PRE-CALCULUS · UNITS 1–9 · QUICK REFERENCE

PRE-CALCULUS

Grades 11–12 · 6 pages

Functions and transformations, inverses, polynomial, rational, exponential and log functions, conics, matrices, parametric equations, series, counting, limits and rates of change.

1 Functions · 2 Transformations · 3 Inverses · 4 Polynomials · 5 Rational · 6 Exp & log · 7 Conics · 8 Matrices · 9 Parametric · 10 Series & binomials · 11 Counting · 12 Limits · 13 Rates

1 FUNCTIONS & THEIR FEATURES

one input, one output, and how the graph behaves

Function	each input has exactly one output
Domain / range	allowed inputs / outputs produced
Zero	an input with f(x) = 0 (an x-intercept)
y-intercept	the point (0, f(0))
Basics	vertical line test, evaluating, f + g: **Algebra §10**

Interval notation

- [a, b] includes the ends; (a, b) leaves them out
- ∞ always takes a parenthesis; ∪ joins pieces

EX $x \le -1$ or $x > 3 \to$ **$(-\infty, -1] \cup (3, \infty)$**

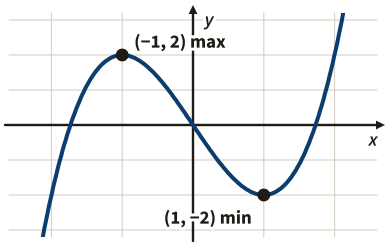
EX $-2 < x \le 5 \to$ **$(-2, 5]$**

Domain of $\frac{\sqrt{(x+3)}}{x-2}$ WORKED

$x + 3 \ge 0 \to x \ge -3$ *even root*

$x \ne 2$ *denominator*

domain **$[-3, 2) \cup (2, \infty)$**



f(x) = x³ − 3x rises on (−∞, −1) and (1, ∞), falls on (−1, 1).

Increasing, decreasing, extrema

- **Increasing** on an interval: $x_1 < x_2$ gives $f(x_1) < f(x_2)$.
- A **local max (min)** is a peak (valley) of nearby values.
- Name intervals by their **x**-values, with open ends.

Symmetry

- **Even**: f(−x) = f(x), mirror in the y-axis.
- **Odd**: f(−x) = −f(x), half-turn about the origin.

EX $x^3 - 3x$: $f(-x) = -x^3 + 3x = -f(x) \to$ **odd**

Difference quotient

The slope between x and x + h; it becomes the derivative in §13.

$$\frac{f(x+h) - f(x)}{h},\ h \ne 0$$

f(x) = x² − 3x WORKED

$f(x+h) - f(x) = 2xh + h^2 - 3h$ *expand, cancel*

$\div h \to$ **$2x + h - 3$**

Piecewise WORKED

$f(x) = x^2$ for $x < 1$; $3 - x$ for $x \ge 1$

$f(-2) =$ **4**, $f(1) =$ **2**, $f(4) =$ **−1** *pick the piece first*

Average rate of change WORKED

$f(x) = x^3 - 3x$ on [0, 2]

$\frac{f(2) - f(0)}{2 - 0} = \frac{2 - 0}{2} =$ **1** *secant slope*

MEMORY TRICK **DR. ZIPS**

Describe any graph in six moves: **D**omain, **R**ange, **Z**eros, **I**ntervals (up or down), **P**eaks and valleys, **S**ymmetry.

2 PARENT FUNCTIONS & TRANSFORMATIONS

a few shapes, moved and stretched

Parent functions (graphs: Algebra §10)

Parent	Domain	Range	Symmetry
x, x^3, $\sqrt[3]{x}$	all reals	all reals	odd
x^2, $\lvert x \rvert$	all reals	$[0, \infty)$	even
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$	none
$\frac{1}{x}$	$x \ne 0$	$y \ne 0$	odd
b^x	all reals	$(0, \infty)$	none
$\log_b x$	$(0, \infty)$	all reals	none

Transformations of y = f(x)

Change	Effect on the graph
f(x) + k	up k (down if k < 0)
f(x − h)	right h (left if h < 0)
−f(x)	reflect over the x-axis
f(−x)	reflect over the y-axis
a · f(x)	vertical stretch by |a|
f(bx)	horizontal shrink by $\frac{1}{b}$

MEMORY TRICK

Inside Opposite, Outside Obvious

Changes **inside** f( ) act on x and run backwards: f(x − 3) moves **right**, f(2x) **shrinks**. Changes **outside** do just what they say.

