TRIGONOMETRY

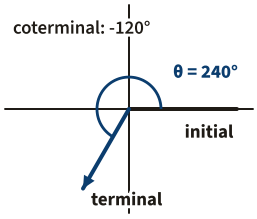
Grades 10–12 · 6 pages

Radians, right triangles, the unit circle, graphs, inverses, identities, equations, oblique triangles, vectors, polar coordinates and complex numbers.

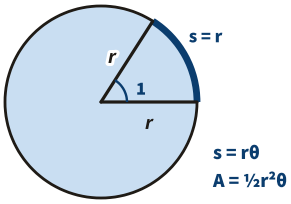
1 Angles · 2 Right triangles · 3 Unit circle · 4 Reference angles · 5 Graphs · 6 Sinusoids · 7 Inverses · 8 Identities · 9 Proofs & equations · 10 Oblique triangles · 11 Vectors · 12 Polar · 13 Complex

1 ANGLES, RADIANS & CIRCULAR MOTION

measure turning two ways



Standard position: vertex at the origin, initial side on the +x-axis; counterclockwise is positive.



1 radian: the arc equals the radius (≈ 57.30°).

Coterminal angles

Same terminal side: add or subtract full turns, $\theta \pm 360°k$ or $\theta \pm 2\pi k$.

EX $840° - 2\cdot360° = \mathbf{120°}$ · $-45° + 360° = \mathbf{315°}$

Degrees–minutes–seconds

$1° = 60'$, $1' = 60''$.

EX $25°30'36'' = 25 + \frac{30}{60} + \frac{36}{3600} = \mathbf{25.51°}$

⚠ Radians in formulas

$s = r\theta$, $A = \frac{1}{2}r^2\theta$ and $v = r\omega$ need θ in **radians**. Convert degrees first.

Converting

Degrees → radians: $\times \frac{\pi}{180}$. Radians → degrees: $\times \frac{180}{\pi}$.

$$\pi \text{ rad} = 180°$$

EX $150° = \frac{5\pi}{6} \cdot \frac{7\pi}{4} = \mathbf{315°} \cdot 1 \text{ rad} \approx \mathbf{57.30°}$

MEMORY TRICK **Want π? Put π on top**

Going **to** radians, multiply by $\frac{\pi}{180}$ (π upstairs). Going **to** degrees, use $\frac{180}{\pi}$ so the π cancels.

Arc length & sector area (θ in radians)

$$s = r\theta \cdot A = \tfrac{1}{2}r^2\theta$$

EX $r = 6$, $\theta = \frac{2\pi}{3}$: $s = \mathbf{4\pi \approx 12.57}$, $A = \mathbf{12\pi \approx 37.70}$

Angular & linear speed

- Angular speed $\omega = \frac{\theta}{t}$ (rad per unit time)
- Linear speed $v = \frac{s}{t} = \mathbf{r\omega}$
- 1 revolution = 2π rad

Bicycle wheel WORKED

$r = 0.35$ m at 300 rev/min

$\omega = 300 \cdot 2\pi \div 60 = 10\pi \approx \mathbf{31.42}$ rad/s — *rev → rad*

$v = 0.35 \cdot 10\pi \approx \mathbf{11.0}$ m/s — *$v = r\omega$*

Common conversions

Deg	Rad	Deg	Rad
30°	π/6	120°	2π/3
45°	π/4	180°	π
60°	π/3	270°	3π/2
90°	π/2	360°	2π

2 RIGHT-TRIANGLE TRIG & APPLICATIONS

six ratios, one triangle

The six ratios (acute θ)

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ · $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ · $\tan\theta = \frac{\text{opp}}{\text{adj}}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}}$ · $\sec\theta = \frac{\text{hyp}}{\text{adj}}$ · $\cot\theta = \frac{\text{adj}}{\text{opp}}$

SOH-CAH-TOA and the special triangles: Geometry Part 2 §12 and Part 1 §3.

MEMORY TRICK **One "co" per pair**

Each reciprocal pair has exactly one **co**: **co**secant ↔ sine, secant ↔ **co**sine, **co**tangent ↔ tangent.

