CALCULUS · UNITS 6–9 · QUICK REFERENCE

Calculus 2

Grades 11–12+ · College | 8 pages

Integration techniques and tables, improper integrals, area, volume, arc length, surface area, work and center of mass, sequences and series with every convergence test, power and Taylor series, parametric and polar curves, and differential equations.

1 Toolkit · 2 By parts · 3 Trig integrals · 4 Trig substitution · 5 Partial fractions · 6 Numerical · 7 Improper · 8 Area & volume · 9 Length, work, mass · 10 Sequences · 11 Series · 12 Convergence tests · 13 Alternating · 14 Power series · 15 Taylor series · 16 Parametric · 17 Vectors & motion · 18 Polar · 19 Diff. equations · 20 Slips & practice

1 INTEGRATION TOOLKIT

basics: Calculus 1 §10–12

Integral table (a > 0; add + C)

$\int f(x)\,dx$	$= F(x) + C$	$\int f(x)\,dx$	$= F(x) + C$	$\int f(x)\,dx$	$= F(x) + C$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\tan x$	$\ln\lvert\sec x\rvert$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a}$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\cot x$	$\ln\lvert\sin x\rvert$	$\frac{1}{\sqrt{(a^2-x^2)}}$	$\arcsin\frac{x}{a}$
e^x; a^x	e^x; $\frac{a^x}{\ln a}$	$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\frac{1}{x\sqrt{(x^2-a^2)}}$	$\frac{1}{a}\operatorname{arcsec}\frac{\lvert x\rvert}{a}$
$\sin x$; $\cos x$	$-\cos x$; $\sin x$	$\csc x$	$-\ln\lvert\csc x+\cot x\rvert$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\lvert\frac{a+x}{a-x}\rvert$
$\sec^2 x$; $\csc^2 x$	$\tan x$; $-\cot x$	$\sin^2 x$	$\frac{x}{2}-\frac{\sin 2x}{4}$	$\frac{1}{\sqrt{(x^2\pm a^2)}}$	$\ln\lvert x+\sqrt{(x^2\pm a^2)}\rvert$
$\sec x\tan x$	$\sec x$	$\cos^2 x$	$\frac{x}{2}+\frac{\sin 2x}{4}$	$\sqrt{(a^2-x^2)}$	$\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\arcsin\frac{x}{a}$
$\csc x\cot x$	$-\csc x$	$\tan^2 x$	$\tan x - x$	$\ln x$; $x\,e^{ax}$	$x\ln x - x$; $\frac{e^{ax}}{a^2}(ax-1)$

Choosing a technique

If you see…	Try	If you see…	Try
a function times its derivative	**u-substitution**	a rational function P(x)/Q(x)	**partial fractions** (§5)
$x\,e^x$, $x\sin x$, $\ln x$, $\arctan x$	**by parts** (§2)	x^2+bx+c that won't factor	**complete the square**
powers of sin, cos, tan, sec	**trig integrals** (§3)	no antiderivative, or only data	**numerical rules** (§6)
$\sqrt{(a^2-x^2)}$, $\sqrt{(a^2+x^2)}$, $\sqrt{(x^2-a^2)}$	**trig substitution** (§4)	an infinite limit or a blow-up	**improper integral** (§7)

u-substitution, changing the limits WORKED

$\int_0^2 x(x^2+1)^3\,dx$: $u = x^2+1$, $du = 2x\,dx$

$x = 0 \to u = 1$; $x = 2 \to u = 5$ *new limits*

$\frac{1}{2}\int_1^5 u^3\,du = \frac{1}{8}(625-1) = \mathbf{78}$

no back-substitution

Recognize a table form WORKED

$\int \frac{e^x}{1+e^{2x}}\,dx$: $u = e^x$, $du = e^x\,dx$

$= \int \frac{du}{1+u^2} = \mathbf{\arctan(e^x) + C}$

Complete the square WORKED

$x^2+4x+13 = (x+2)^2+9$

$\int \frac{dx}{(x+2)^2+9} = \mathbf{\frac{1}{3}\arctan\frac{x+2}{3} + C}$ *a = 3*

2 INTEGRATION BY PARTS

the product rule run backward

Integration by parts

Definite: $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$. Pick u so du is simpler; dv must be easy to integrate.

$$\int u\,dv = uv - \int v\,du$$

MEMORY TRICK **LIATE (Traditional)**

Choose u from the first type present: **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rigonometric, **E**xponential. The rest is dv.

One round WORKED

$\int x\,e^x\,dx$: $u = x$, $dv = e^x\,dx \to du = dx$, $v = e^x$

$= x\,e^x - \int e^x\,dx = \mathbf{e^x(x-1) + C}$

dv = dx WORKED

$\int \ln x\,dx$: $u = \ln x$, $v = x$

$= x\ln x - \int x\cdot\frac{1}{x}\,dx = \mathbf{x\ln x - x + C}$

$\int \arctan x\,dx = \mathbf{x\arctan x - \frac{1}{2}\ln(1+x^2) + C}$

same idea

Cyclic: the integral comes back WORKED

$I = \int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$ *parts once*

$= e^x\sin x - e^x\cos x - I$ *parts again*

$2I = e^x(\sin x - \cos x) \to \mathbf{I = \frac{e^x(\sin x - \cos x)}{2} + C}$

solve for I

Tabular method: $\int x^2\sin x\,dx$

Sign	u and its derivatives	dv and its integrals
+	x^2	$\sin x$
−	$2x$	$-\cos x$
+	2	$-\sin x$
	0	$\cos x$

Multiply down the diagonals with alternating signs: $\mathbf{-x^2\cos x + 2x\sin x + 2\cos x + C}$. Use it when u differentiates to 0.