Point mapping for y = a·f(b(x − h)) + k

Move a few key points of the parent; draw through them.

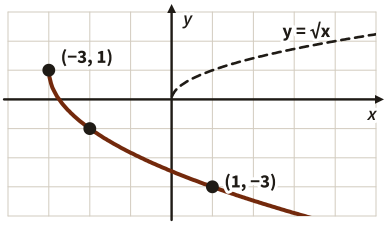
$$(x, y) \to \left(\frac{x}{b} + h,\ ay + k\right)$$

g(x) = −2√(x + 3) + 1 WORKED

left 3, stretch 2, flip, up 1

(0, 0) → (−3, 1); (1, 1) → (−2, −1); (4, 2) → (1, −3)

domain **$[-3, \infty)$**, range **$(-\infty, 1]$**



Dashed: √x. Solid: g(x) = −2√(x + 3) + 1.

Absolute value WORKED

$f(x) = |2x - 4| - 3$

vertex **(2, −3)**; zeros: $|2x - 4| = 3 \to x =$ **0.5, 3.5**

Step function

Greatest integer ⟦x⟧: the largest integer ≤ x.

EX ⟦2.7⟧ = **2**, ⟦−1.2⟧ = **−2**

3 COMPOSITION & INVERSES

functions in a chain, and undoing them

Composition

$(f \circ g)(x) = f(g(x))$: apply g first. The domain keeps x in the domain of g **and** g(x) in the domain of f.

Basic examples: Algebra §10.

$f(x) = \sqrt{x}$, $g(x) = x^2 - 4$ WORKED

$(f \circ g)(x) = \sqrt{(x^2 - 4)}$: domain $(-\infty, -2] \cup [2, \infty)$

$(g \circ f)(x) = x - 4$: domain $[0, \infty)$ *not all reals*

Decompose $h(x) = (3x + 1)^5$ WORKED

inner $g(x) = 3x + 1$, outer $f(x) = x^5$ *$h = f \circ g$*

MEMORY TRICK **f AFTER g**

Read $f \circ g$ as "f **after** g": the function written nearest x acts first.

Operations & domain WORKED

$f(x) = \sqrt{x}$, $g(x) = x - 1$

$(\frac{f}{g})(x) = \frac{\sqrt{x}}{x - 1}$: domain $[0, 1) \cup (1, \infty)$

both domains, $g \neq 0$

Inverse functions

- f^{-1} exists only if f is **one-to-one** (passes the horizontal line test).
- $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
- Domain and range trade places; the graphs mirror in **y = x**.

Inverse of $f(x) = \frac{2x + 1}{x - 3}$ WORKED

$x = \frac{2y + 1}{y - 3} \rightarrow xy - 3x = 2y + 1$ *swap x and y*

$y(x - 2) = 3x + 1 \rightarrow f^{-1}(x) = \frac{3x + 1}{x - 2}$

check: $f^{-1}(5) = \frac{16}{3}$, $f(\frac{16}{3}) = 5$ ✓

One-to-one? WORKED

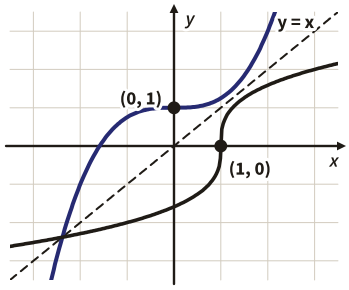
$f(x) = x^2 - 4x$: $f(0) = f(4) = 0$

two inputs, one output

not one-to-one; restrict to $x \geq 2$ *vertex at 2*

MEMORY TRICK **Swap, Solve, Check**

To invert: **S**wap x and y, **S**olve for y, **C**heck that $f(f^{-1}(x)) = x$.



$f(x) = \frac{1}{4}x^3 + 1$ (color) and $f^{-1}(x) = \sqrt[3]{(4(x - 1))}$ (black): (a, b) on $f \leftrightarrow$ (b, a) on f^{-1}.