8-15-17 triangle, θ opposite 8 WORKED

$\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$

$\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$ — *flip each*

Solve the triangle WORKED

$\theta = 28°$, hypotenuse 20

opp $= 20 \sin 28° \approx \mathbf{9.39}$; adj $= 20 \cos 28° \approx \mathbf{17.66}$

other acute angle = **62°**

Find an angle WORKED

opp 7, hyp 12

$\theta = \sin^{-1}(\frac{7}{12}) \approx \mathbf{35.69°}$ — *inverse sine*

Elevation & depression

Both are measured from the **horizontal**. The angle of depression from A to B equals the angle of elevation from B to A (alternate interior angles).

Cliff to boat WORKED

cliff 40 m, depression 12°

distance $= \frac{40}{\tan 12°} \approx \mathbf{188.2}$ m — *$\tan = \text{opp/adj}$*

Bearings

- **Compass bearing**: N 35° E means 35° east of north.
- **True bearing**: clockwise from north (N 35° E = 035°).

EX 12 km on N 35° E: east $12 \sin 35° \approx \mathbf{6.88}$ km, north $12 \cos 35° \approx \mathbf{9.83}$ km

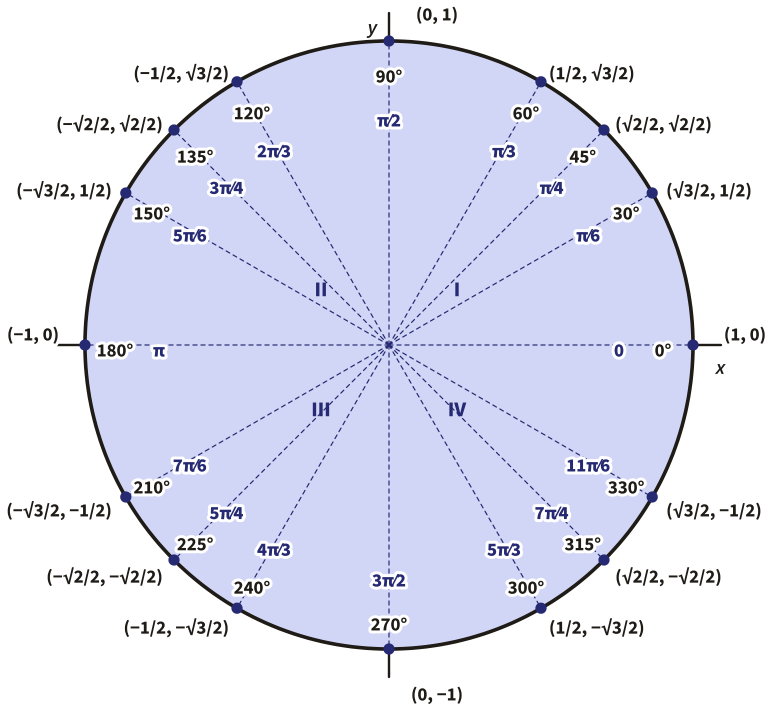
Cofunctions

$\sin\theta = \cos(90° - \theta)$, $\tan\theta = \cot(90° - \theta)$, $\sec\theta = \csc(90° - \theta)$: the two acute angles trade roles.

EX $\sin 28° = \cos 62°$

3 THE UNIT CIRCLE

every point is (cos θ, sin θ)



Inside: degrees, then radians. Outside: (cos θ, sin θ). Mirror quadrant I and change signs.

Definitions

For the point (x, y) where θ meets the circle $x^2 + y^2 = 1$:

$\cos\theta = \mathbf{x}$, $\sin\theta = \mathbf{y}$, $\tan\theta = \frac{y}{x}$

$\sec\theta = \frac{1}{x}$, $\csc\theta = \frac{1}{y}$, $\cot\theta = \frac{x}{y}$

For radius r: $x = r\cos\theta$, $y = r\sin\theta$.

MEMORY TRICK **Cos comes first**

Alphabetical and on the axes: **c**os before **s**in, like **x** before **y**.

Quadrant I values

θ	0	π/6	π/4	π/3	π/2
sin	0	1/2	√2/2	√3/2	1
cos	1	√3/2	√2/2	1/2	0
tan	0	√3/3	1	√3	—

MEMORY TRICK **Count up the roots**

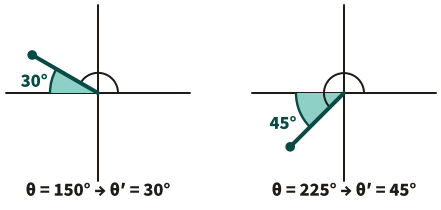
sin of 0°, 30°, 45°, 60°, 90° = $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$; cosine reads the list backward.

