Calculus 1

Grades 11–12+ · 6 pages

Limits and continuity, every derivative rule, rates and approximation, the Mean Value Theorem, curve sketching, optimization, Riemann sums, the Fundamental Theorem, substitution, area, volume and differential equations.

1 Limits · 2 Continuity · 3 Derivative · 4 Rules · 5 Chain & implicit · 6 In context · 7 MVT & tests · 8 Sketching · 9 Optimization · 10 Antiderivatives · 11 Riemann sums · 12 FTC & u-sub · 13 Applications · 14 Diff. equations

1 LIMITS IN DEPTH

first look (tables, graphs, factoring): Pre-Calc §12

Limit laws

If $\lim f = L$ and $\lim g = M$ ($x \to c$), the limit of a sum, difference, product, constant multiple, power or root is that combination of L and M; for a quotient, L/M needs $M \neq 0$.

Polynomials, and rational functions where defined: just substitute.

Read the form

Substitution gives	Meaning	Next step
a number	that is the limit	done
$\frac{0}{0}$	indeterminate	algebra, squeeze, L'Hôpital
$\frac{k}{0}$, $k \neq 0$	infinite limit (VA)	check the sign on each side
$\frac{\infty}{\infty}$	indeterminate	divide by top power

MEMORY TRICK **Plug, Peek, Pick**

Plug in c. **Peek** at the form you get. **Pick** the tool that form calls for: algebra, squeeze, top power or L'Hôpital.

Infinite limits WORKED

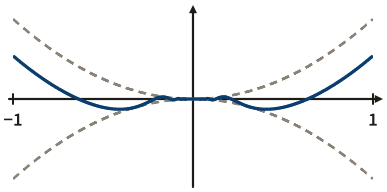
$\lim_{x \to 2^+} \frac{x+1}{x-2} = +\infty$ *top 3, bottom 0^+*

$\lim_{x \to 2^-} \frac{x+1}{x-2} = -\infty$ *bottom 0^-: $x = 2$ is a VA*

Squeeze theorem

The bounds only need to hold near c.

$g \le f \le h$ and $\lim g = \lim h = L$ $\Rightarrow \lim f = L$



$-x^2 \le x^2 \cos(1/x) \le x^2$, so the middle graph is squeezed to 0 at $x = 0$.

Squeeze WORKED

$-1 \le \cos(1/x) \le 1 \to -x^2 \le x^2 \cos(1/x) \le x^2$

both bounds $\to 0$, so $\lim_{x \to 0} x^2 \cos(1/x) = \mathbf{0}$

Trig limits (x in radians)

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

EX $\lim_{x \to 0} \frac{\sin 5x}{3x} = \frac{5}{3} \cdot \lim \frac{\sin 5x}{5x} = \mathbf{\frac{5}{3}}$

At $-\infty$ with a root WORKED

$\lim_{x \to -\infty} \frac{\sqrt{(4x^2+1)}}{x-3}$

$\sqrt{(x^2)} = |x| = -x$ for $x < 0$ *watch the sign*

$= \lim \frac{-\sqrt{(4 + 1/x^2)}}{1 - 3/x} = \mathbf{-2}$

$\infty - \infty$: use the conjugate WORKED

$\lim_{x \to \infty} (\sqrt{(x^2+6x)} - x) = \lim \frac{6x}{\sqrt{(x^2+6x)} + x}$

$= \lim \frac{6}{\sqrt{(1+6/x)} + 1} = \mathbf{3}$

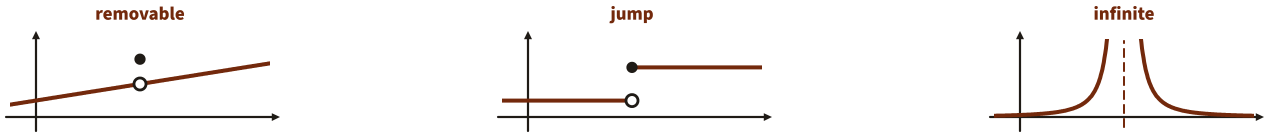
Precise definition

$\lim_{x \to c} f(x) = L$ means: for every $\varepsilon > 0$ there is a $\delta > 0$ with $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

EX $\lim_{x \to 2} (3x - 1) = 5$: $|3x - 6| = 3|x - 2| < \varepsilon$ when **$\delta = \varepsilon/3$**

2 CONTINUITY & THE INTERMEDIATE VALUE THEOREM

no breaks, no jumps, no holes



Continuous at x = c

f(c) is defined, $\lim_{x \to c} f(x)$ exists, and they are equal (Pre-Calc §12: D-E-E).