Restrict, then invert WORKED

$f(x) = (x - 1)^2 + 2$, $x \geq 1$ *one-to-one half*

$f^{-1}(x) = \mathbf{1 + \sqrt{(x - 2)}}$, $x \geq 2$

Inverse from a table WORKED

$f(-1) = 4$, $f(0) = 1$, $f(2) = -3$, $f(5) = -7$

$f^{-1}(4) = \mathbf{-1}$, $f^{-1}(-3) = \mathbf{2}$, $f^{-1}(f(5)) = \mathbf{5}$

read backwards

4 POLYNOMIAL FUNCTIONS

end behavior, zeros, division (factoring: Algebra §4)

End behavior ($a_n x^n$ decides)

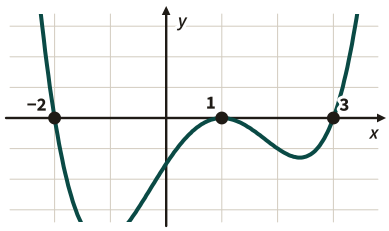
Leading term	a > 0	a < 0
even degree	↖ ↗ both up	↙ ↘ both down
odd degree	↙ ↗ down, up	↖ ↘ up, down

MEMORY TRICK **Even arms Agree**

Even degree: both arms point the **same** way. Odd degree: the arms split. The sign of the leading coefficient sets the right arm.

Zeros & multiplicity

- Factor $(x - c)^m$: **odd** m crosses the x-axis, **even** m touches and turns.
- Degree n: at most n real zeros and at most n − 1 turning points.



$f(x) = \frac{1}{4}(x + 2)(x - 1)^2(x - 3)$: crosses at −2 and 3, touches at 1; degree 4, $a > 0$.

Long division WORKED

$(2x^3 - x^2 + 3x - 4) \div (x^2 + 1)$

$= \mathbf{2x - 1}$, remainder $\mathbf{x - 3}$

so $2x^3 - x^2 + 3x - 4 = (x^2 + 1)(2x - 1) + (x - 3)$

check

3	2	-3	-11	6
		6	9	-6
	2	3	-2	0

Synthetic: $(2x^3 - 3x^2 - 11x + 6) \div (x - 3) = 2x^2 + 3x - 2$, remainder 0, so zeros 3, ½, −2.

Remainder & factor theorems

- P(c) is the remainder of $P(x) \div (x - c)$.
- P(c) = 0 exactly when $(x - c)$ is a factor.

EX $x^4 - 3x^2 + 2x - 5$ at 2: $16 - 12 + 4 - 5 = \mathbf{3}$

Polynomial inequality WORKED

$x^3 - 4x > 0 \rightarrow x(x - 2)(x + 2) > 0$ *factor*

signs on $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$: − + − + *test points*

$\mathbf{(-2, 0) \cup (2, \infty)}$

Counting zeros

- **Fundamental Theorem of Algebra**: degree n has exactly n complex zeros, counting multiplicity.
- **Conjugate pairs**: with real coefficients, a + bi brings a − bi.
- **Descartes' rule**: sign changes in P(x) give the most positive zeros (or fewer by 2s); use P(−x) for negatives.

All zeros of $x^3 - x^2 + 4x - 4$ WORKED

signs + − + −: 3 or 1 positive; P(−x): 0 negative *Descartes*

$x^2(x - 1) + 4(x - 1) = (x - 1)(x^2 + 4)$ *group*

zeros **1, 2i, −2i**

Build from zeros WORKED

degree 3, zeros 2 and 1 + i *so 1 − i too*

$(x - 2)(x^2 - 2x + 2) = \mathbf{x^3 - 4x^2 + 6x - 4}$

Intermediate Value Theorem

If P is continuous and P(a), P(b) have opposite signs, P has a zero between a and b.

EX $x^3 - x - 1$: $P(1) = -1$, $P(2) = 5 \rightarrow$ a zero in (1, 2), $\approx$ **1.3247**

5 RATIONAL FUNCTIONS

a polynomial over a polynomial

MEMORY TRICK

Happy Vets Help Injured Sparrows

Graph in order: **H**oles (canceled factors), **V**ertical asymptotes, **H**orizontal or slant asymptote, **I**ntercepts, **S**ign chart.

Analyze $f(x) = \frac{x^2 - x - 2}{x^2 - 4}$ WORKED

$= \frac{(x-2)(x+1)}{(x-2)(x+2)}$ *factor*

hole at x = 2: $\frac{2+1}{2+2} \to$ **(2, ¾)** *cancel, then plug in*

VA **x = −2**; HA **y = 1** *equal degrees*

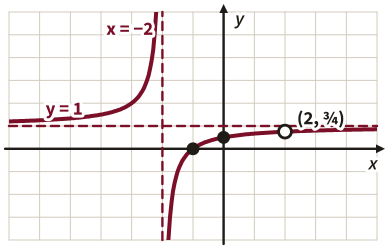
x-int **−1**; y-int **½**

Rational equation WORKED

$\frac{2}{x-1} + \frac{1}{x+1} = \frac{4}{x^2-1}$

× (x − 1)(x + 1): 2(x + 1) + (x − 1) = 4 → x = 1

x = 1 makes a denominator 0: **no solution** *extraneous*



The graph of the analyzed f: a hole, not an asymptote, at x = 2.