Definite WORKED

$\int_1^e \ln x\,dx = [x\ln x - x]_1^e$

$= (e-e) - (0-1) = \mathbf{1}$

3 TRIGONOMETRIC INTEGRALS — identities first, then a substitution

Identities you need

Pythagorean $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$

Half-angle $\sin^2 x = \frac{1 - \cos 2x}{2}$; $\cos^2 x = \frac{1 + \cos 2x}{2}$

Double angle $\sin 2x = 2 \sin x \cos x$

Product to sum $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

Odd sine power WORKED

$\int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$

u = cos x

$= -\int (u^2 - u^4) \, du = \mathbf{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$

Sine–cosine powers

$\int \sin^m x \cos^n x \, dx$	Strategy
m odd	save one sin x; rest to cos with $\sin^2 = 1 - \cos^2$; u = cos x
n odd	save one cos x; rest to sin; u = sin x
both even	half-angle formulas, repeatedly

Different angles: product to sum, e.g. $\int \sin 3x \cos x \, dx = -\frac{\cos 2x}{4} - \frac{\cos 4x}{8} + C$.

Even secant power WORKED

$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$

u = tan x

$= \mathbf{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$

Tangent–secant powers

$\int \tan^m x \sec^n x \, dx$	Strategy
n even	save $\sec^2 x$; rest to tan with $\sec^2 = 1 + \tan^2$; u = tan x
m odd	save sec x tan x; rest to sec; u = sec x
otherwise	identities, or the $\sec^3 x$ result below

$\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$ (by parts).

Even powers WORKED

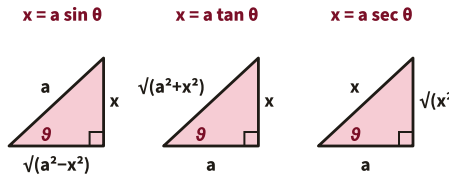
$\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx$ *half-angle*

$= [\frac{x}{2} - \frac{\sin 2x}{4}]_0^{\pi} = \mathbf{\frac{\pi}{2}}$

4 TRIGONOMETRIC SUBSTITUTION — square roots to trig

The three cases

Expression	Substitute	Identity used	dx
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	$a \cos\theta \, d\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	$a \sec^2\theta \, d\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$, $0 \le \theta < \frac{\pi}{2}$ $(x \ge a)$	$\sec^2\theta - 1 = \tan^2\theta$	$a \sec\theta \tan\theta \, d\theta$



Draw the triangle to switch back to x: label the sides from the substitution, then read off any trig ratio.

Sine case WORKED

$\int \frac{dx}{x^2\sqrt{(9 - x^2)}}$: $x = 3 \sin\theta$, $dx = 3 \cos\theta \, d\theta$, $\sqrt{(9 - x^2)} = 3 \cos\theta$

$= \frac{1}{9}\int \csc^2\theta \, d\theta = -\frac{1}{9} \cot\theta + C$

$\cot\theta = \frac{\sqrt{(9 - x^2)}}{x} \rightarrow \mathbf{-\frac{\sqrt{(9 - x^2)}}{9x} + C}$ *triangle*

Tangent case WORKED

$\int_0^2 \frac{dx}{\sqrt{(x^2 + 4)}} = [\ln(x + \sqrt{(x^2 + 4)})]_0^2$ *x = 2 tan ϑ*

$= \ln(2 + 2\sqrt{2}) - \ln 2 = \mathbf{\ln(1 + \sqrt{2}) \approx 0.8814}$

Area of a circle, proved WORKED

$A = 2\int_{-r}^{r} \sqrt{(r^2 - x^2)} \, dx$; $x = r \sin\theta$

$= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = \mathbf{\pi r^2}$ *r = 3 gives 9π*

5 PARTIAL FRACTIONS — split into simple pieces

Steps

- If deg(top) ≥ deg(bottom), **long-divide** first.
- Factor the bottom completely.
- Write one term per factor (table).
- Solve for the constants: cover-up for distinct linear factors, otherwise match coefficients.
- Integrate each piece: ln, power or arctan.

Distinct linear: cover-up WORKED

$\frac{5x - 4}{x^2 - x - 2} = \frac{A}{x - 2} + \frac{B}{x + 1}$

$A = \frac{5(2) - 4}{2 + 1} = 2$; $B = \frac{5(-1) - 4}{-1 - 2} = 3$

cover the factor, plug its root

$\int = \mathbf{2 \ln|x - 2| + 3 \ln|x + 1| + C}$

Setting up

Factor in the bottom	Terms to write
$(x - a)$	$\frac{A}{x - a}$
$(x - a)^k$, repeated	$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_k}{(x - a)^k}$
$x^2 + bx + c$, irreducible	$\frac{Bx + C}{x^2 + bx + c}$
$(x^2 + bx + c)^k$	one $\frac{B_i x + C_i}{(x^2 + bx + c)^i}$ for each $i \le k$

Repeated factor WORKED

$\frac{1}{x^2(x - 1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x - 1}$

$\int = \mathbf{-\ln|x| + \frac{1}{x} + \ln|x - 1| + C}$

Irreducible quadratic WORKED

$\frac{2x^2 + x + 4}{x(x^2 + 4)} = \frac{1}{x} + \frac{x + 1}{x^2 + 4}$ *A = 4/4 = 1, then match*

$\int = \mathbf{\ln|x| + \frac{1}{2}\ln(x^2 + 4) + \frac{1}{2}\arctan\frac{x}{2} + C}$

MEMORY TRICK **Cover it, plug it**

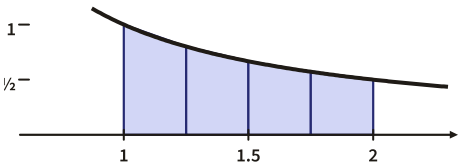
For a factor $(x - a)$: **cover** it, **plug** $x = a$ into the rest. The result is its constant.

6 NUMERICAL INTEGRATION — when antiderivatives fail

The rules

Rule ($\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$)	Formula	Error bound
Midpoint M_n	$\Delta x[f(m_1) + f(m_2) + \dots + f(m_n)]$, m_i = midpoint of strip i	$\frac{K_2(b-a)^3}{24n^2}$
Trapezoid T_n	$\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$	$\frac{K_2(b-a)^3}{12n^2}$
Simpson S_n (n even)	$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n)]$	$\frac{K_4(b-a)^5}{180n^4}$

$K_j \ge |f^{(j)}|$ on [a, b]. T_n = average of the left and right sums; $S_{2n} = (2M_n + T_n)/3$. Simpson is exact for cubics. Concave up: $M_n \le$ exact $\le T_n$.