- Polynomials, e^x, sin x, cos x: continuous everywhere.
- Rational, root, log, tan: continuous on their domains.
- Sums, products, quotients (denominator $\neq 0$) and compositions of continuous functions are continuous.

Fix a removable break WORKED

$g(x) = \frac{x^2 - 1}{x - 1} = x + 1$ for $x \neq 1$

define $g(1) = \mathbf{2}$ *the limit*

MEMORY TRICK **Draw it in one stroke**

If you can trace the graph over an interval without lifting the pencil, f is continuous there. Each break type above is a lift.

Intermediate Value Theorem

f continuous on [a, b] and N between f(a) and f(b) $\Rightarrow$

f(c) = N for some c in (a, b)

Extreme Value Theorem

f continuous on a **closed** interval [a, b] has an absolute maximum and minimum there (found in §7).

Choose k for continuity WORKED

$f(x) = kx^2$ for $x \le 2$; $3x + k$ for $x > 2$

match at 2: $4k = 6 + k$ *left value = right limit*

k = 2

A root of cos x = x WORKED

$h(x) = \cos x - x$: $h(0) = 1$, $h(1) \approx -0.4597$

continuous, signs differ

a root lies in (0, 1); it is $\approx$ **0.7391**

On a closed interval

Continuous on [a, b]: on (a, b), from the right at a, from the left at b.

3 THE DERIVATIVE

slope of the tangent, rate of change right now (intro: Pre-Calc §13)

Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Notation $f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$

Second $f''(x)$, $\frac{d^2y}{dx^2}$: the rate of change of f'

Units output units per input unit (m/s, $ per item)

f(x) = √x WORKED

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

conjugate

$h \to 0$: $\mathbf{f'(x) = \frac{1}{2\sqrt{x}}}$

f(x) = 1/x WORKED

$$\frac{1/(x+h) - 1/x}{h} = \frac{-1}{x(x+h)}$$

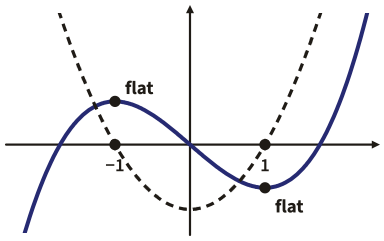
combine

$\mathbf{f'(x) = -\frac{1}{x^2}}$

Tangent & normal lines

$y - f(a) = f'(a)(x - a)$. The normal line has slope $-1/f'(a)$.

EX $f(x) = \sqrt{x}$ at $a = 4$: $f'(4) = \frac{1}{4} \to$ **$y = \frac{1}{4}x + 1$**



$f(x) = x^3 - 3x$ (solid) and $f'(x) = 3x^2 - 3$ (dashed): $f' = 0$ at f's flat points, $f' > 0$ where f rises.

Differentiable vs. continuous

Differentiable at c ⇒ continuous at c, but not the reverse. No derivative at:

- a **corner**: $|x|$ at 0
- a **cusp**: $x^{2/3}$ at 0
- a **vertical tangent**: $\sqrt[3]{x}$ at 0
- any **break** in the graph

Make a piecewise function differentiable WORKED

$f(x) = x^2$ for $x \le 1$; $ax + b$ for $x > 1$

continuous: $1 = a + b$; equal slopes: $2 = a$

value, then slope

$a = 2$, $b = -1$

Estimate from a table WORKED

t: 0, 2, 4, 6 · T: 70, 74, 80, 88

$T'(4) \approx \frac{88 - 74}{6 - 2} =$ **3.5** per unit of t

symmetric difference

Meaning with units WORKED

$C(q)$ = cost in dollars of q items; $C'(100) = 4.5$

the 101st item costs about **$4.50** more

dollars per item

MEMORY TRICK

Three names, one number

Slope of the tangent, instantaneous rate, derivative: three names for the **same** number $f'(a)$.