Horizontal asymptote (degrees n over d)

- n < d: y = 0 · n = d: ratio of leading coefficients · n > d: none (BOBO: Algebra §10)
- A graph may cross its HA; it never crosses a VA.

Slant asymptote

When n = d + 1, divide: the quotient line is the asymptote.

EX $\frac{x^2 + 3x + 1}{x - 1} = x + 4 + \frac{5}{x-1} \to$ **y = x + 4**

Near a vertical asymptote

- Odd multiplicity: the two sides go **opposite** ways (±∞).
- Even multiplicity: both sides go the **same** way.

Partial fractions

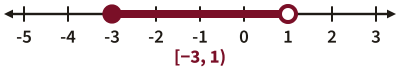
Split a proper fraction over its factored denominator: one constant per linear factor.

EX $\frac{5x-4}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$: x = 2 → A = **2**; x = −1 → B = **3**

Rational inequality $\frac{x+3}{x-1} \le 0$ WORKED

critical points −3 (top = 0) and 1 (undefined)

test −4: +, 0: −, 2: + → **[−3, 1)** *1 never included*

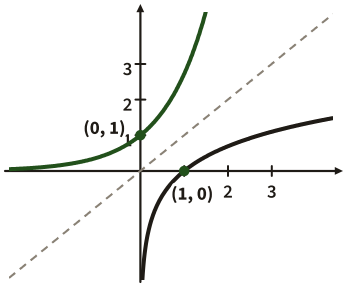


6 EXPONENTIAL & LOGARITHMIC FUNCTIONS

growth, decay and the exponents that measure them

Inverse pair (b > 0, b ≠ 1)

	$y = b^x$	$y = \log_b x$
domain	all reals	(0, ∞)
range	(0, ∞)	all reals
key point	(0, 1)	(1, 0)
asymptote	y = 0	x = 0



e^x (green) and ln x (black) mirror in y = x.

Laws (b, M, N > 0)

- $\log_b MN = \log_b M + \log_b N$; $\log_b \frac{M}{N} = \log_b M - \log_b N$
- $\log_b M^p = p \log_b M$; $\log_b x = \frac{\ln x}{\ln b}$
- $b^{\log_b x} = x$ and $\log_b b^x = x$

More examples: Algebra §10.

MEMORY TRICK **Logs step down**

A log lowers each operation one step: **×** becomes **+**, **÷** becomes **−**, a **power** becomes a **multiplier** in front.

Exponential equations WORKED

$4e^{0.3t} = 20 \to e^{0.3t} = 5 \to t = \frac{\ln 5}{0.3} \approx$ **5.365**

$3^{x+1} = 2^x \to (x+1)\ln 3 = x \ln 2$ *log both sides*

$x = \frac{\ln 3}{\ln 2 - \ln 3} \approx$ **−2.710**

Log equation WORKED

$\log_2 x + \log_2(x-2) = 3 \to x(x-2) = 8$ *combine*

$x^2 - 2x - 8 = 0 \to x = 4$ or −2

x = 4 ($\log_2(-2)$ is undefined) *check*

Quadratic in e^x WORKED

$e^{2x} - 5e^x + 6 = 0 \to (e^x - 2)(e^x - 3) = 0$ *u = e^x*

x = ln 2 ≈ **0.693**, ln 3 ≈ **1.099**

MEMORY TRICK

I Love Solving Carefully

Isolate the power or log, **L**og (or exponentiate) both sides, **S**olve, **C**heck every answer in the original.

Growth & decay

k > 0 growth, k < 0 decay. Doubling time $\frac{\ln 2}{k}$; half-life h: $A = A_0(½)^{t/h}$.

$$\mathbf{A = A_0 e^{kt}}$$

Find k WORKED

5,000 grows to 8,000 in 6 years

$k = \frac{\ln 1.6}{6} \approx$ **0.0783**; doubling ≈ **8.85 yr**

Half-life WORKED

200 g, half-life 8 days, after 20 days

$200(½)^{20/8} \approx$ **35.36 g**

The number e

e = the limit of $(1 + \frac{1}{n})^n \approx 2.71828$. Continuous growth: $A = Pe^{rt}$.