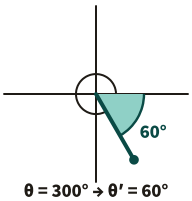
MEMORY TRICK **The left-hand trick**

Palm up, fingers = 0°, 30°, 45°, 60°, 90° from the thumb. Fold one down: $\sin = \frac{\sqrt{(\text{fingers below})}}{2}$, $\cos = \frac{\sqrt{(\text{fingers above})}}{2}$. (Traditional.)

4 REFERENCE ANGLES & SIGNS

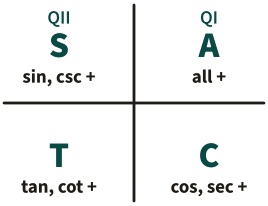
any angle, using quadrant I





Reference angle θ′ (acute, to the x-axis)

Quadrant	θ′ (degrees)	θ′ (radians)
I	θ	θ
II	180° − θ	π − θ
III	θ − 180°	θ − π
IV	360° − θ	2π − θ



Which functions are positive in each quadrant.

MEMORY TRICK

All Students Take Calculus

From QI counterclockwise: **A**ll positive, **S**ine (and csc), **T**angent (and cot), **C**osine (and sec). (Traditional.)

Three-step evaluation

- Find a coterminal angle in [0°, 360°).
- Find θ′ and its quadrant-I value.
- Attach the sign from ASTC.

Evaluate WORKED

$\sin 225°$: θ′ = 45°, QIII → $\mathbf{-\sqrt{2}/2}$	*sin < 0*
$\cos\frac{5\pi}{6}$: θ′ = $\frac{\pi}{6}$, QII → $\mathbf{-\sqrt{3}/2}$	*cos < 0*
$\tan(-60°) = \tan 300°$: QIV → $\mathbf{-\sqrt{3}}$	
$\sec 300° = \frac{1}{\cos 300°} = \frac{1}{1/2} = \mathbf{2}$	*QIV: cos > 0*
$\cos 765° = \cos 45° = \mathbf{\sqrt{2}/2}$	*765 − 720*

From one ratio to all WORKED

$\sin\theta = -\frac{5}{13}$, θ in QIII

$x = -\sqrt{(13^2 - 5^2)} = -12 \rightarrow \cos\theta = \mathbf{-\frac{12}{13}}$

Pythagoras

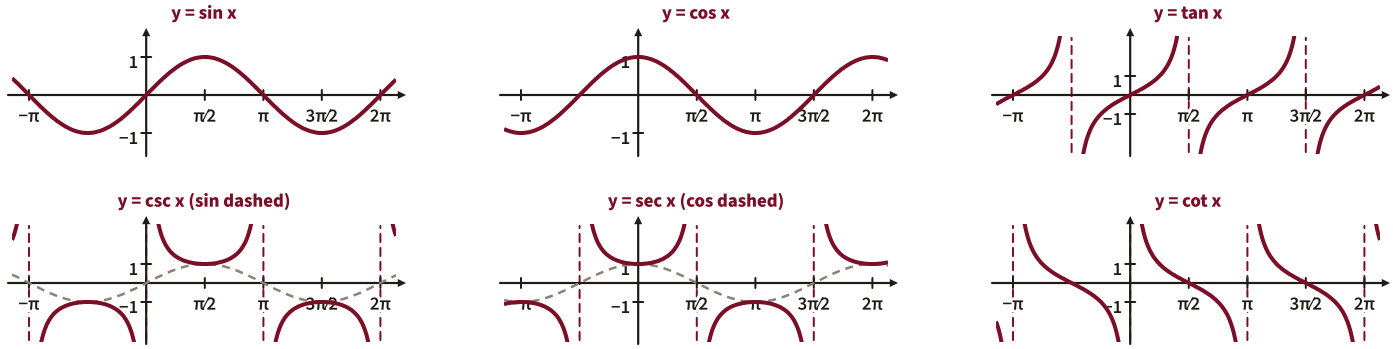
$\tan\theta = \frac{5}{12}$, $\csc\theta = -\frac{13}{5}$

⚠ Quadrantal angles

At 0°, 90°, 180°, 270° there is no reference triangle: read (cos, sin) straight off the axes. tan 90° and sec 90° are **undefined** (x = 0).

