Show answer

$$\frac{2 \cdot 2}{1 - 4} = -\frac{4}{3}$$

7. Medium An irrigation nozzle sprays water at 14 meters per second at an angle θ above level ground. Ignoring air resistance, the water lands a distance $R = \frac{v^2}{g} \cdot 2 \sin \theta \cos \theta$ away, where $v = 14$ and $g = 9.8$. Rewrite R with a double-angle formula and find R exactly, in meters, when $\theta = 15^\circ$.

Show answer

$$R = \frac{v^2}{g} \sin 2\theta = \frac{196}{9.8} \sin 30^\circ = 20 \cdot \frac{1}{2} = 10 \text{ meters}$$

8. Hard Given $\cos \theta = -\frac{7}{25}$ with $180^\circ < \theta < 270^\circ$, find the exact values of $\sin \frac{\theta}{2}$ and $\cos \frac{\theta}{2}$.

Show answer

$$90^\circ < \frac{\theta}{2} < 135^\circ, \text{ Quadrant II. } \sin \frac{\theta}{2} = +\sqrt{\frac{1 + \frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}; \cos \frac{\theta}{2} = -\sqrt{\frac{1 - \frac{7}{25}}{2}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

9. Hard Prove: $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$.

Show answer

$$\cos^4 \theta - \sin^4 \theta = (\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) = (\cos^2 \theta - \sin^2 \theta) \cdot 1 = \cos 2\theta.$$

10. **Hard** In a barn, a beam rises from the floor at an acute angle θ with $\tan \theta = \frac{3}{4}$. A brace is set to bisect the angle between the beam and the floor. Use $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$ to find the exact tangent of the brace's angle with the floor.

Show answer

$\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$, so $\tan \frac{\theta}{2} = \frac{3/5}{9/5} = \frac{1}{3}$ (the brace makes about 18.4° with the floor)

11. **Challenge** Prove: $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

Show answer

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta = 2 \sin \theta \cos^2 \theta + (1 - 2 \sin^2 \theta) \sin \theta \\ &= 2 \sin \theta (1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta. \end{aligned}$$

12. **Challenge** Use the power-reducing formulas twice to write $\sin^4 \theta$ using only first powers of cosines of multiple angles.

Show answer

$$\begin{aligned} \sin^4 \theta &= \left(\frac{1 - \cos 2\theta}{2} \right)^2 = \frac{1 - 2 \cos 2\theta + \cos^2 2\theta}{4}, \text{ and } \cos^2 2\theta = \frac{1 + \cos 4\theta}{2}, \text{ so} \\ \sin^4 \theta &= \frac{3 - 4 \cos 2\theta + \cos 4\theta}{8}. \end{aligned}$$

COMMON MISTAKES

- Writing $\sin 2\theta = 2 \sin \theta$. Test it at $\theta = 90^\circ$: $\sin 180^\circ = 0$, but $2 \sin 90^\circ = 2$. Use $2 \sin \theta \cos \theta$.
- Choosing the half-angle sign from the quadrant of θ . The sign depends on where $\frac{\theta}{2}$ lies; find that interval first by halving the bounds on θ .
- Writing $\sin \frac{\theta}{2} = \frac{1}{2} \sin \theta$. Halving the angle does not halve the sine; use the square-root formula.
- Forgetting to find the missing function first. To use $\sin 2\theta = 2 \sin \theta \cos \theta$ when only $\sin \theta$ is given, find $\cos \theta$ with the Pythagorean identity and the quadrant of θ .

Make it a game • Three Faces of Cosine. In 30 seconds, write all three forms of $\cos^2 \theta$: $\cos^2 \theta - \sin^2 \theta$, $2\cos^2 \theta - 1$, and $1 - 2\sin^2 \theta$. Then parent names a clue, such as 'you know only $\cos \theta = 4/5$.' Choose the best form, compute the double-angle value, and use half-angle for $\sin 22.5$ degrees.

UNIT 5

TRIGONOMETRIC IDENTITIES

An identity is an equation that is true for every allowed angle. In this unit we collect the basic identities (reciprocal, quotient, cofunction and even-odd), prove the Pythagorean identities from Euclid I.47 on the unit circle, and then derive the sum, difference, double-angle and half-angle formulas. Each new formula is proved from the ones before it, so the unit is a small chain of reasoning in the spirit of Euclid's *Elements*.

Along the way you will learn to prove identities properly: start with one side, justify each step, and end with the other side. You will also learn why some tempting steps, such as squaring both sides, can make a false statement look true, and how a single counterexample settles a false claim.

These identities are the working tools for the rest of the course. They let us find exact values like $\sin 15^\circ$ and $\cos 22.5^\circ$, and they are needed to solve trigonometric equations in Unit 6. Ptolemy used the same ideas, in the language of chords, to build the trigonometric table in his *Almagest*.

PREREQUISITES

- Define all six trigonometric functions for a point (x, y) at distance r on the terminal side of an angle (Lesson 3.6).
- Use reference angles and quadrant signs to evaluate trigonometric functions of any angle (Lesson 3.7).
- Know the exact values for multiples of 30° and 45° ($\frac{\pi}{6}$ and $\frac{\pi}{4}$) and the quadrantal angles (Lessons 2.1–2.3 and 3.8).
- Recognize even and odd symmetry in the graphs of sine, cosine and tangent (Lessons 4.1, 4.2 and 4.6).
- Algebra II skills: factoring (including differences of squares), adding rational expressions, and simplifying radicals.
- Use the distance formula and the Pythagorean theorem.

MASTERY CHECKLIST

- ☐ I can state the reciprocal, quotient, cofunction and even–odd identities and explain where each comes from.
- ☐ I can rewrite an expression in sines and cosines and simplify it.
- ☐ I can derive $\sin^2 \theta$ from the Pythagorean theorem and $\cos^2 \theta$ derive the two companion identities from it.
- ☐ I can find all trigonometric values from one value and the quadrant, choosing the correct signs.
- ☐ I can prove an identity by transforming one side into the other, using strategies such as common denominators, factoring and conjugates.
- ☐ I can explain why working on both sides or squaring both sides is not a valid proof, and disprove a false identity with a counterexample.
- ☐ I can derive the formula for $\cos(A \pm B)$ with the distance formula and derive the other sum and difference formulas from it.
- ☐ I can use sum and difference formulas to find exact values such as $\sin \frac{5\pi}{6}$ and $\cos \frac{7\pi}{12}$.
- ☐ I can derive the double-angle formulas and the three forms of $\cos 2\theta$.

- ☐ I can derive and use the half-angle formulas, choosing the sign from the quadrant of

$$\frac{\theta}{2}.$$

- ☐ I can use identities to solve practical problems involving roofs, cranes, slopes and sprinklers.

LESSONS IN THIS UNIT

5.1 Fundamental Identities	50 minutes
5.2 The Pythagorean Identities	45 minutes
5.3 Simplifying and Proving Identities	2 sessions of 45 minutes
5.4 Sum and Difference Formulas	2 sessions of 45 minutes
5.5 Double-Angle and Half-Angle Formulas	2 sessions of 45 minutes

UNIT 5 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but when a problem asks for an exact value, give it in simplified radical or fraction form; you may add a decimal check. Round decimal answers as each problem directs. For proofs, start with one side and show every step. Total: 100 points.

1. **5 pts**

If $\cos \theta = 0.25$, find $\sec \theta$.

2. **5 pts**

Simplify $\sin(-\theta) \csc \theta$ to a number.

3. **6 pts**

Given $\sin \theta = -\frac{8}{17}$ with θ in Quadrant III, find the exact values of $\cos \theta$ and $\tan \theta$.

4. (8 pts)

A path climbs a hillside on Mr. Miller's farm at an acute angle θ with $\tan \theta = \frac{5}{12}$. Ruth walks 65 meters along the path. Use the Pythagorean identities to find $\cos \theta$ and $\sin \theta$, then find the horizontal distance $65 \cos \theta$ and the height gained $65 \sin \theta$, in meters.

5. (6 pts)

Simplify $(1 - \sin^2 \theta)(1 + \tan^2 \theta)$ to a number.

6. (8 pts)

Prove: $\frac{\sin \theta}{1 - \cos \theta} = \frac{1 + \cos \theta}{\sin \theta}$.

7. (6 pts)

Samuel claims that $(\sin \theta - \cos \theta)^2 = 1$ is an identity. Find the exact value of the left side at $\theta = 45^\circ$, and explain whether one such check settles the question.

8. (6 pts)

Find the exact value of $\sin 105^\circ$.

9. (6 pts)

Find the exact value of $\tan 75^\circ$ in simplest form.

10. (8 pts)

Suppose $\sin A = \frac{5}{13}$ and $\cos B = \frac{3}{5}$, with A and B both in Quadrant I. Find the exact values of $\sin(A + B)$ and $\cos(A - B)$.

11. (8 pts)

A crane's 30-meter boom points 60° above horizontal. The operator lowers it by 45° . The height of the tip above the pivot is then $30 \sin(60^\circ - 45^\circ)$. Use a difference formula to find the height exactly, then to the nearest hundredth of a meter.

12. (8 pts)

Given $\cos \theta = \frac{5}{13}$ with θ in Quadrant IV, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

13. (6 pts)

Use a half-angle formula to find the exact value of $\sin 22.5^\circ$.

14. (6 pts)

A garden sprinkler throws water at 14 meters per second at an angle θ above level ground. Ignoring air resistance, the water lands $R = \frac{v^2}{g} \cdot 2 \sin \theta \cos \theta$ meters away, with $v = 14$ and $g = 9.8$. Rewrite R with a double-angle formula and find R when $\theta = 30^\circ$, exactly and to the nearest tenth of a meter.

15. (8 pts)

Prove: $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$.

ANSWER KEY

$$1. \sec \theta = \frac{1}{0.25} = 4$$

$$2. -\sin \theta \cdot \frac{1}{\sin \theta} = -1 \text{ (where } \sin \theta \neq 0 \text{)}$$

$$3. \cos \theta = -\frac{15}{17}, \tan \theta = \frac{8}{15}$$

$$4. \sec^2 \theta = 1 + \frac{25}{144} = \frac{169}{144}, \text{ so } \cos \theta = \frac{12}{13} \text{ and } \sin \theta = \tan \theta \cos \theta = \frac{5}{13}. \text{ Horizontal distance 60 m; height gained 25 m.}$$

$$5. \cos^2 \theta \cdot \sec^2 \theta = 1 \text{ (where } \cos \theta \neq 0 \text{)}$$

$$6. \text{ Multiply the left side by } \frac{1 + \cos \theta}{1 + \cos \theta}: \frac{\sin \theta(1 + \cos \theta)}{1 - \cos^2 \theta} = \frac{\sin \theta(1 + \cos \theta)}{\sin^2 \theta} = \frac{1 + \cos \theta}{\sin \theta}.$$

$$7. \text{ At } 45^\circ \text{ the left side is } \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)^2 = 0 \neq 1. \text{ One counterexample is enough to show it is not an identity. (The correct identity is } (\sin \theta - \cos \theta)^2 = 1 - 2 \sin \theta \cos \theta \text{.)}$$

$$8. \sin(60^\circ + 45^\circ) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.965926$$

$$9. \tan(45^\circ + 30^\circ) = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3} \approx 3.732051$$

$$10. \cos A = \frac{12}{13}, \sin B = \frac{4}{5}, \sin(A + B) = \frac{5}{13} \cdot \frac{3}{5} + \frac{12}{13} \cdot \frac{4}{5} = \frac{63}{65}; \cos(A - B) = \frac{12}{13} \cdot \frac{3}{5} - \frac{4}{5} \cdot \frac{5}{13} = \frac{56}{65}$$

$$11. \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}, \text{ so the height is } \frac{15(\sqrt{6} - \sqrt{2})}{2} \approx 7.76 \text{ meters}$$

$$12. \sin \theta = -\frac{12}{13}, \sin 2\theta = 2 \left(-\frac{12}{13} \right) \frac{5}{13} = -\frac{120}{169}; \cos 2\theta = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$$

$$13. \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2} \approx 0.382683$$

$$14. R = 20 \sin 2\theta = 20 \sin 60^\circ = 10\sqrt{3} \approx 17.3 \text{ meters}$$

$$15. \text{ Use } \sin 2\theta = 2 \sin \theta \cos \theta \text{ and } \cos 2\theta = 2 \cos^2 \theta - 1: \frac{2 \sin \theta \cos \theta}{1 + 2 \cos^2 \theta - 1} = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS

In Unit 1 we used $\sin^{-1}$ to turn a ratio back into an acute angle. Now that sine, cosine and tangent are defined for every angle, that question needs more care. Each function repeats forever, so many angles share the same value. In this unit we learn why each function must be restricted to a carefully chosen interval before it has an inverse, and we study the inverse functions arcsine, arccosine and arctangent: their domains, ranges, graphs and values.

We then turn the question around and solve trigonometric equations, finding every angle that makes an equation true. We begin with basic equations like $2 \sin x - 1 = 0$, write both the solutions in one turn and the general solution, and then solve harder equations by factoring, by using the identities of Unit 5, and by handling multiple angles like $\sin 2x$.

These skills join everything before them. The unit circle of Unit 3 explains why equations have two solutions per turn, the graphs of Unit 4 explain the shapes of the inverse functions, and the identities of Unit 5 become tools for solving. Inverse functions are also how we will find angles in the oblique triangles of Unit 7 and in the surveying, navigation and periodic-motion problems of Unit 8.

PREREQUISITES

- Find inverse ratios of acute angles and solve right triangles (Lesson 1.4).
- Know the exact values of sine, cosine and tangent for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (Lesson 3.8).
- Use reference angles and the signs of the functions by quadrant (Lesson 3.7).
- Recognize the graphs, periods and asymptotes of sine, cosine and tangent (Lessons 4.1, 4.2 and 4.6).
- Use the Pythagorean, sum and double-angle identities (Lessons 5.2, 5.4 and 5.5).
- Factor quadratic expressions and use the zero-product property (Algebra II).

MASTERY CHECKLIST

- ☐ I can explain why sine, cosine and tangent must be restricted before they have inverse functions, and name each restricted interval.
- ☐ I can state the domain and range of $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arctan} x$.
- ☐ I can evaluate inverse trigonometric functions exactly for special values and with a calculator in radians or degrees.
- ☐ I can explain why $\arccos$ of a negative number is in Quadrant II while $\arcsin$ and $\arctan$ of a negative number are in Quadrant IV.
- ☐ I can tell when $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ are correct and evaluate compositions like $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$.

- ☐ I can find values like

$\sin\left(\arccos\frac{3}{5}\right)$ by drawing a right triangle in the correct quadrant.

- ☐ I can solve

$$\sin x = c$$

$$\text{and } \tan x = c$$

and

for $[0, 2\pi)$

every

where

you

solve

using

the

identity

and

the

value

of

the

angle

is

the

same

as

the

angle

.

- ☐ I can solve quadratic-type trigonometric equations by factoring.

- ☐ I can use identities to rewrite an equation in one function, and I factor instead of dividing by an expression that could be zero.

- ☐ I can solve multiple-angle equations like $\sin 2\theta$ by enlarging the interval first.
 $\frac{1}{2}$
- ☐ I can set up and solve word problems with ladders, ramps, rafters, water wheels and daylight models using inverse functions and trigonometric equations.

LESSONS IN THIS UNIT

6.1 Inverse Sine and Restricted Domains	50 minutes
6.2 Inverse Cosine and Inverse Tangent	50 minutes
6.3 Compositions with Inverse Functions	50 minutes
6.4 Solving Basic Trigonometric Equations	50 minutes
6.5 Equations Using Factoring, Identities and Multiple Angles	
2 sessions of 45 minutes	

UNIT 6 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but give exact values wherever a problem asks for them. Unless a problem says otherwise, give angles in radians, and give all solutions in the interval $[0, 2\pi)$. Round as each problem directs; do not round intermediate values. Show your work: setting up each problem correctly earns most of the credit. 100 points total.

1. **5 pts**

Find the exact value of $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ in radians.

2. **5 pts**

Find the exact values in radians: (a) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\arctan \sqrt{3}$.

3. **5 pts**

Use a calculator to find $\arcsin(0.6)$ in degrees, to the nearest tenth of a degree.

4. **5 pts**

A wheelchair ramp at the church entrance rises 2 feet over a horizontal run of 24 feet. Find the angle the ramp makes with the ground, to the nearest tenth of a degree.

5. **5 pts**

Explain why the domain of $\arctan x$ is all real numbers, while the domain of $\arcsin x$ is only $[-1, 1]$.

6. **6 pts**

Find the exact value of $\sin\left(\arccos\left(-\frac{4}{5}\right)\right)$.

7. **6 pts**

Find the exact value of $\arccos\left(\cos \frac{5\pi}{3}\right)$ in radians.

8. **6 pts**

A rafter 26 feet long makes an angle of $\arctan \frac{5}{12}$ with the horizontal. Find the rise of the rafter, $26 \sin\left(\arctan \frac{5}{12}\right)$, exactly.

9. **8 pts**

Solve $2 \cos x - \sqrt{2} = 0$ for all x in $[0, 2\pi)$, giving exact values. Then write the general solution.

10. **8 pts**

Solve $4 \sin x + 1 = 0$ for all x in $[0, 2\pi)$. Round to the nearest thousandth of a radian.

11. 8 pts

The height of a bucket on a water wheel is $h(t) = 7 + 9 \sin\left(\frac{\pi t}{20}\right)$ feet above the water, t seconds after it passes the level of the center going up. During the first turn ($0 \leq t < 40$), at what times is the bucket 11.5 feet above the water? Give exact values and decimals to the nearest tenth of a second.

12. 8 pts

Solve $2 \sin^2 x + \sin x - 1 = 0$ for all x in $[0, 2\pi)$. Give exact values.

13. 8 pts

Solve $\cos 2x = \cos x$ for all x in $[0, 2\pi)$. Give exact values.

14. 8 pts

Solve $\sin 2x = \frac{\sqrt{3}}{2}$ for all x in $[0, 2\pi)$. Give exact values.

15. 9 pts

Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for every x in $[-1, 1]$.

ANSWER KEY

1. $-\frac{\pi}{4}$
2. (a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$
3. $\approx 36.9^\circ$
4. $\arctan \frac{2}{24} = \arctan \frac{1}{12} \approx 4.8^\circ$
5. The domain of an inverse function is the set of outputs of the original function. Sine only takes values between -1 and 1 , so only those numbers have an arcsine. Tangent takes every real value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, since every slope is the tangent of some angle, so every real number has an arctangent.
6. The angle is in Quadrant II with $x = -4$, $r = 5$, so $y = 3$ and the value is $\frac{3}{5}$.
7. $\cos \frac{5\pi}{3} = \frac{1}{2}$, so the value is $\arccos \frac{1}{2} = \frac{\pi}{3}$.
8. The angle sits in a 5-12-13 triangle, so its sine is $\frac{5}{13}$ and the rise is $26 \cdot \frac{5}{13} = 10$ feet.
9. $\cos x = \frac{\sqrt{2}}{2}$, so $x = \frac{\pi}{4}, \frac{7\pi}{4}$; general solution $x = \pm \frac{\pi}{4} + 2k\pi$, k any integer.
10. $\sin x = -0.25$; the reference angle is $\arcsin 0.25 \approx 0.2527$. Sine is negative in Quadrants III and IV, so $x \approx \pi + 0.2527 \approx 3.394$ and $x \approx 2\pi - 0.2527 \approx 6.031$.
11. $\sin\left(\frac{\pi t}{20}\right) = \frac{4.5}{9} = \frac{1}{2}$, so $\frac{\pi t}{20} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Then $t = \frac{10}{3} \approx 3.3$ s and $t = \frac{50}{3} \approx 16.7$ s.
12. $(2 \sin x - 1)(\sin x + 1) = 0$, so $\sin x = \frac{1}{2}$ or $\sin x = -1$: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
13. Use $\cos 2x = 2 \cos^2 x - 1$: $2 \cos^2 x - \cos x - 1 = 0$, so $(2 \cos x + 1)(\cos x - 1) = 0$. Then $\cos x = 1$ gives $x = 0$, and $\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}, \frac{4\pi}{3}$. So $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.
14. With $0 \leq 2x < 4\pi$: $2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$, so $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$.
15. Let $\theta = \arcsin x$, so $\sin \theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $0 \leq \frac{\pi}{2} - \theta \leq \pi$, and by the cofunction identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta = x$. So $\frac{\pi}{2} - \theta$ is the one angle in $[0, \pi]$ whose cosine is x ; that is, $\arccos x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arcsin x$. Adding $\arcsin x$ to both sides gives the result.

6.1 INVERSE SINE AND RESTRICTED DOMAINS

OBJECTIVES

- Explain why the sine function must be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ before it can have an inverse.
- Evaluate $\arcsin x$ exactly for special values and with a calculator for other values, in radians or degrees.
- Graph $y = \arcsin x$ and state its domain, range and symmetry.

WATCH

Inverse Trigonometric Functions (1 of 4: Choosing a Domain for the Inverse of sine)

Eddie Woo · 8:11

<https://www.youtube.com/watch?v=7JXMfs7Mg-8>

WHEN TO WATCH

After reading the Core ideas on restricting the domain, before Example 1.

Note: Woo writes $\sin^{-1}x$; this course also writes $\arcsin x$.

BACKUP VIDEO

Introduction to Inverse Trig Functions (1 of 2): Why a whole new topic?

Eddie Woo · 6:04

<https://www.youtube.com/watch?v=Xx8UxNprQtw>

Before the Core ideas, as a 'why do we need this' preview.

KEY TERMS

one-to-one function

A function that never takes the same output twice. Its graph passes the horizontal line test, and only such a function has an inverse function.

restricted domain

A smaller part of a function's natural domain, chosen so that the function is one-to-one there.

inverse sine

$\arcsin x$ (also written $\sin^{-1} x$) is the one angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin y = x$. It is defined for $-1 \leq x \leq 1$.

principal value

The single value an inverse trigonometric function returns, chosen from its restricted range.

CORE IDEAS

In Lesson 1.4 we used $\sin^{-1}$ to find an acute angle from a ratio. Now sine is defined for every angle, and a problem appears. A function has an inverse only if it is **one-to-one**: each output comes from exactly one input. Sine is not. For example, $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$, and because sine has period 2π , infinitely many other angles give $\frac{1}{2}$ as well. The horizontal line $y = \frac{1}{2}$ crosses the sine graph infinitely often. So the question 'which angle has sine $\frac{1}{2}$?' has no single answer until we agree where to look.

We agree to look in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. On this interval sine rises steadily from -1 to 1 (on the unit circle the point moves up the right half of the circle from the bottom to the top), so it takes every value in $[-1, 1]$ exactly once. The choice is a wise one for several reasons: it contains 0 , it contains all the acute angles (so it agrees with the $\sin^{-1}$ of Unit 1), it has no gaps, and it is centered at 0 , so it keeps the symmetry $\sin(-\theta) = -\sin \theta$.

Definition. $y = \arcsin x$ means $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. The domain is $[-1, 1]$ because sine never leaves that interval, and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, or $[-90^\circ, 90^\circ]$ in degrees. So the answer always lies in Quadrant I, Quadrant IV, or on the axis between them. A positive input gives a Quadrant I angle; a negative input gives a negative angle in Quadrant IV. To evaluate, ask 'which angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has this sine?' For other values use the calculator's $\sin^{-1}$ key, in radian or degree mode as the problem requires.

The graph of an inverse function is the reflection of the original graph across the line $y = x$, because reflecting swaps each point (a, b) to (b, a) . Reflecting the restricted sine curve gives the graph of $y = \arcsin x$ through $\left(-1, -\frac{\pi}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\pi}{6}\right)$, $(0, 0)$, $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and $\left(1, \frac{\pi}{2}\right)$. It is increasing, it stops at the endpoints $x = \pm 1$, it becomes very

steep near those endpoints (where the sine curve was flat), and it is symmetric about the origin: $\arcsin(-x) = -\arcsin x$, so arcsine is an odd function.

KEY FORMULAS

$$y = \arcsin x \iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\text{Domain: } [-1, 1] \quad \text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\arcsin(-x) = -\arcsin x$$

From history. The prefix *arc* comes from the unit circle. On a circle of radius 1 the radian measure of an angle equals the length of its arc, so $\arcsin x$ is literally 'the arc whose sine is x .' The notation $\sin^{-1} x$ was introduced by the English astronomer John Herschel in 1813. Both notations are still in use, which is why your calculator key says $\sin^{-1}$ while many books write $\arcsin$.

WORKED EXAMPLES

Example 1. Find the exact value in radians: (a) $\arcsin \frac{\sqrt{3}}{2}$ (b) $\arcsin \left(-\frac{\sqrt{3}}{2}\right)$.

Show solution

- (a) We need the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{3}}{2}$.
- From the 30-60-90 triangle, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and $\frac{\pi}{3}$ lies in the interval. So

$$\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$
- (b) The input is negative, so the angle must be in $\left[-\frac{\pi}{2}, 0\right)$, that is, in Quadrant IV.
- Since $\sin \left(-\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, we get $\arcsin \left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.
- Note: $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ also have sine $-\frac{\sqrt{3}}{2}$, but they are not in the range of arcsine, so they are not the answer.

Answer: (a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$

Example 2. Samuel leans a 12-foot ladder against the side of the barn. The top of the ladder touches the wall 10.5 feet above the ground. Find the angle the ladder makes with the ground, to the nearest tenth of a degree. Then explain why the calculator gives this angle and not 180° minus it, which has the same sine.

Show solution

1. The ladder is the hypotenuse, and the height on the wall is opposite the angle θ at the ground. So $\sin \theta = \frac{10.5}{12} = 0.875$.
2. Then $\theta = \arcsin(0.875)$. In degree mode the calculator gives $61.04497\dots^\circ$.
3. Rounded to the nearest tenth, $\theta \approx 61.0^\circ$.
4. The angle $180^\circ - 61.04^\circ = 118.96^\circ$ also has sine 0.875. But arcsine returns only the angle in $[-90^\circ, 90^\circ]$, and that is the one we want: the ladder, wall and ground form a right triangle, so the angle at the ground must be acute.

Answer: $\theta \approx 61.0^\circ$

PRACTICE

1. Easy Find the exact value of $\arcsin \frac{1}{2}$ in radians.

Show answer

$$\frac{\pi}{6}$$

2. Easy Find the exact value of $\arcsin(-1)$ in radians.

Show answer

$$-\frac{\pi}{2}$$

3. Easy Find $\arcsin \frac{\sqrt{2}}{2}$ in degrees.

Show answer

$$45^\circ$$

4. **Medium** Find $\arcsin\left(-\frac{1}{2}\right)$ in degrees. Explain why the answer is not 210° or 330° .

Show answer

-30° . Both 210° and 330° have sine $-\frac{1}{2}$, but the range of arcsine is $[-90^\circ, 90^\circ]$, and -30° is the only angle there with that sine.

5. **Medium** Use a calculator to find $\arcsin(0.3)$ in radians, to the nearest thousandth.

Show answer

$$\approx 0.305$$

6. **Medium** Evaluate $\arcsin(1.2)$, or explain why it does not exist.

Show answer

It does not exist. The sine of any angle lies between -1 and 1 , so no angle has sine 1.2 ; the domain of arcsine is $[-1, 1]$.

7. **Medium** State the range of $y = 3 \arcsin x$ as an interval, using exact values.

Show answer

$$\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$$

8. **Hard** Ruth flies a kite on 150 feet of string, held taut. The kite is 95 feet higher than her hands. Find the angle the string makes with the horizontal, to the nearest tenth of a degree.

Show answer

$$\arcsin \frac{95}{150} \approx 39.3^\circ$$

9. **Hard** A rafter 16 feet long rises 6 feet from the top of the wall to the ridge. Find the angle the rafter makes with the horizontal, to the nearest tenth of a degree.

Show answer

$$\arcsin \frac{6}{16} \approx 22.0^\circ$$

10. **Hard** Explain why mathematicians do not restrict sine to $[0, \pi]$ to define arcsine.

Show answer

Sine is not one-to-one on $[0, \pi]$: for example $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6} = \frac{1}{2}$. Also, on $[0, \pi]$ sine only takes values from 0 to 1, so negative inputs would have no arcsine.

On $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ sine takes every value in $[-1, 1]$ exactly once.

11. **Challenge** Find the domain of $f(x) = \arcsin(2x - 1)$.

Show answer

We need $-1 \leq 2x - 1 \leq 1$, so $0 \leq 2x \leq 2$, and the domain is $[0, 1]$.

12. **Challenge** Prove that $\arcsin(-x) = -\arcsin x$ for every x in $[-1, 1]$.

Show answer

Let $\theta = \arcsin x$, so $\sin \theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $\sin(-\theta) = -\sin \theta = -x$, and $-\theta$ also lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since arcsine returns the one angle in that interval whose sine is $-x$, $\arcsin(-x) = -\theta = -\arcsin x$.

COMMON MISTAKES

- Giving an angle outside the range, such as $\arcsin\left(-\frac{1}{2}\right) = \frac{11\pi}{6}$. The answer must lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so negative inputs give negative angles: here $-\frac{\pi}{6}$.
- Reading $\sin^{-1} x$ as $\frac{1}{\sin x}$. The -1 means inverse; the reciprocal of sine is $\csc x$.
- Leaving the calculator in the wrong mode. Check whether the problem asks for radians or degrees before pressing $\sin^{-1}$.
- Trying to evaluate arcsine of a number outside $[-1, 1]$. A calculator error here is correct: no such angle exists.

Make it a game • Principal Value Trap. Draw a sine graph and slide a pencil across horizontally to see it hits many points, so it is not one-to-one. Shade the interval from negative pi over 2 to pi over 2. Parent calls tempting wrong answers for arcsin of negative one half, such as 11 pi over 6. Catch them and give the principal value.

6.2 INVERSE COSINE AND INVERSE TANGENT

OBJECTIVES

- State the restricted domains used to define $\arccos x$ and $\arctan x$, and give the domain and range of each inverse function.
- Evaluate $\arccos x$ and $\arctan x$ exactly for special values and with a calculator for other values.
- Graph $y = \arccos x$ and $y = \arctan x$, including the horizontal asymptotes of arctangent.

WATCH

Inverse Trigonometric Functions (3 of 4: Determining the domain of $\cos^{-1}x$ and $\tan^{-1}x$)

Eddie Woo · 7:32

<https://www.youtube.com/watch?v=fXNcPjWvf9o>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo writes $\cos^{-1}x$ and $\tan^{-1}x$ for $\arccos$ and $\arctan$.

BACKUP VIDEO

Inverse Trigonometric Functions: $\tan^{-1}(x)$

Eddie Woo · 6:39

<https://www.youtube.com/watch?v=R148LmmJdq8>

Before the practice set: the arctangent graph and its asymptotes.

KEY TERMS

inverse cosine

$\arccos x$ (also written $\cos^{-1} x$) is the one angle y in $[0, \pi]$ with $\cos y = x$. It is defined for $-1 \leq x \leq 1$.

inverse tangent

$\arctan x$ (also written $\tan^{-1} x$) is the one angle y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan y = x$. It is defined for every real number x .

horizontal asymptote

A horizontal line that a graph approaches more and more closely as x grows very large or very negative. The graph of $y = \arctan x$ has the horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

CORE IDEAS

Cosine cannot use the interval chosen for sine, because $\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$: cosine is not one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Instead we restrict cosine to $[0, \pi]$. There the point on the unit circle travels over the top half of the circle from $(1, 0)$ to $(-1, 0)$, so its x -coordinate falls steadily from 1 to -1 and takes each value exactly once. This interval also contains all the acute angles. **Definition:** $y = \arccos x$ means $\cos y = x$ and $0 \leq y \leq \pi$. The domain is $[-1, 1]$; the range is $[0, \pi]$, or $[0^\circ, 180^\circ]$. The answer is never negative: a positive input gives a Quadrant I angle, and a negative input gives a Quadrant II angle.

The graph of $y = \arccos x$ is the restricted cosine curve reflected across $y = x$. It passes through $(-1, \pi)$, $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $(1, 0)$, and it is decreasing. It is not symmetric about the origin. Instead it satisfies $\arccos(-x) = \pi - \arccos x$. **Proof:** let $\theta = \arccos x$, so $\cos \theta = x$ and $0 \leq \theta \leq \pi$. By the difference formula, $\cos(\pi - \theta) = -\cos \theta = -x$, and $\pi - \theta$ also lies in $[0, \pi]$. So $\pi - \theta$ is the angle arccosine returns for $-x$.

Tangent has period π and vertical asymptotes at $x = \frac{\pi}{2} + k\pi$; the two at $x = \pm\frac{\pi}{2}$ bound the interval around 0. On the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ it rises from very large negative values to very large positive values and takes every real value exactly once. So we restrict tangent to that interval. The endpoints are left out because tangent is undefined there. **Definition:** $y = \arctan x$ means $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The domain is all real numbers, since any slope is the tangent of some angle; the range is the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Reflecting the restricted tangent curve across $y = x$ turns its vertical asymptotes $x = \pm\frac{\pi}{2}$ into the **horizontal asymptotes** $y = \pm\frac{\pi}{2}$ of $y = \arctan x$. The graph passes through $\left(-1, -\frac{\pi}{4}\right)$, $(0, 0)$, $\left(1, \frac{\pi}{4}\right)$ and $\left(\sqrt{3}, \frac{\pi}{3}\right)$. It rises everywhere, but more and

more slowly, and it is odd: $\arctan(-x) = -\arctan x$. A summary worth memorizing: arcsine and arctangent return angles in Quadrants I and IV; arccosine returns angles in Quadrants I and II.

KEY FORMULAS

$$y = \arccos x \iff \cos y = x \text{ and } 0 \leq y \leq \pi$$

$$y = \arctan x \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\arccos(-x) = \pi - \arccos x, \quad \arctan(-x) = -\arctan x$$

From history. In 1671 the Scottish mathematician James Gregory found the infinite series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ for $-1 \leq x \leq 1$. Putting $x = 1$ gives $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$, a series Leibniz also published. Mathematicians of the Kerala school in India had found the same series earlier. Series like this let people compute angles and π itself to many decimal places long before calculators.

Euclid connection. The identity $\arccos(-x) = \pi - \arccos x$ says the two angles are supplementary. Euclid I.13 shows that the two angles a line makes with another line on which it stands together equal two right angles, and that is exactly the picture: the angles for x and $-x$ sit side by side on the diameter of the unit circle.

WORKED EXAMPLES

Example 1. Find the exact value in radians: (a) $\arccos\left(-\frac{1}{2}\right)$ (b) $\arctan(-\sqrt{3})$.

Show solution

1. (a) We need the angle in $[0, \pi]$ whose cosine is $-\frac{1}{2}$. The input is negative, so the angle is in Quadrant II.
 2. The reference angle with cosine $\frac{1}{2}$ is $\frac{\pi}{3}$. The Quadrant II angle with that reference angle is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. So $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. (Not $-\frac{\pi}{3}$: arccosine is never negative.)
 3. (b) We need the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $-\sqrt{3}$. The input is negative, so the angle is in Quadrant IV.
 4. Since $\tan \frac{\pi}{3} = \sqrt{3}$ and tangent is odd, $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$. So $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$.
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Answer: (a) $\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$

Example 2. A 20-foot ladder leans against a silo, with its foot 6 feet from the base of the silo on level ground. Use inverse cosine to find the angle the ladder makes with the ground, to the nearest tenth of a degree. Then check the answer with inverse tangent.

Show solution

1. The ladder is the hypotenuse and the 6-foot distance is adjacent to the angle θ at the ground, so $\cos \theta = \frac{6}{20} = 0.3$.
2. Then $\theta = \arccos(0.3)$. In degree mode the calculator gives $72.5424\dots^\circ$, so $\theta \approx 72.5^\circ$.
3. Check: by the Pythagorean theorem the top of the ladder is $\sqrt{20^2 - 6^2} = \sqrt{364} \approx 19.08$ feet up the silo.
4. So $\tan \theta = \frac{\sqrt{364}}{6}$, and $\arctan\left(\frac{\sqrt{364}}{6}\right) = 72.5424\dots^\circ$ (enter $\sqrt{364}$ exactly, not the rounded 19.08). The two methods agree.

Answer: $\theta \approx 72.5^\circ$

PRACTICE

1. Easy Find the exact value of $\arccos \frac{1}{2}$ in radians.

Show answer

$$\frac{\pi}{3}$$

2. Easy Find the exact value of $\arctan 1$ in radians.

Show answer

$$\frac{\pi}{4}$$

3. Easy Find the exact value of $\arccos(-1)$ in radians.

Show answer

$$\pi$$

4. Medium Find $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ in degrees.

Show answer

$$135^\circ$$

5. **Medium** Find the exact value of $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ in radians.

Show answer

$$-\frac{\pi}{6}$$

6. **Medium** Use a calculator to find $\arccos(0.2)$ in radians, to the nearest hundredth.

Show answer

$$\approx 1.37$$

7. **Medium** Mr. Miller's barn roof has a pitch of 5 in 12: it rises 5 inches for every 12 inches of horizontal run. Find the angle the roof makes with the horizontal, to the nearest tenth of a degree.

Show answer

$$\arctan \frac{5}{12} \approx 22.6^\circ$$

8. **Hard** A 24-foot ladder leans against the barn wall with its foot 7 feet from the wall on level ground. Find the angle the ladder makes with the ground, to the nearest tenth of a degree.

Show answer

$$\arccos \frac{7}{24} \approx 73.0^\circ$$

9. **Hard** Find $\arctan(1000)$ in radians to the nearest thousandth, and explain why it is so close to $\frac{\pi}{2}$.

Show answer

≈ 1.570 . An angle whose tangent is 1000 is a very steep angle, just short of a right angle; the graph of $y = \arctan x$ approaches its horizontal asymptote $y = \frac{\pi}{2} \approx 1.5708$ but never reaches it.

10. **Hard** Given that $\arccos(0.2) \approx 1.3694$, use the identity $\arccos(-x) = \pi - \arccos x$ to find $\arccos(-0.2)$ to the nearest hundredth, without the inverse cosine key.

Show answer

$$\pi - 1.3694 \approx 1.77$$

11. **Challenge** Prove that $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$.

Show answer

Let $\alpha = \arctan \frac{1}{2}$ and $\beta = \arctan \frac{1}{3}$. By the sum formula, $\tan(\alpha + \beta) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{5/6}{5/6} = 1$. Since $0 < \frac{1}{3} < \frac{1}{2} < 1$, both α and β lie between 0 and $\frac{\pi}{4}$, so $0 < \alpha + \beta < \frac{\pi}{2}$. The only angle in that interval with tangent 1 is $\frac{\pi}{4}$.

12. **Challenge** Find the domain of $g(x) = \arccos\left(\frac{x}{3} - 1\right)$.

Show answer

We need $-1 \leq \frac{x}{3} - 1 \leq 1$, so $0 \leq \frac{x}{3} \leq 2$, and the domain is $[0, 6]$.

COMMON MISTAKES

- Giving a negative answer for arccosine, such as $\arccos\left(-\frac{1}{2}\right) = -\frac{\pi}{3}$. The range of arccosine is $[0, \pi]$, so negative inputs give Quadrant II angles: here $\frac{2\pi}{3}$.
- Using the arcsine range for arccosine, or the other way around. Memorize: arcsine and arctangent give Quadrants I and IV; arccosine gives Quadrants I and II.
- Thinking arctangent is limited to inputs between -1 and 1 . Tangent takes every real value, so $\arctan x$ is defined for all x ; it is the output that is limited, to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- Including the endpoints $\pm\frac{\pi}{2}$ in the range of arctangent. Tangent is undefined there, so these values are never reached; they are asymptotes.

Make it a game • Floor Circle. Tape a big unit circle on the floor. Parent calls $\arcsin(x)$, $\arccos(x)$ or $\arctan(x)$ with a special value. Step onto the arc where the answer lies, and say the angle and the range allowed: right half for arcsin, upper half for arccos. Tangent gives an open interval. Ten correct steps wins.

6.3 COMPOSITIONS WITH INVERSE FUNCTIONS

OBJECTIVES

- Decide when $f(f^{-1}(x)) = x$ and when $f^{-1}(f(x)) = x$ for sine, cosine and tangent, and evaluate such compositions when they do not simplify directly.
- Evaluate expressions such as $\sin(\arccos x)$ exactly by drawing a right triangle in the correct quadrant.
- Rewrite compositions such as $\sin(\arctan x)$ as algebraic expressions in x .

WATCH

Considering Inverse Trigonometric Expressions as Angles in Triangles

Eddie Woo · 7:46

<https://www.youtube.com/watch?v=Zs0vObiAu0o>

WHEN TO WATCH

Before Example 1: drawing the right triangle for $\sin(\arccos x)$.

BACKUP VIDEO

Evaluating Trig & Inv Trig Ratios (1 of 2)

Eddie Woo · 12:30

<https://www.youtube.com/watch?v=wYlPfAcAB20>

Before Example 2, for more evaluation practice.

KEY TERMS

composition

Applying one function to the output of another, as in $\sin(\arccos x)$: first find the angle $\arccos x$, then take its sine.

principal range

The restricted set of angles an inverse function returns: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for arcsine, $[0, \pi]$ for arccosine and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for arctangent.

reference triangle

A right triangle drawn from the terminal side of an angle to the x -axis, with signed legs x and y and hypotenuse $r > 0$.

CORE IDEAS

Function after inverse. $\arcsin x$ is, by definition, an angle whose sine is x . So $\sin(\arcsin x) = x$ every time the left side makes sense, that is, for $-1 \leq x \leq 1$. In the same way $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$, and $\tan(\arctan x) = x$ for every real x . Outside the domain there is nothing to compute: $\sin(\arcsin 2)$ is undefined, because $\arcsin 2$ does not exist.

Inverse after function. The other order is trickier, because the inverse function only returns angles in its principal range. $\arcsin(\sin x) = x$ only when x is already in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Otherwise first compute $\sin x$, then ask which angle *in the range* has that sine. For example, $\sin \frac{5\pi}{6} = \frac{1}{2}$, so $\arcsin\left(\sin \frac{5\pi}{6}\right) = \arcsin \frac{1}{2} = \frac{\pi}{6}$, not $\frac{5\pi}{6}$. Likewise $\arccos(\cos x) = x$ only for $0 \leq x \leq \pi$, and $\arctan(\tan x) = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Mixed compositions with a triangle. To find $\sin\left(\arccos \frac{3}{5}\right)$, name the angle: let $\theta = \arccos \frac{3}{5}$. Then $\cos \theta = \frac{3}{5}$ and θ is in $[0, \pi]$; since the cosine is positive, θ is in Quadrant I. Draw a right triangle with adjacent side 3 and hypotenuse 5. The opposite side is $\sqrt{5^2 - 3^2} = 4$, so $\sin \theta = \frac{4}{5}$. When the input is negative, the range tells you the quadrant and so the signs: arcsine or arctangent of a negative number lies in Quadrant IV ($x > 0, y < 0$), and arccosine of a negative number lies in Quadrant II ($x < 0, y > 0$).