T_4 for $\int_1^2 dx/x$. The curve is concave up, so each trapezoid sits above it: T_4 overestimates.

$\int_1^2 dx/x = \ln 2 \approx 0.693147$

n = 4	Value	Error	Bound
T_4	0.697024	+0.003877	0.0104
M_4	0.691220	−0.001927	0.0052
S_4	0.693254	+0.000107	0.00052

$|f''| = 2/x^3 \le 2 = K_2$; $|f^{(4)}| = 24/x^5 \le 24 = K_4$ on [1, 2].

Strips for error < 0.0001 WORKED

Trapezoid: $\frac{2}{12n^2} < 0.0001 \to$ **n = 41** — *$n^2 > 1666.7$*

Simpson: $\frac{24}{180n^4} < 0.0001 \to$ **n = 8** — *$n^4 > 1333.3$, n even*

7 IMPROPER INTEGRALS — infinite limits or blow-ups

Two types

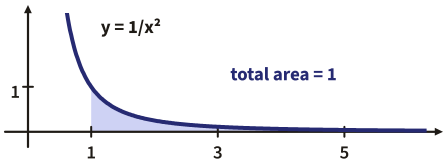
If f blows up at c in [a, b], split at c and use one-sided limits. **Converges** if every limit is finite; otherwise **diverges**.

$$\int_a^\infty f\,dx = \lim_{b\to\infty} \int_a^b f\,dx$$

Worked limits WORKED

$\int_0^\infty e^{-2x}\,dx = \lim \frac{1 - e^{-2b}}{2} = \mathbf{\frac{1}{2}}$

$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} (2 - 2\sqrt{t}) = \mathbf{2}$ — *$p = ½$*



An infinitely long region can have finite area: $\int_1^\infty x^{-2}\,dx = \lim (1 - 1/b) = 1$.

Comparison test

- $0 \le f \le g$: $\int g$ converges ⇒ $\int f$ converges; $\int f$ diverges ⇒ $\int g$ diverges.

EX $0 < e^{-x^2} \le e^{-x}$ on $[1, \infty)$, $\int_1^\infty e^{-x}\,dx = 1/e$: **converges**.

Benchmarks

p-integral	Converges when	Value
$\int_1^\infty \frac{dx}{x^p}$	p > 1	$\frac{1}{p-1}$
$\int_0^1 \frac{dx}{x^p}$	p < 1	$\frac{1}{1-p}$

p = 1 diverges in both: $\int_1^\infty dx/x = \lim \ln b = \infty$. Two-sided: $\int_{-\infty}^\infty dx/(1+x^2) = \pi$ (split at 0).

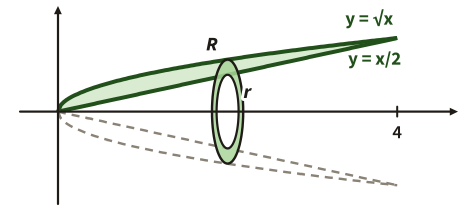
⚠ Hidden blow-up

- $\int_{-1}^1 \frac{dx}{x^2}$ is **not** −2: it blows up at x = 0 and **diverges**.

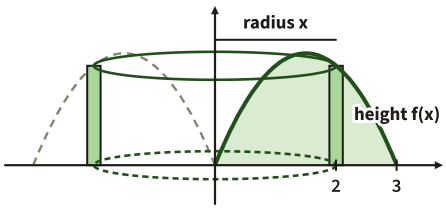
8 AREA & VOLUME — slice, then add slices

Set-ups (Calculus 1 guide §13 has first examples)

Method	Slice	Formula	Method	Slice	Formula
Area (vertical)	top − bottom	$A = \int_a^b$ (top − bottom) dx	Washer	⊥ axis, with a hole	$V = \pi\int (R^2 - r^2)\,dx$
Area (horizontal)	right − left	$A = \int_c^d$ (right − left) dy	Shell	∥ axis	$V = 2\pi\int$ (radius)(height) dx
Disk	⊥ axis, solid	$V = \pi\int R^2\,dx$	Cross sections	⊥ the base	$V = \int A(x)\,dx$



Region between $y = \sqrt{x}$ and $y = x/2$ turned about the x-axis: each slice is a washer, outer radius $R = \sqrt{x}$, inner $r = x/2$.



Region under $y = x(3 - x)$ turned about the y-axis: a shell has radius x, height f(x), thickness dx.

Washer WORKED

Curves meet where $\sqrt{x} = x/2$: x = 0, 4

$V = \pi\int_0^4 (x - \frac{x^2}{4})\,dx = \pi(8 - \frac{16}{3}) = \mathbf{\frac{8\pi}{3}}$ — *$R^2 - r^2$*

Horizontal slices WORKED

Between $x = y^2$ and $x = y + 2$: $y^2 = y + 2 \to y = -1, 2$

$A = \int_{-1}^2 (y + 2 - y^2)\,dy = \mathbf{\frac{9}{2}}$ — *right − left*

Shell WORKED

$V = 2\pi\int_0^3 x \cdot x(3 - x)\,dx = 2\pi(27 - \frac{81}{4})$

$= \mathbf{\frac{27\pi}{2} \approx 42.41}$

About x = −1: radius x + 1 → $V = \mathbf{\frac{45\pi}{2}}$ — *radius from the axis*

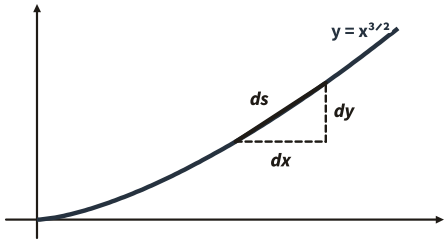
Square cross sections WORKED

Base between $y = x$ and $y = x^2$ on [0, 1]; side $x - x^2$

$V = \int_0^1 (x - x^2)^2\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \mathbf{\frac{1}{30}}$

9 ARC LENGTH, SURFACE AREA, WORK & MASS

more sums of small pieces



Zoom in and a curve is a hypotenuse: $ds = \sqrt{(dx^2 + dy^2)} = \sqrt{(1 + (dy/dx)^2)}\,dx$.