EX $2,000 at 5% for 10 years: $2000e^{0.5} \approx$ **$3,297.44**

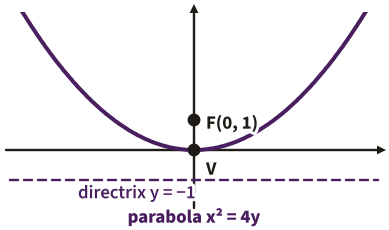
Log scales

Each step of 1 is a factor of 10: pH = −log[H^+], decibels, earthquake magnitude.

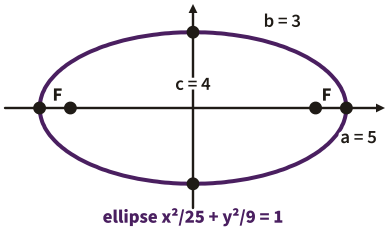
EX $[H^+] = 3.2 \times 10^{-4} \to$ pH ≈ **3.49**

7 CONIC SECTIONS

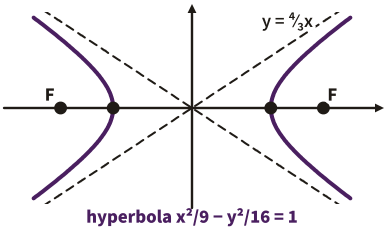
foci, eccentricity and general form (basic forms: Algebra §9)



parabola $x^2 = 4y$



ellipse $x^2/25 + y^2/9 = 1$



hyperbola $x^2/9 - y^2/16 = 1$

Foci and eccentricity (shift by (h, k) for other centers)

Conic (center or vertex at origin)	Focus relation	Eccentricity e	Defining distance
Parabola $x^2 = 4py$	focus (0, p), directrix y = −p	e = 1	to focus = to directrix
Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b	$c^2 = a^2 - b^2$, foci (±c, 0)	$e = \frac{c}{a} < 1$	sum to foci = 2a
Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2 + b^2$, foci (±c, 0)	$e = \frac{c}{a} > 1$	difference = 2a
Circle	both foci at the center	e = 0	to center = r

Ellipse from general form WORKED

$9x^2 + 25y^2 - 36x + 150y + 36 = 0$

$9(x-2)^2 + 25(y+3)^2 = 225$ *complete squares*

$\frac{(x-2)^2}{25} + \frac{(y+3)^2}{9} = 1$

center **(2, −3)**, c = 4: foci **(−2, −3), (6, −3)**, e = **0.8**

Parabola $y^2 = 12x$ WORKED

$4p = 12 \to p = 3$ *opens right*

focus **(3, 0)**, directrix **x = −3**

Hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ WORKED

$c = \sqrt{(9 + 16)} = 5$: foci **(±5, 0)**

e = $\frac{5}{3}$; asymptotes **$y = \pm\frac{4}{3}x$**

MEMORY TRICK **H is for Hypotenuse**

In a **h**yperbola, c is the longest, like a **h**ypotenuse: $c^2 = a^2 + b^2$. In an ellipse, a is the longest: $c^2 = a^2 - b^2$.

Classify $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

- B = 0: AC = 0 parabola; AC > 0 ellipse (circle if A = C); AC < 0 hyperbola.
- Any B: $B^2 - 4AC < 0$ ellipse, = 0 parabola, > 0 hyperbola.

Earth's orbit WORKED

a ≈ 149.6 million km, e ≈ 0.0167 *nearly a circle*

nearest a(1 − e) ≈ **147.1**, farthest a(1 + e) ≈ **152.1** million km

8 SYSTEMS & MATRICES

solve many equations at once (2×2 basics: Algebra §6)

Nonlinear system WORKED

$x^2 + y^2 = 25$, $y = x + 1$

$x^2 + (x+1)^2 = 25 \to x^2 + x - 12 = 0$ *substitute*

(3, 4) and **(−4, −3)**

Matrix arithmetic

- Size m × n = rows × columns. Add same-size matrices entry by entry.
- AB needs columns of A = rows of B; the result is rows(A) × cols(B).
- Row of A times column of B, summed. In general **AB ≠ BA**.