Algebraic forms. The same reasoning works with a variable. Let $\theta = \arccos x$. Then $\cos \theta = x$, and by the Pythagorean identity $\sin \theta = \pm\sqrt{1 - x^2}$. Because θ lies in $[0, \pi]$, where sine is never negative, we take the positive root: $\sin(\arccos x) = \sqrt{1 - x^2}$. In the same way, since arcsine returns angles in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where cosine is never negative, $\cos(\arcsin x) = \sqrt{1 - x^2}$. Always use the range to decide the sign of a square root.

KEY FORMULAS

$$\sin(\arcsin x) = x \quad (-1 \leq x \leq 1), \quad \arcsin(\sin x) = x \quad \left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right)$$

$$\cos(\arccos x) = x \quad (-1 \leq x \leq 1), \quad \arccos(\cos x) = x \quad (0 \leq x \leq \pi)$$

$$\tan(\arctan x) = x \quad (\text{all } x), \quad \arctan(\tan x) = x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$\sin(\arccos x) = \cos(\arcsin x) = \sqrt{1 - x^2}$$

Euclid connection. Every triangle in this lesson is completed with the Pythagorean theorem, Euclid I.47: given two sides of the right triangle, the third is found from $a^2 + b^2 = c^2$. The identity $\sin(\arccos x) = \sqrt{1 - x^2}$ is simply I.47 written for a triangle with hypotenuse 1 and one leg x .

WORKED EXAMPLES

Example 1. Find the exact value: (a) $\sin\left(\arccos \frac{3}{5}\right)$ (b) $\tan\left(\arcsin\left(-\frac{5}{13}\right)\right)$.

Show solution

- (a) Let $\theta = \arccos \frac{3}{5}$. Then $\cos \theta = \frac{3}{5}$, and since the input is positive, θ is in Quadrant I.
- Draw a right triangle with adjacent side 3 and hypotenuse 5. The opposite side is $\sqrt{25 - 9} = 4$. So $\sin \theta = \frac{4}{5}$.
- (b) Let $\phi = \arcsin\left(-\frac{5}{13}\right)$. Then $\sin \phi = -\frac{5}{13}$, and since the input is negative, ϕ is in Quadrant IV, between $-\frac{\pi}{2}$ and 0.
- In Quadrant IV take $y = -5$ and $r = 13$. Then $x = \sqrt{169 - 25} = 12$, positive in Quadrant IV.
- So $\tan \phi = \frac{y}{x} = \frac{-5}{12} = -\frac{5}{12}$.

Answer: (a) $\frac{4}{5}$ (b) $-\frac{5}{12}$

Example 2. Find the exact value in radians: (a) $\arcsin\left(\sin \frac{5\pi}{6}\right)$ (b) $\arctan\left(\tan \frac{3\pi}{4}\right)$. Explain why neither answer equals the angle inside.

Show solution

1. (a) $\frac{5\pi}{6}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so we cannot simply cancel. First compute the inside: $\sin \frac{5\pi}{6} = \frac{1}{2}$.
2. Then $\arcsin \frac{1}{2} = \frac{\pi}{6}$, the angle in the range of arcsine with sine $\frac{1}{2}$.
3. (b) $\frac{3\pi}{4}$ is not in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Compute the inside: $\tan \frac{3\pi}{4} = -1$.
4. Then $\arctan(-1) = -\frac{\pi}{4}$. Check: $-\frac{\pi}{4} = \frac{3\pi}{4} - \pi$, which makes sense because tangent has period π .
5. In both cases the original angle lies outside the principal range, and an inverse function can only return an angle inside its range.

Answer: (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$

PRACTICE

1. Easy Evaluate $\sin(\arcsin 0.7)$.

Show answer

$$0.7$$

2. Easy Evaluate $\arccos\left(\cos \frac{\pi}{5}\right)$ exactly.

Show answer

$$\frac{\pi}{5}, \text{ since } \frac{\pi}{5} \text{ is in } [0, \pi].$$

3. Easy Find the exact value of $\cos\left(\arcsin \frac{1}{2}\right)$.

Show answer

$$\frac{\sqrt{3}}{2}$$

4. **Medium** Find the exact value of $\sin\left(\arccos\frac{5}{13}\right)$.

Show answer

$$\frac{12}{13}$$

5. **Medium** Find the exact value of $\tan\left(\arcsin\left(-\frac{3}{5}\right)\right)$.

Show answer

$$-\frac{3}{4}$$

6. **Medium** Find the exact value of $\arcsin\left(\sin\frac{2\pi}{3}\right)$ in radians.

Show answer

$$\frac{\pi}{3}$$

7. **Medium** Ruth's kite string is 50 meters long and makes an angle of $\arcsin(0.6)$ with the horizontal. Without finding the angle, compute the horizontal distance from her hands to the point directly below the kite, $50\cos(\arcsin 0.6)$, exactly.

Show answer

$\cos(\arcsin 0.6) = 0.8$, so the distance is 40 meters.

8. **Hard** Find the exact value of $\arccos\left(\cos\frac{7\pi}{6}\right)$ in radians, and explain why it is not $\frac{7\pi}{6}$.

Show answer

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$, and $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$. The answer is not $\frac{7\pi}{6}$ because arccosine only returns angles in $[0, \pi]$.

9. **Hard** Find the exact value of $\cos(\arctan(-2))$.

Show answer

The angle is in Quadrant IV with $x = 1$, $y = -2$, $r = \sqrt{5}$, so the value is $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \approx 0.447$.

10. **Hard** A rafter 15 feet long makes an angle of $\arctan \frac{3}{4}$ with the horizontal ceiling joist. Find its horizontal run, $15 \cos \left(\arctan \frac{3}{4} \right)$, and its rise, $15 \sin \left(\arctan \frac{3}{4} \right)$, exactly.

Show answer

The angle sits in a 3-4-5 triangle, so its cosine is $\frac{4}{5}$ and its sine is $\frac{3}{5}$. Run = 12 feet, rise = 9 feet.

11. **Challenge** Show that $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$ for every real number x . Explain why no $\pm$ sign is needed.

Show answer

Let $\theta = \arctan x$, so $\tan \theta = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. On that interval $\cos \theta > 0$. From $1 + \tan^2 \theta = \sec^2 \theta$ we get $\sec \theta = \sqrt{1+x^2}$ (positive root, since $\cos \theta > 0$), so $\cos \theta = \frac{1}{\sqrt{1+x^2}}$. Then $\sin \theta = \tan \theta \cos \theta = \frac{x}{\sqrt{1+x^2}}$. The sign of the result comes from x itself, which is correct: θ is in Quadrant I when $x > 0$ and in Quadrant IV when $x < 0$.

12. **Challenge** Find the exact value of $\sin \left(2 \arccos \frac{1}{3} \right)$.

Show answer

Let $\theta = \arccos \frac{1}{3}$, so $\cos \theta = \frac{1}{3}$ and $\sin \theta = \frac{2\sqrt{2}}{3}$. Then $\sin 2\theta = 2 \sin \theta \cos \theta = \frac{4\sqrt{2}}{9} \approx 0.629$.

COMMON MISTAKES

- Cancelling automatically: writing $\arcsin \left(\sin \frac{5\pi}{6} \right) = \frac{5\pi}{6}$. Cancel only when the angle is already in the principal range; otherwise evaluate the inside first.
- Putting the triangle in the wrong quadrant. For $\arccos$ of a negative number the angle is in Quadrant II, so $x < 0$ and $y > 0$; for $\arcsin$ or $\arctan$ of a negative number it is in Quadrant IV, so $x > 0$ and $y < 0$.
- Writing $\sin(\arccos x) = \pm \sqrt{1-x^2}$. The range of arccosine fixes the sign; here sine is never negative, so only the positive root is correct.
- Trying to simplify $\cos(\arccos 3)$ to 3. The inner expression does not exist, so the whole expression is undefined.

Make it a game • Inverse Riddles. Parent reads riddles: 'I am the angle whose cosine is $\frac{3}{5}$. What is the sine of me?' Sketch a triangle and answer in 30 seconds. Gotcha cards ask for $\arcsin(\sin(5\pi/6))$; beware the answer. Then write two riddles of your own, one with a quadrant trap, for your parent.

6.4 SOLVING BASIC TRIGONOMETRIC EQUATIONS

OBJECTIVES

- Solve equations of the form $\sin x = c$, $\cos x = c$ and $\tan x = c$ for all solutions in $[0, 2\pi)$, using exact values or a calculator.
- Write the general solution of a basic trigonometric equation using the period.
- Solve practical problems, such as the height of a water wheel bucket or hours of daylight, that lead to basic trigonometric equations.

WATCH

Solving Trigonometric Equations (1 of 4: Simplest Example)

Eddie Woo · 4:57

<https://www.youtube.com/watch?v=wn96zuDRUkM>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

Solving Trigonometric Equations (3 of 5: Using the unit circle)

Eddie Woo · 7:07

https://www.youtube.com/watch?v=Q1fPB835T_E

Before the practice set: finding every solution with the unit circle.

KEY TERMS

trigonometric equation

An equation involving trigonometric functions that is true only for certain values of the variable, unlike an identity, which is true for every value.

solutions in one period

The solutions that lie in a single period, usually $[0, 2\pi)$ or $[0^\circ, 360^\circ)$.

general solution

A formula that lists every solution, such as $x = \frac{\pi}{6} + 2k\pi$, where k is any integer.

CORE IDEAS

An identity such as $\sin^2 x + \cos^2 x = 1$ is true for every x . An equation such as $\sin x = \frac{1}{2}$ is true only for some x , and our job is to find them all. Because the functions are periodic, there are infinitely many solutions, so we usually find the ones in one period, $[0, 2\pi)$, and then describe the rest by adding multiples of the period. The method: (1) isolate the trigonometric function; (2) find the reference angle; (3) use the signs by quadrant to find every solution in $[0, 2\pi)$; (4) add multiples of the period for the general solution.

Why two solutions per turn? On the unit circle, $\sin x = c$ asks where the horizontal line $y = c$ meets the circle. If $-1 < c < 1$, it meets the circle at two points, mirror images across the y -axis: if one angle is α , the other is $\pi - \alpha$. For $\cos x = c$, the vertical line $x = c$ meets the circle at two points, mirror images across the x -axis: α and $2\pi - \alpha$ (or $-\alpha$). If $c = \pm 1$ the line just touches the circle, giving one solution, and if $|c| > 1$ the line misses the circle and there is no solution. Tangent is different: its period is π , so $\tan x = c$ has exactly one solution in every half-open interval of length π , giving two solutions in $[0, 2\pi)$ that differ by π .

Using the inverse functions. The inverse function gives one solution; the symmetry gives the other. For $\sin x = c$, let $\alpha = \arcsin c$; the solutions are α and $\pi - \alpha$, adding 2π to any that are negative. For $\cos x = c$, let $\alpha = \arccos c$; the solutions are α and $2\pi - \alpha$. For $\tan x = c$, let $\alpha = \arctan c$; the solutions are α and $\alpha + \pi$, again adding 2π to a negative value.

The general solution follows from the period. If α solves $\sin x = c$, so does $\alpha + 2\pi$, $\alpha - 2\pi$, and so on. We write $x = \alpha + 2k\pi$ or $x = (\pi - \alpha) + 2k\pi$, where k is any integer. For tangent we write $x = \alpha + k\pi$. Always check a solution by substituting it back into the original equation.

KEY FORMULAS

$$\sin x = c : \quad x = \alpha + 2k\pi \text{ or } x = (\pi - \alpha) + 2k\pi, \quad \alpha = \arcsin c$$

$$\cos x = c : \quad x = \pm\alpha + 2k\pi, \quad \alpha = \arccos c$$

$$\tan x = c : \quad x = \alpha + k\pi, \quad \alpha = \arctan c$$

WORKED EXAMPLES

Example 1. Solve $2 \sin x - 1 = 0$. Give all solutions in $[0, 2\pi)$ exactly, then give the general solution.

Show solution

1. Isolate the sine: $2 \sin x = 1$, so $\sin x = \frac{1}{2}$.
2. The reference angle is $\arcsin \frac{1}{2} = \frac{\pi}{6}$.
3. Sine is positive in Quadrants I and II. The Quadrant I solution is $\frac{\pi}{6}$; the Quadrant II solution is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.
4. Check: $2 \sin \frac{5\pi}{6} - 1 = 2 \cdot \frac{1}{2} - 1 = 0$.
5. General solution: $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$, where k is any integer.

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}$; general solution $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$

Example 2. A water wheel on Mr. Miller's farm has radius 8 feet, and its center is 6 feet above the water. It turns once every 60 seconds. The height of one bucket above the water, t seconds after it passes the level of the center on its way up, is $h(t) = 6 + 8 \sin \left(\frac{\pi t}{30} \right)$ feet. During the first turn ($0 \leq t < 60$), at what times is the bucket exactly 10 feet above the water?

Show solution

1. Set $h(t) = 10$: $6 + 8 \sin \left(\frac{\pi t}{30} \right) = 10$, so $\sin \left(\frac{\pi t}{30} \right) = \frac{4}{8} = \frac{1}{2}$.
2. Let $u = \frac{\pi t}{30}$. As t runs over $[0, 60)$, u runs over $[0, 2\pi)$, exactly one turn.
3. In $[0, 2\pi)$, $\sin u = \frac{1}{2}$ gives $u = \frac{\pi}{6}$ or $u = \frac{5\pi}{6}$.
4. Solve for $t = \frac{30u}{\pi}$: $t = \frac{30}{\pi} \cdot \frac{\pi}{6} = 5$ and $t = \frac{30}{\pi} \cdot \frac{5\pi}{6} = 25$.
5. Check: $h(5) = 6 + 8 \sin \frac{\pi}{6} = 6 + 4 = 10$. The bucket is at 10 feet once on the way up (5 s) and once on the way down (25 s).

Answer: $t = 5$ seconds and $t = 25$ seconds

PRACTICE

1. Easy Solve $\sin x = \frac{\sqrt{2}}{2}$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

2. Easy Solve $\cos x = -\frac{1}{2}$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

3. Easy Solve $\tan x = 1$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

4. Medium Solve $2 \cos x + \sqrt{3} = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$\cos x = -\frac{\sqrt{3}}{2}, \text{ so } x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

5. Medium Solve $2 \sin \theta + \sqrt{3} = 0$ for all θ in $[0^\circ, 360^\circ)$. Give answers in degrees.

Show answer

$$\sin \theta = -\frac{\sqrt{3}}{2}, \text{ so } \theta = 240^\circ, 300^\circ$$

6. Medium Solve $\sin x = -1$ in $[0, 2\pi)$, and write the general solution.

Show answer

$$x = \frac{3\pi}{2}; \text{ general solution } x = \frac{3\pi}{2} + 2k\pi, k \text{ any integer}$$

7. Medium Solve $5 \sin x = 2$ for all x in $[0, 2\pi)$. Round to the nearest thousandth of a radian.

Show answer

$$x = \arcsin 0.4 \approx 0.412 \text{ and } x = \pi - \arcsin 0.4 \approx 2.730$$

8. **Hard** Solve $3 \cos x + 1 = 0$ for all x in $[0, 2\pi)$. Round to the nearest thousandth of a radian.

Show answer

$$x = \arccos\left(-\frac{1}{3}\right) \approx 1.911 \text{ and } x = 2\pi - \arccos\left(-\frac{1}{3}\right) \approx 4.373$$

9. **Hard** Solve $\tan x = -2.5$ for all x in $[0, 2\pi)$. Round to the nearest thousandth of a radian.

Show answer

$$\arctan(-2.5) \approx -1.190 \text{ is negative, so add } \pi \text{ and } 2\pi: x \approx 1.951 \text{ and } x \approx 5.093$$

10. **Hard** In a certain town the hours of daylight on day t of the year are modeled by $D(t) = 12 + 3 \sin\left(\frac{2\pi(t - 80)}{365}\right)$. On which two days of the year are there 13.5 hours of daylight? Round to the nearest whole day.

Show answer

$$\sin\left(\frac{2\pi(t - 80)}{365}\right) = \frac{1}{2}, \text{ so } \frac{2\pi(t - 80)}{365} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}. \text{ Then } t = 80 + \frac{365}{12} \approx 110 \text{ and } t = 80 + \frac{5 \cdot 365}{12} \approx 232: \text{ about day 110 and day 232.}$$

11. **Hard** Explain why $\sin x = 0.4$ has two solutions in $[0, 2\pi)$, $\sin x = 1$ has only one, and $\sin x = 1.4$ has none.

Show answer

Solutions are the points where the horizontal line $y = c$ meets the unit circle. The line $y = 0.4$ crosses the circle at two points, one in Quadrant I and one in Quadrant II. The line $y = 1$ only touches the circle at its top point $(0, 1)$, so $x = \frac{\pi}{2}$ is the only solution. The line $y = 1.4$ lies above the circle and never meets it, since sine is never greater than 1.

12. **Challenge** For the water wheel in Worked Example 2, $h(t) = 6 + 8 \sin\left(\frac{\pi t}{30}\right)$.

During the first turn ($0 \leq t < 60$), for what interval of time is the bucket less than 2 feet above the water? (The bucket dips below the surface for part of this time.)

Show answer

$$h(t) = 2 \text{ when } \sin\left(\frac{\pi t}{30}\right) = -\frac{1}{2}, \text{ so } \frac{\pi t}{30} = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}, \text{ giving } t = 35 \text{ and } t = 55.$$

Between these times the sine is less than $-\frac{1}{2}$, so the bucket is below 2 feet for $35 < t < 55$ seconds.

COMMON MISTAKES

- Stopping at the calculator answer. arcsin, arccos and arctan give only one solution; use the quadrant signs to find the second one in $[0, 2\pi)$.
- Using the wrong partner angle. For sine the partner of α is $\pi - \alpha$; for cosine it is $2\pi - \alpha$; for tangent it is $\alpha + \pi$. Picture the unit circle to choose correctly.
- Leaving a negative calculator value as an answer in $[0, 2\pi)$. If $\arcsin c$ or $\arctan c$ is negative, add 2π to bring it into the interval.
- Using period 2π for tangent in the general solution. Tangent repeats every π , so write $x = \alpha + k\pi$; writing $\alpha + 2k\pi$ misses half the solutions.

Make it a game • Button Solutions. Draw a unit circle on paper. Parent names an equation, such as $\sin x = \text{negative square root } 3 \text{ over } 2$. Place buttons at every solution in 0 to 2π within 20 seconds, then write the general solution using the period. Count correct buttons. Beat your own best across ten equations.

6.5 EQUATIONS USING FACTORING, IDENTITIES AND MULTIPLE ANGLES

OBJECTIVES

- Solve quadratic-type trigonometric equations by factoring and the zero-product property.
- Use the Pythagorean and double-angle identities to rewrite an equation in terms of a single trigonometric function.
- Solve equations with multiple angles, such as $\sin 2x = \frac{1}{2}$, by enlarging the interval before dividing.

WATCH

Harder Trigonometric Equations (1 of 3: Equations Reducible to Quadratics)

Eddie Woo · 7:43

<https://www.youtube.com/watch?v=PsvpINYSWqs>

WHEN TO WATCH

Before Example 1 (quadratic-type equations).

Note: Woo's example is a phase change, $\sin(x + 30^\circ)$, in degrees; the same 'adjust the interval' idea applies to $\sin 2x$.

BACKUP VIDEO

Solving Trigonometric Equations (4 of 4: Adjusted Frequency & Phase)

Eddie Woo · 6:47

<https://www.youtube.com/watch?v=4-2O79zTHQs>

Before Example 2: adjusting the interval when the angle is changed.

KEY TERMS

quadratic-type equation

An equation such as $2\sin^2 x - \sin x - 1 = 0$ that is a quadratic in one trigonometric function.

zero-product property

If $AB = 0$, then $A = 0$ or $B = 0$. It lets us split a factored equation into simpler equations.

extraneous solution

A value produced by a solving step (such as squaring both sides) that does not satisfy the original equation.

multiple-angle equation

An equation in which the variable appears as bx inside a trigonometric function, such as $\sin 2x$ or $\tan 3x$.

CORE IDEAS

Quadratic type. In $2 \sin^2 x - \sin x - 1 = 0$, the expression $\sin x$ plays the role of a single unknown. Writing $u = \sin x$ gives $2u^2 - u - 1 = 0$, which factors as $(2u + 1)(u - 1) = 0$. By the zero-product property, $\sin x = -\frac{1}{2}$ or $\sin x = 1$, and each of these is a basic equation from Lesson 6.4. If a factor gives an impossible value, such as $\sin x = 2$, it simply contributes no solutions.

Using identities. When an equation mixes functions, rewrite it in terms of one function. The Pythagorean identity turns $\sin^2 x$ into $1 - \cos^2 x$ (or $\cos^2 x$ into $1 - \sin^2 x$). The double-angle identities replace $\sin 2x$ with $2 \sin x \cos x$ and $\cos 2x$ with $1 - 2 \sin^2 x$ or $2 \cos^2 x - 1$; choose the form that matches the other terms. Then factor.

Never divide both sides by an expression like $\sin x$ that could be zero: that throws away every solution where it is zero. Move everything to one side and factor out the common expression instead.

Multiple angles. To solve $\sin 2x = \frac{1}{2}$ for x in $[0, 2\pi)$, let $u = 2x$. If $0 \leq x < 2\pi$, then $0 \leq u < 4\pi$: two full turns. So we must find every u in $[0, 4\pi)$, not just in $[0, 2\pi)$. That gives four values of u , and dividing by 2 gives four values of x . In general $\sin bx = c$ (with b a positive integer and $-1 < c < 1$) has $2b$ solutions in $[0, 2\pi)$, because bx goes around the circle b times. This matches Unit 4: $y = \sin bx$ has period $\frac{2\pi}{b}$, so it fits b full cycles into $[0, 2\pi)$.

Check every answer. Squaring both sides can add extraneous solutions, and mistakes with the interval can add or lose solutions. Substituting each answer into the original equation is the surest test.

KEY FORMULAS

$$\sin^2 x + \cos^2 x = 1$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

$$0 \leq x < 2\pi \iff 0 \leq bx < 2b\pi \quad (b > 0)$$

From history. Galileo, in his *Two New Sciences* (1638), showed that a projectile fired over level ground travels farthest when it is launched at 45° , and that launch angles equally above and below 45° give equal ranges. In modern notation this follows from the range formula $R = \frac{v^2}{g} \sin 2\theta$ and the multiple-angle equations of this lesson: $\sin 2\theta$ is largest when $2\theta = 90^\circ$, and $\sin 2\theta = \sin(180^\circ - 2\theta)$.

WORKED EXAMPLES

Example 1. Solve $2 \sin^2 x - \sin x - 1 = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show solution

1. Treat $\sin x$ as the unknown and factor: $(2 \sin x + 1)(\sin x - 1) = 0$. (Check by expanding: $2 \sin^2 x - 2 \sin x + \sin x - 1$.)
2. By the zero-product property, $\sin x = -\frac{1}{2}$ or $\sin x = 1$.
3. $\sin x = 1$ gives $x = \frac{\pi}{2}$.
4. $\sin x = -\frac{1}{2}$: the reference angle is $\frac{\pi}{6}$, and sine is negative in Quadrants III and IV, so $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.
5. Check $x = \frac{7\pi}{6}$: $2 \left(\frac{1}{4}\right) - \left(-\frac{1}{2}\right) - 1 = \frac{1}{2} + \frac{1}{2} - 1 = 0$.

Answer: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Example 2. Solve $\sin 2x = \frac{1}{2}$ for all x in $[0, 2\pi)$. Give exact values.

Show solution

1. Let $u = 2x$. Since $0 \leq x < 2\pi$, we need every u with $0 \leq u < 4\pi$.
2. In the first turn, $\sin u = \frac{1}{2}$ gives $u = \frac{\pi}{6}$ and $u = \frac{5\pi}{6}$.
3. In the second turn add 2π : $u = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ and $u = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$.
4. Divide each by 2: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.
5. Check $x = \frac{13\pi}{12}$: $\sin \frac{13\pi}{6} = \sin \left(2\pi + \frac{\pi}{6}\right) = \frac{1}{2}$. Stopping after the first turn would have lost half the solutions.

Answer: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

PRACTICE

1. Easy Solve $\sin x (2 \cos x - 1) = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$\sin x = 0 \text{ gives } 0, \pi; \cos x = \frac{1}{2} \text{ gives } \frac{\pi}{3}, \frac{5\pi}{3}. \text{ So } x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}.$$

2. Easy Solve $\tan^2 x = 3$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$\tan x = \pm\sqrt{3}, \text{ so } x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

3. Easy Solve $2 \cos^2 x - 1 = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$\cos x = \pm \frac{\sqrt{2}}{2}, \text{ so } x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

4. Medium Solve $2 \cos^2 x + 3 \cos x + 1 = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$$(2 \cos x + 1)(\cos x + 1) = 0, \text{ so } \cos x = -\frac{1}{2} \text{ or } \cos x = -1: x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

5. Medium Solve $2 \sin^2 x + 3 \cos x - 3 = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

Replace $\sin^2 x$ with $1 - \cos^2 x$: $-2 \cos^2 x + 3 \cos x - 1 = 0$, or $(2 \cos x - 1)(\cos x - 1) = 0$. So $\cos x = \frac{1}{2}$ or $\cos x = 1$: $x = 0, \frac{\pi}{3}, \frac{5\pi}{3}$

6. Medium Solve $\sin 2x = \cos x$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

$2 \sin x \cos x - \cos x = 0$, so $\cos x (2 \sin x - 1) = 0$. $\cos x = 0$ gives $\frac{\pi}{2}, \frac{3\pi}{2}$; $\sin x = \frac{1}{2}$ gives $\frac{\pi}{6}, \frac{5\pi}{6}$. So $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

7. Medium Samuel kicks a ball from level ground at 15 meters per second. Ignoring air resistance, its range is $R = \frac{v^2}{9.8} \sin 2\theta$ meters, where θ is the launch angle ($0^\circ < \theta < 90^\circ$). At what angle is the range greatest, and what is that range, to the nearest tenth of a meter?

Show answer

R is greatest when $\sin 2\theta = 1$, so $2\theta = 90^\circ$ and $\theta = 45^\circ$. Then $R = \frac{225}{9.8} \approx 23.0$ meters.

8. Hard Solve $\tan 3x = 1$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

Let $u = 3x$, with $0 \leq u < 6\pi$. Then $u = \frac{\pi}{4} + k\pi$ for $k = 0, 1, \dots, 5$, so $x = \frac{\pi}{12} + \frac{k\pi}{3}$: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

9. Hard Solve $\cos 2x + \cos x = 0$ for all x in $[0, 2\pi)$. Give exact values.

Show answer

Use $\cos 2x = 2 \cos^2 x - 1$: $2 \cos^2 x + \cos x - 1 = 0$, so $(2 \cos x - 1)(\cos x + 1) = 0$. Then $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

10. **Hard** A field sprinkler shoots water at 12 meters per second from ground level. Ignoring air resistance, the water lands $R = \frac{v^2}{9.8} \sin 2\theta$ meters away, where θ is the nozzle angle ($0^\circ < \theta < 90^\circ$). Find both nozzle angles that make the water land 10 meters away, to the nearest tenth of a degree.

Show answer

$$\sin 2\theta = \frac{98}{144} \approx 0.6806. \text{ Then } 2\theta \approx 42.887^\circ \text{ or } 2\theta \approx 180^\circ - 42.887^\circ = 137.113^\circ, \\ \text{so } \theta \approx 21.4^\circ \text{ or } \theta \approx 68.6^\circ.$$

11. **Hard** To solve $\sin x \cos x = \sin x$ on $[0, 2\pi)$, Ruth divided both sides by $\sin x$, got $\cos x = 1$, and answered $x = 0$. Explain her error and give all the solutions.

Show answer

Dividing by $\sin x$ assumes $\sin x \neq 0$, which throws away the solutions where $\sin x = 0$. Instead write $\sin x \cos x - \sin x = 0$, so $\sin x (\cos x - 1) = 0$. Then $\sin x = 0$ gives $x = 0, \pi$, and $\cos x = 1$ gives $x = 0$. The solutions are $x = 0$ and $x = \pi$; she lost $x = \pi$.

12. **Challenge** Solve $\sin x + \cos x = 1$ for all x in $[0, 2\pi)$. (Hint: use the sum formula to show $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$.)

Show answer

$$\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = \sqrt{2} \left(\sin x \cdot \frac{\sqrt{2}}{2} + \cos x \cdot \frac{\sqrt{2}}{2} \right) = \sin x + \cos x. \text{ So} \\ \sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \text{ with } \frac{\pi}{4} \leq x + \frac{\pi}{4} < \frac{9\pi}{4}. \text{ Then } x + \frac{\pi}{4} = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}, \text{ so } x = 0 \text{ or } \\ x = \frac{\pi}{2}.$$

COMMON MISTAKES

- Dividing both sides by $\sin x$ or $\cos x$. This loses every solution where that function is zero. Move all terms to one side and factor instead.
- Finding only the solutions in the first turn for a multiple angle. For $\sin 2x = c$ on $[0, 2\pi)$, solve for $2x$ on $[0, 4\pi)$ before dividing by 2.
- Keeping impossible factors. If factoring gives $\cos x = 3$ or $\sin x = -2$, that factor contributes no solutions.
- Forgetting to check after squaring both sides. Squaring can create extraneous solutions; substitute every answer into the original equation.

Make it a game • The Solve Off. Both players solve the same equation in 3 minutes, such as $2 \sin^2 x - \sin x - 1 = 0$ for x between 0 and 2π . Factor, apply the zero-product property and check for extraneous answers. Compare solution sets. One point for finding an error or a missing solution in the other's work.

INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS

In Unit 1 we used $\sin^{-1}$ to turn a ratio back into an acute angle. Now that sine, cosine and tangent are defined for every angle, that question needs more care. Each function repeats forever, so many angles share the same value. In this unit we learn why each function must be restricted to a carefully chosen interval before it has an inverse, and we study the inverse functions arcsine, arccosine and arctangent: their domains, ranges, graphs and values.

We then turn the question around and solve trigonometric equations, finding every angle that makes an equation true. We begin with basic equations like $2 \sin x - 1 = 0$, write both the solutions in one turn and the general solution, and then solve harder equations by factoring, by using the identities of Unit 5, and by handling multiple angles like $\sin 2x$.

These skills join everything before them. The unit circle of Unit 3 explains why equations have two solutions per turn, the graphs of Unit 4 explain the shapes of the inverse functions, and the identities of Unit 5 become tools for solving. Inverse functions are also how we will find angles in the oblique triangles of Unit 7 and in the surveying, navigation and periodic-motion problems of Unit 8.

PREREQUISITES

- Find inverse ratios of acute angles and solve right triangles (Lesson 1.4).
- Know the exact values of sine, cosine and tangent for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (Lesson 3.8).
- Use reference angles and the signs of the functions by quadrant (Lesson 3.7).
- Recognize the graphs, periods and asymptotes of sine, cosine and tangent (Lessons 4.1, 4.2 and 4.6).
- Use the Pythagorean, sum and double-angle identities (Lessons 5.2, 5.4 and 5.5).
- Factor quadratic expressions and use the zero-product property (Algebra II).

MASTERY CHECKLIST

- ☐ I can explain why sine, cosine and tangent must be restricted before they have inverse functions, and name each restricted interval.
- ☐ I can state the domain and range of $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arccot} x$.
- ☐ I can evaluate inverse trigonometric functions exactly for special values and with a calculator in radians or degrees.
- ☐ I can explain why $\arccos$ of a negative number is in Quadrant II while $\arcsin$ and $\arctan$ of a negative number are in Quadrant IV.
- ☐ I can tell when $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ are correct and evaluate compositions like $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$ correctly.

- ☐ I can find values like

$\sin\left(\arccos\frac{3}{5}\right)$ by drawing a right triangle in the correct quadrant.

- ☐ I can solve

$$\sin x = c$$

$$\text{and } \tan x = c$$

and

for $[0, 2\pi)$

every

where

you

solve

using

the

identity

and

the

value

of

the

angle

is

the

same

as

the

angle

.

- ☐ I can solve quadratic-type trigonometric equations by factoring.

- ☐ I can use identities to rewrite an equation in one function, and I factor instead of dividing by an expression that could be zero.

- ☐ I can solve multiple-angle equations like $\sin 2\theta$ by enlarging the interval first.
 $\frac{1}{2}$
- ☐ I can set up and solve word problems with ladders, ramps, rafters, water wheels and daylight models using inverse functions and trigonometric equations.

LESSONS IN THIS UNIT

6.1 Inverse Sine and Restricted Domains	50 minutes
6.2 Inverse Cosine and Inverse Tangent	50 minutes
6.3 Compositions with Inverse Functions	50 minutes
6.4 Solving Basic Trigonometric Equations	50 minutes
6.5 Equations Using Factoring, Identities and Multiple Angles	2 sessions of 45 minutes

UNIT 6 TEST

Time allowed: 60 minutes. A scientific calculator is allowed, but give exact values wherever a problem asks for them. Unless a problem says otherwise, give angles in radians, and give all solutions in the interval $[0, 2\pi)$. Round as each problem directs; do not round intermediate values. Show your work: setting up each problem correctly earns most of the credit. 100 points total.

1. **5 pts**

Find the exact value of $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ in radians.

2. **5 pts**

Find the exact values in radians: (a) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\arctan \sqrt{3}$.

3. **5 pts**

Use a calculator to find $\arcsin(0.6)$ in degrees, to the nearest tenth of a degree.

4. **5 pts**

A wheelchair ramp at the church entrance rises 2 feet over a horizontal run of 24 feet. Find the angle the ramp makes with the ground, to the nearest tenth of a degree.

5. **5 pts**

Explain why the domain of $\arctan x$ is all real numbers, while the domain of $\arcsin x$ is only $[-1, 1]$.

6. **6 pts**

Find the exact value of $\sin\left(\arccos\left(-\frac{4}{5}\right)\right)$.

7. **6 pts**

Find the exact value of $\arccos\left(\cos \frac{5\pi}{3}\right)$ in radians.

8. **6 pts**

A rafter 26 feet long makes an angle of $\arctan \frac{5}{12}$ with the horizontal. Find the rise of the rafter, $26 \sin\left(\arctan \frac{5}{12}\right)$, exactly.

9. **8 pts**

Solve $2 \cos x - \sqrt{2} = 0$ for all x in $[0, 2\pi)$, giving exact values. Then write the general solution.

10. **8 pts**

Solve $4 \sin x + 1 = 0$ for all x in $[0, 2\pi)$. Round to the nearest thousandth of a radian.

11. 8 pts

The height of a bucket on a water wheel is $h(t) = 7 + 9 \sin\left(\frac{\pi t}{20}\right)$ feet above the water, t seconds after it passes the level of the center going up. During the first turn ($0 \leq t < 40$), at what times is the bucket 11.5 feet above the water? Give exact values and decimals to the nearest tenth of a second.

12. 8 pts

Solve $2 \sin^2 x + \sin x - 1 = 0$ for all x in $[0, 2\pi)$. Give exact values.

13. 8 pts

Solve $\cos 2x = \cos x$ for all x in $[0, 2\pi)$. Give exact values.

14. 8 pts

Solve $\sin 2x = \frac{\sqrt{3}}{2}$ for all x in $[0, 2\pi)$. Give exact values.

15. 9 pts

Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for every x in $[-1, 1]$.

ANSWER KEY

1. $-\frac{\pi}{4}$
2. (a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$
3. $\approx 36.9^\circ$
4. $\arctan \frac{2}{24} = \arctan \frac{1}{12} \approx 4.8^\circ$
5. The domain of an inverse function is the set of outputs of the original function. Sine only takes values between -1 and 1 , so only those numbers have an arcsine. Tangent takes every real value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, since every slope is the tangent of some angle, so every real number has an arctangent.
6. The angle is in Quadrant II with $x = -4$, $r = 5$, so $y = 3$ and the value is $\frac{3}{5}$.
7. $\cos \frac{5\pi}{3} = \frac{1}{2}$, so the value is $\arccos \frac{1}{2} = \frac{\pi}{3}$.
8. The angle sits in a 5-12-13 triangle, so its sine is $\frac{5}{13}$ and the rise is $26 \cdot \frac{5}{13} = 10$ feet.
9. $\cos x = \frac{\sqrt{2}}{2}$, so $x = \frac{\pi}{4}, \frac{7\pi}{4}$; general solution $x = \pm \frac{\pi}{4} + 2k\pi$, k any integer.
10. $\sin x = -0.25$; the reference angle is $\arcsin 0.25 \approx 0.2527$. Sine is negative in Quadrants III and IV, so $x \approx \pi + 0.2527 \approx 3.394$ and $x \approx 2\pi - 0.2527 \approx 6.031$.
11. $\sin\left(\frac{\pi t}{20}\right) = \frac{4.5}{9} = \frac{1}{2}$, so $\frac{\pi t}{20} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Then $t = \frac{10}{3} \approx 3.3$ s and $t = \frac{50}{3} \approx 16.7$ s.
12. $(2 \sin x - 1)(\sin x + 1) = 0$, so $\sin x = \frac{1}{2}$ or $\sin x = -1$: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
13. Use $\cos 2x = 2 \cos^2 x - 1$: $2 \cos^2 x - \cos x - 1 = 0$, so $(2 \cos x + 1)(\cos x - 1) = 0$. Then $\cos x = 1$ gives $x = 0$, and $\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}, \frac{4\pi}{3}$. So $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.
14. With $0 \leq 2x < 4\pi$: $2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$, so $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$.
15. Let $\theta = \arcsin x$, so $\sin \theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $0 \leq \frac{\pi}{2} - \theta \leq \pi$, and by the cofunction identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta = x$. So $\frac{\pi}{2} - \theta$ is the one angle in $[0, \pi]$ whose cosine is x ; that is, $\arccos x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arcsin x$. Adding $\arcsin x$ to both sides gives the result.

OBLIQUE TRIANGLES: THE LAWS OF SINES AND COSINES

Most triangles in the real world have no right angle: a field bounded by three fences, a roof truss, the triangle formed by two lookouts and a distant fire. This unit extends trigonometry from right triangles to every triangle. We first find areas with $\frac{1}{2}ab \sin C$ and Heron's formula. Then we prove the two great laws of oblique triangles, the Law of Sines and the Law of Cosines, and learn which one to use for each kind of given data.

As always, we ask why. Each law is proved from earlier work: the Law of Sines by dropping an altitude, and the Law of Cosines by the distance formula and $\sin^2 \theta + \cos^2 \theta = 1$. We meet old friends along the way: Euclid's congruence theorems (I.4, I.8, I.26) explain which data fix a triangle, and his Propositions II.12–13 turn out to be the Law of Cosines in geometric dress. The ambiguous case (SSA) shows how careful reasoning, not just calculation, decides whether data describe zero, one or two triangles.

These tools are the working core of surveying and navigation. Unit 8 applies them directly to angles of elevation, bearings, triangulation and three-dimensional problems, so mastering the methods and the habit of checking answers here will pay off immediately.

PREREQUISITES

- Solve right triangles with sine, cosine and tangent and their inverses (Lessons 1.2–1.4).
- Use reference angles and the fact that $\sin(180^\circ - \theta) = \sin \theta$ and $\cos(180^\circ - \theta) = -\cos \theta$ (Lessons 3.6–3.8).
- Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ (Lesson 5.2).
- Explain why $\sin^{-1}$ returns only angles in $[-90^\circ, 90^\circ]$ and $\cos^{-1}$ returns angles in $[0^\circ, 180^\circ]$ (Lessons 6.1–6.2).
- Use the distance formula and simplify radicals (Algebra).
- Recall the triangle angle sum (Euclid I.32) and the congruence criteria SAS, SSS and ASA from Geometry.

MASTERY CHECKLIST

- ☐ I can derive $\text{Area} = \frac{1}{2}ab \sin C$ and use it for acute and obtuse included angles.
- ☐ I can find a triangle's area from its three sides with Heron's formula.
- ☐ I can work backward from an area to find both possible included angles.
- ☐ I can prove the Law of Sines and use it to solve AAS and ASA triangles.
- ☐ I can decide whether SSA data give zero, one or two triangles by comparing
 - a with h and with $b \sin A$
 - b .
- ☐ I can find both triangles in the ambiguous case and label each solution clearly.
- ☐ I can prove the Law of Cosines and explain how it generalizes the Pythagorean theorem and Euclid II.12–13.
- ☐ I can solve SAS triangles and SSS triangles, finding the largest angle first in SSS.
- ☐ I can name the case of any triangle problem and choose the correct law to start.
- ☐ I can check a solution with the angle sum, side–angle order and the equal ratios of the Law of Sines.

- I can model surveying, carpentry and navigation problems with oblique triangles.

LESSONS IN THIS UNIT

7.1 The Area of a Triangle	50 minutes
7.2 The Law of Sines	55 minutes
7.3 The Ambiguous Case (SSA)	2 sessions of 45 minutes
7.4 The Law of Cosines	55 minutes
7.5 Solving Any Triangle	2 sessions of 45 minutes

UNIT 7 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; set it to degree mode. Show your work, including a labeled sketch for every triangle. Unless a problem says otherwise, round side lengths to the nearest tenth and angles to the nearest tenth of a degree, and do not round intermediate values. Give exact values where asked. Total: 100 points.

1. (5 pts)

Find the exact area of a triangle with sides 9 and 14 and an included angle of 30° .

2. (6 pts)

Use Heron's formula to find the exact area of a triangle with sides 10, 17 and 21.

3. (6 pts)

In triangle ABC , $A = 45^\circ$, $B = 60^\circ$ and $a = 12$. Find b exactly, then to the nearest tenth.

4. (8 pts)

Two points A and B on a straight shore are 800 ft apart. A buoy C floats offshore. The angles measured are $\angle CAB = 62^\circ$ and $\angle CBA = 54^\circ$. How far is the buoy from point A ?

5. (5 pts)

How many triangles have $A = 40^\circ$, $a = 8$ and $b = 10$? Justify your answer.

6. (10 pts)

Solve triangle ABC if $A = 30^\circ$, $a = 7$ and $b = 10$. Find every solution.

7. (7 pts)

In triangle ABC , $b = 10$, $c = 12$ and $A = 75^\circ$. Find a .