Surface area WORKED

$y = \sqrt{x}$ on $[0, 2]$: $f\sqrt{(1 + (f')^2)} = \sqrt{(x + ¼)}$

$S = 2\pi\int_0^2 \sqrt{(x + ¼)}\,dx = \frac{13\pi}{3} \approx \mathbf{13.61}$

Spring (Hooke: F = kx) WORKED

10 N stretches it 0.2 m → $k = 50$ N/m

$W = \int_0^{0.3} 50x\,dx = \mathbf{2.25\ J}$ *stretch 0.3 m*

Formulas

Quantity	Integral
Arc length	$L = \int_a^b \sqrt{(1 + [f'(x)]^2)}\,dx$
Surface area (about the x-axis)	$S = 2\pi\int_a^b f(x)\sqrt{(1 + [f'(x)]^2)}\,dx$
Work (variable force)	$W = \int_a^b F(x)\,dx$
Pumping liquid	$W = \int$ (weight density)(area of slice)(distance lifted) dy
Hydrostatic force	$F = \int$ (weight density)(depth)(width) dy
Mass of a rod	$m = \int_a^b \rho(x)\,dx$; $\bar{x} = \frac{1}{m}\int_a^b x\,\rho(x)\,dx$
Centroid of a region	$\bar{x} = \frac{1}{A}\int x(f - g)\,dx$; $\bar{y} = \frac{1}{A}\int \frac{1}{2}(f^2 - g^2)\,dx$

Arc length WORKED

$y = x^{3/2}$ on $[0, 4]$: $1 + (f')^2 = 1 + \frac{9x}{4}$

$L = \frac{8}{27}[(1 + \frac{9x}{4})^{3/2}]_0^4 = \frac{8}{27}(10\sqrt{10} - 1) \approx \mathbf{9.073}$

Pumping a tank WORKED

Cylinder, radius 2 m, height 5 m, full of water (9,800 N/m³), pumped over the top

$W = \int_0^5 9800 \cdot 4\pi \cdot (5 - y)\,dy = 490{,}000\pi \approx \mathbf{1.54 \times 10^6\ J}$

slice at height y

Rod and centroid WORKED

$\rho = 1 + x$ on $[0, 2]$: $m = 4$, moment $= \frac{14}{3} \to \bar{x} = \mathbf{\frac{7}{6}}$

Under $y = x^2$, $0 \le x \le 1$: $A = \frac{1}{3}$, centroid $(\mathbf{\frac{3}{4}}, \mathbf{\frac{3}{10}})$

10 SEQUENCES

$a_1, a_2, a_3, \ldots$

Key facts

Converges $\lim_{n\to\infty} a_n = L$ (finite); otherwise **diverges**

Function rule if $a_n = f(n)$ and $f(x) \to L$, then $a_n \to L$ (so L'Hôpital may help)

Squeeze $b_n \le a_n \le c_n$ and $b_n, c_n \to L \Rightarrow a_n \to L$

Monotone & bounded an increasing sequence bounded above converges (likewise decreasing and bounded below)

Geometric $r^n \to 0$ if $|r| < 1$; $r^n \to 1$ if $r = 1$; diverges otherwise

Common limits

a_n	Limit	a_n	Limit
$\frac{1}{n^p}$ $(p > 0)$	0	$\frac{\ln n}{n}$	0
$n^{1/n}$	1	$\frac{x^n}{n!}$	0 (any x)
$(1 + \frac{x}{n})^n$	e^x	$\frac{n!}{n^n}$	0

Recursive sequence WORKED

$a_1 = 1$, $a_{n+1} = \sqrt{(2 + a_n)}$: 1, 1.732, 1.932, 1.983, …

increasing and bounded above by 2, so it converges

monotone & bounded

limit $L = \sqrt{(2 + L)} \to L^2 - L - 2 = 0 \to \mathbf{L = 2}$ *L > 0*

Four quick ones WORKED

$\frac{3n^2 - n}{5n^2 + 2} \to \mathbf{\frac{3}{5}}$ *divide by n^2*

$\frac{\ln n}{n} \to \frac{1/x}{1} \to \mathbf{0}$ *L'Hôpital on f(x)*

$(1 + \frac{2}{n})^n \to \mathbf{e^2}$

$(-1)^n$: −1, 1, −1, … **diverges** *no single limit*

MEMORY TRICK **Growth ladder**

For large n: $\ln n \ll n^p \ll b^n\ (b > 1) \ll n! \ll n^n$. The faster one on the bottom of a fraction sends it to 0.

11 INFINITE SERIES

limits of partial sums

Partial sums

$S_N = a_1 + a_2 + \ldots + a_N$. The series **converges** to S if $S_N \to S$. Changing or dropping finitely many terms never changes convergence (only the sum).

$$\mathbf{\Sigma_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N}$$

Geometric series

Diverges if $|r| \ge 1$. a = first term, r = ratio of consecutive terms.

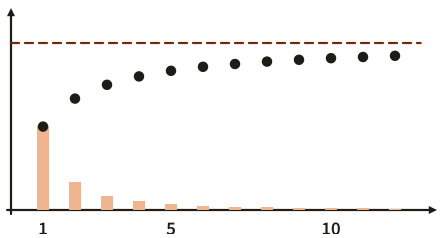
$$\mathbf{\Sigma_{n=0}^{\infty} a\,r^n = \frac{a}{1 - r}\ \text{if}\ |r| < 1}$$

Geometric WORKED

$\Sigma_{n=1}^{\infty} 3(\frac{2}{5})^n$: first term $\frac{6}{5}$, $r = \frac{2}{5}$

sum $= \frac{6/5}{1 - 2/5} = \mathbf{2}$

$0.272727\ldots = \frac{0.27}{1 - 0.01} = \mathbf{\frac{3}{11}}$ *repeating decimal*



Telescoping: $\Sigma\ 1/(n(n + 1)) = \Sigma\ (1/n - 1/(n + 1))$, so $S_N = 1 - 1/(N + 1) \to 1$. Bars: terms; dots: partial sums.