4 DERIVATIVE RULES

c, n and a are constants

Core rules

Rule	Derivative
constant	$(c)' = 0$
power	$(x^n)' = nx^{n-1}$ (any real n)
multiple, sum	$(cf)' = cf'$; $(f \pm g)' = f' \pm g'$
product	$(fg)' = f'g + fg'$
quotient	$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$
chain	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

MEMORY TRICK

Low d-high minus high d-low (traditional)

For $\frac{\text{high}}{\text{low}}$: **low** times d(**high**) minus **high** times d(**low**), all over **low squared**. The minus makes order matter.

Power rule WORKED

$\frac{d}{dx}(4x^3 - \frac{2}{x} + 5\sqrt{x})$

write as powers

$= \mathbf{12x^2 + \frac{2}{x^2} + \frac{5}{2\sqrt{x}}}$

Product WORKED

$\frac{d}{dx}(x^2 \sin x) =$ **$2x \sin x + x^2 \cos x$**

Trigonometric

f	f′	f	f′
sin x	cos x	cos x	−sin x
tan x	$\sec^2 x$	cot x	$-\csc^2 x$
sec x	sec x tan x	csc x	−csc x cot x

MEMORY TRICK **Co- means minus**

The derivatives of the **co**-functions (cos, cot, csc) all start with a **minus** sign.

Exponential & log

f	f′
e^x	e^x
a^x	$a^x \ln a$
ln x	$\frac{1}{x}$ ($\ln\|x\|$ too)
$\log_a x$	$\frac{1}{x \ln a}$

Inverse trig

f	f′
$\sin^{-1} x$	$\frac{1}{\sqrt{(1 - x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1 - x^2)}}$
$\tan^{-1} x$	$\frac{1}{1 + x^2}$
$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{(x^2 - 1)}}$

Quotient WORKED

$\frac{d}{dx}\frac{x+1}{x^2+1} = \frac{(x^2+1) - (x+1)(2x)}{(x^2+1)^2}$

$= \mathbf{\frac{1 - 2x - x^2}{(x^2+1)^2}}$

Exponential, log, inverse trig WORKED

$\frac{d}{dx}(e^x \ln x) =$ **$e^x \ln x + \frac{e^x}{x}$** *product*

$\frac{d}{dx} 3^x =$ **$3^x \ln 3$**

$\frac{d}{dx} \tan^{-1}(2x) = \mathbf{\frac{2}{1 + 4x^2}}$ *chain*

Rules together WORKED

$\frac{d}{dx} x^3e^{2x} = 3x^2e^{2x} + 2x^3e^{2x} =$ **$x^2e^{2x}(3 + 2x)$**

product + chain

$\frac{d}{dx}\sqrt{(x^2 + 9)} = \frac{x}{\sqrt{(x^2 + 9)}}$; at $x = 4$: **$\frac{4}{5}$**

Higher derivatives

Differentiate again: $f(x) = x^4 \to f'' = 12x^2$, $f''' = 24x$.

$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$: every 4th derivative repeats.

5 CHAIN RULE, IMPLICIT & INVERSE DERIVATIVES

functions inside functions, and curves that are not functions

Chain rule

For $y = f(u)$ with $u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

MEMORY TRICK **DOT**

Derivative of the **O**utside (inside left alone), **T**imes the derivative of the inside. Repeat for each layer.

Chain rule WORKED

$\frac{d}{dx}(3x^2 + 1)^5 = 5(3x^2 + 1)^4 \cdot 6x =$ **$30x(3x^2 + 1)^4$**

$\frac{d}{dx}\sin^3(2x) =$ **$6\sin^2(2x)\cos(2x)$** *three layers*

$\frac{d}{dx}e^{-x^2} =$ **$-2x\,e^{-x^2}$**

Chain rule from a table WORKED

$g(1) = 2$, $g'(1) = 3$, $f'(2) = -4$; $h = f \circ g$

$h'(1) = f'(g(1)) \cdot g'(1) = -4 \cdot 3 =$ **-12**

Implicit differentiation

- Differentiate both sides with respect to x.
- Each y-term gets a factor $\frac{dy}{dx}$ (chain rule).
- Collect the $\frac{dy}{dx}$ terms and solve.