8. (7 pts)

A triangle has sides 8, 10 and 15. Find its largest angle and state whether the triangle is acute, right or obtuse.

9. (8 pts)

A tunnel is to be dug straight through a hill. From a point P in the open, a surveyor measures 380 m to one end of the planned tunnel and 510 m to the other end, with an angle of 48° between the two sight lines. How long will the tunnel be?

10. (9 pts)

A pilot flies 300 km in a straight line, turns through an angle of 40° , and flies 200 km farther. How far is the plane from its starting point?

11. (10 pts)

A triangular pasture on Mr. Miller's farm has sides of 50, 70 and 90 yards. Find its largest angle, then find its area in two ways: with $\frac{1}{2}ab \sin C$ and with Heron's formula. Round the area to the nearest square yard.

12. (10 pts)

Starting from the area formula $\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$, prove the Law of Sines. Then explain in one or two sentences why the area formula holds even when the angle used is obtuse.

13. (9 pts)

Let $A = 30^\circ$ and $b = 12$. Find every length a for which exactly one triangle exists. Explain your reasoning.

ANSWER KEY

1. $\frac{1}{2}(9)(14) \sin 30^\circ = 31.5$ square units
2. $s = 24$; $\sqrt{24 \cdot 14 \cdot 7 \cdot 3} = \sqrt{7056} = 84$ square units
3. $b = \frac{12 \sin 60^\circ}{\sin 45^\circ} = 6\sqrt{6} \approx 14.7$
4. $C = 64^\circ$; $AC = \frac{800 \sin 54^\circ}{\sin 64^\circ} \approx 720.1$ ft
5. Two: $h = 10 \sin 40^\circ \approx 6.43$, and $h < a < b$
6. $\sin B = \frac{10 \sin 30^\circ}{7} = \frac{5}{7}$; two triangles. Triangle 1: $B \approx 45.6^\circ$, $C \approx 104.4^\circ$, $c \approx 13.6$.
Triangle 2: $B \approx 134.4^\circ$, $C \approx 15.6^\circ$, $c \approx 3.8$
7. $a^2 = 100 + 144 - 240 \cos 75^\circ \approx 181.883$, so $a \approx 13.5$
8. $\cos C = \frac{64 + 100 - 225}{160} = -0.38125$, so $C \approx 112.4^\circ$; the triangle is obtuse
9. $d^2 = 380^2 + 510^2 - 2(380)(510) \cos 48^\circ \approx 145145.0$, so $d \approx 381.0$ m
10. The angle inside the triangle at the turn is 140° . $d^2 = 300^2 + 200^2 - 2(300)(200) \cos 140^\circ \approx 221925.3$, so $d \approx 471.1$ km
11. $\cos C = \frac{2500 + 4900 - 8100}{7000} = -0.1$, $C \approx 95.7^\circ$. Area = $\frac{1}{2}(50)(70) \sin C = 1750\sqrt{0.99} \approx 1741$ yd²; Heron: $s = 105$, $\sqrt{105 \cdot 55 \cdot 35 \cdot 15} = \sqrt{3031875} \approx 1741$ yd²
12. All three expressions equal the same area, so $\frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$.
Dividing every part by $\frac{1}{2}abc$ gives $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, which is the Law of Sines. When C is obtuse, the altitude from A meets the extension of BC beyond C , and the right triangle formed has angle $180^\circ - C$ at C , so $h = b \sin(180^\circ - C) = b \sin C$, and the formula $\frac{1}{2}ah = \frac{1}{2}ab \sin C$ still holds.
13. $h = 12 \sin 30^\circ = 6$. Exactly one triangle when $a = 6$ (a right triangle, with $B = 90^\circ$) or when $a \geq 12$ (the second crossing of the arc falls at or behind A). For $6 < a < 12$ there are two triangles, and for $a < 6$ there are none

7.1 THE AREA OF A TRIANGLE

OBJECTIVES

- Derive the formula $\text{Area} = \frac{1}{2}ab \sin C$ and use it when two sides and the included angle are known.
- Use Heron's formula to find the area of a triangle from its three sides.
- Work backward from a known area to find a missing side or angle.

WATCH

Area of a Triangle (Deriving the trigonometric formula)

Eddie Woo · 7:31

<https://www.youtube.com/watch?v=EO2wBUqvs6Q>

WHEN TO WATCH

After reading the Core ideas, before Example 1 (deriving $\frac{1}{2}ab \sin C$).

BACKUP VIDEO

How to Find the Area of a Triangle with Trigonometry (Precalculus - Trigonometry 34)

Professor Leonard · 16:19

https://www.youtube.com/watch?v=_i9t-ldjG7o

Before Example 2: Heron's formula is in the second half.

KEY TERMS

oblique triangle

A triangle with no right angle. All three of its angles are acute, or one is obtuse.

altitude

The perpendicular segment from a vertex to the line containing the opposite side; its length is the height for that base.

included angle

The angle formed by two given sides of a triangle; it lies between them, at the vertex they share.

semiperimeter

Half the perimeter of a triangle: $s = \frac{1}{2}(a + b + c)$.

Heron's formula

The area of a triangle with sides a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter.

CORE IDEAS

From now on we label every triangle the same way. The vertices are A, B, C , the angles at those vertices are also called A, B, C , and the side opposite each angle gets the matching small letter: side a is opposite angle A , side b is opposite angle B , and side c is opposite angle C . With this habit, formulas can be read and remembered at a glance.

You already know that the area of a triangle is $\frac{1}{2}(\text{base})(\text{height})$. Trigonometry lets us find the height when we do not know it. Take side $a = BC$ as the base and drop the altitude h from A to line BC . In the right triangle formed with side $b = CA$ as its hypotenuse, $\sin C = \frac{h}{b}$, so $h = b \sin C$. Therefore $\text{Area} = \frac{1}{2}ah = \frac{1}{2}ab \sin C$. In words: the area is half the product of two sides times the sine of the angle between them.

What if C is obtuse? Then the altitude from A falls outside the triangle, on the extension of BC beyond C . The right triangle now contains the angle $180^\circ - C$ at C , so $h = b \sin(180^\circ - C)$. But $\sin(180^\circ - C) = \sin C$ (a reference-angle fact from Unit 3), so again $h = b \sin C$ and the same formula holds. If $C = 90^\circ$, then $\sin C = 1$ and the formula becomes the familiar $\frac{1}{2}ab$ for a right triangle. One formula covers every case.

Sometimes we know all three sides but no angle. Heron's formula handles this directly. Compute the semiperimeter $s = \frac{1}{2}(a + b + c)$; then $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$. The factors $s-a, s-b, s-c$ are always positive for a real triangle, because each side is shorter than the sum of the other two. (Heron's formula can be proved from $\frac{1}{2}ab \sin C$ together with the Law of Cosines, which we meet in Lesson 7.4.)

The area formula can also be run backward. If you know the area and two sides, then $\sin C = \frac{2 \cdot \text{Area}}{ab}$. Be careful: a sine value between 0 and 1 belongs to two angles in a triangle, an acute angle C and its supplement $180^\circ - C$. Both may give the same area, so both may be correct unless other information rules one out.

KEY FORMULAS

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$$

$$s = \frac{1}{2}(a + b + c)$$

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}$$

From history. Heron of Alexandria, a Greek mathematician and engineer of the first century AD, gave the three-sides area formula with a geometric proof in his book *Metrica*, a practical work on measuring areas and volumes. Heron also wrote on surveying instruments and machines, so it is fitting that his formula is still used by surveyors who can measure the sides of a field more easily than its angles.

Euclid connection. Euclid I.41 shows that a triangle is half of a parallelogram on the same base and between the same parallels. That is the root of the $\frac{1}{2}$ in every triangle area formula, including $\frac{1}{2}ab \sin C$.

WORKED EXAMPLES

Example 1. In triangle ABC , $a = 8$ cm, $b = 11$ cm and $C = 40^\circ$. Find the area to the nearest tenth of a square centimeter.

Show solution

1. Angle C is the included angle between sides a and b , so use $\text{Area} = \frac{1}{2}ab \sin C$.
2. $\text{Area} = \frac{1}{2}(8)(11) \sin 40^\circ = 44 \sin 40^\circ$.
3. $\sin 40^\circ \approx 0.6427876$, so $\text{Area} \approx 44(0.6427876) \approx 28.2827$.
4. Rounded to the nearest tenth, the area is about 28.3 square centimeters.

Answer: About 28.3 cm²

Example 2. Mr. Miller's triangular hay field has sides of 120 m, 150 m and 210 m. Find its area exactly, and then to the nearest square meter.

Show solution

1. No angle is given, so use Heron's formula. The semiperimeter is $s = \frac{1}{2}(120 + 150 + 210) = 240$.
2. The three differences are $s - a = 240 - 120 = 120$, $s - b = 240 - 150 = 90$ and $s - c = 240 - 210 = 30$.
3. Area = $\sqrt{240 \cdot 120 \cdot 90 \cdot 30} = \sqrt{77,760,000}$.
4. Simplify: $77,760,000 = 7776 \cdot 10^4$ and $7776 = 1296 \cdot 6 = 36^2 \cdot 6$. So $\sqrt{77,760,000} = 100 \cdot 36\sqrt{6} = 3600\sqrt{6}$.
5. $3600\sqrt{6} \approx 3600(2.4494897) \approx 8818.16$, so the field covers about 8818 square meters (a little less than one hectare).

Answer: $3600\sqrt{6} \text{ m}^2 \approx 8818 \text{ m}^2$

PRACTICE

1. **Easy** Find the exact area of a triangle with sides 6 and 9 and an included angle of 30° .

Show answer

13.5 square units

2. **Easy** Two sides of a triangle are 7 cm and 10 cm, and the angle between them is 65° . Find the area to the nearest tenth.

Show answer

About 31.7 cm^2

3. **Easy** Use Heron's formula to find the area of a triangle with sides 5, 12 and 13. Then check your answer another way.

Show answer

30 square units (it is a right triangle, so $\frac{1}{2} \cdot 5 \cdot 12 = 30$)

4. **Medium** Find the area of a triangle with sides 7, 8 and 9. Give the exact value and the value to the nearest hundredth.

Show answer

$12\sqrt{5} \approx 26.83$ square units

5. **Medium** Ruth is planting a triangular herb garden. Two sides along the fence measure 18 ft and 25 ft, and the fences meet at an angle of 112° . How many square feet of garden will she have? Round to the nearest tenth.

Show answer

About 208.6 ft^2

6. **Medium** Use $\text{Area} = \frac{1}{2}ab \sin C$ to show that an equilateral triangle with side x has area $\frac{\sqrt{3}}{4}x^2$. Then find the exact area when $x = 10$.

Show answer

$$\frac{1}{2}x \cdot x \sin 60^\circ = \frac{\sqrt{3}}{4}x^2; \text{ for } x = 10 \text{ the area is } 25\sqrt{3} \approx 43.30$$

7. **Medium** A triangle has two sides of 12 and 15 and an area of 50. Find every possible measure of the included angle, to the nearest tenth of a degree.

Show answer

$$\sin C = \frac{100}{180} = \frac{5}{9}, \text{ so } C \approx 33.7^\circ \text{ or } C \approx 146.3^\circ$$

8. **Hard** A parallelogram has sides 14 in and 20 in, and one of its angles is 55° . Find its area to the nearest tenth. (Hint: a diagonal splits it into two congruent triangles.)

Show answer

$$14 \cdot 20 \sin 55^\circ \approx 229.4 \text{ in}^2$$

9. **Hard** A surveyor paces off a triangular pasture with sides 260 m, 280 m and 300 m. Find its area in hectares ($1 \text{ ha} = 10,000 \text{ m}^2$).

Show answer

$$s = 420, \text{ Area} = \sqrt{420 \cdot 160 \cdot 140 \cdot 120} = 33,600 \text{ m}^2 = 3.36 \text{ ha}$$

10. **Hard** A regular pentagon is inscribed in a circle of radius 10 cm. By splitting it into five congruent triangles from the center, find its area to the nearest tenth.

Show answer

$$\text{Each central angle is } 72^\circ; \text{ Area} = 5 \cdot \frac{1}{2}(10)(10) \sin 72^\circ = 250 \sin 72^\circ \approx 237.8 \text{ cm}^2$$

11. **Challenge** Explain why $\text{Area} = \frac{1}{2}ab \sin C$ is still correct when angle C is obtuse, even though the altitude falls outside the triangle. Draw the figure and justify every step.

Show answer

Extend side BC beyond C and drop the altitude h from A to that line, meeting it at D . Triangle ACD is right-angled at D , with hypotenuse b and angle $180^\circ - C$ at C (angles on a straight line sum to 180°). So $h = b \sin(180^\circ - C)$. Since supplementary angles have equal sines, $h = b \sin C$. The base is still a , so $\text{Area} = \frac{1}{2}ah = \frac{1}{2}ab \sin C$.

12. **Challenge** A triangle has sides 13, 14 and 15. Find its area with Heron's formula. Then use the area to find the largest angle to the nearest hundredth of a degree, and explain how you know that angle is acute rather than obtuse.

Show answer

Area = 84. For the angle C opposite 15, $\sin C = \frac{2(84)}{13 \cdot 14} = \frac{12}{13}$. Since $13^2 + 14^2 = 365 > 15^2 = 225$, the angle opposite the longest side is acute (by Euclid II.12 an obtuse angle would make $15^2 > 13^2 + 14^2$, and by I.48 a right angle would make them equal), so $C \approx 67.38^\circ$, not 112.62° .

COMMON MISTAKES

- Using an angle that is not between the two given sides. In $\frac{1}{2}ab \sin C$, angle C must be the angle where sides a and b meet. Mark the two sides on your sketch and check that the angle sits between them.
- Leaving the calculator in radian mode. $\sin 40$ in radians is about 0.745, not 0.643. Check the mode before every degree problem.
- Forgetting to halve the perimeter in Heron's formula. s is the semiperimeter; using the full perimeter makes $s - a$, $s - b$, $s - c$ far too large.
- Giving only the acute angle when working backward from an area. If $\sin C = k$ with $0 < k < 1$, then both C and $180^\circ - C$ fit; keep both unless the problem rules one out.

Make it a game • Chair Triangle. Place three chairs in a room to make a triangle and measure the sides with string. Guess the area, then compute it two ways: one half $ab \sin C$ using a protractor to measure an angle, and Heron's formula. Do the two answers agree? Closest guess scores. Explain why Heron's formula needs the semiperimeter.

55 MINUTES

7.2 THE LAW OF SINES

OBJECTIVES

- Prove the Law of Sines for acute and obtuse triangles.
- Solve triangles given two angles and a side (AAS and ASA).
- Apply the Law of Sines to distances that cannot be measured directly.

WATCH

The Sine Rule (1 of 2: What does it actually mean?)

Eddie Woo · 7:50<https://www.youtube.com/watch?v=fB0hLF-t5-w>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo calls it the 'sine rule'.

BACKUP VIDEO

The Sine Rule (2 of 2: Proving & Using the Formula)

Eddie Woo · 10:00https://www.youtube.com/watch?v=DD-Tn_EEjI8

Before Example 2: proving and using the law.

KEY TERMS

Law of Sines

In any triangle, each side divided by the sine of its opposite angle gives the same number: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

ASA

Given data of two angles and the side between them.

AAS

Given data of two angles and a side that is not between them.

solving a triangle

Finding all three sides and all three angles from the data given.

circumcircle

The circle passing through all three vertices of a triangle; its radius is the circumradius R .

CORE IDEAS

Right-triangle ratios need a right angle. For an oblique triangle we build one. In triangle ABC , drop the altitude h from C to line AB . In the right triangle that contains angle A , $h = b \sin A$. In the right triangle that contains angle B , $h = a \sin B$. Both expressions equal the same h , so $b \sin A = a \sin B$. Divide both sides by $\sin A \sin B$ to get $\frac{a}{\sin A} = \frac{b}{\sin B}$.

If angle A is obtuse, the altitude from C falls outside the triangle, on the extension of BA beyond A . The right triangle there contains the angle $180^\circ - A$, so $h = b \sin(180^\circ - A) = b \sin A$, and the argument goes through unchanged. Dropping the altitude from A instead gives $\frac{b}{\sin B} = \frac{c}{\sin C}$ in the same way. Together these give the full Law of Sines. A second proof: the area of the triangle is $\frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$; divide all three by $\frac{1}{2}abc$.

The Law of Sines is a set of proportions, so it solves a triangle whenever we know one side together with its opposite angle, plus one more piece. With AAS or ASA data, first find the third angle from $A + B + C = 180^\circ$. Then every side has its opposite angle known, and one complete ratio $\frac{a}{\sin A}$ lets us find the other sides: for example, $b = \frac{a \sin B}{\sin A}$.

The common ratio has a geometric meaning. Each ratio $\frac{a}{\sin A}$ equals $2R$, the diameter of the triangle's circumcircle. So the Law of Sines says that in one circle, longer chords stand opposite larger angles, in exact proportion to the sines. Use this as a check: the longest side must be opposite the largest angle.

Two angles and any side determine a triangle completely, so AAS and ASA problems always have exactly one solution, provided the two given angles add to less than 180° . (The case of two sides and a non-included angle is trickier; it is the subject of Lesson 7.3.)

KEY FORMULAS

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$\frac{a}{\sin A} = 2R$$

From history. The Law of Sines for plane triangles appears in *De Triangulis Omnimodis* (*On Triangles of Every Kind*), written in 1464 by the German astronomer Regiomontanus and printed in 1533. The book collected the trigonometry of plane and spherical triangles into one orderly treatise, laid out in the style of Euclid with definitions, theorems and proofs. It helped make trigonometry a subject of its own instead of a chapter in astronomy.

Euclid connection. Euclid I.18 and I.19 prove that in any triangle the greater side is opposite the greater angle, and the greater angle is opposite the greater side. The Law of Sines turns this comparison into an exact measurement.

WORKED EXAMPLES

Example 1. In triangle ABC , $A = 40^\circ$, $B = 60^\circ$ and $a = 10$. Solve the triangle. Round sides to the nearest tenth.

Show solution

1. This is AAS data. Find the third angle: $C = 180^\circ - 40^\circ - 60^\circ = 80^\circ$.
2. We know the complete pair $a = 10$ and $A = 40^\circ$, so the common ratio is $\frac{10}{\sin 40^\circ} \approx \frac{10}{0.6427876} \approx 15.55724$. Keep this value in the calculator; do not round it.
3. $b = \frac{10 \sin 60^\circ}{\sin 40^\circ} \approx 15.55724(0.8660254) \approx 13.473$.
4. $c = \frac{10 \sin 80^\circ}{\sin 40^\circ} \approx 15.55724(0.9848078) \approx 15.321$.
5. Check: the largest angle (80°) is opposite the longest side ($c \approx 15.3$), and the smallest angle (40°) is opposite the shortest side ($a = 10$).

Answer: $C = 80^\circ$, $b \approx 13.5$, $c \approx 15.3$

Example 2. Samuel and Ruth stand at points A and B on a straight shoreline, 500 m apart. They sight a fishing boat at C . Samuel measures $\angle CAB = 52^\circ$ and Ruth measures $\angle CBA = 71^\circ$. Find the distance from Samuel to the boat and the distance from the boat to the shoreline, each to the nearest tenth of a meter.

Show solution

1. This is ASA data: two angles and the side $c = AB = 500$ between them. The angle at the boat is $C = 180^\circ - 52^\circ - 71^\circ = 57^\circ$.
2. The distance from Samuel to the boat is $AC = b$, which is opposite angle $B = 71^\circ$. By the Law of Sines, $\frac{b}{\sin 71^\circ} = \frac{500}{\sin 57^\circ}$.
3. $b = \frac{500 \sin 71^\circ}{\sin 57^\circ} \approx \frac{500(0.9455186)}{0.8386706} \approx 563.70$ m.
4. The distance to the shoreline is the perpendicular from C to line AB . In the right triangle with hypotenuse AC and angle 52° at A , that distance is $b \sin 52^\circ \approx 563.7008(0.7880108) \approx 444.20$ m. (Use the unrounded value of b .)
5. So Samuel is about 563.7 m from the boat, and the boat is about 444.2 m from shore.

Answer: $AC \approx 563.7$ m; distance from shore ≈ 444.2 m

PRACTICE

1. Easy In triangle ABC , $A = 30^\circ$, $B = 45^\circ$ and $a = 8$. Find b exactly and to the nearest hundredth.

Show answer

$$b = \frac{8 \sin 45^\circ}{\sin 30^\circ} = 8\sqrt{2} \approx 11.31$$

2. Easy In triangle ABC , $A = 45^\circ$, $B = 30^\circ$ and $a = 10$. Find b exactly.

Show answer

$$b = \frac{10 \cdot \frac{1}{2}}{\frac{\sqrt{2}}{2}} = 5\sqrt{2} \approx 7.07$$

3. Easy In triangle ABC , $A = 35^\circ$, $B = 50^\circ$ and $c = 20$. Find C , a and b . Round sides to the nearest tenth.

Show answer

$$C = 95^\circ, a \approx 11.5, b \approx 15.4$$

4. **Medium** Solve triangle ABC if $A = 25^\circ$, $B = 110^\circ$ and $a = 9$. Round sides to the nearest tenth.

Show answer

$$C = 45^\circ, b \approx 20.0, c \approx 15.1$$

5. **Medium** Solve triangle ABC if $A = 58^\circ$, $C = 47^\circ$ and $b = 30$. Round sides to the nearest tenth.

Show answer

$$B = 75^\circ, a \approx 26.3, c \approx 22.7$$

6. **Medium** Two forest-fire lookout towers, A and B , are 12 miles apart. A ranger in tower A sees smoke at an angle of 41° from the line joining the towers; a ranger in tower B sees it at 63° from the same line. How far is the fire from tower A , to the nearest tenth of a mile?

Show answer

$$\text{The angle at the fire is } 76^\circ; \text{ distance} = \frac{12 \sin 63^\circ}{\sin 76^\circ} \approx 11.0 \text{ miles}$$

7. **Medium** To measure across a river, a surveyor lays out a baseline AB of 200 ft along one bank. A tree C stands on the far bank. She measures $\angle CAB = 68^\circ$ and $\angle CBA = 75^\circ$. Find the distance AC and the width of the river (the perpendicular distance from C to line AB), each to the nearest tenth of a foot.

Show answer

$$C = 37^\circ, AC = \frac{200 \sin 75^\circ}{\sin 37^\circ} \approx 321.0 \text{ ft; width} = AC \sin 68^\circ \approx 297.6 \text{ ft}$$

8. **Hard** A roof truss has a horizontal base of 30 ft. The two rafters make angles of 35° and 50° with the base. How high is the peak above the base, to the nearest tenth of a foot?

Show answer

$$\text{The peak angle is } 95^\circ. \text{ The rafter at the } 35^\circ \text{ end is } \frac{30 \sin 50^\circ}{\sin 95^\circ} \approx 23.07 \text{ ft, so the height is } 23.07 \sin 35^\circ \approx 13.2 \text{ ft}$$

9. **Hard** In triangle ABC , $A = 40^\circ$, $B = 75^\circ$ and $c = 16$. Find the area of the triangle to the nearest tenth.

Show answer

$$C = 65^\circ, a = \frac{16 \sin 40^\circ}{\sin 65^\circ} \approx 11.348, \text{Area} = \frac{1}{2}ac \sin B \approx 87.7 \text{ square units}$$

10. **Hard** Prove the Law of Sines $\frac{a}{\sin A} = \frac{b}{\sin B}$ for a triangle in which angle A is obtuse. Draw the figure and justify each step.

Show answer

Extend side BA beyond A and drop the perpendicular from C to that line at D ; let $CD = h$. In right triangle CDB , $h = a \sin B$. In right triangle CDA , the angle at A is $180^\circ - A$, so $h = b \sin(180^\circ - A) = b \sin A$. Hence $a \sin B = b \sin A$, and dividing by $\sin A \sin B$ gives $\frac{a}{\sin A} = \frac{b}{\sin B}$.

11. **Challenge** Let triangle ABC have circumcircle with center O and radius R , and let angle A be acute. Draw the diameter BD through B . Use Euclid III.21 (angles in the same segment are equal) and III.31 (the angle in a semicircle is right) to prove $\frac{a}{\sin A} = 2R$. Then find R for a triangle with $a = 10$ and $A = 30^\circ$.

Show answer

Angle BDC stands on the same arc BC as angle A , so $\angle BDC = A$ (III.21).

Since BD is a diameter, $\angle BCD = 90^\circ$ (III.31). In right triangle BCD , $\sin A = \frac{BC}{BD} = \frac{a}{2R}$, so $\frac{a}{\sin A} = 2R$. For $a = 10$, $A = 30^\circ$: $2R = 20$, so $R = 10$.

12. **Challenge** From point A on level ground, the angle of elevation to the top T of a radio tower is 32° . After walking 100 m straight toward the tower to point B , the angle of elevation is 47° . Use the Law of Sines in triangle ABT to find the height of the tower to the nearest tenth of a meter.

Show answer

In triangle ABT , $\angle ABT = 133^\circ$, so $\angle ATB = 15^\circ$. $BT = \frac{100 \sin 32^\circ}{\sin 15^\circ} \approx 204.75$ m. Height $= BT \sin 47^\circ \approx 149.7$ m

COMMON MISTAKES

- Pairing a side with the wrong angle. Each side must go with the angle directly across from it. Label the sketch first: a across from A , b across from B , c across from C .

- Trying to use the Law of Sines before any complete pair is known. With ASA data you must first find the third angle; only then does a known side have its known opposite angle.
- Rounding the common ratio or an early side and reusing it. Store full calculator values and round only the final answers.
- Forgetting a final reasonableness check. The longest side must face the largest angle; if it does not, look for a mislabeled side or a calculator in radian mode.

Make it a game • Find the Pair. Parent draws six triangles on cards with different data: ASA, AAS, SAS, SSS and so on. In 5 seconds each, call 'Law of Sines ready' or 'not yet,' and name the matching angle-and-side pair you need. Then solve one ready triangle completely, finding all sides and angles. Beat your score.

7.3 THE AMBIGUOUS CASE (SSA)

OBJECTIVES

- Decide whether SSA data determine zero, one or two triangles by comparing the given side with the height $h = b \sin A$.
- Find every triangle that fits SSA data, using the Law of Sines and the supplement of the first angle found.
- Explain geometrically why SSA is not a congruence test.

WATCH

Sine Rule: The Ambiguous Case

Eddie Woo · 11:25

<https://www.youtube.com/watch?v=io43Zrr2e3s>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo calls it the 'sine rule'.

BACKUP VIDEO

Using the Sine Rule to Find Angles (2 of 3: The Ambiguous Case)

Eddie Woo · 6:26

https://www.youtube.com/watch?v=pd_ywdkpmPk

Before the practice set, for another ambiguous-case example.

KEY TERMS

SSA

Given data of two sides and an angle that is not between them (the angle is opposite one of the given sides).

ambiguous case

The SSA situation, so called because the data may fit no triangle, exactly one triangle, or two different triangles.

height h

For SSA data A, a, b , the length $h = b \sin A$: the shortest distance from vertex C to the line along which side c lies.

supplementary angles

Two angles whose sum is 180° . They have equal sines: $\sin(180^\circ - \theta) = \sin \theta$.

CORE IDEAS

Suppose we know angle A , the side $b = AC$ next to it, and the side $a = BC$ opposite it. Picture building the triangle. Draw angle A with one ray as the line on which B must lie. Mark C on the other ray at distance b from A . Now swing an arc of radius a around C , like a gate swinging on its hinge. Every point where the arc meets the first ray is a possible vertex B . The arc may miss the ray, touch it once, or cross it twice. That is why SSA is called the ambiguous case.

The deciding length is the height $h = b \sin A$, the perpendicular distance from C to the ray. When A is acute: if $a < h$, the arc falls short and there is **no triangle**; if $a = h$, the arc just touches and there is **one right triangle**; if $h < a < b$, the arc crosses the ray twice on the same side of A and there are **two triangles**; if $a \geq b$, the second crossing lands at or behind A , so there is **one triangle**. When A is right or obtuse, there is one triangle if $a > b$ and none if $a \leq b$, because the side opposite the largest angle must be the longest side.

Algebra tells the same story. The Law of Sines gives $\sin B = \frac{b \sin A}{a} = \frac{h}{a}$. If this is greater than 1, no angle works: no triangle. If it equals 1, $B = 90^\circ$. If it is less than 1, there are two candidate angles: $B_1 = \sin^{-1}\left(\frac{b \sin A}{a}\right)$, which is acute, and its supplement $B_2 = 180^\circ - B_1$. A candidate is valid only if $A + B < 180^\circ$, leaving room for a positive angle C . Always test the second candidate; never assume it fails.

When there are two triangles, they share vertex C and the same angle A . Their two positions of B , called B_1 and B_2 , are symmetric about the foot of the altitude from C , because $CB_1 = CB_2 = a$ makes triangle CB_1B_2 isosceles. So the angle at B_2 in the second triangle is the supplement of the angle at B_1 , which is exactly what the algebra predicted.

Solve each valid triangle completely and label the answers clearly: Triangle 1 with B_1, C_1, c_1 and Triangle 2 with B_2, C_2, c_2 . Keep unrounded values from one step to the next.

KEY FORMULAS

$$h = b \sin A$$

$$\sin B = \frac{b \sin A}{a}$$

$$B_2 = 180^\circ - B_1$$

Euclid connection. Euclid proves that a triangle is fixed by SAS (I.4), by SSS (I.8), and by ASA or AAS (I.26). He gives no such theorem for two sides and a non-included angle, and the swinging-arc picture of this lesson shows why: SSA data can describe two different triangles.

WORKED EXAMPLES

Example 1. In triangle ABC , $A = 50^\circ$, $a = 12$ and $b = 8$. How many triangles are possible? Solve each one, rounding angles and sides to the nearest tenth.

Show solution

1. Angle A is acute and $a = 12 \geq b = 8$, so exactly one triangle is possible.
(Check: $h = 8 \sin 50^\circ \approx 6.13$, and a is longer than both h and b .)
2. Law of Sines: $\sin B = \frac{8 \sin 50^\circ}{12} \approx \frac{8(0.7660444)}{12} \approx 0.5106963$.
3. $B_1 = \sin^{-1}(0.5106963) \approx 30.7102^\circ$. The supplement $B_2 \approx 149.2898^\circ$ fails, because $50^\circ + 149.2898^\circ > 180^\circ$.
4. $C = 180^\circ - 50^\circ - 30.7102^\circ \approx 99.2898^\circ$.
5. $c = \frac{12 \sin C}{\sin 50^\circ} \approx \frac{12(0.9868846)}{0.7660444} \approx 15.459$.
6. One triangle: $B \approx 30.7^\circ$, $C \approx 99.3^\circ$, $c \approx 15.5$.

Answer: One triangle: $B \approx 30.7^\circ$, $C \approx 99.3^\circ$, $c \approx 15.5$

Example 2. Mr. Miller is laying out a triangular calf pen. From corner post A , one fence runs 50 m to post C . A second fence line leaves A at an angle of 40° to the first. He has a 35 m length of fence to run from C to a post B somewhere on the second line. Where can post B go? Find every possible length AB and the matching angles, to the nearest tenth.

Show solution

1. Label the data: $A = 40^\circ$, $b = AC = 50$, $a = BC = 35$ (opposite A). This is SSA.
2. Height: $h = 50 \sin 40^\circ \approx 32.139$. Since $h < a < b$ ($32.139 < 35 < 50$), there are two triangles.
3. $\sin B = \frac{50 \sin 40^\circ}{35} \approx 0.9182680$, so $B_1 \approx 66.6742^\circ$ and $B_2 = 180^\circ - B_1 \approx 113.3258^\circ$. Both leave room for C , since $40^\circ + 113.3258^\circ < 180^\circ$.
4. Triangle 1: $C_1 = 180^\circ - 40^\circ - 66.6742^\circ = 73.3258^\circ$, and $c_1 = \frac{35 \sin 73.3258^\circ}{\sin 40^\circ} \approx 52.161$ m.
5. Triangle 2: $C_2 = 180^\circ - 40^\circ - 113.3258^\circ = 26.6742^\circ$, and $c_2 = \frac{35 \sin 26.6742^\circ}{\sin 40^\circ} \approx 24.444$ m.
6. Check: the two positions of B are symmetric about the foot of the altitude from C , which is $50 \cos 40^\circ \approx 38.302$ m from A . Indeed $\frac{1}{2}(52.161 + 24.444) \approx 38.302$.
7. Post B can stand about 52.2 m from A (with $B \approx 66.7^\circ$, $C \approx 73.3^\circ$) or about 24.4 m from A (with $B \approx 113.3^\circ$, $C \approx 26.7^\circ$).

Answer: Two triangles: $AB \approx 52.2$ m with $B \approx 66.7^\circ$, $C \approx 73.3^\circ$; or $AB \approx 24.4$ m with $B \approx 113.3^\circ$, $C \approx 26.7^\circ$

PRACTICE

1. Easy How many triangles have $A = 30^\circ$, $a = 4$ and $b = 8$? Explain.

Show answer

One: $h = 8 \sin 30^\circ = 4 = a$, so the arc just touches and $B = 90^\circ$ (a right triangle)

2. Easy How many triangles have $A = 60^\circ$, $a = 5$ and $b = 10$? Explain.

Show answer

None: $h = 10 \sin 60^\circ \approx 8.66 > 5$, so side a is too short to reach

3. **Easy** How many triangles have $A = 110^\circ$, $a = 7$ and $b = 5$? Explain.

Show answer

One: A is obtuse and $a > b$

4. **Medium** Solve triangle ABC if $A = 110^\circ$, $a = 5$ and $b = 7$, or show that no such triangle exists.

Show answer

No triangle: A is obtuse, so a must be the longest side, but $a < b$. (Also $\sin B = \frac{7 \sin 110^\circ}{5} \approx 1.316 > 1$.)

5. **Medium** In triangle ABC , $A = 35^\circ$, $a = 9$ and $b = 12$. Show that there are two triangles and find both possible measures of B to the nearest tenth of a degree.

Show answer

$h = 12 \sin 35^\circ \approx 6.88 < 9 < 12$; $\sin B \approx 0.7647685$, so $B \approx 49.9^\circ$ or $B \approx 130.1^\circ$

6. **Medium** Solve triangle ABC if $A = 65^\circ$, $a = 20$ and $b = 15$. Round to the nearest tenth.

Show answer

One triangle ($a > b$): $B \approx 42.8^\circ$, $C \approx 72.2^\circ$, $c \approx 21.0$

7. **Medium** Ruth stands at point A on a straight farm road. The barn B is 300 m from her, in a direction making a 28° angle with the road. A 200 m water pipe will run straight from the barn to a point C on the road. How far along the road from A could C be? Give every answer to the nearest tenth of a meter.

Show answer

$h = 300 \sin 28^\circ \approx 140.8 < 200 < 300$, so two positions: $AC \approx 122.9$ m or $AC \approx 406.9$ m

8. **Hard** Explain why SSA data with A acute and $a \geq b$ can never give two triangles.

Show answer

If there were a second triangle, its angle would be $B_2 = 180^\circ - B_1$, which is obtuse. But since $a \geq b$, the larger-side-opposite-larger-angle rule (Euclid I.18–19) gives $A \geq B_2$, so A would be obtuse too, and a triangle cannot have two obtuse angles. In the picture, the arc of radius $a \geq b$ about C meets the ray from A only once, because its second crossing falls at or behind A .

9. **Hard** Let $A = 40^\circ$ and $b = 10$. For which lengths a are there exactly two triangles? Give the endpoints to the nearest hundredth.

Show answer

Two triangles exactly when $10 \sin 40^\circ < a < 10$, that is, about $6.43 < a < 10$

10. **Hard** In triangle ABC , $A = 25^\circ$, $a = 7$ and $b = 12$. Find the area of each possible triangle, to the nearest tenth. List the smaller area first.

Show answer

Two triangles, with $c \approx 6.051$ or $c \approx 15.701$. Areas $= \frac{1}{2}bc \sin A \approx 15.3$ and ≈ 39.8 square units

11. **Challenge** In the two-triangle case, the two possible lengths of side c are c_1 and c_2 . Prove that $c_1 + c_2 = 2b \cos A$. Then check the formula with Worked Example 2.

Show answer

Let D be the foot of the perpendicular from C to the ray. In right triangle ADC , $AD = b \cos A$. Because $CB_1 = CB_2 = a$, triangle CB_1B_2 is isosceles and the altitude CD bisects its base, so B_1 and B_2 lie the same distance d on either side of D . Then $c_1 + c_2 = (AD + d) + (AD - d) = 2AD = 2b \cos A$. Check: $52.1608 + 24.4436 = 76.6044$, and $100 \cos 40^\circ \approx 76.6044$.

12. **Challenge** Solve triangle ABC completely if $C = 48^\circ$, $c = 15$ and $b = 19$. Find every solution, rounding to the nearest tenth.

Show answer

$h = 19 \sin 48^\circ \approx 14.12 < 15 < 19$: two triangles. Triangle 1: $B \approx 70.3^\circ$, $A \approx 61.7^\circ$, $a \approx 17.8$. Triangle 2: $B \approx 109.7^\circ$, $A \approx 22.3^\circ$, $a \approx 7.7$

COMMON MISTAKES

- Stopping after the calculator's answer. $\sin^{-1}$ returns only the acute angle B_1 . Always test $B_2 = 180^\circ - B_1$ by checking whether $A + B_2 < 180^\circ$.
- Accepting a second triangle that does not exist. If $A + B_2 \geq 180^\circ$, there is no room for angle C ; reject B_2 .
- Missing the no-triangle case. If $\frac{b \sin A}{a} > 1$, the calculator gives an error; that error is the answer "no triangle," not a mistake to fix.
- Mixing up the two triangles. Label everything with subscripts (B_1, C_1, c_1 and B_2, C_2, c_2) and finish one triangle before starting the other.

Make it a game · Swing the Strip. On paper, draw angle $A = 30$ degrees with side $b = 10$ along one ray. Use a string of length a as a swinging side from the end of b . Parent picks a : 3, 5, 7 or 12. Predict zero, one or two triangles from $h = b \sin A$, then test by swinging. One point per correct prediction.

55 MINUTES

7.4 THE LAW OF COSINES

OBJECTIVES

- Prove the Law of Cosines with coordinates and connect it to Euclid II.12–13.
- Solve triangles given two sides and the included angle (SAS).
- Solve triangles given three sides (SSS), beginning with the largest angle.

WATCH

The Cosine Rule (1 of 3: Proof of the Formula)

Eddie Woo · 13:24<https://www.youtube.com/watch?v=FeubDMBf2wo>

WHEN TO WATCH

After reading the Core ideas, before Example 1 (Woo's own proof of the law).

Note: Woo calls it the 'cosine rule'; his proof differs from the coordinate proof in this lesson.

BACKUP VIDEO

The Cosine Rule (1 of 2: Introduction & relation to Pythagoras' Theorem)

Eddie Woo · 10:52<https://www.youtube.com/watch?v=s3DvfQbiSic>

Before Example 2: the law as an upgrade of Pythagoras.

KEY TERMS

Law of Cosines

In any triangle, $c^2 = a^2 + b^2 - 2ab \cos C$,
and likewise for the other two sides.

SSS

Given data of all three sides.

SAS

Given data of two sides and the angle
between them (the included angle).

correction term

The quantity $-2ab \cos C$ that adjusts the
Pythagorean sum $a^2 + b^2$: negative
when C is acute, zero when C is right,
positive when C is obtuse.

CORE IDEAS

The Law of Sines needs a side together with its opposite angle. With SAS or SSS data we never have such a pair, so we need a second law. Place triangle ABC on coordinate axes with C at the origin and B on the positive x -axis at $(a, 0)$. Point A is at distance b from the origin, on a ray making angle C with the x -axis, so by the unit-circle definition $A = (b \cos C, b \sin C)$. This works whether C is acute, right or obtuse.

Now find $c = AB$ with the distance formula: $c^2 = (b \cos C - a)^2 + (b \sin C - 0)^2 = b^2 \cos^2 C - 2ab \cos C + a^2 + b^2 \sin^2 C$. Group the b^2 terms and use $\sin^2 C + \cos^2 C = 1$: $c^2 = a^2 + b^2 - 2ab \cos C$. That is the Law of Cosines. Placing a different vertex at the origin gives the other two forms.

Read it as a corrected Pythagorean theorem. When $C = 90^\circ$, $\cos C = 0$ and the law becomes $c^2 = a^2 + b^2$. When C is acute, $\cos C > 0$, so c^2 is less than $a^2 + b^2$; when C is obtuse, $\cos C < 0$, so c^2 is greater. This is exactly what Euclid proved with areas in Book II: Proposition 12 for the obtuse case and Proposition 13 for the acute case.

SAS: use the Law of Cosines to find the third side. Then find a second angle. If you use the Law of Sines for it, choose the angle opposite the **shorter** of the two remaining sides; that angle must be acute, so the calculator's $\sin^{-1}$ value is correct. Finish with the angle sum.

SSS: solve the law for the cosine: $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$. Find the **largest** angle (opposite the longest side) first. Cosine is negative for obtuse angles, so $\cos^{-1}$ tells you plainly whether that angle is obtuse; the two angles left must then be acute, and either law finishes the job safely. Three lengths make a triangle only if each is less than the sum of the other two; otherwise the formula gives a cosine of ± 1 (a flat "triangle," when one length equals the sum of the other two) or outside $[-1, 1]$ (when it is longer).

KEY FORMULAS

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

From history. Euclid stated the content of this law around 300 BC in geometric form, as statements about squares and rectangles, with no cosines at all. The trigonometric form came much later. In France the Law of Cosines is still called the theorem of al-Kashi, after the fifteenth-century Persian astronomer Jamshid al-Kashi, who stated it in a form suited to numerical calculation.

Euclid connection. Euclid II.12 says that in an obtuse triangle the square on the side opposite the obtuse angle exceeds the sum of the squares on the other two sides by twice a certain rectangle; II.13 says the square on a side opposite an acute angle falls short by twice such a rectangle. That rectangle has area $a \cdot b |\cos C|$ in modern terms (side a times the projection of b onto the line of a), so II.12–13 together are the Law of Cosines, and with I.47 they cover every triangle.

WORKED EXAMPLES

Example 1. In triangle ABC , $b = 7$, $c = 10$ and $A = 50^\circ$. Solve the triangle. Round the side to the nearest hundredth and the angles to the nearest tenth of a degree.