Basics

nth-term test $\lim a_n \ne 0 \Rightarrow$ the series **diverges** ($\Sigma \frac{n}{2n + 1}$: terms $\to \frac{1}{2}$). If $\lim a_n = 0$, the test says nothing.

Harmonic series $\Sigma \frac{1}{n}$ diverges, though its terms $\to 0$ ($S_N \approx \ln N$)

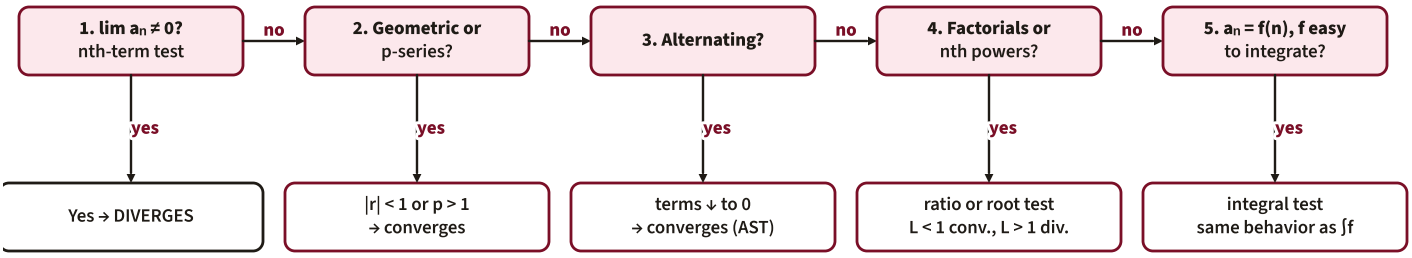
p-series $\Sigma \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \le 1$; $\Sigma \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$ (Euler)

Linearity $\Sigma(c\,a_n \pm b_n) = c\Sigma a_n \pm \Sigma b_n$ when both converge

12 CONVERGENCE TESTS — which test, and when

Test summary

Test	Use on	Condition	Conclusion
nth-term	any series	$\lim a_n \neq 0$	diverges (never proves convergence)
Geometric	$\Sigma\, a\, r^n$	$\lvert r\rvert < 1$ / $\lvert r\rvert \geq 1$	converges to $\frac{a}{1-r}$ / diverges
p-series	$\Sigma\, 1/n^p$	$p > 1$ / $p \leq 1$	converges / diverges
Integral	$a_n = f(n)$, f positive, continuous, decreasing	$\int_1^\infty f(x)\,dx$ finite / infinite	same as the integral; remainder $\int_{N+1}^\infty f \leq R_N \leq \int_N^\infty f$
Direct comparison	$0 \leq a_n \leq b_n$	Σb_n converges / Σa_n diverges	Σa_n converges / Σb_n diverges
Limit comparison	$a_n, b_n > 0$	$\lim a_n/b_n = c$, $0 < c < \infty$	both converge or both diverge
Alternating series	$\Sigma(-1)^n b_n$, $b_n > 0$	b_n decreasing and $b_n \to 0$	converges; $\lvert\text{error}\rvert \leq$ first omitted term
Ratio	factorials, powers like 3^n	$L = \lim \lvert a_{n+1}/a_n\rvert$	$L < 1$ converges absolutely; $L > 1$ diverges; $L = 1$ no information
Root	$a_n = (\ldots)^n$	$L = \lim \lvert a_n\rvert^{1/n}$	same conclusions as the ratio test



Still stuck: compare (direct or limit comparison) with a geometric or p-series that looks like it. For any test, check $\lvert a_n\rvert$ too: $\Sigma\lvert a_n\rvert$ converges ⇒ absolutely convergent; Σa_n converges but $\Sigma\lvert a_n\rvert$ diverges ⇒ conditionally convergent.

Ratio WORKED

$\Sigma \frac{n^2}{3^n}$: $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{3n^2} \to \frac{1}{3}$

$L < 1$: **converges**

Root WORKED

$\Sigma \left(\frac{2n}{3n+1}\right)^n$: root $= \frac{2n}{3n+1} \to \frac{2}{3}$

converges

Integral WORKED

$\Sigma_{n=2}^\infty \frac{1}{n \ln n}$: $\int_2^b \frac{dx}{x \ln x} = \ln(\ln b) - \ln(\ln 2) \to \infty$

$u = \ln x$

diverges

Ratio with factorials WORKED

$\Sigma \frac{n!}{n^n}$: ratio $= \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$

converges

$\Sigma_{n=1}^\infty \frac{3^n}{n!} = \mathbf{e^3 - 1}$ *converges, ratio → 0*

Limit comparison WORKED

$\Sigma \frac{n+1}{n^3+2}$ vs $\Sigma \frac{1}{n^2}$: ratio → 1

$p = 2 > 1$: **converges**

$\Sigma \frac{1}{\sqrt{(n^3+1)}}$ vs $\frac{1}{n^{3/2}}$: **converges**

MEMORY TRICK

Never Guess; Always Run It Carefully

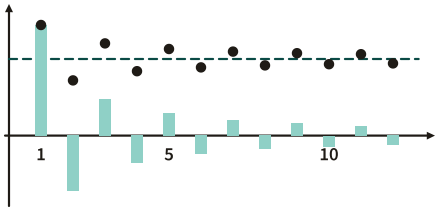
Test order: **N**th-term, **G**eometric/p, **A**lternating, **R**atio/root, **I**ntegral, **C**omparison.

13 ALTERNATING SERIES & ABSOLUTE CONVERGENCE — signs that switch

Alternating series test

Error bound: $\lvert S - S_N\rvert \leq b_{N+1}$, and S lies between consecutive partial sums.

$\Sigma(-1)^{n+1} b_n$ converges if $b_n \downarrow$ and $b_n \to 0$



Alternating harmonic series: partial sums zigzag in toward $\ln 2 \approx 0.693$.