EX $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&0\\1&-1\end{bmatrix} = \begin{bmatrix}\mathbf{4}&\mathbf{-2}\\\mathbf{10}&\mathbf{-4}\end{bmatrix}$

2 × 2 determinant & inverse

For $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, det A = ad − bc. A^{-1} exists only if det A ≠ 0.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

MEMORY TRICK **Swap, Negate, Divide**

For a 2 × 2 inverse: **S**wap the main diagonal, **N**egate the other diagonal, **D**ivide everything by the determinant.

Solve with A^{-1} WORKED

$2x + y = 5$, $3x + 2y = 8$

det = 4 − 3 = 1, $A^{-1} = \begin{bmatrix}2&-1\\-3&2\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} = A^{-1}\begin{bmatrix}5\\8\end{bmatrix} = \begin{bmatrix}\mathbf{2}\\\mathbf{1}\end{bmatrix}$ *x = 2, y = 1*

Cramer's rule

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$: replace that variable's column with the constants.

EX same system: D = 1, $D_x = 10 - 8 = 2$, $D_y = 16 - 15 = 1 \to$ **(2, 1)**

3 × 3 determinant (row 1)

a(ei − fh) − b(di − fg) + c(dh − eg)

EX the coefficient matrix of the elimination system at right: 1(1 − 2) − 1(−2 − 1) + 1(4 + 1) = **7**

Row operations

- Swap two rows · multiply a row by a nonzero number · add a multiple of one row to another.
- Aim for a staircase (row-echelon form), then back-substitute.

3 × 3 by elimination WORKED

$\left[\begin{array}{ccc|c}1&1&1&6\\2&-1&1&3\\1&2&-1&2\end{array}\right]$ *augmented*

$R_2 - 2R_1$: −3y − z = −9; $R_3 - R_1$: y − 2z = −4

y = 2z − 4 → −7z + 12 = −9 → z = 3

(1, 2, 3) *check all three*

⚠ No unique answer

- A row 0 0 0 | c with c ≠ 0: **no solution**.
- A row of all zeros: **infinitely many** (dependent).
- det = 0 means one of these two.

9 PARAMETRIC EQUATIONS & THE TRIG TOOLKIT

x and y driven by a third variable

Parametric curves

$x = f(t)$, $y = g(t)$: each t gives a point, and increasing t gives the curve a direction.

- Line from (x_1, y_1) to (x_2, y_2): $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$
- Circle: $x = h + r\cos t$, $y = k + r\sin t$

Eliminate the parameter WORKED

$x = 2t + 1$, $y = t^2 \to t = \frac{x - 1}{2}$

$y = \frac{(x-1)^2}{4}$, a parabola

$x = 3\cos t$, $y = 2\sin t \to \frac{x^2}{9} + \frac{y^2}{4} = 1$

$\cos^2 + \sin^2 = 1$

Line segment WORKED

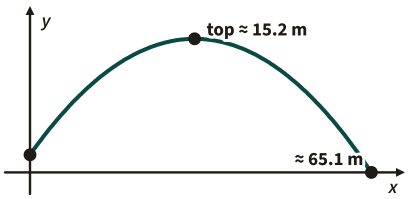
(1, 2) to (5, −6), $0 \le t \le 1$

$x = 1 + 4t$, $y = 2 - 8t$; $t = \frac{1}{2} \to$ **(3, −2)** *midpoint*

Projectile motion

$g \approx 9.8$ m/s² (32 ft/s²); air resistance ignored.

$$x = (v_0 \cos\theta)t$$
$$y = h_0 + (v_0 \sin\theta)t - \tfrac{1}{2}gt^2$$



$v_0 = 25$ m/s at 40°, launched 2 m up.

That launch WORKED

$x = 19.15t$, $y = 2 + 16.07t - 4.9t^2$

top at $t = \frac{16.07}{9.8} \approx 1.64$ s: height ≈ **15.2 m**

$y = 0$ at $t \approx$ **3.40 s**; range ≈ **65.1 m**

quadratic formula

Trig in Pre-Calculus (see the Trigonometry guide)

- Radians and the unit circle: Trig §1, §3
- Sinusoids $y = A\sin(B(x - C)) + D$, period $\frac{2\pi}{B}$: Trig §6
- Inverse trig and identities: Trig §7–9
- Vectors, $u \cdot v = |u||v|\cos\theta$: Trig §11
- Polar graphs; De Moivre $(r\,\mathrm{cis}\,\theta)^n = r^n\,\mathrm{cis}\,n\theta$: Trig §12–13

MEMORY TRICK **The puppeteer t**

One hand (t) works two strings: x and y move separately, and the curve is where the puppet goes.