Show solution

1. This is SAS: angle A lies between sides b and c . Use $a^2 = b^2 + c^2 - 2bc \cos A$.
2. $a^2 = 49 + 100 - 2(7)(10) \cos 50^\circ = 149 - 140(0.6427876) \approx 59.00974$, so $a \approx 7.68178$.
3. The two remaining sides are $b = 7$ and $c = 10$. Angle B is opposite the shorter one, so it must be acute. Law of Sines: $\sin B = \frac{7 \sin 50^\circ}{7.68178} \approx 0.698056$, so $B \approx 44.2712^\circ$.
4. $C = 180^\circ - 50^\circ - 44.2712^\circ \approx 85.7288^\circ$.
5. Check: the largest angle ($C \approx 85.7^\circ$) is opposite the longest side ($c = 10$).

Answer: $a \approx 7.68$, $B \approx 44.3^\circ$, $C \approx 85.7^\circ$

Example 2. Mr. Miller's triangular hay field (Lesson 7.1) has sides 120 m, 150 m and 210 m. Find all three corner angles to the nearest tenth of a degree.

Show solution

1. This is SSS. Label $a = 120$, $b = 150$, $c = 210$, and find the largest angle C first.

$$2. \cos C = \frac{120^2 + 150^2 - 210^2}{2(120)(150)} = \frac{14400 + 22500 - 44100}{36000} = \frac{-7200}{36000} = -0.2.$$

3. The cosine is negative, so C is obtuse: $C = \cos^{-1}(-0.2) \approx 101.5370^\circ$.

$$4. \cos A = \frac{150^2 + 210^2 - 120^2}{2(150)(210)} = \frac{52200}{63000} \approx 0.8285714, \text{ so } A \approx 34.0477^\circ.$$

$$5. B = 180^\circ - 101.5370^\circ - 34.0477^\circ \approx 44.4153^\circ. \text{ (Check: } \cos B = \frac{120^2 + 210^2 - 150^2}{2(120)(210)} = \frac{5}{7}, \text{ and } \cos^{-1} \frac{5}{7} \approx 44.4153^\circ.)$$

6. As a final check, $\frac{1}{2}(120)(150) \sin C = 9000\sqrt{0.96} \approx 8818 \text{ m}^2$, matching Heron's formula in Lesson 7.1.

Answer: $A \approx 34.0^\circ$, $B \approx 44.4^\circ$, $C \approx 101.5^\circ$

PRACTICE

1. **Easy** In triangle ABC , $b = 5$, $c = 8$ and $A = 60^\circ$. Find a exactly.

Show answer

$$a^2 = 25 + 64 - 80 \cdot \frac{1}{2} = 49, \text{ so } a = 7$$

2. **Easy** A triangle has sides 3, 5 and 7. Find its largest angle exactly.

Show answer

$$\cos C = \frac{9 + 25 - 49}{30} = -\frac{1}{2}, \text{ so } C = 120^\circ$$

3. **Easy** In triangle ABC , $b = 12$, $c = 15$ and $A = 38^\circ$. Find a to the nearest tenth.

Show answer

$$a^2 = 369 - 360 \cos 38^\circ \approx 85.316, \text{ so } a \approx 9.2$$

4. **Medium** A triangle has sides $a = 7$, $b = 9$ and $c = 12$. Find all three angles to the nearest tenth of a degree.

Show answer

$$\cos C = -\frac{1}{9}, C \approx 96.4^\circ; \cos A = \frac{22}{27}, A \approx 35.4^\circ; B \approx 48.2^\circ \text{ (indeed } \cos B = \frac{2}{3})$$

5. **Medium** Explain why the Law of Cosines is a generalization of the Pythagorean theorem, and what it says about c^2 compared with $a^2 + b^2$ when C is acute and when C is obtuse.

Show answer

When $C = 90^\circ$, $\cos C = 0$, so $c^2 = a^2 + b^2$, the Pythagorean theorem. When C is acute, $\cos C > 0$ and $c^2 < a^2 + b^2$; when C is obtuse, $\cos C < 0$ and $c^2 > a^2 + b^2$. The term $-2ab \cos C$ measures how far the triangle is from being right-angled at C .

6. **Medium** A surveyor wants the length of a pond. From a point P on dry ground she measures 450 m to one end of the pond and 620 m to the other end, with an angle of 72° between the two sight lines. How long is the pond, to the nearest tenth of a meter?

Show answer

$$d^2 = 450^2 + 620^2 - 2(450)(620) \cos 72^\circ \approx 414468.5, \text{ so } d \approx 643.8 \text{ m}$$

7. **Medium** Two rafters of a barn roof, 16 ft and 20 ft long, meet at the ridge at an angle of 100° . How far apart are their lower ends, to the nearest tenth of a foot?

Show answer

$$d^2 = 256 + 400 - 640 \cos 100^\circ \approx 767.135, \text{ so } d \approx 27.7 \text{ ft}$$

8. **Hard** A baseball diamond is a square with sides of 90 ft. The pitcher's mound is 60.5 ft from home plate, on the diagonal from home plate to second base. How far is the mound from first base, to the nearest tenth of a foot?

Show answer

The angle at home plate between the first-base line and the diagonal is 45° .

$$d^2 = 60.5^2 + 90^2 - 2(60.5)(90) \cos 45^\circ \approx 4059.857, \text{ so } d \approx 63.7 \text{ ft}$$

9. **Hard** Without finding any angle, decide whether a triangle with sides 6, 8 and 11 is acute, right or obtuse. Then find its largest angle to the nearest tenth of a degree.

Show answer

$$6^2 + 8^2 = 100 < 11^2 = 121, \text{ so the angle opposite 11 is obtuse. } \cos C = \frac{36 + 64 - 121}{96} = -0.21875, C \approx 102.6^\circ$$

10. **Hard** The coordinate proof covers every case at once. Give a second proof of $c^2 = a^2 + b^2 - 2ab \cos C$ for an obtuse angle C by dropping the altitude from A to the extension of side BC , in the spirit of Euclid II.12.

Show answer

Let the altitude from A meet line BC extended beyond C at D , with $AD = h$ and $CD = x$. In right triangle ADC , the angle at C is $180^\circ - C$, so $x = b \cos(180^\circ - C) = -b \cos C$ and $h^2 = b^2 - x^2$. In right triangle ADB , $BD = a + x$, so $c^2 = (a + x)^2 + h^2 = a^2 + 2ax + x^2 + b^2 - x^2 = a^2 + b^2 + 2ax$. Substituting $x = -b \cos C$ gives $c^2 = a^2 + b^2 - 2ab \cos C$. The term $2ax$ is Euclid's "twice the rectangle."

11. **Challenge** A parallelogram has sides 10 and 14, and one of its angles is 70° . Find the lengths of both diagonals to the nearest hundredth, shorter first. Then show that the sum of the squares of the diagonals equals the sum of the squares of the four sides.

Show answer

Short diagonal: $d_1^2 = 296 - 280 \cos 70^\circ$, $d_1 \approx 14.15$. Long diagonal (opposite the 110° angle): $d_2^2 = 296 + 280 \cos 70^\circ$, $d_2 \approx 19.79$. Adding, the cosine terms cancel: $d_1^2 + d_2^2 = 592 = 2(10^2 + 14^2)$

12. **Challenge** Two ships leave the same harbor at noon on straight courses that differ by 64° . One sails at 15 knots and the other at 20 knots (1 knot is 1 nautical mile per hour). How far apart are they at 2:30 p.m., to the nearest tenth of a nautical mile? Explain why the distance between them grows at a constant rate, and find that rate.

Show answer

After 2.5 hours they have gone 37.5 and 50 nautical miles. $d^2 = 37.5^2 + 50^2 - 2(37.5)(50) \cos 64^\circ \approx 2262.36$, so $d \approx 47.6$ nautical miles. At any time t the sides are $15t$ and $20t$ with the same 64° angle, so every triangle is similar and $d = t\sqrt{625 - 600 \cos 64^\circ} \approx 19.0t$: the distance grows at about 19.0 knots

COMMON MISTAKES

- Pairing the wrong angle with the side. In $c^2 = a^2 + b^2 - 2ab \cos C$, the side on the left must be opposite the angle on the right, and that angle must be the one between a and b .

- Combining terms too early: $a^2 + b^2 - 2ab \cos C$ is not $(a^2 + b^2 - 2ab) \cos C$. Multiply $2ab \cos C$ first, then subtract.
- Dropping the sign of a negative cosine. For an obtuse angle, $-2ab \cos C$ is positive and makes the opposite side longer. Use parentheses when entering $\cos 100^\circ$ and similar values.
- Finding a small angle first in SSS and then using the Law of Sines for the largest angle. $\sin^{-1}$ cannot detect an obtuse angle. Find the largest angle first with the Law of Cosines.

Make it a game • Hinge Predict. Tape two rulers or pencil lengths of 6 and 8 together at one end like a hinge. Parent sets an angle: 0, 60, 90, 120 or 180 degrees. Predict the third side, then compute it with the Law of Cosines and check with a ruler. What happens at 90 degrees? Explain the correction term.

7.5 SOLVING ANY TRIANGLE

OBJECTIVES

- Classify given triangle data as AAS, ASA, SSA, SAS or SSS and choose the right law to begin.
- Solve any triangle completely, in an order that avoids ambiguity and rounding errors.
- Check a solution for consistency with the angle sum, side-angle order and the Law of Sines.

WATCH

The Cosine Rule (3 of 3: How to tell which rule to use)

Eddie Woo · 9:58

<https://www.youtube.com/watch?v=gR7dAvxc2dI>

WHEN TO WATCH

Before Example 1: deciding which law to use.

Note: Woo says 'sine rule' and 'cosine rule'.

BACKUP VIDEO

Overview of Trigonometric Techniques

Eddie Woo · 12:47

<https://www.youtube.com/watch?v=fMIqJzt2sCk>

Before the practice set, for a review of choosing a rule.

KEY TERMS

case of a triangle

The pattern of the three given parts:
AAS, ASA, SSA, SAS or SSS.

consistency check

A test that a finished solution fits together: the angles sum to 180° , the longest side faces the largest angle, and all three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, $\frac{c}{\sin C}$ agree.

AAA

Three angles and no side. Such data fix only the shape of a triangle, not its size, so they cannot be solved for lengths.

CORE IDEAS

Every triangle problem in this unit gives three parts, at least one of them a side. First sketch the triangle and mark the given parts. Then name the case, because the case tells you where to start. **AAS or ASA:** find the third angle, then use the Law of Sines. **SSA:** use the Law of Sines and test for zero, one or two triangles (Lesson 7.3). **SAS:** use the Law of Cosines for the third side. **SSS:** use the Law of Cosines for the largest angle.

Why this plan? The Law of Sines needs a known side with its known opposite angle, which AAS, ASA and SSA provide (after the angle sum, in the ASA case) but SAS and SSS do not. The Law of Cosines relates three sides and one angle, so it fits SAS and SSS exactly. When both laws could work, prefer the one that cannot mislead you: $\cos^{-1}$ distinguishes acute from obtuse angles, while $\sin^{-1}$ always returns an acute angle.

A safe order for finishing a triangle once one law has started it: after any Law of Cosines step, find the angle opposite the shorter remaining side next (it must be acute), and get the last angle from $A + B + C = 180^\circ$. Carry full calculator values from step to step and round only the final answers.

Finally, check. The angles must add to 180° ; the longest side must face the largest angle and the shortest side the smallest; and the three ratios $\frac{a}{\sin A}, \frac{b}{\sin B}, \frac{c}{\sin C}$ should agree to within rounding. A check that uses a different law from the one you solved with is the strongest, because the same mistake rarely fools two methods. An area computed two ways (with $\frac{1}{2}ab \sin C$ and with Heron's formula) is another good check.

KEY FORMULAS

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$A + B + C = 180^\circ$$

$$\text{Area} = \frac{1}{2}ab \sin C$$

From history. For centuries, surveyors and navigators solved oblique triangles by hand with printed tables of sines and, after the early 1600s, tables of logarithms of sines, which turned multiplication into addition. Careful computers checked every result a second way before trusting it, because a single slip could misplace a coastline on a chart. The habit of checking taught in this lesson comes from that long practical tradition.

Euclid connection. Euclid's congruence theorems match the cases that always give one triangle: SAS (I.4), SSS (I.8), and ASA and AAS (I.26). If the data fix a unique triangle, trigonometry can compute it; SSA, which Euclid does not list, is exactly the case that may not.

WORKED EXAMPLES

Example 1. In triangle ABC , $a = 9$, $B = 72^\circ$ and $c = 14$. Name the case and solve the triangle. Round the side to the nearest hundredth and the angles to the nearest tenth of a degree.

Show solution

1. Angle B lies between sides a and c , so this is SAS. Start with the Law of Cosines.
2. $b^2 = a^2 + c^2 - 2ac \cos B = 81 + 196 - 252 \cos 72^\circ \approx 277 - 252(0.3090170) \approx 199.1277$, so $b \approx 14.11126$.
3. Of the remaining sides, $a = 9$ is shorter than $c = 14$, so angle A must be acute.
 $\sin A = \frac{9 \sin 72^\circ}{14.11126} \approx 0.606573$, so $A \approx 37.3421^\circ$.
4. $C = 180^\circ - 72^\circ - 37.3421^\circ \approx 70.6579^\circ$.
5. Check with a different law: $\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{314.1277}{395.1153} \approx 0.795028$,
which gives $A \approx 37.3421^\circ$ again. The longest side ($b \approx 14.11$) faces the largest angle (72°).

Answer: SAS; $b \approx 14.11$, $A \approx 37.3^\circ$, $C \approx 70.7^\circ$

Example 2. A triangular building lot has corners A , B and C . The surveyor measures $AB = 85$ m, $AC = 110$ m, and the angle at corner B as 70° . Find the remaining angles, the length BC (to the nearest tenth of a meter) and the area of the lot (to the nearest square meter).

Show solution

1. Label: $c = AB = 85$, $b = AC = 110$, $B = 70^\circ$. The known angle B is opposite the known side b , so this is SSA.
2. Since B is acute and $b = 110 \geq c = 85$, there is exactly one triangle.
3. $\sin C = \frac{85 \sin 70^\circ}{110} \approx \frac{79.87387}{110} \approx 0.7261261$, so $C \approx 46.5626^\circ$. (The supplement 133.4374° would make $B + C > 180^\circ$.)
4. $A = 180^\circ - 70^\circ - 46.5626^\circ \approx 63.4374^\circ$.
5. $BC = a = \frac{110 \sin A}{\sin 70^\circ} \approx 104.703$ m.
6. Area $= \frac{1}{2}ac \sin B \approx \frac{1}{2}(104.703)(85)(0.9396926) \approx 4181.5$ m².
7. Check: $\frac{a}{\sin A} \approx 117.06$, $\frac{b}{\sin B} \approx 117.06$, $\frac{c}{\sin C} \approx 117.06$. The ratios agree.

Answer: $C \approx 46.6^\circ$, $A \approx 63.4^\circ$, $BC \approx 104.7$ m, area ≈ 4182 m²

PRACTICE

1. **Easy** Name the case and the law you would use first: (a) $A = 40^\circ$, $B = 65^\circ$, $c = 12$; (b) $a = 7$, $b = 9$, $c = 13$; (c) $A = 35^\circ$, $a = 8$, $b = 10$; (d) $a = 5$, $C = 120^\circ$, $b = 6$.

Show answer

(a) ASA: find C from the angle sum, then the Law of Sines. (b) SSS: Law of Cosines for the largest angle (opposite 13). (c) SSA: Law of Sines, testing for two triangles. (d) SAS: Law of Cosines for c .

2. **Easy** A triangle has sides 5, 6 and 7. Find its smallest angle to the nearest tenth of a degree.

Show answer

The smallest angle is opposite 5: $\cos \theta = \frac{36 + 49 - 25}{84} = \frac{5}{7}$, so $\theta \approx 44.4^\circ$

3. **Easy** In triangle ABC , $A = 72^\circ$, $B = 38^\circ$ and $b = 15$. Find a to the nearest tenth.

Show answer

$$\text{AAS: } a = \frac{15 \sin 72^\circ}{\sin 38^\circ} \approx 23.2$$

4. **Medium** Solve triangle ABC if $a = 14$, $b = 9$ and $C = 110^\circ$. Round to the nearest tenth.

Show answer

$$\text{SAS: } c^2 = 277 - 252 \cos 110^\circ \approx 363.189, c \approx 19.1; B \approx 26.3^\circ \text{ (opposite the shorter side); } A \approx 43.7^\circ$$

5. **Medium** Solve triangle ABC if $A = 52^\circ$, $a = 15$ and $b = 13$. Round to the nearest tenth.

Show answer

$$\text{SSA with } a > b: \text{ one triangle. } B \approx 43.1^\circ, C \approx 84.9^\circ, c \approx 19.0$$

6. **Medium** Samuel solved a triangle and reported $A = 40^\circ$, $B = 60^\circ$, $C = 80^\circ$, $a = 10$, $b = 12$, $c = 15.3$. Check his work with the Law of Sines. Which side is wrong, and what should it be (to the nearest tenth)?

Show answer

$$\frac{a}{\sin A} \approx 15.56 \text{ and } \frac{c}{\sin C} \approx 15.54 \text{ agree, but } \frac{b}{\sin B} \approx 13.86 \text{ does not. Side } b \text{ is wrong: } b = \frac{10 \sin 60^\circ}{\sin 40^\circ} \approx 13.5$$

7. **Medium** Explain why three angles (AAA) can never be solved for side lengths, and why, in an SSS problem, it is wise to find the largest angle first.

Show answer

Triangles with the same three angles are similar (Euclid VI.4) but may be any size, so AAA data fix only the shape; at least one length is needed. In SSS, only the largest angle can be obtuse. Finding it first with $\cos^{-1}$ settles whether it is obtuse; the other two angles are then certainly acute, so the Law of Sines cannot mislead you for them.

8. **Hard** The gable end of a barn is a triangle with a 36 ft base. Its two rafters are 20 ft and 22 ft long. Find the angle each rafter makes with the base (to the nearest tenth of a degree) and the height of the ridge above the base (to the nearest tenth of a foot).

Show answer

SSS. At the end of the 20 ft rafter: $\cos \theta = \frac{36^2 + 20^2 - 22^2}{2(36)(20)} = \frac{1212}{1440}$, $\theta \approx 32.7^\circ$. At the end of the 22 ft rafter: $\cos \phi = \frac{1380}{1584}$, $\phi \approx 29.4^\circ$. Height = $20 \sin \theta \approx 10.8$ ft

9. **Hard** A small plane flies 220 miles in a straight line, turns through 35° , and flies 150 miles more. How far is it from its starting point (to the nearest tenth of a mile), and what angle does the straight line home make with its first leg (to the nearest tenth of a degree)?

Show answer

The angle inside the triangle at the turn is $180^\circ - 35^\circ = 145^\circ$. $d^2 = 220^2 + 150^2 - 2(220)(150) \cos 145^\circ \approx 124964.0$, $d \approx 353.5$ mi. The angle at the start is opposite the 150-mile side, so it is acute: $\sin \theta = \frac{150 \sin 145^\circ}{353.50}$, $\theta \approx 14.1^\circ$

10. **Hard** Solve triangle ABC if $A = 52^\circ$, $a = 11$ and $b = 13$. Find every solution, rounding to the nearest tenth.

Show answer

SSA with $h = 13 \sin 52^\circ \approx 10.24 < 11 < 13$: two triangles. Triangle 1: $B \approx 68.6^\circ$, $C \approx 59.4^\circ$, $c \approx 12.0$. Triangle 2: $B \approx 111.4^\circ$, $C \approx 16.6^\circ$, $c \approx 4.0$

11. **Challenge** A triangle has $a = 8$, $b = 13$ and area 40. Find every possible length of side c , to the nearest tenth.

Show answer

$\sin C = \frac{2(40)}{8 \cdot 13} = \frac{10}{13}$, so $\cos C = \pm \frac{\sqrt{69}}{13}$. Then $c^2 = 233 \mp 16\sqrt{69}$, giving $c \approx 10.0$ (with $C \approx 50.3^\circ$) or $c \approx 19.1$ (with $C \approx 129.7^\circ$)

12. **Challenge** Prove the projection formula $a = b \cos C + c \cos B$ for a triangle in which angles B and C are both acute. Then use it to check the solution of Worked Example 1.

Show answer

Drop the altitude from A to side BC , meeting it at D . Because B and C are acute, D lies between B and C . In right triangle ADC , $CD = b \cos C$; in right triangle ADB , $BD = c \cos B$. So $a = BD + DC = c \cos B + b \cos C$. Check: in Worked Example 1, angles A and C are acute, so the formula with b as the base reads $b = a \cos C + c \cos A \approx 9(0.331208) + 14(0.795028) \approx 2.98087 + 11.13039 = 14.11126$, matching $b \approx 14.11126$.

COMMON MISTAKES

- Reaching for the Law of Sines with SAS or SSS data. Without a known side and its opposite angle, the Law of Sines has two unknowns. Start with the Law of Cosines.
- Using $\sin^{-1}$ for an angle that might be obtuse. After a Law of Cosines step, use the Law of Sines only for the angle opposite the shorter remaining side, or use the Law of Cosines again.
- Treating SSA like the other cases and reporting only one triangle. Whenever the given angle is opposite a given side, test for zero, one or two triangles.
- Rounding intermediate results to one decimal place and reusing them. Errors grow from step to step; store values in the calculator and round only at the end.

Make it a game • Case Sorting Sprint. Make ten data cards: AAS, ASA, SAS, SSS, SSA and AAA. Draw one at a time. In 10 seconds, name the case, the law to start with, and whether it is ambiguous or impossible to determine. Sort the pile, record your time, and race to beat your own personal best.

OBLIQUE TRIANGLES: THE LAWS OF SINES AND COSINES

Most triangles in the real world have no right angle: a field bounded by three fences, a roof truss, the triangle formed by two lookouts and a distant fire. This unit extends trigonometry from right triangles to every triangle. We first find areas with $\frac{1}{2}ab \sin C$ and Heron's formula. Then we prove the two great laws of oblique triangles, the Law of Sines and the Law of Cosines, and learn which one to use for each kind of given data.

As always, we ask why. Each law is proved from earlier work: the Law of Sines by dropping an altitude, and the Law of Cosines by the distance formula and $\sin^2 \theta + \cos^2 \theta = 1$. We meet old friends along the way: Euclid's congruence theorems (I.4, I.8, I.26) explain which data fix a triangle, and his Propositions II.12–13 turn out to be the Law of Cosines in geometric dress. The ambiguous case (SSA) shows how careful reasoning, not just calculation, decides whether data describe zero, one or two triangles.

These tools are the working core of surveying and navigation. Unit 8 applies them directly to angles of elevation, bearings, triangulation and three-dimensional problems, so mastering the methods and the habit of checking answers here will pay off immediately.

PREREQUISITES

- Solve right triangles with sine, cosine and tangent and their inverses (Lessons 1.2–1.4).
- Use reference angles and the fact that $\sin(180^\circ - \theta) = \sin \theta$ and $\cos(180^\circ - \theta) = -\cos \theta$ (Lessons 3.6–3.8).
- Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ (Lesson 5.2).
- Explain why $\sin^{-1}$ returns only angles in $[-90^\circ, 90^\circ]$ and $\cos^{-1}$ returns angles in $[0^\circ, 180^\circ]$ (Lessons 6.1–6.2).
- Use the distance formula and simplify radicals (Algebra).
- Recall the triangle angle sum (Euclid I.32) and the congruence criteria SAS, SSS and ASA from Geometry.

MASTERY CHECKLIST

- ☐ I can derive Area and use it for acute and obtuse $\frac{1}{2}ab \sin C$ included angles.
- ☐ I can find a triangle's area from its three sides with Heron's formula.
- ☐ I can work backward from an area to find both possible included angles.
- ☐ I can prove the Law of Sines and use it to solve AAS and ASA triangles.
- ☐ I can decide whether SSA data give zero, one or two triangles by comparing
 - a with h and with $b \sin A$
 - b .
- ☐ I can find both triangles in the ambiguous case and label each solution clearly.
- ☐ I can prove the Law of Cosines and explain how it generalizes the Pythagorean theorem and Euclid II.12–13.
- ☐ I can solve SAS triangles and SSS triangles, finding the largest angle first in SSS.
- ☐ I can name the case of any triangle problem and choose the correct law to start.
- ☐ I can check a solution with the angle sum, side–angle order and the equal ratios of the Law of Sines.

- I can model surveying, carpentry and navigation problems with oblique triangles.

LESSONS IN THIS UNIT

7.1 The Area of a Triangle	50 minutes
7.2 The Law of Sines	55 minutes
7.3 The Ambiguous Case (SSA)	2 sessions of 45 minutes
7.4 The Law of Cosines	55 minutes
7.5 Solving Any Triangle	2 sessions of 45 minutes

UNIT 7 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; set it to degree mode. Show your work, including a labeled sketch for every triangle. Unless a problem says otherwise, round side lengths to the nearest tenth and angles to the nearest tenth of a degree, and do not round intermediate values. Give exact values where asked. Total: 100 points.

1. 5 pts

Find the exact area of a triangle with sides 9 and 14 and an included angle of 30° .

2. 6 pts

Use Heron's formula to find the exact area of a triangle with sides 10, 17 and 21.

3. 6 pts

In triangle ABC , $A = 45^\circ$, $B = 60^\circ$ and $a = 12$. Find b exactly, then to the nearest tenth.

4. 8 pts

Two points A and B on a straight shore are 800 ft apart. A buoy C floats offshore. The angles measured are $\angle CAB = 62^\circ$ and $\angle CBA = 54^\circ$. How far is the buoy from point A ?

5. (5 pts)

How many triangles have $A = 40^\circ$, $a = 8$ and $b = 10$? Justify your answer.

6. (10 pts)

Solve triangle ABC if $A = 30^\circ$, $a = 7$ and $b = 10$. Find every solution.

7. (7 pts)

In triangle ABC , $b = 10$, $c = 12$ and $A = 75^\circ$. Find a .

8. (7 pts)

A triangle has sides 8, 10 and 15. Find its largest angle and state whether the triangle is acute, right or obtuse.

9. (8 pts)

A tunnel is to be dug straight through a hill. From a point P in the open, a surveyor measures 380 m to one end of the planned tunnel and 510 m to the other end, with an angle of 48° between the two sight lines. How long will the tunnel be?

10. (9 pts)

A pilot flies 300 km in a straight line, turns through an angle of 40° , and flies 200 km farther. How far is the plane from its starting point?

11. (10 pts)

A triangular pasture on Mr. Miller's farm has sides of 50, 70 and 90 yards. Find its largest angle, then find its area in two ways: with $\frac{1}{2}ab \sin C$ and with Heron's formula. Round the area to the nearest square yard.

12. (10 pts)

Starting from the area formula $\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$, prove the Law of Sines. Then explain in one or two sentences why the area formula holds even when the angle used is obtuse.

13. (9 pts)

Let $A = 30^\circ$ and $b = 12$. Find every length a for which exactly one triangle exists. Explain your reasoning.

ANSWER KEY

1. $\frac{1}{2}(9)(14) \sin 30^\circ = 31.5$ square units
2. $s = 24$; $\sqrt{24 \cdot 14 \cdot 7 \cdot 3} = \sqrt{7056} = 84$ square units
3. $b = \frac{12 \sin 60^\circ}{\sin 45^\circ} = 6\sqrt{6} \approx 14.7$
4. $C = 64^\circ$; $AC = \frac{800 \sin 54^\circ}{\sin 64^\circ} \approx 720.1$ ft
5. Two: $h = 10 \sin 40^\circ \approx 6.43$, and $h < a < b$
6. $\sin B = \frac{10 \sin 30^\circ}{7} = \frac{5}{7}$; two triangles. Triangle 1: $B \approx 45.6^\circ$, $C \approx 104.4^\circ$, $c \approx 13.6$.
Triangle 2: $B \approx 134.4^\circ$, $C \approx 15.6^\circ$, $c \approx 3.8$
7. $a^2 = 100 + 144 - 240 \cos 75^\circ \approx 181.883$, so $a \approx 13.5$
8. $\cos C = \frac{64 + 100 - 225}{160} = -0.38125$, so $C \approx 112.4^\circ$; the triangle is obtuse
9. $d^2 = 380^2 + 510^2 - 2(380)(510) \cos 48^\circ \approx 145145.0$, so $d \approx 381.0$ m
10. The angle inside the triangle at the turn is 140° . $d^2 = 300^2 + 200^2 - 2(300)(200) \cos 140^\circ \approx 221925.3$, so $d \approx 471.1$ km
11. $\cos C = \frac{2500 + 4900 - 8100}{7000} = -0.1$, $C \approx 95.7^\circ$. Area = $\frac{1}{2}(50)(70) \sin C = 1750\sqrt{0.99} \approx 1741$ yd²; Heron: $s = 105$, $\sqrt{105 \cdot 55 \cdot 35 \cdot 15} = \sqrt{3031875} \approx 1741$ yd²
12. All three expressions equal the same area, so $\frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$.
Dividing every part by $\frac{1}{2}abc$ gives $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, which is the Law of Sines. When C is obtuse, the altitude from A meets the extension of BC beyond C , and the right triangle formed has angle $180^\circ - C$ at C , so $h = b \sin(180^\circ - C) = b \sin C$, and the formula $\frac{1}{2}ah = \frac{1}{2}ab \sin C$ still holds.
13. $h = 12 \sin 30^\circ = 6$. Exactly one triangle when $a = 6$ (a right triangle, with $B = 90^\circ$) or when $a \geq 12$ (the second crossing of the arc falls at or behind A). For $6 < a < 12$ there are two triangles, and for $a < 6$ there are none

APPLICATIONS

Trigonometry was born as a practical art. Astronomers, navigators and surveyors needed to find distances and heights they could not measure with a rope or chain, and the ratios of a triangle gave them a way. In this unit you put everything from Units 1 through 7 to work: angles of elevation and depression for towers, silos and cliffs; bearings for ships and aircraft; triangulation for rivers, fields and even the size of the earth; right triangles hidden inside three-dimensional roofs, rooms and pyramids; and sinusoidal models for water wheels, tides, daylight and springs.

The mathematics is not new, but the skill is. In each problem you must read a description, draw a careful diagram, decide which triangle or which model fits, and only then compute. That habit of turning a real situation into a clear figure is what Eratosthenes, Snellius and every surveyor and navigator since have practiced, and it is the main goal of this unit.

Unit 8 draws on right-triangle trigonometry (Unit 1), radians and the unit circle (Unit 3), sinusoidal graphs (Unit 4), trigonometric equations (Unit 6) and the Laws of Sines and Cosines (Unit 7). The optional Unit 9 continues these applications with polar coordinates and vectors.

PREREQUISITES

- Solve right triangles with sine, cosine, tangent and their inverses (Lessons 1.3 and 1.4).
- Use exact values for 30° , 45° and 60° (Unit 2).
- Write and interpret sinusoidal equations $y = a \sin(b(x - c)) + d$ and $y = a \cos(b(x - c)) + d$ (Lessons 4.4 and 4.5).
- Solve trigonometric equations on an interval (Lessons 6.4 and 6.5).
- Solve oblique triangles with the Law of Sines and the Law of Cosines, and find areas with $\frac{1}{2}ab \sin C$ and Heron's formula (Unit 7).

MASTERY CHECKLIST

- ☐ I can draw a diagram for a height-and-distance problem and find heights and distances from angles of elevation and depression.
- ☐ I can explain why an angle of depression equals the matching angle of elevation, and solve problems with two observations of one object.
- ☐ I can convert between compass bearings and true three-figure bearings and find back bearings.
- ☐ I can find the interior angle at a turn and solve two-leg navigation problems for distance and bearing.
- ☐ I can find an inaccessible distance, such as the width of a river, from a measured baseline and two angles.
- ☐ I can find the area of a surveyed field by dividing it into triangles, and convert square feet to acres.
- ☐ I can explain Eratosthenes' method for measuring the earth.
- ☐ I can find lengths and angles in boxes, rooms, roofs and pyramids by isolating right triangles.
- ☐ I can solve problems where an object is observed from two points that are not in line with it.
- ☐ I can build a sinusoidal model for a water wheel, tides, daylight hours or a spring, and use it to predict values and find times.

LESSONS IN THIS UNIT

8.1	Angles of Elevation and Depression	50 minutes
8.2	Bearings and Navigation	2 sessions of 45 minutes
8.3	Surveying and Triangulation	2 sessions of 45 minutes
8.4	Problems in Three Dimensions	50 minutes
8.5	Modeling Periodic Motion	2 sessions of 45 minutes

UNIT 8 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; set it to degree mode except where a problem uses radians. Draw a labeled diagram for every word problem. Keep full calculator precision in intermediate steps and round only the final answer as each problem directs. Show your work; answers without supporting work receive partial credit at most. Total: 100 points.

1. 5 pts

From the top of a lighthouse 30 m above the sea, the angle of depression to a sailboat is 12° . How far is the sailboat from the base of the lighthouse, to the nearest tenth of a meter?

2. 5 pts

(a) Write the compass bearing $S40^\circ E$ as a true three-figure bearing. (b) If the bearing of a buoy from a ship is that bearing, what is the bearing of the ship from the buoy?

3. 6 pts

The height of a bucket on a water wheel is $h(t) = 20 - 18 \cos\left(\frac{\pi t}{45}\right)$ feet, with t in seconds. Find the bucket's greatest height, its least height and the period of the wheel.

4. 8 pts

From a point on level ground the angle of elevation to the top of a silo is 30° . From a point 40 ft closer, in line with the first point and the silo, the angle of elevation is 45° . Find the height of the silo exactly and to the nearest tenth of a foot.

5. 6 pts

A ship sails 20 miles on a bearing of 045° , then 15 miles on a bearing of 135° . Find its distance from the start and its bearing from the start, to the nearest tenth of a degree.

6. 8 pts

A triangular hay field has two sides of 400 ft and 350 ft meeting at an angle of 75° . Find its area to the nearest square foot and in acres to the nearest hundredth (1 acre = 43,560 sq ft).

7. 7 pts

A barn loft is 20 ft long, 15 ft wide and 10 ft high. Find the length of a straight pole reaching from one floor corner to the opposite ceiling corner, to the nearest tenth of a foot, and the angle it makes with the floor, to the nearest tenth of a degree.

8. 8 pts

To find the width of a river, a surveyor lays out a baseline $AB = 120$ m along one straight bank and sights a post C on the far bank. She measures $\angle CAB = 62^\circ$ and $\angle CBA = 54^\circ$. Find the width of the river to the nearest tenth of a meter.

9. 7 pts

A town has 16 hours of daylight on its longest day, day 172, and 8 hours on its shortest. Using a period of 365 days, write a model for the hours of daylight $D(t)$ on day t and predict the daylight on day 80, to the nearest tenth of an hour.

10. 10 pts

A pilot flies 120 miles from A to B on a bearing of 060° , then 80 miles from B to C on a bearing of 200° . Find the distance AC to the nearest tenth of a mile and the bearing of C from A to the nearest tenth of a degree.

11. 10 pts

From a point A due west of a tower, the angle of elevation of its top is 40° . From a point B due south of the tower, the angle of elevation is 35° . Points A and B are 150 m apart on level ground. Find the height of the tower to the nearest tenth of a meter.

12. 10 pts

Using the daylight model of Problem 9, on which days of the year does the town have more than 15 hours of daylight? Give the day numbers to the nearest tenth and state how many whole days qualify.

13. 10 pts

An observer measures the angle of elevation α to the top of a tower, then walks a distance d straight toward it on level ground and measures a larger angle of elevation β . Prove that the height of the tower is $h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$.

ANSWER KEY

1. $\frac{30}{\tan 12^\circ} \approx 141.1 \text{ m}$
2. (a) 140° ; (b) 320°
3. Greatest height 38 ft, least height 2 ft, period $\frac{2\pi}{\pi/45} = 90 \text{ s}$
4. Let x be the distance from the nearer point. $h = x$ and $h = (x + 40) \tan 30^\circ$, so $h\sqrt{3} = h + 40$, $h = \frac{40}{\sqrt{3} - 1} = 20(\sqrt{3} + 1) \approx 54.6 \text{ ft}$
5. The turn is 90° , so the distance is $\sqrt{20^2 + 15^2} = 25 \text{ miles}$; bearing = $045^\circ + \tan^{-1}\left(\frac{15}{20}\right) \approx 081.9^\circ$
6. $\frac{1}{2}(400)(350) \sin 75^\circ \approx 67,615 \text{ sq ft} \approx 1.55 \text{ acres}$
7. Floor diagonal = 25 ft; pole = $\sqrt{25^2 + 10^2} = \sqrt{725} \approx 26.9 \text{ ft}$; angle = $\tan^{-1}\left(\frac{10}{25}\right) \approx 21.8^\circ$
8. $C = 64^\circ$; $AC = \frac{120 \sin 54^\circ}{\sin 64^\circ} \approx 108.014 \text{ m}$; width = $AC \sin 62^\circ \approx 95.4 \text{ m}$
9. $D(t) = 4 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12$; $D(80) = 4 \cos\left(-\frac{184\pi}{365}\right) + 12 \approx 11.9 \text{ hours}$
10. Back bearing at B is 240° , so $\angle ABC = 240^\circ - 200^\circ = 40^\circ$. $AC = \frac{\sqrt{120^2 + 80^2 - 2(120)(80) \cos 40^\circ}}{\sin 40^\circ} \approx 78.1 \text{ mi}$. $\sin A = \frac{80 \sin 40^\circ}{78.0509} \approx 0.658839$; A is opposite a side shorter than AB , so it is acute: $A \approx 41.21^\circ$. C lies clockwise from the first leg, so its bearing is $060^\circ + 41.21^\circ \approx 101.2^\circ$.
11. With O the foot of the tower, $OA = h \cot 40^\circ$, $OB = h \cot 35^\circ$ and $\angle AOB = 90^\circ$. So $h^2(\cot^2 40^\circ + \cot^2 35^\circ) = 150^2$, $h = \frac{150}{\sqrt{1.4202766 + 2.0396067}} \approx 80.6 \text{ m}$
12. $4 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12 > 15$ gives $\cos(\cdot) > 0.75$, so $|t - 172| < \frac{365}{2\pi} \cos^{-1}(0.75) \approx 41.985$. Thus $130.0 < t < 214.0$: days 131 through 213, which is 83 days
13. Let x be the distance from the second point to the foot of the tower. The tower is vertical and the ground is level, so both observation triangles are right triangles. From the second point, $\tan \beta = \frac{h}{x}$, so $x = \frac{h}{\tan \beta}$. From the first point, $\tan \alpha = \frac{h}{x + d}$, so $x + d = \frac{h}{\tan \alpha}$. Subtracting, $d = \frac{h}{\tan \alpha} - \frac{h}{\tan \beta} = h \cdot \frac{\tan \beta - \tan \alpha}{\tan \alpha \tan \beta}$.

Since $\beta > \alpha$ and both are acute, $\tan \beta - \tan \alpha > 0$, so we may divide: $h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$.

8.1 ANGLES OF ELEVATION AND DEPRESSION

OBJECTIVES

- Draw an accurate diagram for a height-and-distance problem and identify the right triangle it contains.
- Use angles of elevation and depression with the tangent, sine and cosine ratios to find heights and distances.
- Solve two-observation problems, where one angle is measured from two different points, by writing two equations in two unknowns.

WATCH

Finding an Angle of Elevation

Eddie Woo · 5:27

<https://www.youtube.com/watch?v=0hv6Kz8sXSw>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Neither video works a two-observation problem (Example 2).

BACKUP VIDEO

Angles for Position (Elevation/Depression & Bearings)

Eddie Woo · 11:43

<https://www.youtube.com/watch?v=83t9aU7HFDo>

Before the practice set: elevation, depression and bearings together.

KEY TERMS

line of sight

The straight line from an observer's eye to the object being observed.

angle of elevation

The angle between the horizontal and the line of sight when the observer looks **up** at an object.

angle of depression

The angle between the horizontal and the line of sight when the observer looks **down** at an object. It is always measured from the horizontal, never from the vertical.

clinometer

A simple instrument for measuring angles of elevation and depression, often a protractor with a weighted string or a level and sighting tube.

CORE IDEAS

Most height-and-distance problems hide a right triangle. The horizontal ground (or a horizontal line through the observer's eye) and a vertical object such as a tower, silo or cliff meet at a right angle. The line of sight is the hypotenuse. Once you draw this triangle and mark the known angle, the problem becomes a Unit 1 problem: choose the ratio that links the side you know to the side you want.

An **angle of elevation** is measured *up* from the horizontal; an **angle of depression** is measured *down* from the horizontal. If an observer at A looks down at B with angle of depression θ , then an observer at B looks up at A with angle of elevation θ . The reason is that the horizontal line through A is parallel to the horizontal line through B , and the line of sight is a transversal, so the two angles are alternate angles and are equal. This lets you move the angle into the right triangle you are using.

When the observer's eye is above the ground, add the eye height at the end. If Samuel's eye is 5.5 ft above the ground, the right triangle gives only the height of the object *above his eye*.

Two observations. Often you cannot reach the foot of a tower, so you cannot measure the horizontal distance to it. Instead, measure the angle of elevation α from one point, walk a known distance d straight toward the tower, and measure the larger angle β . Let h be the height and x the unknown distance from the nearer point. Then $h = x \tan \beta$ and $h = (x + d) \tan \alpha$. Setting these equal gives $x \tan \beta = x \tan \alpha + d \tan \alpha$, so $x = \frac{d \tan \alpha}{\tan \beta - \tan \alpha}$ and $h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$. You do not need to memorize this; set up the two equations each time.

Good habits: draw a diagram, label every known and unknown, keep full calculator precision until the last step, and ask whether the answer is reasonable. A larger angle of elevation from the same height must mean you are closer.

KEY FORMULAS

$$\text{height} = (\text{horizontal distance}) \times \tan(\text{angle of elevation})$$

$$\text{horizontal distance} = \frac{\text{height}}{\tan(\text{angle of depression})}$$

$$h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha} \quad (\text{angles } \alpha < \beta \text{ measured } d \text{ apart, in line with the object})$$

From history. For centuries, sailors found their latitude by measuring an angle of elevation: the altitude of the Pole Star or of the noon sun above the horizon. They used instruments such as the mariner's astrolabe and the quadrant, and later the sextant. Surveyors used similar sighting instruments on land to find the heights of hills and buildings they could not climb.

Euclid connection. The equality of the angle of depression from A and the angle of elevation from B is Euclid I.29: a straight line falling on two parallel straight lines makes the alternate angles equal.

WORKED EXAMPLES

Example 1. Standing on level ground 120 ft from the base of a grain silo, Mr. Miller measures the angle of elevation to the top of the silo as 32° . Ignore his height. How tall is the silo, to the nearest tenth of a foot?