Circle $x^2 + y^2 = 25$ WORKED

$2x + 2y\,y' = 0 \rightarrow$ **$y' = -\frac{x}{y}$**

at (3, 4): slope **$-\frac{3}{4}$**

$x^3 + y^3 = 6xy$ at (3, 3) WORKED

$3x^2 + 3y^2y' = 6y + 6xy'$ *product rule on xy*

$y' = \frac{2y - x^2}{y^2 - 2x} = \frac{6 - 9}{9 - 6} =$ **-1**

tangent line: **$y = -x + 6$**

Logarithmic differentiation WORKED

$y = x^x \rightarrow \ln y = x \ln x$ *take ln first*

$\frac{y'}{y} = \ln x + 1 \rightarrow$ **$y' = x^x(\ln x + 1)$**

Derivative of an inverse

where $f(a) = b$: flip the slope at the matching point.

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

EX $f(x) = x^3 + x$, $f(1) = 2$: $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{4}$

Inverse sine, implicitly WORKED

$y = \sin^{-1} x \rightarrow \sin y = x$

$\cos y \cdot y' = 1 \rightarrow y' = \frac{1}{\cos y} =$ **$\frac{1}{\sqrt{(1 - x^2)}}$**

$\cos y = \sqrt{(1 - \sin^2 y)} \geq 0$

Second derivative, implicitly WORKED

$x^2 + y^2 = 25$: $y' = -x/y$

$y'' = -\frac{y - x\,y'}{y^2} = -\frac{x^2 + y^2}{y^3} = -\frac{25}{y^3}$ *substitute y′*

at (3, 4): $y'' =$ **$-\frac{25}{64}$**

6 DERIVATIVES IN CONTEXT

motion, related rates, approximation and L'Hôpital

Motion on a line

Quantity	Meaning
$s(t)$	position
$v(t) = s'(t)$	velocity (signed); speed = $\lvert v \rvert$
$a(t) = v'(t) = s''(t)$	acceleration
$v = 0$	at rest; may turn around

$s(t) = t^3 - 6t^2 + 9t$ WORKED

$v = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$

at rest at **$t = 1, 3$**; $a = 6t - 12$

$t = 0.5$: $v = 3.75$, $a = -9 \rightarrow$ **slowing down**

signs differ

MEMORY TRICK **Same Signs Speed up**

If v and a have the **same** sign, the object speeds up; opposite signs, it slows down.

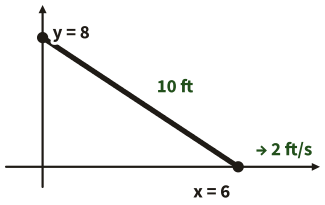
Related rates

- Draw and label; mark what changes.
- Write an equation linking the quantities.
- Differentiate with respect to t.
- Only **then** substitute the known values; solve.

MEMORY TRICK

Dogs Eat Dinner, Sleep Soundly

Draw, **E**quation, **D**ifferentiate, **S**ubstitute, **S**olve, in that order. Substituting before differentiating freezes the rates at 0.



$x^2 + y^2 = 10^2$ at every instant.

Sliding ladder WORKED

10 ft ladder; foot slides out at 2 ft/s; foot 6 ft out

$x^2 + y^2 = 100 \rightarrow y = 8$; $2x\,x' + 2y\,y' = 0$

$y' = -\frac{6 \cdot 2}{8} =$ **-1.5 ft/s** (top slides down)

Filling a cone WORKED

cone tip down, radius 3 ft, height 6 ft: $r = h/2$

$V = \frac{\pi}{12}h^3 \rightarrow V' = \frac{\pi}{4}h^2\,h'$

$V' = 2$ ft³/min, $h = 4$: $h' = \frac{1}{2\pi} \approx$ **0.159 ft/min**

Linear approximation

Near a, $f(x) \approx L(x)$. Differential: $dy = f'(x)\,dx$.

$$L(x) = f(a) + f'(a)(x - a)$$

Approximate WORKED

$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) =$ **2.025** *true value 2.02485*

sphere $r = 10$ cm ± 0.1: $dV = 4\pi r^2\,dr \approx$ **±125.7 cm³**

L'Hôpital's rule

Only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (rewrite $0 \cdot \infty$, $\infty - \infty$ or 1^∞ first); differentiate top and bottom **separately**.

$$\lim \frac{f}{g} = \lim \frac{f'}{g'}$$

L'Hôpital WORKED

$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim \frac{e^x - 1}{2x} = \lim \frac{e^x}{2} =$ **$\frac{1}{2}$** *twice*