Two kinds of convergence

Absolutely convergent — $\Sigma\lvert a_n\rvert$ converges (then Σa_n converges too)

Conditionally convergent — Σa_n converges but $\Sigma\lvert a_n\rvert$ diverges

Rearrangement — a conditionally convergent series can be reordered to sum to any number (Riemann); absolute sums never change

Classify WORKED

$\Sigma \frac{(-1)^{n+1}}{n} = \ln 2$: $\Sigma\, 1/n$ diverges — ***conditional***

$\Sigma \frac{(-1)^{n+1}}{n^2}$: $\Sigma\, 1/n^2$ converges — ***absolute***

$\Sigma \frac{(-1)^n n}{n+1}$: terms do not → 0 — ***diverges***

How many terms? WORKED

$\Sigma \frac{(-1)^{n+1}}{n^3}$ with error < 0.001

need $\frac{1}{(N+1)^3} < 0.001 \to N + 1 \geq 11$

$1/11^3 \approx 0.00075$

N = 10 terms

14 POWER SERIES — a polynomial that never ends

Power series about x = a

Converges for $|x-a| < R$ (R = **radius**), diverges for $|x-a| > R$; **test both endpoints separately** to get the interval. R may be 0 or ∞.

$$\Sigma_{n=0}^{\infty} c_n(x-a)^n$$

Term by term (same R)

- $f'(x) = \Sigma\, n\, c_n(x-a)^{n-1}$
- $\int f(x)\,dx = C + \Sigma\, c_n \dfrac{(x-a)^{n+1}}{n+1}$
- Endpoints may gain or lose convergence: recheck them.

MEMORY TRICK

Ratio finds R; endpoints finish it

The ratio test gives the **radius** but never decides the **endpoints**: test each one by itself.

Radius and interval WORKED

$\Sigma \dfrac{(x-2)^n}{n\cdot 3^n}$: ratio → $\dfrac{|x-2|}{3} < 1$

$R = 3$: open interval $(-1, 5)$

$x = 5$: $\Sigma\, 1/n$ diverges; $x = -1$: $\Sigma\,(-1)^n/n$ converges *endpoints*

interval **[−1, 5)**

Quick radii (ratio test)

Series	Radius
$\Sigma \dfrac{x^n}{n!}$	∞
$\Sigma\, n!\, x^n$	0
$\Sigma \dfrac{n\,x^n}{4^n}$	4

Building from the geometric series

Start	Operation	Result ($\lvert x\rvert < 1$)
$\dfrac{1}{1-x} = \Sigma\, x^n$	$x \to -x^2$	$\dfrac{1}{1+x^2} = \Sigma\,(-1)^n x^{2n}$
$\dfrac{1}{1+x^2}$	integrate	$\arctan x = \Sigma \dfrac{(-1)^n x^{2n+1}}{2n+1}$
$\dfrac{1}{1+x} = \Sigma\,(-1)^n x^n$	integrate	$\ln(1+x) = \Sigma \dfrac{(-1)^{n+1} x^n}{n}$
$\dfrac{1}{1-x}$	differentiate, $\times x$	$\dfrac{x}{(1-x)^2} = \Sigma\, n\, x^n$

At $x = 1$, arctan gives $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \ldots$ (Gregory–Leibniz) and $\ln(1+x)$ gives $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \ldots$.

15 TAYLOR & MACLAURIN SERIES — matching every derivative at one point

Series to know

Function	Maclaurin series	Converges for
e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots = \Sigma \dfrac{x^n}{n!}$	all x
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots = \Sigma \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots = \Sigma \dfrac{(-1)^n x^{2n}}{(2n)!}$	all x
$\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + \ldots = \Sigma\, x^n$	$-1 < x < 1$
$\ln(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots = \Sigma \dfrac{(-1)^{n+1} x^n}{n}$	$-1 < x \le 1$
$\arctan x$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots = \Sigma \dfrac{(-1)^n x^{2n+1}}{2n+1}$	$-1 \le x \le 1$
$(1+x)^k$	$1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \ldots$	$-1 < x < 1$ (binomial)

Taylor series of f about x = a

$a = 0$: **Maclaurin** series. Stopping at degree n gives the Taylor polynomial $T_n(x)$.

$$f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Lagrange error bound

$M \ge |f^{(n+1)}|$ between a and x. For an alternating series, the first omitted term also bounds the error.

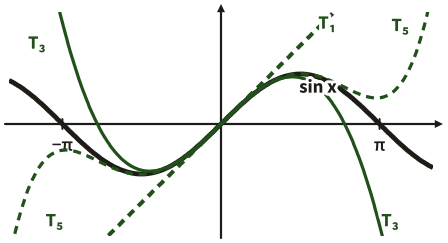
$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

Binomial series WORKED

$\sqrt{(1+x)} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \ldots$ *$k = ½$*

$\sqrt{1.1} \approx 1 + 0.05 - 0.00125 =$ **1.04875**

true 1.048809



Higher-degree Taylor polynomials hug sin x over a wider interval.

Taylor polynomial and error WORKED

$\ln x$ about 1: $T_3 = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3}$

$\ln 1.1 \approx 0.1 - 0.005 + 0.000333 =$ **0.095333**

true 0.095310

$f^{(4)} = -6/x^4$, $M = 6$: $|R_3| \le \dfrac{6(0.1)^4}{24} =$ **0.000025**

actual 0.000023

Integrate a series WORKED

$e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \ldots$ *$x \to -x^2$*

$\int_0^1 e^{-x^2}\,dx \approx 1 - \dfrac{1}{3} + \dfrac{1}{10} - \dfrac{1}{42} + \dfrac{1}{216} \approx$ **0.7475**

true 0.7468

alternating: error $\le \dfrac{1}{1320} \approx 0.00076$

MEMORY TRICK **Odd sine, even cosine**

Sine: odd powers from x. **Cosine**: even powers from 1. Both alternate signs; $\mathbf{e^x}$ keeps every power, all plus.