Show solution

1. Draw the right triangle: the ground from Mr. Miller to the silo is the adjacent side (120 ft), the silo is the opposite side h , and the right angle is at the base of the silo.
2. The ratio linking opposite and adjacent is tangent: $\tan 32^\circ = \frac{h}{120}$.
3. Solve: $h = 120 \tan 32^\circ \approx 120 \times 0.6248694 \approx 74.984$ ft.
4. Round: the silo is about 75.0 ft tall.

Answer: $h = 120 \tan 32^\circ \approx 75.0$ ft

Example 2. Ruth wants the height of a water tower that stands inside a fenced lot, so she cannot reach its base. From a point on level ground the angle of elevation to the top is 25° . She walks 50 m straight toward the tower and measures the angle of elevation again: 40° . Ignoring her height, find the height of the tower to the nearest tenth of a meter.

Show solution

1. Let h be the height and x the distance from the second point to the base of the tower. The first point is $x + 50$ m from the base.
2. From the second point: $h = x \tan 40^\circ$. From the first point: $h = (x + 50) \tan 25^\circ$.
3. Set them equal: $x \tan 40^\circ = (x + 50) \tan 25^\circ$, so $x = \frac{50 \tan 25^\circ}{\tan 40^\circ - \tan 25^\circ}$.
4. Using full calculator values: $\tan 25^\circ \approx 0.4663077$ and $\tan 40^\circ \approx 0.8390996$, so $x \approx \frac{23.31538}{0.3727920} \approx 62.543$ m.
5. Then $h = x \tan 40^\circ \approx 62.543 \times 0.8390996 \approx 52.479$ m.
6. Check with the first point: $(62.543 + 50) \tan 25^\circ \approx 112.543 \times 0.4663077 \approx 52.479$ m. The two agree.
7. The tower is about 52.5 m tall.

Answer: $h = \frac{50 \tan 25^\circ \tan 40^\circ}{\tan 40^\circ - \tan 25^\circ} \approx 52.5$ m

PRACTICE

1. **Easy** When the sun's angle of elevation is 58° , a flagpole casts a shadow 15 ft long on level ground. How tall is the flagpole, to the nearest tenth of a foot?

Show answer

$$h = 15 \tan 58^\circ \approx 24.0 \text{ ft}$$

2. **Easy** From the top of a cliff 80 m above the sea, the angle of depression to a fishing boat is 18° . How far is the boat from the base of the cliff, to the nearest tenth of a meter?

Show answer

$$d = \frac{80}{\tan 18^\circ} \approx 246.2 \text{ m}$$

3. **Easy** A kite string 150 ft long is pulled straight and makes an angle of elevation of 50° with the ground. Ignoring the height of the person holding it, how high is the kite, to the nearest tenth of a foot?

Show answer

$$h = 150 \sin 50^\circ \approx 114.9 \text{ ft}$$

4. **Medium** A fence post 6 ft tall casts a shadow 10 ft long on level ground. Find the angle of elevation of the sun, to the nearest tenth of a degree.

Show answer

$$\theta = \tan^{-1} \left(\frac{6}{10} \right) \approx 31.0^\circ$$

5. **Medium** A lighthouse keeper whose eye is 45 m above the sea sees two ships directly in line on the same side of the lighthouse. The angles of depression are 20° to the farther ship and 32° to the nearer ship. How far apart are the ships, to the nearest tenth of a meter?

Show answer

$$\frac{45}{\tan 20^\circ} - \frac{45}{\tan 32^\circ} \approx 123.64 - 72.02 \approx 51.6 \text{ m}$$

6. **Medium** Samuel stands 60 ft from the base of a silo. His eye is 5.5 ft above the level ground, and the angle of elevation from his eye to the top of the silo is 38° . How tall is the silo, to the nearest tenth of a foot?

Show answer

$$5.5 + 60 \tan 38^\circ \approx 5.5 + 46.88 \approx 52.4 \text{ ft}$$

7. **Medium** An airplane flying at an altitude of 9000 ft begins its approach when the angle of depression from the pilot to the end of the runway is 7° . To the nearest hundred feet, find (a) the horizontal ground distance to the runway and (b) the straight-line distance from the plane to the runway.

Show answer

$$(a) \frac{9000}{\tan 7^\circ} \approx 73,300 \text{ ft}; (b) \frac{9000}{\sin 7^\circ} \approx 73,800 \text{ ft}$$

8. **Hard** From a window 30 ft above the street, the angle of elevation to the top of the building across the street is 40° and the angle of depression to its base is 25° . How tall is the building across the street, to the nearest tenth of a foot?

Show answer

$$\text{Width of street} = \frac{30}{\tan 25^\circ} \approx 64.335 \text{ ft; height} = 30 + 64.335 \tan 40^\circ \approx 30 + 53.98 \approx 84.0 \text{ ft}$$

9. **Hard** From a point on level ground, the angle of elevation to the top of a church steeple is 35° . From a point 40 ft closer, in line with the first point and the steeple, the angle of elevation is 52° . Find the height of the steeple, to the nearest tenth of a foot.

Show answer

$$h = \frac{40 \tan 35^\circ \tan 52^\circ}{\tan 52^\circ - \tan 35^\circ} \approx 61.8 \text{ ft}$$

10. **Challenge** Explain why the angle of depression from the top of a cliff to a boat equals the angle of elevation from the boat to the top of the cliff.

Show answer

Both angles are measured from a horizontal line: one through the top of the cliff and one through the boat. Horizontal lines are parallel. The line of sight crosses both of them, so the angle of depression and the angle of elevation are alternate angles formed by a transversal of two parallel lines, and alternate angles are equal (Euclid I.29).

11. **Challenge** A 20 ft flagpole stands on the flat roof of a building. From a point on level ground, the angle of elevation to the bottom of the flagpole is 40° and the angle of elevation to its top is 45° . How tall is the building, to the nearest tenth of a foot?

Show answer

Let x be the distance to the building and h its height. Then $h = x \tan 40^\circ$ and $h + 20 = x \tan 45^\circ = x$. So $x - x \tan 40^\circ = 20$, $x = \frac{20}{1 - \tan 40^\circ} \approx 124.30 \text{ ft}$, and $h = x - 20 \approx 104.3 \text{ ft}$.

COMMON MISTAKES

- Measuring the angle of depression from the vertical. It is always measured down from the **horizontal**; mark the horizontal line at the observer first, then the angle.

- Putting the angle of depression at the wrong corner of the triangle. Use the alternate-angle fact to move it to the angle of elevation at the object, which sits inside the right triangle.
- Forgetting the observer's eye height. If the angle is measured from the eye, the triangle gives the height above the eye; add the eye height at the end.
- Rounding x in a two-observation problem and then using the rounded value. Keep full calculator precision until the final answer.

Make it a game • Diagram Dictation. Parent reads a word problem: a ship sights a cliff top at 28 degrees, or a lighthouse keeper looks down at a boat. In 30 seconds, sketch the diagram and put the angle in the correct place, measured from the horizontal. Name the right triangle and ratio. Then explain why elevation equals depression.

8.2 BEARINGS AND NAVIGATION

OBJECTIVES

- Read and write directions as compass bearings (such as $N40^{\circ}E$) and as true three-figure bearings (such as 040°), and find back bearings.
- Find the interior angle of a navigation triangle from the bearings of its legs.
- Solve two-leg navigation problems with right-triangle trigonometry and the Laws of Sines and Cosines.

WATCH

Bearings (2 of 3: True bearings)

Eddie Woo · 7:14

https://www.youtube.com/watch?v=-Luy_PRwdpU

WHEN TO WATCH

After reading the Core ideas on true bearings, before Example 1.

BACKUP VIDEO

Bearings (1 of 3: Compass bearings)

Eddie Woo · 6:16

<https://www.youtube.com/watch?v=vukYndoYtLM>

Before Example 1 if compass bearings ($N 40^{\circ} E$ style) are new to you.

KEY TERMS

compass bearing

A direction given by starting at north or south and turning a stated angle toward east or west, such as $N40^{\circ}E$ or $S25^{\circ}W$. The angle is between 0° and 90° .

back bearing

The bearing of the starting point as seen from the destination. It differs from the original bearing by 180° .

true bearing

A direction given as an angle measured **clockwise from north**, from 000° up to (but not including) 360° , written with three figures: east is 090° , south is 180° , west is 270° . Also called a three-figure bearing.

course

The direction in which a ship or aircraft travels, usually given as a true bearing.

nautical mile

A unit of distance used at sea and in the air; one knot is a speed of one nautical mile per hour.

CORE IDEAS

Navigators give directions in two ways. A **compass bearing** starts from north or south and turns toward east or west: $N40^\circ E$ means face north, then turn 40° toward the east. A **true bearing** is a single angle measured clockwise from north and written with three figures: $N40^\circ E$ is 040° , $S25^\circ W$ is $180^\circ + 25^\circ = 205^\circ$, and $N70^\circ W$ is $360^\circ - 70^\circ = 290^\circ$. In this lesson all bearings are measured from true north. Notice that a bearing is measured clockwise from north, while an angle in standard position is measured counterclockwise from the positive x -axis.

Back bearings. If the bearing of B from A is θ , the bearing of A from B is $\theta + 180^\circ$ (subtract 360° if the result passes 360°). The north lines at A and at B are parallel, and the segment AB crosses both, so the angles it makes with them are related by Euclid I.29.

Components. A trip of length d on bearing θ moves $d \cos \theta$ north and $d \sin \theta$ east (a negative value means south or west). The roles of sine and cosine are the reverse of standard position because bearings are measured from north, not from east.

The angle at the turn. A two-leg trip $A \rightarrow B \rightarrow C$ makes a triangle. To find the interior angle at B , draw a north line at B and mark two directions: the back bearing toward A and the bearing of the second leg toward C . The angle between them is $\angle ABC$. Equivalently, if the course changes by t degrees (with $0^\circ < t < 180^\circ$), then $\angle ABC = 180^\circ - t$.

Once you know two sides and the included angle, the Law of Cosines gives the direct distance AC . Then the Law of Sines (or the Law of Cosines again) gives $\angle BAC$, and you add it to or subtract it from the first bearing, depending on which side of the first leg C lies. A sketch settles which. If the turn is exactly 90° , the triangle is right-angled and the Pythagorean theorem and tangent are enough.

KEY FORMULAS

$$\text{back bearing} = \theta \pm 180^\circ$$

$$\text{north component} = d \cos \theta, \quad \text{east component} = d \sin \theta \quad (\theta \text{ a true bearing})$$

$$\angle ABC = 180^\circ - (\text{change of course at } B)$$

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC) \cos B$$

From history. The magnetic compass was in use for navigation in both China and Europe by about the twelfth century. Before satellite navigation, a ship's officer found the ship's position by dead reckoning: from a known starting point, they combined each course steered with the distance run on it, which is exactly the two-leg triangle of this lesson. A compass needle points to magnetic north, which differs from true north, so navigators corrected their compass readings before plotting.

Euclid connection. Back bearings and the angle at a turn rest on Euclid I.29: the north lines through two points are parallel, so a straight course between them makes alternate angles that are equal and interior angles on the same side that sum to 180° .

WORKED EXAMPLES

Example 1. A fishing boat sails 12 miles on a bearing of 070° , then turns and sails 9 miles on a bearing of 160° . How far is the boat from its starting point, and what is its bearing from the start? Give the bearing to the nearest tenth of a degree.

Show solution

1. The course changes from 070° to 160° , a turn of 90° to the right. So the interior angle at the turn is $180^\circ - 90^\circ = 90^\circ$: the triangle is right-angled.
2. Distance: $\sqrt{12^2 + 9^2} = \sqrt{225} = 15$ miles.
3. At the start, the angle between the first leg and the line to the boat satisfies $\tan A = \frac{9}{12}$, so $A = \tan^{-1}(0.75) \approx 36.87^\circ$.
4. The boat turned right (clockwise), so its final position is clockwise from the first leg. Bearing = $070^\circ + 36.87^\circ \approx 106.9^\circ$.

Answer: 15 miles, on a bearing of about 106.9°

Example 2. A small plane flies 200 km from airfield A on a bearing of 040° to town B , then 150 km on a bearing of 115° to town C . Find the distance from A to C to the nearest tenth of a kilometer and the bearing of C from A to the nearest tenth of a degree.

Show solution

1. Angle at B : the back bearing from B to A is $040^\circ + 180^\circ = 220^\circ$, and the bearing from B to C is 115° . So $\angle ABC = 220^\circ - 115^\circ = 105^\circ$. (Check: the course changed by 75° , and $180^\circ - 75^\circ = 105^\circ$.)
2. Law of Cosines: $AC^2 = 200^2 + 150^2 - 2(200)(150) \cos 105^\circ = 62,500 + 60,000(0.2588190) \approx 78,029.14$.
3. So $AC \approx \sqrt{78,029.14} \approx 279.337$ km.
4. Law of Sines for the angle at A : $\sin A = \frac{150 \sin 105^\circ}{279.337} \approx \frac{144.889}{279.337} \approx 0.518689$.
Angle B is obtuse, so angle A must be acute: $A \approx 31.24^\circ$.
5. The plane turned right at B , so C is clockwise from the first leg. Bearing of C from $A = 040^\circ + 31.24^\circ \approx 071.2^\circ$.

Answer: $AC \approx 279.3$ km, on a bearing of about 071.2° from A

PRACTICE

1. **Easy** Write each compass bearing as a true three-figure bearing: (a) $N40^\circ E$, (b) $S25^\circ W$, (c) $N70^\circ W$.

Show answer

(a) 040° , (b) 205° , (c) 290°

2. **Easy** (a) The bearing of a barn from the farmhouse is 065° . What is the bearing of the farmhouse from the barn? (b) The bearing of a buoy from a ship is 310° . What is the bearing of the ship from the buoy?

Show answer

(a) 245° , (b) 130°

3. **Easy** A hiker walks 5 km on a bearing of 150° . How far south and how far east of the starting point is the hiker, to the nearest hundredth of a kilometer?

Show answer

South: $5 \cos 30^\circ \approx 4.33$ km; east: $5 \sin 30^\circ = 2.50$ km

4. **Medium** A sailboat travels 8 miles on a bearing of 030° , then 6 miles on a bearing of 120° . Find its distance from the start and its bearing from the start, to the nearest tenth of a degree.

Show answer

The turn is 90° , so the distance is $\sqrt{8^2 + 6^2} = 10$ miles; bearing = $030^\circ + \tan^{-1}\left(\frac{6}{8}\right) \approx 066.9^\circ$

5. **Medium** Two ships leave the same harbor at the same time. One sails at 15 knots on a bearing of 050° ; the other sails at 20 knots on a bearing of 170° . How far apart are they after 2 hours, to the nearest tenth of a nautical mile?

Show answer

The sides are 30 and 40 nautical miles with included angle 120° : $d = \sqrt{30^2 + 40^2 - 2(30)(40)\cos 120^\circ} = \sqrt{3700} \approx 60.8$ nautical miles

6. **Medium** Fire lookout tower B is 10 miles due east of tower A . A ranger at A sees smoke on a bearing of 052° ; a ranger at B sees the same smoke on a bearing of 310° . How far is the fire from each tower, to the nearest hundredth of a mile?

Show answer

$\angle A = 90^\circ - 52^\circ = 38^\circ$, $\angle B = 310^\circ - 270^\circ = 40^\circ$, angle at the fire = 102° . From A : $\frac{10 \sin 40^\circ}{\sin 102^\circ} \approx 6.57$ mi; from B : $\frac{10 \sin 38^\circ}{\sin 102^\circ} \approx 6.29$ mi

7. **Hard** A ship leaves port and sails 25 miles on a bearing of 300° , then 18 miles on a bearing of 020° . How far is it from port, to the nearest tenth of a mile, and on what bearing must it sail to return directly to port, to the nearest tenth of a degree?

Show answer

Angle at the turn = $120^\circ - 20^\circ = 100^\circ$; distance = $\sqrt{25^2 + 18^2 - 2(25)(18)\cos 100^\circ} \approx 33.2$ mi. The angle at port is $\sin^{-1}\left(\frac{18 \sin 100^\circ}{33.2458}\right) \approx 32.22^\circ$, so the ship's bearing from port is $300^\circ + 32.22^\circ = 332.22^\circ$, and the return bearing is $332.22^\circ - 180^\circ \approx 152.2^\circ$.

8. **Hard** A pilot flies 300 miles from A to B on a bearing of 075° , then 220 miles from B to C on a bearing of 160° . Find the distance from C back to A , to the nearest tenth of a mile, and the bearing the pilot should fly from C to return to A , to the nearest tenth of a degree.

Show answer

$\angle ABC = 255^\circ - 160^\circ = 95^\circ$; $AC = \sqrt{300^2 + 220^2 - 2(300)(220)\cos 95^\circ} \approx 387.2$ mi. $\angle BAC \approx 34.48^\circ$, so the bearing of C from A is $\approx 109.48^\circ$ and the return bearing is $\approx 289.5^\circ$.

9. **Challenge** A ship sails from A to B on bearing α , then from B to C on bearing β , where $\alpha < \beta < \alpha + 180^\circ$ (a turn to the right of less than 180°). Explain why $\angle ABC = 180^\circ - (\beta - \alpha)$.

Show answer

At B , the back bearing toward A is $\alpha + 180^\circ$, because the north lines at A and B are parallel. The bearing toward C is β . Both are measured clockwise from the same north line at B , so the angle between the two directions is $(\alpha + 180^\circ) - \beta = 180^\circ - (\beta - \alpha)$. Since $0^\circ < \beta - \alpha < 180^\circ$, this is between 0° and 180° , so it is the interior angle of the triangle.

10. **Challenge** From point A , a ship's navigator sights a lighthouse on a bearing of 040° . The ship sails 12 miles due east to point B , where the lighthouse bears 340° . (a) How far is the ship from the lighthouse at B ? (b) How close did the ship come to the lighthouse during the run from A to B ? Give both to the nearest tenth of a mile.

Show answer

Angles: at A , $90^\circ - 40^\circ = 50^\circ$; at B , $340^\circ - 270^\circ = 70^\circ$; at the lighthouse, 60° .
 (a) $\frac{12 \sin 50^\circ}{\sin 60^\circ} \approx 10.6$ mi. (b) Both angles at A and B are acute, so the closest point lies between A and B ; the distance is $10.6146 \sin 70^\circ \approx 10.0$ mi.

COMMON MISTAKES

- Measuring a true bearing counterclockwise or from east, as in standard position. True bearings start at north and turn **clockwise**.
- Using the bearings themselves as triangle angles. The interior angle at a turn must be found from the back bearing and the new bearing, or as 180° minus the change of course.

- Forgetting to convert the angle at the start into a bearing. Add or subtract it from the first leg's bearing according to which way the ship turned, and check with a sketch.
- Giving a bearing outside the range 000° to 359° when finding a back bearing. Add or subtract 180° so the result stays in range.

Make it a game • Captain's Chart. Draw a compass rose on paper. Take turns calling legs of a voyage in compass bearings, such as N40 degrees E for 5 nautical miles. Draw each leg with a protractor, convert to three-figure bearings like 040 degrees, and find the back bearing for the whole trip. Closest return course wins.

8.3 SURVEYING AND TRIANGULATION

OBJECTIVES

- Find inaccessible distances, such as the width of a river, from one measured baseline and measured angles.
- Find the area of a triangular or four-sided field from survey measurements by dividing it into triangles.
- Explain how Eratosthenes measured the earth and how a chain of triangles carries one baseline across a whole country.

WATCH

Radial Surveys (2 of 2: Compass)

Eddie Woo · 13:02

<https://www.youtube.com/watch?v=rwFyJ3OvXo8>

WHEN TO WATCH

Before Example 2: a field split into triangles, each solved with the law of cosines.

Note: 'Radial survey' is an Australian classroom method; it uses true bearings and the 'cosine rule'.
No assigned video covers the river-width baseline or Eratosthenes.

BACKUP VIDEO

How does land surveying work?

Practical Engineering · 6:26

<https://www.youtube.com/watch?v=SPCewaAfqPA>

After the lesson: how surveyors triangulate with a theodolite and the Great Trigonometrical Survey of India (the trig is in the first 2 minutes).

KEY TERMS

baseline

A straight line on the ground whose length is measured with great care; every other distance in a survey is computed from it.

triangulation

Finding the positions of distant points by measuring the angles of triangles whose sides join them, starting from one measured baseline.

inaccessible distance

A distance that cannot be measured directly, for example across a river, a canyon or a lake.

acre

A traditional unit of land area equal to 43,560 square feet.

CORE IDEAS

Angles are easy to measure precisely with a sighting instrument; long distances over rough ground are hard to measure. A surveyor therefore measures **one** distance carefully, the baseline, and then measures only angles. If the baseline AB and the angles at A and B toward a far point C are known, the third angle is $C = 180^\circ - A - B$ (Euclid I.32), and the Law of Sines gives the other sides: $AC = \frac{AB \sin B}{\sin C}$ and $BC = \frac{AB \sin A}{\sin C}$. This is an ASA triangle, which you solved in Lesson 7.2.

Width of a river. Lay the baseline along one bank and sight a tree C on the far bank from both ends. After finding AC , the perpendicular distance from C to the bank is $AC \sin A$, because the perpendicular from C to line AB makes a right triangle with hypotenuse AC .

Triangulation. Once AC and BC are known, either one can serve as the baseline of a new triangle. Measuring only angles, a surveyor can carry the work from triangle to triangle across a whole region. Errors still creep in, so surveyors usually measure a second baseline far from the first and compare it with its computed length as a check.

Land areas. Any field with straight sides can be divided by diagonals into triangles. For each triangle use whichever area formula fits the data: $\frac{1}{2}ab \sin C$ for two sides and the included angle, Heron's formula for three sides (Lesson 7.1). For a triangle known by one side c and the two angles A and B at its ends, the Law of Sines gives $b = \frac{c \sin B}{\sin C}$, so the area is $\frac{1}{2}bc \sin A = \frac{c^2 \sin A \sin B}{2 \sin C}$.

Eratosthenes' measurement of the earth. At noon on the summer solstice, the sun was said to shine straight down a well at Syene, so it was directly overhead there. At the same moment at Alexandria, roughly north of Syene, a vertical pole cast a shadow showing that the sun's rays made an angle of about 7.2° with the vertical. The sun is so far away that its rays arrive parallel. At Syene the sun's ray runs straight down the vertical line to the earth's center. The vertical line at Alexandria, extended to the center, crosses both this ray and the parallel ray at Alexandria, so the angle at

the center equals the shadow angle (alternate angles, Euclid I.29). Thus the distance from Syene to Alexandria is $\frac{7.2}{360} = \frac{1}{50}$ of the whole circumference.

KEY FORMULAS

$$AC = \frac{AB \sin B}{\sin C}, \quad C = 180^\circ - A - B$$

$$\text{width of river} = AC \sin A$$

$$\text{Area} = \frac{c^2 \sin A \sin B}{2 \sin C} \quad (\text{one side and the two angles at its ends})$$

$$\text{circumference} = \frac{360^\circ}{\theta} \times (\text{arc distance})$$

From history. Eratosthenes of Cyrene, librarian at Alexandria in the third century BC, estimated the circumference of the earth from the angle of the noon sun at two cities and the distance between them. Around 1615 the Dutch scholar Willebrord Snellius measured a chain of triangles across the Netherlands to find the length of a degree of latitude, an early large-scale use of triangulation. In the eighteenth and nineteenth centuries, great national surveys mapped France, Britain and other lands with networks of triangles; the Great Trigonometrical Survey of India, begun in 1802, carried such a chain across the subcontinent.

Euclid connection. Each surveying triangle uses Euclid I.32, that the angles of a triangle sum to two right angles, to find its third angle. Eratosthenes' argument uses Euclid I.29: the vertical line at Alexandria crosses two parallel sun rays, so the alternate angles it makes with them are equal, and the shadow angle equals the angle at the earth's center.

WORKED EXAMPLES

Example 1. To find the width of a river, a surveyor lays out a baseline AB of 200 m along one straight bank. A tree C stands on the opposite bank. She measures $\angle CAB = 72^\circ$ and $\angle CBA = 58^\circ$. Find the distance AC and the width of the river, each to the nearest tenth of a meter.

Show solution

1. Third angle: $C = 180^\circ - 72^\circ - 58^\circ = 50^\circ$.
2. Law of Sines: $\frac{AC}{\sin 58^\circ} = \frac{200}{\sin 50^\circ}$, so $AC = \frac{200 \sin 58^\circ}{\sin 50^\circ} \approx \frac{169.6096}{0.7660444} \approx 221.410$ m.
3. Drop a perpendicular from C to the bank AB . It forms a right triangle with hypotenuse AC and angle 72° at A , so the width is $AC \sin 72^\circ \approx 221.410 \times 0.9510565 \approx 210.573$ m.
4. Check from the other end: $BC = \frac{200 \sin 72^\circ}{\sin 50^\circ} \approx 248.303$ m, and $BC \sin 58^\circ \approx 210.573$ m. The same.
5. So $AC \approx 221.4$ m and the river is about 210.6 m wide.

Answer: $AC \approx 221.4$ m; width ≈ 210.6 m

Example 2. Mr. Miller's field is a convex quadrilateral $ABCD$. A survey gives $AB = 310$ ft, $BC = 240$ ft, $\angle ABC = 105^\circ$, $CD = 280$ ft and $DA = 350$ ft. Find the area of the field to the nearest square foot and in acres to the nearest hundredth (1 acre = 43,560 sq ft).

Show solution

1. Divide the field along the diagonal AC into triangles ABC and ACD .
2. Triangle ABC (two sides and the included angle): Area =
 $\frac{1}{2}(310)(240) \sin 105^\circ = 37,200(0.9659258) \approx 35,932.44$ sq ft.
3. Diagonal by the Law of Cosines: $AC^2 = 310^2 + 240^2 - 2(310)(240) \cos 105^\circ = 153,700 + 148,800(0.2588190) \approx 192,212.27$, so $AC \approx 438.4202$ ft. Keep these digits.
4. Triangle ACD (three sides) by Heron's formula: $s = \frac{438.4202 + 280 + 350}{2} \approx 534.2101$, and Area = $\sqrt{s(s - 438.4202)(s - 280)(s - 350)} \approx \sqrt{534.2101 \times 95.7899 \times 254.2101 \times 184.2101} \approx 48,951.84$ sq ft.
5. Total: $35,932.44 + 48,951.84 \approx 84,884.28$ sq ft, about 84,884 sq ft.
6. In acres: $\frac{84,884.28}{43,560} \approx 1.95$ acres.

Answer: About 84,884 sq ft, or 1.95 acres

PRACTICE

1. **Easy** Suppose the noon sun's rays make an angle of 7.2° with the vertical at one city while the sun is directly overhead at a second city 800 km due south. Using Eratosthenes' method, estimate the circumference of the earth.

Show answer

$$\frac{360}{7.2} \times 800 = 50 \times 800 = 40,000 \text{ km}$$

2. **Easy** A surveyor lays out a baseline $AB = 100$ m along a riverbank and sights a stake C on the far bank so that $\angle CAB = 90^\circ$. From B , she measures $\angle CBA = 55^\circ$. Find AC , the width of the river, to the nearest tenth of a meter.

Show answer

$$AC = 100 \tan 55^\circ \approx 142.8 \text{ m}$$

3. **Easy** A triangular garden plot has two sides of 120 m and 95 m that meet at an angle of 64° . Find its area to the nearest tenth of a square meter.

Show answer

$$\frac{1}{2}(120)(95) \sin 64^\circ \approx 5123.1 \text{ m}^2$$

4. **Medium** To find the length of a pond, Samuel stands at a point P from which he can walk straight to either end. He measures $PA = 180$ ft, $PB = 145$ ft and $\angle APB = 70^\circ$. How long is the pond, AB , to the nearest tenth of a foot?

Show answer

$$AB = \sqrt{180^2 + 145^2 - 2(180)(145) \cos 70^\circ} \approx 188.6 \text{ ft}$$

5. **Medium** From the ends of a 500 m baseline AB , a surveyor sights a marker on a distant hilltop C . She measures $\angle CAB = 48^\circ$ and $\angle CBA = 65^\circ$. Find the distance AC to the nearest tenth of a meter.

Show answer

$$C = 67^\circ; AC = \frac{500 \sin 65^\circ}{\sin 67^\circ} \approx 492.3 \text{ m}$$

6. **Medium** A triangular pasture has sides 210 ft, 180 ft and 150 ft. Find its area to the nearest square foot and in acres to the nearest hundredth.

Show answer

$$s = 270; \text{Area} = \sqrt{270 \cdot 60 \cdot 90 \cdot 120} = \sqrt{174,960,000} \approx 13,227 \text{ sq ft} \approx 0.30 \text{ acre}$$

7. **Hard** The width of a river is to be found from a baseline $AB = 150$ m laid along one straight bank. A boulder C on the far bank is sighted with $\angle CAB = 40^\circ$ and $\angle CBA = 75^\circ$. Find the width of the river to the nearest tenth of a meter.

Show answer

$$C = 65^\circ; AC = \frac{150 \sin 75^\circ}{\sin 65^\circ} \approx 159.867 \text{ m; width} = AC \sin 40^\circ \approx 102.8 \text{ m}$$

8. **Hard** A four-sided wood lot $ABCD$ is surveyed by measuring the diagonal $AC = 400$ ft and four angles: $\angle BAC = 35^\circ$, $\angle BCA = 50^\circ$, $\angle CAD = 40^\circ$ and $\angle ACD = 45^\circ$, with B and D on opposite sides of AC . Find the area of the lot to the nearest square foot and in acres to the nearest hundredth.

Show answer

Triangle ABC : $B = 95^\circ$, area = $\frac{400^2 \sin 35^\circ \sin 50^\circ}{2 \sin 95^\circ} \approx 35,285.07$ sq ft. Triangle ACD : $D = 95^\circ$, area = $\frac{400^2 \sin 40^\circ \sin 45^\circ}{2 \sin 95^\circ} \approx 36,500.45$ sq ft. Total $\approx 71,786$ sq ft ≈ 1.65 acres.

9. **Challenge** Across a river from a 200 m baseline AB stand a barn C and a windmill D . At A a surveyor measures $\angle BAC = 85^\circ$ and $\angle BAD = 40^\circ$. At B she measures $\angle ABC = 35^\circ$ and $\angle ABD = 90^\circ$. Both C and D are on the same side of AB . Find the distance between the barn and the windmill, to the nearest tenth of a meter.

Show answer

In $\triangle ABC$: $C = 60^\circ$, $AC = \frac{200 \sin 35^\circ}{\sin 60^\circ} \approx 132.4618$ m. In $\triangle ABD$: $D = 50^\circ$, $AD = \frac{200}{\cos 40^\circ} \approx 261.0815$ m. $\angle CAD = 85^\circ - 40^\circ = 45^\circ$, so $CD = \sqrt{AC^2 + AD^2 - 2(AC)(AD) \cos 45^\circ} \approx 191.8$ m.

10. **Challenge** Explain why, in Eratosthenes' method, the angle between the sun's rays and the vertical pole at Alexandria equals the angle at the center of the earth between the lines to Alexandria and Syene.

Show answer

The sun is so far away that its rays reaching Syene and Alexandria are parallel. At Syene the sun is overhead, so the ray there points straight along the vertical line through Syene toward the earth's center. The vertical pole at Alexandria also points toward the center, so the line of the pole extended crosses both parallel rays. The shadow angle at Alexandria and the angle at the center are alternate angles made by this line with the two parallel rays, so they are equal (Euclid I.29). The arc between the cities is therefore the same fraction of the full circle as this angle is of 360° .

COMMON MISTAKES

- Using the given angles at A and B directly in the Law of Sines with the wrong opposite sides. Find the third angle first, then pair each side with the angle opposite it: the baseline is opposite the angle at the far point.
- Reporting AC as the width of the river. AC is a slanted line; the width is the perpendicular distance $AC \sin A$.
- Rounding the diagonal of a field before using it in Heron's formula. Heron's formula subtracts nearly equal numbers, so keep at least four decimal places in the diagonal.
- Dividing by 43,560 when the area is in square meters or square yards. The acre conversion applies only to square feet.

Make it a game • Triangulate the Gap. Pick an object across the room as your inaccessible point. Two players stand at the ends of a measured baseline and sight the object using a paper protractor. Record the two angles and compute the distance with the Law of Sines. Then pace it off to check. Closest estimate wins.

8.4 PROBLEMS IN THREE DIMENSIONS

OBJECTIVES

- Find lengths and angles in boxes, rooms, pyramids and roofs by isolating right triangles that lie in vertical or horizontal planes.
- Find the angle between a line and a horizontal plane, and the angle between a sloping face and its base.
- Solve problems where an object is observed from two points that are not in line with it, such as a flagpole seen from the south and from the east.

WATCH

3-Dimensional Trigonometry: Introductory Example (1 of 2)

Eddie Woo · 13:09

https://www.youtube.com/watch?v=gEcs8YrJ_kg

WHEN TO WATCH

After reading the Core ideas, before Example 1.

BACKUP VIDEO

The Lighthouse (Challenging 3D Trig Question)

Eddie Woo · 11:08

https://www.youtube.com/watch?v=8Ltzu_aFVng

Before Example 2: a harder observer-and-tower problem in 3D.

KEY TERMS

space diagonal

A segment joining two vertices of a box that do not lie on the same face, such as a bottom corner and the opposite top corner.

projection

The "shadow" of a segment on a plane, found by dropping perpendiculars from its endpoints to the plane.

angle between a line and a plane

The angle between the line and its projection on the plane.

angle between two planes

For a sloping face meeting a horizontal base along an edge, the angle between two lines, one in each plane, both perpendicular to that edge at the same point.

line of greatest slope

On a sloping plane, the steepest line, perpendicular to the horizontal lines in that plane.

CORE IDEAS

The secret of three-dimensional problems is to find right triangles and draw each one flat. A vertical line, like a corner post, a flagpole or the height of a pyramid, is perpendicular to every horizontal line through its foot. So any triangle formed by a vertical segment and a horizontal segment from its foot has a right angle there, even when the drawing on paper does not look like a right angle.

The box diagonal. For a box with length l , width w and height h , first find the diagonal of the base: $d = \sqrt{l^2 + w^2}$. The vertical edge of height h stands at one end of this base diagonal, perpendicular to it, so the space diagonal is $D = \sqrt{d^2 + h^2} = \sqrt{l^2 + w^2 + h^2}$. The angle θ between the space diagonal and the base satisfies $\tan \theta = \frac{h}{d}$, because the base diagonal is the projection of the space diagonal on the base.

Sloping faces. To find the angle between a sloping face (a roof or a pyramid's side) and the horizontal, use a line in the face that is **perpendicular to the bottom edge**, such as the slant height of a pyramid or a common rafter of a roof. A line that runs slantwise across the face, such as a pyramid's lateral edge or a hip rafter, rises less steeply, so it makes a smaller angle with the horizontal.

Observations from two directions. When a flagpole of height h is seen from a point A due south of it and from a point B due east of it, the horizontal distances are $h \cot \alpha$ and $h \cot \beta$. These two distances are the legs of a right triangle on the ground with hypotenuse AB , so $(h \cot \alpha)^2 + (h \cot \beta)^2 = AB^2$. One unknown, one equation.

KEY FORMULAS

$$D = \sqrt{l^2 + w^2 + h^2}$$

$$\tan \theta = \frac{h}{\sqrt{l^2 + w^2}} \quad (\text{angle between a box's space diagonal and its base})$$

$$h^2 (\cot^2 \alpha + \cot^2 \beta) = AB^2 \quad (\text{object seen from two points at right angles from its foot})$$

Euclid connection. Euclid treats solid geometry in Book XI. Its third definition says a straight line is at right angles to a plane when it makes right angles with all the straight lines that meet it in that plane, which is exactly why every vertical edge makes a right triangle with any horizontal line through its foot.

WORKED EXAMPLES

Example 1. A rectangular grain bin is 4 m long, 3 m wide and 12 m tall. Find the length of the longest straight rod that fits inside it (from a bottom corner to the opposite top corner) and the angle the rod makes with the floor, to the nearest tenth of a degree.

Show solution

1. Diagonal of the floor: $d = \sqrt{4^2 + 3^2} = 5$ m.
2. The vertical edge at the far corner is perpendicular to the floor diagonal, so the rod is the hypotenuse of a right triangle with legs 5 and 12: $D = \sqrt{5^2 + 12^2} = 13$ m.
3. The floor diagonal is the projection of the rod on the floor, so the angle with the floor satisfies $\tan \theta = \frac{12}{5}$.
4. $\theta = \tan^{-1}(2.4) \approx 67.4^\circ$.

Answer: 13 m, at about 67.4° to the floor

Example 2. From a point A due south of a flagpole, the angle of elevation of its top is 30° . From a point B due east of the flagpole, the angle of elevation is 40° . Points A and B are 100 m apart on level ground. Find the height of the flagpole to the nearest tenth of a meter.

Show solution

1. Let O be the foot of the pole and h its height. In the vertical triangles: $OA = h \cot 30^\circ = h\sqrt{3}$ and $OB = h \cot 40^\circ$.
2. On the ground, OA points south and OB points east, so $\angle AOB = 90^\circ$ and $OA^2 + OB^2 = AB^2$.
3. So $h^2 (3 + \cot^2 40^\circ) = 100^2$. With $\cot 40^\circ \approx 1.1917536$, $\cot^2 40^\circ \approx 1.4202766$.
4. $h = \frac{100}{\sqrt{4.4202766}} \approx \frac{100}{2.1024454} \approx 47.564$ m.
5. Check: $OA \approx 82.383$ m and $OB \approx 56.684$ m, and $82.383^2 + 56.684^2 \approx 10,000$. Correct.
6. The flagpole is about 47.6 m tall.

Answer: $h = \frac{100}{\sqrt{3 + \cot^2 40^\circ}} \approx 47.6$ m

PRACTICE

1. **Easy** A wooden crate is 6 in. by 8 in. by 10 in. Find the length of its space diagonal, exactly and to the nearest tenth of an inch.

Show answer

$$\sqrt{36 + 64 + 100} = \sqrt{200} = 10\sqrt{2} \approx 14.1 \text{ in.}$$

2. **Easy** A cube has edges of length 5. Find the angle between a space diagonal and the base, to the nearest tenth of a degree.

Show answer

$$\tan \theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ so } \theta \approx 35.3^\circ$$

3. **Easy** A common rafter of a barn roof rises 8 ft over a horizontal run of 12 ft. Find the length of the rafter (ignoring any overhang) to the nearest tenth of a foot and the roof's angle with the horizontal to the nearest tenth of a degree.

Show answer

$$\text{Length} = \sqrt{8^2 + 12^2} = \sqrt{208} \approx 14.4 \text{ ft; angle} = \tan^{-1} \left(\frac{8}{12} \right) \approx 33.7^\circ$$

4. **Medium** A pyramid has a square base 16 m on a side and a height of 15 m, with its apex directly above the center of the base. Find (a) the slant height of a face, (b) the angle between a face and the base, and (c) the angle between a lateral edge and the base. Give angles to the nearest tenth of a degree.

Show answer

$$\text{(a) } \sqrt{15^2 + 8^2} = 17 \text{ m; (b) } \tan^{-1} \left(\frac{15}{8} \right) \approx 61.9^\circ; \text{ (c) the half-diagonal of the base is } 8\sqrt{2}, \text{ so } \tan^{-1} \left(\frac{15}{8\sqrt{2}} \right) \approx 53.0^\circ$$

5. **Medium** Point A is 40 m due north of a flagpole, and the angle of elevation of the top from A is 35° . Point B is 60 m due west of the flagpole. Find the height of the flagpole to the nearest tenth of a meter and the angle of elevation of its top from B to the nearest tenth of a degree.

Show answer

$$h = 40 \tan 35^\circ \approx 28.0 \text{ m; from } B: \tan^{-1} \left(\frac{28.0083}{60} \right) \approx 25.0^\circ$$

6. **Medium** Ruth stretches a string in a room 12 ft long, 9 ft wide and 8 ft high, from one corner of the floor to the opposite corner of the ceiling. How long is the string, and what angle does it make with the floor, to the nearest tenth of a degree?

Show answer

$$\text{Floor diagonal} = 15 \text{ ft; string} = \sqrt{15^2 + 8^2} = 17 \text{ ft; angle} = \sin^{-1} \left(\frac{8}{17} \right) \approx 28.1^\circ$$

7. **Hard** A square shed 24 ft on a side has a hip roof shaped like a pyramid, with each of the four roof faces pitched at 30° to the horizontal. Find (a) the height of the peak above the top of the walls, (b) the length of a hip rafter running from a corner of the walls to the peak, both to the nearest tenth of a foot, and (c) the angle the hip rafter makes with the horizontal, to the nearest tenth of a degree.

Show answer

(a) $12 \tan 30^\circ = 4\sqrt{3} \approx 6.9$ ft; (b) the hip rafter runs $12\sqrt{2}$ ft horizontally, so its length is $\sqrt{288 + 48} = \sqrt{336} \approx 18.3$ ft; (c) $\tan^{-1} \left(\frac{4\sqrt{3}}{12\sqrt{2}} \right) = \tan^{-1} \left(\frac{1}{\sqrt{6}} \right) \approx 22.2^\circ$

8. **Hard** From a point A due south of a radio tower, the angle of elevation of the top is 45° . Samuel walks 100 m due east to point B , where the angle of elevation is 30° . Find the height of the tower, exactly and to the nearest tenth of a meter.

Show answer

Let O be the foot and h the height. $OA = h$, $OB = h\sqrt{3}$, and $\angle OAB = 90^\circ$, so $3h^2 = h^2 + 100^2$, $h^2 = 5000$, $h = 50\sqrt{2} \approx 70.7$ m

9. **Challenge** In a box, explain why the triangle formed by a space diagonal, the base diagonal and a vertical edge has its right angle where the vertical edge meets the base, even though it may not look like a right angle in a drawing.

Show answer

The vertical edge is perpendicular to the base plane. A line perpendicular to a plane is perpendicular to every line in that plane passing through its foot (Euclid XI, Definition 3). The base diagonal is such a line, so the angle between the vertical edge and the base diagonal is 90° . A flat drawing distorts angles in depth, so the right angle does not look square on paper.

10. **Challenge** A hillside is a plane sloping at 20° to the horizontal. A farm track runs straight up the hillside, making an angle of 50° (measured on the hillside) with the line of greatest slope. Find the angle the track makes with the horizontal, to the nearest tenth of a degree.