Direction of travel

$x = \cos t$, $y = \sin t$ runs **counterclockwise** from (1, 0); $x = \sin t$, $y = \cos t$ runs **clockwise** from (0, 1). Same circle, different motion.

10 SEQUENCES, SERIES, INDUCTION & BINOMIALS

patterns, sums, proofs and $(a + b)^n$

Sequence a list $a_1, a_2, a_3, \ldots$
Series the sum of its terms
Recursive each term from the one before
Explicit a_n straight from n

Recursive to explicit WORKED

$a_1 = 3$, $a_n = 2a_{n-1} - 1$

3, 5, 9, 17, 33, … → $a_n = 2^n + 1$

Arithmetic & geometric (Algebra §11)

- $a_n = a_1 + (n - 1)d$; $S_n = \frac{n(a_1 + a_n)}{2}$
- $a_n = a_1 r^{n-1}$; $S_n = \frac{a_1(1 - r^n)}{1 - r}$
- Infinite geometric: $S = \frac{a_1}{1 - r}$ if $|r| < 1$, else no sum

Power sums

Sums run k = 1 to n. Constants and sums split: $\Sigma(ca_k + b_k) = c\Sigma a_k + \Sigma b_k$; $\Sigma c = cn$.

$$\Sigma k = \frac{n(n+1)}{2}$$
$$\Sigma k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\Sigma k^3 = \frac{n^2(n+1)^2}{4}$$

Use the formulas WORKED

$\Sigma_{k=1}^{20}(3k - 2) = 3 \cdot 210 - 2 \cdot 20 =$ **590**

$\Sigma_{k=1}^{10} k^2 = \frac{10 \cdot 11 \cdot 21}{6} =$ **385**

Repeating decimal WORKED

$0.2727\ldots = \frac{27}{100} + \frac{27}{10{,}000} + \cdots$ $r = \frac{1}{100}$

$= \frac{0.27}{1 - 0.01} = \frac{27}{99} =$ **$\frac{3}{11}$**

Bouncing ball WORKED

dropped 10 m, each bounce 60% as high

$10 + 2 \cdot \frac{6}{1 - 0.6} = 10 + 30 =$ **40 m** *up and down*

Mathematical induction

- **Base**: show the statement for n = 1.
- **Assume** it holds for n = k.
- **Show** it then holds for n = k + 1.

Prove $1 + 3 + \cdots + (2n - 1) = n^2$ WORKED

$n = 1$: $1 = 1^2$ ✓ *base*

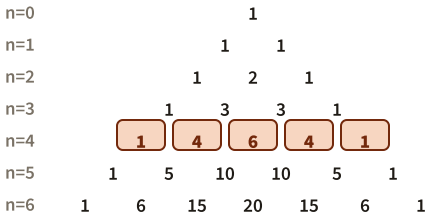
assume $1 + 3 + \cdots + (2k - 1) = k^2$

add $2k + 1$: $k^2 + 2k + 1 =$ **$(k + 1)^2$** ✓

next odd number

MEMORY TRICK **BAS: the dominoes**

Base knocks over the first domino; **A**ssume one falls; **S**how it knocks over the next. Then they all fall.



Pascal's triangle: each entry is the sum of the two above; row n holds ${}_nC_k$.

Binomial theorem

Term k + 1 is ${}_nC_k\, a^{n-k}b^k$.

$$(a + b)^n = \Sigma_{k=0}^{n}\, {}_nC_k\, a^{n-k}b^k$$

Expand $(2x - 1)^4$ WORKED

row 4: 1 4 6 4 1

$16x^4 - 32x^3 + 24x^2 - 8x + 1$

One term WORKED

x^5 term of $(x + 3)^8$: k = 3

${}_8C_3 \cdot 3^3 = 56 \cdot 27 =$ **$1{,}512x^5$**

MEMORY TRICK **Down, up, total n**

Powers of a count **down** from n, powers of b count **up** from 0, and each term's powers add to **n**.

11 COUNTING & PROBABILITY

how many ways, and how likely

Counting principle

m ways, then n ways: m · n ways in all.

EX 3 shirts, 4 pants, 2 pairs of shoes: **24** outfits

Arrangements & selections

n! = n(n − 1)⋯1, 0! = 1. P counts ordered lists; C counts groups.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

Order matters?	Example (from 10 people)	Count
yes: P	president, vice president, secretary	${}_{10}P_3$ = **720**
no: C	a committee of 3	${}_{10}C_3$ = **120**

MEMORY TRICK

P for Position, C for Committee

If swapping two people makes a **new** result (positions), use **P**. If not (a committee), use **C**.