5 GRAPHS OF THE SIX FUNCTIONS

period, domain, range and asymptotes



Key facts (k any integer)

f	Period	Domain	Range	Zeros / asymptotes
sin	2π	all reals	$[-1, 1]$	zeros at $k\pi$
cos	2π	all reals	$[-1, 1]$	zeros at $\frac{\pi}{2} + k\pi$
tan	π	$x \neq \frac{\pi}{2} + k\pi$	all reals	zeros $k\pi$; asymptotes $\frac{\pi}{2} + k\pi$
csc	2π	$x \neq k\pi$	$(-\infty, -1] \cup [1, \infty)$	asymptotes $k\pi$
sec	2π	$x \neq \frac{\pi}{2} + k\pi$	$(-\infty, -1] \cup [1, \infty)$	asymptotes $\frac{\pi}{2} + k\pi$
cot	π	$x \neq k\pi$	all reals	zeros $\frac{\pi}{2} + k\pi$; asymptotes $k\pi$

MEMORY TRICK

Asymptotes live where the denominator dies

tan and sec have asymptotes where **cos = 0**; cot and csc where **sin = 0**.

Symmetry

- cos and sec are **even**: graphs mirror across the y-axis.
- sin, csc, tan, cot are **odd**: symmetric about the origin.

Reciprocal sketch

Draw sin (or cos) lightly; put asymptotes at its zeros; the csc (sec) branches touch its peaks and valleys and open away.

6 SINUSOIDS, TRANSFORMATIONS & HARMONIC MOTION

$y = A \sin(B(x - C)) + D$

The four numbers

Part	Meaning	Formula
$\lvert A\rvert$	amplitude (half the height)	$\frac{\max - \min}{2}$
B	cycles in 2π	period $\frac{2\pi}{B}$ (tan, cot $\frac{\pi}{B}$)
C	phase shift (right if +)	factor B out first
D	midline $y = D$	$\frac{\max + \min}{2}$

MEMORY TRICK **Big B, busy wave**

A bigger **B** squeezes more cycles in: the period **shrinks** to $2\pi/B$.

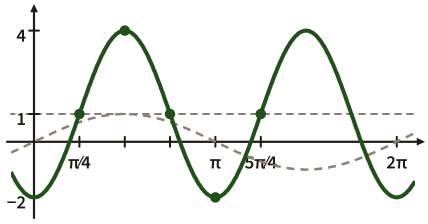
Write the equation WORKED

max 18 at $t = 2$, min 4 at $t = 8$

$A = \frac{18 - 4}{2} = 7$, $D = \frac{18 + 4}{2} = 11$

period $2(8 - 2) = 12 \rightarrow B = \frac{2\pi}{12} = \frac{\pi}{6}$

$y = \mathbf{7\cos(\frac{\pi}{6}(t - 2)) + 11}$ *cos starts at a max*



$y = 3\sin(2x - \pi/2) + 1$ (solid) over $y = \sin x$ (dashed); dots = five key points.

Read the graph WORKED

$3\sin(2x - \frac{\pi}{2}) + 1 = 3\sin(2(x - \frac{\pi}{4})) + 1$

factor B first: shift π/4, not π/2

amplitude **3**, period $\frac{2\pi}{2} = \boldsymbol{\pi}$

shift $\frac{\pi}{4}$ **right**, midline **y = 1**; max 4, min −2

Five key points

Split one period into quarters starting at the shift: midline, max, midline, min, midline (sine). Cosine starts at the max.

EX $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4} \rightarrow y = 1, 4, 1, -2, 1$

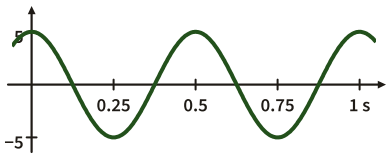
MEMORY TRICK **A-B-C-D**

Amplitude, **B** sets the period, **C** shifts sideways, **D** moves the midline: read them in alphabetical order.

Simple harmonic motion

$d = a\sin\omega t$ or $d = a\cos\omega t$: amplitude $|a|$, period $\frac{2\pi}{\omega}$, frequency $f = \frac{\omega}{2\pi}$ (cycles per unit time).

EX $d = 5\cos 4\pi t$ cm: amplitude **5 cm**, period **0.5 s**, frequency **2 Hz**.



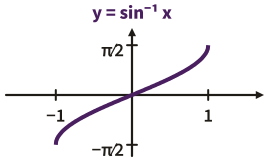
$d = 5\cos 4\pi t$: two full cycles each second.