Show answer

Along a length L of track, the distance moved along the line of greatest slope is $L \cos 50^\circ$, and that line rises at 20° , so the rise is $L \cos 50^\circ \sin 20^\circ$. The track's angle ϕ satisfies $\sin \phi = \sin 20^\circ \cos 50^\circ \approx 0.219846$, so $\phi \approx 12.7^\circ$.

COMMON MISTAKES

- Treating a slanted line as if it were the height. Identify which segment is truly vertical and which lies in the horizontal base before choosing a ratio.
- Measuring the angle of a roof or pyramid face along a lateral edge or hip rafter. The face angle uses a line perpendicular to the bottom edge, such as the slant height or a common rafter.
- Using a base edge instead of the base diagonal when finding the angle between a space diagonal and the base. The projection of the space diagonal on the base is the base diagonal.
- Assuming two observation points and the foot of a pole are in a straight line. Read the directions carefully; south and east give a right angle at the foot, not a straight line.

Make it a game • Space Diagonal Guess. Pick a cereal box or a corner of a room. Guess the space diagonal before measuring length, width and height. Then compute the diagonal with right triangles in vertical and horizontal planes, and the angle it makes with the floor. Check with a string from corner to opposite corner. Closest guess scores.

8.5 MODELING PERIODIC MOTION

OBJECTIVES

- Build a sinusoidal model $y = a \cos(b(t - c)) + d$ from a physical description: a turning wheel, tide times and heights, or the longest and shortest days of the year.
- Use a model to predict a value at a given time and to find the times when a quantity reaches a given level.
- Explain why uniform motion around a circle produces simple harmonic motion when viewed from the side.

WATCH

The Ferris Wheel - Trigonometric Function Model (1 of 3: Setting up the equation).

Eddie Woo · 9:59

<https://www.youtube.com/watch?v=Cu1jg7wLOKk>

WHEN TO WATCH

Before Example 1 (the water wheel works the same way).

Note: Woo builds the model in Desmos and starts from $\cos t$. The same video is the backup for Lesson 4.5.

BACKUP VIDEO

The Ferris Wheel - Trigonometric Function Model (2 of 3: Adjusting the period).

Eddie Woo · 9:13

https://www.youtube.com/watch?v=4_viqcSyc5c

Right after the primary: setting the period inside the cosine.

KEY TERMS

periodic motion

Motion that repeats itself exactly after a fixed interval of time, the period.

frequency

The number of complete cycles per unit of time; it is the reciprocal of the period, $f = \frac{1}{T}$.

midline

The horizontal line $y = d$ halfway between the maximum and minimum values of a sinusoidal model.

simple harmonic motion

Back-and-forth motion whose displacement from a rest position is a sine or cosine function of time, such as a weight bobbing on a spring.

CORE IDEAS

In Lesson 4.5 you wrote sinusoidal equations from graphs. Here the data come from the world. Four numbers fix the model $y = a \cos(b(t - c)) + d$. The **midline** is the average of the maximum and minimum: $d = \frac{\max + \min}{2}$. The **amplitude** is half the distance between them: $|a| = \frac{\max - \min}{2}$. The **period** T is the time for one full cycle, so $b = \frac{2\pi}{T}$. The **shift** c is the time of a maximum when $a > 0$.

Choose the form that makes the shift simplest. If the motion starts at a maximum, use $y = a \cos(bt) + d$. If it starts at a minimum, use $y = -a \cos(bt) + d$ with $a > 0$. If it starts at the midline going up, use $y = a \sin(bt) + d$. All three are the same family of curves; only the starting point differs.

A turning wheel. A point on a wheel of radius r whose center is k units above the ground, turning once every T seconds, has height $h(t) = k - r \cos\left(\frac{2\pi}{T}t\right)$ if it starts at the bottom. This comes straight from the unit circle: after turning through angle θ from the bottom, the point is $r \cos \theta$ below the center.

Simple harmonic motion. Let a point travel around a circle of radius a at a steady angular speed ω , starting on the positive x -axis. At time t it is at $(a \cos \omega t, a \sin \omega t)$. Its shadow on the y -axis therefore moves by $y = a \sin \omega t$: back and forth, fastest at the middle and momentarily still at the ends. A weight on a spring, a swinging pendulum (for small swings) and a vibrating string all move this way, with period $T = \frac{2\pi}{\omega}$.

To find **when** a model reaches a level, set the model equal to that level and solve the trigonometric equation as in Lessons 6.4 and 6.5. A sinusoid crosses a level between its maximum and minimum twice in each period, so look for every solution in the time interval the problem asks about, and check each against the story.

KEY FORMULAS

$$y = a \cos(b(t - c)) + d$$

$$d = \frac{\max + \min}{2}, \quad |a| = \frac{\max - \min}{2}$$

$$b = \frac{2\pi}{T}, \quad f = \frac{1}{T}$$

$$h(t) = k - r \cos\left(\frac{2\pi t}{T}\right) \quad (\text{point on a wheel, starting at the bottom})$$

From history. Water wheels turned mill machinery for many centuries, and a bucket on the rim of such a wheel is a natural example of periodic motion. In 1656 the Dutch scientist Christiaan Huygens built the first pendulum clock, using the steady period of a swinging pendulum to keep far better time than earlier clocks. Harbor towns kept tide tables because high and low water return on a regular schedule, roughly every twelve and a half hours.

WORKED EXAMPLES

Example 1. A water wheel 6 m in diameter turns once every 20 seconds. Its center is 2.5 m above the surface of the stream. A bucket on the rim is at its lowest point at $t = 0$. Write a model for the bucket's height h above the water after t seconds, and find its height at $t = 7$ s to the nearest hundredth of a meter.

Show solution

1. Radius $r = 3$ m, so the amplitude is 3. The midline is the height of the center: $d = 2.5$.
2. Period $T = 20$ s, so $b = \frac{2\pi}{20} = \frac{\pi}{10}$.
3. The bucket starts at the bottom, so use a negative cosine: $h(t) = 2.5 - 3 \cos\left(\frac{\pi t}{10}\right)$. Check: $h(0) = 2.5 - 3 = -0.5$ m, so the bucket dips half a meter into the stream, as it should to fill.
4. At $t = 7$: $h(7) = 2.5 - 3 \cos(0.7\pi)$. Since $\cos(0.7\pi) = \cos 126^\circ \approx -0.5877853$, $h(7) \approx 2.5 + 1.7633558 \approx 4.26$ m.

Answer: $h(t) = 2.5 - 3 \cos\left(\frac{\pi t}{10}\right)$; $h(7) \approx 4.26$ m

Example 2. In a certain northern town the longest day of the year, day 172, has 15.2 hours of daylight and the shortest day has 9.2 hours. Take the period to be 365 days. Write a model $D(t)$ for the hours of daylight on day t , and find on which days the town has more than 14 hours of daylight. Give the day numbers to the nearest tenth, then state the whole days and how many there are.

Show solution

1. Midline: $d = \frac{15.2 + 9.2}{2} = 12.2$. Amplitude: $a = \frac{15.2 - 9.2}{2} = 3$. Period 365, so $b = \frac{2\pi}{365}$. The maximum is on day 172, so $c = 172$.

2. Model: $D(t) = 3 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12.2$.

3. Solve $D(t) > 14$: $\cos\left(\frac{2\pi}{365}(t - 172)\right) > 0.6$.

4. Near the maximum, cosine exceeds 0.6 when its argument is between $-\cos^{-1}(0.6)$ and $\cos^{-1}(0.6)$, where $\cos^{-1}(0.6) \approx 0.9272952$ rad. So $|t - 172| < \frac{365}{2\pi}(0.9272952) \approx 53.868$.

5. Thus $118.1 < t < 225.9$. The whole days with more than 14 hours are days 119 through 225, which is 107 days. (The interval itself is about 107.7 days long.)

Answer: $D(t) = 3 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12.2$; more than 14 hours when $118.1 < t < 225.9$, that is, days 119 through 225 (107 days)

PRACTICE

1. **Easy** A weight on a spring moves according to $y = 5 \sin(4\pi t)$, where y is in centimeters and t in seconds. Find the amplitude, the period and the frequency.

Show answer

Amplitude 5 cm; period $\frac{2\pi}{4\pi} = 0.5$ s; frequency 2 cycles per second

2. **Easy** At a harbor the water is 9 ft deep at high tide and 1 ft deep at low tide. Find the amplitude and the midline of a sinusoidal model for the depth.

Show answer

Amplitude $\frac{9 - 1}{2} = 4$ ft; midline $y = \frac{9 + 1}{2} = 5$ ft

3. **Easy** A weight hanging on a spring is pulled 6 cm below its rest position and released at $t = 0$. It bobs with a period of 2 s. Let d be its displacement in cm above the rest position. Write a model for $d(t)$ and find $d\left(\frac{1}{3}\right)$.

Show answer

$$d(t) = -6 \cos(\pi t); d\left(\frac{1}{3}\right) = -6 \cos \frac{\pi}{3} = -3, \text{ that is, 3 cm below rest}$$

4. **Medium** At a harbor, high tide of 9 ft occurs at 3:00 a.m. and the next low tide of 1 ft occurs at 9:15 a.m. Write a model for the depth h in feet, with t in hours after midnight, and find the depth at noon to the nearest tenth of a foot.

Show answer

$$\text{Half a period is 6.25 h, so } T = 12.5 \text{ h. } h(t) = 4 \cos\left(\frac{2\pi}{12.5}(t - 3)\right) + 5. \text{ At noon,}$$

$$h(12) = 4 \cos(1.44\pi) + 5 \approx 4(-0.18738) + 5 \approx 4.3 \text{ ft}$$

5. **Medium** A Ferris wheel 50 ft in diameter has its center 30 ft above the ground and turns once every 60 seconds. Ruth boards at the bottom at $t = 0$. Write a model for her height and find her height after 20 seconds.

Show answer

$$h(t) = 30 - 25 \cos\left(\frac{\pi t}{30}\right); h(20) = 30 - 25 \cos \frac{2\pi}{3} = 42.5 \text{ ft}$$

6. **Medium** A town has 14.8 hours of daylight on its longest day, day 172, and 9.6 hours on its shortest. Using a period of 365 days, write a model for the hours of daylight and predict the daylight on day 100, to the nearest tenth of an hour.

Show answer

$$D(t) = 2.6 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12.2; D(100) = 2.6 \cos\left(-\frac{144\pi}{365}\right) + 12.2 \approx$$

$$13.0 \text{ hours}$$

7. **Hard** Using the tide model of Problem 4, a fishing boat needs at least 7 ft of water to leave the harbor. Between midnight and 6:00 a.m., during what time interval can it leave? Give the times to the nearest minute.

Show answer

$$4 \cos\left(\frac{2\pi}{12.5}(t - 3)\right) + 5 \geq 7 \text{ gives } \cos(\cdot) \geq \frac{1}{2}, \text{ so } \left|\frac{2\pi}{12.5}(t - 3)\right| \leq \frac{\pi}{3}, |t - 3| \leq$$

$$\frac{12.5}{6} \approx 2.0833 \text{ h. So } 0.9167 \leq t \leq 5.0833: \text{ from 12:55 a.m. to 5:05 a.m.}$$

8. **Hard** An undershot water wheel has radius 4 m and its center is 1.5 m above the water. It turns once every 24 seconds, and a certain paddle is at the top at $t = 0$. Find, to the nearest tenth of a second, (a) when the paddle first enters the water, (b) when it first comes out, and (c) how long it stays underwater in each turn.

Show answer

$$h(t) = 1.5 + 4 \cos\left(\frac{\pi t}{12}\right). \text{ Underwater when } \cos\left(\frac{\pi t}{12}\right) < -0.375.$$

$$\cos^{-1}(-0.375) \approx 1.955193 \text{ rad and } 2\pi - 1.955193 \approx 4.327992 \text{ rad. (a) } t = \frac{12}{\pi}(1.955193) \approx 7.5 \text{ s; (b) } t = \frac{12}{\pi}(4.327992) \approx 16.5 \text{ s; (c) about 9.1 s}$$

9. **Challenge** A peg on the rim of a wheel of radius a turns at a steady angular speed ω , starting at the rightmost point of the wheel. A lamp far to the right throws the peg's shadow onto a vertical wall to the left. Explain why the shadow moves in simple harmonic motion, and find its period.

Show answer

Put the wheel's center at the origin. After time t the peg has turned through angle ωt , so by the unit-circle definition it is at $(a \cos \omega t, a \sin \omega t)$. Light from a lamp far away travels horizontally, so the shadow on the vertical wall keeps only the peg's height: $y = a \sin \omega t$. This is a sine function of time, which is simple harmonic motion with amplitude a . It repeats when ωt increases by 2π , so the period is $\frac{2\pi}{\omega}$.

10. **Challenge** A weight on a spring moves by $d(t) = a \sin(bt)$ cm with a period of 1.6 s. At $t = 0.2$ s its displacement is 3 cm. (a) Find a exactly. (b) Find the first time it reaches a displacement of -2 cm, to the nearest thousandth of a second.

Show answer

$$\begin{aligned} \text{(a) } b &= \frac{2\pi}{1.6} = \frac{5\pi}{4}, \text{ so } 3 = a \sin \frac{\pi}{4} \text{ and } a = 3\sqrt{2} \approx 4.243. \text{ (b) } \sin\left(\frac{5\pi t}{4}\right) = \\ &= -\frac{2}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}. \text{ The displacement is first negative for } 0.8 < t < 1.6, \text{ so } \frac{5\pi t}{4} = \\ &= \pi + \sin^{-1}\left(\frac{\sqrt{2}}{3}\right) \approx 3.632475, \text{ giving } t \approx 0.925 \text{ s} \end{aligned}$$

COMMON MISTAKES

- Using the full distance from maximum to minimum as the amplitude. The amplitude is **half** that distance.

- Using the time from high tide to low tide as the period. That is only half a period; double it.
- Starting a model with $\cos$ when the motion begins at the minimum. Use $-a \cos(bt) + d$, or check by substituting $t = 0$ into your model.
- Finding only one solution when solving for a time. A sinusoid reaches each level between its extremes twice per cycle; find both and keep those that fit the question's time interval.

Make it a game • Clock Tip Model. Imagine a clock whose minute hand is 6 inches long and whose center is 10 inches above a table. Write a cosine model for the height of its tip, with t in minutes. Predict the height at 20 minutes, and find when it first reaches 13 inches. Parent challenges with a new clock.

APPLICATIONS

Trigonometry was born as a practical art. Astronomers, navigators and surveyors needed to find distances and heights they could not measure with a rope or chain, and the ratios of a triangle gave them a way. In this unit you put everything from Units 1 through 7 to work: angles of elevation and depression for towers, silos and cliffs; bearings for ships and aircraft; triangulation for rivers, fields and even the size of the earth; right triangles hidden inside three-dimensional roofs, rooms and pyramids; and sinusoidal models for water wheels, tides, daylight and springs.

The mathematics is not new, but the skill is. In each problem you must read a description, draw a careful diagram, decide which triangle or which model fits, and only then compute. That habit of turning a real situation into a clear figure is what Eratosthenes, Snellius and every surveyor and navigator since have practiced, and it is the main goal of this unit.

Unit 8 draws on right-triangle trigonometry (Unit 1), radians and the unit circle (Unit 3), sinusoidal graphs (Unit 4), trigonometric equations (Unit 6) and the Laws of Sines and Cosines (Unit 7). The optional Unit 9 continues these applications with polar coordinates and vectors.

PREREQUISITES

- Solve right triangles with sine, cosine, tangent and their inverses (Lessons 1.3 and 1.4).
- Use exact values for 30° , 45° and 60° (Unit 2).
- Write and interpret sinusoidal equations $y = a \sin(b(x - c)) + d$ and $y = a \cos(b(x - c)) + d$ (Lessons 4.4 and 4.5).
- Solve trigonometric equations on an interval (Lessons 6.4 and 6.5).
- Solve oblique triangles with the Law of Sines and the Law of Cosines, and find areas with $\frac{1}{2}ab \sin C$ and Heron's formula (Unit 7).

MASTERY CHECKLIST

- ☐ I can draw a diagram for a height-and-distance problem and find heights and distances from angles of elevation and depression.
- ☐ I can explain why an angle of depression equals the matching angle of elevation, and solve problems with two observations of one object.
- ☐ I can convert between compass bearings and true three-figure bearings and find back bearings.
- ☐ I can find the interior angle at a turn and solve two-leg navigation problems for distance and bearing.
- ☐ I can find an inaccessible distance, such as the width of a river, from a measured baseline and two angles.
- ☐ I can find the area of a surveyed field by dividing it into triangles, and convert square feet to acres.
- ☐ I can explain Eratosthenes' method for measuring the earth.
- ☐ I can find lengths and angles in boxes, rooms, roofs and pyramids by isolating right triangles.
- ☐ I can solve problems where an object is observed from two points that are not in line with it.
- ☐ I can build a sinusoidal model for a water wheel, tides, daylight hours or a spring, and use it to predict values and find times.

LESSONS IN THIS UNIT

8.1	Angles of Elevation and Depression	50 minutes
8.2	Bearings and Navigation	2 sessions of 45 minutes
8.3	Surveying and Triangulation	2 sessions of 45 minutes
8.4	Problems in Three Dimensions	50 minutes
8.5	Modeling Periodic Motion	2 sessions of 45 minutes

UNIT 8 TEST

Time allowed: 60 minutes. A scientific calculator is allowed; set it to degree mode except where a problem uses radians. Draw a labeled diagram for every word problem. Keep full calculator precision in intermediate steps and round only the final answer as each problem directs. Show your work; answers without supporting work receive partial credit at most. Total: 100 points.

1. 5 pts

From the top of a lighthouse 30 m above the sea, the angle of depression to a sailboat is 12° . How far is the sailboat from the base of the lighthouse, to the nearest tenth of a meter?

2. 5 pts

(a) Write the compass bearing $S40^\circ E$ as a true three-figure bearing. (b) If the bearing of a buoy from a ship is that bearing, what is the bearing of the ship from the buoy?

3. 6 pts

The height of a bucket on a water wheel is $h(t) = 20 - 18 \cos\left(\frac{\pi t}{45}\right)$ feet, with t in seconds. Find the bucket's greatest height, its least height and the period of the wheel.

4. 8 pts

From a point on level ground the angle of elevation to the top of a silo is 30° . From a point 40 ft closer, in line with the first point and the silo, the angle of elevation is 45° . Find the height of the silo exactly and to the nearest tenth of a foot.

5. 6 pts

A ship sails 20 miles on a bearing of 045° , then 15 miles on a bearing of 135° . Find its distance from the start and its bearing from the start, to the nearest tenth of a degree.

6. 8 pts

A triangular hay field has two sides of 400 ft and 350 ft meeting at an angle of 75° . Find its area to the nearest square foot and in acres to the nearest hundredth (1 acre = 43,560 sq ft).

7. 7 pts

A barn loft is 20 ft long, 15 ft wide and 10 ft high. Find the length of a straight pole reaching from one floor corner to the opposite ceiling corner, to the nearest tenth of a foot, and the angle it makes with the floor, to the nearest tenth of a degree.

8. 8 pts

To find the width of a river, a surveyor lays out a baseline $AB = 120$ m along one straight bank and sights a post C on the far bank. She measures $\angle CAB = 62^\circ$ and $\angle CBA = 54^\circ$. Find the width of the river to the nearest tenth of a meter.

9. 7 pts

A town has 16 hours of daylight on its longest day, day 172, and 8 hours on its shortest. Using a period of 365 days, write a model for the hours of daylight $D(t)$ on day t and predict the daylight on day 80, to the nearest tenth of an hour.

10. 10 pts

A pilot flies 120 miles from A to B on a bearing of 060° , then 80 miles from B to C on a bearing of 200° . Find the distance AC to the nearest tenth of a mile and the bearing of C from A to the nearest tenth of a degree.

11. 10 pts

From a point A due west of a tower, the angle of elevation of its top is 40° . From a point B due south of the tower, the angle of elevation is 35° . Points A and B are 150 m apart on level ground. Find the height of the tower to the nearest tenth of a meter.

12. 10 pts

Using the daylight model of Problem 9, on which days of the year does the town have more than 15 hours of daylight? Give the day numbers to the nearest tenth and state how many whole days qualify.

13. 10 pts

An observer measures the angle of elevation α to the top of a tower, then walks a distance d straight toward it on level ground and measures a larger angle of elevation β . Prove that the height of the tower is $h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$.

ANSWER KEY

1. $\frac{30}{\tan 12^\circ} \approx 141.1 \text{ m}$
2. (a) 140° ; (b) 320°
3. Greatest height 38 ft, least height 2 ft, period $\frac{2\pi}{\pi/45} = 90 \text{ s}$
4. Let x be the distance from the nearer point. $h = x$ and $h = (x + 40) \tan 30^\circ$, so $h\sqrt{3} = h + 40$, $h = \frac{40}{\sqrt{3} - 1} = 20(\sqrt{3} + 1) \approx 54.6 \text{ ft}$
5. The turn is 90° , so the distance is $\sqrt{20^2 + 15^2} = 25 \text{ miles}$; bearing = $045^\circ + \tan^{-1}\left(\frac{15}{20}\right) \approx 081.9^\circ$
6. $\frac{1}{2}(400)(350) \sin 75^\circ \approx 67,615 \text{ sq ft} \approx 1.55 \text{ acres}$
7. Floor diagonal = 25 ft; pole = $\sqrt{25^2 + 10^2} = \sqrt{725} \approx 26.9 \text{ ft}$; angle = $\tan^{-1}\left(\frac{10}{25}\right) \approx 21.8^\circ$
8. $C = 64^\circ$; $AC = \frac{120 \sin 54^\circ}{\sin 64^\circ} \approx 108.014 \text{ m}$; width = $AC \sin 62^\circ \approx 95.4 \text{ m}$
9. $D(t) = 4 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12$; $D(80) = 4 \cos\left(-\frac{184\pi}{365}\right) + 12 \approx 11.9 \text{ hours}$
10. Back bearing at B is 240° , so $\angle ABC = 240^\circ - 200^\circ = 40^\circ$. $AC = \frac{\sqrt{120^2 + 80^2 - 2(120)(80) \cos 40^\circ}}{\sin 40^\circ} \approx 78.1 \text{ mi}$. $\sin A = \frac{80 \sin 40^\circ}{78.0509} \approx 0.658839$; A is opposite a side shorter than AB , so it is acute: $A \approx 41.21^\circ$. C lies clockwise from the first leg, so its bearing is $060^\circ + 41.21^\circ \approx 101.2^\circ$.
11. With O the foot of the tower, $OA = h \cot 40^\circ$, $OB = h \cot 35^\circ$ and $\angle AOB = 90^\circ$. So $h^2(\cot^2 40^\circ + \cot^2 35^\circ) = 150^2$, $h = \frac{150}{\sqrt{1.4202766 + 2.0396067}} \approx 80.6 \text{ m}$
12. $4 \cos\left(\frac{2\pi}{365}(t - 172)\right) + 12 > 15$ gives $\cos(\cdot) > 0.75$, so $|t - 172| < \frac{365}{2\pi} \cos^{-1}(0.75) \approx 41.985$. Thus $130.0 < t < 214.0$: days 131 through 213, which is 83 days
13. Let x be the distance from the second point to the foot of the tower. The tower is vertical and the ground is level, so both observation triangles are right triangles. From the second point, $\tan \beta = \frac{h}{x}$, so $x = \frac{h}{\tan \beta}$. From the first point, $\tan \alpha = \frac{h}{x + d}$, so $x + d = \frac{h}{\tan \alpha}$. Subtracting, $d = \frac{h}{\tan \alpha} - \frac{h}{\tan \beta} = h \cdot \frac{\tan \beta - \tan \alpha}{\tan \alpha \tan \beta}$.

Since $\beta > \alpha$ and both are acute, $\tan \beta - \tan \alpha > 0$, so we may divide: $h = \frac{d \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$.

POLAR COORDINATES AND VECTORS

This optional unit carries the trigonometry of the whole course into two new settings. Polar coordinates locate a point by a direction and a distance, the way a navigator, a radar operator or a surveyor actually sees the world. With them, circles through the pole, the petals of a rose and the heart-shaped cardioid have short, beautiful equations. Students will plot polar points, convert between polar and rectangular form, and graph the classic polar curves using tables and symmetry.

Vectors are quantities with both size and direction: forces, velocities and displacements. Resolving a vector into components is exactly the polar-to-rectangular conversion, so the two halves of the unit share one idea: $x = r \cos \theta$, $y = r \sin \theta$. Students will use components to add forces on a load, steer a boat across a current, find an airplane's true course in a wind, and measure the angle between two directions with the dot product, whose formula comes straight from the Law of Cosines.

The unit draws on the unit circle and exact values (Unit 3), inverse tangent (Unit 6), the Laws of Sines and Cosines (Unit 7) and bearings (Unit 8). It is a natural bridge to physics and to the vectors and polar curves of calculus.

PREREQUISITES

- Evaluate trigonometric functions of any angle in degrees or radians, including exact values for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (Lessons 3.6–3.8).
- Use reference angles and quadrant signs (Lesson 3.7).
- Use $\tan^{-1}$ on a calculator and know its range (Lesson 6.2).
- Apply the Law of Cosines and the Law of Sines (Lessons 7.2 and 7.4).
- Read true bearings (Lesson 8.2).
- Use the Pythagorean theorem and complete the square to find the center and radius of a circle.

MASTERY CHECKLIST

- ☐ I can plot a point (r, θ) , including one with negative r , and give several other polar names for it.
- ☐ I can find the distance between two points given in polar form using the Law of Cosines.
- ☐ I can derive
$$x = r \cos \theta, \text{ in } \theta$$
$$\text{an } r^2 = x^2 + y^2,$$
$$\text{d}$$
$$\text{an } \theta$$
$$\text{d}$$
$$\text{co}$$
$$\text{nv}$$
$$\text{ert}$$
$$\text{po}$$
$$\text{int}$$
$$\text{s}$$
$$\text{bo}$$
$$\text{th}$$
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nt.

- ☐ I can convert simple equations between polar and rectangular form, for example showing that

$r = a$ is a circle.

$$2a \cos \theta$$

- ☐ I can graph a polar equation from a table of values and use the symmetry tests.
- ☐ I can recognize circles, roses, cardioids and limaçons from their equations and predict the number and length of petals or the type of limaçon.
- ☐ I can resolve a vector into components and find a vector's magnitude and direction angle from its components.
- ☐ I can resolve a weight on a ramp into components along and perpendicular to the ramp.
- ☐ I can add vectors to find the resultant of forces or velocities, including a boat in a current and an airplane in a wind.
- ☐ I can prove

$\mathbf{u} \cdot \mathbf{v}$ from the Law of Cosines and use it to find the angle between two

vectors or test for perpendicularity.

LESSONS IN THIS UNIT

9.1 Polar Coordinates

45 minutes

9.2 Converting Between Polar and Rectangular Form

50 minutes

UNIT 9 TEST

Time allowed: 90 minutes. A scientific calculator is allowed; a graphing calculator is not. Show your work. Give exact answers when asked; otherwise round as stated in each problem, without rounding intermediate values. Angles are in degrees unless the problem asks for radians. Total: 100 points.

1. **5 pts**

Rename the point $(-6, 210^\circ)$ with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Give θ in degrees.

2. **6 pts**

Find the exact distance between the polar points $(5, 20^\circ)$ and $(8, 80^\circ)$.

3. **6 pts**

Convert the polar point $\left(8, \frac{4\pi}{3}\right)$ to rectangular coordinates, exactly.

4. **7 pts**

Convert the rectangular point $(-8, 15)$ to polar coordinates with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Round θ to the nearest tenth of a degree.

5. **7 pts**

(a) Convert $x^2 + y^2 = 10y$ to a polar equation solved for r . (b) Convert $r = 3 \sec \theta$ to rectangular form. Describe each graph.

6. **8 pts**

Identify each curve and give the requested feature: (a) $r = 5 \sin 3\theta$ (number and length of petals); (b) $r = 2 + 2 \sin \theta$ (type of curve and its greatest distance from the pole); (c) $r = 4 + \cos \theta$ (type of limaçon).

7. (7 pts)

The limaçon $r = 2 - 4 \cos \theta$ has an inner loop. Find all θ in $[0, 2\pi)$ at which it passes through the pole, exactly in radians.

8. (6 pts)

A vector has magnitude 50 and direction angle 225° . Find its components exactly and to the nearest hundredth.

9. (7 pts)

Find the magnitude and direction angle of $\mathbf{v} = \langle -12, -5 \rangle$, with $0^\circ \leq \theta < 360^\circ$ to the nearest tenth of a degree.

10. (8 pts)

Samuel rolls a 240-lb piano up a ramp inclined at 30° . Ignoring friction, find the force parallel to the ramp needed to hold the piano in place, and the component of its weight pressing against the ramp. Give exact values and decimals to the nearest tenth of a pound.

11. (7 pts)

Find the angle between $\mathbf{u} = \langle 1, 7 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$, exactly in degrees.

12. (9 pts)

A plane heads due east with an airspeed of 180 mph. A wind blows due north at 75 mph. Find the plane's ground speed and its course as a true bearing, to the nearest tenth of a degree.

13. (9 pts)

A ferry moves at 10 mph through still water and must cross a river 1 mile wide, landing directly opposite its dock. The current flows at 5 mph. At what angle upstream of straight across must it aim, what is its speed over the ground, and how many minutes does the crossing take (to the nearest tenth of a minute)?

14. (8 pts)

Prove that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are perpendicular (when neither is $\mathbf{0}$). Use components $\mathbf{u} = \langle a_1, b_1 \rangle$, $\mathbf{v} = \langle a_2, b_2 \rangle$. What does this say about the diagonals of a rhombus?

ANSWER KEY

1. $(6, 30^\circ)$
2. $d^2 = 25 + 64 - 80 \cos 60^\circ = 49$, so $d = 7$.
3. $\left(8 \cos \frac{4\pi}{3}, 8 \sin \frac{4\pi}{3}\right) = (-4, -4\sqrt{3}) \approx (-4, -6.928)$
4. $r = 17$. Quadrant II: $\theta = 180^\circ - \tan^{-1} \frac{15}{8} \approx 180^\circ - 61.93^\circ \approx 118.1^\circ$. The point is about $(17, 118.1^\circ)$.
5. (a) $r^2 = 10r \sin \theta$, so $r = 10 \sin \theta$: a circle through the pole with center $(0, 5)$ and radius 5. (b) $r \cos \theta = 3$, so $x = 3$: a vertical line.
6. (a) A rose with 3 petals, each of length 5. (b) A cardioid, with its cusp at the pole and its greatest distance 4 at $\theta = \frac{\pi}{2}$. (c) A convex limaçon, since $4 \geq 2 \cdot 1$.
7. $2 - 4 \cos \theta = 0$, so $\cos \theta = \frac{1}{2}$: $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.
8. $\langle -25\sqrt{2}, -25\sqrt{2} \rangle \approx \langle -35.36, -35.36 \rangle$
9. $\|\mathbf{v}\| = 13$. Quadrant III: $\theta = 180^\circ + \tan^{-1} \frac{5}{12} \approx 180^\circ + 22.62^\circ \approx 202.6^\circ$.
10. Parallel: $240 \sin 30^\circ = 120$ lb. Perpendicular: $240 \cos 30^\circ = 120\sqrt{3} \approx 207.8$ lb.
11. $\mathbf{u} \cdot \mathbf{v} = -3 + 28 = 25$; $\|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$, $\|\mathbf{v}\| = 5$. $\cos \theta = \frac{25}{25\sqrt{2}} = \frac{\sqrt{2}}{2}$, so $\theta = 45^\circ$.
12. Ground velocity $\langle 180, 75 \rangle$, speed $\sqrt{32,400 + 5625} = \sqrt{38,025} = 195$ mph. It travels $\tan^{-1} \frac{75}{180} \approx 22.6^\circ$ north of east, a true bearing of $90^\circ - 22.62^\circ \approx 067.4^\circ$.
13. $\sin \alpha = \frac{5}{10}$, so aim 30° upstream. Ground speed $10 \cos 30^\circ = 5\sqrt{3} \approx 8.66$ mph.
Time $\frac{1}{5\sqrt{3}}$ h $= 4\sqrt{3} \approx 6.9$ minutes.
14. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (a_1 + a_2)(a_1 - a_2) + (b_1 + b_2)(b_1 - b_2) = a_1^2 - a_2^2 + b_1^2 - b_2^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$. A zero dot product of nonzero vectors means they are perpendicular. A rhombus built on $\mathbf{u}$ and $\mathbf{v}$ has diagonals $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, so the diagonals of a rhombus are perpendicular.

9.1 POLAR COORDINATES

OBJECTIVES

- Plot points given in polar coordinates (r, θ) , including points with negative r and angles in degrees or radians.
- Give several different polar names for the same point, and recognize when two names describe the same point.
- Use the Law of Cosines to find the distance between two points given in polar form.

WATCH

Introduction to Polar Coordinates (Precalculus - Trigonometry 36)

Professor Leonard · 38:06

https://www.youtube.com/watch?v=OL_a6LEdo3M

WHEN TO WATCH

After reading the Core ideas, before Example 1; covers negative r and multiple names for one point (38 min).

Note: Woo's video introduces polar form for complex numbers: he says 'modulus' for r and 'argument' for θ , and $z = x + iy$. Watch the first half.

BACKUP VIDEO

Introducing Polar Form (1 of 3: An alternative coordinate system)

Eddie Woo · 10:53

<https://www.youtube.com/watch?v=tdNKkc6jbr0>

Short preview before the Core ideas: why polar coordinates exist (map and globe analogy).

KEY TERMS

pole

The fixed point from which distances are measured in polar coordinates; it plays the role of the origin.

polar axis

A fixed ray starting at the pole, usually drawn pointing to the right (along the positive x -axis), from which angles are measured.

polar coordinates

A pair (r, θ) locating a point by its directed distance r from the pole and the angle θ measured from the polar axis, counterclockwise when positive.

directed distance

The signed value r : when $r > 0$ the point lies on the terminal side of θ ; when $r < 0$ it lies $|r|$ units out on the opposite ray, in the direction $\theta + 180^\circ$.

CORE IDEAS

Rectangular coordinates tell you how far to go east and how far to go north. Polar coordinates tell you **which way to face and how far to walk**. Stand at the pole, face along the polar axis, turn through the angle θ (counterclockwise if $\theta > 0$, clockwise if $\theta < 0$), then walk r units. This is exactly how a radar operator or a ship's navigator locates an object: by a direction and a range.

Negative r . If $r < 0$, turn through θ as usual, but walk **backward** $|r|$ units. You land on the ray opposite the terminal side, that is, in the direction $\theta + 180^\circ$ (or $\theta + \pi$). So $(-r, \theta)$ and $(r, \theta + \pi)$ are the same point. For example, $(-2, 60^\circ)$ is the same point as $(2, 240^\circ)$, in Quadrant III.

Many names for one point. In rectangular coordinates each point has exactly one name. In polar coordinates each point has infinitely many. Adding a full turn does not move the point, so $(r, \theta) = (r, \theta + 2\pi k)$ for every integer k . Combining this with the rule for negative r gives $(r, \theta) = (-r, \theta + \pi + 2\pi k)$. The pole is special: $(0, \theta)$ is the pole for every angle θ , because walking zero units goes nowhere no matter which way you face.

Distance between two polar points. Let $P = (r_1, \theta_1)$ and $Q = (r_2, \theta_2)$ with $r_1, r_2 > 0$. The pole O , P and Q form a triangle with sides $OP = r_1$, $OQ = r_2$ and included angle $\theta_2 - \theta_1$ at O . By the Law of Cosines, $PQ^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)$. It does not matter whether the angle between the rays is $\theta_2 - \theta_1$ or 360° minus that, or which point is subtracted from which, because cosine is even and has period 360° . If a point has negative r , rename it with positive r first.

KEY FORMULAS

$$(r, \theta) = (r, \theta + 2\pi k) = (-r, \theta + \pi + 2\pi k), \quad k \text{ an integer}$$

$$(0, \theta) \text{ is the pole for every } \theta$$

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$$

From history. Long before anyone wrote (r, θ) , Archimedes described his spiral by a point moving outward along a ray at a steady rate while the ray turns at a steady rate, which is a polar description in words. Isaac Newton and Jakob Bernoulli both used coordinates of this kind in the late seventeenth century. Navigators had long fixed positions the same way, by a direction and a distance from a known landmark.

Euclid connection. Euclid defines a circle (Book I, Definition 15) as the figure whose points are all equally distant from one point, the center. In polar coordinates that definition becomes the simplest possible equation: all points with $r = a$.

WORKED EXAMPLES

Example 1. Plot $P = (4, 150^\circ)$. Then give two other polar names for P : (a) one with $r > 0$ and a negative angle between -360° and 0° ; (b) one with $r < 0$ and $0^\circ \leq \theta < 360^\circ$. Give angles in degrees.

Show solution

1. To plot P , face along the polar axis, turn 150° counterclockwise, and walk 4 units. P lies in Quadrant II.
2. (a) Subtract one full turn: $150^\circ - 360^\circ = -210^\circ$. So $P = (4, -210^\circ)$. Turning 210° clockwise ends on the same ray as turning 150° counterclockwise.
3. (b) To use $r = -4$, point the angle at the **opposite** ray, then walk backward. The opposite ray is $150^\circ + 180^\circ = 330^\circ$. So $P = (-4, 330^\circ)$.
4. Check (b): facing 330° (Quadrant IV) and walking backward 4 units lands in the direction $330^\circ - 180^\circ = 150^\circ$, which is where P is.

Answer: (a) $(4, -210^\circ)$; (b) $(-4, 330^\circ)$

Example 2. A harbor radar places the harbor at the pole with the polar axis pointing due east. One ship is at $(8 \text{ mi}, 40^\circ)$ and another at $(5 \text{ mi}, 100^\circ)$. How far apart are the ships? Give an exact answer.

Show solution

1. The harbor O and the two ships form a triangle. Two sides are the ranges:

$$r_1 = 8 \text{ and } r_2 = 5.$$

2. The angle at the harbor between the two directions is $100^\circ - 40^\circ = 60^\circ$.

3. Law of Cosines: $d^2 = 8^2 + 5^2 - 2(8)(5) \cos 60^\circ = 64 + 25 - 80 \cdot \frac{1}{2} = 89 - 40 = 49$.

4. So $d = \sqrt{49} = 7$.

Answer: The ships are exactly 7 miles apart.

PRACTICE

1. Easy In which quadrant does the point $(5, 200^\circ)$ lie?

Show answer

Quadrant III (between 180° and 270°)

2. Easy Give a name for the point $\left(2, \frac{\pi}{3}\right)$ with $r > 0$ and $2\pi \leq \theta < 4\pi$. Give θ in radians.

Show answer

$$\left(2, \frac{7\pi}{3}\right)$$

3. Easy In which quadrant does the point $(-3, 45^\circ)$ lie?

Show answer

Quadrant III. With $r < 0$ the point lies in the direction $45^\circ + 180^\circ = 225^\circ$.

4. Medium Rename the point $(-5, 60^\circ)$ with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Give θ in degrees.

Show answer

$$(5, 240^\circ)$$

5. **Medium** Rename the point $\left(6, \frac{\pi}{6}\right)$ with $r < 0$ and $-2\pi < \theta \leq 0$. Give θ exactly in radians.

Show answer

$\left(-6, -\frac{5\pi}{6}\right)$, since the opposite ray is $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$ and $\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$.

6. **Medium** Which of the following name the same point as $\left(4, \frac{3\pi}{4}\right)$? (a) $\left(4, -\frac{5\pi}{4}\right)$
(b) $\left(-4, -\frac{\pi}{4}\right)$ (c) $\left(-4, \frac{5\pi}{4}\right)$ (d) $\left(4, \frac{11\pi}{4}\right)$

Show answer

(a), (b) and (d). (a) and (d) differ from $\frac{3\pi}{4}$ by $\mp 2\pi$; (b) points to $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.
But (c) points to $\frac{5\pi}{4} + \pi = \frac{9\pi}{4}$, which is the direction $\frac{\pi}{4}$, so (c) is a different point.

7. **Medium** A lighthouse keeper uses polar coordinates with the lighthouse at the pole and the polar axis due east. A fishing boat is sighted at $(3 \text{ mi}, 50^\circ)$ and a sailboat at $(3 \text{ mi}, 110^\circ)$. How far apart are the boats? Give an exact answer.

Show answer

$d^2 = 9 + 9 - 18 \cos 60^\circ = 9$, so $d = 3$ miles. (The triangle is equilateral.)

8. **Hard** Find the distance between the points $(4, 30^\circ)$ and $(6, 90^\circ)$, exactly and to the nearest hundredth.

Show answer

$d^2 = 16 + 36 - 48 \cos 60^\circ = 28$, so $d = 2\sqrt{7} \approx 5.29$

9. **Hard** Find the distance between the points $\left(5, \frac{\pi}{6}\right)$ and $\left(-3, \frac{5\pi}{6}\right)$, exactly and to the nearest hundredth.

Show answer

Rename the second point: $\left(-3, \frac{5\pi}{6}\right) = \left(3, \frac{11\pi}{6}\right)$, the direction $-\frac{\pi}{6}$. The angle between the rays is $\frac{\pi}{3}$. $d^2 = 25 + 9 - 30 \cdot \frac{1}{2} = 19$, so $d = \sqrt{19} \approx 4.36$

10. **Hard** An airfield radar (polar axis due east) shows one aircraft at range 12 mi and polar angle 20° , and another at range 9 mi and polar angle 140° . How far apart are the aircraft, to the nearest tenth of a mile? (Ignore differences in altitude.)

Show answer

Angle between them 120° . $d^2 = 144 + 81 - 216 \cos 120^\circ = 225 + 108 = 333$, so $d = 3\sqrt{37} \approx 18.2$ mi

11. **Challenge** Explain why $(-r, \theta)$ and $(r, \theta + \pi)$ always name the same point, and why the pole has infinitely many names of the form $(0, \theta)$.

Show answer

Both instructions face along a line through the pole. $(r, \theta + \pi)$ faces the ray opposite to θ and walks forward $|r|$; $(-r, \theta)$ faces θ and walks backward $|r|$, which also moves along the opposite ray. Same ray, same distance, same point. For the pole, walking a distance of 0 leaves you at the pole whatever direction θ you face, so every $(0, \theta)$ is the pole.

12. **Challenge** The point $P = (r, 40^\circ)$ with $r > 0$ is exactly 10 units from $Q = (6, 100^\circ)$. Find r exactly and to the nearest hundredth.

Show answer

$100 = r^2 + 36 - 12r \cos 60^\circ = r^2 - 6r + 36$, so $r^2 - 6r - 64 = 0$ and $r = 3 + \sqrt{73} \approx 11.54$ (the other root, $3 - \sqrt{73}$, is negative).