Limits and identities WORKED

$\lim_{x\to 0} \dfrac{\sin x - x}{x^3} = \lim \dfrac{-x^3/6 + \ldots}{x^3} =$ **$-\dfrac{1}{6}$**

$e \approx \Sigma_{n=0}^{5} \dfrac{1}{n!} =$ **2.716667** *error < 3/6! ≈ 0.0042*

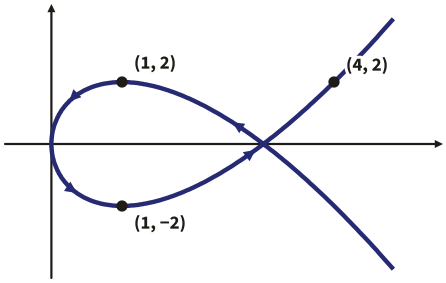
Series show $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler)

16 PARAMETRIC CURVES & MOTION

x and y from t

Formulas

Quantity	Formula
Slope	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $(dx/dt \neq 0)$
Second derivative	$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt}$
Horizontal / vertical tangent	dy/dt = 0 / dx/dt = 0 (not both)
Arc length	$L = \int_\alpha^\beta \sqrt{((dx/dt)^2 + (dy/dt)^2)}\, dt$
Area under the curve	$A = \int y\, dx = \int_\alpha^\beta y(t)\, x'(t)\, dt$
Surface area (x-axis)	$S = 2\pi \int y \sqrt{((dx/dt)^2 + (dy/dt)^2)}\, dt$



$x = t^2$, $y = t^3 - 3t$ for $-2.2 \le t \le 2.2$; arrows show increasing t. The loop closes at (3, 0) ($t = \pm\sqrt{3}$).

Slopes and tangents WORKED

$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$; at t = 2: point (4, 2), slope $\frac{9}{4}$

horizontal at t = ±1: **(1, −2), (1, 2)**; vertical at t = 0: **(0, 0)**

$\frac{d^2y}{dx^2} = \frac{3t^2 + 3}{4t^3}$; at t = 2: $\frac{15}{32}$ (concave up)

Cycloid (one arch) WORKED

$x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$

$L = \int_0^{2\pi} \sqrt{(2 - 2\cos t)}\, dt = \mathbf{8}$

$A = \int_0^{2\pi} (1 - \cos t)^2\, dt = \mathbf{3\pi}$

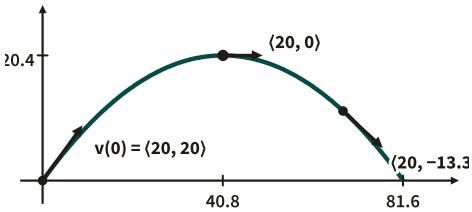
Roberval (area), Wren (length)

17 VECTOR-VALUED FUNCTIONS & MOTION

position moving with t

Definitions

Quantity	Formula
Position	r(t) = ⟨x(t), y(t)⟩; d/dt and ∫ act on each component (∫ adds a constant **vector**)
Velocity, acceleration	v = r′; a = v′ = r″
Speed	$\lvert v\rvert = \sqrt{(x'^2 + y'^2)}$
Unit tangent	T = r′/|r′| (§16 curve at t = 2: ⟨4, 9⟩/√97)
Displacement	$\int_a^b v\, dt = r(b) - r(a)$
Distance	$\int_a^b \lvert v\rvert\, dt$
Circular motion	r = ⟨R cos ωt, R sin ωt⟩: speed Rω; |a| = Rω² toward the center



Projectile: r(t) = ⟨20t, 20t − 4.9t²⟩ m. Velocity arrows (scaled) keep their horizontal part 20 m/s; the vertical part shrinks, hits 0 at the top, then turns negative.

Projectile WORKED

a = ⟨0, −9.8⟩, v(0) = ⟨20, 20⟩ m/s, r(0) = ⟨0, 0⟩

v = ⟨20, 20 − 9.8t⟩; r = ⟨20t, 20t − 4.9t²⟩ *integrate twice*

top at $t = \frac{20}{9.8} \approx 2.04$ s, height ≈ **20.4 m** *vertical v = 0*

lands at t ≈ 4.08 s, **81.6 m** away *y = 0*

From velocity to position WORKED

$v(t) = \langle t^2, t\rangle$, $r(0) = \langle 1, 2\rangle \to r(t) = \langle 1 + \frac{t^3}{3}, 2 + \frac{t^2}{2}\rangle$

speed at t = √3: $\sqrt{(9 + 3)} = \mathbf{2\sqrt{3}}$

distance, $0 \le t \le \sqrt{3}$: $\int t\sqrt{(t^2 + 1)}\, dt = \frac{7}{3}$ *$u = t^2 + 1$*

18 POLAR CURVES

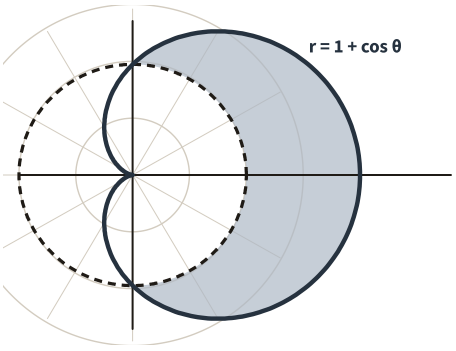
radius r at angle θ

Conversions

To rectangular $x = r\cos\theta$, $y = r\sin\theta$

To polar $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

Slope $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$



Cardioid r = 1 + cos θ with the circle r = 1 (dashed); shaded: inside the cardioid, outside the circle.

Common graphs

Curve	Equation	Shape
Circle	r = a; r = 2a cos θ; r = 2a sin θ	center (0,0); (a, 0); (0, a)
Cardioid	r = a(1 ± cos θ)	heart, through the pole
Limaçon	r = a + b cos θ	inner loop if |a| < |b|
Rose	r = a cos nθ	n petals (n odd), 2n (n even)
Lemniscate	$r^2 = a^2 \cos 2\theta$	figure eight
Spiral	r = aθ	Archimedes

Calculus in polar

Quantity	Formula
Area	$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$
Area between	$A = \frac{1}{2}\int_\alpha^\beta (R^2 - r^2)\, d\theta$
Arc length	$L = \int_\alpha^\beta \sqrt{(r^2 + (dr/d\theta)^2)}\, d\theta$

Areas WORKED

Whole cardioid: $\frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2\, d\theta = \frac{3\pi}{2}$

Shaded: $\frac{1}{2}\int_{-\pi/2}^{\pi/2} [(1 + \cos\theta)^2 - 1]\, d\theta = 2 + \frac{\pi}{4}$

curves meet at $\vartheta = \pm\pi/2$

Cardioid length: $\int_0^{2\pi} \sqrt{(2 + 2\cos\theta)}\, d\theta = \mathbf{8}$

MEMORY TRICK **Half r squared**

A polar region is a fan of thin wedges, each ½r² dθ. Integrate between the angles that bound it.