$\lim_{x \to 0^+} x \ln x = \lim \frac{\ln x}{1/x} = \lim (-x) =$ **0**

$0 \cdot \infty \rightarrow \infty/\infty$

$\lim_{x \to \infty} (1 + \frac{3}{x})^x =$ **e^3** *take ln: $1^\infty \rightarrow 0/0$*

7 MEAN VALUE THEOREM, EXTREMA & DERIVATIVE TESTS

what f′ and f″ say about f

Mean Value Theorem

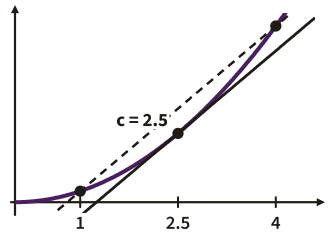
f continuous on [a, b] and differentiable on (a, b) ⇒ for some c in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's theorem

Same conditions plus f(a) = f(b): then f′(c) = 0 for some c in (a, b).

EX $x^3 - 4x$ on $[-2, 2]$: $3c^2 - 4 = 0 \to c = \pm\frac{2}{\sqrt{3}} \approx \pm 1.155$



$f(x) = x^2$ on [1, 4]: secant slope 5; the tangent at c = 2.5 is parallel.

MEMORY TRICK

The speedometer moment

Drive 120 miles in 2 hours and at **some** instant the speedometer reads exactly **60 mph**. That instant is the c of the MVT.

Critical number — c in the domain with f′(c) = 0 or f′(c) undefined

Local extremum — highest or lowest nearby (only at critical numbers)

Absolute extremum — highest or lowest on the whole interval

Closed-interval method WORKED

$f(x) = x^3 - 3x^2 + 1$ on $[-1, 4]$: $f' = 3x(x - 2)$

$f(-1) = -3$, $f(0) = 1$, $f(2) = -3$, $f(4) = 17$

critical numbers and ends

abs. max **17** at x = 4; abs. min **−3** at x = −1 and 2

First derivative test

- f′ changes + to −: **local max**
- f′ changes − to +: **local min**
- no sign change: neither

Concavity & inflection

- f″ > 0: concave **up** (holds water)
- f″ < 0: concave **down**
- Inflection point: concavity changes (f″ changes sign).

Second derivative test

At f′(c) = 0: f″(c) > 0 → **local min**; f″(c) < 0 → **local max**; f″(c) = 0 → no conclusion (use the first test).

Both tests on $f(x) = x^3 - 12x$ WORKED

$f' = 3(x^2 - 4)$: critical **±2**; rises on $(-\infty, -2)$ and $(2, \infty)$

$f'' = 6x$: f″(−2) < 0 → local max **(−2, 16)**

f″(2) > 0 → local min **(2, −16)**; inflection **(0, 0)**

MEMORY TRICK

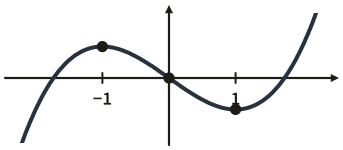
Plus smiles, minus frowns

f″ **positive** bends up like a smile, so a flat spot there is a **minimum**; f″ **negative** frowns, giving a **maximum**.

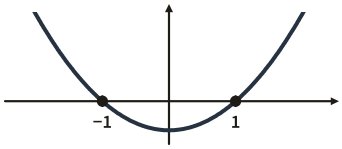
8 CURVE SKETCHING

put every clue on one graph

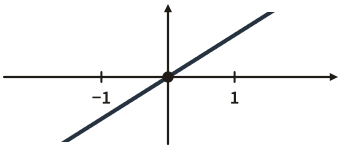
f(x) = x³ − 3x



f′(x) = 3x² − 3: zeros at the turns



f″(x) = 6x: zero at the inflection



MEMORY TRICK

Do I See All Features, Sir?

Domain, **I**ntercepts, **S**ymmetry, **A**symptotes, **F**irst derivative (rise, fall, extrema), **S**econd derivative (concavity, inflection). Then plot.