COMMON MISTAKES

- Plotting $(-r, \theta)$ on the terminal side of θ . A negative r means walk backward: the point is on the opposite ray, in the direction $\theta + 180^\circ$.
- Changing the sign of r without adding 180° (or π) to the angle, or adding 180° without changing the sign of r . Do both together, or neither.
- Measuring angles clockwise when θ is positive. Positive angles turn counterclockwise from the polar axis, just as in standard position.
- In the distance formula, using a point with negative r as if r were positive. Rename it with positive r first, so the triangle's sides and angle are correct.

Make it a game • Walk the Polar Point. Stand at a taped pole facing the polar axis. Parent calls a point, such as $(3, 135 \text{ degrees})$. Turn through the angle, then walk 3 steps. For negative r , such as $(-2, 45 \text{ degrees})$, turn and walk backward. Then give three other names for each point. Ten correct walks wins.

9.2 CONVERTING BETWEEN POLAR AND RECTANGULAR FORM

OBJECTIVES

- Derive the conversion formulas $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$ from a right triangle.
- Convert points from polar to rectangular form and from rectangular to polar form, choosing the angle in the correct quadrant.
- Convert simple equations (lines, circles and conics through simple substitutions) from one form to the other.

WATCH

How to Convert From Polar Coordinates to Rectangular Coordinates (Precalculus - Trigonometry 37).

Professor Leonard · 34:24

<https://www.youtube.com/watch?v=nyab0CZK2BE>

WHEN TO WATCH

After reading the Core ideas, before Example 1.

Note: Woo's video is about complex numbers (modulus–argument form); the conversion formulas are the same.

BACKUP VIDEO

Introducing Polar Form (2 of 3: Relationship to rectangular form).

Eddie Woo · 9:32

<https://www.youtube.com/watch?v=kCsuJ-Yo8n8>

Short alternative before Example 1: $x = r \cos \theta$, $y = r \sin \theta$.

KEY TERMS

rectangular coordinates

The familiar (x, y) coordinates measured along two perpendicular axes; also called Cartesian coordinates.

conversion formulas

The equations $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$ that translate between polar and rectangular coordinates when the pole is the origin and the polar axis is the positive x -axis.

polar equation

An equation in r and θ whose graph is the set of all points having at least one polar name that satisfies it.

CORE IDEAS

Put the pole at the origin and the polar axis along the positive x -axis. A point P with polar name (r, θ) , $r > 0$, lies on the terminal side of θ at distance r from the origin. By the definition of sine and cosine for a point at distance r (Lesson 3.6), $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$. Multiply through by r : $x = r \cos \theta$ and $y = r \sin \theta$. These also hold when $r < 0$, because replacing (r, θ) by $(-r, \theta + \pi)$ changes the sign of both r and the cosine (and sine), so the products are unchanged.

Polar to rectangular is direct: substitute into $x = r \cos \theta$ and $y = r \sin \theta$. **Rectangular to polar** takes more care. The distance is $r = \sqrt{x^2 + y^2}$ (the Pythagorean theorem). The angle satisfies $\tan \theta = \frac{y}{x}$, but tangent repeats every 180° , so $\tan^{-1} \frac{y}{x}$ only gives an angle in Quadrant I or IV. **Always look at the signs of x and y first.** If the point is in Quadrant II or III, add 180° (or π) to the calculator's answer. If it is in Quadrant IV and you need $0^\circ \leq \theta < 360^\circ$, add 360° (or 2π). If $x = 0$, use $\theta = 90^\circ$ or 270° directly.

Converting equations. To go from polar to rectangular, try to create the groupings $r \cos \theta$, $r \sin \theta$ and r^2 , often by multiplying both sides by r . For example, $r = 6 \sin \theta$ becomes $r^2 = 6r \sin \theta$, which is $x^2 + y^2 = 6y$. Completing the square gives $x^2 + (y - 3)^2 = 9$, a circle of radius 3 centered at $(0, 3)$. (Multiplying by r adds the pole as a solution, but the pole is already on the graph, since $r = 0$ when $\theta = 0$.)

To go from rectangular to polar, substitute $x = r \cos \theta$ and $y = r \sin \theta$ and solve for r if you can. A vertical line $x = a$ becomes $r \cos \theta = a$, or $r = a \sec \theta$. A circle centered at the origin, $x^2 + y^2 = a^2$, becomes simply $r = a$. A line through the origin, $y = mx$,

becomes $\theta = \tan^{-1} m$. Each form has its strengths: circles centered at the pole and lines through the pole are simplest in polar form, while most other lines are simplest in rectangular form.

KEY FORMULAS

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x} \quad (x \neq 0)$$

$$\theta = \tan^{-1} \frac{y}{x} \text{ in Quadrant I (add } 360^\circ \text{ in Quadrant IV for } 0^\circ \leq \theta < 360^\circ); \quad \theta = \tan^{-1} \frac{y}{x} + 180^\circ \text{ in Quadrants II and III}$$

From history. René Descartes' *La Géométrie* (1637) showed how to describe curves by equations relating two lengths, the beginning of what we now call rectangular or Cartesian coordinates. Polar coordinates came into regular use soon after, and the conversion formulas let mathematicians move a curve into whichever system makes it simpler.

Euclid connection. The formula $r^2 = x^2 + y^2$ is the Pythagorean theorem, Euclid I.47, applied to the right triangle with legs $|x|$ and $|y|$ and hypotenuse r . Euclid III.31 (the angle in a semicircle is a right angle) explains why $r = 2a \cos \theta$ is a circle: see practice problem 12.

WORKED EXAMPLES

Example 1. (a) Convert the polar point $\left(6, \frac{5\pi}{6}\right)$ to rectangular coordinates, exactly. (b) Convert the rectangular point $(-2, -2)$ to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$, exactly in radians.

Show solution

1. (a) $x = 6 \cos \frac{5\pi}{6} = 6 \left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$ and $y = 6 \sin \frac{5\pi}{6} = 6 \cdot \frac{1}{2} = 3$. The point is $(-3\sqrt{3}, 3) \approx (-5.196, 3)$, in Quadrant II as expected.

2. (b) $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$.

3. $\tan \theta = \frac{-2}{-2} = 1$. A calculator gives $\tan^{-1} 1 = \frac{\pi}{4}$, but that is Quadrant I. The point $(-2, -2)$ has both coordinates negative, so it is in Quadrant III. Add π :
 $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

4. Check: $2\sqrt{2} \cos \frac{5\pi}{4} = 2\sqrt{2} \left(-\frac{\sqrt{2}}{2}\right) = -2$, and likewise $y = -2$.

Answer: (a) $(-3\sqrt{3}, 3)$; (b) $\left(2\sqrt{2}, \frac{5\pi}{4}\right)$

Example 2. (a) Convert $r = 6 \sin \theta$ to rectangular form and identify the graph. (b) Convert the line $2x - 3y = 6$ to polar form, solved for r .

Show solution

1. (a) Multiply both sides by r to create familiar groupings: $r^2 = 6r \sin \theta$.
2. Substitute $r^2 = x^2 + y^2$ and $r \sin \theta = y$: $x^2 + y^2 = 6y$.
3. Complete the square in y : $x^2 + y^2 - 6y + 9 = 9$, so $x^2 + (y - 3)^2 = 9$. This is a circle with center $(0, 3)$ and radius 3. It passes through the pole, which is why multiplying by r added nothing new.
4. (b) Substitute $x = r \cos \theta$ and $y = r \sin \theta$: $2r \cos \theta - 3r \sin \theta = 6$.
5. Factor out r : $r(2 \cos \theta - 3 \sin \theta) = 6$, so $r = \frac{6}{2 \cos \theta - 3 \sin \theta}$.
6. Check one point: at $\theta = 0$, $r = 3$, the rectangular point $(3, 0)$, and indeed $2(3) - 3(0) = 6$.

Answer: (a) $x^2 + (y - 3)^2 = 9$, a circle with center $(0, 3)$ and radius 3; (b)

$$r = \frac{6}{2 \cos \theta - 3 \sin \theta}$$

PRACTICE

1. **Easy** Convert the polar point $\left(4, \frac{\pi}{3}\right)$ to rectangular coordinates, exactly.

Show answer

$$(2, 2\sqrt{3}) \approx (2, 3.464)$$

2. **Easy** Convert the rectangular point $(0, -5)$ to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$. Give θ exactly in radians.

Show answer

$$\left(5, \frac{3\pi}{2}\right)$$

3. **Easy** Convert each polar equation to rectangular form: (a) $r = 5$; (b) $\theta = \frac{\pi}{3}$.

Show answer

(a) $x^2 + y^2 = 25$, a circle of radius 5 centered at the origin; (b) $\frac{y}{x} = \tan \frac{\pi}{3}$, so $y = \sqrt{3}x$, a line through the origin (since r may be negative, the whole line is included).

4. **Medium** Convert the rectangular point $(-1, \sqrt{3})$ to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$. Give θ exactly in radians.

Show answer

$r = 2$; Quadrant II with reference angle $\frac{\pi}{3}$, so $\theta = \frac{2\pi}{3}$. The point is $\left(2, \frac{2\pi}{3}\right)$.

5. **Medium** Convert the rectangular point $(3, -4)$ to polar coordinates with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Round θ to the nearest tenth of a degree.

Show answer

$r = 5$. $\tan^{-1}\left(-\frac{4}{3}\right) \approx -53.1^\circ$, which is already in Quadrant IV; add 360° : $\theta \approx 306.9^\circ$. The point is about $(5, 306.9^\circ)$.

6. **Medium** Convert the rectangular point $(-5, -12)$ to polar coordinates with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Round θ to the nearest tenth of a degree.

Show answer

$r = 13$. $\tan^{-1}\frac{12}{5} \approx 67.4^\circ$ is in Quadrant I, but the point is in Quadrant III, so $\theta \approx 67.38^\circ + 180^\circ \approx 247.4^\circ$. The point is about $(13, 247.4^\circ)$.

7. **Medium** Samuel is surveying a field with an instrument set up at a corner post. He sights a second fence post 85.0 m away at an angle of 37.0° north of the east fence line. Taking the instrument as the origin, east as the positive x -axis and north as the positive y -axis, find how far east and how far north the second post is, each to the nearest tenth of a meter.

Show answer

$x = 85 \cos 37^\circ \approx 67.9$ m east; $y = 85 \sin 37^\circ \approx 51.2$ m north

8. **Hard** Convert $r = 4 \cos \theta$ to rectangular form. Show that the graph is a circle and give its center and radius.

Show answer

$r^2 = 4r \cos \theta$, so $x^2 + y^2 = 4x$. Then $x^2 - 4x + 4 + y^2 = 4$, so $(x - 2)^2 + y^2 = 4$: a circle with center $(2, 0)$ and radius 2.

9. **Hard** Write the parabola $y = x^2$ in polar form, solved for r .

Show answer

$$r \sin \theta = r^2 \cos^2 \theta. \text{ Dividing by } r \text{ (the pole is still included, at } \theta = 0\text{): } r = \frac{\sin \theta}{\cos^2 \theta} = \tan \theta \sec \theta.$$

10. **Hard** A survey drone lifts off from Mr. Miller's barn and hovers at a point 150 m west and 80 m south of the barn. Using the barn as the pole and due east as the polar axis, give the drone's polar coordinates with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Give r exactly and θ to the nearest tenth of a degree. (Ignore its height.)

Show answer

Rectangular point $(-150, -80)$. $r = \sqrt{22,500 + 6400} = \sqrt{28,900} = 170$ m.
 Quadrant III: $\theta = \tan^{-1} \frac{80}{150} + 180^\circ \approx 28.07^\circ + 180^\circ \approx 208.1^\circ$. So about $(170 \text{ m}, 208.1^\circ)$.

11. **Challenge** Convert $r = \frac{2}{1 - \cos \theta}$ to rectangular form and identify the curve.

Show answer

$r - r \cos \theta = 2$, so $r = 2 + x$. Square: $x^2 + y^2 = x^2 + 4x + 4$, so $y^2 = 4x + 4 = 4(x + 1)$. This is a parabola opening to the right with vertex $(-1, 0)$. (Squaring adds nothing new here: on the curve $r = 2 + x > 0$ always.)

12. **Challenge** Let $a > 0$, let O be the pole and $A = (2a, 0)$. Use Euclid III.31 (the angle in a semicircle is a right angle) to prove that every point P on the circle with diameter OA satisfies $r = 2a \cos \theta$, where (r, θ) is the polar name of P . Then confirm the result by converting $r = 2a \cos \theta$ to rectangular form.

Show answer

If $P \neq O$, A lies on the circle with diameter OA , then $\angle OPA = 90^\circ$ by III.31. In right triangle OPA , the hypotenuse is $OA = 2a$ and the angle at O is θ , so $OP = OA \cos \theta$, that is, $r = 2a \cos \theta$ (for points below the axis, θ is negative and $\cos \theta$ is unchanged). The endpoints check too: A is $\theta = 0$, $r = 2a$, and O is $\theta = \frac{\pi}{2}$, $r = 0$. Algebraically, $r^2 = 2ar \cos \theta$ gives $x^2 + y^2 = 2ax$, or $(x - a)^2 + y^2 = a^2$: center $(a, 0)$, radius a , the same circle.

COMMON MISTAKES

- Trusting the calculator's $\tan^{-1} \frac{y}{x}$ without checking the quadrant. For $(-5, -12)$ it gives 67.4° , a Quadrant I angle. Plot the point first and add 180° when $x < 0$.

- Writing $r = x^2 + y^2$ instead of $r^2 = x^2 + y^2$. The distance is the square root:
 $r = \sqrt{x^2 + y^2}$.
- Mixing degree and radian modes. If the angle is $\frac{5\pi}{6}$, your calculator must be in radian mode before computing $\cos \theta$.
- Trying to substitute for $\cos \theta$ alone when converting an equation. Look for the groupings $r \cos \theta$, $r \sin \theta$ and r^2 ; multiply both sides by r to create them.

Make it a game • Translation Desk. You are a translator at a border between polar and rectangular. Parent speaks a point in one language, such as (4, 60 degrees) or (-3, 3). Translate in 20 seconds using $x = r \cos \theta$, $y = r \sin \theta$ and $r^2 = x^2 + y^2$. Beware the quadrant trap: (-3, 3) is at 135 degrees. Score ten.

9.3 GRAPHING POLAR EQUATIONS

OBJECTIVES

- Graph polar equations by making a table of values and plotting points, including points with negative r .
- Use the three symmetry tests to cut the work of graphing in half or in quarters.
- Recognize and sketch the standard polar curves: circles, rose curves, cardioids and limaçons, predicting the number and length of petals and the shape of a limaçon from its equation.

WATCH

How to Graph Basic Polar Equations (Precalculus - Trigonometry 41)

Professor Leonard · 24:48

<https://www.youtube.com/watch?v=d-KXFtkLZVA>

WHEN TO WATCH

Before Example 1.

BACKUP VIDEO

How to Graph Advanced Polar Equations with Symmetry (Precalculus - Trigonometry 42)

Professor Leonard · 46:38

<https://www.youtube.com/watch?v=I9BKbq2MVfg>

Before Example 2: symmetry tests and roses (long).

KEY TERMS

rose curve

The graph of $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$ for an integer $n \geq 2$; it has n petals when n is odd and $2n$ petals when n is even, each of length $|a|$.

cardioid

A heart-shaped curve with equation $r = a(1 \pm \cos \theta)$ or $r = a(1 \pm \sin \theta)$; it passes through the pole with a point (cusp) there.

limaçon

The graph of $r = a + b \cos \theta$ or $r = a + b \sin \theta$ ($a, b > 0$); it has an inner loop if $a < b$, is a cardioid if $a = b$, has a dimple if $b < a < 2b$, and is convex if $a \geq 2b$. The name comes from the French word for snail.

symmetry test

A substitution that, if it produces an equivalent equation, guarantees that the graph has a certain symmetry.

CORE IDEAS

Plotting from a table. Choose angles that make the trig function easy, such as multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Compute r for each, plot each point (r, θ) (remember: negative r goes on the opposite ray), and join them in order of increasing θ . Think of a lighthouse beam sweeping around while a bead slides in and out along it; the bead traces the curve. Pay special attention to the angles where $r = 0$ (the curve passes through the pole) and where $|r|$ is largest (the curve is farthest out).

Symmetry tests. (1) **Polar axis:** if replacing θ by $-\theta$ gives an equivalent equation, the graph is symmetric about the polar axis, because $(r, -\theta)$ is the mirror image of (r, θ) in that axis. Any equation in $\cos \theta$ alone passes, since cosine is even. (2) **The line $\theta = \frac{\pi}{2}$:** replace θ by $\pi - \theta$; equations in $\sin \theta$ alone pass, since $\sin(\pi - \theta) = \sin \theta$. (3)

The pole: replace r by $-r$. These tests are sufficient but not necessary: a graph can be symmetric even when a test fails, because each point has many polar names (see practice problem 12).

Circles and roses. $r = a$ is a circle of radius $|a|$ about the pole. $r = 2a \cos \theta$ and $r = 2a \sin \theta$ are circles of diameter $|2a|$ passing through the pole (Lesson 9.2). For the rose $r = a \cos(n\theta)$, $|r|$ reaches its largest value $|a|$ when $n\theta$ is a multiple of π , giving petal tips at $\theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots$. If n is even there are $2n$ distinct petals. If n is odd, the second half-turn retraces the first, so there are only n petals. The sine version is the same rose rotated.

Cardioids and limaçons. In $r = a + b \cos \theta$ with $a, b > 0$, the distance swings between $a + b$ (at $\theta = 0$) and $a - b$ (at $\theta = \pi$). If $a < b$, then r becomes negative for some angles and the curve makes an **inner loop** through the pole. If $a = b$, then r just touches 0 at $\theta = \pi$: a **cardioid** with a cusp at the pole. If $a > b$, the curve never reaches the pole; it has a **dimple** when $a < 2b$ and is **convex** (egg-shaped) when $a \geq 2b$. With $\sin \theta$ in place of $\cos \theta$, the same shapes point up or down instead of left or right.

KEY FORMULAS

$r = a$ (circle about the pole), $r = 2a \cos \theta$, $r = 2a \sin \theta$ (circles through the pole)

$r = a \cos(n\theta)$, $r = a \sin(n\theta)$: n petals if n odd, $2n$ petals if n even, each of length $|a|$

$r = a + b \cos \theta$ ($a, b > 0$) : $a < b$ inner loop; $a = b$ cardioid; $b < a < 2b$ dimpled; $a \geq 2b$ convex

Symmetry: $\theta \rightarrow -\theta$ (polar axis); $\theta \rightarrow \pi - \theta$ ($\theta = \frac{\pi}{2}$); $r \rightarrow -r$ (pole)

From history. Jakob Bernoulli so admired the logarithmic spiral, whose polar equation is $r = ae^{b\theta}$, that he asked for it to be carved on his tombstone in Basel; the stonemason carved an Archimedean spiral instead. The Italian mathematician Guido Grandi studied the rose curves in the early eighteenth century and named them rhodonea, from the Greek word for rose.

WORKED EXAMPLES

Example 1. Graph the cardioid $r = 2 + 2 \cos \theta$. Find its largest distance from the pole and the points where it crosses the line $\theta = \frac{\pi}{2}$ (the y -axis), in rectangular coordinates.

Show solution

1. Symmetry: replacing θ by $-\theta$ gives $2 + 2 \cos(-\theta) = 2 + 2 \cos \theta$, the same equation. So the graph is symmetric about the polar axis, and we only need $0 \leq \theta \leq \pi$, then reflect.
2. Table: $\theta = 0: r = 4$. $\theta = \frac{\pi}{3}: r = 2 + 1 = 3$. $\theta = \frac{\pi}{2}: r = 2$. $\theta = \frac{2\pi}{3}: r = 2 - 1 = 1$.
 $\theta = \pi: r = 0$.
3. Plot these points and join them smoothly: the curve starts at $(4, 0)$ on the polar axis, sweeps up through $(0, 2)$, and comes into the pole at $\theta = \pi$. Reflect across the polar axis for the lower half. The pointed cusp at the pole and the round bulge on the right give the heart shape.
4. Largest distance: $\cos \theta \leq 1$, so $r \leq 4$, reached at $\theta = 0$.
5. Crossings of the y -axis: at $\theta = \frac{\pi}{2}$, $r = 2$, the point $(0, 2)$; at $\theta = \frac{3\pi}{2}$, $r = 2$, the point $(0, -2)$. (The curve also touches the y -axis at the pole.)

Answer: A cardioid symmetric about the polar axis with its cusp at the pole; largest distance 4 (at $\theta = 0$); it crosses the y -axis at $(0, 2)$ and $(0, -2)$ and also passes through the pole.

Example 2. Graph the rose $r = 3 \cos 2\theta$. How many petals does it have, how long is each, and in which directions do the petal tips point?

Show solution

1. Symmetry: $\cos(2(-\theta)) = \cos 2\theta$, so the graph is symmetric about the polar axis. Also $\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos 2\theta$, so it is symmetric about $\theta = \frac{\pi}{2}$. With both, it is symmetric about the pole too.
2. Table for $0 \leq \theta \leq \frac{\pi}{2}$: $\theta = 0: r = 3$. $\theta = \frac{\pi}{12}: r = 3 \cos \frac{\pi}{6} = \frac{3\sqrt{3}}{2} \approx 2.60$. $\theta = \frac{\pi}{6}: r = 1.5$. $\theta = \frac{\pi}{4}: r = 0$. $\theta = \frac{\pi}{3}: r = -1.5$. $\theta = \frac{\pi}{2}: r = -3$.
3. From $\theta = 0$ to $\frac{\pi}{4}$ we trace the upper half of a petal along the polar axis, ending at the pole. From $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r is negative, so the points land on the opposite rays, between $\frac{5\pi}{4}$ and $\frac{3\pi}{2}$. At $\theta = \frac{\pi}{2}$, $r = -3$ is the point 3 units straight down: the tip of the bottom petal.
4. Continuing (or using the symmetries) gives four petals of length 3 with tips at the rectangular points $(3, 0)$, $(0, -3)$, $(-3, 0)$ and $(0, 3)$. This agrees with the rule: $n = 2$ is even, so there are $2n = 4$ petals of length $|a| = 3$.

Answer: Four petals, each of length 3, with tips pointing along the positive and negative x - and y -axes, at $(3, 0)$, $(0, 3)$, $(-3, 0)$ and $(0, -3)$.

PRACTICE

1. Easy Complete the table for $r = 4 \cos \theta$ at $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, giving exact values.

Show answer

$$r = 4, 2\sqrt{3} \approx 3.464, 2, 0$$

2. Easy Describe the graph of $r = 5 \sin \theta$: give the center (in rectangular coordinates) and radius.

Show answer

A circle through the pole with diameter 5 along the y -axis: center $(0, 2.5)$, radius 2.5. (From $x^2 + y^2 = 5y$.)

3. **Easy** Give the number of petals and the petal length for (a) $r = 4 \sin 3\theta$; (b) $r = 2 \cos 4\theta$.

Show answer

(a) 3 petals ($n = 3$ is odd), each of length 4; (b) 8 petals ($n = 4$ is even), each of length 2.

4. **Medium** Ruth lays out a four-leaf flower bed whose border follows $r = 6 \cos 2\theta$, with r in feet and the pole at the center of the bed. How far from the center does each leaf reach, and what is the distance from the tip of one leaf to the tip of the opposite leaf?

Show answer

Each leaf reaches 6 ft from the center; opposite tips are 12 ft apart.

5. **Medium** Test $r = 1 + 2 \sin \theta$ for symmetry about the line $\theta = \frac{\pi}{2}$, and name the type of curve.

Show answer

Replace θ by $\pi - \theta$: $1 + 2 \sin(\pi - \theta) = 1 + 2 \sin \theta$, the same equation, so the graph is symmetric about $\theta = \frac{\pi}{2}$. Since $a = 1 < b = 2$, it is a limaçon with an inner loop.

6. **Medium** For the cardioid $r = 3 - 3 \cos \theta$, find r at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. On which side of the pole is the cusp, in which direction does the curve bulge, and how far from the pole does the curve reach?

Show answer

$r = 0, 3, 6, 3$. The cusp is at the pole, on the right side of the curve (it is reached at $\theta = 0$), and the curve bulges to the left, reaching 6 units from the pole at $\theta = \pi$.

7. **Medium** Classify each limaçon as having an inner loop, being a cardioid, dimpled, or convex: (a) $r = 3 + 2 \cos \theta$; (b) $r = 5 + 2 \cos \theta$; (c) $r = 2 + 4 \sin \theta$; (d) $r = 4 - 4 \sin \theta$.

Show answer

(a) dimpled ($2 < 3 < 4$); (b) convex ($5 \geq 4$); (c) inner loop ($2 < 4$); (d) cardioid ($a = b = 4$).

8. **Hard** The limaçon $r = 1 + 2 \cos \theta$ passes through the pole. Find all θ in $[0, 2\pi)$ where $r = 0$, exactly in radians, and say for which θ the inner loop is traced.

Show answer

$\cos \theta = -\frac{1}{2}$, so $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Between these angles $r < 0$, and the inner loop is traced.

9. **Hard** Find the points where the circles $r = 2 \sin \theta$ and $r = 2 \cos \theta$ intersect. Give the intersection away from the pole in polar form with $0 \leq \theta < \pi$, exactly.

Show answer

Setting $2 \sin \theta = 2 \cos \theta$ gives $\tan \theta = 1$, $\theta = \frac{\pi}{4}$, $r = \sqrt{2}$: the point $(\sqrt{2}, \frac{\pi}{4})$, which is $(1, 1)$ in rectangular form. The circles also meet at the pole, which the first reaches at $\theta = 0$ and the second at $\theta = \frac{\pi}{2}$, so solving the equations alone misses it.

10. **Hard** A cardioid microphone, often used on a church pulpit or a stage, picks up sound with a sensitivity pattern $r = 1 + \cos \theta$, where θ is measured from the direction the microphone faces. Its greatest sensitivity is straight ahead. At what angles θ in $[0^\circ, 360^\circ)$ is the sensitivity exactly half the greatest sensitivity? Give angles in degrees.

Show answer

The greatest value is $r = 2$ at $\theta = 0^\circ$. Half of that is 1: $1 + \cos \theta = 1$, so $\cos \theta = 0$ and $\theta = 90^\circ$ or 270° (directly to either side).

11. **Challenge** Explain why the rose $r = a \cos(n\theta)$ has only n petals when n is odd, but $2n$ petals when n is even. (Hint: compare the point at $\theta + \pi$ with the point at θ .)

Show answer

At $\theta + \pi$, $r = a \cos(n\theta + n\pi)$. If n is odd, $\cos(n\theta + n\pi) = -\cos(n\theta)$, so the point is $(-r, \theta + \pi)$, which is the same point as (r, θ) . Thus the second half-turn retraces the first, and the $2n$ half-petal sweeps of a full turn only make n distinct petals. If n is even, $\cos(n\theta + n\pi) = \cos(n\theta)$, so the point is $(r, \theta + \pi)$, the reflection of (r, θ) through the pole, a new point. No retracing occurs and all $2n$ petals are distinct.

12. **Challenge** Show that replacing θ by $-\theta$ in $r = \sin 2\theta$ does **not** give an equivalent equation, yet the graph is still symmetric about the polar axis. (Hint: the point $(-r, \pi - \theta)$ is also the mirror image of (r, θ) in the polar axis.)

Show answer

Replacing θ by $-\theta$ gives $r = \sin(-2\theta) = -\sin 2\theta$, a different equation, so the test fails. But $(-r, \pi - \theta)$ is the mirror image of (r, θ) : it is the point $(r, -\theta)$ renamed. Substituting it: $-r = \sin(2\pi - 2\theta) = -\sin 2\theta$, so $r = \sin 2\theta$, the original equation. Hence the mirror image of every point on the graph is also on the graph. The tests are sufficient, not necessary, because each point has many polar names.

COMMON MISTAKES

- Plotting negative values of r on the terminal side of θ . When $r < 0$, plot the point on the opposite ray; this is what creates the inner loop of a limaçon and half the petals of a rose with even n .
- Counting $2n$ petals for every rose. When n is odd, the curve retraces itself and has only n petals. Check by sketching one full turn.
- Concluding that a graph has no symmetry because a test failed. A failed test proves nothing; check a few points or try the alternate test with $(-r, \pi - \theta)$.
- Finding intersections only by setting the two equations equal. That misses the pole (and sometimes other points) when the curves pass through it at different angles. Always check the pole separately and look at a sketch.

Make it a game • Predict the Petals. Parent writes a polar equation, such as $r = 2 \cos 3\theta$ or $r = 1 + \cos \theta$. Before plotting, predict the shape and the number of petals, and the symmetry test it passes. Then plot points every 30 degrees on polar paper to check. One point per correct prediction; beat your best score.

9.4 VECTORS: MAGNITUDE, DIRECTION AND COMPONENTS

OBJECTIVES

- Resolve a vector given by magnitude and direction into horizontal and vertical components using $\cos \theta$ and $\sin \theta$.
- Find the magnitude and direction angle of a vector from its components, placing the angle in the correct quadrant.
- Use components to work with vectors between two points, scalar multiples, unit vectors, and practical quantities such as forces on a ramp and an aircraft's velocity.

WATCH

Projectile Motion: Simple Worked Example (1 of 4: Resolving Initial Velocity).

Eddie Woo · 10:41

<https://www.youtube.com/watch?v=poCBM0P6u3s>

WHEN TO WATCH

Before Example 1: splitting a velocity into $V \cos \theta$ and $V \sin \theta$.

Note: Woo's context is a projectile, not a ramp. 3Blue1Brown is conceptual and has no trig.

BACKUP VIDEO

Vectors | Chapter 1, Essence of linear algebra

3Blue1Brown · 9:52

https://www.youtube.com/watch?v=fNk_zzaMoSs

Before the Core ideas, as a picture of what a vector and its components are.

KEY TERMS

vector

A quantity with both size and direction, such as a force, a velocity or a displacement, drawn as an arrow. Two arrows with the same length and the same direction represent the same vector, wherever they are drawn.

scalar

A quantity with size only, such as a temperature, a mass or a speed; a plain real number.

magnitude

The length of a vector $\mathbf{v}$, written $\|\mathbf{v}\|$.
 For a force it is the strength in pounds or newtons; for a velocity it is the speed.

direction angle

The angle θ a vector makes with the positive x -axis, measured counterclockwise, usually with $0^\circ \leq \theta < 360^\circ$.

components

The horizontal and vertical parts a and b of a vector, written $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$.

unit vector

A vector of magnitude 1. The standard unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

CORE IDEAS

Some quantities are fully described by a number: the temperature is 70°F . Others need a direction too: a wind of 20 mph **from the west** is not the same as a wind of 20 mph from the north. Such quantities are **vectors**. We draw a vector as an arrow from an initial point to a terminal point. Only the length and direction matter, so we may slide the arrow anywhere without changing it. Placing its tail at the origin is the most convenient choice, and then the vector is described by the coordinates of its tip, $\langle a, b \rangle$.

Resolving into components. If $\mathbf{v}$ has magnitude $\|\mathbf{v}\|$ and direction angle θ , its tip sits at distance $\|\mathbf{v}\|$ from the origin on the terminal side of θ . This is just a point in polar coordinates, so by Lesson 9.2, $a = \|\mathbf{v}\| \cos \theta$ and $b = \|\mathbf{v}\| \sin \theta$. The signs take care of themselves: a vector pointing into Quadrant II automatically gets a negative horizontal component.

Magnitude and direction from components. Going back, $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ by the Pythagorean theorem, and $\tan \theta = \frac{b}{a}$. As with polar coordinates, sketch the vector first: if it points into Quadrant II or III, add 180° to the calculator's $\tan^{-1} \frac{b}{a}$; if into Quadrant IV, add 360° to get an angle in $[0^\circ, 360^\circ)$. The vector from $P(x_1, y_1)$ to $Q(x_2, y_2)$ is $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

Scalar multiples and unit vectors. Multiplying by a scalar k multiplies each component: $k\langle a, b \rangle = \langle ka, kb \rangle$. Then $\|k\mathbf{v}\| = \sqrt{k^2a^2 + k^2b^2} = |k| \|\mathbf{v}\|$, so $k\mathbf{v}$ is $|k|$ times as long, pointing the same way if $k > 0$ and the opposite way if $k < 0$. Dividing a

nonzero vector by its own magnitude gives the unit vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ in its direction, which equals $\langle \cos \theta, \sin \theta \rangle$, a point on the unit circle.

Forces on a ramp. A load of weight W resting on a ramp inclined at angle α is pulled straight down. Resolve the weight not along x and y but along the ramp and perpendicular to it. The angle between the weight vector and the perpendicular to the ramp equals α (their sides are perpendicular to the ramp and to the ground, respectively), so the component pulling the load **down the ramp** is $W \sin \alpha$ and the component **pressing into the ramp** is $W \cos \alpha$. Without friction, a force of $W \sin \alpha$ up the ramp holds the load still. A gentle ramp (α small) needs only a small force; that is why ramps make work easier.

KEY FORMULAS

$$\mathbf{v} = \langle a, b \rangle = \langle \|\mathbf{v}\| \cos \theta, \|\mathbf{v}\| \sin \theta \rangle = a\mathbf{i} + b\mathbf{j}$$

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}, \quad \tan \theta = \frac{b}{a}$$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

$$k\langle a, b \rangle = \langle ka, kb \rangle, \quad \|k\mathbf{v}\| = |k| \|\mathbf{v}\|, \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$\text{On a ramp of angle } \alpha : \quad W_{\parallel} = W \sin \alpha, \quad W_{\perp} = W \cos \alpha$$

From history. The Flemish engineer Simon Stevin studied the balance of weights on inclined planes in his book on statics, published in 1586. Using his famous picture of a loop of balls draped over a triangular prism, he showed why the force needed to hold a load on a slope depends on the slope, and he is often credited with an early form of the rule that combines forces by the triangle or parallelogram. The modern notation of vectors with components came much later, in the nineteenth century.

Euclid connection. Euclid I.33 proves that the straight lines joining the ends of equal and parallel straight lines (on the same side) are themselves equal and parallel, and I.34 proves that the opposite sides of a parallelogram are equal. This is why an arrow may be slid parallel to itself without changing the vector: the original arrow, the moved arrow and the two segments joining their ends form a parallelogram, so the arrows have the same length and direction.

WORKED EXAMPLES

Example 1. (a) A vector $\mathbf{v}$ has magnitude 10 and direction angle 150° . Find its components exactly. (b) Find the magnitude and direction angle of $\mathbf{w} = \langle -7, 24 \rangle$, with the angle to the nearest tenth of a degree.

Show solution

1. (a) $a = 10 \cos 150^\circ = 10 \left(-\frac{\sqrt{3}}{2} \right) = -5\sqrt{3} \approx -8.660$ and $b = 10 \sin 150^\circ = 10 \cdot$

$\frac{1}{2} = 5$. So $\mathbf{v} = \langle -5\sqrt{3}, 5 \rangle$. The horizontal component is negative, as it should be for a vector pointing into Quadrant II.

2. (b) Magnitude: $\|\mathbf{w}\| = \sqrt{(-7)^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$.

3. Direction: $\mathbf{w}$ points left and up, into Quadrant II. The calculator gives

$\tan^{-1} \frac{24}{-7} \approx -73.74^\circ$, a Quadrant IV angle, which is wrong. Add 180° : $\theta \approx 106.26^\circ$, which rounds to 106.3° .

4. Check: $25 \cos 106.26^\circ \approx -7.00$ and $25 \sin 106.26^\circ \approx 24.00$.

Answer: (a) $\mathbf{v} = \langle -5\sqrt{3}, 5 \rangle \approx \langle -8.660, 5 \rangle$; (b) $\|\mathbf{w}\| = 25$, $\theta \approx 106.3^\circ$

Example 2. A barrel of cider weighing 300 lb rests on a loading ramp inclined at 20° to the ground. Ignoring friction, find the component of its weight that pulls it down the ramp (the force needed to hold it in place) and the component pressing it against the ramp. Round to the nearest pound.

Show solution

1. The weight is a vector of magnitude 300 lb pointing straight down. Resolve it along the ramp and perpendicular to the ramp.
2. The angle between the weight and the perpendicular to the ramp equals the ramp angle, 20° , because the weight is perpendicular to the ground and the perpendicular to the ramp is perpendicular to the ramp; turning both lines through 90° gives back the ground and the ramp, which meet at 20° .
3. Down the ramp: $W \sin 20^\circ = 300 \times 0.342020 \approx 102.61$ lb, which rounds to 103 lb.
4. Into the ramp: $W \cos 20^\circ = 300 \times 0.939693 \approx 281.91$ lb, which rounds to 282 lb.
5. Check with the Pythagorean theorem: $102.606^2 + 281.908^2 \approx 10,528 + 79,472 = 90,000 = 300^2$. A force of only about 103 lb up the ramp holds a 300 lb barrel.

Answer: About 103 lb down the ramp and about 282 lb into the ramp.

PRACTICE

1. Easy Find the magnitude of $\mathbf{v} = \langle 6, 8 \rangle$.

Show answer

$$\|\mathbf{v}\| = \sqrt{36 + 64} = 10$$

2. Easy A vector has magnitude 12 and direction angle 60° . Find its components exactly.

Show answer

$$\langle 12 \cos 60^\circ, 12 \sin 60^\circ \rangle = \langle 6, 6\sqrt{3} \rangle \approx \langle 6, 10.392 \rangle$$

3. **Easy** Find the component form and the magnitude of $\overrightarrow{PQ}$ for $P(2, -1)$ and $Q(7, 11)$.

Show answer

$$\overrightarrow{PQ} = \langle 5, 12 \rangle; \|\overrightarrow{PQ}\| = 13$$

4. **Medium** Find the magnitude and direction angle of $\langle 4, -3 \rangle$, with $0^\circ \leq \theta < 360^\circ$, to the nearest tenth of a degree.

Show answer

Magnitude 5. Quadrant IV: $\tan^{-1}\left(-\frac{3}{4}\right) \approx -36.9^\circ$, so $\theta \approx 323.1^\circ$.

5. **Medium** A vector has magnitude 20 and direction angle 215° . Find its components to the nearest hundredth.

Show answer

$$\langle 20 \cos 215^\circ, 20 \sin 215^\circ \rangle \approx \langle -16.38, -11.47 \rangle$$

6. **Medium** An airplane flies at 400 mph on a true bearing of 120° . With east as the positive x -axis and north as the positive y -axis, find the east and north components of its velocity exactly. (Hint: a bearing is measured clockwise from north, so the direction angle is $90^\circ - 120^\circ = -30^\circ$.)

Show answer

East: $400 \cos(-30^\circ) = 200\sqrt{3} \approx 346.4$ mph; north: $400 \sin(-30^\circ) = -200$ mph (that is, 200 mph southward).

7. **Medium** Find the unit vector in the direction of $\mathbf{v} = \langle 5, -12 \rangle$. Give exact components.

Show answer

$$\|\mathbf{v}\| = 13, \text{ so } \mathbf{u} = \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \approx \langle 0.3846, -0.9231 \rangle$$

8. **Hard** Samuel and his father load a 150-lb crate of tools into a trailer up a ramp inclined at 25° . Ignoring friction, what force parallel to the ramp is needed to keep the crate from sliding back, and what is the component of the weight pressing against the ramp? Round each to the nearest tenth of a pound.

Show answer

Parallel: $150 \sin 25^\circ \approx 63.4$ lb; perpendicular: $150 \cos 25^\circ \approx 135.9$ lb

9. **Hard** Let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$. Find $3\mathbf{u} - 2\mathbf{v}$, its magnitude, and its direction angle in $[0^\circ, 360^\circ)$ to the nearest tenth of a degree.

Show answer

$3\mathbf{u} - 2\mathbf{v} = \langle 9, -6 \rangle - \langle -6, 2 \rangle = \langle 15, -8 \rangle$. Magnitude $\sqrt{225 + 64} = 17$. Quadrant IV: $\theta = 360^\circ + \tan^{-1}\left(-\frac{8}{15}\right) \approx 360^\circ - 28.07^\circ \approx 331.9^\circ$.

10. **Hard** A vector $\mathbf{v}$ has magnitude 9, a vertical component of -4.5 , and a negative horizontal component. Find its direction angle in $[0^\circ, 360^\circ)$ and its exact horizontal component.

Show answer

$\sin \theta = \frac{-4.5}{9} = -\frac{1}{2}$ with $\cos \theta < 0$, so θ is in Quadrant III: $\theta = 210^\circ$. Horizontal component $9 \cos 210^\circ = -\frac{9\sqrt{3}}{2} \approx -7.794$.

11. **Challenge** Prove that $\|k\mathbf{v}\| = |k| \|\mathbf{v}\|$ for every scalar k and vector $\mathbf{v} = \langle a, b \rangle$. Then explain why $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector whenever $\mathbf{v} \neq \mathbf{0}$.

Show answer

$\|k\mathbf{v}\| = \sqrt{(ka)^2 + (kb)^2} = \sqrt{k^2(a^2 + b^2)} = \sqrt{k^2} \sqrt{a^2 + b^2} = |k| \|\mathbf{v}\|$, since $\sqrt{k^2} = |k|$. Taking $k = \frac{1}{\|\mathbf{v}\|}$ (positive, and defined because $\mathbf{v} \neq \mathbf{0}$) gives

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{1}{\|\mathbf{v}\|} \cdot \|\mathbf{v}\| = 1.$$

12. **Challenge** Ruth pulls a sled of firewood with a rope held at an angle θ above the horizontal. The tension in the rope is 40 lb. For the sled to move, the horizontal component of the pull must be at least 30 lb. What is the largest angle θ she can use? Give the answer in degrees, to the nearest tenth.

Show answer

$40 \cos \theta \geq 30$, so $\cos \theta \geq 0.75$ and $\theta \leq \cos^{-1} 0.75 \approx 41.4^\circ$.

COMMON MISTAKES

- Accepting the calculator's $\tan^{-1} \frac{b}{a}$ as the direction angle in every case. Sketch the vector; if $a < 0$, add 180° .
- Swapping sine and cosine. With the direction angle measured from the positive x -axis, the horizontal component uses cosine and the vertical uses sine. On a ramp,

the component along the slope uses $\sin \alpha$; check that it becomes 0 when the ramp is flat.

- Using a bearing as if it were a direction angle. Bearings are measured clockwise from north; direction angles counterclockwise from east. Convert with $\theta = 90^\circ - \beta$, then adjust by 360° if needed.
- Subtracting in the wrong order for $\overrightarrow{PQ}$. It is terminal minus initial: $Q - P$, not $P - Q$.