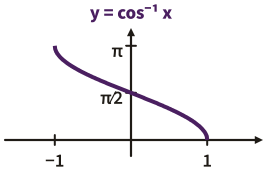
Explore

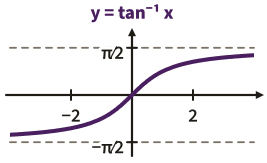
Add sine waves and you can draw almost any curve: Fourier: circles that draw (schoollaunch.org).

7 INVERSE TRIG FUNCTIONS

angles from ratios, on restricted ranges







Domains and ranges

Function	Domain	Range (the answer)	Quadrants
$\sin^{-1} x$ (arcsin)	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	I, IV
$\cos^{-1} x$ (arccos)	$[-1, 1]$	$[0, \pi]$	I, II
$\tan^{-1} x$ (arctan)	all reals	$(-\frac{\pi}{2}, \frac{\pi}{2})$	I, IV
$\cot^{-1} x$	all reals	$(0, \pi)$	I, II
$\sec^{-1} x$	$\lvert x\rvert \geq 1$	$[0, \pi]$, $\neq \frac{\pi}{2}$	I, II
$\csc^{-1} x$	$\lvert x\rvert \geq 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\neq 0$	I, IV

MEMORY TRICK

Sine & tangent answer right, cosine answers up

$\sin^{-1}$ and $\tan^{-1}$ give angles on the **right** half (QI or QIV); $\cos^{-1}$ gives angles on the **top** half (QI or QII).

Evaluate WORKED

$\sin^{-1}(-\frac{1}{2}) = \mathbf{-\frac{\pi}{6}}$ — *QIV, not 7π/6*

$\cos^{-1}(-\frac{\sqrt{2}}{2}) = \mathbf{\frac{3\pi}{4}}$ — *QII*

$\tan^{-1}(\sqrt{3}) = \mathbf{\frac{\pi}{3}}$

⚠ **$\sin^{-1} x \neq 1/\sin x$**

The −1 means **inverse function**. The reciprocal is $(\sin x)^{-1} = \csc x$.

Compositions WORKED

$\sin(\sin^{-1} 0.3) = \mathbf{0.3}$ — *inside the domain*

$\sin^{-1}(\sin\frac{5\pi}{6}) = \sin^{-1}(\frac{1}{2}) = \mathbf{\frac{\pi}{6}}$ — *not 5π/6*

$\cos(\tan^{-1}\frac{3}{4}) = \mathbf{\frac{4}{5}}$ — *draw a 3-4-5 triangle*

Triangle method

For $\cos(\tan^{-1} u)$: let $\theta = \tan^{-1} u$, draw a right triangle with $\tan\theta = u$, find the third side, read the ratio. Signs follow the range.

8 IDENTITIES

true for every angle where both sides are defined

Reciprocal & quotient

$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean

÷ $\cos^2\theta$: $\mathbf{\tan^2\theta + 1 = \sec^2\theta}$ · ÷ $\sin^2\theta$: $\mathbf{1 + \cot^2\theta = \csc^2\theta}$

$$\mathbf{\sin^2\theta + \cos^2\theta = 1}$$

MEMORY TRICK

One identity, three copies

Learn $\sin^2 + \cos^2 = 1$; divide by $\cos^2$ or $\sin^2$ to get the other two.

Even / odd & cofunction

- $\cos(-\theta) = \cos\theta$, $\sec(-\theta) = \sec\theta$ (**even**)
- $\sin(-\theta) = -\sin\theta$; tan, csc, cot also odd
- $\sin(\frac{\pi}{2} - \theta) = \cos\theta$, $\tan(\frac{\pi}{2} - \theta) = \cot\theta$, $\sec(\frac{\pi}{2} - \theta) = \csc\theta$

Product-to-sum

- $\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- $\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

Sum & difference

- $\mathbf{\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B}$
- $\mathbf{\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B}$
- $\mathbf{\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}}$

EX $\sin 75° = \sin(45° + 30°) = \mathbf{\frac{\sqrt{6} + \sqrt{2}}{4}} \approx \mathbf{0.9659}$; $\tan 15° = \tan(45° - 30°) = \mathbf{2 - \sqrt{3}} \approx \mathbf{0.2679}$

MEMORY TRICK

Sine mixes, cosine matches

Sine **mixes** (sin·cos) and **keeps** the sign; cosine **matches** (cos·cos, sin·sin) and **flips** it.