$f(x) = x^4 - 4x^3$ WORKED

domain all reals; x-intercepts 0, 4

$f' = 4x^2(x - 3)$: critical 0, 3

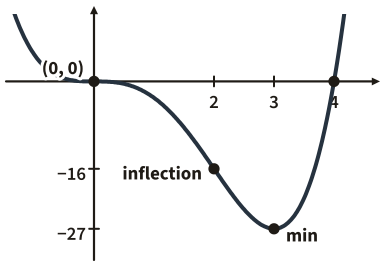
$f'' = 12x(x - 2)$: f″ = 0 at 0, 2

local min **(3, −27)**; inflection **(0, 0)**, **(2, −16)**

x = 0 is not an extremum

Sign chart

Interval	f′	f″	Graph
(−∞, 0)	−	+	falls, cup up
(0, 2)	−	−	falls, cup down
(2, 3)	−	+	falls, cup up
(3, ∞)	+	+	rises, cup up



$f(x) = x^4 - 4x^3$: flat but not turning at 0; minimum at 3.

Reading the graph of f′

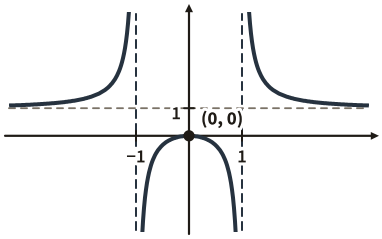
- f′ above the axis ↔ f rising; below ↔ f falling
- f′ crosses zero ↔ f has a local max or min
- f′ has a peak or valley ↔ f has an inflection point

With asymptotes: $f(x) = \frac{x^2}{x^2 - 1}$ WORKED

VA **x = ±1**; HA **y = 1**; even function

$f' = \frac{-2x}{(x^2 - 1)^2}$: rises for x < 0, falls for x > 0 → local max **(0, 0)**

$f'' = \frac{6x^2 + 2}{(x^2 - 1)^3}$: cup up for |x| > 1, down for |x| < 1; no inflection



$f(x) = x^2/(x^2 - 1)$: the dashed lines are the asymptotes.

9 OPTIMIZATION

the best size, shape or price

MEMORY TRICK **SOCKS**

Sketch, write the **O**bjective, write the **C**onstraint, **K**ill a variable with it, **S**olve $f' = 0$ and check ends or f''. Answer with units.

Fence along a river WORKED

600 m of fence, three sides: $A = x(600 - 2x)$

$A' = 600 - 4x = 0 \to x = 150$ *$A'' = -4 < 0$: max*

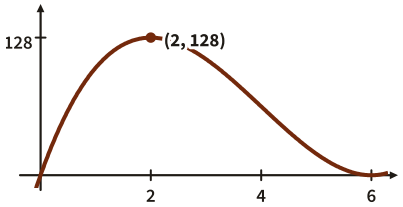
150 m × 300 m, area **45,000 m²**

Open box from a 12 in square WORKED

cut x-inch corners: $V = x(12 - 2x)^2$, $0 < x < 6$

$V' = (12 - 2x)(12 - 6x) = 0 \to x = 2$

$V = 2 \cdot 8^2 =$ **128 in³**



Box volume V(x): the peak is the answer.

Best price WORKED

q items sell at $p = 40 - 0.5q$ dollars

$R = q(40 - 0.5q)$; $R' = 40 - q = 0 \to q = 40$

price **$20**, revenue **$800** *$R'' = -1 < 0$*

Cheapest can, 500 cm³ WORKED

$S = 2\pi r^2 + 2\pi rh$, $h = \frac{500}{\pi r^2} \to S = 2\pi r^2 + \frac{1000}{r}$

$S' = 4\pi r - \frac{1000}{r^2} = 0 \to r = \sqrt[3]{(\frac{250}{\pi})} \approx$ **4.30 cm**

$h = 2r \approx$ **8.60 cm** *height = diameter*

Closest point on $y = \sqrt{x}$ to (3, 0) WORKED

minimize $D = (x - 3)^2 + x$ *square of distance*

$D' = 2(x - 3) + 1 = 0 \to x = 2.5$

point **$(2.5, \sqrt{2.5}) \approx (2.5, 1.58)$**

10 ANTIDERIVATIVES

undo the derivative

Indefinite integral

F is an antiderivative of f if $F' = f$. Then $\int f(x)\,dx = F(x) + C$: every antiderivative differs by a constant.