Make it a game • Walk the Components. Mark east and north on the floor. Parent gives a vector: magnitude 10 steps at 30 degrees north of east. Compute the horizontal and vertical components, then walk the east leg and the north leg and mark your spot. Walk straight to check. Then reverse: given components, find magnitude and direction angle.

9.5 VECTOR APPLICATIONS

OBJECTIVES

- Add vectors by components or by the parallelogram law to find the resultant of several forces or velocities.
- Solve practical problems with resultants: forces on a load, a boat crossing a river with a current, and an airplane flying in a wind.
- Prove that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ and use the dot product to find the angle between two vectors and to test for perpendicularity.

WATCH

Concurrent Forces - Non-Equilibrium (3 of 3: Finding magnitude & direction)

Eddie Woo · 6:47

<https://www.youtube.com/watch?v=zuKqtuWJz8I>

WHEN TO WATCH

Before Example 1: magnitude and direction of a resultant force.

Note: Woo's dot-product lesson is framed as extending to 3D; the 2D formula is reviewed at the start. Neither video covers the boat or plane problems.

BACKUP VIDEO

Dot Product (1 of 2: Geometric interpretation)

Eddie Woo · 15:14

<https://www.youtube.com/watch?v=UqpY07EiwCU>

Before the Core ideas on the dot product: what $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ means.

KEY TERMS

resultant

The single vector equal to the sum of two or more vectors; for forces, the one force that has the same effect as all of them together.

parallelogram law

To add $\mathbf{u}$ and $\mathbf{v}$ drawn from a common point, complete the parallelogram; the diagonal from that point is $\mathbf{u} + \mathbf{v}$. Equivalently, place the tail of $\mathbf{v}$ at the tip of $\mathbf{u}$ (tip-to-tail).

equilibrium

The state of an object at rest (or moving steadily) when the resultant of all forces on it is the zero vector.

orthogonal

Perpendicular; nonzero vectors are orthogonal exactly when their dot product is 0.

dot product

For $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$, the number $\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$.

heading and course

The heading is the direction a boat or plane is pointed; the course (or track) is the direction it actually travels over the ground once the current or wind is added.

CORE IDEAS

Adding vectors. Geometrically, place $\mathbf{v}$ with its tail at the tip of $\mathbf{u}$; the arrow from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$ is $\mathbf{u} + \mathbf{v}$. Drawing both from one point and completing the parallelogram gives the same diagonal. In components the rule is simple: $\langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 + a_2, b_1 + b_2 \rangle$. So the standard method is: resolve every vector into components (Lesson 9.4), add the components, then find the magnitude and direction of the sum.

Two vectors and the Law of Cosines. When only two vectors are added, the triangle method is often quicker. If the angle between $\mathbf{u}$ and $\mathbf{v}$ (tails together) is θ , then in the tip-to-tail triangle the angle opposite the resultant is $180^\circ - \theta$, so $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. The Law of Sines then gives the direction of the resultant.

Boats and airplanes. A boat's velocity over the ground is its velocity through the water plus the velocity of the current. An airplane's ground velocity is its air velocity (airspeed in the heading direction) plus the wind velocity. To travel straight across a river, the pilot must aim partly upstream so that the upstream part of the boat's velocity cancels the current. For a body at rest, such as a sign hanging from two cables, the forces are in **equilibrium**: their components add to zero both horizontally and vertically.

The dot product and angles. Let θ be the angle between nonzero vectors $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ drawn from the origin. The third side of the triangle they form is $\mathbf{u} - \mathbf{v}$. By the Law of Cosines, $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. In components the left side is $(a_1 - a_2)^2 + (b_1 - b_2)^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(a_1 a_2 + b_1 b_2)$.

Comparing the two expressions: $a_1a_2 + b_1b_2 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. So $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$, and since $\cos 90^\circ = 0$, two nonzero vectors are perpendicular exactly when $\mathbf{u} \cdot \mathbf{v} = 0$.

KEY FORMULAS

$$\langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 + a_2, b_1 + b_2 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (\theta = \text{angle between } \mathbf{u} \text{ and } \mathbf{v})$$

$$\mathbf{u} \cdot \mathbf{v} = a_1a_2 + b_1b_2 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}, \quad \mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0$$

$$\text{Equilibrium: } \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}$$

From history. Navigators solved current-and-wind problems long before vectors had a name, by drawing triangles on charts: the ship's heading and speed on one side, the drift of the current on another, and the actual track as the third. The dot product, with its link to the cosine of the angle between two directions, came out of the nineteenth-century work that turned these drawings into an algebra of vectors.

Euclid connection. The parallelogram law rests on Euclid I.34: the opposite sides of a parallelogram are equal, so the side drawn from the tip of $\mathbf{u}$ is a copy of $\mathbf{v}$, and tip-to-tail addition and the parallelogram diagonal agree. The dot-product formula is the Law of Cosines, whose geometric form is Euclid II.12–13.

WORKED EXAMPLES

Example 1. Two tractors pull a stump with chains. Using east as the positive x -axis, the first pulls with 600 lb in the direction 30° and the second with 800 lb in the direction 120° . (a) Use the dot product to show that the chains are perpendicular. (b) Find the magnitude and direction of the resultant force, with the direction to the nearest tenth of a degree.

Show solution

1. Components: $\mathbf{F}_1 = \langle 600 \cos 30^\circ, 600 \sin 30^\circ \rangle = \langle 300\sqrt{3}, 300 \rangle$ and $\mathbf{F}_2 = \langle 800 \cos 120^\circ, 800 \sin 120^\circ \rangle = \langle -400, 400\sqrt{3} \rangle$.
2. (a) $\mathbf{F}_1 \cdot \mathbf{F}_2 = (300\sqrt{3})(-400) + (300)(400\sqrt{3}) = -120,000\sqrt{3} + 120,000\sqrt{3} = 0$.
The chains are perpendicular, as the directions 30° and 120° suggest.
3. (b) Add: $\mathbf{R} = \langle 300\sqrt{3} - 400, 300 + 400\sqrt{3} \rangle \approx \langle 119.615, 992.820 \rangle$.
4. Magnitude: $\|\mathbf{R}\|^2 = (300\sqrt{3} - 400)^2 + (300 + 400\sqrt{3})^2 = (270,000 - 240,000\sqrt{3} + 160,000) + (90,000 + 240,000\sqrt{3} + 480,000) = 1,000,000$, so $\|\mathbf{R}\| = 1000$ lb. (Since the chains are perpendicular, this is just $\sqrt{600^2 + 800^2}$.)
5. Direction: $\mathbf{R}$ is in Quadrant I, so $\theta = \tan^{-1} \frac{992.820}{119.615} \approx 83.13^\circ$. Check: the resultant lies $\tan^{-1} \frac{800}{600} \approx 53.13^\circ$ beyond the first chain, and $30^\circ + 53.13^\circ = 83.13^\circ$.

Answer: (a) $\mathbf{F}_1 \cdot \mathbf{F}_2 = 0$, so the chains are perpendicular. (b) The resultant is 1000 lb in the direction $\approx 83.1^\circ$.

Example 2. A motorboat moves at 6 mph through still water. It must cross a river 0.5 mile wide whose current flows at 3 mph, and land directly opposite its starting point. At what angle upstream (from the line straight across) must the pilot aim, what is the boat's speed over the ground, and how long does the crossing take? Give the angle in degrees and the time to the nearest tenth of a minute.

Show solution

1. Let the line straight across be the y -axis and let the current flow in the positive x direction: current $\mathbf{c} = \langle 3, 0 \rangle$.
2. Aim the boat at angle α upstream of straight across: boat velocity through the water $\mathbf{b} = \langle -6 \sin \alpha, 6 \cos \alpha \rangle$.
3. To go straight across, the ground velocity $\mathbf{b} + \mathbf{c}$ must have x -component 0: $-6 \sin \alpha + 3 = 0$, so $\sin \alpha = \frac{1}{2}$ and $\alpha = 30^\circ$.
4. Ground speed: the y -component, $6 \cos 30^\circ = 3\sqrt{3} \approx 5.196$ mph. (Check with the right triangle: $\sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$.)
5. Time: $\frac{0.5}{3\sqrt{3}} = \frac{\sqrt{3}}{18} \approx 0.096225$ hour. Multiply by 60: $\frac{10}{\sqrt{3}} \approx 5.7735$ minutes, which rounds to 5.8 minutes.

Answer: Aim 30° upstream; ground speed $3\sqrt{3} \approx 5.20$ mph; the crossing takes about 5.8 minutes.

PRACTICE

1. Easy Let $\mathbf{u} = \langle 3, -1 \rangle$ and $\mathbf{v} = \langle -5, 4 \rangle$. Find $\mathbf{u} + \mathbf{v}$ and its magnitude, exactly.

Show answer

$$\mathbf{u} + \mathbf{v} = \langle -2, 3 \rangle; \|\mathbf{u} + \mathbf{v}\| = \sqrt{13} \approx 3.606$$

2. Easy Find $\langle 2, 5 \rangle \cdot \langle -3, 4 \rangle$.

Show answer

$$-6 + 20 = 14$$

3. Easy Are $\langle 6, -4 \rangle$ and $\langle 2, 3 \rangle$ perpendicular? Justify with the dot product.

Show answer

$$\text{Yes: } \langle 6, -4 \rangle \cdot \langle 2, 3 \rangle = 12 - 12 = 0.$$

4. **Medium** Find the angle between $\mathbf{u} = \langle 3, 1 \rangle$ and $\mathbf{v} = \langle 1, 2 \rangle$, exactly in degrees.

Show answer

$$\cos \theta = \frac{3 + 2}{\sqrt{10}\sqrt{5}} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}, \text{ so } \theta = 45^\circ.$$

5. **Medium** Find the angle between $\mathbf{u} = \langle 2, 0 \rangle$ and $\mathbf{v} = \langle -1, \sqrt{3} \rangle$, exactly in degrees.

Show answer

$$\cos \theta = \frac{-2}{2 \cdot 2} = -\frac{1}{2}, \text{ so } \theta = 120^\circ.$$

6. **Medium** Two of Mr. Miller's horses drag a stone sled. One pulls with 400 lb straight ahead and the other with 300 lb at right angles to the first. Find the magnitude of the resultant and the angle it makes with the 400-lb force, to the nearest tenth of a degree.

Show answer

$$\sqrt{400^2 + 300^2} = 500 \text{ lb, at } \tan^{-1} \frac{300}{400} \approx 36.9^\circ \text{ from the 400-lb force.}$$

7. **Medium** A small plane heads due north with an airspeed of 200 mph. A wind blows due east at 50 mph. Find the plane's ground speed to the nearest tenth of a mph and its course as a true bearing to the nearest tenth of a degree.

Show answer

Ground velocity $\langle 50, 200 \rangle$. Ground speed $\sqrt{42,500} = 50\sqrt{17} \approx 206.2$ mph.

Course: $\tan^{-1} \frac{50}{200} \approx 14.0^\circ$ east of north, a bearing of about 014.0°.

8. **Hard** A 200-lb sign hangs from two cables, each making an angle of 40° with the horizontal, one to the left and one to the right. Find the tension in each cable to the nearest tenth of a pound.

Show answer

By symmetry both tensions equal T . Horizontal parts cancel. Vertical:

$$2T \sin 40^\circ = 200, \text{ so } T = \frac{100}{\sin 40^\circ} \approx 155.6 \text{ lb.}$$

9. **Hard** Forces of 50 N and 30 N act on a bolt at an angle of 60° to each other. Find the magnitude of the resultant exactly, and the angle between the resultant and the 50-N force to the nearest tenth of a degree.

Show answer

$\|\mathbf{R}\|^2 = 2500 + 900 + 2(50)(30) \cos 60^\circ = 4900$, so $\|\mathbf{R}\| = 70$ N. Placing the 50-N force along the x -axis, $\mathbf{R} = \langle 65, 15\sqrt{3} \rangle$, so $\cos \alpha = \frac{65}{70} = \frac{13}{14}$ and $\alpha \approx 21.8^\circ$.

10. **Hard** Ruth rows a boat at 4 mph, aiming straight across a river 0.4 mile wide. The current flows at 3 mph. Find her speed over the ground, the angle between her actual path and the line straight across (to the nearest tenth of a degree), and how far downstream she lands.

Show answer

Ground speed $\sqrt{4^2 + 3^2} = 5$ mph. Angle $\tan^{-1} \frac{3}{4} \approx 36.9^\circ$ downstream of straight across. The crossing takes $\frac{0.4}{4} = 0.1$ h, during which the current carries her $3(0.1) = 0.3$ mile downstream.

11. **Challenge** Prove that for nonzero vectors $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ with angle θ between them, $a_1a_2 + b_1b_2 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. Then explain why $\mathbf{u} \cdot \mathbf{v} = 0$ means the vectors are perpendicular.

Show answer

Draw $\mathbf{u}$ and $\mathbf{v}$ from the origin; the segment joining their tips is $\mathbf{u} - \mathbf{v} = \langle a_1 - a_2, b_1 - b_2 \rangle$. The Law of Cosines gives $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. Expanding in components, $\|\mathbf{u} - \mathbf{v}\|^2 = a_1^2 - 2a_1a_2 + a_2^2 + b_1^2 - 2b_1b_2 + b_2^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(a_1a_2 + b_1b_2)$. Setting the two equal and cancelling gives $a_1a_2 + b_1b_2 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$. (If the vectors point the same or opposite ways there is no triangle, but the formula still checks directly with $\theta = 0^\circ$ or 180° .) Since the magnitudes are not zero, $\mathbf{u} \cdot \mathbf{v} = 0$ forces $\cos \theta = 0$, and for $0^\circ \leq \theta \leq 180^\circ$ that means $\theta = 90^\circ$.

12. **Challenge** A crane hook is held still by three ropes. Measuring directions from the positive x -axis, the first rope pulls with 40 lb at 0° and the second with 40 lb at 120° . Find the magnitude and direction angle of the third rope's pull, with the angle in $[0^\circ, 360^\circ)$.

Show answer

$\mathbf{F}_1 + \mathbf{F}_2 = \langle 40, 0 \rangle + \langle -20, 20\sqrt{3} \rangle = \langle 20, 20\sqrt{3} \rangle$. For equilibrium $\mathbf{F}_3 = \langle -20, -20\sqrt{3} \rangle$, with magnitude $\sqrt{400 + 1200} = 40$ lb and direction 240° (Quadrant III, reference angle 60°).

COMMON MISTAKES

- Adding magnitudes instead of vectors. Forces of 400 lb and 300 lb at right angles give 500 lb, not 700 lb. Add components, then find the magnitude.
- Using the minus sign of the Law of Cosines for a resultant. With θ the angle between the vectors placed tail to tail, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$, because the triangle's angle is $180^\circ - \theta$. Check: perpendicular vectors give the Pythagorean theorem.
- Confusing heading with course. The heading is where the boat or plane points; the course is where it actually goes after the current or wind is added. Read carefully which one the problem gives and which it asks for.
- Stopping at $\cos \theta$, or forgetting to divide by the magnitudes. The dot product is a number, not an angle: divide it by $\|\mathbf{u}\| \|\mathbf{v}\|$ to get $\cos \theta$, then apply $\cos^{-1}$, which always gives an angle between 0° and 180° . A negative dot product means the angle is obtuse.

Make it a game • Arrow Add. Cut paper arrows to scale for forces or velocities: 3 units east and 4 units north. Place them head to tail and predict the resultant's length and direction from components before measuring. Check with a ruler and protractor. Then plan a boat crossing a river with a current: heading, speed, and landing point.

POLAR COORDINATES AND VECTORS

This optional unit carries the trigonometry of the whole course into two new settings. Polar coordinates locate a point by a direction and a distance, the way a navigator, a radar operator or a surveyor actually sees the world. With them, circles through the pole, the petals of a rose and the heart-shaped cardioid have short, beautiful equations. Students will plot polar points, convert between polar and rectangular form, and graph the classic polar curves using tables and symmetry.

Vectors are quantities with both size and direction: forces, velocities and displacements. Resolving a vector into components is exactly the polar-to-rectangular conversion, so the two halves of the unit share one idea: $x = r \cos \theta$, $y = r \sin \theta$. Students will use components to add forces on a load, steer a boat across a current, find an airplane's true course in a wind, and measure the angle between two directions with the dot product, whose formula comes straight from the Law of Cosines.

The unit draws on the unit circle and exact values (Unit 3), inverse tangent (Unit 6), the Laws of Sines and Cosines (Unit 7) and bearings (Unit 8). It is a natural bridge to physics and to the vectors and polar curves of calculus.

PREREQUISITES

- Evaluate trigonometric functions of any angle in degrees or radians, including exact values for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (Lessons 3.6–3.8).
- Use reference angles and quadrant signs (Lesson 3.7).
- Use $\tan^{-1}$ on a calculator and know its range (Lesson 6.2).
- Apply the Law of Cosines and the Law of Sines (Lessons 7.2 and 7.4).
- Read true bearings (Lesson 8.2).
- Use the Pythagorean theorem and complete the square to find the center and radius of a circle.

MASTERY CHECKLIST

- ☐ I can plot a point (r, θ) , including one with negative r , and give several other polar names for it.
- ☐ I can find the distance between two points given in polar form using the Law of Cosines.
- ☐ I can derive
 - $x = r \cos \theta$ and $y = r \sin \theta$
 - $r^2 = x^2 + y^2$

ra

nt.

- ☐ I can convert simple equations between polar and rectangular form, for example showing that

$r = a$ is a circle.

$2a \cos \theta$

- ☐ I can graph a polar equation from a table of values and use the symmetry tests.
- ☐ I can recognize circles, roses, cardioids and limaçons from their equations and predict the number and length of petals or the type of limaçon.
- ☐ I can resolve a vector into components and find a vector's magnitude and direction angle from its components.
- ☐ I can resolve a weight on a ramp into components along and perpendicular to the ramp.
- ☐ I can add vectors to find the resultant of forces or velocities, including a boat in a current and an airplane in a wind.
- ☐ I can prove

$\mathbf{u} \cdot \mathbf{v}$ from the Law of Cosines and use it

$\mathbf{v}$ to find the angle between two

$\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ or test for perpendicularity.

LESSONS IN THIS UNIT

9.1 Polar Coordinates

45 minutes

9.2 Converting Between Polar and Rectangular Form

50 minutes

UNIT 9 TEST

Time allowed: 90 minutes. A scientific calculator is allowed; a graphing calculator is not. Show your work. Give exact answers when asked; otherwise round as stated in each problem, without rounding intermediate values. Angles are in degrees unless the problem asks for radians. Total: 100 points.

1. **5 pts**

Rename the point $(-6, 210^\circ)$ with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Give θ in degrees.

2. **6 pts**

Find the exact distance between the polar points $(5, 20^\circ)$ and $(8, 80^\circ)$.

3. **6 pts**

Convert the polar point $\left(8, \frac{4\pi}{3}\right)$ to rectangular coordinates, exactly.

4. **7 pts**

Convert the rectangular point $(-8, 15)$ to polar coordinates with $r > 0$ and $0^\circ \leq \theta < 360^\circ$. Round θ to the nearest tenth of a degree.

5. **7 pts**

(a) Convert $x^2 + y^2 = 10y$ to a polar equation solved for r . (b) Convert $r = 3 \sec \theta$ to rectangular form. Describe each graph.

6. **8 pts**

Identify each curve and give the requested feature: (a) $r = 5 \sin 3\theta$ (number and length of petals); (b) $r = 2 + 2 \sin \theta$ (type of curve and its greatest distance from the pole); (c) $r = 4 + \cos \theta$ (type of limaçon).

7. (7 pts)

The limaçon $r = 2 - 4 \cos \theta$ has an inner loop. Find all θ in $[0, 2\pi)$ at which it passes through the pole, exactly in radians.

8. (6 pts)

A vector has magnitude 50 and direction angle 225° . Find its components exactly and to the nearest hundredth.

9. (7 pts)

Find the magnitude and direction angle of $\mathbf{v} = \langle -12, -5 \rangle$, with $0^\circ \leq \theta < 360^\circ$ to the nearest tenth of a degree.

10. (8 pts)

Samuel rolls a 240-lb piano up a ramp inclined at 30° . Ignoring friction, find the force parallel to the ramp needed to hold the piano in place, and the component of its weight pressing against the ramp. Give exact values and decimals to the nearest tenth of a pound.

11. (7 pts)

Find the angle between $\mathbf{u} = \langle 1, 7 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$, exactly in degrees.

12. (9 pts)

A plane heads due east with an airspeed of 180 mph. A wind blows due north at 75 mph. Find the plane's ground speed and its course as a true bearing, to the nearest tenth of a degree.

13. (9 pts)

A ferry moves at 10 mph through still water and must cross a river 1 mile wide, landing directly opposite its dock. The current flows at 5 mph. At what angle upstream of straight across must it aim, what is its speed over the ground, and how many minutes does the crossing take (to the nearest tenth of a minute)?

14. (8 pts)

Prove that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are perpendicular (when neither is $\mathbf{0}$). Use components $\mathbf{u} = \langle a_1, b_1 \rangle$, $\mathbf{v} = \langle a_2, b_2 \rangle$. What does this say about the diagonals of a rhombus?

ANSWER KEY

1. $(6, 30^\circ)$
2. $d^2 = 25 + 64 - 80 \cos 60^\circ = 49$, so $d = 7$.
3. $\left(8 \cos \frac{4\pi}{3}, 8 \sin \frac{4\pi}{3}\right) = (-4, -4\sqrt{3}) \approx (-4, -6.928)$
4. $r = 17$. Quadrant II: $\theta = 180^\circ - \tan^{-1} \frac{15}{8} \approx 180^\circ - 61.93^\circ \approx 118.1^\circ$. The point is about $(17, 118.1^\circ)$.
5. (a) $r^2 = 10r \sin \theta$, so $r = 10 \sin \theta$: a circle through the pole with center $(0, 5)$ and radius 5. (b) $r \cos \theta = 3$, so $x = 3$: a vertical line.
6. (a) A rose with 3 petals, each of length 5. (b) A cardioid, with its cusp at the pole and its greatest distance 4 at $\theta = \frac{\pi}{2}$. (c) A convex limaçon, since $4 \geq 2 \cdot 1$.
7. $2 - 4 \cos \theta = 0$, so $\cos \theta = \frac{1}{2}$: $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.
8. $\langle -25\sqrt{2}, -25\sqrt{2} \rangle \approx \langle -35.36, -35.36 \rangle$
9. $\|\mathbf{v}\| = 13$. Quadrant III: $\theta = 180^\circ + \tan^{-1} \frac{5}{12} \approx 180^\circ + 22.62^\circ \approx 202.6^\circ$.
10. Parallel: $240 \sin 30^\circ = 120$ lb. Perpendicular: $240 \cos 30^\circ = 120\sqrt{3} \approx 207.8$ lb.
11. $\mathbf{u} \cdot \mathbf{v} = -3 + 28 = 25$; $\|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$, $\|\mathbf{v}\| = 5$. $\cos \theta = \frac{25}{25\sqrt{2}} = \frac{\sqrt{2}}{2}$, so $\theta = 45^\circ$.
12. Ground velocity $\langle 180, 75 \rangle$, speed $\sqrt{32,400 + 5625} = \sqrt{38,025} = 195$ mph. It travels $\tan^{-1} \frac{75}{180} \approx 22.6^\circ$ north of east, a true bearing of $90^\circ - 22.62^\circ \approx 067.4^\circ$.
13. $\sin \alpha = \frac{5}{10}$, so aim 30° upstream. Ground speed $10 \cos 30^\circ = 5\sqrt{3} \approx 8.66$ mph. Time $\frac{1}{5\sqrt{3}}$ h $= 4\sqrt{3} \approx 6.9$ minutes.
14. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (a_1 + a_2)(a_1 - a_2) + (b_1 + b_2)(b_1 - b_2) = a_1^2 - a_2^2 + b_1^2 - b_2^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$. A zero dot product of nonzero vectors means they are perpendicular. A rhombus built on $\mathbf{u}$ and $\mathbf{v}$ has diagonals $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, so the diagonals of a rhombus are perpendicular.

MIDTERM EXAM

Covers Units 1–4. Time: 90 minutes.

You have 90 minutes. A scientific calculator is allowed; notes, books and graphing calculators are not. Check that your calculator is in degree or radian mode as each problem requires. Show all work, including a labeled sketch for every triangle and word problem; a correct setup earns partial credit. Give exact values (fractions, radicals and multiples of π) when a problem asks for them. Otherwise keep full calculator accuracy and round only the final answer, to the precision stated in the problem. Proofs must give a reason for every step.

1. **4 pts** Lesson 1.2

In right triangle ABC with the right angle at C , the legs are $a = 8$ and $b = 15$. Find the hypotenuse c , then give $\sin A$, $\cos A$ and $\tan A$ as fractions in lowest terms.

2. **5 pts** Lesson 1.3

A 20-foot ladder leans against the side of Mr. Miller's barn and makes an angle of 72° with the level ground. How high up the wall does the top of the ladder reach? Round to the nearest tenth of a foot.

3. **5 pts** Lesson 1.4

Samuel is building a ramp up to a porch. The ramp rises 2.5 ft over a horizontal run of 30 ft. Find the angle the ramp makes with the ground, to the nearest tenth of a degree, and the length of the ramp surface, to the nearest tenth of a foot.

4. **5 pts** Lesson 1.5

If $\tan \theta = \frac{5}{12}$ and θ is acute, find the exact values of the other five trigonometric ratios of θ .

5. **4 pts** Lesson 2.1

Ruth's square garden has a diagonal path 10 m long. Find the exact side length of the garden, then approximate it to the nearest hundredth of a meter.

6. **4 pts** Lesson 2.2

In a 30° - 60° - 90° triangle the shorter leg is 6. Find the exact lengths of the longer leg and the hypotenuse.

7. **5 pts** Lesson 2.3

Prove that $\sin \theta = \cos(90^\circ - \theta)$ for every acute angle θ . Use a right triangle in your proof.

8. **4 pts** Lesson 2.3

Find the angle θ , in degrees, for which $\sin(3\theta) = \cos(\theta + 10^\circ)$, given that 3θ and $\theta + 10^\circ$ are both acute.

9. **4 pts** Lesson 3.1

Find the angle in $[0^\circ, 360^\circ)$ that is coterminal with -870° , and name the quadrant of its terminal side.

10. **4 pts** Lesson 3.3

(a) Convert 225° to radians, exact. (b) Convert $\frac{7\pi}{9}$ radians to degrees.

11. **6 pts** Lesson 3.4

A center-pivot irrigation pipe 400 m long pivots about one end and sweeps through an angle of 135° . (a) How far does the outer end travel, to the nearest meter? (b) What area of the field is watered, to the nearest square meter?

12. **5 pts** Lesson 3.5

A tractor's rear wheel has a diameter of 1.6 m and turns at 50 revolutions per minute. Find the tractor's speed in kilometers per hour, to the nearest tenth.

13. **6 pts** Lesson 3.6

The point $(-5, 12)$ lies on the terminal side of an angle θ in standard position. Find r and the exact values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

14. **5 pts** Lesson 3.7

Given $\cos \theta = -\frac{3}{5}$ and θ in Quadrant III, find the exact values of $\sin \theta$, $\tan \theta$ and $\sec \theta$.

15. **5 pts** Lesson 3.8

Find the exact values: (a) $\cos \frac{5\pi}{6}$ (b) $\tan \frac{7\pi}{4}$ (c) $\sec \frac{4\pi}{3}$ (d) $\sin \frac{3\pi}{2}$

16. **4 pts** Lesson 4.6

For $y = \tan \left(\frac{x}{2} \right)$, find the period and the vertical asymptotes in the interval $[0, 4\pi]$. Give exact values in radians.

17. **5 pts** Lesson 4.3

For $y = -4 \cos \left(\frac{\pi x}{3} \right)$, find the amplitude, the period, and the first $x > 0$ at which the graph reaches its maximum value.

18. **7 pts** Lesson 4.4

For $y = 3 \sin \left(2 \left(x - \frac{\pi}{6} \right) \right) - 1$, give the amplitude, period, phase shift, vertical shift (midline), maximum value and minimum value.

19. **8 pts** Lesson 4.5

Challenge. At a harbor the water is 14 ft deep at high tide at 4:00 a.m. and 4 ft deep at the next low tide at 10:12 a.m. Assume the depth varies sinusoidally with a period of 12.4 hours. (a) Write an equation $y = a \cos(b(t - c)) + d$ for the depth y in feet, where t is hours after midnight. (b) Find the depth at noon, to the nearest tenth of a foot.

20. **5 pts** Lesson 4.7

Consider $y = 2 \csc x$ on $[0, 2\pi]$. (a) List the vertical asymptotes. (b) Give the local minimum point on $(0, \pi)$ and the local maximum point on $(\pi, 2\pi)$. (c) State the range.

ANSWER KEY

1. $c = 17$; $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$
2. $20 \sin 72^\circ \approx 19.0$ ft
3. $\theta = \tan^{-1} \left(\frac{2.5}{30} \right) \approx 4.8^\circ$; length $= \sqrt{30^2 + 2.5^2} \approx 30.1$ ft
4. $\sin \theta = \frac{5}{13}$, $\cos \theta = \frac{12}{13}$, $\csc \theta = \frac{13}{5}$, $\sec \theta = \frac{13}{12}$, $\cot \theta = \frac{12}{5}$
5. $\frac{10}{\sqrt{2}} = 5\sqrt{2} \approx 7.07$ m
6. Longer leg $6\sqrt{3} \approx 10.39$, hypotenuse 12
7. Let $\theta = A$ in right triangle ABC with the right angle at C . The angles sum to 180° (Euclid I.32), so $B = 90^\circ - A$. Side a is opposite A and adjacent to B , and c is the hypotenuse. So $\sin A = \frac{a}{c}$ and $\cos B = \frac{a}{c}$. Therefore $\sin A = \cos B = \cos(90^\circ - A)$.
8. The angles are complementary: $3\theta + \theta + 10^\circ = 90^\circ$, so $\theta = 20^\circ$
9. $-870^\circ + 3 \cdot 360^\circ = 210^\circ$; Quadrant III
10. (a) $\frac{5\pi}{4}$ rad ≈ 3.927 rad; (b) 140°
11. $\theta = \frac{3\pi}{4}$. (a) $s = r\theta = 300\pi \approx 942$ m; (b) $A = \frac{1}{2}r^2\theta = 60000\pi \approx 188,496$ m²
12. $\omega = 100\pi$ rad/min and $r = 0.8$ m, so $v = r\omega = 80\pi$ m/min $= 4.8\pi$ km/h ≈ 15.1 km/h
13. $r = 13$; $\sin \theta = \frac{12}{13}$, $\cos \theta = -\frac{5}{13}$, $\tan \theta = -\frac{12}{5}$
14. $\sin \theta = -\frac{4}{5}$, $\tan \theta = \frac{4}{3}$, $\sec \theta = -\frac{5}{3}$
15. (a) $-\frac{\sqrt{3}}{2}$ (b) -1 (c) -2 (d) -1
16. Period 2π ; asymptotes $x = \pi$ and $x = 3\pi$
17. Amplitude 4; period $\frac{2\pi}{\pi/3} = 6$; the graph starts at its minimum -4 when $x = 0$, so the first maximum ($y = 4$) is half a period later, at $x = 3$
18. Amplitude 3; period π ; phase shift $\frac{\pi}{6}$ to the right; midline $y = -1$; maximum 2; minimum -4
19. (a) $a = \frac{14 - 4}{2} = 5$, $d = \frac{14 + 4}{2} = 9$, $b = \frac{2\pi}{12.4} \approx 0.5067$, $c = 4$: $y = 5 \cos \left(\frac{2\pi}{12.4}(t - 4) \right) + 9$; (b) $y(12) = 5 \cos \left(\frac{16\pi}{12.4} \right) + 9 \approx 5.9$ ft

20. (a) $x = 0, x = \pi, x = 2\pi$; (b) local minimum $\left(\frac{\pi}{2}, 2\right)$, local maximum $\left(\frac{3\pi}{2}, -2\right)$;
(c) $(-\infty, -2] \cup [2, \infty)$

FINAL EXAM

Covers Units 1–8 (Unit 9 is not tested). Time: 2 hours.

You have 2 hours. A scientific calculator is allowed; notes, books and graphing calculators are not. Check degree or radian mode before each problem. Show all work, with a labeled sketch for every triangle and word problem; a correct setup and method earn partial credit. Give exact values (fractions, radicals and multiples of π) when a problem asks for them. Otherwise keep full calculator accuracy and round only the final answer, to the precision stated in the problem. Proofs must work on one side only and give a reason for each step.

1. **3 pts** Lesson 2.2

Find the exact value of $\tan 30^\circ \cdot \sec 30^\circ$.

2. **4 pts** Lesson 1.4

Solve right triangle ABC with $C = 90^\circ$, $A = 34^\circ$ and hypotenuse $c = 50$. Round sides to the nearest tenth.

3. **4 pts** Lesson 1.5

If $\cos \theta = \frac{2}{7}$ and θ is acute, find the exact values of $\csc \theta$ and $\cot \theta$ with rationalized denominators.

4. **3 pts** Lesson 3.4

The minute hand of the clock in a church tower is 15 cm long. How far does its tip travel in 20 minutes? Give the exact answer and round to the nearest tenth of a centimeter.

5. **4 pts** Lesson 3.8

Find the exact values: (a) $\sin \frac{11\pi}{6}$ (b) $\cos \frac{3\pi}{2}$ (c) $\tan \left(-\frac{2\pi}{3}\right)$

6. **3 pts** Lesson 3.5

At a sawmill, a belt connects a pulley of radius 4 in to a pulley of radius 10 in. The small pulley turns at 300 revolutions per minute. How fast does the large pulley turn, in revolutions per minute?

7. **4 pts** Lesson 4.4

For $y = 2 \cos(3x + \pi) + 1$, give the amplitude, period, phase shift, midline, maximum and minimum. (Hint: factor the inside first.)

8. **5 pts** Lesson 4.5

In a northern town the longest day of the year (day 172) has 15.2 hours of daylight and the shortest day has 9.0 hours. Using a period of 365 days, write $H(t) = a \cos(b(t - c)) + d$ for the hours of daylight on day t , then predict the daylight on day 80 to the nearest tenth of an hour.

9. **3 pts** Lesson 4.6

For $y = \tan(2x)$, find the period and the vertical asymptotes in $[0, \pi)$, exact in radians.

10. **3 pts** Lesson 5.2

Simplify $\frac{1 - \cos^2 x}{\sin x \cos x}$ to a single trigonometric function.

11. **5 pts** Lesson 5.3

Prove the identity $\sec x - \cos x = \sin x \tan x$. Work on one side only.

12. **4 pts** Lesson 5.4

Use a sum formula to find the exact value of $\sin 75^\circ$.

13. **5 pts** Lesson 5.5

Given $\sin \theta = \frac{3}{5}$ with θ in Quadrant II, find the exact values of $\sin 2\theta$, $\cos 2\theta$ and $\cos \frac{\theta}{2}$.

14. **3 pts** Lesson 6.2

Find the exact values in radians: (a) $\arccos\left(-\frac{1}{2}\right)$ (b) $\arctan(-\sqrt{3})$

15. **4 pts** Lesson 6.3

Find the exact values: (a) $\sin\left(\arccos\left(-\frac{5}{13}\right)\right)$ (b) $\tan\left(\arcsin\frac{3}{5}\right)$

16. **4 pts** Lesson 6.4

Solve $2 \cos x + \sqrt{3} = 0$ on $[0, 2\pi)$. Give exact answers in radians.

17. (5 pts) Lesson 6.5

Solve $2 \sin^2 x - \sin x - 1 = 0$ on $[0, 2\pi)$. Give exact answers in radians.

18. (4 pts) Lesson 7.1

A triangular field on Mr. Miller's farm has two fences of 120 m and 150 m that meet at an angle of 48° . Find the area of the field to the nearest square meter.

19. (4 pts) Lesson 7.2

In triangle ABC , $A = 40^\circ$, $B = 65^\circ$ and $a = 12$. Find C , b and c . Round sides to the nearest tenth.

20. (5 pts) Lesson 7.3

Challenge. In triangle ABC , $A = 35^\circ$, $a = 8$ and $b = 12$. How many triangles fit these data? Find every possible measure of angle B , to the nearest tenth of a degree.

21. (4 pts) Lesson 7.4

A triangle has sides 7, 8 and 13. Find its largest angle exactly, in degrees.

22. (3 pts) Lesson 8.1

From a point on level ground 60 ft from the base of a grain silo, the angle of elevation to the top of the silo is 52° . Find the height of the silo to the nearest tenth of a foot.

23. (5 pts) Lesson 8.2

A ship leaves port and sails 30 nautical miles on a bearing of 060° , then turns and sails 45 nautical miles on a bearing of 130° . (a) How far is the ship from port, to the nearest tenth of a nautical mile? (b) What is the bearing from port to the ship, to the nearest degree?

24. (4 pts) Lesson 8.3

To measure the width of a river, a surveyor lays out a baseline AB of 250 m along one straight bank. A tree C stands on the far bank. The angle between the baseline and the line of sight to the tree is 72° at A and 58° at B . Find AC and the width of the river (the perpendicular distance from C to line AB), each to the nearest meter.

25. **5 pts** Lesson 8.5

Challenge. A water wheel of radius 3 m has its center 1 m above the surface of the stream. It turns counterclockwise once every 12 seconds. Paddle tip P is at the top of the wheel at $t = 0$. (a) Write $h(t)$, the height of P above the water in meters after t seconds. (b) Find the first time $t > 0$ when P reaches the water, to the nearest hundredth of a second.

ANSWER KEY

$$1. \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} = \frac{2}{3}$$

$$2. B = 56^\circ, a = 50 \sin 34^\circ \approx 28.0, b = 50 \cos 34^\circ \approx 41.5$$

$$3. \text{The opposite side is } \sqrt{49-4} = 3\sqrt{5}, \text{ so } \csc \theta = \frac{7}{3\sqrt{5}} = \frac{7\sqrt{5}}{15} \approx 1.0435 \text{ and } \cot \theta = \frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{15} \approx 0.2981$$

$$4. \theta = \frac{20}{60} \cdot 2\pi = \frac{2\pi}{3}, \text{ so } s = 15 \cdot \frac{2\pi}{3} = 10\pi \approx 31.4 \text{ cm}$$

$$5. (a) -\frac{1}{2} (b) 0 (c) \sqrt{3}$$

$$6. \text{Both rims move at the belt's linear speed, so } 4 \cdot 300 = 10\omega \text{ and } \omega = 120 \text{ rpm}$$

$$7. y = 2 \cos \left(3 \left(x + \frac{\pi}{3} \right) \right) + 1: \text{amplitude } 2, \text{ period } \frac{2\pi}{3}, \text{ phase shift } \frac{\pi}{3} \text{ to the left,} \\ \text{midline } y = 1, \text{ maximum } 3, \text{ minimum } -1$$

$$8. a = \frac{15.2 - 9.0}{2} = 3.1, d = \frac{15.2 + 9.0}{2} = 12.1, b = \frac{2\pi}{365}, c = 172: H(t) = \\ 3.1 \cos \left(\frac{2\pi}{365}(t - 172) \right) + 12.1; H(80) \approx 12.1 \text{ hours}$$

$$9. \text{Period } \frac{\pi}{2}; \text{ asymptotes } x = \frac{\pi}{4} \text{ and } x = \frac{3\pi}{4}$$

$$10. \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$$

$$11. \sec x - \cos x = \frac{1}{\cos x} - \cos x \text{ (reciprocal identity)} = \frac{1 - \cos^2 x}{\cos x} \text{ (common} \\ \text{denominator)} = \frac{\sin^2 x}{\cos x} \text{ (Pythagorean identity)} = \sin x \cdot \frac{\sin x}{\cos x} = \sin x \tan x \\ \text{(quotient identity)}$$

$$12. \sin(45^\circ + 30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.9659$$

$$13. \cos \theta = -\frac{4}{5}; \sin 2\theta = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5} \right) = -\frac{24}{25}; \cos 2\theta = 1 - 2 \cdot \frac{9}{25} = \frac{7}{25}; \frac{\theta}{2} \text{ is between} \\ 45^\circ \text{ and } 90^\circ, \text{ so } \cos \frac{\theta}{2} = \sqrt{\frac{1 - 4/5}{2}} = \frac{\sqrt{10}}{10} \approx 0.3162$$

$$14. (a) \frac{2\pi}{3} (b) -\frac{\pi}{3}$$

$$15. (a) \frac{12}{13} \text{ (the angle is in Quadrant II, where sine is positive)} (b) \frac{3}{4}$$

$$16. \cos x = -\frac{\sqrt{3}}{2}, \text{ so } x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

$$17. (2 \sin x + 1)(\sin x - 1) = 0, \text{ so } \sin x = 1 \text{ or } \sin x = -\frac{1}{2}: x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

18. $\frac{1}{2}(120)(150) \sin 48^\circ \approx 6688 \text{ m}^2$
19. $C = 75^\circ$; $b = \frac{12 \sin 65^\circ}{\sin 40^\circ} \approx 16.9$; $c = \frac{12 \sin 75^\circ}{\sin 40^\circ} \approx 18.0$
20. $\sin B = \frac{12 \sin 35^\circ}{8} \approx 0.8604$. Both $B \approx 59.4^\circ$ and $B \approx 120.6^\circ$ work, since $35^\circ + 120.6^\circ < 180^\circ$: two triangles
21. The largest angle is opposite the side 13: $\cos C = \frac{7^2 + 8^2 - 13^2}{2 \cdot 7 \cdot 8} = -\frac{56}{112} = -\frac{1}{2}$, so $C = 120^\circ$
22. $60 \tan 52^\circ \approx 76.8 \text{ ft}$
23. The angle at the turn is $180^\circ - 70^\circ = 110^\circ$. (a) $d^2 = 30^2 + 45^2 - 2(30)(45) \cos 110^\circ$, so $d \approx 62.0$ nautical miles; (b) by the Law of Sines the angle at port is about 43.0° , so the bearing is about $060^\circ + 43^\circ = 103^\circ$
24. $C = 180^\circ - 72^\circ - 58^\circ = 50^\circ$; $AC = \frac{250 \sin 58^\circ}{\sin 50^\circ} \approx 277 \text{ m}$; width = $AC \sin 72^\circ \approx 263 \text{ m}$
25. (a) $h(t) = 3 \cos\left(\frac{\pi t}{6}\right) + 1$; (b) $\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{3}$, so $t = \frac{6}{\pi} \cos^{-1}\left(-\frac{1}{3}\right) \approx 3.65 \text{ s}$