Slope and conversion WORKED

$r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$: r = 1, r′ = −1 → $\frac{dy}{dx} = \mathbf{1}$

$r = 4\sin\theta \to x^2 + y^2 = 4y \to \mathbf{x^2 + (y - 2)^2 = 4}$

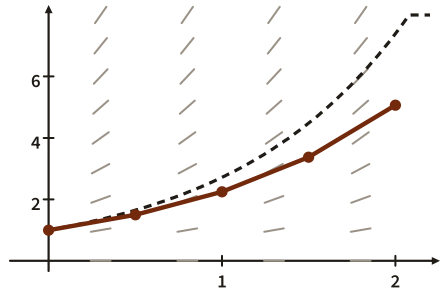
circle, radius 2

19 DIFFERENTIAL EQUATIONS

basics: Calculus 1 §14

Separable: dy/dx = g(x) h(y)

- Move y-terms to dy and x-terms to dx: $\frac{dy}{h(y)} = g(x)\,dx$.
- Integrate both sides; one + C.
- Use the initial condition; solve for y if you can.



y′ = y, y(0) = 1, h = 0.5: Euler's polygon (solid) reaches 5.06 at x = 2; the true solution e^x (dashed) reaches 7.39.

MEMORY TRICK

Step from where you stand

Use the slope at the point you are **standing** on, never the point you are stepping to.

Separate WORKED

$\frac{dy}{dx} = 2x(1 + y^2)$, y(0) = 0

$\frac{dy}{1 + y^2} = 2x\,dx \rightarrow \arctan y = x^2 + C$, C = 0

$y = \tan(x^2)$ *for $|x| < \sqrt{(\pi/2)}$*

Euler's method

Step along the tangent line at the **current** point; smaller h means smaller error.

$$y_{n+1} = y_n + h \cdot F(x_n, y_n)$$

Euler for y′ = x + y, y(0) = 1, h = 0.1

n	x_n	y_n	slope $x_n + y_n$
0	0	1	1
1	0.1	1.1	1.2
2	0.2	**1.22**	

True $y = 2e^x - x - 1$ gives y(0.2) ≈ 1.2428.

Logistic numbers WORKED

K = 1000, P_0 = 100, k = 0.5 → A = 9

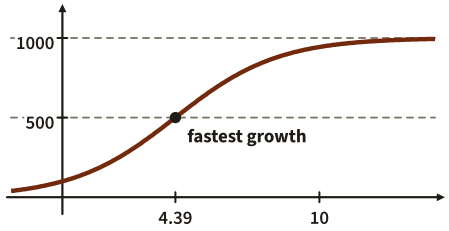
$P(4) = \frac{1000}{1 + 9e^{-2}} \approx$ **450.9**

P = 500 when $9e^{-0.5t} = 1$: t = 2 ln 9 ≈ **4.39**

P = K/2

Models

Model	Equation	Solution
Exponential growth/decay	$\frac{dy}{dt} = ky$	$y = y_0e^{kt}$
Newton's cooling	$\frac{dT}{dt} = -k(T - T_s)$	$T = T_s + (T_0 - T_s)e^{-kt}$
Logistic	$\frac{dP}{dt} = kP(1 - \frac{P}{K})$	$P = \frac{K}{1 + Ae^{-kt}}$, $A = \frac{K - P_0}{P_0}$



Logistic $P = 1000/(1 + 9e^{-0.5t})$: steepest at P = K/2, leveling off at K.

Cooling WORKED

Coffee: 90 °C → 70 °C in 10 min, room 20 °C

$e^{-10k} = 5/7$

T = 50 °C at $t = 10 \cdot \frac{\ln(3/7)}{\ln(5/7)} \approx$ **25.2 min**

20 COMMON SLIPS & PRACTICE

check yourself

Twelve problems (answers checked by the build)

#	Problem	Answer	#	Problem	Answer
1	$\int x \cos x\,dx$	**$x \sin x + \cos x + C$**	7	Radius of $\Sigma \frac{n\,x^n}{4^n}$	**R = 4**
2	$\int \frac{dx}{x^2 - 9}$	**$\frac{1}{6}\ln\lvert\frac{x-3}{x+3}\rvert + C$**	8	Maclaurin series of $x\,e^x$	**$x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$**
3	$\int_0^{\pi/2} \sin^3 x\,dx$	**$\frac{2}{3}$**	9	Area inside r = 2 sin θ	**π**
4	$\int_2^{\infty} \frac{dx}{x^3}$	**$\frac{1}{8}$**	10	Length of $y = \frac{2}{3}x^{3/2}$, 0 ≤ x ≤ 3	**$\frac{14}{3}$**
5	$\Sigma_{n=0}^{\infty} (-\frac{2}{5})^n$	**$\frac{5}{7}$**	11	Euler: y′ = y − x, y(0) = 2, h = 0.5, two steps	**y(1) ≈ 4.25**
6	Does $\Sigma \frac{n+1}{n^3+2}$ converge?	**yes**	12	Volume: $y = e^x$, 0 ≤ x ≤ 1, about the x-axis	**$\frac{\pi(e^2 - 1)}{2} \approx 10.04$**

Common slips

Slip	Fix
changing variables but keeping the old limits	convert the limits, or switch back to x
missing a blow-up inside [a, b]	look for zeros of the denominator first
"lim $a_n = 0$, so Σa_n converges"	the nth-term test only proves divergence
skipping the endpoints	test each endpoint series separately
shell radius measured from the y-axis for a different axis	radius = distance to the axis of rotation
partial fractions on an improper fraction	long-divide first
polar area without the ½ or with the wrong θ-limits	$A = \frac{1}{2}\int r^2\,d\theta$ between the right angles
forgetting + C, or the constant vector in motion problems	add it, then use the initial condition
ratio test L = 1 read as "converges"	L = 1 decides nothing: try another test
degrees in a series or derivative	radians always