Initial value problem WORKED

$f'(x) = 3x^2 + 2$, $f(1) = 5$

$f = x^3 + 2x + C$; $1 + 2 + C = 5 \to C = 2$

$f(x) = x^3 + 2x + 2$

From acceleration WORKED

$a = -9.8$ m/s², $v(0) = 20$, $s(0) = 2$

$v =$ **$-9.8t + 20$**, $s =$ **$-4.9t^2 + 20t + 2$**

one constant per step

Basic antiderivatives

$\int f(x)\,dx$	$= F(x) + C$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x; a^x	e^x; $\frac{a^x}{\ln a}$
$\cos x$; $\sin x$	$\sin x$; $-\cos x$
$\sec^2 x$; $\csc^2 x$	$\tan x$; $-\cot x$
$\sec x \tan x$; $\csc x \cot x$	$\sec x$; $-\csc x$
$\frac{1}{1+x^2}$; $\frac{1}{\sqrt{(1-x^2)}}$	$\tan^{-1} x$; $\sin^{-1} x$

MEMORY TRICK

Up one, divide by it, add C

Power rule backwards: raise the power **up one**, **divide** by the new power, then **+ C**. (It fails only for x^{-1}, which gives $\ln|x|$.)

Term by term WORKED

$\int(6x^2 - \frac{4}{x} + 3e^x)\,dx =$ **$2x^3 - 4\ln|x| + 3e^x + C$**

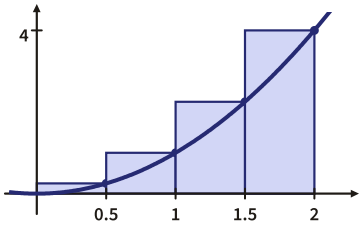
$\int(\sqrt{x} + \frac{1}{x^2})\,dx =$ **$\frac{2}{3}x^{3/2} - \frac{1}{x} + C$** *powers ½ and −2*

11 RIEMANN SUMS & THE DEFINITE INTEGRAL

area as a limit of rectangles

Riemann sums

Split [a, b] into n pieces of width $\Delta x = \frac{b - a}{n}$; add f(sample) · Δx. Samples: left ends, right ends or midpoints. Trapezoid = average of left and right.



Right-endpoint rectangles for x^2 on [0, 2], n = 4: $R_4 = 3.75$.

x^2 on [0, 2], n = 4, Δx = 0.5 WORKED

$L_4 = 0.5(0 + 0.25 + 1 + 2.25) =$ **1.75**

$R_4 = 0.5(0.25 + 1 + 2.25 + 4) =$ **3.75**

$M_4 =$ **2.625**; $T_4 =$ **2.75**; exact $\frac{8}{3} \approx 2.667$

MEMORY TRICK

Rising: left lags, right runs ahead

For an **increasing** f, left sums **under**estimate and right sums **over**estimate. Falling f: the reverse.

Trapezoid rule

$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$

EX x^2 on [0, 2], n = 4: $0.25[0 + 2(0.25 + 1 + 2.25) + 4] =$ **2.75**

Definite integral

Net signed area: area above the x-axis minus area below.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Sigma\, f(x_i^*)\,\Delta x$$

Properties

- $\int_a^a f = 0$; $\int_b^a f = -\int_a^b f$
- $\int_a^b f + \int_b^c f = \int_a^c f$
- $\int(cf \pm g) = c\int f \pm \int g$

By geometry WORKED

$\int_{-3}^{3} \sqrt{(9 - x^2)}\,dx =$ half a circle of radius 3 $=$ **$\frac{9\pi}{2}$**

$\int_0^4 (x - 2)\,dx =$ **0**

equal triangles above and below

12 THE FUNDAMENTAL THEOREM & SUBSTITUTION

derivatives and integrals undo each other

FTC, part 1

With a function as the upper limit: $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\,g'(x)$.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

EX $\frac{d}{dx}\int_0^{x^2} \cos t\,dt = \mathbf{2x\cos(x^2)}$

FTC, part 2

F is any antiderivative of f; the + C cancels.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Evaluate WORKED

$\int_1^3 (2x+1)\,dx = [x^2+x]_1^3 = 12 - 2 = \mathbf{10}$

$\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = \mathbf{2}$

$\int_1^e \frac{1}{x}\,dx = \mathbf{1}$

Net change

$\int_a^b F'(x)\,dx = F(b) - F(a)$: integrating a rate gives the total change.

u-substitution

- Let u = the inside function; du = u′(x) dx.
- Rewrite everything in u (change the limits too).
- Integrate, then return to x if indefinite.