Double angle

$\mathbf{\cos 2\theta} = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$$\mathbf{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$\mathbf{\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

EX $\sin\theta = \frac{3}{5}$ in QII, so $\cos\theta = -\frac{4}{5}$: $\sin 2\theta = \mathbf{-\frac{24}{25}}$, $\cos 2\theta = 1 - 2(\frac{9}{25}) = \mathbf{\frac{7}{25}}$

Simplify WORKED

$\frac{\sin 2x}{1 + \cos 2x} = \frac{2\sin x\cos x}{2\cos^2 x} = \mathbf{\tan x}$

double angle ×2

Half angle (sign from the quadrant of θ/2)

$\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$

$$\mathbf{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}}$$

$$\mathbf{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}}$$

EX $\cos 22.5° = \sqrt{\frac{1 + \sqrt{2}/2}{2}} = \frac{\sqrt{(2 + \sqrt{2})}}{2} \approx \mathbf{0.9239}$

Power reducing

$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$

$$\mathbf{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$\mathbf{\cos^2 x = \frac{1 + \cos 2x}{2}}$$

Sum-to-product

- $\sin A + \sin B = 2\sin\frac{A + B}{2}\cos\frac{A - B}{2}$
- $\sin A - \sin B = 2\cos\frac{A + B}{2}\sin\frac{A - B}{2}$
- $\cos A + \cos B = 2\cos\frac{A + B}{2}\cos\frac{A - B}{2}$
- $\cos A - \cos B = -2\sin\frac{A + B}{2}\sin\frac{A - B}{2}$

9 PROVING IDENTITIES & SOLVING EQUATIONS

an identity is always true; an equation is true for some angles

Proof strategy

- Work on **one side** only (usually the messier one).
- Rewrite in sin and cos; combine fractions; factor.
- Use a Pythagorean identity to trade $\sin^2$ and $\cos^2$.
- Multiply by a conjugate $(1 \pm \sin x)$ when you see $1 \pm \sin x$.

MEMORY TRICK

Never cross the equals sign

In a proof, do not add, multiply or square both sides: transform one side until it **matches** the other.

Prove sec x − cos x = sin x tan x WORKED

$\sec x - \cos x = \frac{1}{\cos x} - \cos x$ *reciprocal*

$= \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$ *Pythagorean*

$= \sin x \cdot \frac{\sin x}{\cos x} =$ **sin x tan x** ✓ *quotient*

Prove $(1 - \cos^2 x)(1 + \cot^2 x) = 1$ WORKED

$= \sin^2 x \cdot \csc^2 x$ *two Pythagorean forms*

$= \sin^2 x \cdot \frac{1}{\sin^2 x} =$ **1** ✓

Solving steps

- Isolate the trig function (or factor).
- Find solutions in one period, using reference angles and ASTC.
- General solution: add $2\pi k$ (sin, cos, sec, csc) or πk (tan, cot).

Linear WORKED

$2 \sin x - 1 = 0 \to \sin x = \frac{1}{2}$

$[0, 2\pi)$: $x =$ **$\frac{\pi}{6}, \frac{5\pi}{6}$** *QI, QII*

all: $\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$

Quadratic WORKED

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$ *factor*

$\cos x = 1$ or $-\frac{1}{2} \to x =$ **0, $\frac{2\pi}{3}, \frac{4\pi}{3}$**

Use an identity WORKED

$\sin 2x = \cos x \to \cos x(2 \sin x - 1) = 0$ *double angle*

$x =$ **$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$**

Multiple angle WORKED

$\tan 3x = 1$ on $[0, \pi)$

$3x = \frac{\pi}{4} + k\pi \to x = \frac{\pi}{12} + \frac{k\pi}{3}$

$x =$ **$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$**

Calculator angle WORKED

$\sin x = 0.4$ on $[0, 2\pi)$

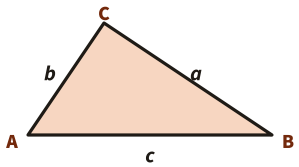
$x \approx$ **0.4115** and $\pi - 0.4115 \approx$ **2.7301** *QI and QII*

⚠ Lost and fake solutions

- Do not divide by cos x: factor it out, or you lose cos x = 0.
- After squaring both sides, check every answer in the original.