Repeated letters WORKED

arrangements of BANANA — *3 A, 2 N*

$\frac{6!}{3!\,2!}$ = **60**

Probability rules

- $P(A) = \frac{\text{favorable outcomes}}{\text{all equally likely outcomes}}$; P(not A) = 1 − P(A)
- P(A or B) = P(A) + P(B) − P(A and B)
- Independent: P(A and B) = P(A) · P(B)

EX two dice, sum 7: $\frac{6}{36} = \frac{1}{6}$; at least one 6: $1 - (\frac{5}{6})^2 = \frac{11}{36}$

Probability by counting WORKED

6 red, 4 blue marbles; draw 2

P(both red) = $\frac{{}_6C_2}{{}_{10}C_2} = \frac{15}{45} = \frac{1}{3}$

12 LIMITS & CONTINUITY

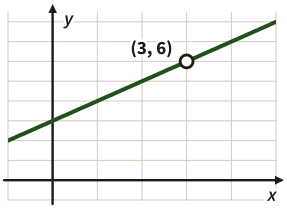
where a function is heading

What a limit means

$\lim_{x\to c} f(x) = L$: f(x) gets as close to L as we like for x close enough to c, from both sides. The value f(c) does not matter.

$\frac{x^2-9}{x-3}$ near 3 → 6

x	2.9	2.99	2.999	3.001	3.01	3.1
f(x)	5.9	5.99	5.999	6.001	6.01	6.1



$y = \frac{x^2-9}{x-3}$: a line with a hole at x = 3.

One-sided limits

The limit exists only if the left and right limits agree.

EX x + 1 for x < 2, x^2 for x ≥ 2: left **3**, right **4** → **no limit** (a jump)

MEMORY TRICK

Please Fix Really Carefully

Plug in first. If you get $\frac{0}{0}$: **F**actor, **R**ationalize, or **C**ombine fractions, then plug in again.

Factor WORKED

$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} (x+3) =$ **6**

Rationalize WORKED

$\lim_{x\to 0} \frac{\sqrt{(x+4)}-2}{x} = \lim_{x\to 0} \frac{1}{\sqrt{(x+4)}+2} = \frac{1}{4}$

multiply by the conjugate

Combine fractions WORKED

$\lim_{x\to 0} \frac{1/(x+2) - 1/2}{x} = \lim_{x\to 0} \frac{-1}{2(x+2)} = -\frac{1}{4}$

At infinity; a special limit

- Divide by the highest power: $\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \frac{3}{2}$ (the HA).
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (x in radians)

Continuity at x = c

f(c) is **defined**, $\lim_{x\to c} f(x)$ **exists**, and the two are **equal**.

Breaks: removable (hole), jump, infinite (vertical asymptote).

MEMORY TRICK **D-E-E**

Continuous means **D**efined, **E**xists, **E**qual: three checks, in that order.

13 FROM AVERAGE TO INSTANTANEOUS RATE

the first step into calculus

The derivative

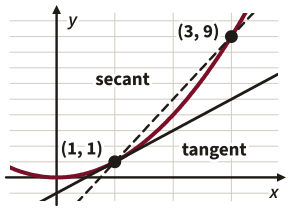
Secant slope (average rate, §1) → tangent slope (instantaneous rate) as h → 0.

$$ f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} $$

Power rule preview

$\frac{d}{dx} x^n = nx^{n-1}$, term by term (proved in Calculus).

EX $\frac{d}{dx}(3x^4 - 5x + 2) =$ **$12x^3 - 5$**



$y = x^2$: secant slope 4 from 1 to 3; tangent slope 2 at x = 1.

MEMORY TRICK

Zoom until straight

Zoom in on a smooth curve until it looks like a line. That line's slope is the derivative.

From the definition WORKED

$f(x) = x^2$: $\frac{(x+h)^2 - x^2}{h} = 2x + h \to$ **$f'(x) = 2x$**

at x = 3: slope 6, tangent **$y = 6x - 9$**

$y - 9 = 6(x - 3)$

Velocity WORKED

height $s(t) = -16t^2 + 64t$ (ft)

average on [0, 1]: $\frac{48 - 0}{1}$ = **48 ft/s**

$\frac{s(1+h) - s(1)}{h} = 32 - 16h \to$ **32 ft/s** at t = 1

instantaneous