MEMORY TRICK

Inside and its derivative

Look for a function **inside** another with its **derivative** (up to a constant) standing outside. That inside is u.

Substitution WORKED

$\int 2x(x^2+1)^4\,dx = \frac{(x^2+1)^5}{5} + C$ *u = x² + 1*

$\int x\,e^{x^2}\,dx = \mathbf{\tfrac{1}{2}e^{x^2} + C}$ *du = 2x dx*

$\int_0^1 x\sqrt{1-x^2}\,dx = \tfrac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{3}$

u = 1 − x², limits 1 → 0

13 APPLICATIONS OF THE INTEGRAL

average value, motion, area and volume

Average value

Some c in [a, b] has $f(c) = f_{avg}$ (MVT for integrals).

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EX x^2 on [0, 3]: $\frac{1}{3}\cdot 9 = \mathbf{3}$, reached at $c = \mathbf{\sqrt{3}}$

Displacement vs distance WORKED

$v(t) = t^2 - 4t + 3$ on [0, 3] (v < 0 on (1, 3))

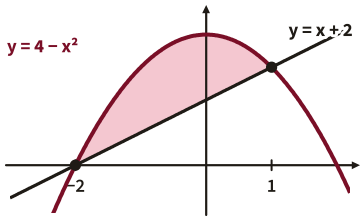
displacement $\int_0^3 v\,dt = \mathbf{0}$

distance $\int_0^3 |v|\,dt = \frac{4}{3} + \frac{4}{3} = \mathbf{\frac{8}{3}}$ *split where v = 0*

Area between curves

Or ∫ [right − left] dy. Find a and b where the curves meet.

$$A = \int_a^b [\text{top} - \text{bottom}]\,dx$$



Region between $y = 4 - x^2$ and $y = x + 2$.

That region WORKED

meet: $4 - x^2 = x + 2 \to x = -2, 1$

$\int_{-2}^1 (2 - x - x^2)\,dx = \mathbf{\frac{9}{2}}$ *top − bottom*

MEMORY TRICK

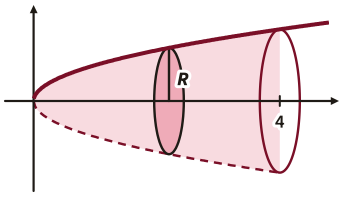
Top minus bottom

Area: **top** curve minus **bottom** curve, left to right. Volume: square **first**, then subtract, $\pi R^2 - \pi r^2$, never $\pi(R - r)^2$.

Disks & washers (about the x-axis)

r = 0 gives a disk. R = outer radius, r = inner radius.

$$V = \pi\int_a^b [R(x)^2 - r(x)^2]\,dx$$



$y = \sqrt{x}$ on [0, 4] spun about the x-axis; each slice is a disk of radius $R = \sqrt{x}$.

Volumes WORKED

disk: $\pi\int_0^4 (\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \mathbf{8\pi \approx 25.13}$

washer, y = x over $y = x^2$ on [0, 1]: $\pi\int_0^1 (x^2 - x^4)\,dx = \mathbf{\frac{2\pi}{15} \approx 0.419}$

14 DIFFERENTIAL EQUATIONS

equations for an unknown function

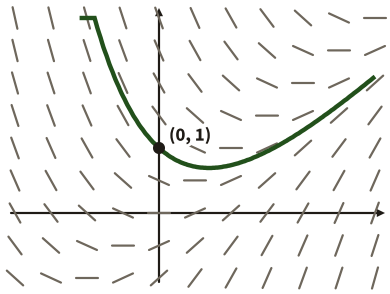
Differential equation relates y, its derivatives and x

General solution a family, with + C

Particular solution the one member fitting an initial condition

Separation of variables

- Put every y on the dy side and every x on the dx side.
- Integrate both sides (one + C is enough).
- Solve for y; use the initial condition.



Slope field for $y' = x - y$; the curve through (0, 1) is $y = x - 1 + 2e^{-x}$.

MEMORY TRICK

SIP

Separate, **I**ntegrate, **P**lug in the initial value. Only then is the answer one curve instead of a family.

Separate WORKED

$\frac{dy}{dx} = xy,\ y(0) = 3 \to \frac{dy}{y} = x\,dx$

$\ln|y| = \frac{x^2}{2} + C \to \mathbf{y = 3e^{x^2/2}}$ *y(0) = 3*