10 OBLIQUE TRIANGLES & AREA

any triangle, no right angle needed



Side a faces angle A, and so on.

Law of Sines (AAS, ASA, SSA)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

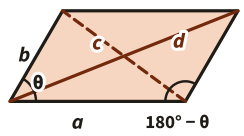
Law of Cosines (SAS, SSS)

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Parallelogram law

$$c^2 + d^2 = 2(a^2 + b^2)$$



Law of Cosines on c (angle θ) and d (180° − θ); $\cos(180° - \theta) = -\cos\theta$, so the 2ab cos θ terms cancel. **EX** 5, 3, 60°: 19 + 49 = 2(25 + 9) ✓

MEMORY TRICK **Sines need a pair**

Sines need a **matched pair**: a side and its opposite angle. No pair (SAS, SSS)? Use cosines.

ASA: A = 40°, B = 75°, a = 10 WORKED

C = 180° − 40° − 75° = **65°**

$b = \frac{10 \sin 75°}{\sin 40°} \approx$ **15.03**; c ≈ **14.10** *law of sines*

SSS: 7, 9, 12 (largest angle first) WORKED

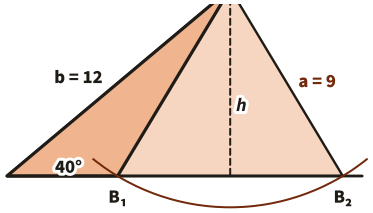
$\cos C = \frac{49 + 81 - 144}{126} = -\frac{1}{9}$

C ≈ **96.38°** (obtuse)

SAS WORKED

b = 10, c = 14, A = 35°

$a^2 = 100 + 196 - 280 \cos 35° \to a \approx$ **8.16**



A = 40°, b = 12, a = 9: side a lands at B_1 or B_2.

SSA: the ambiguous case (A obtuse: 1 if a > b, else 0)

A acute, h = b sin A	Triangles
a < h	**0**
a = h	1 (right)
h < a < b	**2**
a ≥ b	1

Two triangles WORKED

$h \approx 7.71 < 9 < 12$; $\sin B = \frac{12 \sin 40°}{9} \approx 0.8571$

B ≈ **58.99°** (C ≈ 81.01°, c ≈ 13.83) or **121.01°** (C ≈ 18.99°, c ≈ 4.56)

Area

ASA: $K = \frac{a^2 \sin B \sin C}{2 \sin A}$

$$K = \tfrac{1}{2}ab \sin C$$

Heron's formula (SSS)

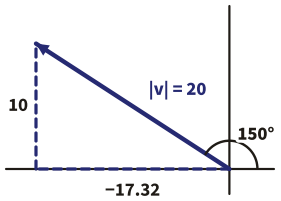
$$K = \sqrt{(s(s-a)(s-b)(s-c))}$$

$$s = \frac{a + b + c}{2}$$

EX 7, 9, 12: s = 14, $K = \sqrt{(14 \cdot 7 \cdot 5 \cdot 2)} = 14\sqrt{5} \approx$ **31.30** (= ½·7·9·sin C ✓)

11 VECTORS

magnitude and direction



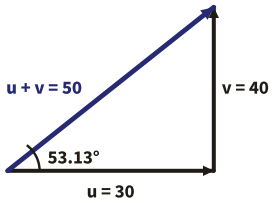
$v = \langle 20\cos 150°, 20\sin 150° \rangle = \langle -17.32, 10 \rangle$.

Component form

- $v = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$
- $|v| = \sqrt{(a^2 + b^2)}$; direction $\tan\theta = \frac{b}{a}$ (check the quadrant)
- $v = \langle |v|\cos\theta, |v|\sin\theta \rangle$; unit vector $\frac{v}{|v|}$

Operations

- Add/subtract componentwise; kv scales the length by $|k|$.
- Dot product: $u \cdot v = u_1v_1 + u_2v_2 = |u||v|\cos\theta$
- $u \perp v$ exactly when $u \cdot v = 0$



Tip to tail (memory trick): start the second where the first ends. 30 + 40 → 50 at 53.13°.

Angle between WORKED

$u = \langle 3, 4 \rangle$, $v = \langle 5, -12 \rangle$

$u \cdot v = 15 - 48 = -33$; $|u| = 5$, $|v| = 13$

$\cos\theta = -\frac{33}{65} \to \theta \approx$ **120.51°**

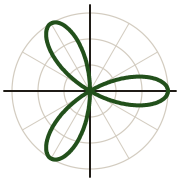
Plane and wind WORKED

plane $\langle 0, 200 \rangle$ mph + wind $\langle 30, 0 \rangle = \langle 30, 200 \rangle$ *add*

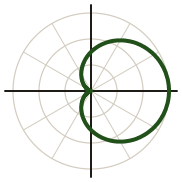
speed ≈ **202.24 mph**, bearing $\tan^{-1}(\frac{30}{200}) \approx$ **N 8.53° E**

12 POLAR COORDINATES & GRAPHS

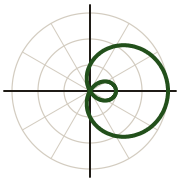
distance and angle instead of x and y



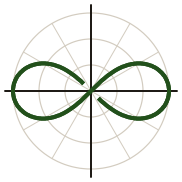
$r = 3\cos 3\theta$



$r = 1.5(1 + \cos\theta)$



$r = 1 + 2\cos\theta$



$r^2 = 9\cos 2\theta$

Conversions

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

Convert WORKED

$(4, \frac{2\pi}{3}) \to (4\cos 120°, 4\sin 120°) =$ **$(-2, 2\sqrt{3})$**

$(-3, 3) \to r = 3\sqrt{2} \approx 4.24$, $\theta =$ **$\frac{3\pi}{4}$** *QII*

Many names, one point

$(r, \theta) = (r, \theta + 2\pi k) = (-r, \theta + \pi)$. A negative r points the opposite way.

EX $r = 4\cos\theta \to r^2 = 4r\cos\theta \to x^2 + y^2 = 4x$: circle, center (2, 0), radius 2

MEMORY TRICK

Odd roses keep, even roses double

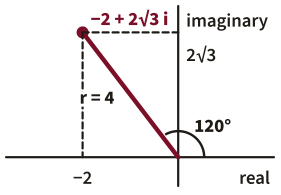
$r = a\cos n\theta$: odd n gives **n** petals, even n gives **2n**.

Polar graph families

Equation	Graph
$r = a$; $\theta = \alpha$	circle at the pole; line through the pole
$r = a\cos\theta$, $r = a\sin\theta$	circle of diameter $\|a\|$ through the pole
$r = a\cos n\theta$	rose: n petals (n odd), 2n (n even)
$r = a(1 \pm \cos\theta)$	cardioid
$r = a + b\cos\theta$	limaçon; inner loop if $\|a\| < \|b\|$
$r^2 = a^2\cos 2\theta$	lemniscate
$r = a\theta$	spiral

13 COMPLEX NUMBERS & DE MOIVRE

rotate and scale in the complex plane



$-2 + 2\sqrt{3}\,i = 4\,\text{cis}\,120°$.

Trigonometric (polar) form

$r = |z| = \sqrt{(a^2 + b^2)}$, $\tan\theta = \frac{b}{a}$ (check the quadrant)

$$z = a + bi$$

$$= r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$$

Multiply & divide

- $z_1z_2 = r_1r_2\,\text{cis}(\theta_1 + \theta_2)$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$

EX $2\,\text{cis}\,30° \cdot 3\,\text{cis}\,60° = 6\,\text{cis}\,90° =$ **6i**

MEMORY TRICK

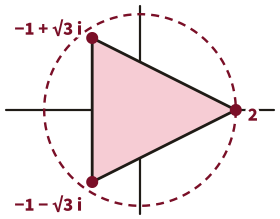
Multiply lengths, add angles

Multiplying complex numbers **multiplies** their lengths and **adds** their angles.

De Moivre's theorem

$$[r\,\text{cis}\,\theta]^n = r^n\,\text{cis}\,n\theta$$

EX $(1 + i)^8 = (\sqrt{2}\,\text{cis}\,45°)^8 = 16\,\text{cis}\,360° =$ **16**



Cube roots of 8, 120° apart.

nth roots

The n roots of $r\,\text{cis}\,\theta$: $\sqrt[n]{r} \cdot \text{cis}(\frac{\theta + 360°k}{n})$, k = 0, 1, …, n − 1. They sit evenly around a circle.

EX $8 = 8\,\text{cis}\,0° \to 2\,\text{cis}\,0°, 2\,\text{cis}\,120°, 2\,\text{cis}\,240°$ = **$2, -1 \pm \sqrt{3}\,i$**